UPPSALA DISSERTATIONS IN MATHEMATICS

112

Combinatorial and analytical problems for fractals and their graph approximations

Konstantinos Tsougkas

Department of Mathematics
Uppsala University
UPPSALA 2019

Dissertation presented at Uppsala University to be publicly examined in Polhemsalen, Ångströmlaboratoriet, Lägerhyddsvägen 1, Uppsala, Friday, 15 February 2019 at 13:15 for the degree of Doctor of Philosophy. The examination will be conducted in English. Faculty examiner: Professor Ben Hambly (Mathematical Institute, University of Oxford).

Abstract
Tsougkas, K. 2019. Combinatorial and analytical problems for fractals and their graph approximations. *Uppsala Dissertations in Mathematics* 112. 37 pp. Uppsala: Department of Mathematics. ISBN 978-91-506-2739-8.

The recent field of analysis on fractals has been studied under a probabilistic and analytic point of view. In this present work, we will focus on the analytic part developed by Kigami. The fractals we will be studying are finitely ramified self-similar sets, with emphasis on the post-critically finite ones. A prototype of the theory is the Sierpinski gasket. We can approximate the finitely ramified self-similar sets via a sequence of approximating graphs which allows us to use notions from discrete mathematics such as the combinatorial and probabilistic graph Laplacian on finite graphs. Through that approach or via Dirichlet forms, we can define the Laplace operator on the continuous fractal object itself via either a weak definition or as a renormalized limit of the discrete graph Laplacians on the graphs.

The aim of this present work is to study the graphs approximating the fractal and determine connections between the Laplace operator on the discrete graphs and the continuous object, the fractal itself.

In paper I, we study the number of spanning trees on the sequence of graphs approximating a self-similar set admitting spectral decimation.

In paper II, we study harmonic functions on p.c.f. self-similar sets. Unlike the standard Dirichlet problem and harmonic functions in Euclidean space, harmonic functions on these sets may be locally constant without being constant in their entire domain. In that case we say that the fractal has a degenerate harmonic structure. We prove that for a family of variants of the Sierpinski gasket the harmonic structure is non-degenerate.

In paper III, we investigate properties of the Kusuoka measure and the corresponding energy Laplacian on the Sierpinski gaskets of level k.

In papers IV and V, we establish a connection between the discrete combinatorial graph Laplacian determinant and the regularized determinant of the fractal itself. We establish that for a certain class of p.c.f. fractals the logarithm of the regularized determinant appears as a constant in the logarithm of the discrete combinatorial Laplacian.

Keywords: Fractal graphs, energy Laplacian, Kusuoka measure

Konstantinos Tsougkas, Department of Mathematics, Box 480, Uppsala University, SE-75106 Uppsala, Sweden.

ISSN 1401-2049
ISBN 978-91-506-2739-8
urn:nbn:se:uu:diva-369918 (http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-369918)

Dedicated to my parents.

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I Anema, J. A., Tsougkas, K. Counting spanning trees on fractal graphs and their asymptotic complexity. Journal of Physics A: Mathematical and Theoretical, Volume 49, Number 35, 2016

II Tsougkas, K. Non-degeneracy of the harmonic structure on Sierpiński gaskets. To appear in the Journal of Fractal Geometry.

III Öberg, A., Tsougkas, K. The Kusuoka measure and energy Laplacian on level-k Sierpinski gaskets. To appear in the Rocky Mountain Journal of Mathematics.

IV Chen, J.P., Teplyaev, A., Tsougkas, K. Regularized Laplacian determinants of self-similar fractals. Letters in mathematical physics 108, no. 6 (2018): 1563-1579

V Tsougkas, K. Connections between discrete and regularized determinants on fractals. Manuscript.

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Contents

1. Introduction

The word fractal comes from the Latin word frāctus, which means broken or shattered and was coined by Benoit Mandelbrot in 1975. In recent years fractals have been widely studied in mathematics not only due to their abstract and beautiful mathematical nature but also because of their real world applications such as in computer graphics, compression algorithms, in music, telecommunications by developing fractal antennae and recently even in medicine.

The field of fractal geometry studies a lot of properties of these objects. Perhaps the most important concept when it comes to the study of fractal geometry is that of dimension. Let a set $S \subset \mathbb{R}^n$, or even a subset of a more general metric space, and let $N(r)$ denote the minimum number of boxes of side length r needed to cover that set. One rather intuitive way to think of dimension is in terms of how the number $N(r)$ scales as the side length of the boxes decreases. Obviously, the smaller boxes we have, the more of them we need to use. It becomes intuitively clear, at least when it comes to thinking of a set of an integer dimension d that it would scale as $N(r) \sim r^{-d}$. This gives rise to defining the box-dimension of a set S as the value

$$d = \lim_{r \to 0^+} \frac{\log N(r)}{\log \frac{1}{r}}$$

if the limit exists. This definition is also equivalent to using balls instead of boxes. However, this notion has some drawbacks and does not always provide us with the numerical value that we would expect some sets to have. As an example, the box counting dimension of a countable set of points may not always be zero, despite a point having zero dimension in itself. A finer version of dimension is the Hausdorff-Besicovitch dimension, most commonly referred to as just the Hausdorff dimension of a set.

Let (X,d) be a complete metric space. For any bounded set $A \subset X$, let

$$H^s_\delta = \inf\{\sum_{i>0} diam(E_i)^s : A \subset \bigcup_{i>0} E_i,\ diam(E_i) \leqslant \delta\}.$$

Then it is clear that as δ increases we allow for more sets so the infimum becomes smaller and thus H^s_δ is decreasing in δ so the limit

$$H^s(A) = lim_{\delta \to 0^+} H^s_\delta(A)$$

exists. We call this the s-dimensional Hausdorff measure. For $0 \leqslant s < t$ we have that $H^t_\delta(E) \leqslant \delta^{t-s} H^s_\delta(E)$ and there exists a unique value such that

$\sup\{s : H^s(E) = \infty\} = \inf\{s : H^s(E) = 0\}$. This $s \in [0, +\infty]$ is called the Hausdorff dimension of E. The s-dimensional Hausdorff measure where s is the Hausdorff dimension of the set is simply called the Hausdorff measure on the set.

For every non-empty open subset U of $\mathbb{R}^n$ we have that its Hausdorff dimensions equals n, in which case the Hausdorff measure is just a renormalized version of the Lebesgue measure. However, for fractal sets that have zero Lebesgue measure, the Hausdorff measure provides us with a non-trivial measure. As an example, the Hausdorff dimension of the Cantor set is $s = \frac{\log 2}{\log 3}$ and the corresponding dimensional Hausdorff measure allows us to have a measure on the Cantor set which we can use to integrate against if we want to perform analysis on the Cantor set.

There exist various techniques that allow us to calculate the dimensions of sets and fractal objects. A comprehensive summary of the field of geometry on fractals is given in [16]. In general, it is highly non-trivial to calculate the Hausdorff dimension of a fractal and there are many open problems in the area. Even relatively simple looking fractals may have a Hausdorff dimension that is very difficult to calculate. There has also been research in algorithms providing us with numerical approximations of the Hausdorff measure of some sets.

It is perhaps interesting to mention that there does not exist a universal rigorous definition of what a fractal is. A popular expression is that “you know a fractal when you see one". Of course, aside from the complete lack of mathematical rigor of this statement, it is not even completely accurate since sometimes objects that may not appear as fractals fall under the category of some of the most common definitions of subcategories of fractals. One such example is the closed unit interval $[0,1]$, which is not our first thought of a fractal, but in fact is a self-similar set. Attempts to give precise mathematical definitions of fractals seem to either be too narrow such that they miss cases of objects that should “obviously be" fractals or are so broad that contain far too many objects that perhaps weren’t intended to be included. One very common misconception often found online in popular science articles is that fractals are geometric objects of non-integer dimension. However, it is very easy to construct “obviously fractal" objects with integer dimension, the Sierpiński tetrahedron being such an example. Nevertheless, in this present work, we will study self-similar sets which have a precise and rigorous definition and provide a wide and rich framework to work with and are often what people think of as a prototype example of fractals. So, from now on whenever the word fractal is used here, it will be taken interchangeably to mean self-similar set.

Intuitively, self-similar sets are geometric objects such that “as we zoom in we see copies of themselves", in other words they exhibit a version of scaling self-similarity. They can either be subsets of $\mathbb{R}^n$ in which case may be visual-

ized, or even abstract metric spaces. Their existence comes from the following theorem, proven by Hutchinson in [25].

Theorem 1.0.1. *If we have a complete metric space (X,d) and $F_i : X \to X$ are contractions for $i = 1,2,\ldots m$, then there exist a unique non-empty compact subset K of X that satisfies*

$$K = F_1(K) \cup \cdots \cup F_m(K).$$

Then K is called the self-similar set with respect to $\{F_1, F_2, \ldots F_m\}$.

The proof of existence comes from applying Banach's fixed point theorem to a suitable metric space equipped with a specific metric called the Hausdorff metric. There exists a very easy methodology to calculate the Hausdorff dimension of self-similar sets satisfying the following condition. We say that the contractions $\{F_i\}_{i=1}^m$ satisfy the open set condition if there exists an open set U such that $F_i(U)$ are disjoint for all $i = 1,\ldots m$ and $\bigcup_{i=1}^m F_i(U) \subset U$. Then the Hausdorff dimension of K is the unique real solution d_h to the following equation due to Moran

$$\sum_{i=1}^{m} r_i^{d_h} = 1$$

where r_i is the contraction ratio of the maps F_i. We can also classify two important families of self-similar sets, the finitely ramified and post-critically finite ones.

Definition 1.0.2. *We call K a post-critically finite set (p.c.f.) if K is connected and there exists a finite set V_0 called the boundary, such that for words $w \neq w'$ with $|w| = |w'|$ we have*

$$F_w K \cap F_{w'} K \subset F_w V_0 \cap F_{w'} V_0$$

with disjoint intersection from V_0 and also each of the boundary points is the fixed point of one of the maps F_i.

The finitely ramified ones are those that may be disconnected by removing a finite number of points. There is also a condition of symmetry, and the self-similar sets satisfying them are called fully symmetric, which will be useful later on when studying the spectrum of the Laplacian on fractals. Combining the above in a compact definition we have the following.

Definition 1.0.3. *K is a fully symmetric finitely ramified self-similar set, if K is a compact connected metric space with injective contraction maps $\{F_i\}_{i=1}^m$ such that*

$$K = F_1(K) \cup \cdots \cup F_m(K),$$

and the following three conditions hold:

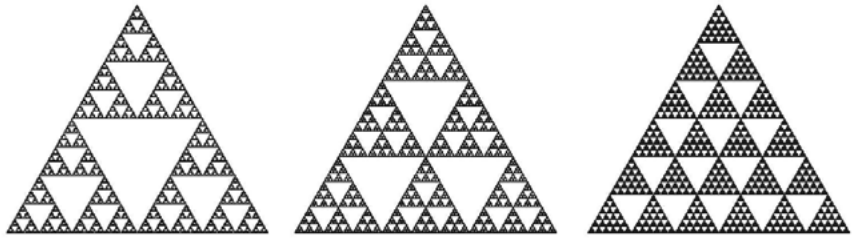

Figure 1.1. Sierpiński gaskets of level 2, 3, and 6.

1. *there exist a finite subset V_0 of K such that*

$$F_j(K) \cap f_k(K) = f_j(V_0) \cap F_k(V_0)$$

   for $j \neq k$ (this intersection may be empty);
2. *if $v_0 \in F_0 \cap F_j(K)$ then v_0 is the fixed point of F_j;*
3. *there is a group $\mathcal{G}$ of isometries of K that has a doubly transitive action on V_0 and is compatible with the self-similar structure $\{F_i\}_{i=1}^m$, which means that for any j and any $g \in \mathcal{G}$ there exist a k such that*

$$g^{-1} \circ F_j \circ g = F_k.$$

A post-critically finite set is also finitely ramified but the converse does not hold. We will now present some examples.

- The unit interval $I = [0,1]$.
  We can construct the unit interval as a self-similar set. It can be obtained from the iterated function system $F_0(x) = \frac{x}{2}$ and $F_1(x) = \frac{x+1}{2}$. It is the unique non-empty compact set satisfying $I = F_0 I \cup F_1 I$. It is also possible to obtain it with a different iterated function system, for example $F_0(x) = \frac{x}{3}, F_1(x) = \frac{x+1}{3}, F_2(x) = \frac{x+2}{3}$. We can create as such an infinite family of IFS all providing us with the unit interval showing that the construction of a self-similar set is not necessarily unique.

- The Cantor set.
  The middle third Cantor set is obtained by the maps $F_0(x) = \frac{x}{3}$ and $F_1(x) = \frac{x+2}{3}$. The Cantor set is not p.c.f as it is not connected. Its Hausdorff dimension is $\frac{\log 2}{\log 3}$.

- The Sierpiński gaskets of level k.
  The most widely studied self-similar set in the fractal setting is the Sierpiński gasket. Starting with an equilateral triangle in $\mathbb{R}^2$, with vertices v_1, v_2, v_3 we can take $\frac{k(k+1)}{2}$ contraction maps F_i with ratio $\frac{1}{k}$ giving us the Sierpiński gaskets of level k, denoted by SG_k and shown in Figure

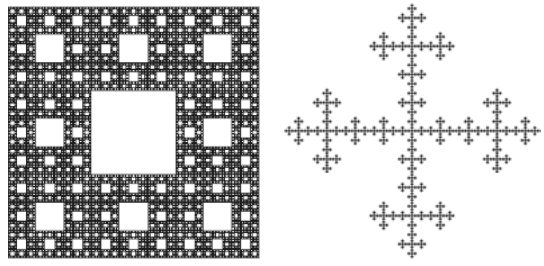

Figure 1.2. The Sierpiński carpet and the Vicsek set.

1.1. The familiar Sierpiński gasket SG is essentially SG_2 with the contraction maps $F_i(x) = \frac{x+v_i}{2}$ for $i = 0,1,2$. The Hausdorff dimension of SG_k is

$$s = 1 + \frac{\log(k+1) - \log 2}{\log k}.$$

giving the value $\frac{\log 3}{\log 2}$ for the regular SG.

- Higher dimensional Sierpiński gaskets SG_d^k.
  We can also define higher d-dimensional analogues of the Sierpiński gaskets of level k. The Sierpiński tetrahedron SG_3^2 is such an example consisting of four maps of contraction ratio $\frac{1}{2}$ giving a pyramid-like structure. Solving Moran's equation we notice that its Hausdorff dimension is $d_h = 2$ giving an easy example of a clearly fractal looking object with integer dimension.
- The Vicsek set.
  This is another example of a p.c.f. set which is obtained by five contraction mappings of ratio $\frac{1}{3}$.
- The post critically infinite Sierpiński gasket.
  This example of a self-similar set obtained by nine contractions denoted in Figure 1.3 is post critically infinite but nonetheless is still finitely ramified.
- The Sierpiński carpet.
  This is an example of a self-similar set obtained by eight contractions of ratio $\frac{1}{3}$. This fractal is not p.c.f., in fact it is even an infinitely ramified fractal since adjacent cells intersect on continuous line segments and thus we cannot make them disconnected by removing finitely many points. Analysis here is significantly harder to be performed due to the infinite ramification property.

There exist also non-deterministic versions of the above [20, 21, 22] and variations based on the Hanoi attractor [1]. We can also study products of fractals such as in [41]. Another variation are the so-called fractalfolds which can be obtained by taking copies of fractals. An example of a fractafold is the

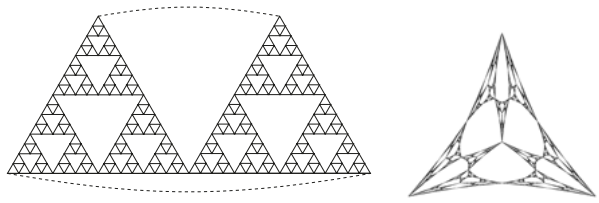

Figure 1.3. The double Sierpiński gasket and the post critically infinite one.

double Sierpiński gasket in Figure 1.3. where we take two copies of the regular *SG* and glue them together at their boundary points, becoming a fractal without boundary. Having studied their geometrical properties, it becomes important that some sort of analysis needs to be developed on these objects. This means that we will need to define the Laplace operator.

1.1 Graph theoretic approach

A very fruitful approach of doing analysis on fractals is through the eyes of graph theory. A graph is an ordered pair $G = (V,E)$ consisting of the set V of vertices and the set E of edges comprised by pairs of vertices. The set of edges E is a multiset. We may also assign a direction to each edge giving us directed graphs, and the vertices of a graph may be finite or infinitely many. For our purposes we will only study finite undirected graphs, where the edges have no orientation.

We begin with some important notions in graph theory. The degree $deg(v)$ of a vertex v is the number of edges connecting this vertex with other vertices, where loops count as having degree two. A graph is called simple if it has no multiple edges connecting two vertices or any loops connecting an edge with itself. A regular graph is a graph such that all its vertices have the same degree. We call a graph complete if every two of its vertices are connected by an edge. A graph is called planar if it can be drawn in the plane such that no two edges intersect each other with the exception at their end points.

On a simple graph, we can associate certain matrices to it that encode information about the underlying graph, the branches of mathematics studying these are algebraic and spectral graph theory [7, 8]. These matrices may also be defined with slight variations on multigraphs (non-simple graphs) too but we omit the details because we will not encounter multigraphs in our investigations here in this present work. All the matrices we will define below are $|V| \times |V|$ real square matrices. The degree matrix D of a graph is the diagonal matrix defined as $D = (d_{ij})$ where $d_{ij} = 0$ for $i \neq j$ and $d_{ii} = deg(v_i)$. The adjacency matrix is defined as $A = (a_{ij})$ where $a_{ij} = 1$ if there is an edge connecting the vertices v_i with v_j and $a_{ij} = 0$ otherwise. Then on the diagonal all entries are zero since loops do not exist on simple graphs. It becomes

clear by the definition that the adjacency matrix is symmetric. The combinatorial graph Laplacian may then be defined as $\Delta = D - A$. We can also define variants of the combinatorial graph Laplacian. The two most studied ones are the probabilistic graph Laplacian $L = D^{-1}\Delta$. In operator form we can simply write this as $\Delta u(x) = \sum_{y \sim x}(u(x) - u(y))$ where u is a function $u : V \to \mathbb{R}$ or $Lu(x) = \frac{1}{deg(x)} \sum_{y \sim x}(u(x) - u(y))$ and the notation $y \sim x$ means that the two vertices are adjacent.

Note that there is some ambiguity in the literature when it comes to the terminology of the probabilistic Laplacian and some authors refer to a different object, as well as often referring to $-\Delta$ as the Laplace operator. For example, in [13] there is a different definition of the probabilistic graph Laplacian. It can immediately be seen that the combinatorial graph Laplacian is symmetric whereas the probabilistic one is not. Thus the combinatorial graph Laplacian has real eigenvalues, and in fact 0 is always an eigenvalue corresponding to the piecewise constant functions on each connected component. Thus in simple connected graphs, 0 will always be a simple eigenvalue. However, it can also be shown that for the probabilistic graph Laplacian its eigenvalues are also real and again the same property holds for the 0 eigenvalue.

We call a graph G a tree if it is connected and contains no cycles, or equivalently if any two vertices can be connected by a unique simple path. A subgraph of a graph is another graph obtained by a subset of the vertices and edges of the original graph. A spanning tree is a subgraph of G that is a tree and includes all vertices of G. Spanning trees are widely studied in graph theory as they have real world applications as well, such as in computer science where various algorithms utilize them. One of the most widely studied topics in graph theory is the enumeration of spanning trees, or in other words, how many spanning trees exist in a finite graph. Of course a graph which is not connected has zero spanning trees, and there exists a related concept, that of spanning forests. For our purposes here, we will restrict our attention to simple connected graphs. Let $\tau(G)$ denote the number of spanning trees of the graph G. It is immediate to see that on the cyclic graph C_n we can take any spanning tree and rotate it n times so we have $\tau(C_n) = n$. It was proven by Cayley that on the complete graph K_n we have $\tau(K_n) = n^{n-2}$. It is of interest to find a formula for the number of spanning trees of a general graph G. This was obtained by Kirchhoff in the celebrated Matrix-Tree theorem. Specifically, let G be a connected graph with n vertices and let $\lambda_1, \ldots, \lambda_{n-1}$ be its non-zero eigenvalues. Then

$$\tau(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i.$$

Since 0 is always a simple eigenvalue in that case, we can abuse notation and denote the product of non-zero eigenvalues as the determinant of the Laplace matrix.

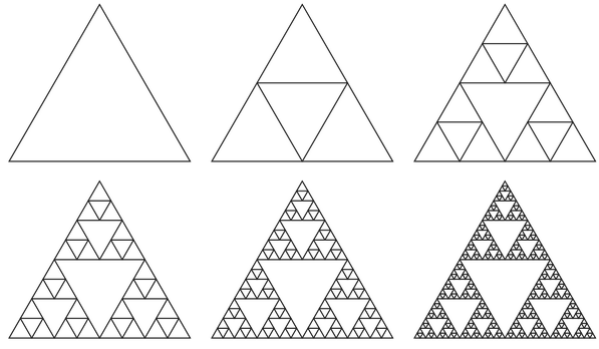

Figure 1.4. Graph approximations of the Sierpiński gasket

Obviously, graphs have a very wide area of applications. In physics and chemistry graphs can model molecules, in computer science measuring information flow and data structures, traffic networks, and linguistics. Nowadays especially popular are social media, where people can be modelled as vertices and friendships as edges connecting people. The graph Laplacian has also many application in computer graphics, in animation and video rendering, combinatorial optimization, machine learning and even in chemistry and protein structures [52]. Thus the field of graph theory and spectral graph theory have been considerably studied and developed and there exists extensive literature on these topics.

Now, to apply the above in our fractal setting. We can approximate the self-similar set through a sequence of graphs, often called "pre-fractals", "self-similar graphs" or just "fractal graphs". We start with the complete graph G_0 on the vertex set V_0 and then take m copies of it, where m is the number of contractions F_i of the self-similar structure, to create the G_1 graph with specific vertex identifications. Continuing inductively, we obtain G_n from m copies of G_{n-1} and we thus obtain our sequence of pre-fractal graphs. For more information we refer the reader to [2, 42]. In Figure 1.4, we present the approximating graphs for the Sierpiński gasket.

This gives us a plethora of fractal graphs to study relating to many self-similar sets. Notice, that the graph sequence is not unique to the self-similar set itself, and it depends on the iterated function system as well. For example, there are many iterated function systems that give rise to the unit intereval, and thus this gives rise to different possible graph approximations. Various graph theoretic properties have been studied for fractal graphs. When it comes to spanning tree enumeration on fractal graphs, considerable work has been done in [10, 11, 40, 45, 46, 47, 48, 49, 50, 53].

1.2 The Laplace operator and analysis on fractals

Fractal sets are “too rough" to define the classical Laplace operator as is the case for open subsets of $\mathbb{R}^n$. We can still however define the Laplace operator on fractals through various approaches, such as a probabilistic one studied by [3, 19, 31, 34] and an analytic one, via the use of Dirichlet forms, originating from the work of Kigami [27, 28, 29]. In this thesis we will focus on Kigami's analytic approach. We will use the slightly different convention of [42], that is the same in spirit to that of Kigami, but is perhaps more suited for considering things from the graph theoretic point of view. On the graph approximation G_n, we set $V_n = \bigcup_i F_i V_{n-1}$ and $V_* = \bigcup_i V_i$. Now, for functions $u, v : V_n \to \mathbb{R}$ we can define the energy form

$$\mathcal{E}_n(u,v) = \sum_{y \sim_n x} c_n(x,y)(u(x)-u(y))(v(x)-v(y))$$

where the notation $y \sim_n x$ denotes that the vertices x and y are adjacent in the graph G_n and $c_n(x,y)$ is the conductance of the edge connecting the vertices x and y. We want to create an energy form on the fractal K and whether that can be done is a difficult problem depending on K and the choices of $c_n(x,y)$. Some requirements need to be satisfied, but for our purposes here we will not focus on this renormalizaation problem. However, we will instead focus here in the cases where this is possible and all the conductances satisfy $c_n(x,y) = r^{-n}$ where r is the so-called renormalization constant. We will also restrict our attention in the cases where $0 < r < 1$ which we refer as a regular harmonic structure. We denote $\mathcal{E}_n(u) = \mathcal{E}_n(u,u)$. If we have a function $u : V_n \to \mathbb{R}$ there exists a unique way to extend it to V_{n+1} such that its energy is minimized. This is called the harmonic extension, and in that case $\mathcal{E}_{n+1}(u) = \mathcal{E}(u)$. So, given a function with initial values on V_0 there exists a unique way to extend it harmonically at every level. Such a function will be called harmonic function. Then the energy of a function $u : K \to \mathbb{R}$ is given by

$$\mathcal{E}(u) = \lim_{m \to \infty} \mathcal{E}_m(u).$$

We will study functions of finite energy, i.e. the vector space

$$dom\mathcal{E} = \{u : K \to \mathbb{R} : \mathcal{E}(u) < \infty\}.$$

It can be seen that functions of finite energy are continuous and thus since K is compact, uniformly continuous. The space of functions of finite energy modulo constants is a Hilbert space with the energy inner product.

Renormalization and harmonic functions are connected with electric networks and random walks on graphs. Having the standard energy form $\mathcal{E}$ on a graph G, we can define the effective resistance between two vertices as

$$R(u,v) = (min\{\mathcal{E}(h) : h(u) = 0 \text{ and } h(v) = 1\})^{-1}.$$

The function satisfying that minimum is in fact going to be the harmonic function with those two boundary values. The effective resistance, also called as resistance metric is a metric on the graph. The probabilistic interpretation of harmonic functions is interpreting the values on the vertices of the graph as probabilities of a random walk on the graph. Take a subset $S \subset G$ of vertices, and denote it as the boundary of the graph. Then if we solve the Dirichlet problem on the graph

$$\Delta h(v) = 0 \text{ for all } v \in G \setminus S \text{ and } h_{|S} = g$$

we will have that the harmonic function h can be written as

$$h(u) = \sum_{v \in S} g(v) Prob(u \to v)$$

where $Prob(u \to v)$ denotes the probability that a random walk on the graph G starting at u first hits v before any other vertices of S. We refer the reader for more details to [9, 15].

If $w = (w_1, \ldots, w_n)$ is a finite word, where $w_i \in \{1, \ldots, m\}$ we define the map $F_w = F_{w_1} \circ \cdots \circ F_{w_n}$. We call $F_w K$ a cell of level $n = |w|$, where $|w|$ is the length of the word. We can now construct a measure on our self-similar set K. The standard measure is a special case of a self-similar measure created in the following way. Assign probability weights μ_i with $\sum_{i=0}^{m} \mu_i = 1$ with each $\mu_i > 0$ and set $\mu(F_w K) = \prod_{i=0}^{|w|} \mu_{w_i}$. For the standard measure we just assign all $\mu_i = 1/m$. On K for the standard invariant measure μ we have $\mu(F_w F_i K) = \frac{1}{m}\mu(F_w K), \quad i = 0, 1, 2, \ldots, m$ for any word w. The self-similar identity $\mu(A) = \sum_i \mu_i \mu(F_i^{-1} A)$ holds for set $A \subset K$. In fact, the standard measure is none other than a renormalized version of the Hausdorff measure.

Using the measure μ we can study integrals so as to do analysis on the fractal. Our functions are uniformly continous because the set K is compact, so we simply define integration as

$$\int_K f \mathrm{d}\mu = \lim_{m \to \infty} \sum_{|w|=m} f(x_w) \mu(F_w K).$$

Having the energy form and the integrals at our disposal, we can now define the Laplace operator on the fractal itself.

Definition 1.2.1. *Let $u \in dom\mathcal{E}$ and f be a continuous function. Then, $u \in dom\Delta_\mu$ and $\Delta_\mu u = f$ if*

$$\mathcal{E}(u, v) = -\int_K f v \mathrm{d}\mu \text{ for all } v \in dom\mathcal{E}_0$$

where $dom\mathcal{E}_0$ denotes the functions of finite energy that vanish on the boundary.

Introducing the notion of normal derivatives we can modify the above notion to define the Neumann Laplacian as well. This is refered to as the weak definition of the Laplacian, we may also construct a pointwise definition for the standard self-similar measure by using the graph Laplacians. Specifically, we define the combinatorial graph Laplacian as

$$\Delta_m u(x) = \sum_{y \sim_m x} (u(y) - u(x)) \text{ for } x \in V_m \setminus V_0.$$

The harmonic functions then at every level satisfy $\Delta_m h = 0$. Notice that our definition of harmonic functions is in fact slightly different than the pure graph theoretic version often found in the literature. In spectral graph theory the harmonic functions are those that $\Delta h = 0$, where Δ is the graph Laplacian defined above as $\Delta = D - A$, and on finite graphs are only piecewise constant on each connected component of the graph. This difference comes from the fact that the graph Laplacian on the fractal graphs was not defined on the boundary, so as to allow us flexibility when it comes to boundary conditions such as Dirichlet or Neumann. Here in the fractal setting, by harmonic functions we refer to the ones solving the Dirichlet problem

$$\Delta_m h = 0 \text{ for all } m \geq 1 \text{ and } h_{|V_0} = g$$

and thus do not necessarily satisfy the Laplace equation, in the graph theoretic sense, on the boundary. There exists an algorithmic approach to calculating the values of a harmonic function on V_m. This is a local extension algorithm, meaning that knowing the boundary values on any cell of level m we can extend it to a cell of level $m+1$ and thus inductively everywhere. Starting from the boundary values at V_0 we can solve the linear system of equations at V_1 giving us the harmonic extension on the first level. Then at every next level, we are in the same situation as before with m cells and new boundary values. Encoding this information into m matrices A_i and thinking of the values of h on V_0 as a vector we have that $h_{|F_i V_0} = A_i h_{|V_0}$ which then inductively gives us that the values on any given cell are obtained by

$$h_{|F_w V_0} = A_w h_{|V_0} \text{ where } A_w = A_{w_m} \cdots A_{w_2} A_{w_1}.$$

These are called harmonic extension matrices, and for example in the Sierpiński gasket they are

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}, A_1 = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}, A_2 = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 1 \end{pmatrix}$$

giving us the so-called "$\frac{1}{5} - \frac{2}{5}$" harmonic extension rule, meaning that on $m+1$ level the value of a harmonic function is $\frac{2}{5}$ times the value of the sum of its closest vertices plus $\frac{1}{5}$ the value of the opposite vertex. Harmonic functions

satisfy the maximum principle, meaning that the maximum and minimum values are on the boundary V_0. If the matrices A_i are invertible then we have a non-degenerate harmonic structure which implies that non-constant harmonic functions cannot be locally constant on any cell. This is not always the case, for example the Vicsek set in Figure 1.2 has a degenerate harmonic structure.

Now, for $x \in V_m$ let $\psi_x^{(m)} : K \to \mathbb{R}$ defined as $\psi_x^{(m)}(x) = 1$, $\psi_x^{(m)}(y) = 0$ for $y \in V_m \setminus \{x\}$ and then extended harmonically to K. The pointwise definition for the Laplacian then becomes

$$\Delta_\mu u(x) = \lim_{m\to\infty} \left(\int_K \psi_x^{(m)} \mathrm{d}\mu \right)^{-1} \Delta_m u(x).$$

with uniform convergence on $V_* \setminus V_0$. When we will be using the standard self-similar measure μ we will omit it from the notation and simply write Δ. Then, the harmonic functions that we defined above are exactly the ones such that $\Delta h = 0$. Now, it may not be at this point clear that the space $dom\Delta_\mu$ is in fact a rich space to study. However, indeed $dom\Delta_\mu$ contains a lot of functions. We will omit the details of the construction of Green's function, but through it we can use the following theorem which shows us the richness of $dom\Delta_\mu$.

Theorem 1.2.2. *The Dirichlet problem*

$$-\Delta_\mu u = f \ , \ u_{|_{V_0}} = 0$$

has a unique solution in $dom\Delta_\mu$ for any continuous f, given by

$$u(x) = \int_K G(x,y) f(y) \mathrm{d}\mu(\mathrm{y})$$

where $G(x,y)$ is the Green's function. If we don't have Dirichlet boundary conditions then the solution is given by

$$u(x) = \int_K G(x,y) f(y) \mathrm{d}\mu(\mathrm{y}) + h(x)$$

where $h(x)$ is a harmonic function with the same boundary values as u.

One big disadvantage of $dom\Delta_\mu$ is that it's not closed under multiplication. Specifically, for $u \in dom\Delta_\mu$ it is proven in [6] that $u^2 \notin dom\Delta_\mu$. This however is specifically for the measure μ. We can define different measures that will give rise to a different Laplacian that will not necessarily suffer from this drawback. Specifically, Kusuoka in [31, 32] defined the measure ν, now referred to as the Kusuoka measure. We take a look first at the energy measures of a function $u \in dom\mathcal{E}$. Let the energy measure ν_u be

$$\nu_u(F_w K) = r^{-|w|} \mathcal{E}(u \circ F_w).$$

Then, let the harmonic functions $h_i(q_j) = \delta_{ij}$ for $q_j \in V_0$ and $i = 1, \ldots, |V_0|$. If the harmonic extension matrices A_i are invertible then the energy measures ν_{h_i} have full support. We define the Kusuoka measure as

$$\nu = \nu_{h_1} + \cdots + \nu_{h_{|V_0|}}.$$

It can be shown that every energy measure is absolutely continuous with respect to the Kusuoka measure. Moreover, on the Sierpiński gasket the Kusuoka measure ν is singular with respect to the self-similar measure μ. The Kusuoka measure gives rise to the energy Laplacian Δ_ν, defined in the weak sense exactly as before, now integrating against the Kusuoka measure. However, this energy Laplacian lacks scaling self-similarity and becomes a significantly harder object to study.

Now, for example if $|V_0| = 3$ such as in the case of SG_k, we can also define the Kusuoka measure in terms of an orthonormal basis of harmonic functions $\{h_1, h_2\}-$modulo constants. This gives the same version of the measure as above up to renormalization with a constant and in fact is independent of the choice of orthonormal basis. Then we have that if $u \in dom\Delta_\nu$ then

$$\Delta u^2 = 2u\Delta_\nu u + 2\frac{d\nu_u}{d\nu}$$

where $\frac{d\nu_u}{d\nu}$ is the Radon-Nikodym derivative. This is among one of the reasons that the energy Laplacian is of interest to study and therefore despite some of its disadvantages such as the lack of self-similarity, it behaves better in some regards. An important formula connecting energy measures, energy forms and integration of functions $u, v \in dom\mathcal{E}$ is the carré du champs formula

$$\int_K f \mathrm{d}\nu_{u,v} = \frac{1}{2}\mathcal{E}(fu, v) + \frac{1}{2}\mathcal{E}(u, fv) - \frac{1}{2}\mathcal{E}(f, uv).$$

There has also been a study of the Kusuoka measure from the ergodic point of view in [26].

1.3 Regarding the spectrum

It is interesting to obtain explicit knowledge of the spectrum of the Laplace operator with respect to various boundary conditions, such as the Dirichlet and Neumann. Depending on the measure used this may not always be possible. For example, the Kusuoka measure gives rise to a Laplace operator that, at the time of this writing, its spectrum is unknown. However, for the self-similar measure, we are able to use its discrete graph approximations to study the spectrum on the actual fractal. This technique is called spectral decimation, first studied in [5, 37], and then in considerable more detail by Fukishima and Shima for the d-dimensional Sierpiński gaskets in [18, 38, 39]. It was later

expanded significantly to a wide class of p.c.f fractals in [2, 36, 43]. For self-similar sets of a specific type, see more [36], we have that the spectrum is obtained through a technique called spectral decimation which can roughly be described as follows. The spectrum of the Laplace operator on the fractal is given by a renormalized limit of a rational function pre-images of eigenvalues on the discrete sequence of graphs approximating it. Moreover, the spectrum on the graph G_{n+1} can be decomposed into "initial eigenvalues" which may appear at every level, and continued eigenvalues which are pre-images of eigenvalues at G_n under the rational function, which is often in fact a polynomial. The set of initial eigenvalues is finite, and the rational function is fixed. This means that every eigenvalue on the graphs is either one of the initial eigenvalues, or a pre-image of an initial eigenvalue. This approach allows us also to construct the eigenfunctions recursively, but some care needs to be taken because not every pre-image of an eigenvalue is allowed to be taken, we have the so-called forbidden eigenvalues. For more details we refer to [2, 36, 43]. So for the fractal itself, as in [14, 44], we say that a fractal Laplacian admits spectral decimation if its eigenvalues are of the form

$$\lambda = c^m \lim_{n\to\infty} c^n R^{-n}(w)$$

where $w \in W$ is the finite set and the branches of the pre-images are taken in such a way such that the limit exists. We will describe very briefly the process here for simplicity only on the Sierpiński gasket for the combinatorial graph Laplacian with Neumann boundary conditions.

The Neumann boundary condition corresponds to imagining that the graph is embedded in a larger graph by reflecting in each boundary vertex and using the eigenvalue equation on the even extension. The spectral decimation polynomial is given by $R(z) = z(5-z)$. The exceptional set of forbidden eigenvalues is $E = \{2,5,6\}$. In multiset notation, on G_0 the spectrum of $\Delta_0 = \{0,6,6\}$. On G_1 it is $\{0,3,3,6,6,6\}$. In particular, at level n every eigenvalue of Δ_n is either 0 or 6 or obtained as a pre-image under R of these, under the condition that we do not encounter a forbidden eigenvalue. We have that $R^{-1}(\{0\}) = \{0,5\}$ and $R^{-1}(\{6\}) = \{2,3\}$ and these are the only cases when we encounter forbidden eigenvalues. Then the spectrum at level n is given by

$$\sigma(\Delta_n) = \{0,6\} \cup \bigcup_{i=0}^{n-1} R^{-i}(\{3\}) \cup \bigcup_{i=0}^{n-2} R^{-i}(\{5\}).$$

The eigenvalue 0 is always a simple eigenvalue and the eigenvalues 5 and 6 appear at every level and with high multiplicities. When we take the pre-images we have two branches

$$\lambda_n = \frac{5 + \varepsilon_n \sqrt{25 - 4\lambda_{n-1}}}{2}$$

where $\varepsilon_n \in \{-1,1\}$. For the Laplace operator $-\Delta$ on the fractal itself, we can define

$$\lambda = \frac{3}{2} \lim_{n\to\infty} 5^n \lambda_n$$

where the limit in the sequence $\{\lambda_n\}_{n\geq n_0}$ is taken for all but a finite number of $\varepsilon_n = -1$, and n_0 is the generation of birth. This procedure gives us the spectrum.

In the year 1966, Kac presented a paper with the title of the now famous question "Can one hear the shape of a drum?". This question is interpreted to mean whether knowledge of the eigenvalues $\{\lambda_n\}$ of the Dirichlet Laplace operator Δ on some bounded domain $U \subset \mathbb{R}^d$ is enough to determine the geometry and shape of the domain. The answer to this question is negative, i.e. there exist isospectral domains even in $\mathbb{R}^2$, however this question has motivated a large amount of research focusing on heat kernels and the eigenvalue counting function. We define the eigenvalue counting function as

$$N(x) = \#\{n \in \mathbb{N} : \lambda_n \leq x\}.$$

A famous result by Weyl is that for sufficiently regular bounded open domains we have

$$\lim_{x\to\infty} \frac{N(x)}{x^{\frac{d}{2}}} = \frac{\omega_d}{(2\pi)^d} V$$

where V is the volume of the domain and ω_d the volume of the unit ball in $\mathbb{R}^d$. This essentially means that by hearing the drum, while it doesn't give us knowledge of its exact shape, we still obtain some information about its geometry such as its d-dimensional volume.

In analysis on fractals, there is a sharp contrast with that of $\mathbb{R}^d$. Specifically, it was shown in [30] that the situation for the analogue of Weyl's result is different. For example, for the Dirichlet or Neumann Laplacian on the Sierpiński gasket, it was shown in [30] that there exists a $\frac{\log 5}{2}$-periodic discontinuous function G such that $0 < \inf G < \sup G < \infty$ and

$$N(x) = x^{\frac{d_s}{2}} G(\log x/2) + O(1)$$

giving us that the limit $\lim_{x\to\infty} N(x)x^{-\frac{d_s}{2}}$ does not converge. It is also interesting to note that the term d_s is different than what we may have expected considering the $\mathbb{R}^n$ case where the equivalent term is that of the dimension n. Now, d_s referred to as the spectral dimension, is actually different than the Hausdorff dimension. These quantities are connected via the Einstein relation $2d_s = d_h d_w$ where d_h is the Hausdorff dimension, d_s the spectral dimension and d_w the walk dimension. For more details we refer the reader to [17].

There exists another notable difference between $\mathbb{R}^n$ and analysis on fractals. In analysis on fractals we have the existence of joint Dirichlet-Neumann eigenvalues and specifically, localized eigenfunctions. In fact, the initial eigenvalues are usually joint Dirichlet-Neumann ones and appear with very high

multiplicities. The reason for the high multiplicities is that if the support of an eigenfunction is in a very small cell, we can essentially move it around to create many of them. That is why they are also referred to as localized eigenfunctions. Their existence is in sharp contrast to analysis on $\mathbb{R}^n$ because eigenfunctions are functions that are analytic and therefore cannot be zero on open sets.

Spectral decimation as in [2, 36] for fully symmetric fractals is valid in general for the probabilstic graph Laplacian and not the combinatorial one. The Laplace operator is usually studied under Neumann or Dirichlet boundary conditions and these conditions affect the spectrum. If the graph approximations consist of k-regular graphs, or perhaps non-regular graphs on the boundary that become regular under the Neumann conditions, then we may also use for our purposes the combinatorial graph Laplacian since it is essentially the same up to a k renormalization with the probabilistic graph Laplacian so again we have spectral decimation. We can then define $\mathcal{L} = \lim_{n\to\infty} c^n \mathcal{L}_n$ where the $c > 1$ is the so-called time-scaling factor and $\mathcal{L}_n u(x) = \frac{1}{deg(x)} \sum_{y\sim x}(u(x) - u(y)$ the probabilistic graph Laplacian on the graph G_n. Let $\mathcal{L} = -\Delta$. Its spectrum is discrete with eigenvalues of the form

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots < \infty$$

with ∞ the only accumulating point and 0 being an eigenvalue only in the Neumann case corresponding to the constant functions. We are interested in assigning meaning to the product $\prod_{n=1}^{\infty} \lambda_n$ which would be the determinant of the operator. Of course this product is actually infinite, but we would still like to interpret it as a real value.

In general, some meaning may be given to infinite divergent sums through the process of regularization. For example, it is clear that the sum

$$\sum_{n=0}^{\infty} 2^n = +\infty$$

However it is useful sometimes, for example in theoretical physics, to attempt to assign real values to such a sum. We know that for $|z| < 1$ it holds that $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$ and through the uniqueness of meromorphic continuation we can abuse notation and evaluate the meromorphic continuation of the function $f : B(0,1) \to \mathbb{C},\ f(z) = \sum_{n=0}^{\infty} z^n$ as $\tilde{f}(z) = \frac{1}{1-z}$ at $z = 2$ to obtain that $\sum_{n=0}^{\infty} 2^n = -1$. Of course, this is only a formal expression. Something similar to the above can also be done for $\prod_{n=1}^{\infty} \lambda_n$ using the so-called spectral zeta function. The spectral zeta function is defined as $\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s}$ for $Re(s)$ sufficiently large so that the sum converges and where the 0 eigenvalue is omitted in the Neumann case. The spectral zeta function may be meromorphically extended to the entire complex plane and its poles are referred to as complex dimensions. Now, under the assumption that the spectral zeta function has no poles on the imaginary axis, we can still assign it a real value through the

following formal manipulations. We notice that

$$\zeta'_{\mathcal{L}}(s) = \left(\sum_{i=1}^{\infty} \frac{1}{\lambda_i^s}\right)' = -\sum_{i=1}^{\infty} \frac{1}{\lambda_i^s} \log \lambda_i.$$

Then we can evaluate at $s = 0$ to obtain

$$\zeta'_{\mathcal{L}}(0) = -\sum_{i=1}^{\infty} \log \lambda_i = \log \prod_{i=1}^{\infty} \lambda_i = -\log \det \mathcal{L},$$

and therefore we can now define the so-called regularized determinant as $\det \mathcal{L} = e^{-\zeta'_{\mathcal{L}}(0)}$. This zeta regularization can also be applied to increasing non vanishing sequences of numbers $\{\lambda_n\}_n$. For example, for the sequence $\{n\}_n$ we see that using the above and Riemann's zeta function we can obtain the following formula

$$\prod_{n=1}^{\infty} n = \sqrt{2\pi}.$$

We can also define the so-called polynomial zeta functions. Let $R(z) = a_d x^d + \cdots + cx$ be a polynomial that has real coefficients with $d \geq 2$ which satisfies $R(0) = 0$ and $R'(0) = c > 1$. The spectral decimation polynomial satisfies these assumptions. We call Φ the entire function that satisfies the functional equation

$$\Phi(\lambda z) = R(\Phi(z)) \text{ with } \Phi(0) = 0, \Phi'(0) = 1.$$

We can now define the so-called polynomial zeta functions as

$$\zeta_{\Phi,w}(s) = \sum_{\substack{\Phi(-\mu)=w \\ \mu>0}} \mu^{-s}$$

which can also equivalently be stated as

$$\zeta_{\Phi,w}(s) = \lim_{n\to\infty} \sum_{z\in R^{-n}(w)} (\lambda^n z)^{-s}.$$

These zeta functions have been defined and studied in [14], [44] and are crucial in the meromorphic extension of the spectral zeta functions of the Laplace operator $\mathcal{L}$. They can be meromorphically extended to the entire complex plane and none of their poles lie on the imaginary axis. Specifically, all their poles are simple and lie on the imaginary line $\mathrm{Re}(s) = \frac{\log d}{\log \lambda}$. We refer the reader to [14, 44]. There is also another type of zeta functions studied on fractals, those on fractal strings, see [33].

It often is that we have for some constant α that $\mathcal{L} = \alpha \lim_n \mathcal{L}_n$. For example in the Sierpinski gasket, in the literature we usually have $\mathcal{L} = 6 \lim_n \mathcal{L}_n$. However, it can also be that α varies in the literature. This is mostly based

on what convention the authors prefer to use. Then, for example in the unit interval, it just is that

$$\mathcal{L} = -\Delta = -\frac{d}{dx^2} = \lim_n \Delta_n = 2\lim_n \mathcal{L}_n.$$

These differences in normalization are at first trivial, making absolutely no difference in the development of the general theory. However, they become more important once we start considering spectral zeta functions. This means that now the eigenvalues are truly of the form

$$\lambda = \alpha c^m \lim_{n\to\infty} c^n R^{-n}(w)$$

and thus the spectral zeta functions may differ from each other like

$$\zeta_1(s) = \alpha^{-s}\zeta_2(s)$$

This distinction becomes important when we attempt to investigate connections between discrete and regularized determinants. For our calculations, we will be using $\alpha = 1$.

2. Summary of the results

2.1 Summary of paper I

In paper I, we provide a formula to determine the number of spanning trees on the graph approximations of a post critically finite self-similar fractal admitting spectral decimation. Specifically, it is shown that the number of spanning trees $\tau(G_n)$ equals

$$\tau(G_n) = \left| \left(\frac{\prod_{i=1}^{|V_n|} d_i}{\sum_{i=1}^{|V_n|} d_i} \right) \prod_{\alpha \in A} \alpha^{\alpha_n} \left[\prod_{\beta \in B} \left(\beta^{\Sigma k=0^n \beta_n^k} \left(\frac{-Q(0)}{P_d} \right)^{\sum_{k=0}^{n} \beta_n^k \left(\frac{d^k-1}{d-1} \right)} \right) \right] \right|.$$

This formula is essentially a calculation of the product of the non-zero eigenvalues of the probabilistic graph Laplacian. Then, the spanning tree evaluation is based on Kirchhoff's Matrix-Tree theorem in its probabilistic graph Laplacian version. Moreover, we provide a proof showing why the asymptotic complexity constant c exists for the fractal graphs based only on their self-similarity without using the machinery of [35] as well as also provide the following lower and upper bounds for it. Specifically,

$$\frac{(|V_0|-2)\log|V_0|}{|V_0|-1} \leqslant c \leqslant \log\left(\frac{(m-1)|V_0|(|V_0|-1)}{|V_1|-|V_0|} \right). \tag{2.1.1}$$

We conclude the paper with a plethora of examples of specific fractal graphs where we calculate the number of the spanning trees of their graph approximations.

2.2 Summary of paper II

In paper II, we study the harmonic structure of p.c.f. self-similar fractals and focus mostly on the Sierpiński gaskets of level k denoted in Figure 1.1. Harmonic functions are functions that minimize the energy at each level, and can easily be studied by evaluating their values on the discrete graph approximating the fractal. This allows us to solve the Dirichlet problem of the combinatorial graph Laplacian on the graphs themselves. As mentioned in the introduction, due to the self-similarity of the fractal graphs, there exists an algorithmic approach to evaluating the values of harmonic functions on each junction point. There exist the so-called harmonic extension matrices A_i such that the values of the harmonic function may be given by

$$\begin{pmatrix} h(F_i q_1) \\ h(F_i q_2) \\ h(F_i q_3) \end{pmatrix} = A_i \begin{pmatrix} h(q_1) \\ h(q_2) \\ h(q_3) \end{pmatrix}.$$

There exists a notable difference between harmonic functions in $\mathbb{R}^n$ and harmonic functions on self-similar sets. In $\mathbb{R}^n$ we know that if a harmonic function defined on a bounded domain Ω is constant on a non-empty open subset of Ω, then it is constant everywhere in Ω. Specifically, when it comes to solving the Dirichlet problem, for a continuous $g : \partial\Omega \to \mathbb{R}$,

$$\Delta h = 0 \text{ and } h|_{\partial\Omega} = g$$

then, if g is constant, h is also constant and if g is not constant then h cannot be locally constant on any open set $\emptyset \neq U \subset \Omega$. This property fails however for some fractals. We can have a non-constant $g : V_0 \to \mathbb{R}$ that gives rise to a harmonic h that is constant on a cell $F_i K$. This is equivalent to a harmonic extension matrix A_i being singular. In that case we say that the fractal has a degenerate harmonic structure. In fact, most self-similar sets have a degenerate harmonic structure which is problematic since many results in the theory require a non-degenerate harmonic structure in the assumption. In [23, 24], Hino has conjectured that for the d-dimensional Sierpiński gaskets of level k the harmonic structure is non-degenerate for any $d \geq 2$ and $k \geq 2$. Our main result is to use Tutte's spring theorem, proven in [51], to give an affirmative answer to this conjecture in the two dimensional case. Specifically, we obtain the following.

Theorem 2.2.1. *For every $k \geq 2$ the harmonic structure on the two dimensional SG_k is non-degenerate.*

We also then study the problem in more generality, to other p.c.f. fractals besides the SG_k. We provide the following connectivity criterion which can be applied in the first graph approximation of the self-similar set which shows when the harmonic structure is degenerate.

Proposition 2.2.2. *Let K be a post-critically finite self-similar set with $|V_0| = 3$ such that its modified first graph approximation $\tilde{G}_1$ is a 3-connected planar graph so that boundary vertices q_1, q_2, q_3 create a facial cycle of the graph. Then, if every edge has a strictly positive weight, the harmonic structure of K is non-degenerate.*

2.3 Summary of paper III

In paper III, we survey and extend properties of the Kusuoka measure and the corresponding energy Laplacian on the Sierpiński gaskets of level k. The stan-

dard self-similar measure, while being easier to study, gives rise to a Laplacian that has domain not closed under multiplication. One way to navigate around this problem is to use the energy Laplacian which is defined in the exact same weak definition but now by integration against the Kusuoka measure. On the SG_k gaskets, the Kusuoka measure is defined as

$$\nu = \nu_{h_1} + \nu_{h_2}$$

where $\{h_1, h_2\}$ is an orthonormal basis of harmonic functions modulo constants. We give the following probabilistic interpretation of the Laplacian pointwise formula.

Proposition 2.3.1. *For a fully symmetric p.c.f. nested self-similar fractal with a regular harmonic structure, the pointwise formula for the Laplacian is given by*

$$\Delta_\mu u(x) = |V_0|(|V_0|-1) \lim_{m\to\infty} \left(\frac{H(q_1,q_2)}{|V_0|-1} \right)^m \Delta_m u(x)$$

where $H(q_1,q_2)$ is evaluated on the first graph approximation.

We also provide a variable weight 'self-similar' formula for the scaling of the Laplacian similar to that in [4].

Theorem 2.3.2. *The energy Laplacian Δ_ν satisfies*

$$\Delta_\nu(u \circ F_j) = rQ_j(\Delta_\nu u) \circ F_j \tag{2.3.1}$$

ν-almost everywhere.

We then show that the only functions in the domain of both the standard and the energy Laplacian are the harmonic functions.

2.4 Summary of paper IV

In paper IV, we investigate connections between the regularized Laplacian determinant and the discrete combinatorial one. For the combinatorial graph Laplacian we know that 0 is always a simple eigenvalue, so we abuse notation and denote as its determinant the product of its non-zero eigenvalues. Now, for a self-adjoint operator with a countably infinite sequence of eigenvalues $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ with $\lambda_n \to \infty$, it is possible to define the regularized determinant as explained in the introduction above as $\det \mathcal{L} = e^{-\zeta'_{\mathcal{L}}(0)}$ where $\zeta_{\mathcal{L}}(s)$ is the spectral zeta function of the Laplace operator. In that case, it was proven for discrete tori in [12] that the logarithm of the regularized determinant appears as a constant in the logarithm of the discrete combinatorial determinants.

Our goal in this paper is to establish a similar connection in the fractal case. We investigate the Diamond fractal, the double N-dimensional Sierpiński gaskets, and the double pq-model of the unit interval. We obtained a similar connection as in [12]. Specifically, we show the following.

Theorem 2.4.1. *For the discrete combinatorial graph Laplacian determinant of the double SG^N, we have that*

$$\log\det\Delta_n = c_a|V_n| + n\log(N+2) + \log\det\mathcal{L},$$

where c_a is the asymptotic complexity constant which is

$$c = \frac{N-2}{N}\log 2 + \frac{N-2}{N-1}\log N + \frac{N-2}{N(N-1)}\log(N+2).$$

The following can also be stated for the double pq-model. The pq-model is essentially the unit interval, viewed as a self-similar set using three contractions, and a graph Laplacian viewed as a specifically weighed random walk. The double model is just as in the Sierpiński gasket case, taking two copies and gluing them at the boundary. For more details, we refer the reader to [44].

Theorem 2.4.2. *For the double pq-model we have that*

$$\log\det\Delta_n = |V_n|(\log 2 + \frac{\log(pq)}{2}) + n\log\frac{(1-q^2)(1-p^2)}{(pq)^2} + \log\det\mathcal{L}$$

2.5 Summary of paper V

In paper V, we have the exact same setting as in paper IV. Now however, instead of performing our calculations on specific self-similar sets, we expand our results in a more abstract setting to self-similar fractafolds satisfying specific conditions. We establish the same connection between the logarithm of the discrete combinatorial graph Laplacian and the regularized one. Specifically, we have the following theorem.

Theorem 2.5.1. *Let a p.c.f. self-similar fractafold such that the number of vertices in its graph approximations is of exponential form. Then its spectral zeta function has no poles on the imaginary axis. Moreover, for the combinatorial graph Laplacian we have that*

$$\log\det\Delta_n = c_a|V_n| + nj\log\lambda + \log\det\mathcal{L}$$

where $\det\mathcal{L}$ *is the regularized determinant, λ the time-scaling constant, c_a the asymptotic complexity constant and $j \in \{0,1,2\}$.*

3. Summary in Swedish

Det aktuella forskningsområdet *Analys på fraktaler* har studerats från både analytisk och probabilistisk synvinkel. Detta arbete är fokuserat på det analytiska anagreppssätt som har utvecklats av Kigami. Fraktalerna som studeras är ändligt ramifierade självlikformiga mängder, med betoning på postkritiskt ändliga mängder. En viktig prototyp är Sierpińskitriangeln. Man kan approximera ändligt ramifierade mängder med en följd av approximerande grafer G_n som gör det möjligt att använda begrepp från diskret matematik, som den kombinatoriska och probabilistiska graf-Laplacen på ändliga grafer. Genom denna approach, eller via Dirichletformer, kan man definiera Laplaceoperatorn på den kontinuerliga fraktalen, antingen genom en svag definition på formen

$$\mathcal{E}(u,v) = -\int_K fv\mathrm{d}\mu \text{ for all } v \in dom_0\mathcal{E}$$

eller som ett renormaliserat gränsvärde av diskreta graf-Laplacer på graferna G_n:

$$\Delta_\mu u(x) = \lim_{m\to\infty}\left(\int_K \psi_x^{(m)}\mathrm{d}\mu\right)^{-1}\Delta_m u(x).$$

Syftet med föreliggande arbete är att studera graferna som approximerar fraktalen och att studera sambandet mellan Lapalaceoperatorn på de diskreta graferna och på fraktalen. Precis som i teorin för spektral grafteori är man intresserad av spektrumet för den fraktala Laplaceoperatorn. Det kan göras med hjälp av en metod som kallas spektral decimering, som i allt väsentligt delar upp spektrumet av Laplacen i två delar, de initiala och de kontinuerliga egenvärdena. De initiala dyker upp på varje nivå av approximation och med höga multipliciteter. De kontinuerliga erhålls genom iteration av urbilden av en fix rationell funktion applicerad på de initiala. Denna iterativa metod ger en fullständig kunskap om spektrumet. I graferna kommer 0 alltid att vara ett enkelt egenvärde och genom att viss begreppslig förskjutning, kallar man produkten av de egenvärden som inte är noll för determinanten av Laplaceoperatorn.

I första artikeln studeras antalet uppspännande träd på en följd av grafer som approximerar en fraktal som tillåter spektral decimering. En formel för antalet ges genom en beräkning av produkten av icke-triviala egenvärden av den probabilistiska graf-Laplacen och med hjälp av en variant av Kirchoffs sats för en probabilistisk graf-Laplace. Ett bevis ges också för att den asymptotiska komplexitetskonstanten existerar genom att enbart utnyttja självlikformigheten och dessutom ges en undre begränsning för denna. Artikeln avslutas med några viktiga exempel.

I den andra artikeln studeras harmoniska funktioner på postkritiskt ändliga fraktaler. I motsats till teorin för harmoniska funktioner i Euklidiska rum kan man ha icke-triviala harmoniska funktioner som är lokalt konstanta. I sådana fall sägs fraktalen ha en degenerande harmonisk struktur. Det finns flera viktiga exempel (Hexagasket, Viscek-mängden). Men självlikformiga mängder har inte alltid denna egenskap. Exempelvis har Sierpińskitriangeln inga icke-triviala lokalt konstanta harmoniska funktioner. Hino gav en förmodan om att för en familj av varianter av (utvidgade) Sierpińskitrianglar (level-k Sierpiński gaskets) så är de alla icke-degenererande, precis som Sierpińskitriangeln och denna förmodan bevisas i artikeln.

I den tredje artkeln undersöks egenskaper hos Kusuoka-måttet och motsvarande energi-Laplace på den familj av utvidgade Sierpińskitriangler som nämndes ovan. En probabilistisk tolkning ges av den punktvisa formeln för Laplaceoperatorn och vi ger även ett "själviklformigt" uttryck för en laplace med variabla vikter. Därefter visas att de enda funktioner som finns i domänen till den vanliga Laplacen och energi-Laplacen är de harmoniska funtionerna.

I artikel fyra och fem visar vi på sambandet mellan determinanten för den diskreta kombinatoriska graf-Laplacen och den motsvarande regulariserade determinanten för fraktalen. Vi visar att för en viss klass av postkritiskt ändliga fraktaler så förekommer logaritmen av den regulariserade determinanten som en konstant i logaritmen av den diskreta kombinatoriska Laplacen. Detta har tidigare observerats för torusen och vårt resultat är det motsvarande för det fraktala fallet.

4. Acknowledgements

First of all, I want to thank my advisor Anders Karlsson. I am immensely grateful for your constant encouragment, support and that you never stopped believing in me. Your positive words, guidance and advice on life have shaped me in many ways. I am not the same person as I was before I started my Ph.D, and a big part of that I owe to you. Thank you for everything and especially for your friendship.

I also want to offer my deepest gratitude to my second advisor Anders Öberg for always being there for me and helping me. You were always kind and patient with me and I have enjoyed our discussions and I have learned a lot from you. I also want to thank you for being the one to introduce me to the beautiful field of analysis on fractals and for pushing me to achieve my best.

My first real Mathematical step was in the University of Athens where I first encountered the beauty of Mathematics and got my curiosity intrigued to explore the topic further in Uppsala. I will always look back at it with fond memories. I am extremely grateful to the many people I have met since then along the way during my Ph.D. journey in universities outside of Uppsala. From extremely productive Mathematical discussions, to all kinds of topics, you have shown me how amazing and kind the Mathematical community can be. The list is long to enumerate here, but a very special thank you to Anders J., Bob, Elmar, Joe, Sasha, and Uta. I call myself extremely lucky to have met you.

I want to give my thanks to all my colleagues and friends in the Mathematics Department of Uppsala University for making it a fantastic work environment to be at and a wonderful community to be a part of. Special thanks to Anna B., Andrea, Andreas, Arianna, Björn, Colin, Dan, Elin, Erik, Filipe, Linnea, Marta, Natasha, Nikos, Sam, Viktoria, Hannah, Sebastian, Tilo, and Yu. You have made the Ph.D. life amazing and unforgettable. Jakob, sharing an office with you has been quite the experience to say the least. Thank you for always making it a fun place to be at, where we talked about everything under the sun, for challenging me and for being a great friend.

I am indebted to the Mathematics Department of Uppsala University for all the help that has been given to me. I especially want to thank Alma, Andreas S., Cecilia, Fredrik, Jörgen, Lina, Thomas, Walter, and Warwick. Thank you all for making our department such a great one and making my life and studies there very enjoyable. I am very grateful to the Mathematics Department of Cornell University for its hospitality during one of my most productive and fun semesters. Spending there that time has been an incredible experience. Special

thanks to Jason for very helpful discussions and a fruitful collaboration and to Jose for being an amazing friend who was always there, be it to talk Math, have some beers or go travelling and exploring together. I am also grateful for the support and hospitality from the Mathematics Departments of the Universities of Connecticut, Geneva, Stuttgart and Tübingen during my visits there.

My friends outside the Mathematics community have always been there to keep me grounded to reality, and help me remember that there is more to life. Special thanks to Alexandra, Dimitris A., Dimitris K., Dimitris T., Elio, Esli, Evaggelia, Ilektra, Konstantina, Laora, Maria S., Petros, Sofia, Stelios, and Thanasis. It is great to be surrounded by such wonderful people. Cristina, thank you for pushing me to actually do this. I doubt if I would ever have started this were it not for your support and encouragement.

Most importantly I want to thank my parents for their continuous love and support and for all you have taught me over the years. Without your love this work would not have been possible. Words can't encompass how deeply grateful I am. Thank you for everything.

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