Cluster Tilting for Representation-Directed Algebras

Laertis Vaso

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This licentiate dissertation consists of the following two papers, referred to by their roman numerals.


INTRODUCTION

An (associative) algebra over a field $K$ is a unital ring which is also a $K$-vector space such that the ring multiplication is compatible with multiplication with scalars. From the point of view of representation theory, one way of studying a finite-dimensional algebra $\Lambda$ is to describe the category $\text{mod} \Lambda$ of finitely generated $\Lambda$-modules. Although describing $\text{mod} \Lambda$ can be very hard in general, there are some cases which are easier to work with. One such case is when $\Lambda$ is representation-finite, that is when there exist finitely many indecomposable $\Lambda$-modules up to isomorphism. In this case, classical Auslander-Reiten theory, developed by Auslander and Reiten in the late 1970’s, gives a combinatorial approach to describing all $\Lambda$-modules and all morphisms between them. Another case where we can obtain some results is when $\Lambda$ is hereditary, that is when every submodule of a projective module is projective. Expressed in another language, the hereditary algebras are those which have global dimension at most 1. In particular, the module categories of algebras which are both representation-finite and hereditary are very well understood.

More recently, Iyama introduced a higher-dimensional analogue of Auslander-Reiten theory for algebras of arbitrary global dimension, based on replacing the focus from the whole module category $\text{mod} \Lambda$ to a subcategory $C \subset \text{mod} \Lambda$ with suitable homological properties called an $n$-cluster tilting subcategory ([Iya06], [Iya07], [Iya08]). For $n > 1$ it is unknown if there exists an $n$-cluster tilting subcategory $C$ containing infinitely many nonisomorphic indecomposable modules and so all known examples can be thought of as higher dimensional versions of representation-finite algebras. When an $n$-cluster tilting subcategory $C$ exists, one can use techniques analogous to classical Auslander-Reiten to study $C$.

If $\text{gl. dim} \Lambda = d$, then we have that $n > d$ implies that $\Lambda$ is semisimple, and so we can restrict to the case $n \leq d$. If in particular there exists a $d$-cluster tilting subcategory $C$, then $C$ can be thought of as a generalization of the module category of a hereditary algebra.

Note that the existence of an $n$-cluster tilting subcategory, for any $n \leq d$, is far from guaranteed. Moreover, there are many known examples in the literature concerning the case $n = d$, while there are very few examples known when $n < d$. This is the main motivation for the work in this dissertation.

The main aim of this dissertation is to find examples of $n$-cluster tilting subcategories for algebras of global dimension $d$, for any possible pair $(n, d)$ with $n \leq d$. The first step to managing this is to find new examples of $n$-cluster tilting subcategories in general. Since the problem of the existence of an $n$-cluster tilting subcategory
for an arbitrary algebra $\Lambda$ is hopeless, we restrict ourselves to a particular class of algebras called *representation-directed algebras*. These are algebras for which there are no cycles of nonzero nonisomorphisms between indecomposable $\Lambda$-modules. In particular these are representation-finite and hence they are quite well understood. The second step is to extrapolate from these new examples a way that allows us to find even more examples but where the global dimension can be easily computed.

In paper I we give a characterization of $n$-cluster tilting subcategories for representation-directed algebras based on the so-called $n$-*Auslander-Reiten translations*. This characterization allows us to reduce the problem of existence of $n$-cluster tilting subcategories for representation-directed algebras to classical Auslander-Reiten theory. We apply this characterization to classify acyclic Nakayama algebras with homogeneous relations which admit an $n$-cluster tilting subcategory in terms of a simple numerical equation. Furthermore, using this classification we also find all Nakayama algebras of global dimension $d < \infty$ which admit a $d$-cluster tilting subcategory.

In paper II we introduce a method to iteratively construct $n$-cluster tilting subcategories for representation-directed algebras. The first step in doing this is defining the process of *gluing* of representation-directed algebras. This process takes us inputs two representation-directed algebras $A$ and $B$, a uniserial projective $A$-module $P$ and a uniserial injective $B$-module $I$ satisfying certain conditions, and it outputs a new representation-directed algebra $\Lambda := B^P \bowtie^I A$, with the advantage that the representation theory of $\Lambda$ is completely described by the representation theories of $A$ and $B$. The second step is introducing the concept of $n$-fractured subcategories. These are defined by generalizing the characterization of $n$-cluster tilting subcategories for representation-directed algebras which we obtained in paper I. Under reasonable compatibility conditions, we show that if $A$ and $B$ admit $n$-fractured subcategories, then the glued algebra $\Lambda := B^P \bowtie^I A$ also admits an $n$-fractured subcategory. Under some conditions, repeatedly gluing algebras results in an algebra with an honest $n$-cluster tilting subcategory. One such case is when the $n$-fractured subcategories of $A$ and $B$ are actual $n$-cluster tilting subcategories and the modules $P$ and $I$ are simple. In this case, computing the global dimension of $\Lambda$ is easy. As a result, we show that if $n$ is odd and $d \geq n$ then there exists an algebra admitting an $n$-cluster tilting subcategory and having global dimension $d$. We show the same result if $n$ is even and $d$ is odd or $d \geq 2n$.

It should be noted that except for providing the aforementioned examples where the interest lies in the pair $(n, d)$, the methods developed in this licentiate dissertation can be used to provide examples of algebras with many other interesting properties that also admit $n$-cluster tilting subcategories. One such example is the existence of a bound quiver algebra $\Lambda = KQ/I$ such that $Q$ has an arbitrary number of sinks and sources and $\Lambda$ admits a 2-cluster tilting subcategory.

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n-Cluster tilting subcategories of representation-directed algebras

Laertis Vaso

Department of Mathematics, Uppsala University, P.O. Box 480, 751 06 Uppsala, Sweden

ABSTRACT

We give a characterization of n-cluster tilting subcategories of representation-directed algebras based on the n-Auslander–Reiten translations. As an application we classify acyclic Nakayama algebras with homogeneous relations which admit an n-cluster tilting subcategory. Finally, we classify Nakayama algebras of global dimension d < ∞ which admit a d-cluster tilting subcategory.

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1. Introduction

In representation theory of finite dimensional algebras, one aims to understand the modules over an algebra and the homomorphisms between them. In the case of a representation-finite algebras, classical Auslander–Reiten theory gives a complete picture of the module category, see for example [1]. In Osamu Iyama’s higher-dimensional Auslander–Reiten theory, introduced in [6] and [7], one replaces the module category with a subcategory with suitable homological properties called an n-cluster tilting subcategory, where n is a positive integer.

If an n-cluster tilting subcategory exists, it behaves similarly to the module category from the perspective of Auslander–Reiten theory. In particular, it contains all the projective and injective modules and there are many higher-dimensional analogues of classical notions. For instance, n-almost split sequences and the n-Auslander–Reiten translations τ_n and τ^-n become almost split sequences and the Auslander–Reiten translations τ and τ^- when n = 1.

If an n-cluster tilting subcategory admits an additive generator M, then M is called an n-cluster tilting module and we say that the algebra is weakly n-representation-finite. If moreover n is equal to the global dimension d of the algebra, the d-cluster tilting subcategory is unique and we say that the algebra is d-representation-finite. In Theorem 3.1 of [5] it is shown that d-representation-finite algebras play the role of hereditary representation-finite algebras in higher-dimensional Auslander–Reiten theory.

E-mail address: laertis.vaso@math.uu.se.

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Since the existence of an \( n \)-cluster tilting subcategory is far from guaranteed, it is natural to ask under which conditions an \( n \)-cluster tilting subcategory exists. We study this question in the case of representation-directed algebras and give the following characterization.

**Theorem 1.** Assume \( \Lambda \) is a representation-directed algebra and let \( \mathcal{C} \) be a full subcategory of \( \text{mod}\Lambda \), closed under direct sums and summands. Denote by \( \mathcal{C}_P \) and \( \mathcal{C}_I \) the sets of isomorphism classes of indecomposable nonprojective respectively noninjective \( \Lambda \)-modules in \( \mathcal{C} \). Then \( \mathcal{C} \) is an \( n \)-cluster tilting subcategory if and only if the following conditions hold:

1. \( \Lambda \in \mathcal{C} \),
2. \( \tau_n \) and \( \tau_n^{-1} \) induce mutually inverse bijections

\[
\begin{align*}
\mathcal{C}_P & \leftrightarrow \mathcal{C}_I, \\
\tau_n & \leftrightarrow \tau_n^{-1}
\end{align*}
\]

3. \( \Omega^i M \) is indecomposable for all \( M \in \mathcal{C}_P \) and \( 0 < i < n \),
4. \( \Omega^{-i} N \) is indecomposable for all \( N \in \mathcal{C}_I \) and \( 0 < i < n \).

**Remark 1.** Let us make three remarks about Theorem 1:

(a) (1) and (2) are known to be necessary for any finite dimensional algebra ([7], Theorem 2.8). Moreover, (3) and (4) are also necessary for any finite dimensional algebra by Corollary 3.3.

(b) Let \( \mathcal{C} \) be an \( n \)-cluster tilting subcategory of \( \text{mod}\Lambda \) where \( \Lambda \) is representation-directed, and \( M \in \mathcal{C} \) be indecomposable. By representation-directedness, (2)–(4) imply that \( \tau_n^{-i} M = 0 \) and \( \tau_j^i M = 0 \) for \( i \) and \( j \) large enough. Then (2) implies that \( M = \tau_n^{-N} P \) for some projective indecomposable module \( P \) and some \( N \geq 0 \). Using (1) and (2) we conclude that \( \mathcal{C} = \text{add} \left( \bigoplus_{i \geq 0} (\tau_n^{-r} \Lambda) \right) \).

(c) We do not know of any examples where conditions (1) and (2) hold and either of conditions (3) and (4) fails.

As an application, we characterize the acyclic Nakayama algebras with homogeneous relations which admit an \( n \)-cluster tilting subcategory.

**Theorem 2.** Let \( Q_m \) be the quiver

\[
Q_m : \quad m \xrightarrow{a_{m-1}} m - 1 \xrightarrow{a_{m-2}} m - 2 \xrightarrow{a_{m-3}} \cdots \xrightarrow{a_2} 2 \xrightarrow{a_1} 1.
\]

Then \( KQ_m / (\text{rad} \ KQ_m)^1 \) admits an \( n \)-cluster tilting subcategory if and only if \( l = 2 \) and \( m = nk + 1 \) for some \( k \geq 0 \) or \( n \) is even and \( m = 2l + 1 + k(nl - l + 2) \) for some \( k \geq 0 \).

Cyclic Nakayama algebras with homogeneous relations which admit \( n \)-cluster tilting subcategories are classified by Darpö and Iyama in [4]. The case \( l = 2 \) in Theorem 2 was first considered by Jasso in [8], Proposition 6.2. Moreover Iyama and Oppermann completely classify 2-representation finite acyclic Nakayama algebras in [5], Theorem 3.12. It turns out that \( d \)-representation-finite Nakayama algebras arise only as acyclic Nakayama algebras with homogeneous relations. Therefore, we also give a complete classification of \( d \)-representation-finite Nakayama algebras.

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Theorem 3. Let $\Lambda$ be a Nakyama algebra of global dimension $d < \infty$. The following are equivalent.

(i) $\Lambda$ is $d$-representation-finite.
(ii) $\Lambda = KQ_m/(\text{rad } KQ_m)^l$ and $d$ is even or $l = 2$.
(iii) $\Lambda = KQ_m/(\text{rad } KQ_m)^l$ and $l \mid m - 1$ or $l = 2$.

Then $d = 2^{m-1}$.

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2. Preliminaries

Throughout the paper, $K$ will be a field and $\Lambda$ a finite dimensional unital associative $K$-algebra. We denote by $\mod \Lambda$ the category of finitely generated right $\Lambda$-modules and in the following we say module instead of right $\Lambda$-module. We will denote by $d$ the global dimension of $\Lambda$ and by $D$ the duality $\Hom(\cdot, K)$.

Recall that if $M$ is an indecomposable nonprojective module, then there exists an almost split sequence

$$0 \to \tau M \to E \to M \to 0$$

in $\mod \Lambda$ and, similarly, if $N$ is an indecomposable noninjective module, then there exists an almost split sequence

$$0 \to N \to F \to \tau^- N \to 0$$

where $\tau$ and $\tau^-$ are the Auslander–Reiten translations. In particular, we have the Auslander–Reiten formulas

$$\Ext^1_\Lambda(M, N) \cong D\Hom(\tau^- N, M) \cong D\Hom(N, \tau M).$$

For further details we refer to chapter IV in [3].

Let $X \in \mod \Lambda$. We will denote by $\Omega X$ the syzygy of $X$, that is the kernel of $P \to X$, where $P$ is the projective cover of $X$ and by $\Omega^- X$ the cosyzygy of $X$, that is the cokernel of $X \to I$ where $I$ is the injective hull of $X$. Note that $\Omega X$ and $\Omega^- X$ are unique up to isomorphism. Following [7], we denote by $\tau_n$ and $\tau_n^-$ the $n$-Auslander–Reiten translations defined by $\tau_n X = \tau(\Omega^{n-1} X)$ and $\tau_n^- X = \tau^-(\Omega^{(n-1)} X)$.

In this paper, all subcategories considered will be full and closed under direct sums and summands. Let $\mathcal{C}$ be a subcategory of $\mod \Lambda$. A morphism $f : M \to X$ with $X \in \mathcal{C}$ is called a left $\mathcal{C}$-approximation if 

$$\homc(X, X') \to \homc(M, X')$$

is surjective for any $X' \in \mathcal{C}$; if moreover for any $M \in \mod \Lambda$ there exists a left $\mathcal{C}$-approximation, we say that $\mathcal{C}$ is covariantly finite. Dually we define a right $\mathcal{C}$-approximation and a contravariantly finite subcategory. If $\mathcal{C}$ is both covariantly and contravariantly finite, we say that $\mathcal{C}$ is functorially finite. Functorially finite subcategories were first introduced in [2].

A morphism $f : M \to N$ in $\mod \Lambda$ will be called left minimal if whenever $f$ is isomorphic to $M \xrightarrow{(f_1, 0)} N_1 \oplus N_2$, we have $N_2 = 0$; if $f$ is also a left $\mathcal{C}$-approximation, we will say that $f$ is a minimal left approximation. Dually we define right minimal morphisms and minimal right approximations. It is well-known that minimal approximations are unique up to isomorphism.

For the rest of the paper $n$ will be a positive integer. The following definition is due to Iyama ([7], [6]).
Definition 2.1. We call a subcategory $\mathcal{C}$ of $\text{mod}\Lambda$ an $n$-cluster tilting subcategory if it is functorially finite and

$$\mathcal{C} = \mathcal{C}^+ = \mathcal{C}^-, $$

where

$$\mathcal{C}^+ := \{X \in \text{mod}\Lambda \mid \text{Ext}^i_{\Lambda}(\mathcal{C}, X) = 0 \text{ for all } 0 < i < n\},$$

$$\mathcal{C}^- := \{X \in \text{mod}\Lambda \mid \text{Ext}^i_{\Lambda}(X, \mathcal{C}) = 0 \text{ for all } 0 < i < n\}.$$

Our main result is inspired by the following necessary condition for $n$-cluster tilting subcategories due to Iyama.

Proposition 2.2. ([7], Theorem 2.8) Let $\mathcal{C}$ be an $n$-cluster tilting subcategory of $\text{mod}\Lambda$. Then $\tau_n$ and $\tau_n^-$ induce mutually inverse bijections

$$\mathcal{C}_P \xrightarrow{\tau_n} \mathcal{C}_I \xleftarrow{\tau_n^-} \mathcal{C}_P.$$

For $M \in \text{mod}\Lambda$ we denote by $\text{add}M$ the subcategory of $\text{mod}\Lambda$ containing all modules isomorphic to direct summands of finite direct sums of $M$. Note that $\text{add}M$ is always functorially finite. Hence $\text{add}M$ is an $n$-cluster tilting subcategory if and only if $\text{add}M = \text{add}M^+ = \text{add}M^-$. In that case we will call $M$ an $n$-cluster tilting module. Observe that if $\Lambda$ is representation-finite, then any additive subcategory of $\text{mod}\Lambda$ is of the form $\text{add}M$ for some $M \in \text{mod}\Lambda$. Moreover it is clear from the definition that any $n$-cluster tilting subcategory contains $\Lambda$ and $D\Lambda$.

If there exists an $n$-cluster tilting subcategory with $n > d$, then $\text{Ext}^i_{\Lambda}(D\Lambda, \Lambda) = 0$ for all $0 < i < \infty$, so $\Lambda$ is semisimple. Therefore, when $\Lambda$ is not semisimple, we have $n \leq d$. Observe also that $\text{mod}\Lambda$ is the unique 1-cluster tilting subcategory of $\text{mod}\Lambda$ so in the following we assume $2 \leq n \leq d$.

To keep track of the $\text{Ext}_{\Lambda}$-vanishing conditions in Definition 2.1, it is useful to consider the following sets of modules.

Definition 2.3. Let $X \in \text{mod}\Lambda$. Define the left ($\text{Ext}_{\Lambda}^n$-)support of $X$, denoted $\mathcal{L}S_n(X)$, to be

$$\mathcal{L}S_n(X) = \{Y \in \text{mod}\Lambda \mid \exists 0 < i < n : \text{Ext}^i_{\Lambda}(X, Y) \neq 0\}.$$

Similarly, define the right ($\text{Ext}_{\Lambda}^n$-)support of $X$, denoted $\mathcal{R}S_n(X)$, to be

$$\mathcal{R}S_n(X) = \{Y \in \text{mod}\Lambda \mid \exists 0 < i < n : \text{Ext}^i_{\Lambda}(Y, X) \neq 0\}.$$

A path from $M_0$ to $M_t$ in $\text{mod}\Lambda$ is a sequence of nonzero nonisomorphisms $f_k : M_{t-1} \rightarrow M_k$ between indecomposable modules $M_0, M_1, \cdots, M_t$ for $t \geq 1$. We define the relation $M \leq N$ on indecomposable modules $M$ and $N$ as the transitive hull of $\text{Hom}_{\Lambda}(M, N) \neq 0$. Then $M \leq N$ if and only if $M \cong N$ or there is a path from $M$ to $N$.

$\Lambda$ is called representation-directed if there is no path from $M$ to $N$ in $\text{mod}\Lambda$ with $M \cong N$. Note that representation-directed algebras are representation-finite; for a proof and more details on paths and representation-directed algebras we refer to [3]. Note also that in this case $M \leq N$ and $N \leq M$ implies $M \cong N$. Therefore, we will write $M < N$ if $M \leq N$ and $M \ncong N$. In the following lemma we collect some basic results that will be used throughout.
Lemma 2.4. Let $M, N \in \text{mod}\Lambda$ be indecomposable and $n \geq 2$. Then,

(i) $M$ is projective if and only if $\mathcal{LS}_n(M) = \emptyset$,

(ii) $N \in \mathcal{LS}_n(M)$ if and only if $\mathcal{RS}_n(N) = \emptyset$,

(iii) if $0 < k < n$ and $\Omega^{k-1}M$ is nonprojective, then $\Omega^k M \in \mathcal{LS}_n(M)$,

(iv) if $0 < k < n$ and $\Omega^{-(k-1)}N$ is noninjective, then $\Omega^{-k} N \in \mathcal{RS}_n(N)$,

(v) if $X$ is an indecomposable summand of $\Omega M$, then $X \leq M$,

(vi) if $Y$ is an indecomposable summand of $\Omega^{-} N$, then $N \leq Y$.

If in addition $\Lambda$ is representation-directed,

(vi) $M \leq N$ and $N \leq M$ imply $M \cong N$,

(vi) if $\tau M \neq 0$, then $\tau N \leq M$,

(vi) if $\tau^{-} N \neq 0$, then $N \leq \tau^{-} N$,

(vi) if $X$ is an indecomposable summand of $\Omega M$, then $X \leq \tau M$,

(vi) if $Y$ is an indecomposable summand of $\Omega^{-} N$, then $\tau^{-} N \leq Y$.

Proof. Statements (i), (i') and (ii) follow immediately from the definitions. Statement (iii) follows by noticing $\text{Ext}_\Lambda^1(M, \Omega^k M) = \text{Ext}_\Lambda^1(\Omega^{k-1} M, \Omega^k M) \neq 0$. Statement (iv) follows because if $P$ is the projective cover of $M$ then there exists some indecomposable summand $P'$ of $P$ with $X \leq P'$ and since $P' \subseteq M$, it follows that $X \leq M$. Statement (v) follows since otherwise there are paths from $M$ to $N$ and from $N$ to $M$. Statement (vi) follows since there is a path from $\tau M$ to $M$ and $\tau M \not\cong M$ for representation-directed algebras. Statement (vii) follows because if $P$ is the projective cover of $M$ and $0 \to \Omega M \to P \to M \to 0$ is a short exact sequence, then we have

$$D\text{Hom}_\Lambda(X, \tau M) \cong \text{Ext}_\Lambda^1(M, X) \cong \text{Hom}_\Lambda(\Omega M, X)/\text{Im}(\_ \circ \tau) \neq 0,$$

where the last inequality follows since the canonical projection $\Omega M \to X$ does not factor through $\tau$, as it is a radical morphism. Statements (iii'), (iv'), (vi') and (vii') follow similarly to (iii), (iv), (vi) and (vii) respectively. □

3. $n$-Cluster tilting subcategories of representation-directed algebras

The aim of this section is to give a proof of Theorem 1. Subsection 3.1 collects some preliminary results while subsection 3.2 uses those results to provide the proof. Before we proceed, let us give an outline of the strategy of the proof.

Theorem 1 asserts that $\mathcal{C}$ is an $n$-cluster tilting subcategory if and only if conditions (1)–(4) in that theorem hold. As we mentioned in Remark 1, conditions (1) and (2) are known to be necessary without assuming $\Lambda$ to be representation-directed. Proposition 3.1, and its Corollaries 3.2 and 3.3 connect conditions (3) and (4) to $\text{Ext}_\Lambda$-vanishing. In particular, Corollary 3.3 shows that conditions (3) and (4) are also necessary without assuming representation-directedness. This establishes one direction of the equivalence in Theorem 1.

For the other direction, first notice that as we mentioned in Remark 1, if $\Lambda$ is a representation-directed algebra, the only candidate for an $n$-cluster tilting subcategory is $\mathcal{C} = \text{add} \left( \bigoplus_{r \geq 0} \left( \tau^{-r}\Lambda \right) \right)$. By applying $n$-Auslander–Reiten duality (Theorem 1.5 in [6]) and in particular Proposition 3.4 we show that if the conditions (1)–(4) hold, then $\mathcal{C} \subseteq C^{\perp_n} = \perp_n C$. To show $C^{\perp_n} \subseteq \mathcal{C}$ we take $X \in C^{\perp_n}$ and consider the
sequence \( \tau^n_i X \). We show that by directedness we have that \( \tau^n_{i-1} X \) is projective for some \( i \geq 1 \) and, using Lemma 3.6, we show that \( X \cong \tau^{n-1}_i \tau^n_{i-1} X \in \mathcal{C} \).

We proceed to show the technical results needed to justify the steps outlined in the above strategy.

3.1. Preparation

In this section we present the results that will be used in the proof of Theorem 1. In the first part of this section we only assume that \( \Lambda \) is a finite dimensional algebra. In fact we show that the conditions (1)–(4) in Theorem 1 are necessary without the use of the assumption of representation-directedness. In the second part of this section, we additionally assume that \( \Lambda \) is representation-directed to prove some technical results which are used in the reverse implication.

We begin by giving a necessary condition for the existence of an \( n \)-cluster tilting subcategory. We thank Steffen Oppermann for suggesting the proof of the following result.

**Proposition 3.1.** Let \( \Lambda \) be a finite dimensional algebra.

(a) Let \( M \in \text{mod}\Lambda \) be indecomposable and nonprojective and let \( P \) be the projective cover of \( M \). If \( \Omega M \) is decomposable, then \( \text{Ext}^1_\Lambda(M, P) \neq 0 \).

(b) Let \( N \in \text{mod}\Lambda \) be indecomposable and noninjective and let \( I \) be the injective hull of \( N \). If \( \Omega^{-1} N \) is decomposable, then \( \text{Ext}^1_\Lambda(I, N) \neq 0 \).

**Proof.** We only prove (a); (b) is proved similarly. Assume towards a contradiction that \( \Omega M = X_1 \oplus X_2 \) with \( X_1 \neq 0 \) and \( X_2 \neq 0 \) (in particular, \( M \) is not projective) and \( \text{Ext}^1_\Lambda(M, P) = 0 \). Consider the short exact sequence \( 0 \rightarrow \Omega M \xrightarrow{\iota} P \xrightarrow{\rho} M \rightarrow 0 \); by applying \( \text{Hom}_\Lambda(\cdot, P) \) we get the long exact sequence

\[
0 \rightarrow \text{Hom}_\Lambda(M, P) \xrightarrow{\alpha_p} \text{Hom}_\Lambda(P, P) \xrightarrow{\alpha_\iota} \text{Hom}_\Lambda(\Omega M, P) \rightarrow \text{Ext}^1_\Lambda(M, P) \rightarrow \cdots.
\]

By our assumption, \( \text{Ext}^1_\Lambda(M, P) = 0 \) so that \( \alpha \circ \iota \) is surjective. Hence \( \iota \) is a left (addP)-approximation. Moreover, it is minimal left for if \( P_1 \oplus P_2 \) is a direct sum decomposition of \( P \) such that \( \iota \) is isomorphic to \( \Omega M \xrightarrow{\iota_1} P_1 \oplus P_2 \), then \( P_2 \) is a direct summand of \( M \), and since \( M \) is not projective and indecomposable, it follows that \( P_2 = 0 \). Now let \( f_1 : X_1 \rightarrow P' \) and \( f_2 : X_2 \rightarrow P'' \) be minimal left (addP)-approximations. Then \( f_1 \oplus f_2 \) is a minimal left (addP)-approximation of \( X_1 \oplus X_2 \), and therefore it is isomorphic to \( \iota \) as a map. As \( P \) is the projective cover of \( M \), we have that \( f_1 \) and \( f_2 \) are both monomorphisms but not isomorphisms. Hence coker \( f_1 \neq 0 \) and coker \( f_2 \neq 0 \). But then \( M = \text{coker } f_1 \oplus \text{coker } f_2 \) contradicts \( M \) being indecomposable. \( \square \)

We have two immediate corollaries.

**Corollary 3.2.** Let \( \Lambda \) be a finite dimensional algebra. Then

(a) If \( M \in \text{mod}\Lambda \) is indecomposable nonprojective such that \( \mathcal{L} S_n(M) = \mathcal{R} S_n(\tau_n M) \), then \( \Omega^i M \) and \( \Omega^{-i} \tau_n M \) are indecomposable for all \( 0 < i < n \).

(b) If \( N \in \text{mod}\Lambda \) is indecomposable noninjective such that \( \mathcal{R} S_n(N) = \mathcal{L} S_n(\tau_n^{-1} N) \), then \( \Omega^{-i} N \) and \( \Omega^i \tau_n^{-1} N \) are indecomposable for all \( 0 < i < n \).

**Proof.** We only prove (a); (b) is proved similarly. Since \( M \) is nonprojective, \( \mathcal{L} S_n(M) \neq \emptyset \) by Lemma 2.4(i). Then, by assumption, we have that \( \mathcal{R} S_n(\tau_n M) \neq \emptyset \) so \( \tau_n M \) is nonzero and even noninjective (Lemma 2.4(i')). In particular, \( \Omega^1 M \neq 0 \) for \( 0 < i < n \).
Let us now prove that $\Omega^{-i}\tau_n M \neq 0$ for $0 < i < n$. Assume towards a contradiction that there is $0 < i < n$, such that $\Omega^{-i}\tau_n M = 0$, while $\Omega^{-(i-1)}\tau_n M \neq 0$. In particular $1 < i$, since $\tau_n M$ is noninjective. Then $\Omega^{-(i-1)}\tau_n M$ is injective and nonzero, so that $\Omega^{-(i-2)}\tau_n M$ is noninjective. By Lemma 2.4(iii’), $\Omega^{-(i-1)}\tau_n M \in RS_n(\tau_n M) = LS_n(M)$, which contradicts $\Omega^{-(i-1)}\tau_n M$ being injective.

Next, assume towards a contradiction that $\Omega^i M$ is indecomposable for some $0 < i < n$ minimal. Then $\Omega^{-i} M$ is indecomposable nonprojective, since $\Omega^i M \neq 0$. Let $P$ be the projective cover of $\Omega^{-i} M$. By Proposition 3.1(a), $\text{Ext}^1_\Lambda(\Omega^{-i} M, P) \neq 0$ and so $\text{Ext}^1_\Lambda(M, P) \neq 0$. But then $P \in LS_n(M) = RS_n(\tau_n M)$, which contradicts $P$ being projective. Hence $\Omega^i M$ is indecomposable for $0 < i < n$. Similarly, using Proposition 3.1(b) we prove that $\Omega^{-i} \tau_n M$ is indecomposable for $0 < i < n$. \hfill $\square$

**Corollary 3.3.** Let $\Lambda$ be a finite dimensional algebra and $C$ be an $n$-cluster tilting subcategory of mod$\Lambda$. Then

(a) $\Omega^i M$ is indecomposable for all $M \in C_P$ and $0 < i < n$,
(b) $\Omega^{-i} N$ is indecomposable for all $N \in C_I$ and $0 < i < n$.

**Proof.** This follows from Theorem 1.5 in [6] and Corollary 3.2. More precisely, in the case of (a) we have from the aforementioned theorem that $LS_n(M) = RS_n(\tau_n M)$ and so Corollary 3.2 implies then that $\Omega^i M$ is indecomposable. The case (b) is similar. \hfill $\square$

Corollary 3.3 gives a necessary condition for a subcategory $C$ to be $n$-cluster tilting: the syzygy and a cosyzygy of an indecomposable module in $C$ must be either indecomposable or 0. In particular, we have now proved that if $C$ is an $n$-cluster tilting subcategory then (1)–(4) in Theorem 1 hold, since (1) is immediate by the definition, (2) follows from Proposition 2.2 and (3) and (4) from Corollary 3.3. More generally, we have shown that conditions (1)–(4) being necessary is true for any finite dimensional algebra, since we have not used representation-directedness yet.

In the rest of this section we develop the tools needed for the reverse implication. In the next proposition we apply the $n$-Auslander–Reiten duality to gain control of the left and right support of an indecomposable module.

**Proposition 3.4.** Let $\Lambda$ be a finite dimensional algebra.

(a) Let $N \in \text{mod}\Lambda$ be indecomposable. Then if $RS_n(N) \neq LS_n(\tau_n N)$, there exists an indecomposable injective module $I$ such that $I \in RS_n(N)$.
(b) Let $M \in \text{mod}\Lambda$ be indecomposable. Then if $LS_n(M) \neq RS_n(\tau_n M)$, there exists an indecomposable projective module $P$ such that $P \in LS_n(M)$.

**Proof.** This follows from Theorem 1.5 in [6]. More precisely, in the case (a) we have from the aforementioned theorem that if we assume $DA \notin RS_n(N)$, then $\text{Ext}^n_{\Lambda^{-1}}(Z, N) \cong D\text{Ext}^1_{\Lambda}(\tau^{-1} N, Z)$ for any $Z \in \text{mod}\Lambda$, which in turn implies $RS_n(N) = LS_n(\tau_n N)$. The case (b) is similar. \hfill $\square$

From now on we will additionally assume that $\Lambda$ is representation-directed. We begin with the following easy lemma.

**Lemma 3.5.** Let $\Lambda$ be a representation-directed algebra.

(a) Let $N, X \in \text{mod}\Lambda$ be indecomposable such that $X \in RS_n(N)$ and $\tau^{-1} N \neq X$. Then $\tau^{-1} N < X$ and $N < X$.
(b) Let $M, Y \in \text{mod}\Lambda$ be indecomposable such that $Y \in LS_n(M)$ and $Y \neq \tau M$. Then $Y < \tau M$ and $Y < M$.
**Proof.** We only prove (a); (b) is proved similarly. Since $X \in \mathcal{R}S_n(N)$, there exists $0 < j < n$ such that $\text{Ext}^j_\Lambda(X, N) \neq 0$. Using dimension shift and the Auslander–Reiten formula we have

$$D\text{Hom}_\Lambda(\tau^{-\Omega^{-(j-1)}N}, X) \cong \text{Ext}^1_\Lambda(X, \Omega^{-(j-1)}N) \cong \text{Ext}^j_\Lambda(X, N) \neq 0.$$ 

If $j = 1$, we have $N < \tau^{-N} < X$ since $\tau^{-N} \not\simeq X$ and the result is proved. Now that assume $j \geq 2$. Then for some indecomposable summand $Y$ of $\tau^{-\Omega^{-(j-1)}N}$ we have $Y \leq X$. Then $\tau Y$ is an indecomposable summand of $\Omega^{-\tau^{-1}}N$. Then, by Lemma 2.4(iv'), there exists an indecomposable summand $Z$ of $\Omega^{-N}$ such that $Z \leq \tau Y$. By Lemma 2.4(vi), we have $\tau^{-N} \leq Z$. All together, we have

$$N < \tau^{-N} \leq Z \leq \tau Y < Y \leq X,$$

as required. \(\square\)

Next we give sufficient conditions for $\tau_n \tau_n^{-1} N = N$ as well as $\tau_n^{-1} \tau_n M = M$ in the case of representation-directed algebras, to be used in the proof of Theorem 1.

**Lemma 3.6.** Let $\Lambda$ be a representation-directed algebra.

(a) Let $N \in \text{mod}\Lambda$ be indecomposable noninjective. Then $\mathcal{R}S_n(N) = \mathcal{L}S_n(\tau_n^{-1}N)$ implies $\tau_n \tau_n^{-1} N = N$.

(b) Let $M \in \text{mod}\Lambda$ be indecomposable nonprojective. Then $\mathcal{L}S_n(M) = \mathcal{R}S_n(\tau_n^{-1}M)$ implies $\tau_n^{-1} \tau_n M = M$.

**Proof.** We only prove (a); (b) is proved similarly. As $N$ is noninjective, we have that $\mathcal{R}S_n(N) \neq \emptyset$ and since $\mathcal{L}S_n(\tau_n^{-1}N) \neq \emptyset$, it follows that $\tau_n^{-1} N \neq 0$. Let $0 < i < n$. First note that $\Omega^{-i}N$ and $\Omega^{-i} \tau_n^{-1} N$ are indecomposable by Corollary 3.2. In particular, $\tau_n^{-1} N$ is indecomposable as well.

Therefore, it is enough to show that $\Omega^{n-1} \tau_n^{-1} N \cong \tau^{-N}$. We will show this by showing $\tau^{-N} \leq \Omega^{n-1} \tau_n^{-1} N$ and $\Omega^{n-1} \tau_n^{-1} N \leq \tau^{-N}$.

Since $\Omega^{n-1} \tau_n^{-1} N \neq 0$, we have $\Omega^{n-1} \tau_n^{-1} N \in \mathcal{L}S_n(\tau_n^{-1} N) = \mathcal{R}S_n(N)$ and so by Lemma 3.5 we have $\tau^{-N} \leq \Omega^{n-1} \tau_n^{-1} N$.

Now, since $\tau^{-N} \in \mathcal{R}S_n(N) = \mathcal{L}S_n(\tau_n^{-1} N)$ there exists some $j$ such that $\text{Ext}^j_\Lambda(\tau_n^{-1} N, \tau^{-N}) \neq 0$. In particular $\text{Hom}_\Lambda(\Omega^j \tau_n^{-1} N, \tau^{-N}) \neq 0$ so

$$\Omega^{n-1} \tau_n^{-1} N \leq \Omega^j \tau_n^{-1} N \leq \tau^{-N},$$

which finishes the proof. \(\square\)

### 3.2. Proof of Theorem 1

With the preparation from the previous section, we can give a proof of the following more general form of Theorem 1.

**Theorem 1.** Let $\Lambda$ be a representation-directed algebra and $\mathcal{C}$ be a subcategory of $\text{mod}\Lambda$. Then the conditions (a), (b) and (c) are equivalent.

(a) (a1) $\Lambda \in \mathcal{C}$,

(a2) $\tau_n$ and $\tau_n^{-1}$ induce mutually inverse bijections.
(a3) \( \Omega M \) is indecomposable for all \( M \in \mathcal{C}_p \) and \( 0 < i < n \),

(a4) \( \Omega^{-i}N \) is indecomposable for all \( N \in \mathcal{C}_l \) and \( 0 < i < n \).

(b1) \( \Lambda \in \mathcal{C} \),

(b2) for all \( M \in \mathcal{C}_p \), we have \( \tau_n M \in \mathcal{C} \) and \( \mathcal{L}_n(M) = \mathcal{R}_n(\tau_n M) \),

(b3) for all \( N \in \mathcal{C}_l \), we have \( \tau_n^{-1}N \in \mathcal{C} \) and \( \mathcal{R}_n(N) = \mathcal{L}_n(\tau_n^{-1}N) \).

(c) \( \mathcal{C} \) is an n-cluster tilting subcategory.

**Proof.** First note that as we mentioned before we have already proved (c) implies (a) by Corollary 3.3 and Proposition 2.2. Next note that (b2) implies (a3) and (b3) implies (a4) by Corollary 3.2. Moreover (b2) and (b3) imply (a2) by Lemma 3.6. This shows that (b) implies (a). Next we will prove (a) implies (b) and finally (a) and (b) imply (c).

(a) implies (b): We only prove (a) implies (b3); (a) implies (b2) is similar and (a) implies (b1) is clear.

Let \( N \in \mathcal{C}_l \). Then \( \tau_n^{-1}N \in \mathcal{C}_p \) by (a2), so it remains to show that \( \mathcal{R}_n(N) = \mathcal{L}_n(\tau_n^{-1}N) \). Assume instead that \( \mathcal{R}_n(N) = \mathcal{L}_n(\tau_n^{-1}N) \) and we will reach a contradiction. Our strategy will be to construct an infinite sequence \( (\tau_k)_k \geq 1 \in \mathcal{C} \) of indecomposable nonisomorphic modules, contradicting the fact that \( \Lambda \) is representation-directed. The sequence \( X_k \) will be defined using \( \tau_n \) and injective modules, which are all in \( \mathcal{C} \) as (a1) and (a2) imply \( \text{DA} \in \mathcal{C} \) since \( \Lambda \) is representation-directed and so representation-finite.

By Proposition 3.4(a) there exists an injective indecomposable module \( I \in \mathcal{R}_n(N) \). More generally, there is a sequence \( X_k \), \( k \geq -1 \), satisfying:

- \( X_{-1} = \tau_n^{-1}N \),
- \( X_0 = I \),
- \( X_k = \tau_n X_{k-2} \) if \( \mathcal{R}_n(\tau_n X_{k-1}) = \mathcal{L}_n(X_{k-1}) \),
- \( X_k \in \mathcal{R}_n(\tau_n X_{k-1}) \) and \( X_k \) indecomposable injective if \( \mathcal{R}_n(\tau_n X_{k-1}) \neq \mathcal{L}_n(X_{k-1}) \).

In particular, \( X_k \in \mathcal{C} \) for all \( k \geq -1 \). We claim that \( X_k \in \mathcal{R}_n(\tau_n X_{k-1}) \) for all \( k \geq 0 \). We will prove this by induction. For \( k = 0 \) we have

\[
X_0 = I \in \mathcal{R}_n(N) \overset{(a2)}{=} \mathcal{R}_n(\tau_n^{-1}N) = \mathcal{R}_n(\tau_n X_{-1}).
\]

For the induction step, assume that \( X_k \in \mathcal{R}_n(\tau_n X_{k-1}) \). We want to prove \( X_{k+1} \in \mathcal{R}_n(\tau_n X_k) \). If \( \mathcal{R}_n(\tau_n X_k) \neq \mathcal{L}_n(X_k) \), then \( X_{k+1} \) is an indecomposable injective module in \( \mathcal{R}_n(\tau_n X_k) \) by construction. Otherwise, \( \mathcal{R}_n(\tau_n X_k) = \mathcal{L}_n(X_k) \) and \( X_{k+1} = \tau_n X_k \). By induction assumption we have \( X_k \in \mathcal{R}_n(\tau_n X_{k-1}) \) and so

\[
X_{k+1} = \tau_n X_{k-1} \in \mathcal{L}_n(X_k) = \mathcal{R}_n(\tau_n X_k)
\]

as required. In particular, \( X_k \in \mathcal{C}_p \) for all \( k \).

Next we use \( X_k \in \mathcal{R}_n(\tau_n X_{k-1}) \) to show that \( X_k < X_{k-1} \). Since \( X_k \in \mathcal{R}_n(\tau_n X_{k-1}) \), there exists some \( 0 < i < n \) with \( \text{Ext}_A^i(X_k, \tau_n X_{k-1}) \neq 0 \). In particular, \( \text{Hom}_A(X_k, \Omega^{-i} \tau_n X_{k-1}) \neq 0 \). Since \( \tau_n X_{k-1} \in \mathcal{C}_l \), we have that \( \Omega^{-i} \tau_n X_{k-1} \) is indecomposable by (a4) and so \( X_k \leq \Omega^{-i} \tau_n X_{k-1} \). Since \( \Omega^{-i} \tau_n X_{k-1} \) is indecomposable for \( i < j < n \),

\[
X_k \leq \Omega^{-i} \tau_n X_{k-1} \leq \Omega^{-(n-1)} \tau_n X_{k-1} < \tau^{-i} \Omega^{-(n-1)} \tau_n X_{k-1}.
\]
Since by (a2) we have \( \tau_n^k X_{k-1} = X_{k-1} \), we get \( X_k < X_{k-1} \). So, the sequence \( X_k \) is an infinite sequence of indecomposable modules such that
\[
\cdots < X_k < X_{k-1} < \cdots < X_1 < X_0
\]

which contradicts the fact that \( \Lambda \) is representation-directed and representation-finite.

(a) and (b) imply (c): We first show \( \perp n \mathcal{C} = \perp n \mathcal{C}. \) We have
\[
\begin{align*}
\perp n \mathcal{C} &= \{ X \in \text{mod} \Lambda \mid \text{Ext}^{i}(\mathcal{C}, X) = 0 \text{ for all } 0 < i < n \} \\
&= \{ X \in \text{mod} \Lambda \mid \text{Ext}^{i}(M, X) = 0 \text{ for all } 0 < i < n \text{ and } M \in \mathcal{C} \} \\
&= \{ X \in \text{mod} \Lambda \mid \text{Ext}^{i}(M, X) = 0 \text{ for all } 0 < i < n \text{ and } M \in \mathcal{C}_P \} \\
&= \{ X \in \text{mod} \Lambda \mid X \notin \mathcal{LS}_n(M) \text{ for all } M \in \mathcal{C}_P \} \\
&= \{ X \in \text{mod} \Lambda \mid X \in \mathcal{LS}_n(M)^c \text{ for all } M \in \mathcal{C}_P \} \\
&= \bigcap_{M \in \mathcal{C}_P} \mathcal{LS}_n(M)^c.
\end{align*}
\]

Similarly,
\[ \perp n \mathcal{C} = \bigcap_{N \in \mathcal{C}_I} \mathcal{RS}_n(N)^c. \]

Hence
\[ \perp n \mathcal{C} = \bigcap_{M \in \mathcal{C}_P} \mathcal{LS}_n(M)^c = \bigcap_{M \in \mathcal{C}_P} \mathcal{RS}_n(M)^c = \bigcap_{N \in \mathcal{C}_I} \mathcal{RS}_n(N)^c = \perp n \mathcal{C}. \]

It remains to show that \( \mathcal{C} = \perp n \mathcal{C} \). Let us first show that \( \perp n \mathcal{C} \subseteq \mathcal{C} \). Let \( X \in \perp n \mathcal{C} \) and without loss of generality we can assume that \( X \) is indecomposable (otherwise use additivity of \( \text{Ext}^i \)). Moreover, if \( X \) is projective then \( X \in \mathcal{C} \) by (a1) so we further assume that \( X \) is nonprojective. Consider the sequence \( \tau_n X \) for \( k \geq 0 \). We consider two possible cases. In the first case we show that \( X \in \mathcal{C} \), while in the second we reach a contradiction.

Case 1: \( \mathcal{LS}_n(\tau_n^{k-1}X) = \mathcal{RS}_n(\tau_n^kX) \) for all \( k \geq 1 \). Then, since \( \Lambda \) is representation-directed, there exists some minimal \( l \) such that \( \tau_n^l X = 0 \). Since
\[ \mathcal{LS}_n(0) = \mathcal{RS}_n(\tau_n^l X) = \mathcal{LS}_n(\tau_n^{l-1}X), \]

\( \tau_n^{l-1}X \) is projective, and so \( \tau_n^{l-1}X \in \mathcal{C} \). Since \( l \) was minimal, we have \( \tau_n^{l-1}X \neq 0 \). Consider the modules \( \Omega^i \tau_n^kX \) where \( 0 < i < n \) and \( 0 \leq k < l - 2 \). Since \( \tau_n^{l-1}X \neq 0 \), they are all nonprojective. Using Corollary 3.2 and induction on \( k \) we find that they are all indecomposable. Hence \( \tau_n^{l-1}X = \tau^k \tau_n^{l-2}X \) is also indecomposable. Since \( \mathcal{RS}_n(\tau_n^{l-1}X) = \mathcal{LS}_n(\tau_n^{l-2}X) \neq \emptyset \) (because \( \tau_n^{l-2}X \) is nonprojective), it follows that \( \tau_n^{l-1}X \) is noninjective and so \( \tau_n^{l-1}X \in \mathcal{C}_I \). On the other hand, Lemma 3.6 implies \( \tau_n^{-l} \tau_n^kX \cong \tau_n^{k-1}X \) for all \( 1 \leq k \leq l - 1 \), and so \( \tau_n^{-l} \tau_n^{l-1}X = X \). Since \( \tau_n^{l-1}X \in \mathcal{C}_I \), it follows that \( X \in \mathcal{C} \).

Case 2: There exists some \( m \geq 1 \) such that \( \mathcal{LS}_n(\tau_n^{m-1}X) \neq \mathcal{RS}_n(\tau_n^mX) \). In particular, \( \tau_n^{m-1}X \neq 0 \). Pick \( l \) minimal such that \( \mathcal{LS}_n(\tau_n^{l-1}X) \neq \mathcal{RS}_n(\tau_n^kX) \). Then we have \( \mathcal{LS}_n(\tau_n^{k-1}X) = \mathcal{RS}_n(\tau_n^kX) \) for all \( 0 < k < l \), and as in Case 1, \( \tau_n^{k-1}X \) is indecomposable nonprojective. Moreover, Lemma 3.6 implies \( \tau_n^{-k} \tau_n^{l-1}X = \tau_n^{-k}X \) and so \( \tau_n^lX \) is noninjective for \( 0 < k < l \). In particular, by Proposition 3.4(b), there exists an indecomposable projective module \( P \) such that \( P \in \mathcal{LS}_n(\tau_n^{l-1}X) \). Then \( \tau_n^{l-1}X \in \mathcal{RS}_n(P) = \mathcal{LS}_n(P) \) by (b3). Equivalently, \( \tau_n P \in \mathcal{RS}_n(\tau_n^{l-1}X) = \mathcal{LS}_n(\tau_n^{l-2}X) \). Repeating this argument, we get \( \tau_n^{l-2}X \in \mathcal{RS}_n(\tau_n^{l-3}X) = \mathcal{LS}_n(\tau_n^{l-4}X) \). Repeating this argument, we get \( \tau_n^{l-3}X \in \cdots \in \mathcal{RS}_n(\tau_n^{l-k}X) = \mathcal{LS}_n(\tau_n^{l-(k+1)}X) \). Repeating this argument, we get
\[ \mathcal{LS}_n(X) \text{. Set } \tau_n^{-(l-1)} P = N; \text{ then } N \in \mathcal{C}_I \text{ and } X \in \mathcal{RS}_n(N), \text{ contradicting } X \in \mathcal{C} = \bigcap_{N \in \mathcal{C}_I} \mathcal{RS}_n(N)^c \text{ and so Case 2 is impossible.} \]

Finally, let us show that \( \mathcal{C} \subseteq \mathcal{C}^{-1} \). Assume towards a contradiction that \( Y \in \mathcal{C} \) is indecomposable but there exists some \( M \in \mathcal{C}_P \) such that \( Y \in \mathcal{LS}_n(M) = \mathcal{RS}_n(\tau_n M) \). By representation-directedness and because of (a2), there exists some minimal \( l \) such that \( \tau_n^l Y \) or \( \tau_n^l M \) is indecomposable projective. Since \( \tau_n M \in \mathcal{LS}_n(Y) \), \( Y \) is nonprojective, and so \( \tau_n M \in \mathcal{LS}_n(Y) = \mathcal{RS}_n(\tau_n Y) \) by (b2), or \( \tau_n Y \in \mathcal{LS}_n(\tau_n M) = \mathcal{RS}_n(\tau_n^2 M) \). Repeating this argument we get \( \tau_n^l Y \) or \( \tau_n^l M \) is indecomposable projective or \( \tau_n^{l+1} M = 0 \). \( \square \)

**Example 3.7.** Let us give an example of a 2-cluster tilting subcategory using Theorem 1. Let \( \Lambda \) be the path algebra of the quiver with relations

\[
\begin{array}{c}
\circ \\
\circ \circ \\
\circ \circ \circ \\
\end{array}
\]

Note that \( \Lambda \) is representation-directed, special biserial and that indecomposable modules are determined uniquely by their dimension vectors. The Auslander–Reiten quiver of \( \Lambda \) is

Let \( M \) be the direct sum of all encircled modules. Note that their syzygies and cosyzygies are indecomposable or zero and computing \( \tau_2 \) and \( \tau_2^- \) applied to them we get

\[
\begin{array}{c}
\begin{array}{c}
0 \ 0 \ \overline{1} \ 0 \\
0 \ \overline{1} \ 0 \ 0 \\
0 \ 0 \ 0 \ 0 \\
\end{array}
\end{array}
\]

If we let \( \mathcal{C} = \text{add} M \), conditions (a) of Theorem 1 are satisfied for \( \mathcal{C} \) and so \( \mathcal{C} \) is a 2-cluster tilting subcategory. A simple computation shows that \( \text{gl.dim.} \Lambda = 3 \); as far as we know this is the first example of an algebra with global dimension 3 that admits a 2-cluster tilting subcategory.

**4. n-Cluster tilting subcategories of acyclic Nakayama algebras with homogeneous relations**

**4.1. Motivation**

In this section we aim to use Theorem 1 to produce examples of \( n \)-cluster tilting subcategories for representation-directed algebras. We begin with a necessary condition.

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Proposition 4.1. Let $Q$ be a connected quiver with $m$ vertices, $\Lambda = KQ/I$ where $I$ is an admissible ideal and $n \geq 2$. Let $k$ be a vertex in $Q_0$, which is a sink or a source such that the full subquiver of $Q$ with vertex set $Q_0 \setminus \{k\}$ is disconnected. Then $\Lambda = KQ/I$ admits no $n$-cluster tilting subcategory.

Proof. Let us prove the proposition when $k$ is a sink; the other case is similar. Let $Q'$ be the full subquiver of $Q$ with vertex set $Q_0 \setminus \{k\}$ and write $Q'$ as the disjoint union of two nonempty quivers $Q_A$ and $Q_B$. Without loss of generality, assume $(Q_A)_0 = \{1, \ldots, k-1\}$ and $(Q_B)_0 = \{k+1, \ldots, m\}$. Consider the indecomposable projective module $P(k)$ corresponding to the vertex $k$. Its dimension vector is

$$\dim P(k) = \begin{pmatrix} a_1 \\ \vdots \\ a_{k-1} \\ b_{k+1} \\ \vdots \\ b_m \end{pmatrix}.$$ 

Moreover it is noninjective and its injective hull, $I(k)$, has $I(k)_k = K$ since $k$ is a sink. Furthermore, in $\dim I(k)$, there is at least one nonzero entry in a position $i < k$ since there exists an arrow from a vertex in $Q_A$ to $k$. Similarly, there is at least one nonzero entry in a position $j > k$. Therefore we have

$$\dim I(k) = \begin{pmatrix} a_1 \\ \vdots \\ a_{k-1} \\ b_{k+1} \\ \vdots \\ b_m \end{pmatrix}, \quad \dim \Omega^- P(k) = \begin{pmatrix} a_1 \\ \vdots \\ a_{k-1} \\ 0 \\ \vdots \\ b_m \end{pmatrix},$$ 

where $(a_1, \ldots, a_{k-1}) \neq (0, \ldots, 0)$ and $(b_{k+1}, \ldots, b_m) \neq (0, \ldots, 0)$. Let $\Omega^- P(k) = (M_i, \phi_\alpha)_{i \in Q_0, \alpha \in Q_1}$. Let $f = (f_i)_{i \in Q_0}$ where $f_i : M_i \to M_i$ is identity if $i < k$ and zero otherwise. Note that $f \neq 0$ and $f \neq \text{Id}$. We will prove that $f$ is an endomorphism of $(M_i, \phi_\alpha)$. Let $\alpha : a \to b$ be an arrow in $Q$. Note that we cannot have $a < k < b$ or $b < k < a$ since $Q'$ is disconnected and we cannot have $a = k$ since $k$ is a sink. We need to show that

$$\phi_\alpha f_a = f_b \phi_\alpha. \quad (4.1)$$

If $a, b < k$, $f_a = f_b = \text{Id}$ and (4.1) becomes $\phi_\alpha = \phi_\alpha$. If $k < a, b$ then $f_a = f_b = 0$ and (4.1) becomes $0 = 0$. If $b = k$, then since $M_k = 0$, we have that $\phi_\alpha = 0$ and (4.1) becomes $0 = 0$ again. Hence $f \in \text{End}(\Omega^- P(k))$ with $f^2 = f$ but $f \neq 0$ and $f \neq \text{Id}$, and so $\text{End}(\Omega^- P(k))$ is not local, which implies that $\Omega^- P(k)$ is not indecomposable. Since any $n$-cluster tilting subcategory must contain the projective modules, $\Lambda$ doesn’t admit an $n$-cluster tilting subcategory by Corollary 3.3. $\square$

Example 4.2. Let $Q$ be a quiver with underlying graph the Dynkin diagram $A_m$ for $m \geq 3$, with nonlinear orientation. Pick any source or sink $k$ with degree 2. Then Proposition 4.1 implies that there exists no $n$-cluster tilting $KQ/I$-module for $I$ an admissible ideal of $KQ$.

Example 4.2 suggests that perhaps the simplest class of representation-directed algebras for which one should try to find $n$-cluster tilting subcategories is quotients of the path algebra of the quiver.
by an admissible ideal. Such algebras are called acyclic Nakayama and for more details on them we refer to [3].

4.2. Computations

In this section we will consider acyclic Nakayama algebras with homogeneous relations. That is, for $m \geq 3$ and $l \geq 2$, we will denote $\Lambda_{m,l} = KQ_m/(\text{rad } KQ_m)^l$. As we will see later, it turns out that this is a necessary condition for a Nakayama algebra to be $d$-representation finite. Since our main tool will be Theorem 1, we will need to compute syzygies, cosyzygies and $n$-Auslander–Reiten translations for $\Lambda_{m,l}$-modules.

Recall that the isomorphism classes of the indecomposable modules for any acyclic Nakayama algebra can be described by the representations $M(i,j)$ of the form

\[
\begin{array}{cccccccc}
0 & 0 & \cdots & 0 & K & 1 & \cdots & 1 \\
& & & & & & & \\
m & & & & & & & \\
\end{array}
\]

with $M(i,j)I = 0$ ([3], Gabriel’s Theorem). We will use the convention that $M(i,j) = 0$ if the coordinates $(i,j)$ do not define a module. In particular, for $\Lambda_{m,l}$-modules we have $M(i,j) \neq 0$ if and only if $1 \leq i \leq m$, $1 \leq j \leq l$ and $2 \leq i+j \leq m+1$. For a vertex $k \in Q_0$, we will denote by $P(k)$ respectively $I(k)$ the corresponding indecomposable projective respectively injective $\Lambda_{m,l}$-module. In the rest of this section, all modules will be $\Lambda_{m,l}$-modules.

Lemma 4.3. Let $M(i,j) \neq 0$. Then

(a) $P(k) = \begin{cases} 
M(1,k) & \text{if } 1 \leq k \leq l-1, \\
M(1+k-l,l) & \text{if } l \leq k \leq m.
\end{cases}$

(b) $I(k) = \begin{cases} 
M(k,l) & \text{if } 1 \leq k \leq m-l+1, \\
M(k,m+1-k) & \text{if } m-l+2 \leq k \leq m.
\end{cases}$

(c) $M(i,j)$ is both projective and injective if and only if $j = l$ and $1 \leq i \leq m-l+1$.

Proof. (c) follows immediately by (a) and (b). We only prove (a); (b) is proved similarly. Note that for $1 \leq k \leq l-1$, $P(k)$ as a representation is isomorphic to

\[
\begin{array}{cccccccc}
0 & 0 & \cdots & 0 & K & 1 & \cdots & 1 \\
& & & & & & & \\
m & & & & & & & \\
\end{array}
\]

which is precisely $M(1,k)$. Similarly, when $l \leq k \leq m$, $P(k)$ is isomorphic to

\[
\begin{array}{cccccccc}
0 & 0 & \cdots & 0 & K & 1 & \cdots & 1 \\
& & & & & & & \\
m & & & & & & & \\
\end{array}
\]

which is precisely $M(1+k-l,l)$.

Next we wish to compute syzygies and cosyzygies of the indecomposable $\Lambda_{m,l}$-modules.
Lemma 4.4. Let \( M(i, j) \neq 0 \). Then

(a) If \( M(i, j) \) is nonprojective,

\[
\Omega M(i, j) = \begin{cases} 
M(1, i - 1) & \text{if } i + j \leq l, \\
M(i + j - l, l - j) & \text{if } l + 1 \leq i + j.
\end{cases}
\]

(b) If \( M(i, j) \) is noninjective,

\[
\Omega^{-1} M(i, j) = \begin{cases} 
M(i + j, l - j) & \text{if } 1 \leq i \leq m - l + 1, \\
M(m + i - l + j, l - j) & \text{if } m - l + 2 \leq i \leq m.
\end{cases}
\]

Proof. We only prove (a); (b) is proved similarly. Assume first that \( l + 1 \leq i + j \) and consider the following commutative diagram

\[
\begin{array}{ccccccc}
M(i + j - l, l - j) : & 0 & \to & \cdots & \to & 0 & \to & \cdots & \to & 0 \\
\downarrow & & & & & & & & & \\
M(i + j - l, l) : & 0 & \to & \cdots & \to & K & \to & \cdots & \to & K \\
\downarrow & & & & & & & & & \\
M(i, j) : & 0 & \to & \cdots & \to & K & \to & \cdots & \to & K \\
\downarrow & & & & & & & & & \\
& & & & & & & & & \\
\end{array}
\]

where the arrows \( K \to K \) are the identity and all other arrows are the zero map. Then

\[ 0 \to M(i + j - l, l - j) \overset{u}{\to} M(i + j - l, l) \overset{s}{\to} M(i, j) \to 0 \]

is a short exact sequence. Since \( M(i + j - l, l) = P(i + j - 1) \) by Lemma 4.3, we have \( \Omega M(i, j) = M(i + j - l, l - j) \). If \( i + j \leq l \), similarly we have the short exact sequence \( 0 \to M(1, i - 1) \to M(1, i + j - 1) \to M(i, j) \to 0 \) with \( M(1, i + j - 1) = P(i + j - 1) \) and so \( \Omega M(i, j) = M(1, i - 1) \). □

Corollary 4.5. Let \( M(i, j) \neq 0 \) with \( j < l \) and \( k \geq 1 \). Then

(a) Denote \( \Omega^k M(i, j) = M(i_k, j_k) \) and assume that \( l + 1 \leq i_{k-1} + j_{k-1} \). Then

\[
\Omega^k M(i, j) = \begin{cases} 
M \left( i + j - \frac{k+1}{2}l, l - j \right) & \text{if } k \text{ is odd,} \\
M \left( i - \frac{k}{2}l, j \right) & \text{if } k \text{ is even.}
\end{cases}
\]

(b) Denote \( \Omega^{-k} M(i, j) = M(i_k, j_k) \) and assume that \( i_{k-1} \leq m - l + 1 \). Then

\[
\Omega^{-k} M(i, j) = \begin{cases} 
M \left( i + j + \frac{k-1}{2}l, l - j \right) & \text{if } k \text{ is odd,} \\
M \left( i + \frac{k}{2}l, j \right) & \text{if } k \text{ is even.}
\end{cases}
\]
Proof. Immediate by using Lemma 4.4 and induction on $k$. □

Proposition 4.6. Let $M(i, j) \neq 0$ and $M(i, j+1) \neq 0$. Then the sequence

$$0 \to M(i, j) \xrightarrow{[p]} M(i, j+1) \oplus M(i+1, j-1) \xrightarrow{[-t, q]} M(i+1, j) \to 0$$

is almost split, where $r$, $t$ are the natural inclusions, $p$, $q$ the natural projections, and by convention $M(i, 0) = 0$.

Proof. This follows from Theorem V.4.1 in [3] by noting that $\text{rad}^t M(i', j') = M(i', j' - t)$ for any $t \geq 0$ and any indecomposable $\Lambda_{m, r}$-module $M(i', j')$. □

Lemma 4.7. Let $M(i, j) \neq 0$. Then

(a) If $M(i, j)$ is nonprojective, $\tau(M(i, j)) = M(i - 1, j)$.
(b) If $M(i, j)$ is noninjective, $\tau^{-1}(M(i, j)) = M(i + 1, j)$.

Proof. Immediate by Proposition 4.6 and by uniqueness, up to isomorphism, of almost split sequences (see [3], Chapter IV.1). □

Lemma 4.8. Let $M(i, j) \neq 0$. Then

(a) If $M(i, j)$ is nonprojective, we have

$$\tau_n M(i, j) = \begin{cases} M(i + j - \frac{n}{2}l - 1, l - j) & \text{if } n \text{ is even}, \\ M(i - \frac{n-1}{2}l - 1, j) & \text{if } n \text{ is odd}. \end{cases}$$

(b) If $M(i, j)$ is noninjective, we have

$$\tau_n^{-1} M(i, j) = \begin{cases} M(i + j + \frac{n-2}{2}l + 1, l - j) & \text{if } n \text{ is even}, \\ M(i + \frac{n-1}{2}l + 1, j) & \text{if } n \text{ is odd}. \end{cases}$$

Proof. Immediate by Corollary 4.5 and Lemma 4.7. Recall that by convention $M(i, j) \neq 0$ if and only if $1 \leq i \leq m$, $1 \leq j \leq l$ and $2 \leq i + j \leq m + 1$. □

4.3. Proof of Theorem 2

With our basic computations done, we are ready to prove Theorem 2.

Theorem 2. $\Lambda_{m, l}$ admits an $n$-cluster tilting subcategory if and only if one of the following holds:

(i) $l = 2$ and $m = nk + 1$ for some $k \geq 0$, or
(ii) $n$ is even and $m = \frac{n}{2}l + 1 + k(nl - l + 2)$ for some $k \geq 0$.

Proof. For the case $l = 2$ we refer to Proposition 6.2 in [8]. Assume now that $l \geq 3$. Set

$$C = \text{add} \left( \bigoplus_{r=0}^{\infty} \tau_n^{-r} \Lambda_{m, l} \right).$$
By Remark 1(b) it is enough to prove that $C$ satisfies condition (a) of Theorem 1 if and only if $n$ is even and $m = \frac{n}{2} l + 1 + k (nl - l + 2)$ for some $k \geq 0$.

Assume first that $C$ satisfies condition (a) and $n$ is odd and we will reach a contradiction. Using Lemma 4.8 and an easy induction we can show that if $n$ is odd and $j < l$, we have

$$\tau^{-k}_n(M(1, j)) = M \left( 1 + k \left( \frac{n-1}{2} l + 1 \right), j \right).$$

Since $l \geq 3$, $M(1, 1)$ and $M(1, 2)$ are indecomposable projective noninjective by Lemma 4.3. Therefore, by condition (a2) of Theorem 1 there exist integers $k_1, k_2 > 0$ such that $\tau^{-k_1}_n M(1, 1)$ and $\tau^{-k_2}_n M(1, 2)$ are indecomposable injective. Computing

$$\tau^{-k_1}_n M(1, 1) = M \left( 1 + k_1 \left( \frac{n-1}{2} l + 1 \right), 1 \right),$$

$$\tau^{-k_2}_n M(1, 2) = M \left( 1 + k_2 \left( \frac{n-1}{2} l + 1 \right), 2 \right),$$

and using Lemma 4.3 we find that

$$M \left( 1 + k_1 \left( \frac{n-1}{2} l + 1 \right), 1 \right) = M(m, 1),$$

$$M \left( 1 + k_2 \left( \frac{n-1}{2} l + 1 \right), 2 \right) = M(m - 1, 2)$$

are the only possibilities. In particular,

$$1 + k_1 \left( \frac{n-1}{2} l + 1 \right) = m, 1 + k_2 \left( \frac{n-1}{2} l + 1 \right) = m - 1$$

which imply $(k_1 - k_2) \left( \frac{n-1}{2} l + 1 \right) = 1$, contradicting $n > 1$.

Hence, $n$ must be even; an easy induction here shows that for $j < l$ we have

$$\tau^{-k}_n(M(1, j)) = \begin{cases} M \left( 1 + j + k \frac{n-2}{2} l + k \left( \frac{n-2}{2} l + 1 \right), j \right) & \text{if } k \text{ is odd}, \\ M \left( 1 + k \frac{n-2}{2} l + k \left( \frac{n-2}{2} l + 1 \right), j \right) & \text{if } k \text{ is even}. \end{cases}$$

As before, $\tau^{-k}_n M(1, 1)$ and $\tau^{-k}_n M(1, 2)$ must be indecomposable injective for some integers $k_1, k_2 > 0$. If we assume that $k_1$ and $k_2$ have different parities or are both even, we reach a contradiction as in the case of $n$ being odd. Therefore $k_1$ must be odd and we have

$$\tau^{-k}_n M(1, 1) = M \left( 2 + k \frac{n-1}{2} l + k_1 \left( \frac{n-2}{2} l + 1 \right), l - 1 \right) = M(m + 2 - l, l - 1)$$

as the only possibility by Lemma 4.3. This implies $2 + k_1 \frac{n-1}{2} l + k_1 \left( \frac{n-2}{2} l + 1 \right) = m + 2 - l$ or equivalently $m = \frac{n}{2} l + 1 + \frac{k_1-1}{2} (nl - l + 2)$ so we get the result for $k = \frac{k_1-1}{2}$.

Now, assume that $n$ is even and that $m = \frac{n}{2} l + 1 + k (nl - l + 2)$ and we will show that condition (a) of Theorem 1 holds for $C$. (a1) holds by construction. Note that by Lemma 4.8, $\tau^{-k}_n M(1, j)$ is indecomposable or zero. For $s = 2k + 1$ we have

$$\tau^{-s}_n M(1, j) = M \left( 1 + j + \frac{s-1}{2} l + s \left( \frac{n-2}{2} l + 1 \right), l - j \right) = M(m + 1 + j - l, l - j),$$
which is injective by Lemma 4.3. Therefore \(\tau_{n-k}^n M(1, j)\) is nonzero for \(0 \leq k \leq s\), it is projective for \(k = 0\) and injective for \(k = s\). Then (a2) holds since by Lemma 4.8, we have that \(\tau_n \tau_n^M(i, j) = M(i, j)\) if \(\tau_n M(i, j) \neq 0\) and \(\tau_n^M M(i, j) = M(i, j)\) if \(\tau_n M(i, j) \neq 0\). Finally (a3) and (a4) hold by Lemma 4.4 and the proof is complete. \(\square\)

**Example 4.9.** For \(m = 9, l = 3, n = 2\) and \(k = 1\) the Auslander–Reiten quiver of \(\Lambda_{9,3} = KQ_9/(\text{rad } KQ_9)^3\) is

\[
\begin{array}{cccccccc}
1,3 & 2,3 & 3,3 & 4,3 & 5,3 & 6,3 & 7,3 \\
1,2 & 2,2 & 3,2 & 4,2 & 5,2 & 6,2 & 7,2 & 8,2 \\
1,1 & 2,1 & 3,1 & 4,1 & 5,1 & 6,1 & 7,1 & 8,1 & 9,1 \\
\end{array}
\]

where we write \((i, j)\) instead of \(M(i, j)\). The circled modules are the indecomposable summands of the 2-cluster tilting module of \(\Lambda_{9,3}\) and they satisfy

\[
\begin{array}{cccccccc}
1,1 & 2,2 & 3,2 & 4,2 & 5,2 & 6,2 & 7,2 & 8,2 \\
1,2 & 2,1 & 3,1 & 4,1 & 5,1 & 6,1 & 7,1 & 8,1 & 9,1 \\
\end{array}
\]

**5. \(d\)-Representation-finite Nakayama algebras**

In this section we classify the Nakayama algebras admitting a \(d\)-cluster tilting subcategory, where \(d = \text{gl.dim} \Lambda\). Even though cyclic Nakayama algebras are not representation-directed, we include a proof that shows that no cyclic Nakayama algebra is \(d\)-representation-finite to present the full classification. Note that the following proposition shows that the homogeneous case of the previous chapter plays a special role.

**Proposition 5.1.** Let \(\Lambda\) be a Nakayama algebra and assume that \(\Lambda\) admits a \(d\)-cluster tilting subcategory. Then \(\Lambda = \Lambda_{m,l}\), for some \(m, l\).

**Proof.** Let us first assume that \(\Lambda = KQ_m/I\) is an acyclic Nakayama algebra that admits a \(d\)-cluster tilting subcategory \(\mathcal{C}\). Assume to a contradiction that \(I \neq (\text{rad } KQ_m)^3\). In particular, \(I \neq \{0\}\). Then we claim that there exist some \(x\) and \(y\) such that at least one of the two following cases is true:

(a) \(M(x + 1, y)\) and \(M(x + 1, y + 1)\) are projective and \(x \geq 1\),
(b) \(M(x - 1, y + 1)\) and \(M(x, y)\) are injective and \(x + y < m + 1\).

To see this, notice that there exists a path \(\alpha_1 \cdots \alpha_{i-k} \in I\) such that either \(\alpha_{i+1} \cdots \alpha_{i-k+1} I \neq I\) or \(\alpha_{i-1} \cdots \alpha_{i-k-1} I \neq I\). In the first case, the coordinates \((i-k, k+2)\) do not correspond to an indecomposable module and it follows that \(M(i-k+1, k+1) \cong P_{i+1}\) and \(M(i-k+1, k+2) \cong P_{i+2}\) satisfy the assumptions of case (a). Similarly, in the second case \(M(i-k, k+1) \cong I(i-k)\) and \(M(i-k-1, k+2) \cong I(i-k-1)\) satisfy the assumptions of case (b). Let us prove that case (a) leads to a contradiction; the case (b) is similar.

Since the ideal \(I\) is admissible, we have \(y \geq 2\). In this case, the relevant part of the Auslander–Reiten quiver looks like
where the cross indicates the absence of the indecomposable module $M(x, y + 1)$, while the dotted lines indicate the presence of some indecomposable modules along them. In other words, since $x \geq 1$, we have that $(x + 1, y)$ is not in the same diagonal as $(1, 1)$.

Since $M(x + 1, y + 1) \neq 0$, we have that $M(x + 1, y)$ is noninjective. Then, by Proposition 2.2, $\tau^{-1}(M(x + 1, y)) = N$ is an indecomposable nonprojective module and moreover, by the same proposition, $\tau(N) = M(x + 1, y)$. By applying $\tau^{-}$ on this we get

$$M(x + 2, y) = \tau^{-}(M(x + 1, y)) = \tau^{-}(\Omega^d(N)),$$

so that

$$M(x + 2, y) = \Omega^{d-1}(N).$$

We have $\text{pd}(\Omega^{d-1}N) \leq 1$, since otherwise we would have $\text{pd}N > d$. Moreover, $\Omega^{d-1}N$ is not projective since $\tau(N) \neq 0$, so $\text{pd}(\Omega^{d-1}N) = 1$. Therefore $\text{pd}(M(x + 2, y)) = 1$. But since $M(x + 1, y + 1)$ is projective, the short exact sequence

$$0 \to M(x + 1, 1) \to M(x + 1, y + 1) \to M(x + 2, y) \to 0$$

implies that $\Omega M(x + 2, y) = M(x + 1, 1)$. But $M(x + 1, 1)$ is nonprojective, since it is a simple module different than $M(1, 1)$ which contradicts $\text{pd}M(x + 2, y) = 1$.

To complete the proof, it remains to show that a cyclic Nakayama algebra $\tilde{\Lambda}$ with $\text{gl.dim} \tilde{\Lambda} = \tilde{d} < \infty$ admits no $\tilde{d}$-cluster tilting subcategory. This case is very similar to the previous one so we omit most of the details.

Let $\tilde{\Lambda} = K\tilde{Q}_m/I$ be a cyclic Nakayama algebra where $\tilde{Q}_m$ is the quiver

Then $I \neq (\text{rad} \tilde{Q}_m)^I$ since otherwise $\tilde{\Lambda}$ is self-injective and thus of infinite global dimension. Then there exists an indecomposable projective noninjective module $\tilde{P}$ and we must have $\text{pd}(\tau^{-}\tilde{P}) = 1$ as in the previous case. Similarly to the previous case, it is not difficult to see that $\Omega \tau^{-}\tilde{P}$ is simple, which is a contradiction since there exists no simple projective $\tilde{\Lambda}$-module. $\square$
5.1. Global dimension of $\Lambda_{m,l}$

Since given $m$ and $l$ we know by Theorem 2 when $\Lambda_{m,l}$ admits an $n$-cluster tilting subcategory, it is enough to see what the global dimension of $\Lambda_{m,l}$ is and then check under what conditions on $m$ and $l$ we have $n = d$.

**Proposition 5.2.** Let $\Lambda = \Lambda_{m,l}$. Then

(a) Let $M(x, y) \neq 0$ and assume $x = 1$ or $y = l$. Then $pdM(x, y) = 0$.

(b) Let $M(x, y) \neq 0$ and assume $x > 1$ and $y \neq l$. Write $x - 2 = ql + r$ with $0 \leq r < l$. We have

$$pdM(x, y) = \begin{cases} 2q + 1 & \text{if } y < l - r, \\ 2q + 2 & \text{if } y \geq l - r. \end{cases} \quad (5.1)$$

(c) Let $m - 1 = q'l + r'$, $0 \leq r' \leq l - 1$. Then

$$pdM(m + 1 - j, j) = \begin{cases} \left\lfloor \frac{m-1}{l} \right\rfloor + \left\lfloor \frac{m-1}{l} \right\rfloor & \text{if } r' = 0 \text{ or } j \leq r', \\ \left\lfloor \frac{m-1}{l} \right\rfloor & \text{otherwise}. \end{cases} \quad (5.2)$$

(d) $gl.dim.\Lambda = \left\lfloor \frac{m-1}{l} \right\rfloor + \left\lfloor \frac{m-1}{l} \right\rfloor$.

**Proof.**

(a) Follows immediately from Lemma 4.3 since $M(x, y)$ is projective.

(b) Throughout, we use

$$pdM(x, y) = pd\Omega(M(x, y)) + 1. \quad (5.3)$$

We first prove (5.1) for $x + y \leq l$. In that case $x - 2 < x + y \leq l$ so that $q = 0$ and $r = x - 2$. Then by Lemma 4.4 $\Omega M(x, y) = M(1, x-1)$ which is projective by Lemma 4.3. Therefore, $pdM(x, y) = 1 = 0q + 1$ as required, since $y < l - (x - 2)$.

Now we use induction on $x + y \geq l$. The base case was just proved. Assume that (5.1) holds when $l \leq x + y \leq k - 1$. Let $M(x, y)$ be such that $x + y = k$. Since $x + y \geq l + 1$, Lemma 4.4 implies $\Omega M(x, y) = M(x + y - l, l - y)$. In particular, (5.1) holds for $\Omega M(x, y)$ by induction assumption. Let $x - 2 = ql + r$ and assume first that $y < l - r$. We calculate

$$x + y - l - 2 = ql + r + 2 + y - l - 2 = (q - 1)l + r + y,$$

where $r + y < r + l - y = l$, so $x + y - l - 2 = q'l + r'$ with $q' = q - 1$, $r' = r + y$. To apply (5.1) to $\Omega M(x, y)$, we need to compare $l - y$ with $l - r'$ so from $0 \geq -r$ we get

$$l - y \geq l - r - y = l - (r + y) = l - r',$$

and thus by (5.1) we have $pdM(x + y - l, l - y) = 2q' + 2$. Then, we have

$$pdM(x, y) = pdM(x + y - l, l - y) + 1 = 2q' + 2 + 1 = 2(q - 1) + 2 = 2q + 1,$$

as required.
For the last case, let \( x - 2 = ql + r \) and \( y \geq l - r \). Now we have
\[
x + y - l - 2 = ql + r + 2 + y - l - 2 = (q - 1)l + l + (r + y - l) = ql + (r + y - l).
\]
Since \( l - r \leq y < l \), we get
\[
0 \leq r + y - l < r \leq l - 1.
\]
So \( x + y - l - 2 = ql + r' \) with \( r' = r + y - l \). We compare \( l - y \) with \( l - r' = 2l - r - y \), so from \( l > r \) we get
\[
2l - r - y > l - y
\]
or
\[
l - r' > l - y.
\]
So \( \text{pdM}(x + y - l, l - y) = 2q + 1 \) by (5.1) and (5.3) now gives
\[
\text{pdM}(x, y) = \text{pdM}(x + y - l, l - y) + 1 = 2q + 1 + 1 = 2q + 2
\]
as required.

(c) We will prove (c) using (b). Let \( m + 1 - j - 2 = ql + r \) so that \( m - 1 = ql + j + r \) for \( 0 \leq r \leq l - 1 \). Assume first that \( j < l - r \) so that \( q' = q \) and \( r' = j + r < l \). Then \( r' \neq 0 \) and \( j \leq j + r = r' \), so that
\[
\left\lfloor \frac{m - 1}{l} \right\rfloor + \left\lfloor \frac{m - 1}{l} \right\rfloor = \left\lfloor \frac{ql + j + r}{l} \right\rfloor + \left\lfloor \frac{ql + j + r}{l} \right\rfloor
\]
\[
= 2q + \left\lfloor \frac{j + r}{l} \right\rfloor + \left\lfloor \frac{j + r}{l} \right\rfloor 0 \leq j + r < l
\]
\[
= 2q + \left\lfloor \frac{j + r}{l} \right\rfloor + \left\lfloor \frac{j + r}{l} \right\rfloor 0 \leq j + r < l
\]
\[
= 2q + 0 + 1 = 2q + 1 \quad (5.1) \quad \text{pdM}(m + 1 - j, j),
\]
as required.

Assume now that \( j \geq l - r \) so that \( j + r \geq l \) and
\[
m - 1 = ql + r + j = ql + l + r + j - l = (q + 1)l + (r + j - l) = q'l + r'.
\]
Note that \( j \leq r' \) gives \( l \leq r \), a contradiction, so \( j > r' \). If \( r' = 0 \) we have
\[
\left\lfloor \frac{m - 1}{l} \right\rfloor + \left\lfloor \frac{m - 1}{l} \right\rfloor = \left\lfloor \frac{q'l}{l} \right\rfloor + \left\lfloor \frac{q'l}{l} \right\rfloor
\]
\[
= 2q' = 2(q + 1) = 2q + 2 \quad (5.1) \quad \text{pdM}(m + 1 - j, j),
\]
as required. Finally, if \( r' \neq 0 \) we have
\[
\left\lfloor \frac{m - 1}{l} \right\rfloor + \left\lfloor \frac{m - 1}{l} \right\rfloor - 1 = \left\lfloor \frac{q'l + r'}{l} \right\rfloor + \left\lfloor \frac{q'l + r'}{l} \right\rfloor - 1
\]
\[
= 2q' + \left\lfloor \frac{r + j - l}{l} \right\rfloor + \left\lfloor \frac{r + j - l}{l} \right\rfloor 0 \leq r + j - l < l
\]
\[
= 2q + 2 + 0 + 1 - 1 = 2q + 2 \quad (5.1) \quad \text{pdM}(m + 1 - j, j),
\]
which completes the proof of (c).
(d) Note that by (c) we have \( \text{gl.dim}\Lambda_{m,l} \leq \left\lceil \frac{m-1}{l} \right\rceil + \left\lceil \frac{m-1}{l} \right\rceil \), since \( M(m + 1 - j, j) \) are exactly the injective non-projective \( \Lambda_{m,l} \)-modules. Since \( \text{pd}M(m, 1) = \left\lceil \frac{m-1}{l} \right\rceil + \left\lceil \frac{m-1}{l} \right\rceil \) by (c), the result follows. \( \square \)

5.2. Proof of Theorem 3

Now we are ready for the classification of the acyclic Nakayama algebras which are \( d \)-representation finite.

**Theorem 3.** If \( \text{gl.dim}\Lambda = d \), then \( \Lambda \) admits a \( d \)-cluster tilting subcategory \( \mathcal{C} \) if and only if \( \Lambda = \Lambda_{m,l} \) and \( l \mid m - 1 \) or \( l = 2 \). Moreover, in that case, \( \mathcal{C} = \text{add}(\Lambda \oplus DA) \) and \( d = 2^{m-1} \).

**Proof.** For the case \( l = 2 \) we refer to Proposition 6.2 in [8], so we assume \( l \geq 3 \).

Assume first that \( \Lambda = \Lambda_{m,l} \) and \( l \mid m - 1 \). Then, by Proposition 5.2, we have \( d = 2^{m-1} \). Then, Theorem 2 implies that \( \Lambda = \Lambda_{d_{l+1},l} \) admits a \( d \)-cluster tilting subcategory.

Assume now that \( \Lambda \) admits a \( d \)-cluster tilting subcategory. Then \( \Lambda = \Lambda_{m,l} \) by Proposition 5.1. By Theorem 2 we get

\[
m = \frac{d}{2}l + 1 + k(dl - l + 2)
\]

for some \( k \geq 0 \) and \( d \) must be even. By Proposition 5.2 we have that \( d \) is even if and only if \( d = 2^{m-1} \) which implies \( l \mid m - 1 \). Finally, a direct computation using Lemma 4.8 gives \( \tau_d^{-1}M(1,j) = M(m+1-l+j,l-j) \) for any \( 1 \leq j \leq l - 1 \). Since \( M(1,j) \) and \( M(m+1-l+j,l-j) \) are the indecomposable projective noninjective respectively injective nonprojective modules, we have \( \tau_d^{-k}\Lambda = 0 \) for \( k \geq 2 \). Hence

\[
\mathcal{C} = \text{add}\left( \bigoplus_{r=0}^{\infty} \tau_d^{-r}\Lambda \right) = \text{add}(\Lambda \oplus \tau_d^{-1}\Lambda) = \text{add}(\Lambda \oplus DA),
\]

which finishes the proof. \( \square \)

**Example 5.3.** As an example, let \( n = 4 \), and \( l = 4 \). Since we want \( n = d \), we must have \( m = \frac{4}{2}4 + 1 = 9 \). Then the Auslander–Reiten quiver of \( \Lambda_{9,4} \) is

\[
\begin{align*}
\begin{array}{cccccccc}
(1,4) & (2,4) & (3,4) & (4,4) & (5,4) & (6,4) \\
(1,3) & (2,3) & (3,3) & (4,3) & (5,3) & (6,3) & (7,3) \\
(1,2) & (2,2) & (3,2) & (4,2) & (5,2) & (6,2) & (7,2) & (8,2) \\
(1,1) & (2,1) & (3,1) & (4,1) & (5,1) & (6,1) & (7,1) & (8,1) & (9,1)
\end{array}
\end{align*}
\]

where the direct sum of all encircled modules is a 4-cluster tilting module.

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GLUING OF $n$-CLUSTER TILTING SUBCATEGORIES FOR REPRESENTATION-DIRECTED ALGEBRAS

LAERTIS VASO

Abstract. Given $n < d < \infty$, we investigate the existence of algebras of global dimension $d$ which admit an $n$-cluster tilting subcategory. We construct many such examples using representation-directed algebras. First, given two representation-directed algebras $A$ and $B$, a projective $A$-module $P$ and an injective $B$-module $I$ satisfying certain conditions, we show how we can construct a new representation-directed algebra $\Lambda := B \oplus I \oplus A$ in such a way that the representation theory of $\Lambda$ is completely described by the representation theories of $A$ and $B$. Next we introduce $n$-fractured subcategories which generalize $n$-cluster tilting subcategories for representation-directed algebras. We then show how one can construct an $n$-cluster tilting subcategory for $\Lambda$ by using $n$-fractured subcategories of $A$ and $B$. As an application of our construction, we show that if $n$ is odd and $d \geq n$ then there exists an algebra admitting an $n$-cluster tilting subcategory and having global dimension $d$. We show the same result if $n$ is even and $d$ is odd or $d \geq 2n$.

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Introduction

For a representation-finite algebra $\Lambda$, classical Auslander-Reiten theory gives a complete description of the module category $\text{mod} \Lambda$, see for example [ARS95]. However in general the whole module category of an algebra is very hard to study. In Osamu Iyama’s higher-dimensional Auslander-Reiten theory ([Iya07], [Iya08]) one replaces the focus from $\text{mod} \Lambda$ to a suitable subcategory $\mathcal{C} \subseteq \text{mod} \Lambda$ satisfying certain homological properties. Such a subcategory $\mathcal{C}$ is called an $n$-cluster tilting subcategory for some positive integer $n$; if moreover $\mathcal{C} = \text{add}(M)$ for some $M \in \text{mod} \Lambda$, then $M$ is called an $n$-cluster tilting module.

An $n$-cluster tilting subcategory $\mathcal{C}$ displays many higher-dimensional analogues to the classical Auslander-Reiten theory: the notions of $n$-Auslander-Reiten translations, $n$-almost split sequences and
$n$-Auslander-Reiten duality correspond to the classical Auslander-Reiten translations, almost split sequences and Auslander-Reiten duality when $n = 1$. However, in general it is not easy to find $n$-cluster tilting subcategories. If we set $d := \text{gl.dim} \Lambda$, then the existence of an $n$-cluster tilting subcategory for $n > d$ implies that $\Lambda$ is semisimple. Hence we may restrict to the case $n \leq d$.

The extreme case $n = d$ is of special interest and has been studied extensively before, for example in [IO13] and [HI10b]. If $C$ is given by a $d$-cluster tilting module $M$, it follows that $C$ is unique and given by

$$C = \text{add}\{\tau^{-i}_d(\Lambda) \mid i \geq 0\},$$

where $\tau^{-i}_d$ denotes the $d$-Auslander-Reiten translation. In this case $\Lambda$ is called $d$-representation-finite (see [HI10a], [IO10]). It is an open question whether all $d$-cluster tilting subcategories are given by $d$-cluster tilting modules. Nevertheless, if we assume the existence of a $d$-cluster tilting module $M$ we can obtain further results about $C = \text{add}(M)$. In particular in this case $C$ is directed if and only if $\text{add}(\Lambda)$ is directed. Furthermore it is asked in [HO14] if the mere existence of a $d$-cluster tilting module implies that $\text{add}(\Lambda)$ is directed.

Cases where $n < d$ have also been studied before. For the case where $\Lambda$ is selfinjective, and so $d = \infty$, see for example [EH08] and [DI17]. Note that in this case $C$ is never directed. A class of examples satisfying $n \leq d < \infty$ with $d \in n\mathbb{Z}$ first appeared in [Vas16] and many more were constructed recently in [JK17]. To our knowledge, the only known example where $n \nmid d$ appears in [Vas18] for $n = 2$ and $d = 3$.

Recall that an algebra $\Lambda$ is called representation-directed if there exists no sequence of nonzero nonisomorphisms $f_k : M_k \to M_{k+1}$ between indecomposable modules $M_0, \ldots, M_t$ with $M_0 \cong M_t$. We have the following characterization of $n$-cluster tilting subcategories for representation-directed algebras.

**Theorem 0.1.** [Vas18 Theorem 1] Assume that $\Lambda$ is a representation-directed algebra and let $C$ be a subcategory of $\text{mod} \Lambda$. Let $\mathcal{P}$ be the full subcategory of projective $\Lambda$-modules and let $\mathcal{C}_\mathcal{P}$ respectively $\mathcal{C}_\mathcal{I}$ be the sets of isomorphism classes of indecomposable nonprojective respectively noninjective $\Lambda$-modules. Then $C$ is an $n$-cluster tilting subcategory if and only if the following conditions hold:

1. $\mathcal{P} \subseteq C$,
2. $\tau_n$ and $\tau^{-n}_n$ induce mutually inverse bijections

$$\mathcal{C}_\mathcal{P} \xleftarrow{\tau_n} \mathcal{C}_\mathcal{I},$$

3. $\Omega^i(M)$ is indecomposable for all indecomposable $M \in \mathcal{C}_\mathcal{P}$ and $0 < i < n$,
4. $\Omega^{-i}(N)$ is indecomposable for all indecomposable $N \in \mathcal{C}_\mathcal{I}$ and $0 < i < n$.

Using this characterization, it is easy to check the existence of $n$-cluster tilting subcategories. Moreover, in this case $\Lambda$ is representation-finite and so any $n$-cluster tilting subcategory admits an additive generator. Finally, since $\text{mod} \Lambda$ is directed, we have that $C$ is also directed. As a consequence, it turns out that there is a unique choice for $C$. It follows that one of the simplest cases to consider when trying to find $n$-cluster tilting subcategories is that of $\Lambda$ being representation-directed.

In this paper we address the question of whether for a pair of positive integers $(n,d)$ with $n < d$ there exists an algebra $\Lambda$ of global dimension $d$, admitting an $n$-cluster tilting subcategory; we call such an algebra $(n,d)$-representation-finite. We show that for $n$ odd and any $d$ we can find an $(n,d)$-representation-finite algebra. Moreover, for $n$ even and $d$ odd or $d \geq 2n$ we again answer the question affirmatively.

To construct $(n,d)$-representation-finite algebras we first introduce the method of gluing. Our method takes as input a representation-directed algebra $\Lambda$ with a certain kind of projective module $P$ and a representation-directed algebra $B$ with a certain kind of injective module $I$ and returns a new representation-directed algebra $\Lambda := B \oplus P \oplus I A$. The representation theory of $\Lambda$ can be completely...
described in terms of the representation theories of $A$ and $B$. In general there may be several choices of $P$ and $I$, but choosing $P$ and $I$ to be simple modules always works.

If $A$ and $B$ admit $n$-cluster tilting subcategories, in general it is not true that $B^\tau A$ admits an $n$-cluster tilting subcategory. To this end we modify the characterization of $n$-cluster tilting subcategories given by Theorem 0.1 and introduce the more general notion of $n$-fractured subcategories. We show that under some compatibility conditions gluing of algebras admitting $n$-fractured subcategories gives rise to an algebra admitting an $n$-fractured subcategory. Moreover, by repeating this process sufficiently many times, one can arrive to an algebra which admits an actual $n$-cluster tilting subcategory, as desired.

Let us call an algebra $\Lambda$ strongly $(n,d)$-representation-directed if $\Lambda$ is representation-directed and $(n,d)$-representation-finite. As a corollary of our previous results we show that if $A$ is strongly $(n,d_1)$-representation-directed, $B$ is strongly $(n,d_2)$-representation-directed, $P$ is a simple projective $A$-module and $I$ is a simple injective $B$-module then $\Lambda = B^\tau A$ is strongly $(n,d)$-representation-directed for some $d$. By iterating this result, many new examples can be constructed. Moreover, while the global dimension $d$ of $\Lambda$ in general is difficult to compute, we show that in some simple cases we have $d = d_1 + d_2$.

This paper is divided in four parts. In the first part of the paper we introduce some basic notation and give a motivating example in detail. In the second part, given two representation-directed algebras $A$ and $B$, we describe our method of gluing of algebras and the associated results. In the third part we introduce $n$-fractured subcategories and describe how they are affected by gluing under certain conditions. In the fourth part of this paper we use these constructions to prove our results about the existence of $(n,d)$-representation-finite algebras. Most results are proved using standard techniques of representation theory: see for example the books [ARS95], [ASS06] as well as the survey article [Rin16]. Many examples are given throughout.

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1. Part I: Preliminaries

1.1. Conventions. Let us introduce some conventions and notation that we use throughout this paper. Let $K$ be an algebraically closed field and $n \geq 1$ an integer. In this paper by an algebra $\Lambda$ we mean a basic finite-dimensional unital associative algebra over $K$ and by $\Lambda$-module we mean a right $\Lambda$-module. We denote the category of right $\Lambda$-modules by mod $\Lambda$. We will write $\mathbb{mod}$ when the algebra is not clear from the context.

For a quiver $Q$ we will denote by $Q_0$ the set of vertices and by $Q_1$ the set of arrows. For an arrow $\alpha \in Q_1$ we will denote by $s(\alpha)$ its source and by $t(\alpha)$ its target. We concatenate paths in quivers from the right to the left, that is if $\alpha_i \in Q_1$ for $1 \leq i \leq n$ are arrows in $Q$, then $\alpha_1 \alpha_2 \cdots \alpha_{n-1} \alpha_n$ is a path in $Q$ if $s(\alpha_i) = t(\alpha_{i-1})$.

By a subcategory of an additive category we always mean a full subcategory closed under isomorphisms, direct sums and summands unless specified otherwise.

Now let $\mathcal{A}_i \subseteq \mathbb{mod} \Lambda$ be subcategories and $M_j \in \mathbb{mod} \Lambda$ be modules indexed by some $i \in I$ and $j \in J$. We set

- $\mathcal{A}_i$ — the set of isomorphism classes of indecomposable modules in $\mathcal{A}_i$,
- $|\mathcal{A}_i|$ — the cardinality of $\mathcal{A}_i$,
- $\text{add}(\mathcal{A}_i)$ — the subcategory of $\mathbb{mod} \Lambda$ containing all direct sums of modules $M$ such that $M \in \mathcal{A}_i$ for some $i \in I$,
- $\text{add}(M_i)$ — the subcategory of $\mathbb{mod} \Lambda$ containing all direct sums of direct summands of $M_i$,
- $\text{add}(\mathcal{A}_i, M_j)_{i \in I, j \in J} := \text{add}(\mathcal{A}_i)\text{add}(M_j)$,
- $\text{Sub}(\mathcal{A}_i)$ — the subcategory of $\mathbb{mod} \Lambda$ containing all submodules of modules in $\mathcal{A}_i$,
- $\text{Sub}(M_j) := \text{Sub}(\text{add}(M_j))$,
For the algebra $\Lambda$, we denote by $D$ the standard duality $D = \text{Hom}_K(-, K)$ between $\text{mod } \Lambda$ and $\text{mod } \Lambda^{op}$. By $\nu$ we denote the Nakayama functor $\nu = D \text{Hom}_\Lambda(-, \Lambda) : \text{mod } \Lambda \rightarrow \text{mod } \Lambda$. By an ideal of $\Lambda$ we mean a two-sided ideal, unless mentioned otherwise. For $X \in \text{mod } \Lambda$ we will denote by $\Omega(X)$ the syzygy of $X$, that is the kernel of $P \rightarrow X$, where $P$ is the projective cover of $X$ and by $\Omega_2(X)$ the cosyzygy of $X$, that is the cokernel of $X \hookrightarrow I$ where $I$ is the injective hull of $X$. Note that $\Omega(X)$ and $\Omega^-(X)$ are unique up to isomorphism. We will denote by $\tau$ and $\tau^-$ the Auslander-Reiten translations. Following [Vas18], we denote by $\tau^+_n$ and $\tau^-_n$ the $n$-Auslander-Reiten translations defined by $\tau^+_n(X) = \tau^n \Omega(X)$ and $\tau^-_n(X) = \tau^n \Omega^-(X)$.

Let $\phi : \Lambda \rightarrow \Gamma$ be an algebra homomorphism. We denote by $\phi_* : \text{mod } \Gamma \rightarrow \text{mod } \Lambda$ the restriction of scalars functor that turns a $\Gamma$-module $M$ into a $\Lambda$-module via $m \cdot \lambda = m \cdot \phi(\lambda)$ for $m \in M$ and $\lambda \in \Lambda$. We denote by $\phi^* : \text{mod } \Lambda \rightarrow \text{mod } \Gamma$ the induced module functor, given by $\phi^*(-) = - \otimes_{\Lambda} \Gamma$. Finally, we denote by $\phi^! : \text{mod } \Lambda \rightarrow \text{mod } \Gamma$ the coinduced module functor, given by $\phi^!(*) = \text{Hom}_\Lambda(\Gamma, -)$. Note that $(\phi^*, \phi_*)$ and $(\phi_*, \phi^!)$ form adjoint pairs. We denote by $A_h$ the quiver

$$1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \rightarrow \cdots \rightarrow h - 1 \xrightarrow{\alpha_{h-1}} h.$$ 

It is well known that the Auslander-Reiten quiver $\Gamma(KA_h) := \triangle(h)$ of $KA_h$ is

$$\triangle(h) :$$

$$[t_1 KA_h]$$

$$| [t_2 KA_h] \quad [\tau^-(t_2 KA_h)]$$

$$| [t_3 KA_h] \quad [\tau^-(t_3 KA_h)] \quad [\tau^2(t_3 KA_h)]$$

$$| [t_4 KA_h] \quad [\tau^-(t_4 KA_h)] \quad [\tau^-(t_4 KA_h)] \quad [\tau^2(t_3 KA_h)]$$

$$| [t_5 KA_h] \quad [\tau^-(t_5 KA_h)] \quad [\tau^2(t_3 KA_h)] \quad [\tau^2(t_3 KA_h)]$$

$$| [t_6 KA_h] \quad [\tau^-(t_6 KA_h)] \quad [\tau^2(t_3 KA_h)] \quad [\tau^2(t_3 KA_h)]$$

$$| [t_7 KA_h] \quad [\tau^-(t_7 KA_h)] \quad [\tau^2(t_3 KA_h)] \quad [\tau^2(t_3 KA_h)]$$

$$| [t_8 KA_h] \quad [\tau^-(t_8 KA_h)] \quad [\tau^2(t_3 KA_h)] \quad [\tau^2(t_3 KA_h)]$$

where we denote by $t_i$ the trivial path at the vertex $i \in A_h$. The shape of the above quiver will appear throughout our investigation. For more details about quivers with relations and Auslander-Reiten theory, we refer to [ASS06].

We also introduce some notation from [Vas18]. Let $\Lambda = K\mathbb{A}_m/I$ where $I$ is an admissible ideal. Then $\Lambda$ is called an acyclic Nakayama algebra and its representation theory is well known, see for example [ASS06] Chapter V. In particular, recall that the isomorphism classes of the indecomposable modules of $\Lambda$ can be described by the representations $M(i, j)$ of the form

$$0 \xrightarrow{1} 0 \xrightarrow{0} \cdots 0 \xrightarrow{0} 0 \xrightarrow{m-(i-1)-(j-1)} K \xrightarrow{1} \cdots \xrightarrow{1} K \xrightarrow{1} 0 \xrightarrow{0} \cdots 0 \xrightarrow{0} 0 \xrightarrow{m}$$

with $M(i, j)I = 0$ ([ASS06] Gabriel’s Theorem). We will use the convention that $M(i, j) = 0$ if the coordinates $(i, j)$ do not define a $\Lambda$-module. When drawing the Auslander-Reiten quiver of $\Lambda$ we will simply write $(i, j)$ for the vertex $[M(i, j)]$. Furthermore, for a vertex $k \in A_m$, we will denote by $P(k)$ respectively $I(k)$ the corresponding indecomposable projective respectively injective $\Lambda$-module.

For $m \geq h$ we further set $\Lambda_{m,h} := K\mathbb{A}_m/\text{rad}(K\mathbb{A}_m)^h$. In particular, for $\Lambda_{m,h}$-modules we have $M(i, j) \neq 0$ if and only if $1 \leq i \leq m$, $1 \leq j \leq h$ and $2 \leq i + j \leq m + 1$. With this notation, we also have $KA_h \cong K\mathbb{A}_h/\text{rad}(K\mathbb{A}_h)^h$. 

\begin{align}
\begin{array}{cccccccc}
0 & \rightarrow & \cdots & \rightarrow & 0 & \rightarrow & K & \rightarrow & 0 \\
1 & & & & \rightarrow & 1 & & \rightarrow & 0 \\
0 & & & & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 0 \\
m-(i-1)-(j-1) & & & & \rightarrow & 1 & & \rightarrow & 0 \\
m & & & & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 0 \\
m & & & & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 0 \\
m & & & & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 0 \\
m & & & & \rightarrow & 0 & \rightarrow & \cdots & \rightarrow & 0 \\
\end{array}
\end{align}

(1.1)
Following [Mil71], given two algebra homomorphisms \( f : A \to C \) and \( g : B \to C \), we define the pullback algebra \((\Lambda, \phi, \psi)\) of \( A \xrightarrow{f} C \xleftarrow{g} B\) to be the subalgebra
\[
\Lambda = \{(a, b) \in A \times B \mid f(a) = g(b)\} \subseteq A \times B,
\]
with \( \phi : \Lambda \to A \) and \( \psi : \Lambda \to B \) being induced by the natural projections. When clear from context, we will identify the pullback \((\Lambda, \phi, \psi)\) with the underlying algebra \( \Lambda \). Notice that whenever we have \( f(a) = g(b) \) for some \( a \in A \) and \( b \in B \), then there exists a unique \( \lambda \in \Lambda \) such that \( \phi(\lambda) = a \) and \( \psi(\lambda) = b \). It turns out that this is the pullback in the category of \( K\)-algebras. Notice that if \( g \) is a surjection, then so is \( \phi \) (but the converse is not true). In this case, this diagram is called a Milnor square of algebras (see [Mil71]).

1.2. A motivating example. Let us first give a motivating example that illustrates the theory that is developed in this paper.

Example 1.1. Let \( B = \Lambda_{0,4} \). Since \( B \) is representation-directed, the only possible candidate for a 2-cluster tilting subcategory is
\[
\mathcal{C}_B = \text{add} \left( \bigoplus_{i \geq 0} \tau_{-i}^-\Lambda \right).
\]
Let us draw the Auslander-Reiten quiver \( \Gamma(B) \) of \( B \) and encircle the vertices corresponding to indecomposable modules in \( \mathcal{C}_B \):

\[
\begin{array}{cccccccc}
\end{array}
\]
where the dotted lines denote the Auslander-Reiten translations. A necessary condition for \( \mathcal{C}_B \) to be a 2-cluster tilting subcategory is that applying \( \tau_{-}^-\) to a noninjective indecomposable module in \( \mathcal{C}_B \) should return a noninjective indecomposable module in \( \mathcal{C}_B \). Since we have \( \tau_{-}^- (M(7,1)) = 0 \) and \( \tau_{-}^- (M(7,2)) = 0 \), with \( M(7,1) \) and \( M(7,2) \) both being noninjective, we conclude that \( \mathcal{C}_B \) is not a 2-cluster tilting subcategory. Let us denote the full subquiver of \( \Gamma(B) \) containing the vertices \( \{(7,1), (7,2), (7,3), (8,1), (8,2), (9,1)\} \) by \( \triangle^{(7,3)} \). Notice that as quivers we have \( \triangle^{(7,3)} \cong \triangle(3) \).

Next let \( A = \Lambda_{6,5} \) and let \( \mathcal{C}_A = \text{add} \left( \bigoplus_{i \geq 1} \tau_{-i}^-A \right) \). As before we draw the Auslander-Reiten quiver \( \Gamma(A) \) of \( A \) and encircle the indecomposable modules in \( \mathcal{C}_A \):

\[
\begin{array}{cccccccc}
1 & 5 & & & 2 & 5 & & & 1 & 4 & & & 2 & 4 & & & 3 & 4 & & & 4 & 3 & & & 5 & 2 & & & 6 & 1,
\end{array}
\]
In this case \( \mathcal{C}_A \) is a 2-cluster tilting subcategory. Let us denote the full subquiver of \( \Gamma(A) \) containing the vertices \( \{(1,1), (1,2), (1,3), (2,1), (2,2), (3,1)\} \) by \( (1,3)\triangle \). Notice that again we have \( (1,3)\triangle \cong \triangle(3) \).
Let us now consider the algebra $\Lambda$ given by the quiver with relations

$$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7 \rightarrow 8 \rightarrow 9 \rightarrow 10 \rightarrow 11 \rightarrow 12,$$

where the dotted lines indicate zero relations. Then $\Lambda$ can be seen as a certain pullback diagram $A \xrightarrow{f} KA_3 \xrightarrow{g} B$. Let $C_\Lambda = \text{add} \left( \bigoplus_{i \geq 1} T_2^{-1} \Lambda \right)$. As before we draw the Auslander-Reiten quiver $\Gamma(\Lambda)$ of $\Lambda$ and encircle the indecomposable modules in $C_\Lambda$:

Notice that we can identify $\Gamma(\Lambda)$ with the amalgamated sum $\Gamma(B) \amalg \Gamma(A)$, under the identification $\Delta^{(7,3)} \equiv \Delta(3) \equiv (1,3)\Delta$. Under this identification we also see that much of the representation theory of $\Lambda$ is given by the representation theory of $B$ and $A$: indecomposable $\Lambda$-modules correspond to indecomposable $B$-modules or to indecomposable $A$-modules, almost split sequences in mod $\Lambda$ correspond to almost split sequences either in mod $B$ or in mod $A$ and similarly for syzygies and cosyzygies of indecomposable $A$-modules. Moreover $C_\Lambda$ is the additive closure of $C_A$ and $C_B$ viewed inside mod $\Lambda$. Notice that the indecomposable modules in $C_A$ and $C_B$ corresponding to the identified part match.

In this case $C_\Lambda$ turns out to be a 2-cluster tilting subcategory. In particular, in mod $\Lambda$ we have $\tau_2^{-1}(M(7,1)) \cong M(9,4)$ and $\tau_2^{-1}(M(7,2)) \cong M(10,3)$, since these functors can be computed in the subquiver corresponding to mod $A$.

In Example 1.1 we managed to get a 2-cluster tilting subcategory by identifying the “problematic” piece $\Delta^{(7,3)}$ of $\Gamma(B)$ with the “well-behaved” piece $(1,3)\Delta$ of $\Gamma(A)$. In this paper we explain how this process can be defined rigorously and under which conditions it can be applied.

2. Part II: Gluing

2.1. Glued subcategories. Let us first recall some definitions from [ASS1]. Let $\Lambda$ be an algebra and $\mathcal{A} \subseteq \mathcal{L} \subseteq \text{mod } \Lambda$ be subcategories. Recall that a morphism $g : M \rightarrow N$ in $\mathcal{A}$ is called right almost split if $g$ is not a retraction and any non-retraction $v : V \rightarrow N$ with $V \in \mathcal{A}$ factors through $g$. Dually we can define left almost split morphisms. A short exact sequence $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ in $\mathcal{A}$ is called an almost split sequence if $L$ is indecomposable and $g$ is right almost split or equivalently if $N$ is indecomposable and $f$ is left almost split.

A module $P$ in $\mathcal{A}$ is called $\mathcal{A}$-projective if $\text{Ext}^i_\mathcal{A}(P, \mathcal{A}) = 0$ for all $i > 0$ and a module $I$ in $\mathcal{A}$ is called $\mathcal{A}$-injective if $\text{Ext}^i_\mathcal{A}(\mathcal{A}, I) = 0$ for all $i > 0$. We say that $\mathcal{A}$ has almost split sequences if for any non-$\mathcal{A}$-projective indecomposable module $N \in \mathcal{A}$ there is an almost split sequence in $\mathcal{A}$ ending at $N$ and for any non-$\mathcal{A}$-injective indecomposable module $L \in \mathcal{A}$ there is an almost split sequence starting at $L$.

Next we introduce the notion of gluing of subcategories.

**Definition 2.1.** Assume that there exist subcategories $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{L}$ such that the following are satisfied.

(i) $\mathcal{A}$ and $\mathcal{B}$ have almost split sequences,

(ii) $\mathcal{L} = \text{add}(\mathcal{A}, \mathcal{B}),$

(iii) If $M \in \mathcal{A} \setminus \mathcal{B}$ and $M$ is indecomposable, then $\text{Hom}_{\mathcal{A}}(M, B) = 0$, 


(iv) If \( N \in \mathcal{B} \) and \( M \in \mathcal{A} \), then for all \( g : N \to M \), there exists an \( X \in \mathcal{A} \cap \mathcal{B} \) such that \( g = g_1 \circ g_2 \) for some \( g_1 : X \to N \) and \( g_2 : M \to X \).

In that case \( \mathcal{L} \) is called the **gluing of** \( \mathcal{B} \) and \( \mathcal{A} \) and we write \( \mathcal{L} = \mathcal{B} \triangle \mathcal{A} \). Note that gluing is not a commutative operation.

**Theorem 2.2.** Assume that \( \mathcal{L} = \mathcal{B} \triangle \mathcal{A} \).

(i) If \( N \in \mathcal{A} \) and \( 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \) is an almost split sequence in \( \mathcal{A} \), then it is also an almost split sequence in \( \mathcal{L} \).

(ii) If \( N \in \mathcal{B} \) and \( 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0 \) is an almost split sequence in \( \mathcal{B} \), then it is also an almost split sequence in \( \mathcal{L} \).

**Proof.**

(i) Since \( L \) and \( N \) are indecomposable in \( \mathcal{A} \), they are also indecomposable in \( \mathcal{L} \). Hence it is enough to show that \( g \) is right almost split in \( \mathcal{L} \). Clearly \( g \) is not a retraction in \( \mathcal{L} \). Let \( v : V \to N \) be a morphism in \( \mathcal{L} \) which is not a retraction and without loss of generality assume that \( V \) is indecomposable. By Definition 2.1(ii), we have that \( V \in \mathcal{A} \) or \( V \in \mathcal{B} \). If \( V \in \mathcal{A} \), then \( v \) factors through \( g \) because \( g \) is right almost split in \( \mathcal{A} \). If \( V \in \mathcal{B} \setminus \mathcal{A} \), then by Definition 2.1(iv) there exists some \( X \in \mathcal{A} \cap \mathcal{B} \) such that \( v = v_1 \circ v_2 \) with \( v_1 : X \to N \) and \( v_2 : V \to X \).

Note that \( v_1 \) is not a retraction since \( \text{Hom}_\mathcal{A}(N,X) = 0 \) by Definition 2.1(iii). Hence there exists \( h : X \to M \) such that \( v_1 = g \circ h \). But then \( v = g \circ (h \circ v_2) \) as required.

(ii) Similarly to (i), let \( v : V \to N \) be a nonzero non-retraction in \( \mathcal{L} \). Since \( \text{Hom}_\mathcal{A}(V,B) \neq 0 \), we have that \( V \not\in \mathcal{A} \setminus \mathcal{B} \) by Definition 2.1(iii). Hence \( V \in \mathcal{B} \) and \( v \) factors through \( g \) since \( g \) is right almost split in \( \mathcal{B} \).

\[ \square \]

2.2. **Glued representation-directed algebras.** Throughout this subsection let \( \Lambda \) be a representation-directed algebra.

2.2.1. **Abutments.** If \( M \) is a \( \Lambda \)-module, then \( M \) is said to be **uniserial** if it has a unique composition series. In this case \( M \) has simple top and socle and hence \( M \) is indecomposable. Being uniserial is equivalent to the **radical series**

\[ 0 \subseteq \cdots \subseteq \text{rad}^2(M) \subseteq \text{rad}(M) \subseteq M \]

being a composition series of \( M \) [ASS06] Lemma V.2.2.]. Clearly submodules of uniserial modules are uniserial.

**Definition 2.3.** Let \( \Lambda \) be a representation-directed algebra. We call a uniserial projective \( \Lambda \)-module \( P \) a **left abutment** if every submodule of \( P \) is projective and for any indecomposable projective \( \Lambda \)-module \( P' \) not isomorphic to a submodule of \( P \), we have that all morphisms \( U \to P' \) with \( U \subseteq P \) factor through \( P \).

We call an indecomposable injective \( \Lambda \)-module \( I \) a **right abutment** if \( D(I) \) is a left abutment as a \( \Lambda^{\text{op}} \)-module.

Let \( P \) be a left abutment with composition series

\[ 0 \subseteq P_h \subseteq \cdots \subseteq P_2 \subseteq P_1 = P. \]

Then the modules \( P_i \) are also uniserial and so indecomposable. Hence there exist primitive orthogonal idempotents \( e_1, \ldots, e_h \) such that \( P_1 \cong e_1 \Lambda \) and hence the composition series of \( P \) corresponds to a diagram

\[ 0 \xrightarrow{f_h} e_h \Lambda \xrightarrow{f_{h-1}} \cdots \xrightarrow{f_2} e_2 \Lambda \xrightarrow{f_1} e_1 \Lambda, \]

where \( f_i \in \text{Hom}_\Lambda(e_{i+1}, e_i \Lambda) = e_i \Lambda e_{i+1} \). We call such a choice of \( (e_i, f_i)_{i=1}^h \) a **realization of the left abutment** \( P \) and we denote \( e_i = \sum_{i=1}^h e_i \). Note that \( h \) is the length \( l(P) \) of \( P \) and that \( f_h = 0 \). We will call \( h \) the **height of the left abutment** \( P \).
For a right abutment $I$ such that $D(I)$ has a realization $(e_{h-i+1}, f_{h-i-1})_{i=1}^h$, we call $(e_i, D(f_{i-1}))_{i=1}^h$ a realization of the right abutment $I$ and $h$ the height of the right abutment $I$. Diagrammatically, we have a sequence of factor modules

$$D(\Lambda e_h) \xrightarrow{g_{h-1}} D(\Lambda e_{h-1}) \xrightarrow{g_{h-2}} \cdots \xrightarrow{g_1} D(\Lambda e_1) \xrightarrow{g_0} 0,$$

where $g_0 = 0$.

Note that simple projective modules are the same as left abutments of height 1 and simple injective modules are the same as right abutments of height 1. Note also that since $\Lambda$ is representation-directed, there exists at least one simple projective module and one simple injective module.

The following lemma is an easy observation.

**Lemma 2.4.** Let $\Lambda$ be a representation-directed algebra.

(a) If $P$ is a left abutment realized by $(e_i, f_i)_{i=1}^h$, then

$$\dim_K (\text{Hom}_\Lambda(e_i, \Lambda)) = \begin{cases} 1 & \text{if } i \geq j, \\ 0 & \text{if } i < j. \end{cases}$$

In particular, $\{f_i \circ \cdots \circ f_{i+k}\}$ is a basis of $\text{Hom}_\Lambda(e_{i+k+1}, e_i)$. 

(b) If $I$ is a right abutment realized by $(e_i, g_i)_{i=1}^h$, then

$$\dim_K (\text{Hom}_\Lambda(D(I), D(I))) = \begin{cases} 1 & \text{if } i \geq j, \\ 0 & \text{if } i < j. \end{cases}$$

In particular, $\{g_i \circ \cdots \circ g_{i+k}\}$ is a basis of $\text{Hom}_\Lambda(D(I_{i+k+1}), D(I_i))$.

**Proof.** We only prove (a); then (b) follows from the definition and (a). If $i = j$ then by [ASS05, Proposition IX.1.4] we have $\text{End}_\Lambda(e_i, e_i) \cong K$. Notice that by definition, if $i > j$, we have $\text{Hom}_\Lambda(e_i, e_j) \neq 0$. Since $\Lambda$ is representation-directed, it follows that for $i < j$ we have $\text{Hom}_\Lambda(e_i, e_j) = 0$. It remains to show that for $i > j$, we have $\text{Hom}_\Lambda(e_i, e_j) \cong K$. Since the morphism $f_i : e_{i+1} \Lambda \rightarrow e_i \Lambda$ corresponds to the radical inclusion $\text{rad}(e_i \Lambda) \subseteq e_i \Lambda$, it follows that any homomorphism $g_i : e_{i+1} \Lambda \rightarrow e_i \Lambda$ factors through $f_i$. Since $\text{End}_\Lambda(e_i, e_i) \cong K$, it follows that $g_i = \lambda f_i$ for some $\lambda \in K$ and so $\text{Hom}_\Lambda(e_{i+1}, e_i) \cong K$. The result follows by a simple induction. \qed

**Remark 2.5.** The requirement of $P$ being a left abutment in Lemma 2.4(a) is stronger than what is used in the proof. Specifically, Lemma 2.4(a) holds for any uniserial projective module such that all submodules are projective and dually for Lemma 2.4(b).

The following Lemma describes abutments in terms of quivers with relations.

**Proposition 2.6.** Let $\Lambda$ be a representation-directed algebra.

(a) $P$ is a left abutment realized by $(e_i, f_i)_{i=1}^h$ if and only if there exists a presentation $KQ_{A, I}$ of $\Lambda$ such that $Q_A$ is of the form

![Diagram of quiver](image)

no path of the form $\alpha_i \cdots \alpha_{i+k}$ is in $I$, and there exists an isomorphism $\Phi : KQ_{A, I} \xrightarrow{\sim} \Lambda$ such that for $1 \leq i \leq h$ we have $\Phi(e_i) = e_i$ and for $1 \leq i \leq h - 1$ we have $\Phi(\alpha_i) = f_i$, where $e_i$ is the idempotent of $KQ_{A, I}$ corresponding to the vertex $i$. 
(b) \( J \) is a right abutment realized by \((e_i, g_{i-1})_{i=1}^h \) if and only if there exists a presentation \( KQ_\Lambda/I \) of \( \Lambda \) such that \( Q_\Lambda \) is of the form

\[
\begin{array}{cccccccc}
1 & \rightleftharpoons & 2 & \rightarrow & 3 & \rightarrow & \cdots & \rightarrow & h-1 & \rightarrow & h \\
\end{array}
\]

no path of the form \( \alpha_i \cdots \alpha_{i+k} \) is in \( I \), and there exists an isomorphism \( \Phi : KQ_\Lambda/I \rightarrow \) such that for \( 1 \leq i \leq h \) we have \( \Phi(e_i) = e_i \) and for \( 1 \leq i \leq h-1 \) we have \( \Phi(\alpha_i) = g_i \), where \( e_i \) is the idempotent of \( KQ_\Lambda/I \) corresponding to the vertex \( i \).

**Proof.** (a) Throughout this proof let \( e' = 1_\Lambda - e_i \) and identify \( \text{Hom}_\Lambda(e_i, e_j) \) with \( e_j \Lambda e_i \). In particular, we have that \( \Lambda = e_i \Lambda \oplus e' \Lambda \).

Assume first that \( P \) is a left abutment realized by \((e_i, f_i)_{i=1}^h \) and let \( KQ_\Lambda/I \) be a presentation of \( \Lambda \). In \( Q_\Lambda \) we have the vertices \( Q_{e_i} := \{1, \ldots, h\} \), corresponding to the idempotents \( e_1, \ldots, e_h \), and we set \( Q_{e'} := (Q_{\Lambda})_0 \setminus Q_{e_i} \). We first claim that \( e_i \Lambda e' = 0 \). Equivalently, it is enough to show that \( \text{Hom}_\Lambda(e' \Lambda, e_\Lambda) = 0 \). Assume to a contradiction that there exists a nonzero morphism \( g : e' \Lambda \rightarrow e_i \Lambda \). By the uniqueness of the composition series of \( e_i \Lambda \) it follows that \( \text{rad}(e_i \Lambda) = e_{i+1} \Lambda \) and hence \( g \) factors through \( e_{i+1} \Lambda \). Continuing inductively, we find that \( g \) factors through \( e_h \Lambda \) which is a simple projective module. Since \( e' \Lambda \) is projective and \( e_h \Lambda \notin \text{add}(e' \Lambda) \), we have a contradiction.

Since \( e_i \Lambda e' = 0 \) and since the ideal \( I \) is admissible, it follows that there exists no arrow from \( Q_{e_i} \) to \( Q_{e'} \). Next, let us compute the arrows from \( Q_{e_i} \) to \( Q_{e'} \). Since \( e_h \Lambda \) is a simple projective module, \( h \) is a sink. Let now \( 1 \leq i \leq h-1 \). Since \( \text{rad}(e_i \Lambda) = e_{i+1} \Lambda \), it follows that \( e_i \Lambda/e_{i+1} \Lambda \cong \text{top}(e_i \Lambda) = S_i \) and hence for any \( 1 \leq j \leq h \) we have

\[
e_i(\text{rad}(\Lambda)/\text{rad}^2(\Lambda))e_j = (\text{rad}(e_i \Lambda)/\text{rad}^2(e_i \Lambda))e_j \cong S_{i+1}e_j = \begin{cases} S_{i+1} & \text{if } j = i + 1, \\ 0 & \text{otherwise}. \end{cases}
\]

In particular, \( \dim_K(e_i(\text{rad}(\Lambda)/\text{rad}^2(\Lambda))e_j) = \delta_{i,j-1} \) since \( \Lambda \) is basic. Since the number of arrows \( i \rightarrow j \) in \( Q_\Lambda \) is the same as the dimension of \( e_i(\text{rad}(\Lambda)/\text{rad}^2(\Lambda))e_j \), we conclude that there is exactly one arrow with source \( i \) and its target is \( i+1 \). We denote this arrow by \( \alpha_i \). Since the morphisms \( f_i \) are in \( \text{rad}_\Lambda(e_{i+1} \Lambda, e_i \Lambda) \) (because they are irreducible morphisms), there exists some isomorphism \( \Phi : KQ_\Lambda/I \rightarrow \Lambda \) that maps \( e_i \mapsto e_i \) and \( e_i(KQ_\Lambda/I) \ni \alpha_i \mapsto f_i \in e_i \Lambda e_{i+1} \).

Next we want to compute the arrows from \( Q_{e'} \) to \( Q_{e_i} \). Assume that there exists such an arrow \( \alpha \in e_s \Lambda e_i \) for some \( s < 1 \) and some \( s \in Q_{e'} \). Then this corresponds to some nonzero \( f_s \in \text{rad}_\Lambda(e_i \Lambda, e_s \Lambda) \). By the factorization property of the abutment, and using Lemma 2.4(a) of [2], we have that \( f_s = g \circ \lambda(f_{i-1} \cdots f_{i-1}) \) for some \( g \in \text{hom}(e_1 \Lambda, e_s \Lambda) \) and some nonzero \( \lambda \in K \).

Hence we have that \( \alpha = \lambda \Phi^{-1}(g) \alpha_1 \cdots \alpha_{i-1} \) or that \( \alpha - \lambda \Phi^{-1}(g) \alpha_1 \cdots \alpha_{i-1} \in I \), which means that removing the arrow \( \alpha \) from \( Q_\Lambda \) gives an isomorphic presentation. Therefore, we can pick a quiver \( Q_\Lambda \) such that there are no arrows from \( Q_{e'} \) to \( Q_{e_i} \).

Finally, assume to a contradiction that \( \Lambda \cong KQ_\Lambda/I \) and \( \alpha_1 \cdots \alpha_{i+k} \in I \) with \( i + k \) maximal. Then it is easy to check by a direct computation that \( \text{rad}(e_i \Lambda) \neq e_{i+1} \Lambda \), which contradicts \( e_i \Lambda \) being a left abutment. Hence no path of the form \( \alpha_1 \cdots \alpha_{i+k} \) is in \( I \).

For the other direction, in the given presentation of \( \Lambda \), we have by a direct computation that \( e_i \Lambda \) is a simple projective module and for \( 1 \leq i \leq h-1 \) we have \( \text{rad}(e_i \Lambda) \cong e_{i+1} \Lambda \). Therefore the element \( e_i \in e_i \Lambda e_{i+1} = \text{Hom}_\Lambda(e_{i+1} \Lambda, e_i \Lambda) \) corresponds to the inclusion \( \text{rad}(e_i \Lambda) \subseteq e_i \Lambda \). Hence the radical series of \( e_1 \Lambda \) is its composition series and so \( e_1 \Lambda \) is uniserial. Moreover this...
composition series corresponds to the diagram

\[ 0 \rightarrow e_k \Lambda \xrightarrow{\alpha_{k-1}} \cdots \xrightarrow{\alpha_2} e_2 \Lambda \xrightarrow{\alpha_1} e_1 \Lambda. \]

Since there are no other arrows with target \( j \) for \( 2 \leq j \leq h \), then for \( k \not\in \{1, \ldots, h\} \) we have

\[ \text{Hom}_\Lambda(e_j \Lambda, e_k \Lambda) = e_k \Lambda e_j = e_k \Lambda e_1 \cdots e_{j-1} e_j = \text{Hom}_\Lambda(e_1 \Lambda, e_k \Lambda) a_1 \cdots a_{j-1}. \]

It follows that \( e_1 \Lambda \) is a left abutment with realization \( (e_i, \alpha_i)_{i=1}^h \), so the claim is satisfied for \( e_i = e_1 \) and \( \alpha_i = f_i \).

(b) This follows immediately from the definition and (a), since \( Q_{\Lambda^{op}} = Q_{\Lambda}^{op} \).

\[ \square \]

Proposition 2.6 shows that abutments are linearly oriented arms in the sense of Ringel [Rin16].

**Remark 2.7.** It follows from Proposition 2.6 that if \( (e_i, f_i)_{i=1}^h \) is a realization of a left abutment of height \( h \), then \( (e_i, g_i)_{i=1}^h \) is a realization of a left abutment of height \( h - k + 1 \), for any \( 1 \leq k \leq h \). In particular, a submodule of a left abutment is also a left abutment.

Similarly, if \( (e_i, g_{i-1})_{i=1}^h \) is a realization of a right abutment of height \( h \), then \( (e_i, f_i)_{i=1}^h \) is a realization of a right abutment of height \( k \), for any \( 1 \leq k \leq h \). In particular, a quotient module of a right abutments is also a right abutment.

If \( \Lambda \) is given by a quiver with relations, it is easy to find all abutments using Proposition 2.6 as the following examples show.

**Example 2.8.** Let \( B \) be given by the quiver with relations

\[
\begin{align*}
1 & \rightarrow 2 \rightarrow 3 \rightarrow 4 \\
1' & \rightarrow 2' \rightarrow 3'.
\end{align*}
\]

Then by Proposition 2.6 the left abutments are \( P(5), P(6) \) and \( P(7) \) with heights 3, 2 and 1 respectively, and the right abutments are \( I(3), I(2), I(1), I(3'), I(2') \) and \( I(1') \) with heights 3, 2, 1, 3, 2 and 1 respectively.

**Example 2.9.** It follows from Proposition 2.6(a) that the algebra \( KA_h \) has exactly \( h \) left abutments, namely \( \{ t_i KA_h \}_{i=1}^h \) and that the height of \( t_i KA_h \) is \( h - i + 1 \). By Proposition 2.6(b) the algebra \( KA_h \) has exactly \( h \) right abutments, namely \( \{ D(KA_h t_j) \}_{j=1}^h \) and the height of \( D(KA_h t_j) \) is \( i \). In particular, \( t_1 KA_h \cong D(KA_h t_1) \) is both a left and a right abutment.

By the same proposition it follows that if an algebra \( \Lambda \) admits a module \( M \) that is both a left and a right abutment, then \( \Lambda \cong KA_h \) and \( M \) is the unique indecomposable projective-injective \( KA_h \)-module. In particular, \( M \) has the same height \( h \) as a left and a right abutment.

We have the following important Corollary.

**Corollary 2.10.** Let \( U \) be a left abutment realized by \( (e_i, f_i)_{i=1}^h \) (respectively a right abutment realized by \( (e_i, g_{i-1})_{i=1}^h \) and let \( \Phi : KQ_{\Lambda}/I \rightarrow \Lambda \) be as in Proposition 2.6(a) (respectively as in Proposition 2.6(b)). Let \( \pi \) be the epimorphism \( \pi : KQ_{\Lambda}/I \rightarrow KA_h \) given by identifying the full subquiver with vertices \( Q_{e_i} \) with \( A_h \). Then the morphism \( \pi \circ \Phi^{-1} \) is independent of the choice of \( \Phi \) and it satisfies \( \pi \circ \Phi^{-1}(e_i) = t_i \) for \( 1 \leq i \leq h \) and \( \pi \circ \Phi^{-1}(f_i) = \alpha_i \) (respectively \( \pi \circ \Phi^{-1}(g_i) = \alpha_i \)) for \( 1 \leq i \leq h - 1 \).

**Proof.** Let us assume that \( U \) is a left abutment and \( \Phi \) is as in Proposition 2.6(a); the other case is similar. Notice that we have a short exact sequence

\[
0 \rightarrow \left( 1 - \sum_{i=1}^h e_i \right) \rightarrow KQ_{\Lambda}/I \xrightarrow{\pi} KA_h \rightarrow 0,
\]
In particular, \( \pi \left( \sum_{i=1}^{h} \epsilon_i \right) = 1_{KA_h} \). Let \( \Phi, \Psi : KQ\Lambda / I \to \Lambda \) be isomorphisms satisfying \( \Phi(\epsilon_i) = e_i = \Psi(\epsilon_i) \) and \( \Phi(\alpha_i) = f_i = \Psi(\alpha_i) \). By the description in Proposition 2.6(a) we have that \( \Phi^{-1}(e,a) = \Psi^{-1}(e,a) \) for all \( a \in \Lambda \). It follows that

\[
\pi \circ \Phi^{-1}(a) = \pi(\Phi^{-1}(a)) = \pi \left( \sum_{i=1}^{h} \epsilon_i \right) \pi(\Phi^{-1}(a)) = \pi \left( \sum_{i=1}^{h} \epsilon_i \right) \Phi^{-1}(a) = \pi \left( \Phi^{-1}(e,a) \right) = \pi \circ \Psi^{-1}(a),
\]

which proves that \( \pi \circ \Phi^{-1} = \pi \circ \Psi^{-1} \). The equalities \( \pi \circ \Phi^{-1}(e_i) = t_i \) and \( \pi \circ \Phi^{-1}(f_i) = \alpha_i \) are immediate by definition. \( \square \)

Corollary 2.10 justifies the following definition.

**Definition 2.11.** For a left abutment \( P \) realized by \( (e_i, f_i) \) \( \sum_{i=1}^{h} \epsilon_i \) (respectively a right abutment \( I \) realized by \( (e_i, g_{i-1}) \) \( \sum_{i=1}^{h} \epsilon_i \)) we denote the epimorphism \( \pi \circ \Phi^{-1} : \Lambda \to KA_h \) by \( f_P \) (respectively \( g_I \)) and we call it the footing at \( P \) (respectively \( I \)).

An easy consequence of Definition 2.11 is the following.

**Corollary 2.12.** Let \( \Lambda \) be a representation-directed algebra.

(a) If \( P \) is a left abutment realized by \( (e_i, f_i) \) \( \sum_{i=1}^{h} \epsilon_i \), then \( f_P(e, \lambda) = 0 \) implies \( e, \lambda = 0 \).

(b) If \( I \) is a right abutment realized by \( (e_i, g_{i-1}) \) \( \sum_{i=1}^{h} \epsilon_i \), then \( g_I(\lambda e_i) = 0 \) implies \( \lambda e_i = 0 \).

**Proof.** We only prove (a); (b) is similar. Assume to a contradiction that \( f_P(e, \lambda) = 0 \) but \( e, \lambda \neq 0 \). Since \( \Phi^{-1} \) is an isomorphism, it follows that \( \Phi^{-1}(e, \lambda) \neq 0 \). By Proposition 2.6(a), we have that

\[
\Phi^{-1}(e, \lambda) = \left( \sum_{i=1}^{h} \epsilon_i \right) \Phi^{-1}(\lambda) \neq 0.
\]

By the same Lemma and the definition of \( \pi \), it follows that

\[
\pi \left( \sum_{i=1}^{h} \epsilon_i \right) \Phi^{-1}(\lambda) \neq 0.
\]

But since \( f_P(e, \lambda) = \pi \circ \Phi^{-1}(e, \lambda) \), we have reached a contradiction. \( \square \)

The following Lemma describes abutments in terms of the Auslander-Reiten quiver \( \Gamma(\Lambda) \) of \( \Lambda \).

**Proposition 2.13.** Let \( \Lambda \) be a representation-directed algebra and \( \Gamma(\Lambda) \) be its Auslander-Reiten quiver.
(a) $P = e_1 \Lambda$ is a left abutment realized by $(e_i, f_i)_{i=1}^h$, if and only if
\[ P_{\Delta} : \]

\[
\begin{array}{c}
\tau \tau^{-1} (e_2) \\
\tau \tau^{-2} (e_3) \\
\tau^{-1} (e_2) \\
\tau^{-2} (e_3) \\
\tau^{-1} (e_1) \\
\tau^{-2} (e_2) \\
\tau^{-3} (e_1) \\
\tau^{-2} (e_3) \\
\tau^{-3} (e_2) \\
\tau^{-3} (e_3) \\
\end{array}
\]

is a full subquiver of $\Gamma(\Lambda)$, there are no other arrows in $\Gamma(\Lambda)$ going into $P_{\Delta}$ and, moreover, all northeast arrows are monomorphisms, all southeast arrows are epimorphisms and all modules in the same row have the same dimension. In particular, $\tau^{-i} (e_h \Lambda)$ is the simple top of $e_{h-i} \Lambda$ for $1 \leq i \leq h - 1$. We call $P_{\Delta}$ the foundation of $P$.

(b) $I = D(\Lambda e_h)$ is a right abutment realized by $(e_i, g_i)_{i=1}^h$, if and only if
\[ I_{\Delta} : \]

\[
\begin{array}{c}
\tau D(\Lambda e_{h-1}) \\
\tau^2 D(\Lambda e_{h-2}) \\
\tau^3 D(\Lambda e_{h-3}) \\
\tau^4 D(\Lambda e_{h-4}) \\
\tau^3 D(\Lambda e_{h-5}) \\
\tau^2 D(\Lambda e_{h-6}) \\
\tau D(\Lambda e_{h-7}) \\
\tau D(\Lambda e_{h-8}) \\
\tau D(\Lambda e_{h-9}) \\
\tau D(\Lambda e_{h}) \\
\end{array}
\]

is a full subquiver of $\Gamma(\Lambda)$, there are no other arrows in $\Gamma(\Lambda)$ leaving $I_{\Delta}$ and, moreover, all northeast arrows are monomorphisms, all southeast arrows are epimorphisms and all modules in the same row have the same dimension. In particular, $\tau^i D(\Lambda e_1)$ is the simple socle of $D(\Lambda e_{i+1})$ for $1 \leq i \leq h - 1$. We call $I_{\Delta}$ the foundation of $I$.

**Proof.** We only prove (a); (b) is similar. Assume first that $P_{\Delta}$ is a full subquiver of $\Gamma(\Lambda)$ satisfying the required properties. Since all northeast arrows are monomorphisms and there are no other arrows going into $P_{\Delta}$, it follows that $e_1 \Lambda$ is uniserial. For the factorization property, let $P'$ be an indecomposable projective module such that there exists a nonzero homomorphism $\phi : e_1 \Lambda \to P'$ and $P' \not\subseteq e_1 \Lambda$ for some $1 \leq i \leq h$. Set $J_i = \tau^{-(h-i)} (e_{h-i+1} \Lambda)$ and $\mathcal{J} := \{J_i\}_{i=1}^h$, that is $\mathcal{J}$ is the additive closure of all indecomposable modules $X$ such that $[X]$ appears in the rightmost southeast diagonal of $P_{\Delta}$. Since the only indecomposable projective modules $Y$ with $[Y] \in P_{\Delta}$ are isomorphic to submodules of $e_1 \Lambda$, it follows that $[P'] \not\subseteq P_{\Delta}$. Since there are no arrows going into $P_{\Delta}$, the only arrows going out of $P_{\Delta}$ have one of the vertices $\{[J_1], \ldots, [J_h]\}$ as a source. Hence $\phi$ factors through $\mathcal{J}$ so that $\phi = g_1 \circ g_2$
with \( g_2 : e_1 \Lambda \to N \) and \( g_1 : N \to P' \) for some \( N \in \mathcal{J} \). Moreover, since there are no other arrows going out of \( P^\Delta \setminus \{[J_1], \ldots, [J_h]\} \), all squares in \( P^\Delta \) correspond to almost split sequences and hence they are commutative. It follows that any morphism from \( e_1 \Lambda \) to \( \mathcal{J} \) factors through \( e_1 \Lambda \). Hence the morphism \( g_2 \) factors through \( e_1 \Lambda \) which shows that \( \phi : e_1 \Lambda \to P' \) factors through \( e_1 \Lambda \), as required.

For the other direction we use induction on \( h \geq 1 \). If \( h = 1 \) then \( e_1 \Lambda \) is a simple projective module and so there are no irreducible morphisms in \( \Gamma(\Lambda) \) into \( e_1 \Lambda \). Assume the result is true for \( h = k \) and we will prove it for \( h = k + 1 \). By induction hypothesis, and since by Remark \( \textbf{2.7} \) we have that \( e_2 \Lambda \) is also a left abutment of height \( h - 1 \), it follows that

\[
\begin{array}{c}
\varepsilon_{2^\Delta} : \\
\end{array}
\]

is also a full subquiver of \( \Gamma(\Lambda) \) and there are no other arrows in \( \Gamma(\Lambda) \) going into \( e_2 \Lambda^\Delta \). Since \( e_1 \Lambda \) is uniserial, we have \( e_2 \Lambda \cong \text{rad}(e_1 \Lambda) \) and so there is an arrow \([e_2 \Lambda] \to [e_1 \Lambda] \) in \( \Gamma(\Lambda) \). We claim that this and the arrow \([e_2 \Lambda] \to [\tau^-(e_3 \Lambda)] \) are the only arrows in \( \Gamma(\Lambda) \) starting from \([e_2 \Lambda] \). To see this, note that any other arrow starting from \([e_2 \Lambda] \) corresponds to the inclusion of \( e_2 \Lambda \) into an indecomposable projective module \( P' \), since there are no other arrows going into \([e_2 \Lambda] \). But then this would correspond to some irreducible homomorphism that would not factor through \( e_1 \Lambda \), contradicting the fact that \( e_1 \Lambda \) is a left abutment. Hence there is an almost split sequence

\[
0 \longrightarrow e_2 \Lambda \longrightarrow e_1 \Lambda \oplus \tau^-(e_3 \Lambda) \longrightarrow \tau^-(e_2 \Lambda) \longrightarrow 0.
\]

Then a similar argument shows that there are exactly two arrows from \([\tau^-(e_j \Lambda)] \) for \( 3 \leq j \leq h \), exactly as required.

Since \( e_1 \Lambda \) is uniserial, we know that \( \dim_K(e_{h-i} \Lambda) = i + 1 \). Since almost split sequences are exact sequence, it easily follows from simple dimension arguments that northeast arrows are monomorphisms, southeast arrows are epimorphisms and along the same row the dimensions remain the same. In particular, the last row has only simple modules, and since there is always an epimorphism \( e_{h-i} \Lambda \to \tau^{-1}(e_h \Lambda) \) in \( P^\Delta \), the result follows.

If \( P \) is a left abutment of \( \Lambda \) we set

\[
\mathcal{F}_P := \{X \in \text{mod} \Lambda \mid [X] \in P^\Delta\}.
\]

Similarly, if \( I \) is a right abutment of \( \Lambda \) we set

\[
\mathcal{G}_I := \{X \in \text{mod} \Lambda \mid [X] \in I^\Delta\}.
\]

Using Proposition \( \textbf{2.13} \) it can be shown that

\[
\mathcal{F}_P = \text{Fac(Sub}(P)), \quad \mathcal{G}_I = \text{Sub(Fac}(I)).
\]

The following corollary shows that every abutment gives rise to an example of glued subcategories.

**Corollary 2.14.** Let \( \Lambda \) be a representation-directed algebra.

(a) Let \( P \) be a left abutment of \( \Lambda \). Then \( \text{mod} \Lambda = \mathcal{F}_P \triangle \text{mod}(\Lambda) \).

(b) Let \( I \) be a right abutment of \( \Lambda \). Then \( \text{mod} \Lambda = (\text{mod} \Lambda) \triangle \mathcal{G}_I \).
Proof. Follows immediately by Proposition \ref{prop1}.

Example 2.15. Let \( B \) be as in Example \ref{ex1}. Then the Auslander-Reiten quiver \( \Gamma(B) \) of \( B \) is

\[
\begin{array}{c}
\overset{5}{\overset{7}{6}} \overset{4}{\overset{5}{6}} \overset{3}{\overset{2}{3}} \overset{2}{\overset{1}{3}} \overset{3}{\overset{2}{3}} \overset{2}{\overset{1}{3}} \overset{3}{\overset{2}{3}} \overset{2}{\overset{1}{3}} \\
\end{array}
\]

where the labels indicate the composition series of the corresponding indecomposable modules. We can see the foundations \( \overset{7}{\overset{6}{5}} \overset{5}{\overset{6}{7}} \) of the left abutments and the foundations \( \overset{\Delta}{\overset{\Delta}{\Delta}} \) of the right abutments that were computed in Example \ref{ex1}.

The following corollaries will be used later.

Corollary 2.16. Let \( \Lambda \) be a representation-directed algebra.

(a) Let \( P \) be a left abutment of \( \Lambda \) and \( M \in \mathcal{F}_P \). Then \( \text{proj.dim}(M) \leq 1 \).

(b) Let \( I \) be a right abutment of \( \Lambda \) and \( N \in \mathcal{G}_I \). Then \( \text{inj.dim}(N) \leq 1 \).

Proof. We only prove (a); (b) is similar. Without loss of generality, we may assume that \( M \) is indecomposable. Then by Proposition \ref{prop2} the projective cover of \( M \) is also in \( \mathcal{F}_P \) since there are no other arrows in \( \overset{P}{\Delta} \). Since the only indecomposable projective modules in \( \mathcal{F}_P \) are in \( \text{Sub}(P) \), it follows that \( \Omega(M) \) is either projective or zero, as required.

Corollary 2.17. Let \( \Lambda \) be a representation-directed algebra.

(a) Let \( P \) be a left abutment realized by \( (e_i, f_i)_{i=1}^h \). Then for every \( \lambda \in \Lambda \) we have \( e_i \lambda = e_i \lambda e_i \).

(b) Let \( I \) be a right abutment realized by \( (e_i, g_{i-1})_{i=1}^h \). Then for every \( \lambda \in \Lambda \) we have \( e_i \lambda e_i = e_i \lambda e_i \).

Proof. We only prove (a); (b) is similar. Rewriting \( e_i \lambda = e_i \lambda e_i \) as \( e_i \lambda (1 - e_i) = 0 \), we see that it is enough to show that \( e_i \Lambda(1 - e_i) = 0 \). We have

\[
e_i \Lambda(1 - e_i) \cong \bigoplus_{i \in \{1, \ldots, h\}} e_i \Lambda e_i \cong \bigoplus_{i \in \{1, \ldots, h\}} \text{Hom}_{\Lambda}(e_i \Lambda, e_i \Lambda) = 0,
\]

where the last equality follows from Proposition \ref{prop2} since there is no arrow going into \( P \Delta \) in \( \Gamma(\Lambda) \).

Corollary 2.18. Let \( \Lambda \) be a representation-directed algebra.

(a) Let \( P \) be a left abutment realized by \( (e_i, f_i)_{i=1}^h \) and let \( M \in \mathcal{F}_P \). Then for every \( m \in M \) we have \( me_i = m \).

(b) Let \( I \) be a right abutment realized by \( (e_i, g_{i-1})_{i=1}^h \) and let \( N \in \mathcal{G}_I \). Then for every \( n \in N \) we have \( ne_i = n \).

Proof. We only prove (a); (b) is similar. If for a module \( X \) we have that \( xe_i = x \) holds for all \( x \in X \) then it clearly holds for all submodules and epimorphic images of \( X \), so by Proposition \ref{prop2} it is enough to show (a) for \( M = e_1 \Lambda \). But this follows immediately by Corollary \ref{cor1}.
2.2.2. Gluing via pullbacks. In this section, we fix two representation-directed algebras $A$ and $B$, such that $A$ admits a left abutment $P$ realized by $(e_i, f_i)_{i=1}^h$ and $B$ admits a right abutment $I$ realized by $(e_i, g_{i-1})_{i=1}^h$. Notice that $P$ and $I$ have the same height. Accordingly, we have footing maps $f_P : A \to KA_h$ and $g_I : B \to KA_h$. Suggestively for what follows, we write

$$1_A = e_{h-l+1} + \cdots + e_{0} + e_1 + \cdots + e_h$$

$$1_B = \epsilon_1 + \cdots + \epsilon_h + \epsilon_{h+1} + \cdots + \epsilon_m$$

where all $e_i$'s and $\epsilon_i$'s are primitive orthogonal idempotents. Furthermore, when clear from context, we will use the notation $1_C$ for both $\epsilon_\ast = \sum_{i=1}^h \epsilon_i$ and $\epsilon_\ast = \sum_{i=1}^h \epsilon_i$.

**Definition 2.19.** We call the pullback $\Lambda$ of $A \xleftarrow{f_P} KA_h \xrightarrow{g_I} B$ the gluing of $A$ and $B$ along $P$ and $I$ and we denote it by $\Lambda := B \xrightarrow{f_P} A$.

Notice that since pullbacks are associative, the gluing is associative too.

**Example 2.20.** Let $A$ be a representation-directed algebra and $P$ be a left abutment of $A$ of height $h$. Let $B = KA_h$ and let $I = I(h)$ be the unique indecomposable injective-projective $B$-module. By Example 2.9 we have that $I$ is a right abutment of $KA_h$ of height $h$. The identity map $\text{Id}_{KA_h} : KA_h \to KA_h$ is the unique $K$-algebra morphism that satisfies the conditions of Corollary 2.10 and so the footing at $I$ is $g_I = \text{Id}_{KA_h}$. It is easy to see that the pullback of $A \xleftarrow{f_P} KA_h \xrightarrow{\text{Id}_{KA_h}} KA_h$ is $(A, \text{Id}_{A}, f_P)$ and so $A = KA_h \xleftarrow{P(h)} A$. Similarly, if $I$ is a right abutment of $A$ of height $h$ and $P(h)$ is the unique left abutment of $KA_h$ of height $h$, we have $A = A \xrightarrow{P(h)} KA_h$.

In the following and when $I$ and $P$ are clear from context we will simply call $\Lambda$ the *gluing of $A$ and $B$* and denote it by $\Lambda := B \xrightarrow{f_P} A$. That is, we have the following pullback diagram

$$\begin{array}{ccc}
\Lambda & \xrightarrow{\psi} & B \\
\phi \downarrow & & \downarrow g_I \\
A & \xleftarrow{f_P} & KA_h.
\end{array}$$

For convenience, let us recall the functors defined by $\phi$ and $\psi$ on the corresponding module categories.

$$\begin{array}{ccc}
\text{mod } A & \xleftarrow{\phi_*} & \text{mod } A, \\
\phi^i \downarrow & & \downarrow \phi^i \\
\text{mod } \Lambda & \xleftarrow{\psi_*} & \text{mod } B. \\
\psi^i \downarrow & & \downarrow \psi^i
\end{array}$$

Since $\phi$ and $\psi$ are epimorphisms, it follows that $\phi_\ast$ and $\psi_\ast$ are full and faithful. Our aim in this section is to show that if $\Lambda = B \triangleleft A$ then $\text{mod } \Lambda = (\text{mod } B) \triangleleft (\text{mod } A)$, where we identify $\text{mod } A$ and $\text{mod } B$ with their images under $\phi_\ast$ and $\psi_\ast$ respectively. To this end, we need to verify that $\text{mod } A$ and $\text{mod } B$ satisfy the conditions of Definition 2.1.

**Definition 2.21.** Let $X \in \text{mod } \Lambda$. We will say that $X$ is supported in $A$ if $\phi_\ast \phi^i(X) \cong X$ and that $X$ is supported in $B$ if $\psi_\ast \psi^i(X) \cong X$.

Before we proceed to our main result, we will need a series of technical lemmas.

**Lemma 2.22.** (a) If $X \in \text{mod } A$, then $\phi^i \phi_\ast(X) \cong X$. In particular, $\phi_\ast(X)$ is supported in $A$.

(b) If $Y \in \text{mod } B$, then $\psi^i \psi_\ast(Y) \cong Y$. In particular, $\psi_\ast(Y)$ is supported in $B$.

**Proof.** We only prove (a); (b) is similar. Since $(\phi_\ast, \phi^i)$ form an adjoint pair where the left adjoint is full and faithful, the components of the unit map $\eta : X \to \phi^i \phi_\ast(X)$ are isomorphisms (see [Mac98, page 90]).

\[\square\]
Lemma 2.23. Let \( \varepsilon_i \in \Lambda \) be defined by

\[
\varepsilon_i = \begin{cases} 
(e_i, 0) & \text{for } h - l + 1 \leq i \leq 0, \\
(e_i, e_i) & \text{for } 1 \leq i \leq h, \\
(0, e_i) & \text{for } h + 1 \leq i \leq m.
\end{cases}
\]

Then the set \( \{\varepsilon_i | h - l + 1 \leq i \leq m\} \) is a set of orthogonal idempotents in \( \Lambda \) such that

\[ 1_{\Lambda} = \varepsilon_{h-l+1} + \cdots + \varepsilon_0 + \varepsilon_1 + \cdots + \varepsilon_h + \varepsilon_{h+1} + \cdots + \varepsilon_m \]

and

\[ \phi(\varepsilon_i) = \begin{cases} 
\varepsilon_i & \text{if } h - l + 1 \leq i \leq h, \\
0 & \text{otherwise}
\end{cases} \quad \psi(\varepsilon_i) = \begin{cases} 
\varepsilon_i & \text{if } 1 \leq i \leq m, \\
0 & \text{otherwise}.
\end{cases} \]

Proof. It is a simple calculation to check that the \( \varepsilon_i \)'s satisfy the requirements. \( \square \)

In the following we set

\[ \varepsilon_A = \varepsilon_{h-l+1} + \cdots + \varepsilon_h, \quad \varepsilon_B = \varepsilon_1 + \cdots + \varepsilon_m, \quad \varepsilon_C = \varepsilon_1 + \cdots + \varepsilon_h, \]

\[ \varepsilon_{A'} = \varepsilon_A - \varepsilon_C, \quad \varepsilon_{B'} = \varepsilon_B - \varepsilon_C. \]

We also introduce some notation to simplify expressions later in this section. We will denote \( [A', B'] := \{A', 1, \ldots, h, B'\} \) and we order the set \( [A', B'] \) by \( A' < 1 < \cdots < h < B' \). In particular, we have

\[ \varepsilon_{A} = \sum_{i \in [A', B']} \varepsilon_i. \]

The following technical lemma will be used to show that the idempotents \( \varepsilon_i \) are also primitive, among other things.

Lemma 2.24. (a) If \( \phi(\lambda) = 0 \), then \( \lambda \varepsilon_A = 0 \).
(b) If \( \psi(\lambda) = 0 \), then \( \varepsilon_B \lambda = 0 \).

Proof. We only prove (a); (b) is similar. Let \( \lambda = (a, b) \in \Lambda \). Since \( \phi(\lambda) = 0 \), we have that \( a = 0 \). Therefore

\[ \lambda \varepsilon_A = (0, b)(1_{A'}, \varepsilon_i) = (0, b \varepsilon_i). \]

By definition of the pullback, we have \( g_i(b \varepsilon_i) = f_i(0) = 0 \). By Corollary 2.12(b), this implies that \( b \varepsilon_i = 0 \) and so \( \lambda \varepsilon_A = 0 \). \( \square \)

Lemma 2.25. The set \( \{\varepsilon_i\}_{i=h-l+1}^{m} \) is a complete set of primitive orthogonal idempotents of \( \Lambda \).

Proof. By Lemma 2.23 it is enough to show that for \( h - l + 1 \leq i \leq m \), the idempotent \( \varepsilon_i \) is primitive. Since the idempotents \( \{\varepsilon_i\}_{h-l+1 \leq i \leq h} \) and \( \{\varepsilon_i\}_{1 \leq i \leq m} \) are primitive, it is enough to show that

\[ \varepsilon_i \Lambda \varepsilon_i \cong \begin{cases} 
e_i A_c_i & \text{for } h - l + 1 \leq i \leq 0, \\
\varepsilon_i B_{e_i} & \text{for } 1 \leq i \leq m.
\end{cases} \]

For \( h - l + 1 \leq i \leq 0 \), define the \( K \)-algebra morphism \( \phi_i : \varepsilon_i \Lambda \varepsilon_i \to \varepsilon_i A \varepsilon_i \) by \( \varepsilon_i \lambda \varepsilon_i \mapsto \phi(\varepsilon_i \lambda \varepsilon_i) = \varepsilon_i \phi(\lambda) \varepsilon_i \). If \( \varepsilon_i \phi(\lambda) \varepsilon_i = 0 \), then \( \varepsilon_i \lambda \varepsilon_i \Lambda \varepsilon_i = 0 \) by Lemma 2.24. But \( \varepsilon_i \lambda \varepsilon_i \Lambda \varepsilon_i = \varepsilon_i \lambda \varepsilon_i \) since \( h - l + 1 \leq i \leq 0 \), showing injectivity of \( \phi_i \). Surjectivity follows by surjectivity of \( \phi \) and so \( \phi_i \) is an isomorphism. Similarly, we can show that for \( 1 \leq i \leq m \), the \( K \)-algebra morphism \( \psi_i : \varepsilon_i \Lambda \varepsilon_i \to \varepsilon_i B \varepsilon_i \) defined by \( \varepsilon_i \lambda \varepsilon_i \mapsto \varepsilon_i \psi(\lambda) \varepsilon_i \) is an isomorphism. \( \square \)

Lemma 2.26. Let \( M \in \text{mod} \Lambda \).

(a) \( m \varepsilon_A = m \) for all \( m \in M \) if and only if \( M \) is supported in \( A \).
(b) \( m \varepsilon_B = m \) for all \( m \in M \) if and only if \( M \) is supported in \( B \).
Proof. We only prove (a); (b) is similar. Assume first that \( M \) is supported in \( A \). Then we have that \( \phi_*(\text{Hom}_A(A, M)) \cong M \) via the map \( (f : A \to M) \mapsto f(1_A) \). Hence for any \( m \in M \) there exists \( f_m : A \to M \) such that \( f_m(1_A) = m \). Then

\[
m \epsilon_A = f_m(1_A) \epsilon_A = f_m(1_A \epsilon_A) = f_m(1_A \phi(\epsilon_A)) = f_m(1_A 1_A) = f_m(1_A) = m,
\]
as required.

In the other direction we want to show that \( \phi_* (\text{Hom}_A(A, M)) \cong M \).

The induced \( \Lambda \)-module homomorphism map \( (f : A \to M) \mapsto f(1_A) \) is clearly injective. Let us show that it is also surjective. For \( m \in M \) define the map \( f_m : A \to M \) by \( f_m(a) = m \lambda_a \) where \( \lambda_a \in \Lambda \) and \( \phi(\lambda_a) = a \). To show that this is well-defined, let \( \phi(\lambda_a) = \phi(\lambda'_a) = a \). Then \( \phi(\lambda_a - \lambda'_a) = 0 \) and so by Lemma 2.24 we have that \( (\lambda_a - \lambda'_a) \epsilon_A = 0 \). Then

\[
m(\lambda_a - \lambda'_a) = (m(\lambda_a - \lambda'_a)) \epsilon_A = m((\lambda_a - \lambda'_a) \epsilon_A) = m0 = 0,
\]
as required. Clearly \( f_m \) is \( K \)-linear and it is a \( \Lambda \)-module homomorphism since for any \( \lambda \in \Lambda \) and \( a \in \Lambda \) we have

\[
f_m(a \lambda) = f_m(a \phi(\lambda)) = m(\lambda a \lambda) = (m \lambda a) \lambda = f_m(a) \lambda,
\]
which completes the proof.

We have the following immediate corollary.

**Corollary 2.27.** Let \( M \in \text{mod} \Lambda \) and suppose that \( M \) is supported in \( A \) (respectively \( B \)).

(a) All submodules of \( M \) are supported in \( A \) (respectively \( B \)). In particular, \( \text{rad} M \) and \( \text{soc} M \) are supported in \( A \) (respectively \( B \)).

(b) All quotient modules of \( M \) are supported in \( A \) (respectively \( B \)). In particular, \( \text{top} M \) is supported in \( A \) (respectively \( B \)).

**Proof.** Immediate by Lemma 2.26.

We can now identify the indecomposable projective and injective \( \Lambda \)-modules.

**Proposition 2.28.**

(a) \( \epsilon_i \Lambda \) is supported in \( A \) for \( h - l + 1 \leq i \leq 0 \) and is supported in \( B \) for \( 1 \leq i \leq m \).

(b) \( D(\epsilon_i \Lambda) \) is supported in \( A \) for \( h - l + 1 \leq i \leq h \) and is supported in \( B \) for \( h + 1 \leq i \leq m \).

**Proof.** We only prove (a); (b) is similar. Let \( \lambda = (a, b) \in \Lambda \). For \( h - l + 1 \leq i \leq 0 \) we have

\[
(\epsilon_i, \lambda)(1_A - \epsilon_A) = ((\epsilon_i, 0)(a, b))(0, 1_B - 1_C) = (0, 0),
\]
so \( \epsilon_i \lambda = (\epsilon_i, \lambda) \epsilon_A \) and thus \( \epsilon_i \Lambda \) is supported in \( A \) by Lemma 2.26.

For \( h + 1 \leq i \leq m \) we have

\[
(\epsilon_i, \lambda)(1_A - \epsilon_B) = (0, \epsilon_i)(a, b)(1_A - 1_C, 0) = (0, 0),
\]
so \( \epsilon_i \lambda = (\epsilon_i, \lambda) \epsilon_B \) and thus \( \epsilon_i \Lambda \) is supported in \( B \) by Lemma 2.26.

For \( 1 \leq i \leq h \) we have

\[
(\epsilon_i, \lambda)(1_A - \epsilon_B) = (\epsilon_i, \epsilon_i)(a, b)(1_A - 1_C, 0) = (\epsilon_i a(1_A - 1_C), 0).
\]

By the definition of \( \Lambda \) we have

\[
f_P(\epsilon_i a(1_A - 1_C)) = (h(0)) = 0,
\]
and by Corollary 2.12(a) this implies \( \epsilon_i a(1_A - 1_C) = 0 \). So we have \( \epsilon_i \lambda = (\epsilon_i, \lambda) \epsilon_B \) and thus \( \epsilon_i \Lambda \) is supported in \( B \) by Lemma 2.26.

**Corollary 2.29.** Let \( s_A \) be the number of simple projective \( A \)-modules up to isomorphism and \( t_A \) be the number of simple injective \( A \)-modules up to isomorphism. Similarly define \( s_B, t_B, s_A \) and \( t_A \). Then

\[
(s_A, t_A) = (s_A + s_B - 1, t_A + t_B - 1).
\]
Proof. Let us show that $s_A = s_A + s_B - 1$; that $A = A + B - 1$ is proved similarly. Let $h - l + 1 \leq i \leq m$. If $1 \leq i \leq m$, then $\varepsilon_i \Lambda$ is a simple projective $\Lambda$-module if and only if $\varepsilon_i B$ is a simple projective $B$-module by Proposition 2.28(a). Since $\{\varepsilon_i\}_{1 \leq i \leq m}$ is a complete set of primitive orthogonal idempotents for $B$, it follows that there are exactly $s_B$ simple projective $\Lambda$-modules $\varepsilon_i \Lambda$ for $1 \leq i \leq m$.

Similarly, if $h - l + 1 \leq i \leq 0$ then $\varepsilon_i \Lambda$ is a simple projective $\Lambda$-module if and only if $\varepsilon_i \Lambda$ is a simple projective $\Lambda$-module. By Proposition 2.13(a) it follows that if $1 \leq i \leq h$, then $\varepsilon_i \Lambda$ is simple if and only if $i = h$. Since $\{\varepsilon_i\}_{h - l + 1 \leq i \leq h}$ is a complete set of primitive orthogonal idempotents for $A$, it follows that there are exactly $s_A - 1$ simple projective $\Lambda$-modules $\varepsilon_i \Lambda$ for $h - l + 1 \leq i \leq 0$.

Finally, since $\{\varepsilon_i\}_{h - l + 1 \leq i \leq m}$ is a complete set of primitive orthogonal idempotents for $A$ by Lemma 2.25 it follows that there are exactly $s_B + (s_A - 1)$ simple projective $\Lambda$-modules, as required. □

Corollary 2.30. For every $m \in \operatorname{rad}(\varepsilon_i \Lambda)$ we have $m \varepsilon B = m$.

Proof. By Proposition 2.28 and Corollary 2.27 we have $\operatorname{rad}(\varepsilon_i \Lambda) \cong \psi_* (\operatorname{rad}(\varepsilon_i B))$. In particular, $\operatorname{rad}(\varepsilon_i \Lambda)$ is supported in $B$. Hence for $m \in \operatorname{rad}(\varepsilon_i \Lambda)$ we have $m \varepsilon B = m$ by Lemma 2.26 and it is enough to show that $m \varepsilon C = 0$. Since $m \in \operatorname{rad}(\varepsilon_i \Lambda) \cong \psi_* (\operatorname{rad}(\varepsilon_i B))$, we may assume that $m = \varepsilon_i b$ for some $\varepsilon_i b \in \operatorname{rad}(\varepsilon_i B)$. By Proposition 2.6(b), it follows that $\varepsilon_i b \varepsilon C = 0$. Then, since $\varepsilon_i b \in \psi_* (\operatorname{rad}(\varepsilon_i B))$ we have

$$(\varepsilon_i b) \varepsilon C = (\varepsilon_i b) \psi (\varepsilon C) = \varepsilon_i b \varepsilon C = 0,$$

as required. □

The following lemma contains important information about the directedness that is required to prove $\operatorname{mod} \Lambda = (\operatorname{mod} B) \Delta (\operatorname{mod} A)$.

Lemma 2.31. Let $i, j \in [A', B']$:

(i) $\varepsilon_i \Lambda \varepsilon_j = 0$ and $\varepsilon_j \Lambda \varepsilon_i = 0$.

(ii) If $1 \leq i, j \leq h$, then $\dim_K (\varepsilon_i \Lambda \varepsilon_j) = \begin{cases} 1 & \text{if } i \leq j, \\ 0 & \text{if } i > j. \end{cases}$

(iii) If $i < j$, then $\varepsilon_i \Lambda \varepsilon_j = 0$.

(iv) $\Lambda \cong \bigoplus_{\varepsilon_i \Lambda \varepsilon_j} \varepsilon_i \Lambda \varepsilon_j$.

(v) If $1 \leq j \leq k \leq i \leq h$, then $\varepsilon_j \Lambda \varepsilon_i \cong \varepsilon_j \Lambda \varepsilon_k \Lambda \varepsilon_i$.

(vi) If $1 \leq i \leq h$, then $\varepsilon_i \Lambda \varepsilon_k \varepsilon_j \Lambda \varepsilon_i \cong \varepsilon_i \Lambda \varepsilon_k \Lambda \varepsilon_i$.

Proof.

(i) Immediate since $\varepsilon_i = (1_A - 1_C, 0)$ and $\varepsilon_j = (0, 1_B - 1_C)$.

(ii) Let $\nu_B : \operatorname{mod} B \to \operatorname{mod} B$ be the Nakayama functor. Recall that $\nu_B$ induces an equivalence between the subcategories of projective and injective $B$-modules. Thus we have

$$\dim_K (\operatorname{Hom}_\Lambda(\varepsilon_j \Lambda, \varepsilon_i \Lambda)) \cong \dim_K (\operatorname{Hom}_B(\varepsilon_j B, \varepsilon_i B)) \quad \text{(Proposition 2.28)}$$

$$\cong \dim_K (\operatorname{Hom}_B(\nu_B(\varepsilon_j B), \nu_B(\varepsilon_i B)))$$

$$\cong \dim_K (\operatorname{Hom}_B(D(B \varepsilon_j), D(B \varepsilon_i)))$$

$$= \begin{cases} 1 & \text{if } i \leq j, \\ 0 & \text{if } i > j. \end{cases} \quad \text{(Proposition 2.13)}$$

(iii) If $i = A'$ and $j = B'$ the result follows from (i). If $1 \leq i < j \leq h$ the result follows from (ii).

If $i = A'$ and $1 \leq j \leq h$, then for any $\lambda = (a, b) \in \Lambda$ we have

$$\varepsilon_j \lambda \varepsilon_i = (\varepsilon_j, \varepsilon_i)(a, b)(1_A - 1_C, 0) = (\varepsilon_i a(1_A - 1_C), 0).$$

In particular, $\psi(\varepsilon_i \lambda \varepsilon_i) = 0$ and so $\varepsilon_j B(\varepsilon_j \lambda \varepsilon_i) = 0$ by Lemma 2.24. But

$$\varepsilon_j \lambda \varepsilon_i = (\varepsilon_j B \varepsilon_i) \lambda \varepsilon_i = \varepsilon_j B(\varepsilon_j \lambda \varepsilon_i) = 0$$

and so $\varepsilon_j \lambda \varepsilon_i = 0$. If $1 \leq i \leq h$ and $j = B'$, we similarly show that for $\lambda \in \Lambda$ we have

$$\phi(\varepsilon_j \lambda \varepsilon_i) = 0$$

and so $\varepsilon_j \lambda \varepsilon_i = 0$, from which the result follows.
Proposition 2.32. Let \( M \in \text{mod} \Lambda \). Then \( M \cong X \oplus Y \) for some \( X, Y \in \text{mod} \Lambda \) where \( X \) is supported in \( A \) and \( Y \) is supported in \( B \).

Proof. Let us pick a sequence of nonzero morphisms

\[
\varepsilon_1 \Lambda \xrightarrow{p_1} \cdots \xrightarrow{p_{h-1}} \varepsilon_h \Lambda \xleftarrow{p_h} \text{rad}(\varepsilon_h \Lambda),
\]

where \( p_h \) corresponds to the inclusion of the radical of \( \varepsilon_h \Lambda \). By applying \( \text{Hom}_\Lambda(-, M) \), and since \( \text{Hom}_\Lambda(\varepsilon_i \Lambda, M) \cong M_{\varepsilon_i} \), we get the commutative diagram

\[
\begin{array}{c}
\text{Hom}_\Lambda(\varepsilon_1 \Lambda, M) \xrightarrow{s_1} \cdots \xrightarrow{s_{h-1}} \text{Hom}_\Lambda(\varepsilon_h \Lambda, M) \xrightarrow{s_h} \text{Hom}_\Lambda(\text{rad}(\varepsilon_h \Lambda), M),
\end{array}
\]

where

- \( s_i(m_{\varepsilon_i}) = p_m(\varepsilon_i \lambda) = m_{\varepsilon_i} \lambda \),
- \( s_i^{-1}(\chi : \varepsilon_i \Lambda \to M) = \chi(\varepsilon_i) = \chi(\varepsilon_i) \varepsilon_i \),
- \( q_i(m_{\varepsilon_i}) = m_{\varepsilon_{i+1}} \varepsilon_{i+1} \) for \( 1 \leq i \leq h-1 \),
- \( [q_h(m_{\varepsilon_h})](\varepsilon_h \lambda) = m_{\varepsilon_h} \lambda \) for \( \lambda \in \text{rad}(\Lambda) \).

Let \( U_h = \ker q_h \) and for \( 1 \leq i \leq h-1 \) define \( U_i = q_i^{-1}(U_{i+1}) \). Moreover for \( 1 \leq i \leq h \) let \( V_i \) be such that \( q_{i-1}(V_i) \subseteq V_i \) and \( M_{\varepsilon_i} = U_i \oplus V_i \). Set \( U = \bigoplus_{i=1}^h U_i \), \( V = \bigoplus_{i=1}^h V_i \), \( X = M_{\varepsilon_A} \oplus U \) and \( Y = V \oplus M \varepsilon_B \). Then clearly \( M \cong X \oplus Y \) as vector spaces and it remains to show that both \( X \) and \( Y \) are submodules of \( M \) since by construction it is clear that \( X \) is supported in \( A \) and \( Y \) is supported in \( B \).

Before we show that \( X \) and \( Y \) are submodules of \( M \), we first claim that \( U_h \) satisfies \( U_h \varepsilon_B = 0 \). To show this, let \( u \varepsilon_h \in U_h \) and \( \lambda \in \Lambda \). If \( \lambda \in \text{rad}(\Lambda) \) then by construction we have \( u \varepsilon_h \lambda = [q_h(u \varepsilon_h)](\varepsilon_h \lambda) = 0 \). If \( \lambda \) is not in the radical of \( \Lambda \), then we can write \( \lambda = \sum_{i=h-1}^m c_i \varepsilon_i + \mu \) with \( \mu \in \text{rad}(\Lambda) \), since \( \varepsilon_i \) are the only elements in \( \Lambda \) that act nontrivially on simple \( \Lambda \)-modules. Then

\[
\begin{align*}
u \varepsilon_h \lambda \varepsilon_B &= u \varepsilon_h \left( \sum_{i=h-1}^m c_i \varepsilon_i \right) \varepsilon_B + u \varepsilon_h \mu \varepsilon_B = u0 + 0 = 0,
\end{align*}
\]

since \( \varepsilon_h \left( \sum_{i=h-1}^m c_i \varepsilon_i \right) \varepsilon_B = \varepsilon_h \varepsilon_B = 0 \) and \( \mu \in \text{rad}(\Lambda) \). Hence, the claim is proved.

Now let us show that \( X = M_{\varepsilon_A} \oplus U \) is a submodule of \( M \). First let \( m_{\varepsilon_A} \in M_{\varepsilon_A} \) and \( \lambda \in \Lambda \). Then by Lemma 2.33(iv) we have \( \lambda = \varepsilon_x \varepsilon_y \) with \( A' \leq x \leq y \leq B' \). We need to show that \( m_{\varepsilon_A} \lambda = m_{\varepsilon_{A'}} \varepsilon_x \varepsilon_y \in X \) for all \( A' \leq x \leq y \leq B' \). If \( A' < x \) then \( m(\varepsilon_x \varepsilon_y) = m \varepsilon_x \varepsilon_y = 0 \), so we assume that \( x = A' \). If \( y = A' \), then \( m_{\varepsilon_{A'}} \lambda_{\varepsilon_A} \in M_{\varepsilon_A} \), while if \( y = B' \), then \( m_{\varepsilon_{A'}} \varepsilon_B = 0 \) by Lemma 2.33(i). It remains to check the case \( 1 \leq y \leq h \). Since \( m_{\varepsilon_A} \varepsilon_y \in M_{\varepsilon_A} \), it is enough to show that \( q_h \circ \cdots \circ q_y(m_{\varepsilon_A} \varepsilon_y) = 0 \) since then \( m_{\varepsilon_A} \varepsilon_y \in U' \). We have \( s_y(m_{\varepsilon_A} \varepsilon_y) = p_{m_{\varepsilon_A}} \lambda \) and for any \( n \in \text{rad}(\varepsilon_A \Lambda) \) we have \( n = n_{\varepsilon_B} \) by Corollary 2.30. Hence for any \( n \in \text{rad}(\varepsilon_A \Lambda) \) we have

\[
\begin{align*}
[q_h \circ \cdots \circ q_y(m_{\varepsilon_A} \varepsilon_y)](n) &= s_y(m_{\varepsilon_A} \varepsilon_y) \circ p_y \circ \cdots \circ p_h(n_{\varepsilon_B}) \\
&= p_{m_{\varepsilon_A}} \lambda \circ p_y \circ \cdots \circ p_h(n_{\varepsilon_B}) \\
&= m_{\varepsilon_A} \lambda \varepsilon_y \circ \cdots \circ p_h(n) \varepsilon_B \\
&= 0,
\end{align*}
\]

The following proposition is the most important step in showing the main result in this section.
where the last equality comes from Lemma 2.31). Next let \( u \varepsilon_i \in U_i \) and \( \lambda \in \Lambda \). Again by Lemma 2.31(iv) we have \( \lambda = \varepsilon_x \lambda \varepsilon_y \) with \( x \leq y \leq B' \) and it is enough to show that \( u \varepsilon_i \lambda = u \varepsilon_i \varepsilon_x \lambda \varepsilon_y \in X \). We can assume that \( x = i \) since otherwise \( u(\varepsilon_i \varepsilon_x) \lambda \varepsilon_y = u0 \lambda \varepsilon_y = 0 \). If \( y < B' \), then \( u \varepsilon_i \lambda \varepsilon_y \in M \varepsilon_y \) and it is enough to show \( u \varepsilon_i \lambda \varepsilon_y \in U_y \). Since \( \varepsilon_i \lambda \varepsilon_y \cong \text{Hom}_\Lambda(\varepsilon_y \Lambda, \varepsilon_i \Lambda) \cong K \), there exists some \( k \in K \) such that \( p_i \cdots p_y(\varepsilon_y) = k \varepsilon_i \lambda \varepsilon_y \).

Then

\[
q_{y-1} \cdots q_i(u \varepsilon_i) = s_{y-1}^{-1}(s_i(u \varepsilon_i) \circ p_i \cdots p_{y-1})
\]

\[
= s_y^{-1}(p_u \circ p_i \cdots p_{y-1}(\varepsilon_y))
\]

\[
= u(p_u \circ p_i \cdots p_{y-1} \varepsilon_y)
\]

\[
= u(k \varepsilon_i \lambda \varepsilon_y)
\]

\[
= k(u \varepsilon_i \lambda \varepsilon_y).
\]

Since the left hand side is in \( U_y \) by construction, we have \( k(u \varepsilon_i \lambda \varepsilon_y) \in U_y \) as required. If \( y = B' \) then by Lemma 2.31(vi) we have \( u \varepsilon_i \lambda \varepsilon_y = u \varepsilon_i \lambda \varepsilon_y \varepsilon_y \in B' \). Using the same argument as before, we can show that \( u \varepsilon_i \lambda \varepsilon_y \in U_y \). But then by the claim that we proved at the start of this proof we have that \( u \varepsilon_i \lambda \varepsilon_y \varepsilon_y \lambda \varepsilon_y = 0 \in X \). Hence we have showed that \( X \) is a submodule of \( M' \).

It remains to show that \( Y = V \oplus M \varepsilon_B \) is a submodule of \( M \). First, let \( \varepsilon_i \in V_i \) and \( \lambda \in \Lambda \). As in the previous cases, we can assume that \( \lambda = \varepsilon_i \lambda \varepsilon_y \) with \( i \leq y \leq B' \). If \( y = B' \) then \( \varepsilon_i \lambda \varepsilon_B \in M \varepsilon_B \) and so again \( \varepsilon_i \lambda \varepsilon_y \in Y \).

Finally, if \( m \varepsilon_B \in M \varepsilon_B \) and \( \lambda \in \Lambda \) we have \( m \varepsilon_B \lambda = 0 \) unless \( \lambda = \varepsilon_B \lambda \varepsilon_B \), in which case \( m \varepsilon_B \lambda \in M \varepsilon_B \). This shows that \( Y \) is a submodule of \( M \) and concludes the proof.

**Corollary 2.33.** If \( M \in \text{mod} \Lambda \) is indecomposable, then \( M \) is supported in \( A \) or \( M \) is supported in \( B \).

**Proof.** Immediate by Proposition 2.32. \( \square \)

**Lemma 2.34.** Let \( M \in \text{mod} \Lambda \).

(a) Assume that \( \text{Hom}_\Lambda(\varepsilon_i \Lambda, M) \neq 0 \). If \( M \) is supported in \( A \), then \( h - l + 1 \leq i \leq h \). If \( M \) is supported in \( B \), then \( 1 \leq i \leq m \).

(b) Assume that \( \text{Hom}_\Lambda(M, D(\varepsilon_i \Lambda)) \neq 0 \). If \( M \) is supported in \( A \), then \( h - l + 1 \leq i \leq h \). If \( M \) is supported in \( B \), then \( 1 \leq i \leq m \).

**Proof.** We only prove (a); (b) is similar. Let \( 0 \neq \zeta \in \text{Hom}_\Lambda(\varepsilon_i \Lambda, M) \). Then \( \zeta(\varepsilon_i) = m \neq 0 \), since otherwise for some \( \lambda \in \Lambda \) we have

\[
0 \neq \zeta(\varepsilon_i \lambda) = \zeta(\varepsilon_i 1 \lambda) = \zeta(\varepsilon_i 1 \lambda) \lambda = \zeta(\varepsilon_i) \lambda = 0 \lambda = 0.
\]

Assume that \( M \) is supported in \( A \). Then by Lemma 2.26 we have that \( m = m \varepsilon_A \) and so

\[
0 \neq m = m \varepsilon_A = \zeta(\varepsilon_i) \varepsilon_A = \zeta(\varepsilon_i \varepsilon_A).
\]

It follows that \( h - l + 1 \leq i \leq h \) since otherwise \( \varepsilon_i \varepsilon_A = 0 \). Similarly, assume that \( M \) is supported in \( B \). Then

\[
0 \neq m = m \varepsilon_B = \zeta(\varepsilon_i) \varepsilon_B = \zeta(\varepsilon_i \varepsilon_B)
\]

and so \( 1 \leq i \leq m \). \( \square \)

**Lemma 2.35.** Let \( M \in \text{mod} \Lambda \) be indecomposable. Then the following are equivalent.

(a) \( M \) is supported in both \( A \) and \( B \),
Proof. We only show that (a) and (b) are equivalent; to show that (a) and (c) are equivalent is similar. If \( M \) is supported in both \( A \) and \( B \), then it follows from Lemma \ref{lem:adm} that \( m \in C = m \) for every \( m \in C \). In particular, if \( S_\lambda(i) \) is the simple \( \Lambda \)-module corresponding to the idempotent \( \varepsilon_i \), we have \( \text{Hom}_\Lambda(M, S_\lambda(i)) = 0 \) and \( \text{Hom}_\Lambda(S_\lambda(i), M) = 0 \) for \( i \notin \{1, \ldots, h\} \). Hence we have that \( \text{Hom}_\Lambda(\phi'(M), S_\lambda(i)) = 0 \) and \( \text{Hom}_\Lambda(S_\lambda(i), \phi(M)) = 0 \) for \( i < 1 \). But this implies that \( [\phi'(M)] \in \mathcal{F}_\Delta \) by Proposition \ref{prop:adm} and so \( M \in \phi_\ast(\mathcal{F}_\Delta) \).

If \( M \in \phi_\ast(\mathcal{F}_\Delta) \) then by Corollary \ref{cor:adm} for every \( m \in M \), we have \( m \in C = m \) and so \( M \) is supported in both \( A \) and \( B \) by Lemma \ref{lem:adm}. \( \square \)

To simplify notation in the rest of this section, let us denote the subcategories \( \psi_\ast(\text{mod} \ A) \) \( \subseteq \text{mod} \ A \) and \( \phi_\ast(\text{mod} \ B) \) \( \subseteq \text{mod} \ B \) by \( \text{mod} \ A \) and \( \text{mod} \ B \), respectively. Now we are ready to show the main result for this section.

**Proposition 2.36.** If \( \Lambda = B \triangleleft A \), then \( \text{mod} \Lambda = (\text{mod} \ A)_\ast \triangleleft (\text{mod} \ A)_\ast \).

Proof. We need to check conditions (i)—(iv) of Definition \ref{def:adm} with \( A = (\text{mod} \ A)_\ast \) and \( B = (\text{mod} \ B)_\ast \). Condition (i) is immediate. Condition (ii) follows from Proposition \ref{prop:adm}, since if \( M \in \text{mod} \Lambda \) is indecomposable, then either \( M \) is supported in \( A \) or \( M \) is supported in \( B \) by Corollary \ref{cor:adm}.

For condition (iii) let \( M \in (\text{mod} \ A)_\ast \setminus (\text{mod} \ B)_\ast \) be indecomposable and assume to a contradiction that for some \( N \in (\text{mod} \ B)_\ast \), there exists a nonzero morphism \( g : M \rightarrow N \). In particular, we have that \( M \rightarrow \text{Im} \ g \hookrightarrow N \) and so \( \text{Im} \ g \in (\text{mod} \ A)_\ast \cap (\text{mod} \ B)_\ast = \phi_\ast(\mathcal{F}_\Delta) \). Hence there exists a morphism \( \phi'(M) \rightarrow \phi'(\text{Im} \ g) \) in \( \text{mod} \ A \) with \( \phi'(\text{Im} \ g) \in \mathcal{F}_\Delta \). By Proposition \ref{prop:adm}, this means that \( \phi'(M) \in \mathcal{F}_\Delta \). But by Corollary \ref{cor:adm}, this implies that \( M \in \phi_\ast(\mathcal{F}_\Delta) = (\text{mod} \ A)_\ast \cap (\text{mod} \ B)_\ast \), which contradicts \( M \in (\text{mod} \ A)_\ast \setminus (\text{mod} \ B)_\ast \).

For condition (iv) notice that any \( g : N \rightarrow M \) with \( N \in (\text{mod} \ B)_\ast \) and \( M \in (\text{mod} \ A)_\ast \) factors as \( N \rightarrow \text{Im} \ g \hookrightarrow M \) and \( \text{Im} \ g \) is in \( (\text{mod} \ A)_\ast \cap (\text{mod} \ B)_\ast \) by Corollary \ref{cor:adm}. \( \square \)

The following corollaries describe the representation theory of \( \Lambda \) in terms of the representation theory of \( A \) and \( B \) and will be particularly useful in the following section.

**Corollary 2.37.** \( \Lambda \) is representation-directed.

Proof. Let \( f_0 : Y_0 \rightarrow Y_1 \) be a nonzero morphism between indecomposable modules \( Y_0, Y_1 \in \text{mod} \ A \). We need to show that there exists a chain of nonzero nonisomorphisms \( f_i : Y_i \rightarrow Y_{i+1}, 1 \leq i \leq k \) with \( Y_{k+1} \cong Y_0 \).

Assume to a contradiction that such a chain exists. If all \( Y_i \) are supported in \( B \), then this gives rise to a chain of indecomposable \( B \)-module nonzero nonisomorphisms \( \psi^*(f_i) : \psi^*(Y_i) \rightarrow \psi^*(Y_{i+1}) \) for \( 0 \leq i \leq k \) such that \( \psi^*(Y_{k+1}) \cong \psi^*(Y_0) \), which contradicts the fact that \( B \) is representation-directed.

Hence there exists some minimal \( j \) such that \( Y_j \) is not supported in \( B \). Then \( Y_j \) is supported in \( A \) by Corollary \ref{cor:adm}. Since \( Y_j \) is supported in \( A \) and not in \( B \), and since \( \text{mod} \Lambda = (\text{mod} \ B)_\ast \triangleleft (\text{mod} \ A)_\ast \), it follows that \( Y_j \) is supported in \( A \) and not in \( B \) for all \( i \geq j \) by Definition \ref{def:adm}(iii). Since \( Y_{k+1} \cong Y_0 \) and \( j \) was minimal, it follows that \( j = 0 \) and that all \( Y_i \) are supported in \( A \). Then this gives rise to a chain of indecomposable \( A \)-module nonzero nonisomorphisms \( \phi(f_i) : \phi(Y_i) \rightarrow \phi(Y_{i+1}) \) for \( 0 \leq i \leq k \) such that \( \phi(Y_{k+1}) \cong \phi(Y_0) \), which contradicts the fact that \( A \) is representation-directed. \( \square \)

**Corollary 2.38.** Let \( M \in \text{mod} \Lambda \) be indecomposable.

(a1) If \( M \in (\text{mod} \ A)_\ast \setminus (\text{mod} \ B)_\ast \), then \( \tau(M) \cong \phi_\ast \tau \phi'(M) \) and \( \Omega(M) \cong \phi_\ast \psi_\ast \).

(a2) If \( M \in (\text{mod} \ B)_\ast \), then \( \tau(M) \cong \psi_\ast \psi \phi'(M) \) and \( \Omega(M) \cong \psi_\ast \psi \phi'(M) \).

(b1) If \( M \in (\text{mod} \ B)_\ast \setminus (\text{mod} \ A)_\ast \), then \( \tau(M) \cong \psi_\ast \psi \phi'(M) \) and \( \Omega(M) \cong \psi_\ast \psi \phi'(M) \).

(b2) If \( M \in (\text{mod} \ A)_\ast \), then \( \tau(M) \cong \phi_\ast \phi \phi'(M) \) and \( \Omega(M) \cong \phi_\ast \phi \phi'(M) \).

Proof. We only prove (a1) and (a2); (b1) and (b2) are similar. The claims about \( \tau \Lambda \) follow immediately by Theorem \ref{thm:adm} and so we only show the claims about the syzygy.
If $M \in \left(\text{mod } \Lambda\right)_a \setminus \left(\text{mod } B\right)_a$ and $\text{Hom}_A(\varepsilon \Lambda, M) \neq 0$, then $h - l + 1 \leq i \leq h$ by Lemma 2.34. Moreover, since $M \in \left(\text{mod } \Lambda\right)_a \setminus \left(\text{mod } B\right)_a$, we have that $M \not\in \mathcal{F}_P$ by Corollary 2.35. In particular, $\text{Hom}_A(M, S(i)) = 0$ for $1 \leq i \leq h$ by Proposition 2.19. Therefore, if $\varepsilon \Lambda$ is a summand of the projective cover of $M$, then $h - l + 1 \leq i \leq 0$. But then $\varepsilon \Lambda$ is supported in $A$ by Proposition 2.26 and so the projective cover of $M$ is supported in $A$. In particular, we can compute the syzygy of $M$ by viewing $M$ as an $A$-module instead.

If $M \in \left(\text{mod } B\right)_a$, then again by Lemma 2.34 the projective cover of $M$ is supported in $B$ and the result follows as in the previous case. \qed

**Corollary 2.39.** Let $\text{gl. dim}(\Lambda) = d$, $\text{gl. dim}(A) = d_1$ and $\text{gl. dim}(B) = d_2$. Then

$$\max\{d_1, d_2\} \leq d \leq d_1 + d_2.$$  

**Proof.** Let $M \in \text{mod } \Lambda$ and set $U^k = \phi_*(\Omega^k(\text{mod } A))$. Since $\text{gl. dim}(A) = d_1$, we have that $U^k = 0$ for $k > d_1$. We claim that

$$\Omega^j(M) \subseteq \text{add}(U^k, \left(\text{mod } B\right)_a) \text{ for any } j \geq 0. \quad (2.1)$$

We prove (2.1) by induction. The base case $j = 0$ follows immediately by Proposition 2.36. For the induction step, assume that (2.1) holds for $j = k$ and we will show that it holds for $j = k + 1$. Then we have that

$$\Omega^{k+1}(M) \cong X \oplus Y,$$

with $X \in U^k$ and $Y \in \left(\text{mod } B\right)_a$. By Proposition 2.36 we can write $X \cong X_1 \oplus X_2$ with $X_1 \in \left(\text{mod } A\right)_a \setminus \left(\text{mod } B\right)_a$ and $X_2 \in \left(\text{mod } B\right)_a$. Then by Corollary 2.38(a1) we have that $\Omega(X_1) \in U^{k+1}$ and by Corollary 2.38(a2) we have that $\Omega(X_2), \Omega(Y) \in \left(\text{mod } B\right)_a$. Hence

$$\Omega^{k+1}(M) \cong (\Omega(X_1)) \oplus (\Omega(X_2) \oplus Y) \in \text{add}(U^{k+1}, \left(\text{mod } B\right)_a),$$

and the induction step is proved.

Let us now show that $d \leq d_1 + d_2$. If we have $\Omega^d(M) = 0$, then $\text{proj. dim}(M) \leq d_1 \leq d_1 + d_2$. If $\Omega^d(M) \neq 0$, then by (2.1) we can write $\Omega^d(M) \cong U \oplus V$ with $U \in \text{add}(U^k, \left(\text{mod } B\right)_a)$ and $V \in \left(\text{mod } B\right)_a$. Then

$$\text{proj. dim}(M) = d_1 + \max\{\text{proj. dim}(U), \text{proj. dim}(V)\}.$$  

Since $\phi^d(U) \in \Omega^d(\text{mod } A)$, it follows that $\phi^d(U)$ is a projective $A$-module and hence $\Omega^d(U) = 0$. Since $U \not\in \left(\text{mod } B\right)_a$, Corollary 2.38(a1) gives $\Omega(U) = 0$ and so $\text{proj. dim}(U) = 0$. Since $V \in \left(\text{mod } B\right)_a$, we can compute the projective resolution of $V$ in mod $B$ by Corollary 2.38(a2). In particular we have that $\text{proj. dim}(V) \leq \text{gl. dim}(B) = d_2$. It follows that

$$\text{proj. dim}(M) = d_1 + \text{proj. dim}(V) \leq d_1 + d_2,$$

and since $M$ was arbitrary, we conclude that $d \leq d_1 + d_2$.

Next, let us now show that $d_2 \leq d$. Let $N$ be a $B$-module with $\text{proj. dim}(N) = d_2$. Then $\psi_*(N)$ is a $\Lambda$-module and by Corollary 2.38(a2) we have that $\text{proj. dim}(\psi_*(N)) = d_2$. Hence $d_2 \leq d$. Finally, let us show that $d_1 \leq d$. Similarly to before, let $L$ be an $A$-module with $\text{inj. dim}(L) = d_1$. Then $\phi_*(L)$ is a $\Lambda$-module and by Corollary 2.38(b2) we have that $\text{inj. dim}(\phi_*(L)) = d_1$, which completes the proof. \qed

**Corollary 2.40.** For the Auslander-Reiten quiver of $\Lambda$ we have, as quivers, $\Gamma(\Lambda) = \Gamma(B) \coprod_{\Delta} \Gamma(A)$, where the righthand side denotes the amalgamated sum under the identification $\Delta = \phi_*(P\Delta) = \psi_*(\Delta')$. Moreover, in this identification, the vertex $[M] \in \Gamma(\Lambda)$ corresponds to the vertex $[\phi(M)]$ in $\Gamma(A)$ if $M$ is supported in $A$ and to the vertex $[\psi_*(M)]$ in $\Gamma(B)$ if $M$ is supported in $B$.

**Proof.** Immediate by Proposition 2.36 and Theorem 2.2, since almost split sequences in $\Gamma(\Lambda)$ correspond to almost split sequences in either $\Gamma(A)$ or $\Gamma(B)$. The vertex identification follows from Proposition 2.32. \qed
Example 2.41. Let \( B \) be as in Example 2.8 and let \( A = \Lambda_{4,3} \). Let \( I = I(3) = \frac{1}{3} \in \text{mod } B \) be the indecomposable injective \( B \)-module corresponding to the vertex 3 of \( Q_B \) and \( P = P(1) = \frac{1}{5} \in \text{mod } A \) be the indecomposable injective corresponding to the vertex 1 of \( Q_A \). Then \( I \) is a right abutment of \( B \) and \( P \) is a left abutment of \( A \), both of height 3. Hence the gluing \( \Lambda = B \triangleright A \) is defined and a simple computation shows that \( \Lambda \) can be given by the quiver with relations

\[
\begin{array}{ccccccc}
0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 4 \\
1' & \rightarrow & 2' & \rightarrow & 3' & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7.
\end{array}
\]

The Auslander-Reiten quiver \( \Gamma(B) \) of \( B \) was computed in example 2.15. The Auslander-Reiten quiver \( \Gamma(A) \) of \( A \) is

\[
\begin{array}{ccccccc}
2 & \rightarrow & 3 & \rightarrow & 1 & \rightarrow & 0 \\
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 0 \\
3 & \rightarrow & 2 & \rightarrow & 1 & \rightarrow & 0 \\
5 & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 0 \\
4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 0 \\
2' & \rightarrow & 3' & \rightarrow & 1' & \rightarrow & 0 \\
5' & \rightarrow & 6' & \rightarrow & 7 & \rightarrow & 0 \\
4' & \rightarrow & 5' & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 0 \\
2'' & \rightarrow & 3'' & \rightarrow & 1'' & \rightarrow & 0 \\
5'' & \rightarrow & 6'' & \rightarrow & 7 & \rightarrow & 0 \\
4'' & \rightarrow & 5'' & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 0.
\end{array}
\]

Using Corollary 2.40 we conclude that the Auslander-Reiten quiver \( \Gamma(\Lambda) \) of \( \Lambda \) is \( \Gamma(A) \) \( \coprod \) \( \Delta \) \( \Gamma(B) \) or

\[
\begin{array}{ccccccc}
2 & \rightarrow & 3 & \rightarrow & 1 & \rightarrow & 0 \\
1 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 0 \\
3 & \rightarrow & 2 & \rightarrow & 1 & \rightarrow & 0 \\
5 & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 0 \\
4 & \rightarrow & 5 & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 0 \\
2' & \rightarrow & 3' & \rightarrow & 1' & \rightarrow & 0 \\
5' & \rightarrow & 6' & \rightarrow & 7 & \rightarrow & 0 \\
4' & \rightarrow & 5' & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 0 \\
2'' & \rightarrow & 3'' & \rightarrow & 1'' & \rightarrow & 0 \\
5'' & \rightarrow & 6'' & \rightarrow & 7 & \rightarrow & 0 \\
4'' & \rightarrow & 5'' & \rightarrow & 6 & \rightarrow & 7 & \rightarrow & 0.
\end{array}
\]

where the intersection of \( \phi_\ast(\Gamma(A)) \) and \( \psi_\ast(\Gamma(B)) \) is exactly \( \Delta(3) \).

Example 2.42. Let \( A \) be a representation-directed algebra with a left abutment \( P \) of height \( h \). Then, by Example 2.20, we have \( A = KA_h \triangleright P \triangleright A \) and so by Corollary 2.40 we have \( \Gamma(\Lambda) = \Gamma(A) \coprod \Delta(h) \) under the identification \( (\text{Id}_A)_\ast P_\Delta = (f_p)_\ast(\Delta(h)) \). Hence we can view any \( A \)-module \( T \in \mathcal{F}_P \) as a \( KA_h \)-module via the functor \( f_\ast P \). Similarly, if \( I \) is a left abutment of \( A \) of height \( h \), any \( A \)-module \( X \in \mathcal{G}_I \) can be viewed as a \( KA_h \)-module through the identification \( A = A \triangleright P \triangleright A \) and the corresponding functor \( g_\ast I \).

We finish this section with a corollary that describes the connection between abutments of \( \Lambda \) and abutments of \( A \) and \( B \).

Corollary 2.43. Let \( \Lambda = B \triangleright A \) and let \( h - l + 1 \leq i \leq m \).

(a1) If \( h - l + 1 \leq i \leq h \) and \( D(Ae_i) \) is a right abutment of \( A \), then \( D(\Lambda e_i) \) is a right abutment of \( \Lambda \).

(a2) If \( h - l + 1 \leq i \leq 0 \) and \( e_i A \) is a left abutment of \( A \) such that \( \mathcal{F}_{e_i A} \cap \mathcal{F}_P = 0 \), then \( e_i \Lambda \) is a left abutment of \( \Lambda \).

(b1) If \( 1 \leq i \leq m \) and \( e_i B \) is a left abutment of \( B \), then \( e_i \Lambda \) is a left abutment of \( \Lambda \).
(b2) If \( h + 1 \leq i \leq m \), and \( D(Bi) \) is a right abutment of \( B \) such that \( G_{D(Bi)} \cap G_I = 0 \), then \( D(\Lambda_i) \) is a right abutment of \( \Lambda \).

(c1) If \( \varepsilon_i \Lambda \) is a left abutment of \( \Lambda \), then \( \varepsilon_i A \) is a left abutment of \( A \) if \( h - l + 1 \leq i \leq 0 \) and \( \varepsilon_i B \) is a left abutment of \( B \) if \( 1 \leq i \leq m \).

(c2) If \( D(\Lambda_i) \) is a right abutment of \( \Lambda \), then \( D(\Lambda e_i) \) is a right abutment of \( A \) if \( h - l + 1 \leq i \leq h \) and \( D(Be_i) \) is a right abutment of \( B \) if \( h + 1 \leq i \leq m \).

**Proof.** Let us indicatively show (a2) and (c1); the rest are similar. For (a2) notice that since \( h - l + 1 \leq i \leq 0 \), we have that \( \varepsilon_i \Lambda \) is supported in \( A \) by Proposition 2.28. In particular, by Lemma 2.23 and the definition of \( \phi \), we have \( \phi(\varepsilon_i \Lambda) = \varepsilon_i A \). Since \( P_{\varepsilon_i A} \cap P \neq 0 \), it follows from Proposition 2.13 that the two subquivers \( \varepsilon_i A \triangle \) and \( \varepsilon_i B \triangle \) of \( \Gamma(\Lambda) \) are disjoint. Hence by Corollary 2.40 it follows that \( \phi_*(\varepsilon_i A \triangle) \) is of the form \( \varepsilon_i A \triangle \) as in Proposition 2.13 and hence \( \varepsilon_i \Lambda \) is a left abutment of \( \Lambda \).

For (c1) notice that \( \phi(\varepsilon_i A) \cong \varepsilon_i A \) for \( h - l + 1 \leq i \leq 0 \) and \( \psi(\varepsilon_i A) \cong \varepsilon_i B \) for \( 1 \leq i \leq m \) again by Proposition 2.28 and Lemma 2.23. Then by Proposition 2.27, the whole foundation \( \varepsilon_i \Lambda \triangle \) is supported either in \( A \) or in \( B \), respectively. Then by Corollary 2.40 the image of \( \varepsilon_i \Lambda \triangle \) under \( \phi \), respectively \( \psi \), is of the form \( \varepsilon_i A \triangle \) respectively \( \varepsilon_i B \triangle \) and hence a left abutment of \( A \) respectively \( B \) by Proposition 2.13.

\[ \square \]

### 3. Part III: Fractures

In this section we will show how to use gluing to construct many examples of representation-directed algebras admitting \( n \)-cluster tilting subcategories. In subsection 3.1 we introduce the building blocks of our construction. In subsection 3.2 we show how the construction works. In subsection 3.3 we are interested in a special case of our construction which we can solve completely.

#### 3.1. Fractured subcategories

First, let us introduce some notation. Let \( \Lambda \) be a representation-directed algebra. We set

\[ P_{\Lambda} = P := \text{add}(\Lambda), \quad P_{\Lambda}^{ab} = P^{ab} := \{ P \in P \mid P \text{ is a left abutment of } \Lambda \}, \]

\[ I_{\Lambda} = I := \text{add}(D(\Lambda)), \quad I_{\Lambda}^{ab} = I^{ab} := \{ I \in I \mid I \text{ is a right abutment of } \Lambda \}. \]

By Proposition 2.13 it follows that for \( [P], [Q] \in P^{ab} \) the relation

\[ [P] \leq [Q] \text{ if and only if } P_P \subseteq P_Q \]

is a partial order. Similarly we define \( \leq \) on \( I^{ab} \). We will refer to elements of those sets as **maximal** or **minimal** with respect to these partial orders. We set

\[ P_{\Lambda}^{\text{mab}} = P^{\text{mab}} := \{ P \in P^{ab} \mid P \text{ is maximal} \}, \quad I_{\Lambda}^{\text{mab}} = I^{\text{mab}} := \{ I \in I^{ab} \mid I \text{ is maximal} \}. \]

The following important definition is due to Iyama ([Iya08], [Iya07]).

**Definition 3.1.** We call a subcategory \( \mathcal{C} \) of \( \mod \Lambda \) an **\( n \)-cluster tilting subcategory** if

\[ \mathcal{C} = \mathcal{C}^{\bot n} = \bot n \mathcal{C}, \]

where

\[ \mathcal{C}^{\bot n} := \{ X \in \mod \Lambda \mid \text{Ext}^i(\mathcal{C}, X) = 0 \text{ for all } 0 < i < n \}, \]

\[ \bot n \mathcal{C} := \{ X \in \mod \Lambda \mid \text{Ext}^i(X, \mathcal{C}) = 0 \text{ for all } 0 < i < n \}. \]

It is clear from the definition that \( \mod \Lambda \) is the unique 1-cluster tilting subcategory of \( \Lambda \). In the following we will assume that \( n \geq 2 \). Observe that since \( \Lambda \) is representation-finite, then any additive subcategory of \( \mod \Lambda \) is of the form \( \text{add}(M) \) for some \( M \in \mod \Lambda \). In this case we call \( M \) an **\( n \)-cluster tilting module**.

Note that \( n \)-cluster tilting subcategories are usually defined in more general settings by adding the requirement of functorial finiteness, but since \( \text{add}(M) \) is always functorially finite we can use the above definition.
Before we proceed, let us introduce one more piece of notation. Let $\mathcal{C}$, $\mathcal{V}$ be subcategories of $\text{mod } \Lambda$. We set $\mathcal{C} \setminus \mathcal{V}$ to be the additive closure of all indecomposable modules $X \in \mathcal{C}$ such that $X \notin \mathcal{V}$. With this in mind we recall the following characterization of $n$-cluster tilting subcategories for representation-directed algebras.

**Theorem 3.2.** [Vas18, Theorem 1] Assume that $\Lambda$ is a representation-directed algebra and let $\mathcal{C}$ be a subcategory of $\text{mod } \Lambda$. Then $\mathcal{C}$ is an $n$-cluster tilting subcategory if and only if the following conditions hold:

1. $\mathcal{P} \subseteq \mathcal{C}$,
2. $\tau_n$ and $\tau_{-n}$ induce mutually inverse bijections

$$
\begin{array}{c}
\mathcal{C} \setminus \mathcal{P} \\
\tau_n \\
\mathcal{C} \setminus \mathcal{I} \\
\tau_{-n}
\end{array}
$$

3. $\Omega^i(M)$ is indecomposable for all indecomposable $M \in \mathcal{C} \setminus \mathcal{P}$ and $0 < i < n$.
4. $\Omega^{-i}(N)$ is indecomposable for all indecomposable $N \in \mathcal{C} \setminus \mathcal{I}$ and $0 < i < n$.

Let $P_1$ be a maximal left abutment of $\Lambda$ with composition series

$$0 \subseteq P_h \subseteq \cdots \subseteq P_2 \subseteq P_1.$$  

We want to use Theorem 3.2 to generalize the definition of an $n$-cluster tilting subcategory so that we can replace the module $P = P_1 \oplus \cdots \oplus P_{h-1} \oplus P_h \in \mathcal{P}$ with a suitable module $T = T_1 \oplus \cdots \oplus T_{h-1} \oplus T_h \in \mathcal{F}_{P_1}$ instead. Since we want to generalize the definition of $n$-cluster tilting, we also want to have $\text{Ext}^i_{\Lambda}(T, T) = 0$ for $0 < i < n$. Since by Corollary 2.16 we have that $\text{proj} \, \dim(T) \leq 1$, this simplifies to $\text{Ext}^1_{\Lambda}(T, T) = 0$. Since $T \in \mathcal{F}_{P_1}$, if we view $T$ as a $KA_h$-module via $f_{P_1}^*$, we conclude that $f_{P_1}^*(T)$ should be a tilting $KA_h$-module. Tilting modules of $KA_h$ were classified in [HR81]. The following Proposition asserts that a basic tilting module of $KA_h$ has the correct number of indecomposable summands, which is necessary for our construction to work.

**Proposition 3.3.** [HR81, paragraph (4.1)] Let $T$ be a basic tilting module of $KA_h$. Then $T$ has exactly $h$ indecomposable summands.

Let $P$ be a left abutment of $\Lambda$. Recall that by Example 2.42 we can view a $\Lambda$-module $T$ in $\mathcal{F}_P$ as a $KA_h$-module via $(f_P)^*$ and dually for right abutments. With this in mind, we give the following definition.

**Definition 3.4.** Let $\Lambda$ be a representation-directed algebra.

(a) Let $P$ be a maximal left abutment of $\Lambda$ realized by $(e_i, f_i)_{i=1}^h$. A fracture of $P$ is a module $T \in \mathcal{F}_P$ such that $f_P(T) := T^i$ is a basic tilting $KA_h$-module. The level of $T$, denoted $\text{lvl}(T)$, is defined to be the number

$$\text{lvl}(T) := \max \{i \in \{1, \ldots, h\} \mid e_{-i+1} \Lambda \notin \text{add}(T)\} + 1.$$ 

(b) Let $I$ be a maximal right abutment of $\Lambda$ realized by $(e_i, g_i)_{i=1}^h$. A fracture of $I$ is a module $T \in \mathcal{G}_I$ such that $g_I(T) := T^i$ is a basic tilting $KA_h$-module. The level of $T$, denoted $\text{lvl}(T)$, is defined to be the number

$$\text{lvl}(T) := \max \{i \in \{1, \ldots, h\} \mid D(\Lambda e_i) \notin \text{add}(T)\} + 1.$$ 

The following lemma collects some basic information about fractures.

**Lemma 3.5.** Let $\Lambda$ be a representation-directed algebra.
(a) Let $T$ be a fracture of a maximal left abutment $P$, realized by $(e_i, f_i)_{i=1}^h$. Then $T$ has $h$ indecomposable summands and $\text{proj. dim}(T) \leq 1$.

(b) Let $T$ be a fracture of a maximal right abutment $I$, realized by $(e_i, g_i)_{i=1}^h$. Then $T$ has $h$ indecomposable summands and $\text{inj. dim}(T) \leq 1$.

**Proof.** Follows immediately by Corollary 2.16 and Proposition 3.3. \qed

**Example 3.6.** For a maximal left abutment $P$ realized by $(e_i, f_i)_{i=1}^h$, there exists a unique (up to isomorphism) fracture of $P$ that is projective, namely $T = \bigoplus_{i=1}^h e_i \Lambda$. To see that this is a fracture, notice that

$$T' = \bigoplus_{i=1}^h \phi_i(e_i \Lambda) = \bigoplus_{i=1}^h t_i K A_h = K A_h$$

is a tilting module. The fact that $T$ is the unique projective fracture of $P$ follows by Lemma 3.5.

Similarly, if $I$ is a right abutment realized by $(e_i, g_i-1)_{i=1}^h$, then $T = \bigoplus_{i=1}^h D(\Lambda e_i)$ is the unique fracture of $I$ that is injective.

**Definition 3.7.** Let $\Lambda$ be a representation-directed algebra.

(a) A **left fracturing** $T_L$ of $\Lambda$ is a module

$$T_L = \bigoplus_{[P] \in \mathcal{P}^{\text{mab}}} T^{(P)},$$

where $T^{(P)}$ is a fracture of $P$. We set $\mathcal{P}^L := \text{add} \left\{ \mathcal{P}^P, T^L \right\}$.

(b) A **right fracturing** $T_R$ of $\Lambda$ is a module

$$T_R = \bigoplus_{[I] \in \mathcal{I}^{\text{mab}}} T^{(I)},$$

where $T^{(I)}$ is a fracture of $I$. We set $\mathcal{I}^R := \text{add} \left\{ \mathcal{I}^I, T^R \right\}$.

(c) A **fracturing** of $\Lambda$ is a pair $(T_L, T_R)$ where $T_L$ is a left fracturing of $\Lambda$ and $T_R$ is a right fracturing of $\Lambda$.

Notice that if $(T_L, T_R)$ is a fracturing of $\Lambda$, then we have by Proposition 3.3 that $|\mathcal{P}^L| = |\mathcal{P}|$ and $|\mathcal{I}^R| = |\mathcal{I}|$. In particular, we always have $|\mathcal{P}^L| = |\mathcal{I}^R|$.

**Lemma 3.8.** Let $(T_L, T_R)$ be a fracturing of $\Lambda$. Then

(a) The following are equivalent
   (a1) $\mathcal{P}^L = \mathcal{P}$,
   (a2) $T_L$ is projective,
   (a3) $T^L \cong \bigoplus_{[P] \in \mathcal{P}^{\text{mab}}} P$.

(b) The following are equivalent
   (b1) $\mathcal{I}^R = \mathcal{I}$,
   (b2) $T_R$ is injective,
   (b3) $T^R \cong \bigoplus_{[I] \in \mathcal{I}^{\text{mab}}} I$.

**Proof.** We only prove (a); (b) is similar. First we show (a1) implies (a2). If $\mathcal{P}^L = \mathcal{P}$ then every module in $\mathcal{P}^L$ is projective. In particular, $T^L$ is projective. To see that (a2) implies (a3), first notice that if $T^L = \bigoplus_{Q \in \mathcal{P}^{\text{mab}}} T^{(Q)}$ is projective then $T^{(Q)}$ is projective for every maximal left abutment $Q$ of $\Lambda$. By Example 3.6 this implies that every indecomposable submodule of $Q$ is isomorphic to a summand of
Lemma 3.8. Following the definition, we also have
\[ T(\mathcal{P}(\Lambda)) = \tau \oplus \frac{3}{2} \oplus \frac{1}{3} \] and \( T(\mathcal{I}(\Lambda)) = 2' \oplus \frac{3'}{3} \oplus \frac{1'}{3'} \).

By construction, \( T(\mathcal{P}(\Lambda)) \) is the unique (up to isomorphism) projective fracture of \( \mathcal{P}(\Lambda) \) and the modules \( T(\mathcal{I}(\Lambda)) \) and \( T(\mathcal{I}(\Lambda)) \) are fractures of \( I(1) \) and \( I(1') \) respectively. Then \( (T(\mathcal{L}(\Lambda)), T(\mathcal{R}(\Lambda))) \) is a fracturing of \( \Lambda \), where \( T(\mathcal{L}(\Lambda)) = T(\mathcal{P}(\Lambda)) \) and \( T(\mathcal{R}(\Lambda)) = T(\mathcal{I}(\Lambda)) \). Since \( T(\mathcal{P}(\Lambda)) \) is projective, we have \( \mathcal{P}(\Lambda) = \text{add}(B) \) by Lemma 3.8. Following the definition, we also have
\[ T^R = \text{add} \left( \frac{5}{7} \oplus \frac{3}{6} \oplus \frac{4}{5} \oplus \frac{3}{4} \oplus T(\mathcal{I}(\Lambda)) \oplus T(\mathcal{I}(\Lambda)) \right). \]

Definition 3.10. Assume that \( \Lambda \) is a representation-directed algebra with a fracturing \( (\mathcal{L}(\Lambda), \mathcal{R}(\Lambda)) \) and \( \mathcal{C} \) be subcategory of \( \text{mod} \, \Lambda \). Then \( \mathcal{C} \) is called a \( (\mathcal{L}(\Lambda), \mathcal{R}(\Lambda), n) \)-fractured subcategory if

1. \( \mathcal{P}(\mathcal{C}) \subseteq \mathcal{C} \),
2. \( \tau_n \) and \( \tau_n^{-1} \) induce mutually inverse bijections
   \[ \mathcal{C} \backslash \mathcal{P}(\mathcal{C}) \leftrightarrow \mathcal{C} \backslash \mathcal{I}(\mathcal{C}), \]
3. \( \Omega^i(M) \) is indecomposable for all indecomposable \( M \in \mathcal{C} \backslash \mathcal{P}(\mathcal{C}) \) and \( 0 < i < n \),
4. \( \Omega^{-i}(N) \) is indecomposable for all indecomposable \( N \in \mathcal{C} \backslash \mathcal{I}(\mathcal{C}) \) and \( 0 < i < n \).

Notice that conditions (1) and (2) in the above definition imply that \( \mathcal{I} \subseteq \mathcal{C} \), since \( |\mathcal{P}(\mathcal{C})| = |\mathcal{I}| \).

Proposition 3.11. Let \( \Lambda \) be a representation-directed algebra and \( (\mathcal{L}(\Lambda), \mathcal{R}(\Lambda)) \) be a fracturing of \( \Lambda \). Let \( \mathcal{C} \) be \( (\mathcal{L}(\Lambda), \mathcal{R}(\Lambda), n) \)-fractured subcategory of \( \text{mod} \, \Lambda \) for \( n \geq 2 \). Then \( \mathcal{C} \) is an \( n \)-cluster tilting subcategory if and only if \( \mathcal{L}(\Lambda) \cong \mathcal{P}(\mathcal{C}) \) and \( \mathcal{R}(\Lambda) \cong \mathcal{I}(\mathcal{C}) \).

Proof. If \( \mathcal{L}(\Lambda) \cong \mathcal{P}(\mathcal{C}) \) and \( \mathcal{R}(\Lambda) \cong \mathcal{I}(\mathcal{C}) \), then Lemma 3.8 implies that \( \mathcal{P}(\mathcal{C}) = \mathcal{P} \) and \( \mathcal{R}(\Lambda) = \mathcal{I} \). Then Theorem 3.2 implies that \( \mathcal{C} \) is an \( n \)-cluster tilting subcategory.

Assume now that \( \mathcal{C} \) is an \( n \)-cluster tilting subcategory of \( \Lambda \) and we will show \( \mathcal{P}(\mathcal{C}) = \mathcal{P} \). (the proof of \( \mathcal{R}(\Lambda) = \mathcal{I} \) is similar). By Lemma 3.8, it is enough to show that \( \mathcal{L}(\Lambda) \) is projective. Assume to a contradiction that \( \mathcal{L}(\Lambda) \) is not projective. Then \( \text{proj.dim}(\mathcal{L}(\Lambda)) = 1 \) by Lemma 3.5. Hence \( \text{Ext}_\Lambda^1(\mathcal{L}(\Lambda), \mathcal{P}) \neq 0 \) which contradicts \( \mathcal{L}(\Lambda) \in \mathcal{C} \).

Proposition 3.11 motivates the following definition.

Definition 3.12. Let \( \Lambda \) be a representation-directed algebra with a fracturing \( (\mathcal{L}(\Lambda), \mathcal{R}(\Lambda)) \) and \( \mathcal{C} \) be a \( (\mathcal{L}(\Lambda), \mathcal{R}(\Lambda), n) \)-fractured subcategory. Then \( \mathcal{C} \) will be called a left \( n \)-cluster tilting subcategory if \( \mathcal{L}(\Lambda) \cong \mathcal{P}(\mathcal{C}) \) and a right \( n \)-cluster tilting subcategory if \( \mathcal{R}(\Lambda) \cong \mathcal{I}(\mathcal{C}) \).
Example 3.13. Let $\Lambda = KA_h$ and $n \geq 2$. Then $\Lambda$ admits a unique maximal left abutment, namely $P(1)$. Moreover, $P(1) = I(h)$ is the unique maximal right abutment of $\Lambda$ as well. A fracturing of $\Lambda$ is then a pair $(T^L, T^R)$ of tilting $KA_h$-modules. Since $\tau_n(M) = 0 = \tau_{-n}(M)$ for any $M \in \mod \Lambda$, it is immediate from the definition that $C \subseteq \mod \Lambda$ is a $(T^L, T^R, n)$-fractured subcategory of $\Lambda$ if and only if $T^L = T^R$ and $C = P^L = T^L$.

Example 3.14. Let $B$ and $(T^L_B, T^R_B)$ be as in Example 3.9. Let $C_B = \text{add} \left( \bigoplus_{i \geq 0} \tau_{-i}^2(B) \right)$. By computing $\tau_{-2}$, we find that $C_B = \text{add} \left( \Lambda \oplus \frac{4}{5} \oplus \frac{4}{5}' \oplus 2' \oplus 3 \right)$. A simple calculation verifies that $C_B$ is a $(T^L_B, T^R_B, 2)$-fractured subcategory. Since $T^\Lambda_B$ is projective, it is a left $n$-cluster tilting subcategory. For the convenience of the reader who might want to verify those claims, we give the Auslander-Reiten quiver of $B$ where we encircle the indecomposable modules which are in $C_B$:

3.2. Main construction. Our aim is to glue algebras admitting fractured subcategories in such a way that the resulting algebra also admits a corresponding fractured subcategory. To this end we need to first describe how to glue algebras with a fracturing. So let us fix two representation-directed algebras $A$ and $B$ with fracturings $(T^L_A, T^R_A)$ respectively $(T^L_B, T^R_B)$ and set $\Lambda = B \otimes_{A'} A$ where $Q$ is a left abutment of $A$ and $J$ is a right abutment of $B$, both of the same height $h$.

Let $P$ be a maximal left abutment of $\Lambda$. Then either $\phi^*(P)$ is a left abutment of $A$ or $\psi^*(P)$ is a left abutment of $B$ by Corollary 2.43(c1). Moreover, in either case it is clearly maximal by Corollary 2.40. Set

$$T^*(P) = \begin{cases} \phi^* \left( T^A_{(\phi^*(P))} \right) & \text{if } P \text{ is supported in } A, \\ \psi^* \left( T^B_{(\psi^*(P))} \right) & \text{if } P \text{ is supported in } B. \end{cases}$$

and

$$T^L_A = \bigoplus_{[P] \in \text{Fract}_\Lambda} T^*(P).$$

Observe that if $P$ is supported in $A$, then by construction the composition $\mod \Lambda \xrightarrow{(-)^*} \mod A \xrightarrow{(-)} \mod KA_h$ maps $T^*(P)$ to a basic tilting $KA_h$-module and similarly if $P$ is supported in $B$. 

Dually, maximal right abutments of \( A \) correspond to maximal right abutments of \( A \) or of \( B \) and so we set

\[
T_i^R = \bigoplus_{[\ell] \in \hat{X}_A} T_i^R \quad \text{if } i \text{ is supported in } A, \\
T_i^R = \bigoplus_{[\ell] \in \hat{X}_B} T_i^R \quad \text{if } i \text{ is supported in } B,
\]

and let \( T_i \) be the unique maximal right abutment of \( \Lambda \) of height \( h \). Let \( \Lambda = KA_B \) and let \( I = I(h) \) be the unique maximal right abutment of \( B \). Then by Example 2.20 we have that \( \Lambda \cong \Lambda = \sim B \). Let \( T_i \) be a tilting \( \sim B \)-module and let \( (T, T) \) be a fracturing of \( B \). Let also \( (T_i, T_i) \) be a fracturing of \( A \) such that we have

\[
T_i^Q \cap F_Q = \add(T_i) \quad \text{as } i \text{ is a right abutment of } \Lambda.
\]

Then it is easy to check that

\[
(T_i^L, T_i^R) = (T_i^L, T_i^R) \cap \sim B = (T_i^L, T_i^R) \quad \text{as } i \text{ is a right abutment of } \Lambda.
\]

In particular, \( (T_i^L, T_i^R) \) is a fracturing of \( B \)-module \( \Lambda \).

Furthermore, let \( h \geq 2 \). If \( C_A \) is a \( (T_i^L, T_i^R, n) \)-fractured subcategory of \( A \) and \( C_B = T \) is the unique \( (T, T, n) \)-fractured subcategory of \( B \) (see Example 3.15), then it follows that

\[
\add\{\phi_a(C_A), \psi_x(C_B)\} = C_A
\]

is a \( (T_i^L, T_i^R, n) \)-fractured subcategory. Similar results hold for \( \Lambda = A \)-fractured subcategory in the case where \( i \) is a maximal right abutment of \( A \) of height \( h \) and \( B = KA_B \).

The following theorem shows how to use a \( (T_i^L, T_i^R, n) \)-fractured subcategory and a \( (T_i^L, T_i^R, n) \)-fractured subcategory to construct a \( (T_i^L, T_i^R, n) \)-fractured subcategory under a compatibility condition.

**Theorem 3.16.** Let \( n \geq 2 \). Let \( A \) be a representation-directed algebra with a fracturing \( (T_i^L, T_i^R) \) and let \( C_A \) be a \( (T_i^L, T_i^R, n) \)-fractured subcategory. Let \( Q \in \mod A \) be a maximal left abutment of \( h \) and \( P \in \mod \Lambda \) be a left abutment of \( h \). Moreover, let \( L \) be a representation-directed algebra with a fracturing \( (T_i^L, T_i^R) \) and let \( C_B \) be a \( (T_i^L, T_i^R, n) \)-fractured subcategory. Let \( J \in \mod B \) be a maximal right abutment and let \( I \in \mod \Lambda \) be a right abutment of \( h \).

Assume that

\[
f_T^P \left( \add(T_i^Q) \cap F_Q \right) = g_T^I \left( \add(T_i^J) \cap G_I \right).
\]

Then \( (T_i^L, T_i^R) \) is a fracturing of \( B = B \)-module \( A \) and \( C_A = \add\{\phi_a(C_A), \psi_x(C_B)\} \) is a \( (T_i^L, T_i^R, n) \)-fractured subcategory.

**Proof.** Without loss of generality, we may assume that both \( A \neq KA_B \) and \( B \neq KA_B \) since we considered these cases in Example 3.15.

We need to prove conditions (1)–(4) of Definition 3.10. We pick idempotents of \( A, B \) and \( \Lambda \) as in Lemma 2.23. For condition (1) we need to show that \( P \subseteq C_A \), or equivalently

\[
\add\{P_a, P_b, T_i\}_\Lambda \subseteq \add\{\phi_a(C_A), \psi_x(C_B)\}.
\]

By the construction of \( T_i^L \) and since \( T_i^L \in \add(C_A) \) and \( T_i^B \in \add(C_B) \), it follows that \( T_i^L \in C_A \). Then, if \( \varepsilon_i \Lambda \) is an indecomposable projective \( \Lambda \)-module, it is enough to show the following two claims: first that if \( \varepsilon_i \Lambda \) is not a left abutment, then \( \varepsilon_i \Lambda \in C_A \) and second that if \( \varepsilon_i \Lambda \) is a left abutment, then either \( \varepsilon_i \Lambda \notin C_A \) or \( \varepsilon_i \Lambda \in \add(T_i^L) \).
Claim 1: Assume first that \( \varepsilon_i \Lambda \) is not a left abutment. If \( 1 \leq i \leq m \), then \( \varepsilon_i \Lambda \) is supported in \( B \) by Proposition \ref{P:1} and so \( \psi^* (\varepsilon_i \Lambda) = \varepsilon_i B \). By Corollary \ref{C:2} it follows that \( \varepsilon_i B \) is not a left abutment of \( B \) and so \( \varepsilon_i B \in \mathcal{C}_B \). Hence
\[
\psi_*(\varepsilon_i B) = \varepsilon_i \Lambda \in \psi_*(\mathcal{C}_B) \subseteq \mathcal{C}_\Lambda,
\]
as required.

If \( h - l + 1 \leq i \leq m \), then \( \varepsilon_i \Lambda \) is supported in \( A \) and \( \phi^!(\varepsilon_i \Lambda) = \varepsilon_i A \). If \( \varepsilon_i A \) is not a left abutment of \( A \), then a similar argument as before shows that \( \varepsilon_i \Lambda \in \mathcal{C}_A \). If \( \varepsilon_i A \) is a left abutment of \( A \), then we must have that \( \phi_* (\mathcal{F}_{\varepsilon_i A} \cap \mathcal{F}_P) \neq 0 \), since this intersection being zero implies via Corollary \ref{C:2}(a2) that \( \varepsilon_i \Lambda \) is a left abutment, contradicting our assumption. In particular, we have \( \mathcal{F}_{\varepsilon_i A} \cap \mathcal{F}_{\varepsilon_i A} \neq 0 \), since by assumption \( P \cong \varepsilon_i A \). Since \( i < 1 \), by Proposition \ref{P:2} we have that \( \mathcal{F}_{\varepsilon_i A} \subseteq \mathcal{F}_{\varepsilon_i A} \) and that \( \varepsilon_i \Lambda \) is a full subquiver of \( \varepsilon_i \Lambda \). In particular, \( \varepsilon_i A \) and \( \varepsilon_i A \) are both abutments appearing in the same radical series of the maximal abutment \( Q \) and the height of \( \varepsilon_i A \) is greater than \( h \). Since \( h \geq \text{lvl}(T^Q_A) \), it follows from the definition of the level that \( \varepsilon_i \Lambda \) is an indecomposable summand of \( T^Q_A \). Since \( \varepsilon_i A \in \text{add}(T^Q_A) \subseteq \mathcal{C}_A \), it follows that \( \varepsilon_i \Lambda \in \mathcal{C}_A \), as required. This shows the first claim.

Claim 2: Assume now that \( \varepsilon_i \Lambda \) is a left abutment and that \( \varepsilon_i \Lambda \in \mathcal{C}_A \). Then it is enough to show that \( \varepsilon_i \Lambda \in \text{add}(T^A\Lambda) \). Since \( \varepsilon_i \Lambda \in \text{add}(\phi_*(\mathcal{C}_A), \psi_*(\mathcal{C}_B)) \) and \( \varepsilon_i \Lambda \) is indecomposable, then either \( \phi^!(\varepsilon_i \Lambda) \in \mathcal{C}_A \) or \( \psi^!(\varepsilon_i \Lambda) \in \mathcal{C}_B \). In the first case \( \phi^!(\varepsilon_i \Lambda) \) is a left abutment by Corollary \ref{C:2}(c1) and so \( \psi^!(\varepsilon_i \Lambda) \in \text{add}(T^A\Lambda) \). Similarly in the second case \( \psi^*(\varepsilon_i \Lambda) \in \text{add}(T^B\Lambda) \). Let \( Y > \varepsilon_i \Lambda \) be the unique (up to isomorphism) maximal left abutment greater than \( \varepsilon_i \Lambda \).

If \( h + 1 \leq i \leq m \), then \( \varepsilon_i \Lambda \) is supported in \( B \) by Proposition \ref{P:1} and \( \mathcal{F}_{\varepsilon_i A} \) is supported in \( B \) by Proposition \ref{P:2}. In this case, we have \( \psi^*(\varepsilon_i A) = \varepsilon_i B \in \text{add}(T^B\Lambda) \). Moreover, Corollary \ref{C:2} and Proposition \ref{P:2} imply that \( \psi^*(Y) \) is again a maximal left abutment of \( B \). Hence \( \psi_*(T^B\psi^*(Y)) \in \text{add}(T^B\Lambda) \) by the definition of gluing of fracturings. Since \( \psi^*(Y) \) is maximal, we have \( \varepsilon_i B < \psi^*(Y) \) which implies that \( \varepsilon_i B \in \text{add}(T^B\psi^*(Y)) \) because \( \varepsilon_i B \in \text{add}(T^B\Lambda) \). Then
\[
\varepsilon_i \Lambda = \psi_*(\varepsilon_i B) \in \text{add}(\psi_*(T^B\psi^*(Y))) \subseteq \text{add}(T^B\Lambda),
\]
as required.

If \( 1 \leq i \leq h \), then we will reach a contradiction. In particular, since \( \varepsilon_i \Lambda \) is a left abutment, Proposition \ref{P:2} and Corollary \ref{C:2} imply that \( B \cong K \Lambda_b \), which contradicts our assumption that \( B \neq K \Lambda_b \).

If \( h - l + 1 \leq i \leq 0 \) then we proceed as in the case \( h + 1 \leq i \leq m \) to show that
\[
\varepsilon_i \Lambda = \phi_*(\varepsilon_i A) \in \text{add}(\phi_*(T^A\phi^!(Y))) \subseteq \text{add}(T^A\Lambda),
\]
which proves the second claim.

Hence, condition (1) is satisfied. Conditions (3) and (4) follow immediately by Corollary \ref{C:2} and the corresponding conditions being true for \( \mathcal{C}_A \) and \( \mathcal{C}_B \).

It remains to show that condition (2) holds for \( \mathcal{C}_A \). To simplify notation a bit, in the rest of the proof we will write \( \tau_n \) instead of \( (\tau_n)_A, (\tau_n)_B \) and similarly for \( \tau_n^- \), since the subscript will always be clear from the context.

Let \( M \in \mathcal{C}_A \mathcal{P}_A^- \) and we will show that \( \tau_n(M) \in \mathcal{C}_A \mathcal{P}_A^- \) and \( \tau_n(M) \cong M \); the dual fact that if \( N \in \mathcal{C}_A \mathcal{P}_A^- \) then \( \tau_n^-(N) \in \mathcal{C}_\Lambda \mathcal{P}_A^- \) and \( \tau_n\tau_n^-(N) \cong N \) can be shown similarly.

Since \( M \) is indecomposable, it follows that \( M \in \text{add}(\phi_*(\mathcal{C}_A)) \) or \( M \in \text{add}(\psi_*(\mathcal{C}_B)) \). Therefore, \( \phi^!(M) \in \mathcal{C}_A \) or \( \psi^!(M) \in \mathcal{C}_B \) and in both cases the module remains indecomposable.

Assume first that \( \psi^!(M) \in \mathcal{C}_B \). We claim that \( \psi^!(M) \in (\mathcal{C}_B)_{\mathcal{P}_B^-} \). Assume the opposite to a contradiction. Then \( \psi^!(M) \in T^B_{\mathcal{P}_B^-} = \text{add}(\{P_B \setminus \mathcal{P}_B, T^B_{\mathcal{P}_B^-} \}) \). If \( \psi^!(M) \) is projective but not a left abutment, then we reach a contradiction since \( M \) is also projective but not a left abutment by Proposition \ref{P:1} and Corollary \ref{C:2}. Hence \( M \in \text{add}(T^B_{\mathcal{P}_B^-}) \) and so \( M \in \text{add}(T^B_{\mathcal{P}_B^-}) \) for some maximal left abutment \( Z \). It follows from Corollary \ref{C:2} and Proposition \ref{P:2} that \( \psi_*(Z) \) is a maximal left abutment of \( \Lambda \) unless
B \cong KA_h and Z \cong I \not\cong J. But by our assumption B \not\cong KA_h and so \psi^*(Z) is also a maximal left abutment of \Lambda. It follows that
\[ M \cong \psi_*\psi^*(M) \in \text{add}(\psi_*(T_B^Z)) \subseteq \text{add}(T^Z_A), \]
contradicting \( M \not\in \mathcal{C}_P^A \).

Hence we have \( \psi^*(M) \in (\mathcal{C}_B)_{\mathcal{P}_B^A} \). Since \( \mathcal{C}_B \) is a \((T_B^L, T_B^R, n)\)-fractured subcategory, it follows that \( \tau_n \psi^*(M) \in (\mathcal{C}_A)_{\mathcal{P}_A^A} \). A similar argument as before shows that \( \psi_*((\mathcal{C}_A)_{\mathcal{P}_A^A}) \subseteq (\mathcal{C}_A)_{\mathcal{P}_A^A} \). Moreover, by Corollary 2.38 it follows that \( \tau_n(M) \cong \psi_*\tau_n \psi^*(M) \) and so \( \tau_n(M) \in (\mathcal{C}_A)_{\mathcal{P}_A^A} \). The previous argument shows also that we can compute
\[ \tau_n \tau_n M \cong \psi_* \tau_n \psi_* \tau_n \psi^*(M) \cong \psi_* \tau_n \psi^*(M) \cong \psi_* \psi^*(M) \cong M, \]
as required.

Finally, it remains to check the case \( \phi^*(M) \in \mathcal{C}_A \). As before we can easily show that \( \phi^*(M) \in (\mathcal{C}_A)_{\mathcal{P}_A^A} \). Here we distinguish two cases. If \( \phi^*(M) \not\in \text{add}(T_A^{(Q)}) \cap \mathcal{F}_P \), then \( \tau_n \) can be computed inside \( \mod A \) as per Corollary 2.38 and the previous case. On the other hand, if \( \phi^*(M) \in \text{add}(T_A^{(Q)}) \cap \mathcal{F}_P \), it follows from Corollary 2.40 that \( M \) is supported both in \( A \) and in \( B \). In particular, viewing \( M \) as a \( KA_h \)-module via the compositions \( \mod \Lambda \to \mod A \to \mod KA_h \) and \( \mod A \to \mod B \to \mod KA_h \) produces the same module by Corollary 2.40. Hence the compatibility condition \( \tau_2 \) implies that \( \phi^*(M) \in \text{add}(T_B^{(Q)}) \cap \mathcal{G}_J \). In particular \( \psi^*(M) \in \mathcal{C}_B \), in which case we showed that condition \( \tau_2 \) is satisfied. This completes the proof.

The following corollary of Theorem 3.16 is of particular interest.

**Corollary 3.17.** Let \( n \geq 1 \). Let \( A \) be a strongly \((n, d_1)\)-representation-directed algebra and \( B \) be a strongly \((n, d_2)\)-representation-directed algebra. Let \( P \) be a simple projective \( A \)-module and \( I \) be a simple injective \( B \)-module. Then \( \Lambda = B \check{\otimes} A \) is a strongly \((n, d)\)-representation-directed algebra for some \( d \) with \( \max\{d_1, d_2\} \leq d \leq d_1 + d_2 \).

**Proof.** First we have that \( \Lambda \) is representation-directed by Corollary 2.37 and that \( \max\{d_1, d_2\} \leq d \leq d_1 + d_2 \) by Corollary 2.39. It remains to show that \( \Lambda \) admits an \( n \)-cluster tilting subcategory \( \mathcal{C}_A \). If \( n = 1 \), then \( \mathcal{C}_A = \mod A \). Assume that \( n \geq 2 \). By Proposition 3.11, we have that there exists a \((P^A, I^A, T_A^{(Q)})\)-fractured subcategory \( \mathcal{C}_A \) of \( \mod A \) and a \((P^B, I^B, T_B^{(Q)})\)-fractured subcategory \( \mathcal{C}_B \) of \( B \). If \( Q \) is a maximal left abutment of \( A \), then the corresponding tilting module \( T_A^{(Q)} \) for the fracturing \( (P^A, I^B) \) is projective, since \( P^A \) is projective. It follows that \( \text{lvl}(T_A^{(Q)}) = 1 \). Similarly if \( J \) is a maximal right abutment of \( B \), we have that \( \text{lvl}(T_B^{(J)}) = 1 \).

Since \( P \) and \( I \) are simple, we have that both \( P \) and \( I \) have height 1. In particular, if \( Q \) is a maximal left \( A \)-abutment with \( \mathcal{F}_P \subseteq \mathcal{F}_Q \) and \( J \) is a maximal right \( B \)-abutment with \( \mathcal{G}_J \subseteq \mathcal{G}_I \), it follows that
\[ f_P^*(\text{add}(T_A^{(Q)}) \cap \mathcal{F}_P) \cong \text{add}(S(1)) \cong g_I^*(\text{add}(T_B^{(J)}) \cap \mathcal{G}_I), \]
where \( S(1) \) is the unique simple \( KA_h \)-module. Hence, it follows by Theorem 3.16 that \( T_A^L, T_B^R, \mathcal{C}_A = \text{add} \{\phi_*(\mathcal{C}_A), \psi_*(\mathcal{C}_B)\} \) is a \((T_A^L, T_B^R, n)\)-fractured subcategory. It remains to show that \( T_A^L \cong P^A \) and \( T_B^R \cong I^B \). Let us only show the first isomorphism; the other follows by similar arguments.

We have
\[ T_A^L = \bigoplus_{[R] \in \mathcal{P}_A^A} T_A^L \]
where
\[ T_A^L = \begin{cases} \phi_*\left(T_A^{\phi_*(R)}\right) & \text{if } R \text{ is supported in } A, \\ \psi_*\left(T_B^{\psi_*(R)}\right) & \text{if } R \text{ is supported in } B, \end{cases} \]
so it is enough to show that \( T^*_B^{(R)} \) is projective. If \( R \) is supported in \( B \), then \( T^B_B^{(R)} \) is a projective \( B \)-module by assumption, and so its image under \( \phi_* \) is a projective \( \Lambda \)-module by Proposition \[2.28\].

If \( R \) is supported in \( A \), there is no arrow going into the triangle \( \phi(R) \Delta \). Then by Corollary \[2.40\], there is no arrow going into the triangle \( \phi(R) \Delta \). Let \( h' \) be the height of \( R \). Since \( T^A_A^{(R)} \) is projective by assumption, it is the unique projective fracture corresponding to the triangle \( \phi(R) \Delta \). That is, \( T^A_A^{(R)} \) is isomorphic to a direct sum \( \bigoplus_{i=1}^{h'} T_i \), where \( \{ [T_i] \}_{i=1}^{h'} \) are all the different leftmost vertices in the triangle \( \phi(R) \Delta \). Hence lifting this through \( \phi_* \), we again get a direct sum corresponding to the leftmost vertices of the triangle \( R \Delta \), which is a projective \( \Lambda \)-module and the proof is complete. \( \square \)

Since representation-directed algebras always have simple projective and injective modules, Corollary \[3.17\] can be used to construct arbitrarily many \( n \)-cluster tilting subcategories from known \( n \)-cluster tilting subcategories of representation-directed algebras.

We describe the next simplest case of using Theorem \[3.16\]. First we need to have \( T_A^R = I_A \) and \( T_A^L \) to have exactly one fracture \( T_A^L(Q) \) corresponding to a maximal left abutment \( Q \) that is not projective. Similarly we need to have \( T_B^R = P_L \) and \( T_B^R \) to have exactly one fracture \( T(B) \) corresponding to a maximal right abutment \( J \) that is not injective. Then, after gluing the resulting subcategory will be a \((P_A, I_A, n)\)-fractured subcategory or equivalently an \( n \)-cluster tilting subcategory.

Even if there are more nonprojective fractures chosen for the left fracturing of \( A \) (or similarly noninjective fractures chosen for the right fracturing of \( B \)), by the construction of gluing one can glue at each fracture independently. Say we have an algebra \( \Lambda \) and that at each nonprojective fracture we glue by a left \( n \)-cluster tilting subcategory, while at each noninjective fracture we glue by a right \( n \)-cluster tilting subcategory and each gluing is compatible as per the requirements of Theorem \[3.16\].

Then the result will be an algebra such that the gluing of all the fractured subcategories is an \( n \)-cluster tilting subcategory. We illustrate with a detailed example.

**Example 3.18.** Let \( B \), \( (T_B^L, T_B^R) \) and \( C_B \) be as in Example \[3.14\]. Recall that \( C_B \) was obtained by repeatedly applying \( \tau_2 \) starting from \( B \) and \( C_B \) is a left \( n \)-cluster tilting subcategory and there are two noninjective fractures in \( T_B^R \), namely

\[
T^{(I(3))_B} = 3_B \oplus 2_B^3 \oplus 1_B^3 \quad \text{and} \quad T^{(I(3))_B} = 2_B^3 \oplus 2_B^3 \oplus 1_B^3 .
\]

We want to glue two appropriate algebras with \( B \), one alongside \( 1_B^3 \) and one alongside \( 2_B^3 \). Let us start with \( 1_B^3 \). Consider the algebra \( A \) as in Example \[2.41\]. It is easy to see that \( A \) admits a 2-cluster tilting subcategory, given by \( C_A = \text{add}(A \oplus D(A)) \). Then by Proposition \[3.11\] we have that \( C_A \) is \((T_A^L, T_A^R, 2)\)-fractured subcategory where

\[
T^{(P(1)_A)} = 3_A \oplus 2_A^3 \oplus 1_A^3.
\]

Hence, viewing \( T^{(I(3)_B)} \) and \( T^{(P(1)_A)} \) as \( K_A \)-modules via the respective functors, we have that they coincide since

\[
g^{I(3)_B}(3_B \oplus 2_B^3 \oplus 1_B^3) \cong 1_{K_A} \cong f^{P(1)_A}(3_A \oplus 2_A^3 \oplus 1_A^3) .
\]

In particular, by Theorem \[3.16\] the algebra \( \Lambda_1 = B^{P(1)_A \Delta^{(3)_B}} \) admits a \((T_A^L, T_A^R, 2)\)-fractured subcategory \( C_{\Lambda_1} \), where \((T_A^L, T_A^R) = (T_B^L, T_B^R) = T^{(P(1)_A \Delta^{(3)_B}}(T_A^L, T_A^R) \). Viewing the Auslander-Reiten quivers of \( B \) and \( A \) embedded in the Auslander-Reiten quiver of \( \Lambda_1 \) we can find an additive generator of \( C_{\Lambda_1} \). If we denote the indecomposable modules in the 2-fractured subcategories by encircling the
corresponding vertices we have:

\[ \Gamma(B) \text{ and indecomposables in } \mathcal{C}_B \]

\[ \Gamma(A) \text{ and indecomposables in } \mathcal{C}_A, \]

and after gluing we get

\[ \Gamma(\Lambda_1) \text{ and indecomposables in } \mathcal{C}_{\Lambda_1}. \]

In particular, \( \mathcal{C}_{\Lambda_1} \) is a 2-left cluster tilting subcategory, as expected. Moreover, \( \Lambda_1 \) has two maximal right abutments, namely \( I(2)_{\Lambda_1} \) and \( I(3')_{\Lambda_1} \). The fracture corresponding to the first one is injective, while the fracture of the second one is

\[ T(I(3')_{\Lambda_1}) = 2'_{\Lambda_1} \oplus 2'_{3'}_{\Lambda_1} \oplus 1'_{3'}_{\Lambda_1}, \]

which is noninjective. Hence we want to glue at \( I(3')_{\Lambda_1} \). Let \( C \) be the algebra given by the quiver with relations

\[ -2' \longrightarrow -1' \longrightarrow 0' \longrightarrow 1' \longrightarrow 2' \longrightarrow 3'. \]

Then the Auslander-Reiten quiver \( \Gamma(C) \) of \( C \) is

\[ \begin{align*}
1' & \quad 2' \\
2' & \quad 1' \\
0' & \quad 1' \\
-1' & \quad 0' \\
-2' & \quad -1' \\
3' & \quad 2' \\
0' & \quad -1' \\
-2' & \quad -1' \end{align*} \]
Hence by Proposition 2.13 there is a unique maximal left abutment, namely \( P(1')_C = \frac{1'}{3'} C \) and a unique maximal right abutment, namely \(-\frac{1'}{3'} C\). It follows that \((T^L_C, T^R_C)\) is a fracturing of \( C \), where
\[
T^L_C = 2' C \oplus 3' C \oplus \frac{1'}{3'} C \quad \text{and} \quad T^R_C = -\frac{2'}{1'} C \oplus -2' C.
\]
It is easy to see that \( C \) has a \((T^L_C, T^R_C, 2)\)-fractured subcategory such that the gluing
\[
\Lambda_2 = \Lambda_1^P(1')C_{\Delta(\Lambda_1)}C
\]
is compatible according to Theorem 3.16. Hence the gluing of the subcategories \( \mathcal{C}_{\Lambda_1} \) and \( \mathcal{C}_C \) is a 2-cluster tilting subcategory. Concretely, the Auslander-Reiten quivers of \( \Lambda_1 \) and \( C \) along with their 2-fractured subcategories are

\[
\Gamma(\Lambda_1) \quad \text{and indecomposables in} \quad \mathcal{C}_{\Lambda_1}
\]
the algebra \( \Lambda_2 \) is given by the quiver with relations
\[
0 \rightarrow 1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 6 \rightarrow 7,
\]
and the Auslander-Reiten quiver \( \Gamma(\Lambda_2) \) of \( \Lambda_2 \) with the 2-cluster tilting subcategory of \( \mathcal{C}_{\Lambda_2} \) is

\[
\Gamma(\Lambda_2) \quad \text{and indecomposables in} \quad \mathcal{C}_{\Lambda_2}
\]

Remark 3.19. The algebra of Example 3.18 and Corollary 3.17 give rise to algebras with many interesting properties. For instance let us consider the number of sinks and sources in the quiver of an algebra. Let \( A = \Lambda_2 \) where \( \Lambda_2 \) is as in Example 3.18. As we can see in Example 3.18 by the
Auslander-Reiten quiver of $A$, there exist exactly $s_A = 1$ simple projective $A$-modules and exactly $t_A = 2$ simple injective $A$-modules. Let $B$ be a representation-directed algebra which admits a 2-cluster tilting subcategory and assume that there exist $s_B$ simple projective $B$-modules and $t_B$ simple injective $B$-modules. By gluing at any simple projective $A$-module and any simple injective $B$-module we get the algebra $B^{(1)} = B \triangleleft A$. By Corollary 3.17 we have that $B^{(1)}$ admits a 2-cluster tilting subcategory. By Corollary 2.29 we have that $(s_B, t_B) = (s_B, t_B + 1)$. Continuing inductively, let $B^{(i)}$ be a sequence of algebras defined by $B^{(i)} = B^{(i-1)} \triangleleft A$ where the gluing is done over any simple projective $A$-module and any simple injective $B^{(i-1)}$-module. Then we get that $B^{(i)}$ admits a 2-cluster tilting subcategory and $(s_B, t_B) = (s_B, t_B + i)$.

A similar argument shows that if we let $B_{(j)}$ be a sequence of algebras defined by $B_{(1)} = A_{op} \triangleleft B$ and $B_{(j)} = A_{op} \triangleleft B_{(j-1)}$, where all gluings are done over simple modules, then again $B_{(j)}$ admits a 2-cluster tilting subcategory and $(s_B, t_B) = (s_B + j, t_B)$. More generally, we have that

$$(s_B, t_B) = (s_B + j, t_B + i).$$

In particular, by choosing $(s_B, t_B) = (1, 1)$ (for example, $B = KA_h / \text{rad}(KA_h)^h$ for some $h \geq 3$), we have that for any pair $(s, t)$ with $s, t \geq 1$ there exists an algebra $\Lambda$ such that $\Lambda$ admits a 2-cluster tilting subcategory and $(s, t) = (s, t)$. Since the number of simple projective $\Lambda$-modules corresponds to the number of sinks in the quiver of $\Lambda$ and the number of simple injective $\Lambda$-modules corresponds to the number of sources in the quiver of $\Lambda$, it follows that for any given pair of numbers $(s, t)$, there exists a quiver $Q$ with $s$ sinks and $t$ sources and a bounded quiver algebra $\Lambda = KQ/I$ such that $\Lambda$ admits a 2-cluster tilting subcategory. Note that by construction the number of vertices of the quiver of $\Lambda$ is of the order of $s + t$ but can be made arbitrarily large.

In Example 3.18 it was not clear how one should find the algebras $A$ and $C$. They depended on the type of fractures that the algebra $B$ had and clearly they are not unique since we can always glue at simple modules via Corollary 3.17. The fractures in this example corresponded to slice modules of $KA_2$ and we will see in section 3.3 how we can find appropriate algebras to glue at this case. More generally we have the following question.

**Question 1.** Let $T = \bigoplus_{i=1}^n T_i$ be a basic tilting module of $KA_h$ and $n \geq 2$.

(a) Can we find a representation-directed algebra $A$ with a right fracturing $T_A^R = \bigoplus_{[J] \in \text{I}^{\text{ab}}} T_A^{(J)}$

such that there exists a maximal right abutment $I$ of $B$ with $g_I^* (T_I^{(J)}) \cong T$, for every maximal right abutment $J$ with $J \neq I$ we have that $T_I^{(J)}$ is injective and, moreover, there exists a $(P_A^{(ab)}, T_A^R, n)$-fractured subcategory?

(b) Can we find a representation-directed algebra $A$ with a left fracturing $T_A^L = \bigoplus_{[Q] \in \text{I}^{\text{ab}}} T_A^{(Q)}$

such that there exists a maximal left abutment $P$ of $A$ with $f_P^* (T_P^{(Q)}) \cong T$, for every maximal left abutment $Q$ with $Q \neq P$ we have that $T_P^{(Q)}$ is projective and, moreover, there exists a $(T_A^{L}, I_A^{(ab)}, n)$-fractured subcategory?

If we can answer Question (a) (respectively (b)) affirmatively we will say that we can complete $T$ on the left (respectively right). Notice that by symmetry we can complete $T$ on the left if and only if we can complete $D(T)$ on the right by taking $A = B^{op}$. In particular, if $T \cong D(T)$, then if we can answer Question (a) affirmatively, by Theorem 3.16 we conclude that the algebra $B^{D(T)\triangleleft B^{op}}$ admits an $n$-cluster tilting subcategory and similarly if we answer question 2. We illustrate this situation with an example.

**Example 3.20.** Let $A$ be given by the quiver with relations

$$
\begin{align*}
&\begin{array}{c}
-3 \\
-5
\end{array} \quad \begin{array}{c}
-2 \\
-4
\end{array} \quad \begin{array}{c}
-1 \\
0
\end{array} \quad \begin{array}{c}
1 \\
2
\end{array} \quad \begin{array}{c}
3
\end{array}
\end{align*}
$$

Then $A$ is a representation-directed algebra which admits a 2-cluster tilting subcategory. Note that by construction the number of vertices of the quiver of $\Lambda$ is $Q$ to the number of sources in the quiver of $\Lambda$, it follows that for any given pair of numbers $(s, t)$, there exists a quiver $Q$ with $s$ sinks and $t$ sources and a bound quiver algebra $\Lambda = KQ/I$ such that $\Lambda$ admits a 2-cluster tilting subcategory. Note that by construction the number of vertices of the quiver of $\Lambda$ is of the order of $s + t$ but can be made arbitrarily large.
By Proposition 2.6 there is a unique maximal left abutment, namely $P(1)_A = \frac{1}{3}$. If we set $T^R_A = I^{ab}$ to be an injective right fracturing of $A$ and

$$T^L_A = T^{(P(1)_A)} = \frac{1}{3} A \oplus \frac{1}{2} A \oplus \frac{1}{2} A,$$

then $(T^L_A, T^R_A)$ is a fracturing of $A$ and there exists a $(T^L_A, T^R_A, 3)$-fractured subcategory $C_A$. The Auslander-Reiten quiver $\Gamma(A)$ as well as the indecomposable modules in $C_A$ are

$$\Gamma(A) \text{ and indecomposables in } C_A.$$

In particular, $C_A$ is a right 3-cluster tilting subcategory. Notice that the fracture appearing in the foundation of $P(1)_A$ is symmetric in the sense that we have

$$f^{(P(1)_A)}_P \left( T^{(P(1)_A)}_A \right) \cong D_{K A_3} \left( f^{(P(1)_A)}_P \left( T^{(P(1)_A)}_A \right) \right).$$

Then the algebra $B = A^{op}$ is given by the quiver with relations

$$7 \to 6 \to 5 \to 4 \leftrightarrow 3 \leftrightarrow 2 \leftrightarrow 1.$$

and there exists a unique maximal right abutment of $B$, namely $I(3)_B$. Then, for the choice of fracturing $(P^{(I(3)_B)}_B, T^{(I(3)_B)}_B)$ with

$$g^{(I(3)_B)}_I \left( T^{(I(3)_B)}_B \right) \cong f^{(P(1)_A)}_P \left( T^{(P(1)_A)}_A \right)$$

there exists a left 3-cluster tilting subcategory $C_B = D(C_A)$. Hence we can apply Theorem 3.16. The algebra $\Lambda = B^{P(1)_A \Delta I(3)_B} A$ is given by the quiver with relations

$$-3 \leftrightarrow -2 \leftrightarrow -1 \leftrightarrow 0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow 3 \leftrightarrow 4 \leftrightarrow 5 \leftrightarrow 6 \leftrightarrow 7.$$

Then the Auslander-Reiten quivers of $A$ and $B$ along with their 3-fractured subcategories are

$$\Gamma(A) \text{ and indecomposables in } C_A.$$

$$\Gamma(B) \text{ and indecomposables in } C_B.$$
and the Auslander-Reiten quiver of \( \Lambda \) with its 3-cluster tilting subcategory is

\[
\begin{array}{cccccccccccc}
\otimes & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
\downarrow & & & & & & & & & & & \\
\otimes & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

\( \Gamma(\Lambda) \) and indecomposables in \( \mathcal{C}_\Lambda \).

### 3.3. The case of slice modules

In this section, we answer Question \( \square \) positively in the case of \( T \) being a slice module. We begin with the definition of slice modules for \( KA_h \); for the general definition of slice modules we refer to [HRS1].

**Definition 3.3.21.** A set \( \{T_1, \ldots, T_h\} \) of distinct indecomposable \( KA_h \)-modules is called a slice of \( KA_h \) if they all have different lengths, and if \( l(T_i) = l(T_j) - 1 \) then either there exists a monomorphism \( T_i \hookrightarrow T_j \) or an epimorphism \( T_j \twoheadrightarrow T_i \). In this case we call \( T = \bigoplus_{i=1}^h T_i \) a slice module and \( \text{add}(T) \) a slice subcategory.

Since the possible lengths of \( T_i \) are \( 1 \) to \( h \), we can assume without loss of generality that for a slice of \( KA_h \), we have \( l(T_i) = i \). If we denote the indecomposable \( KA_h \)-modules by \( M(i, j) \) as in (1,1), it follows that a slice of \( KA_h \) is a set of modules \( \{M(1,1), M(i_2,2), \ldots, M(i_h,h)\} \) such that \( i_k = i_{k-1} \) or \( i_k = i_{k-1} - 1 \). In particular, \( i_h = 1 \) and \( i_{h-1} = 1 \) or \( i_{h-1} = 2 \).

**Definition 3.3.22.** Let \( \Lambda \) be a representation-directed algebra and let \( T \) be a fracture corresponding to a maximal abutment of \( \Lambda \). We will say that \( T \) is a slice fracture if \( T \) viewed as a \( KA_h \)-module is a slice module.

Our aim is to answer Question \( \square \) affirmatively when \( T \) is a slice module. Notice that if \( T \) is a slice module, then \( D(T) \) is also a slice module. Hence by symmetry it is enough to answer Question \( \square \) \( a \).

The following computational lemma will be used.

**Lemma 3.3.23.** [Vas18, Lemma 4.8] Let \( M(i, j) \neq 0 \) be a \( \Lambda_{m,h} \)-module. Then

(a) If \( M(i, j) \) is nonprojective, we have

\[
\tau_n(M(i, j)) = \begin{cases} 
M(i + j + \frac{n}{2}h - 1, h - j) & \text{if } n \text{ is even}, \\
M(i + \frac{n-1}{2}h - 1, j) & \text{if } n \text{ is odd}.
\end{cases}
\]

(b) If \( M(i, j) \) is noninjective, we have

\[
\tau_n^{-1}(M(i, j)) = \begin{cases} 
M(i + j + \frac{n-2}{2}h + 1, h - j) & \text{if } n \text{ is even}, \\
M(i + \frac{n+1}{2}h + 1, j) & \text{if } n \text{ is odd}.
\end{cases}
\]

Before we proceed with the main result of this section, let us explain how Lemma 3.23 will be used. By Proposition 2.6 there is a unique maximal left abutment of \( \Lambda_{m,h} \), namely \( P(m - h + 1) = M(1, h) \). Moreover, the Auslander-Reiten quiver of \( \Lambda_{m,h} \) is a subquiver of \( \Delta(m) \) where we remove all the vertices corresponding to indecomposable modules of length at least \( h - 1 \). Described otherwise, it is the same as the quiver \( \Delta(h) \) with the addition of more diagonals on the right hand side of the same height \( h \).

Let \( T \) be a slice fracture of \( P(m - h) \) and let \( \hat{T} \oplus P(m - h + 1) \cong T \). Then Lemma 3.23 implies that the action of \( \tau_n^{-1} \) translates \( \hat{T} \) through the Auslander-Reiten quiver of \( \Gamma(\Lambda_{m,h}) \) without changing
its shape. In other words, for $m$ large enough we have the following pictures:

\[
\begin{align*}
\tau_n & \quad \tau_n(\hat{T}) & \quad \tau_n^{-2}(\hat{T}) \\
\tau_n & \quad \tau_n & \quad \tau_n & \quad \tau_n
\end{align*}
\]

$n$ odd:

$n$ even:

In the above pictures the thick lines represent the indecomposable summands of $\hat{T}$ in the foundation of $P(m-h+1)$. Notice that in the case of $n$ being even $\hat{T}$ is reflected horizontally at every application of $\tau_n$. Moreover, the module $P(m-h+1)$, which would be at the top of the slice, is injective and so $\tau_n(P(m-h+1)) = 0$. Additionally, the above applications of $\tau_n$ are invertible by $\tau_n$. The idea of the proof is that by choosing $m$ correctly we can stop precisely at the point where the thick diagonal aligns with the end of $\Lambda_{m,h}$. Then we can remove these aligned modules from our slice and consider the leftover piece as a slice of smaller height. Finally, an induction on the height of the slice will give us the proof. Let us illustrate with a concrete example.

**Example 3.24.** Consider the slice module $T$ of $KA_5$ given by the following encircled modules in $\Gamma(KA_5)$:

\[
\begin{align*}
(1,1) & \quad (2,1) & \quad (3,1) & \quad (4,1) & \quad (5,1) \\
(1,2) & \quad (2,2) & \quad (3,2) & \quad (4,2) & \quad (5,2) \\
(1,3) & \quad (2,3) & \quad (3,3) & \quad (4,3) & \quad (5,3) \\
(1,4) & \quad (2,4) & \quad (3,4) & \quad (4,4) & \quad (5,4) \\
(1,5) & \quad (2,5) & \quad (3,5) & \quad (4,5) & \quad (5,5)
\end{align*}
\]

Assume we want to complete $T$ on the right for $n = 4$. Let $m > 5$ and consider the maximal left $\Lambda_{m,5}$-abutment $P(m-4) = M(1,5)$. By gluing

\[
KA_5 P(m-4) \Delta^{\Gamma(5)} \Lambda_{m,5} = \Lambda_{m,5}
\]

and assuming $m$ is large enough, Lemma 3.23 gives

\[
\tau_4^{-1}(M(1,4)) = M(11,1), \quad \tau_4^{-1}(M(1,3)) = M(10,2), \quad \tau_4^{-1}(M(2,2)) = M(10,3), \quad \tau_4^{-1}(M(2,1)) = M(9,4).
\]

Hence for $m = 12$ we have the following picture of $\Gamma(A_{12,5})$:

where the encircled modules form a $(T, \tau_4^{-1}(\hat{T}) \oplus M(8,5), 4)$-fractured subcategory $C$. Notice that the top row of the above Auslander-Reiten quiver has only projective-injective modules and so is included
automatically in \( C \). In this setup the modules \( M(8, 5), M(9, 4) \) and \( M(10, 3) \) are injective and hence we opt to glue at the one with the smallest height, that is the module \( I(3) = M(10, 3) \). Then the module

\[
T' = M(10, 3) \oplus M(10, 2) \oplus M(11, 1)
\]

corresponds to a slice of \( KA_3 \). Hence it is enough to show that we can complete this particular slice. After renumbering the vertices and by a similar computation as before we see that the gluing \( KA_3^{(6)} A_{8,3} = A_{8,3} \) gives the following Auslander-Reiten quiver

\[
\begin{array}{cccccccc}
(1, 3) & (2, 3) & (3, 3) & (4, 3) & (5, 3) & (6, 3) \\
(1, 2) & (2, 2) & (3, 2) & (4, 2) & (5, 2) & (6, 2) \\
(1, 1) & (2, 1) & (3, 1) & (4, 1) & (5, 1) & (6, 1) \\
\end{array}
\]

where again the encircled modules form a 4-fractured subcategory. Here we see as before that the module \( M(7, 2) \oplus M(7, 1) \) also corresponds to a slice of \( KA_2 \). After renumbering the vertices and computing as before we see that for the gluing \( KA_2^{(4)} A_{5,2} \) we get the Auslander-Reiten quiver

\[
\begin{array}{cccc}
(1, 2) & (2, 2) & (3, 2) & (4, 2) \\
(1, 1) & (2, 1) & (3, 1) & (4, 1) \\
\end{array}
\]

where now the encircled modules form a 4-cluster tilting subcategory. As a consequence, the algebra

\[
\Lambda_{12,5}^{(6)} A_{8,3} \Lambda_{5,2}^{(4)}
\]

admits a right 4-cluster tilting subcategory.

**Proposition 3.25.** Let \( \{M(i_1, 1), \ldots, M(i_h, h)\} \) be a slice of \( KA_h \) and \( T = \bigoplus_{k=1}^{h} M(i_k, k) \). Then we can complete \( T \) on the left and on the right.

**Proof.** As mentioned before, by symmetry, it is enough to show that we can complete \( T \) on the left. We will use induction on \( h \). For \( h = 1 \) we have only one indecomposable \( KA_1 \)-module, say \( N \), and so \( \text{add}(T) = \text{add}(N) \) is an \( n \)-cluster tilting subcategory for any \( n \). For the induction step, assume that we can complete any slice of \( KA_{h-1} \) on the left and we will show that we can complete any slice of \( KA_h \) on the left.

We consider the two possible cases \( i_{h-1} = 1 \) and \( i_{h-1} = 2 \) separately. For the case \( i_{h-1} = 1 \) notice that by identifying the subcategory \( F_{M(1, h-1)} \) with the category \( \text{mod}(KA_{h-1}) \) via the footing \( f := f_{M(1, h-1)} \), the set \( \{f'(M(i_1, 1)), \ldots, f'(M(i_{h-1}, h-1))\} \) becomes a slice of \( KA_{h-1} \). Hence by induction hypothesis we can find a representation-directed algebra \( B' \) satisfying the conditions of Question \[1].

In particular, \( B' = B' \bigoplus M(1, h-1) \) admits a \( (P_{B'}) \)-fractured subcategory \( C_{B'} \), with the only noninjective fraction of \( T_{B'} \) being \( T_{B'}^{(1)} \bigoplus M(i_1, k) \). Now consider the gluing \( B \bigoplus M(1, h-1) \) into \( KA_h \). By Corollary 2.40, we see that a complete set of nonsimorphic indecomposable \( B \)-modules supported in \( KA_h \) and not in \( B' \) is \( \{\phi_*(M(1, h)), \phi_*(M(2, h-1)), \ldots, \phi_*(M(h, 1))\} \). Moreover these are all injective and linearly ordered with \( \phi_*(M(1, h)) \) being maximal, while \( \phi_*(M(1, h)) \) is the only projective. Then

\[
T_\ast := \psi_*(T_{B'}^{(1)}) \bigoplus \phi_*(M(1, h)) \cong \bigoplus_{i=k}^{h-1} \phi_*(M(i_k, k)) \bigoplus \phi_*(M(1, h)) \cong \phi_*(T)
\]

is a fracture of \( \phi_*(M(1, h)) \) such that \( \phi_*(T_\ast) \cong T \). Let \( C_B := \text{add}\{\psi_*(C_{B'}), \phi_*(M(1, h))\} \). Since \( \phi_*(M(1, h)) \) is projective and injective and since \( \psi_*(C_{B'}) \) is supported only in \( B' \) and is a \( (P_{B'}, T_{B'}, n) \)-fractured subcategory, it follows from Corollary 2.38 that \( C_B \) is a \( (P_{B'}, \psi_*(T_{B'}) \bigoplus \phi_*(M(1, h)), n) \)-fractured subcategory as required.

For the case \( i_{h-1} = 2 \) we consider the cases \( n \) being odd and \( n \) being even separately.
For the case $n$ being odd, first we glue $\left(\Lambda_{h + \frac{n-1}{2}, h, h}\right)^{P(1)\Delta(k)} K\Lambda_h$. Since this is a trivial gluing as in Example 2.20, the resulting algebra is isomorphic to $\Lambda_{h + \frac{n-1}{2}, h, h}$ again. It is a simple computation to see that viewing the modules $M(i, k)$ as $\Lambda_{h + \frac{n-1}{2}, h, h}$-modules we have

$$\phi_*(M(i, k)) \cong M\left(i_k + \frac{n-1}{2}, h, k\right).$$

Computing $\tau_n$ by using Lemma 3.23 gives

$$\tau_n \left( M\left(i_k + \frac{n-1}{2}, h, k\right) \right) = M(i_k - 1, k).$$

In particular we have $\tau_n (M(2 + \frac{n-1}{2}, h, h - 1)) = M(1, h - 1)$, which is a left abutment of $\Lambda_{h + \frac{n-1}{2}, h, h}$ of height $h - 1$. Moreover, now the set $\{\tau_n (M(i_k + \frac{n-1}{2}, h, k))\}^{h-1}_{i=1} \subseteq F_{M(1, h - 1)}$ is a fracture which is a slice, viewed as a $\Lambda_{A_{h-1}}$-module. Hence by induction hypothesis we can complete on the left using some algebra $B'$ admitting a left $n$-cluster tilting subcategory $C_{B'} = \text{add}(X)$; call the resulting algebra $B$. Set

$$C_B := \text{add} \left( X \oplus \bigoplus_{h=1}^{h-1} \left( M\left(i_k + \frac{n-1}{2}, h, k\right) \right) \right) \oplus \left( \bigoplus_{h=1}^{h-1} \left( \tau_n \left( M\left(i_k + \frac{n-1}{2}, h, k\right) \right) \right) \right) \oplus \left( \bigoplus_{i=1}^{n-h+1} \left( M(i, h) \right) \right),$$

where the modules $M(i, h)$ appearing here are the projective-injective $\Lambda_{h + \frac{n-1}{2}, h, h}$-modules. Since all syzygies and cosyzygies of indecomposable modules in $\Lambda_{h + \frac{n-1}{2}, h, h}$ are indecomposable and since by Lemma 3.23 we have

$$\tau_n \tau_n \left( M\left(i_k + \frac{n-1}{2}, h, k\right) \right) = M\left(i_k + \frac{n-1}{2}, h, k\right),$$

it follows that $C_B$ is a left $n$-cluster tilting subcategory of $\Lambda_l$.

Finally, for the case $n$ being even, a similar computation shows that

$$\Lambda_{\frac{(n+2)h}{2} - i_1 + 1, h} = \Lambda_{\frac{(n+2)h}{2} - i_1 + 1, h}^{P(1)\Delta(k)} K\Lambda_h$$

admits a $(T^L, T^R, n)$-fractured subcategory with $\phi'(T^R) \cong T$ and $\psi'(T^L)$ being a slice module of height $h - 1$. Then a similar induction as in the case of $n$ being odd completes the proof.

4. PART IV: $(n, d)$-REPRESENTATION-FINITE ALGEBRAS

In this section we will construct examples of $(n, d)$-representation-finite algebras. Specifically, if $n$ is odd we will construct an $(n, d)$-representation-finite algebra for any $d \geq n$ and if $n$ is even we will construct an $(n, d)$-representation-finite algebra for any $d$ odd or $d \geq 2n$.

In our constructions we will again use acyclic Nakayama algebras. Recall that the Auslander-Reiten quiver $\Gamma(K\Lambda_h/I)$ of a quotient of $K\Lambda_h$ by an admissible ideal is a full subquiver of $\Delta(h)$ with the property that if a vertex $a$ is not in $\Gamma(K\Lambda_h/I)$, then all targets of arrows with source $a$ are not in $\Gamma(K\Lambda_h/I)$. Recall also that acyclic Nakayama algebras can be classified by Kupisch series, first introduced in Kupisch series.

**Definition 4.1.** A tuple $(d_1, \ldots, d_m) \in \mathbb{Z}^n_{\geq 0}$ is called an $m$-Kupisch series if

- $d_m = 1$,
- $d_i \geq 2$ for all $1 \leq i \leq m$ and
- $d_{i-1} - 1 \leq d_i$ for all $2 \leq i \leq m$. 
Then the correspondence between \(m\)-Kupisch series and acyclic Nakayama algebras is given by
\[
\{m\text{-Kupisch series } (d_1, \ldots, d_m) \} \rightarrow \{KA_m/I \text{ with } I = (\alpha_1 \cdots \alpha_{i+d_i-1}) \mid 1 \leq i \leq m) \},
\]
where in the above description of \(I\) paths not belonging to \(A_m\) should just be ignored. Using this identification, we will identify a Kupisch series with the corresponding acyclic Nakayama algebra.

Moreover, given an \(m\)-Kupisch series we explain how to describe the Auslander-Reiten quiver of the corresponding acyclic Nakayama algebra. First, to each number \(d_i\) we will identify a Kupisch series with the corresponding acyclic Nakayama algebra.

\[
\sum_{i=1}^n \text{ind}(d_i).
\]

We will also use the following lemma for computing syzygies, cosyzygies and Auslander-Reiten translations of modules over acyclic Nakayama algebras.

**Lemma 4.2.** Let \(\Lambda\) be an acyclic Nakayama algebra and \(M(i, j) \neq 0\) be a \(\Lambda\)-module. Then
\[
\begin{align*}
(a) & \quad \text{If } M(i, j) \text{ is nonprojective, then } \\
& \quad \tau(M(i, j)) \cong M(i - 1, j), \\
& \quad \Omega(M(i, j)) \cong M(i + j - u_{i+j}, u_{i+j} - j)
\end{align*}
\]
where \(u_s = \max\{j \in \{1, \ldots, n\} \mid (i, j) \in \Gamma(\Lambda) \text{ and } i + j = s\}\).

\[
\begin{align*}
(b) & \quad \text{If } M(i, j) \text{ is noninjective, then } \\
& \quad \tau^{-1}(M(i, j)) \cong M(i + 1, j), \\
& \quad \Omega^{-1}(M(i, j)) \cong M(i + v_{i+j}, v_{i+j} - j)
\end{align*}
\]
where \(v_s = \max\{j \in \{1, \ldots, n\} \mid (i, j) \in \Gamma(\Lambda) \text{ and } i = s\}\).

**Proof.** The claims about the Auslander-Reiten translations are [Vas18 Lemma 4.7]. The claim about syzygy in (a) follows by noticing that \(M(i + j - u_{i+j}, u_{i+j})\) is a projective cover of \(M(i, j)\) and the claim about cosyzygy in (b) follows by noticing that \(M(i, v_i)\) is an injective envelope of \(M(i, j)\).

In the rest of this and the next subsection we will only draw Auslander-Reiten quivers of Nakayama algebras. When drawing such an Auslander-Reiten quiver we will only draw the vertices as the arrows can be inferred by our conventions. Moreover, we will denote by \(h^{(k)}\) a sequence \(h, h, \ldots, h\) where \(h\) appears \(k\) times in a Kupisch series. We give an example using this notation.

**Example 4.3.** The acyclic Nakayama algebra \(\Lambda_{m,h}\) corresponds to the \(m\)-Kupisch series
\[
(h, \ldots, h, h - 1, \ldots, 2, 1) \in \mathbb{Z}_{\geq 0}^m,
\]
or with our notation \((h^{(m-h+1)}, h - 1, \ldots, 2, 1)\). For \(m = 15\) and \(h = 3\), the Auslander-Reiten quiver of \(\Lambda_{15,3}\) is
\[
\begin{array}{c}
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

The algebras \(\Lambda_{m,2}\) are of special interest. Specifically, we will use the following Proposition.

**Proposition 4.4.** \(\begin{align*}
(a) & \quad \text{gl. dim } (\Lambda_{m,2}) = m - 1, \text{ the projective dimension of } I(1) \text{ is } m - 1 \text{ and the } \\
& \quad \text{injective dimension of } P(m) \text{ is } m - 1, \\
(b) & \quad \Lambda_{m,2} \text{ admits a unique basic } (m - 1)\text{-cluster tilting module } M, \\
(c) & \quad M \cong \Lambda_{m,2} \oplus I(1) \cong P(m) \oplus D(\Lambda_{m,2}).
\end{align*}\)

**Proof.** Part (a) of the Lemma is well-known; for a proof see [Vas18 Proposition 5.2]. Parts (b) and (c) follow immediately as a special case of [Jas16 Proposition 6.2].

The following theorem will be our main tool in constructing examples of \((n, d)\)-representation-finite algebras.
Theorem 4.5. (a) Let $A$ be a strongly $(n,d)$-representation-directed algebra and assume that there exists a simple projective $A$-module $P$ with injective dimension $d$. Then $\Lambda = \Lambda_{n+1,2} P_{\Delta I(1)} A$ is strongly $(n,d+1)$-representation-directed and there exists a simple projective $\Lambda$-module $P'$ with injective dimension $d+n$.

(b) Let $B$ be a strongly $(n,d)$-representation-directed algebra and assume that there exists a simple injective $A$-module $I$ with projective dimension $d$. Then $\Lambda = B P_{\Delta I(1)} A_{n+1,2}$ is strongly $(n,d+1)$-representation-directed and there exists a simple injective $\Lambda$-module $P'$ with projective dimension $d+n$.

Proof. We only prove (a); (b) is similar. By Proposition 4.4 we have that $\Lambda_{n+1,2}$ is strongly $(n,n)$-representation-directed. Moreover, $I(1)$ is a simple injective $\Lambda_{n+1,2}$-module since the vertex 1 is a source. Hence by Corollary 3.17 it follows that $\Lambda_{n+1,2} P_{\Delta I(1)} A$ admits an $n$-cluster tilting subcategory and has global dimension at most $d+n$. To finish the proof it is enough to show that there exists a simple projective $\Lambda$-module $P'$ with injective dimension $d+n$.

Notice that by Corollary 2.40 we have $\psi_*(I(1)) \cong \phi_*(P)$ and that this is the only module in $\psi_*(\text{mod} \Lambda_{n+1,2}) \cap \phi_*(\text{mod} A)$. Let $P' := \psi_*(P(n+1))$, where $P(n+1)$ is the simple projective $\Lambda_{n+1,2}$-module corresponding to the vertex $n+1$. Since $\dim_K (P(n+1)) = 1$ it follows that $\dim_K (P') = 1$ and so $P'$ is simple. Moreover, since $P(n+1)$ is projective, it follows that $P'$ is also projective by Proposition 2.28.

It is a simple computation to see that $\Omega^{-n}(P(n+1)) \cong I(1)$ (for example, see [Vas18] Corollary 4.5). Moreover, by Corollary 2.38(b1) we have

$$\Omega^{-n}(P') = \Omega^{-n} \psi_*(P(n+1)) \cong \psi_*(I(1)).$$

In particular, this implies that

$$\text{inj. dim}(P(n+1)) = n + \text{inj. dim}(\psi_*(I(1))).$$

Since $\psi_*(I(1)) \cong \phi_*(P)$ is supported in $A$, it follows from Corollary 2.38(b2) that the injective dimension of $\phi_*(P)$ is the same as the injective dimension of $P$, which is $d$ by assumption. Hence the injective dimension of $P'$ is $d+n$ which completes the proof. □

In particular we have the following corollary.

Corollary 4.6. (a) Let $A$ be a strongly $(n,d)$-representation-directed algebra and assume that there exists a simple projective $A$-module $P$ with injective dimension $d$. Let $\Lambda$ be the algebra

$$\Lambda = \Lambda_{n+1,2} P_{\Delta I(1)} A_{n+1,2} P_{\Delta I(1)} A_{n+1,2} \cdots P_{\Delta I(1)} A_{n+1,2} P_{\Delta I(1)} A,$$

where there appear $k$ terms $\Lambda_{n+1,2}$ on the right-hand side of the above expression. Then $\Lambda$ is strongly $(n,kn+d)$-representation-directed.

(b) Let $B$ be a strongly $(n,d)$-representation-directed algebra and assume that there exists a simple injective $B$-module $I$ with projective dimension $d$. Let $\Lambda$ be the algebra

$$\Lambda = B P_{\Delta I} A_{n+1,2} P_{\Delta I} A_{n+1,2} P_{\Delta I} A_{n+1,2} \cdots P_{\Delta I} A_{n+1,2},$$

where there appear $k$ terms $\Lambda_{n+1,2}$ on the right-hand side of the above expression. Then $\Lambda$ is strongly $(n,kn+d)$-representation-directed.

Proof. Follows immediately by applying Theorem 4.5 $k$ times. □

As an immediate application of Corollary 4.6 we can now recover [Jas16] Proposition 6.2(i):

Example 4.7. It is easy to see (for example using Corollary 2.40) that

$$\Lambda_{kn+1,2} \cong \Lambda_{n+1,2} P_{\Delta I(1)} A_{n+1,2} P_{\Delta I(1)} A_{n+1,2} \cdots P_{\Delta I(1)} A_{n+1,2},$$

where there appear $k$ terms $\Lambda_{n+1,2}$ on the right hand side of the above expression. Hence by Corollary 4.6 it follows that the algebra $\Lambda_{kn+1,2}$ is $(n, kn)$-representation-finite.

Hence we have examples of $(n, kn)$-representation-finite algebras for any $k$. Our construction of examples of $(n, d)$-representation-finite algebras for $d \neq kn$ will follow the same spirit.
4.1. The case of $n$ being odd. Let us start with the case of $n$ being odd. Given $n$, we will construct a strongly $(n,d)$-representation-directed algebra for any $n \leq d \leq 2n-1$. Moreover, each such algebra will admit a simple projective module of injective dimension $d$. Then by applying Corollary 4.6 we obtain an example of an $(n,d)$-representation-directed algebra for any $d$.

Consider the following motivating example.

**Example 4.8.** Let $\Lambda$ be the Nakayama algebra given by the 15-Kupisch series $(2,3^{(11)},2^{(2)},1)$ and let $C_\Lambda = \bigoplus (\tau^{-i}\Lambda)$. Then the Auslander-Reiten quiver $\Gamma(\Lambda)$ of $\Lambda$ is

$$
\bullet \cdots \bullet
$$

where the bold vertices denote the indecomposable modules belonging to $C_\Lambda$. Then using Theorem 3.2 we can see that $C_\Lambda$ is a 9-cluster tilting subcategory, and a simple calculation shows that

$$d = \text{gl. dim}(\Lambda) = \text{proj. dim}(I(1)) = 10.$$

Hence $\Lambda$ is a $(9,10)$-representation-finite algebra. Following the same notation we have the following list of $(9,d)$-representation-finite algebras for different global dimensions $d<18$:

<table>
<thead>
<tr>
<th>$\Lambda$ (as a Kupisch series)</th>
<th>$\Gamma(\Lambda)$ and indecomposables in $C_\Lambda$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2^{(2)}, 3^{(2)}, 4^{(3)}, 5^{(13)}, 4^{(4)}, 3^{(3)}, 2^{(3)}, 1)$</td>
<td>$\bullet \cdots \bullet$</td>
<td>11</td>
</tr>
<tr>
<td>$(2^{(3)}, 3^{(8)}, 2^{(4)}, 1)$</td>
<td>$\bullet \cdots \bullet$</td>
<td>12</td>
</tr>
<tr>
<td>$(2^{(4)}, 3^{(2)}, 4^{(14)}, 3^{(3)}, 2^{(3)}, 1)$</td>
<td>$\bullet \cdots \bullet$</td>
<td>13</td>
</tr>
<tr>
<td>$(2^{(5)}, 3^{(5)}, 2^{(6)}, 1)$</td>
<td>$\bullet \cdots \bullet$</td>
<td>14</td>
</tr>
<tr>
<td>$(2^{(6)}, 3^{(13)}, 2^{(3)}, 1)$</td>
<td>$\bullet \cdots \bullet$</td>
<td>15</td>
</tr>
<tr>
<td>$(2^{(7)}, 3^{(2)}, 2^{(8)}, 1)$</td>
<td>$\bullet \cdots \bullet$</td>
<td>16</td>
</tr>
<tr>
<td>$(2^{(8)}, 3^{(13)}, 2, 1)$</td>
<td>$\bullet \cdots \bullet$</td>
<td>17</td>
</tr>
</tbody>
</table>

Moreover, in each of the above examples we have that the $d = \text{proj. dim}(I(1))$. Hence it follows from Corollary 4.6 that there exist a $(9,d)$-representation-finite algebra for any $d \geq 9$.

With Example 4.8 in mind, we have the following proposition.

**Proposition 4.9.** Let $n$ be odd and $n < d < 2n$.

(a) If $d$ is even, then the acyclic Nakayama algebra $\Lambda$ with Kupisch series

$$
(2^{(d-n)}, 3^{(3(n-\frac{d}{2})-1)}, 2^{(d-n+1)}, 1)
$$

is $(n,d)$-representation-finite.

(b) If $d$ is odd and $d \neq 2n-1$, set

$$h = n - \frac{d-1}{2} \quad \text{and} \quad s = \left( \frac{d-n}{2} \right) (h+1) + 2h.$$
Then the acyclic Nakayama algebra $\Lambda$ with Kupisch series
\[
\left(2^{(d-n)}, 3^{(2)}, 4^{(3)}, \ldots, h^{(h-1)}, (h + 1)^{(h)}, h^{(h)}, (h - 1)^{(h-1)}, \ldots, 3^{(3)}, 2^{(3)}, 1\right)
\]
is $(n, d)$-representation-finite and we have $\text{proj.dim}(I(1)) = d$.

(c) If $d = 2n - 1$, then the acyclic Nakayama algebra $\Lambda = \Lambda_{d(\frac{2n+1}{3}), 3}$ is $(n, d)$-representation-finite and we have $\text{proj.dim}(I(1)) = d$.

Proof.

(a) The Auslander-Reiten quiver $\Gamma(\Lambda)$ in this case has the form
\[
\begin{array}{cccccccc}
Q_1 & \cdot & \cdot & \cdots & \cdot & Q_2 & \cdot & \cdot & \cdots & \cdot & J_1 & \cdot & \cdots & \cdot & J_2,
\end{array}
\]
where $Q_1 \cong M(1, 1)$, $Q_2 \cong M(d - n + 1, 2)$, $J_1 \cong M(2n - \frac{d}{2}, 2)$, $J_2 \cong M(n + \frac{d}{2}, 1, 1)$ and the bold vertices correspond to the indecomposable projective-injective $\Lambda$-modules. Moreover, the vertices $Q_1$ and $Q_2$ correspond to the indecomposable projective noninjective $\Lambda$-modules and the vertices $J_1$ and $J_2$ correspond to the indecomposable injective nonprojective $\Lambda$-modules. Using Lemma 1.2 we compute
\[
\tau_n^{-1}(Q_1) = \tau^{-\Omega^{-(n-1)}(Q_1)} \cong \tau^{-\Omega^{-2n-d-1}(M(1, 1))} \cong \tau^{-\Omega^{-2n-d-1}(M(d - n + 1, 1))} \cong \tau^{-\left(M\left(2n - \frac{d}{2}, 1, 2\right)\right)} \cong J_1.
\]

Similar computations show that
\[
\tau_n^{-1}(Q_2) \cong J_2, \quad \tau_n(J_1) \cong Q_1, \quad \tau_n(J_2) \cong Q_2, \quad \Omega^{n+1}(J_1) \cong 0, \Omega^d(J_2) \cong Q_1.
\]

Hence Theorem 3.2 implies that
\[
\mathcal{C} = \text{add}(\Lambda \oplus \tau_n^{-1}\Lambda) = \text{add}(\Lambda \oplus J_1 \oplus J_2)
\]
is an $n$-cluster tilting subcategory. Moreover, since
\[
\text{proj.dim}(J_1) < d \quad \text{and} \quad \text{proj.dim}(J_2) = d
\]
and $J_1$ and $J_2$ are the only indecomposable injective nonprojective $\Lambda$-modules, we have $\text{gl.dim}(\Lambda) = d$.

For the final part, we have that $J_2 \cong M(n + \frac{d}{2} + 1, 1) \cong I(1)$ and hence $\text{proj.dim}(I(1)) = d$.

(b) The Auslander-Reiten quiver $\Gamma(\Lambda)$ in this case has the form
\[
\begin{array}{cccccccc}
Q_1 & \cdot & \cdot & \cdots & \cdot & Q_2 & \cdot & \cdot & \cdots & \cdot & Q_3 & \cdots & \cdots & \cdot & Q_n & \cdots & \cdots & \cdot & J_1 & \cdot & \cdots & \cdot & J_2 & \cdots & \cdots & \cdot & J_{h-1} & \cdot & \cdots & \cdot & J_h,
\end{array}
\]
where
- $Q_1 \cong M(1, 1)$,
- $Q_i \cong M\left(\frac{i(i+1)}{2}, i\right)$, for $2 \leq i \leq h$,
- $J_i \cong M\left(\frac{h(h-1)}{2} + s + 2 + (i-1)h + \frac{(i-2)(i-1)}{2}, h - i + 1\right)\cong M\left(\frac{(h+i-2)(h+i-1)}{2} + s + 2, h - i + 1\right)$, for $1 \leq i \leq h - 1$,
- $J_h \cong M\left((h-1)(2h-3) + s + (d - n) + 3, 1\right)$,
The Auslander-Reiten quiver $\Gamma(\Lambda)$ in this case has the form

\[
N \cong M \left( \frac{(h+1)(h+2)}{2} + \frac{(d-n-2)(h+1)}{2} + 1, 1 \right)
= M \left( \frac{(2n-d+3)(d+1)}{8} + 1, 1 \right).
\]

Moreover, the bold vertices correspond to the indecomposable projective-injective $\Lambda$-modules, the vertices $Q_i$ correspond to the indecomposable projective noninjective $\Lambda$-modules and the vertices $J_i$ correspond to the indecomposable injective nonprojective $\Lambda$-modules. Using Lemma 4.2, we compute

\[
\tau_n^{-}(Q_i) \cong \begin{cases} 
N & \text{if } i = 1, \\
J_{i-1} & \text{if } 2 \leq i \leq h, \\
Q_{i+1} & \text{if } 1 \leq i \leq h-1, \\
J_h & \text{if } i = h,
\end{cases}
\]

\[
\tau_n^{-}(J_i) \cong \begin{cases} 
Q_1 & \text{if } 1 \leq i \leq h-1, \\
N & \text{if } i = h.
\end{cases}
\]

It follows from Theorem 3.2 that

\[
\mathcal{C} = \text{add}(\Lambda \oplus \tau_n^{-}(\Lambda) \oplus \tau_n^{-}(\Lambda)) = \text{add}(\Lambda \oplus N \oplus D(\Lambda))
\]

is an $n$-cluster tilting subcategory. For the computation of the global dimension, notice that again using Lemma 4.2 as well as the previous computations, for $1 \leq i \leq h-1$ we have

\[
\Omega^{n+1}(J_i) \cong \Omega^2 \Omega^n(J_i) \cong \Omega^2 \tau_n^{-}(Q_i+1) \cong \Omega^2 \tau_n^{-}(M \left( \frac{i(i+1)}{2} + 2, i+1 \right)) \cong \Omega^2 \left( M \left( \frac{i(i+1)}{2} + 2, i+1 \right) \right) \cong Q_i.
\]

A similar computation shows that $\Omega^d(J_h) \cong Q_h$. Since $n$ and $d$ are both odd and since $Q_i$ is projective for any $i$ it follows that

\[
\text{gl. dim}(\Lambda) = \max\{\text{proj. dim}(J_i) \mid 1 \leq i \leq h\} = d.
\]

Finally, since $I(1) = J_n$, we also have $\text{proj. dim}(I(1)) = d$.

(c) The Auslander-Reiten quiver $\Gamma(\Lambda)$ in this case has the form

\[
Q_2 \rightarrow \cdots \rightarrow Q_1 \rightarrow \cdots \rightarrow J_1 \rightarrow \cdots \rightarrow J_2,
\]

where $Q_1 \cong M(1, 1)$, $Q_2 \cong M(1, 2)$, $N \cong M(\frac{n}{2}(n-1)+2, 1)$, $J_1 \cong M(\frac{n}{2}(n-1)+2, 2)$, $J_2 \cong M(\frac{n}{2}(n-1)+3, 1)$ and the bold vertices correspond to the indecomposable projective-injective $\Lambda$-modules. Moreover the vertices $Q_i$ correspond to the indecomposable projective noninjective $\Lambda$-modules and the vertices $J_i$ correspond to the indecomposable injective nonprojective modules. It follows from Lemma 3.23 that

\[
\tau_n^{-}(Q_1) \cong N, \quad \tau_n^{-}(Q_2) \cong J_1, \quad \tau_n^{-}(N) \cong J_2,
\]

and

\[
\tau_n^{-}(N) \cong Q_1, \quad \tau_n^{-}(J_1) \cong Q_2, \quad \tau_n^{-}(J_2) \cong N,
\]

and hence by Theorem 3.2 we have that

\[
\mathcal{C} = \text{add}(\Lambda \oplus \tau_n^{-}(\Lambda) \oplus \tau_n^{-}(\Lambda)) = \text{add}(\Lambda \oplus N \oplus D(\Lambda))
\]

is an $n$-cluster tilting subcategory. Similar computations as above show that

\[
\Omega^n(J_1) \cong Q_1, \quad \Omega^d(J_2) \cong J_1,
\]

from which it follows that $\text{gl. dim}(\Lambda) = d$. Since $J_2 = I(1)$, the proof is complete.

\[\square\]

**Corollary 4.10.** Let $n$ be odd and $d \geq n$. There exists an $(n,d)$-representation-finite algebra $\Lambda$. 
Proof. Write \( d = kn + d' \) for some \( k \geq 1 \) and \( 0 \leq d' \leq n - 1 \). If \( d' = 0 \) then \( \Lambda \) exists by Example 4.7. If \( 0 < d' < n \), let \( \Lambda' \) be an \((n, d')\)-representation-finite algebra as in Proposition 4.9. Then \( \Lambda' \) satisfies the assumptions of Corollary 4.6(b) and so there exists an algebra \( \Lambda \) which is \((n, kn + d')\)-representation-finite as required. \( \square \)

4.2. The case of \( n \) being even. In this case we have the following families of \((n, d)\)-representation-finite algebras.

**Proposition 4.11.** Let \( n \) be even and \( 0 < k < n \).

(a) If \( k \) is even, then the acyclic Nakayama algebra \( \Lambda \) with Kupisch series
\[
\left( 2^{(k)}, 3^{(\frac{3n-k}{2}-1)}, 2^{(k+1)}, 1 \right)
\]
is \((n, n + k)\)-representation-finite and we have \( \text{proj. dim}(I(1)) = n + k \).

(b) If \( k \) is odd and \( k \neq n - 1 \), then the acyclic Nakayama algebra \( \Lambda \) with Kupisch series
\[
\left( 2^{(k)}, 3^{(n - \frac{k+1}{2})}, 2^{(k+1)}, 1 \right)
\]
is \((n, 2n + k)\)-representation-finite and we have \( \text{proj. dim}(I(1)) = n + k \).

(c) If \( k = n - 1 \), then the acyclic Nakayama algebra \( \Lambda = \Lambda_{\frac{n}{2}, 3} \) with Kupisch series
\[
\left( 3^{(2\varphi - 2)}, 2, 1 \right)
\]
is \((n, 2n + k)\)-representation-finite and we have \( \text{proj. dim}(I(1)) = n + k \).

**Proof.** The proof is similar to the proof of Proposition 4.9. Computations are done using Lemma 4.2.

(a) The Auslander-Reiten quiver \( \Gamma(\Lambda) \) in this case has the form
\[
\bullet \quad \ldots \quad \bullet \quad Q_1 \quad \ldots \quad Q_2 \quad \ldots \quad \bullet \quad J_1 \quad \ldots \quad \bullet \quad J_2,
\]
where \( Q_1 \cong M(1, 1) \), \( Q_2 \cong M(k + 1, 2) \), \( J_1 \cong M \left( \frac{3n-k}{2} \right) \), \( J_2 \cong M \left( \frac{3n+k}{2} + 1, 1 \right) \) and the bold vertices correspond to the indecomposable projective-injective \( \Lambda \)-modules. Moreover, the vertices \( Q_1 \) and \( Q_2 \) correspond to the indecomposable projective noninjective \( \Lambda \)-modules, the vertices \( J_1 \) and \( J_2 \) correspond to the indecomposable injective nonprojective \( \Lambda \)-modules and for \( i = 1, 2 \) we have
\[
\tau^-_n(Q_i) \cong J_i, \quad \tau_n(J_i) \cong Q_i.
\]
Hence by Theorem 3.3 we have that \( \mathcal{C} = \text{add}(\Lambda \oplus D(\Lambda)) \) is an \( n \)-cluster tilting subcategory. Finally, we have \( I(1) \cong J_2 \) and
\[
d = \text{gl. dim}(\Lambda) = \text{proj. dim}(J_2) = n + k.
\]

(b) The Auslander-Reiten quiver \( \Gamma(\Lambda) \) in this case has the form
\[
\bullet \quad \ldots \quad \bullet \quad Q_1 \quad \ldots \quad Q_2 \quad \ldots \quad \bullet \quad N_1 \quad \ldots \quad N_2 \quad \ldots \quad \bullet \quad J_1 \quad \ldots \quad \bullet \quad J_2,
\]
where \( Q_1 \cong M(1, 1) \), \( Q_2 \cong M(k + 1, 2) \), \( N_1 \cong M \left( \frac{3n-k+1}{2} \right) \), \( N_2 \cong M \left( \frac{3n+k+1}{2} + 1, 1 \right) \) and the bold vertices correspond to the indecomposable projective-injective \( \Lambda \)-modules. Moreover, the projective noninjective indecomposable modules are \( Q_1 \) and \( Q_2 \), the injective nonprojective indecomposable modules are \( J_1 \) and \( J_2 \) and for \( i = 1, 2 \) we have
\[
\tau^-_n(Q_i) \cong N_i, \quad \tau^-_n(N_i) \cong J_i, \quad \tau_n(N_i) \cong Q_i, \quad \tau_n(J_i) \cong N_i.
\]
Hence by Theorem 3.3 we have that \( \mathcal{C} = \text{add}(\Lambda \oplus D(\Lambda) \oplus N_1 \oplus N_2) \) is an \( n \)-cluster tilting subcategory. Finally, we have \( I(1) = J_2 \) and
\[
d = \text{gl. dim}(\Lambda) = \text{proj. dim}(J_2) = n + k.
\]
(c) The Auslander-Reiten quiver $\Gamma(\Lambda)$ in this case has the form

$$Q_2 \cdot \cdot \cdot \cdot \cdot N_1 \cdot \cdot \cdot \cdot \cdot K_2 \cdot \cdot \cdot \cdot \cdot J_1 \cdot J_2,$$

where $Q_1 \cong M (1,1)$, $Q_2 \cong M (1,2)$, $N_1 \cong M (\frac{d}{2},2)$, $N_2 \cong (\frac{d}{2} + 1,1)$, $K_1 \cong M (3n,1)$, $K_2 \cong M (3n,2)$, $J_1 \cong M (\frac{d}{2} - 1,2)$, $J_2 \cong M (\frac{d}{2} + 1)$ and the bold vertices correspond to the indecomposable projective-injective $\Lambda$-modules. Moreover, the vertices $Q_1$ and $Q_2$ correspond to the indecomposable projective noninjective $\Lambda$-modules, the vertices $J_1$ and $J_2$ correspond to the indecomposable injective nonprojective $\Lambda$-modules and for $i = 1, 2$ we have

$$\tau_n(Q_i) \cong N_i, \quad \tau_n(N_i) \cong K_i, \quad \tau_n(K_i) \cong J_i,$$

Hence by Theorem 3.2 we have that $\mathcal{C} = \text{add}(\Lambda \oplus D(\Lambda) \oplus N_1 \oplus N_2 \oplus K_1 \oplus K_2)$ is an $n$-cluster tilting subcategory. Finally, we have $I(1) = J_2$ and

$$d = \text{gl. dim}(\Lambda) = \text{proj. dim}(J_2) = 3n - 1.$$

\[\square\]

**Corollary 4.12.** Let $n$ be even and $d \geq 2n$. There exists an $(n,d)$-representation-finite algebra $\Lambda$.

**Proof.** Write $d = kn + d'$ for some $k \geq 2$ and $0 \leq d' \leq n - 1$. If $d' = 0$ then $\Lambda$ exists by Example 4.7. If $0 < d' < n$, let $\Lambda'$ be an $(n,d')$-representation-finite algebra as in Proposition 4.11. Then $\Lambda'$ satisfies the assumptions of Corollary 4.6(b) and so there exists an algebra $\Lambda$ which is $(n, kn + d')$-representation-finite as required. \[\square\]

Let us give an example in this case as well.

**Example 4.13.** Let $n = 6$. Using Proposition 4.11 and Example 4.7 we have the following list of $(6,d)$-representation-finite algebras $\Lambda$ where the 6-cluster tilting subcategories are denoted by the bold vertices in the Auslander-Reiten quivers:

<table>
<thead>
<tr>
<th>$\Lambda$ (as a Kupisch series)</th>
<th>$\Gamma(\Lambda)$ and indecomposables in $\mathcal{C}_\Lambda$</th>
<th>$d$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(2^{(6)}, 1)$</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td>$(2^{(2)}, 3^{(15)}, 2^{(2)}, 1)$</td>
<td></td>
<td>13</td>
</tr>
<tr>
<td>$(2^{(2)}, 3^{(5)}, 2^{(3)}, 1)$</td>
<td></td>
<td>8</td>
</tr>
<tr>
<td>$(2^{(3)}, 3^{(12)}, 2^{(4)}, 1)$</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>$(2^{(4)}, 3^{(2)}, 2^{(5)}, 1)$</td>
<td></td>
<td>10</td>
</tr>
<tr>
<td>$(3^{(25)}, 2, 1)$</td>
<td></td>
<td>17</td>
</tr>
</tbody>
</table>

Using Corollary 4.6 and the above list, we obtain a $(6,d)$-representation-finite algebra for any $d \geq 12$.

**4.3 Main result.** In this short section we summarize the results of section 4 in the following theorem.

**Theorem 4.14.** Let $n$ be a positive integer and $d \geq n$.

(a) If $n$ is odd, then there exists an $(n,d)$-representation-finite algebra.

(b) If $n$ is even, and $d$ is even or $d \geq 2n$, then there exists an $(n,d)$-representation-finite algebra.

**Proof.** Follows immediately by Corollary 4.10, Corollary 4.12 and Proposition 4.11. \[\square\]
Remark 4.15. Let us note that Theorem 4.14 is not sharp in the sense that there exist algebras that are \((n,d)\)-representation-finite where \(n\) is even, while \(n < d < 2n\) and \(d\) is odd. For example, in \[\text{Vaso}18\] Example 3.8 it was shown that the path algebra of the quiver with relations

\[
\begin{array}{cccccccc}
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\circ & \circ & \circ & \circ & \circ & \circ & \circ & \circ \\
\end{array}
\]

is \((2,3)\)-representation-finite.

References


