Quantum corrections to central charges and supersymmetric Casimir energy in AdS$_3$/CFT$_2$

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ABSTRACT: We study the Casimir energy of bulk fields in AdS$_3$ and its relation to subleading terms in the central charge of the dual CFT$_2$. Computing both sides of the standard CFT$_2$ relation $E = -c/12$ independently we show that this relation is not necessarily satisfied at the level of individual bulk supergravity states, but in theories with sufficient supersymmetry it is restored at the level of bulk supermultiplets. Assuming only $(0, 2)$ supersymmetry (or more), we improve the situation by relating quantum corrections to the central charge and the supersymmetric Casimir energy which in turn is related to an index. These relations adapt recent progress on the AdS$_5$/CFT$_4$ correspondence to AdS$_3$/CFT$_2$ holography. We test our formula successfully in several examples, including the $(0, 4)$ MSW theory describing classes of 4D black holes and the large $(4, 4)$ theory that is interesting for higher spin holography. We also make predictions for the subleading central charges in several recently proposed $(2, 2)$ dualities where the CFT$_2$ is not yet well-understood.

Contents

1   Introduction

More than 20 years after the advent of the AdS/CFT correspondence, "tests" of holographic duality rarely challenge the conjecture itself, but nevertheless regularly motivate investigations in quantum field theory that are interesting in their own right. Reproducing the trace anomaly presents an important "test" of holography in this sense. On the boundary, the anomalous breaking of conformal symmetry due to an inert background geometry is parametrized by central charges that are usually well understood even for finite rank N of the gauge group. In the dual bulk theory the leading contribution to the anomaly for large rank N of the gauge symmetry is due to a universal boundary term that is classical in nature [1]. For our purposes, the interesting bulk effect is the term that is at the subleading order in $1/N$ and due to one-loop contributions from bulk excitations. The computation of such quantum effects is well understood in simple cases but the elaborate spectrum required for precision holography presents challenges. There has been significant progress on this problem in the case of AdS_5/CFT_4 correspondence and it has largely been solved [2–9]. The purpose of this paper is to address the situation in the case of AdS_3/CFT_2 correspondence, focusing on bulk issues.

As a matter of principle, one-loop quantum effects in the bulk theory include all modes in the theory, including modes from string theory (or some other UV completion) that might not be well-understood. Therefore, it is important to identify the settings where there is sufficient SUSY that it can be justified to ignore unknown parts of the spectrum, leaving only the known SUGRA states. Our goal is to achieve agreements in *all* situations where the legitimacy of focusing on SUGRA states is guaranteed by SUSY, though *not necessarily* in situations where SUSY is too weak, except perhaps for occasional "accidents" (that might well be due to some interesting symmetry beyond SUSY that we have not recognized).

This goal may seem modest but we identify some obstacles. A key challenge is that there is an infinite tower of Kaluza-Klein states so, effectively, the theory is higher dimensional. Renormalization of the theory in AdS_3 and the subsequent sum over the KK tower does not necessarily agree with renormalization of the theory in higher dimensions.

A specific computational scheme was introduced by Beccaria, Macorini, and Tseytlin (BMT) [10]. It amounts to computing the subleading Casimir energy δE by summing over the ground state energy $\frac{1}{2}\omega$ for full KK towers of chiral primaries in supergravity with renormalization in the ζ-function scheme, and then applying the standard CFT$_2$ relation $\delta E = -\delta c/12$ to extract the subleading central charge. BMT tested their prescription successfully in the cases of non-chiral theories with $(4, 4)$ SUSY.

We apply the BMT prescription to compute δE in situations with less SUSY. We also introduce an independent algorithm to compute δc holographically so we can test the relation $\delta E = -\delta c/12$ at the level of individual bulk representations. These tests are largely successful, but not always so.

For example, in theories with fewer than four supercharges the relation is not satisfied. However, this "failure" is not unexpected because, as we show, in this case long multiplets contribute to both δE and δc. Unknown massive fields would be organized in such multiplets and so our supergravity computation is insufficient for an agreement. The AdS/CFT correspondence does not require agreement for individual bulk fields so, in principle, it could be that the sum over all KK fields would restore the relation $\delta E = -\delta c/12$. However, even if such agreement could be established it would anyway be satisfying only after also explaining why long multiplets cancel between themselves. It is an additional and independent concern that the ζ-function regularization may not respect supersymmetry.

To address the situation in a principled manner we consider a different approach, inspired by progress on the AdS$_5$/CFT$_4$ correspondence over the last few years. Assuming at least $(0, 2)$ SUSY, one can define a right-handed superconformal index $\mathcal{I}^R_{\text{s.p.}}(q)$ which counts all the bulk single-particle states that are annihilated by one of the supercharges and hence vanishes when evaluated on long representations.[1] The main result of the present paper is a simple relation between the high-temperature behavior of this single-particle index and the subleading central charge of the boundary CFT. More precisely, we show that the one-loop correction to the left-moving bulk central charge is given by

$$\delta c_L = 12 \lim_{\beta \to 0} \frac{\mathrm{d}}{\mathrm{d}\beta} \mathcal{I}^R_{\text{s.p.}}(q = e^{-\beta}). \tag{1.1}$$

The expression subject to the limit $\beta \to 0$ is a meromorphic function of β that should be renormalized by omission of pole terms prior to the limit. An analogous expression for δc_R applies to theories with $(2, 0)$ SUSY.

To arrive at the formula (1.1), we leverage the notion of supersymmetric Casimir energy E^L_{SUSY}, related to c_L via $E^L_{SUSY} = -c_L/24$. That a supersymmetric version of the Casimir energy can be defined from the superconformal index was first proposed in [11, 12] at the level of single-letter indices in free field theory. We apply this idea to the bulk single-particle index in order to extract the subleading E^L_{SUSY}, and hence the central charge δc_L.

We successfully test the simple relation (1.1) in section 5, in the context of several well-known AdS$_3$/CFT$_2$ dualities. We also make predictions for the subleading central charges in several other cases where the bulk index is known but the boundary CFT is not.

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[1] The superconformal index is essentially the spectral flow of the elliptic genus to the NS-NS sector; see section 4 for the explicit definitions.

This paper is organized as follows. In section 2 we review some aspects of the supersymmetric Casimir energy E_{SUSY} that were previously developed in the context of the $\mathrm{AdS}_5/\mathrm{CFT}_4$ correspondence. It allows a concise expression of the bulk quantum corrections to the Weyl anomaly coefficients a and c obtained in [7, 9]. This will serve as an introduction to the techniques we employ in $\mathrm{AdS}_3/\mathrm{CFT}_2$ holography and motivate our formula (1.1). In section 3 we discuss possible relations between the bulk Casimir energy of supergravity states in AdS_3 and the central charge of the dual 2D CFT. We refine the discussion by defining left-moving and right-moving Casimir energies and attempt to relate these to the corresponding left and right-moving central charges, emphasizing the relevance of a supersymmetric spectrum to the success or failure of such relations at the level of individual bulk multiplets. In section 4 we derive the supersymmetric Casimir energy from the superconformal index and compute from it the left-moving central charge of $\mathcal{N} = (0, 2)$ theories. In order to provide a consistency check of our approach in this minimal supersymmetric context we additionally find the subleading *right-moving* central charge δc_R by computing one-loop Chern-Simons levels δk_R for the $\mathrm{U}(1)_R$ gauge field, which are related by $\delta c_R = -3\delta k_R$. We also discuss analogous relations in theories with more supercharges. In section 5 we test the results from section 4 in particular examples of $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality, including the $(0, 4)$ MSW theory describing classes of 4D black holes and the large $(4, 4)$ theory that is interesting for higher spin holography. We also offer predictions for quantum corrections to the central charge in several $(2, 2)$ cases where the supersymmetric index of bulk supergravity states is known but the dual CFT_2 is not. We conclude with a summary and a discussion of outlooks in section 6.

2 Supersymmetric Casimir energy and the index in $\mathrm{AdS}_5/\mathrm{CFT}_4$

Before turning to $\mathrm{AdS}_3/\mathrm{CFT}_2$ and the justification of the relation (1.1) it is instructive to review the $\mathrm{AdS}_5/\mathrm{CFT}_4$ case.

The supersymmetric partition function Z_{SUSY}, defined as the path integral with appropriate boundary conditions, is proportional to the index $\mathcal{I}$, defined in terms of a trace over the states that respect supersymmetry. The factor of proportionality defines the supersymmetric Casimir energy E_{SUSY} through [11–16].

$$Z_{\mathrm{SUSY}} = e^{-\beta E_{\mathrm{SUSY}}} \mathcal{I} \ . \tag{2.1}$$

In view of its definition, E_{SUSY} plays well with supersymmetry, while the conventional Casimir energy might not.

Like the Casimir energy in 2D, the supersymmetric Casimir energy in 4D is related to the central charges of the theory. Specifically, it can be written in terms of the Weyl anomaly coefficients a and c of an $\mathcal{N} = 1$ SCFT as [13–16]

$$E_{\mathrm{SUSY}} = \frac{8}{3}(b_1 + b_2)(a - c) + \frac{8}{27}\frac{(b_1 + b_2)^3}{b_1 b_2}(3c - 2a). \tag{2.2}$$

Here b_1 and b_2 parametrize families of supersymmetric boundary conditions or, equivalently, the fugacities $t = e^{(b_1+b_2)\beta}$ and $y = e^{(b_1-b_2)\beta}$ that appear in the superconformal index introduced in (2.6) below.

The leading semi-classical saddle point determines central charges of $\mathcal{O}(N^2)$. From the bulk point of view the subleading $\mathcal{O}(1)$ contributions are quantum corrections. The superconformal index provides a useful regulator for determining these quantum corrections. In particular, the subleading central charges — given below in (2.3) — can be computed by extracting the subleading supersymmetric Casimir energy from the single-trace superconformal index [7, 9].

In AdS$_5$, bulk excitations can be organized into multiplets of the superconformal algebra. Representations are labeled by the conformal dimension Δ, a U(1)$_R$ charge r, and two SU(2) spins j_1 and j_2. The central charges are only quantum corrected by loops of shortened multiplets [5, 8]. The contribution from a single short multiplet is given by [6, 8]

$$
\begin{aligned}
c_{\text{chiral}} = -c_{\text{SLII}} = (-1)^{2j_1+2j_2}\frac{9}{128}(2j_1+1)\left(\Delta - \frac{r}{2} - 1\right) \\
\times \left[\left(\Delta - \frac{r}{2}\right)\left(\Delta - \frac{r}{2} - 2\right) + \frac{4}{9}j_1(j_1+1) + \frac{4}{9}\right],
\end{aligned}
$$

$$
\begin{aligned}
a_{\text{chiral}} = -a_{\text{SLII}} = (-1)^{2j_1+2j_2}\frac{9}{128}(2j_1+1)\left(\Delta - \frac{r}{2} - 1\right) \\
\times \left[\left(\Delta - \frac{r}{2}\right)\left(\Delta - \frac{r}{2} - 2\right) - \frac{4}{3}j_1(j_1+1) + \frac{2}{3}\right],
\end{aligned}
\tag{2.3}
$$

where chiral multiplets satisfy the shortening condition

$$
\Delta = \frac{3r}{2}, \qquad j_2 = 0,
\tag{2.4}
$$

and semi-long II (SLII) multiplets satisfy

$$
\Delta = \frac{3r}{2} + 2j_2 + 2.
\tag{2.5}
$$

Multiplets with conserved currents are included as the special case of SLII multiplets with the additional condition $\frac{3}{2}r - j_1 - j_2 = 0$. The fact that the contributions are equal and opposite for the two types of shortened multiplets is consistent with the recombination rules for forming a long multiplet out of a chiral and an SLII multiplet along with the fact that long multiplets do not contribute to the central charges. The central charges also receive contributions from anti-chiral and SLI multiplets; these are given by (2.3) with the replacements $j_1 \leftrightarrow j_2$ and $r \to -r$.

The right-handed $\mathcal{N} = 1$ superconformal index is given by the trace

$$
\mathcal{I}^R(t, y) = \text{Tr}(-1)^F t^{-(\Delta - \frac{r}{2})} y^{2j_1}.
\tag{2.6}
$$

This only receives contributions from states which satisfy $\Delta - \frac{3}{2}r - 2j_2 = 0$. As these states are not present in long multiplets, the index only receives contributions from shortened representations of the superconformal algebra. Henceforth we will drop the R superscript. Analogous results for the left-handed index follow from the replacements $j_1 \leftrightarrow j_2$ and $r \to -r$.

The theories we consider are effectively free at large N, so multiparticle states are generated from single particle states by simple combinatorics. In AdS$_5$/CFT$_4$ this means

we can consider single-particle states in AdS which are dual to single-trace operators in the CFT. In this case the full multi-particle index $\mathcal{I}(t, y)$ is simply related to the single-particle index by means of the plethystic exponential

$$\mathcal{I}(t, y) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \mathcal{I}_{s.p.}(t^k, y^k)\right), \tag{2.7}$$

where the single-particle index is given by the same trace in (2.6), except restricted to single-particle states

$$\mathcal{I}_{s.p.}(t, y) = \text{Tr}_{s.p.}(-1)^F t^{-(\Delta - \frac{r}{2})} y^{2j_1}. \tag{2.8}$$

The single-particle superconformal index for individual short multiplets is

$$\mathcal{I}_{s.p.}^{\text{chiral}}(t, y) = -\mathcal{I}_{s.p.}^{\text{SLII}}(t, y) = (-1)^{2(j_1 + j_2)} \frac{t^{-(\Delta - \frac{r}{2})} \chi_{j_1}(y)}{(1 - t^{-1}y)(1 - t^{-1}y^{-1})}. \tag{2.9}$$

where

$$\chi_j(y) = \frac{y^{j + \frac{1}{2}} - y^{-j - \frac{1}{2}}}{y^{\frac{1}{2}} - y^{-\frac{1}{2}}},$$

is the SU(2) character.

The free-field description allows us to compute the supersymmetric Casimir energy by extracting the term that is linear in β from the single-particle index. In particular, by inserting the index for a shortened multiplet given in (2.9) into

$$E_{\text{SUSY}} = -\frac{1}{2} \lim_{\beta \to 0} \partial_\beta \mathcal{I}_{s.p.}(t = e^{(b_1 + b_2)\beta}, y = e^{(b_1 - b_2)\beta})\Big|^{\text{finite}}. \tag{2.10}$$

we recover the result in (2.2) with a and c given by (2.3).[2] Thus we can utilize the superconformal index as a method of computing the central charges. This is the main idea that we will adapt to the context of AdS$_3$/CFT$_2$ in section 4.

The result in (2.10) can alternatively be understood as a regularized sum of supersymmetric energies of states [11, 12], with the derivative with respect to β in (2.10) bringing down the supersymmetric energy of a single-particle state. In appendix B we discuss the relation between this type of regularization and the zeta-function regularization in the context of AdS$_3$/CFT$_2$. This explains why the term in the single-particle index that is linear in β is equivalent to the $\mathcal{O}(1)$ contribution to the supersymmetric Casimir energy as defined in (2.1) for large-N theories.

3 Central charge and the Casimir energy in AdS$_3$/CFT$_2$

In two-dimensional conformal field theories, the standard (i.e. non-supersymmetric) Casimir energy is simply related to the central charge by [17, 18]

$$E_c = -\frac{c}{12}. \tag{3.1}$$

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[2]For the precise numerical agreement note that the Casimir energy in (2.2) is written in terms of the full central charge, which receives contributions also from anti-chiral and SLI multiplets. These satisfy $a_{\text{chiral}}(\Delta, j, 0, r) = a_{\text{anti-chiral}}(\Delta, 0, j, -r)$ and $a_{\text{SLII}}(\Delta, j_1, j_2, r) = a_{\text{SLI}}(\Delta, j_2, j_1, -r)$ so they introduce an overall factor of two.

In this section, we explore the connection between these two quantities from a holographic point of view. As in the previous section, our focus is on the $\mathcal{O}(1)$ contributions arising from one-loop effects in the bulk. As reviewed below, both quantities can be obtained from the one-loop partition function, which can be decomposed as a sum over the Kaluza-Klein spectrum. From this point of view, it was noted in [10] that the Casimir energy/central charge relation does not generally hold for the holographic contribution of individual states in the bulk. Nevertheless, since (3.1) must hold for all two-dimensional conformal field theories, it must somehow be recovered after summing over the Kaluza-Klein tower.

Some of the best developed examples of AdS$_3$/CFT$_2$ duality have $\mathcal{N} = (4, 4)$ supersymmetry and [10] found that, with this much supersymmetry, the relation (3.1) does hold, after all. It is satisfied on an individual multiplet by multiplet basis, even before the sum over the Kaluza-Klein states is taken. However, AdS$_3$/CFT$_2$ correspondence applies more generally and it is interesting to find criteria for the viability of a holographic description. Thus we explore theories with various amounts of supersymmetry, starting with $\mathcal{N} = (0, 2)$.

3.1 The holographic Weyl anomaly

The holographic Weyl anomaly encoded in the leading central charge $c^{(0)}$ can be obtained from the logarithmically divergent part of the bulk action [1]. For our purposes it is convenient to express the result as [19, 20]

$$c^{(0)} = 6\pi\ell^3 \frac{I_{\text{on-shell}}}{\text{vol}(AdS_3)}, \tag{3.2}$$

where $I_{\text{on-shell}}$ is the classical action

$$I = \frac{1}{16\pi G_3} \int \sqrt{g}\mathcal{L}_3, \tag{3.3}$$

evaluated on-shell. For Einstein gravity with a cosmological constant the on-shell Lagrangian is given by $\mathcal{L}_3 = -R + 2\Lambda = 4/\ell^2$, which yield the Brown-Henneaux central charge $c^{(0)} = 3\ell/2G_3$.

We can generalize the classical bulk result to include quantum corrections by modifying the action $I \to I + \delta I_{\text{eff}}$ so that the central charge receives a correction

$$\delta c = 6\pi\ell^3 \frac{\delta I_{\text{eff}}}{\text{vol}(AdS_3)}. \tag{3.4}$$

Here we take the one-loop effective action

$$\delta I_{\text{eff}} = (-1)^{2s}\frac{1}{2} \log \det D, \tag{3.5}$$

where s is the spin and D is the appropriate differential operator appearing in the bulk field's kinetic term. The statistical factor $(-1)^{2s}$ takes into account the fact that fermionic and bosonic determinants go in the numerator and denominator of the partition function, respectively.

The one-loop effective action is divergent, and can be regulated by the spectral zeta function. For an AdS$_3$ representation labeled by SO$(2,2)$ quantum numbers (Δ, s) we find

$$\delta I_{\text{eff}} = -\frac{1}{2}(-1)^{2s}\zeta'(0, D)$$
$$= \frac{1}{6}(-1)^{2s}\nu(\nu^2 - 3s^2)\log R, \tag{3.6}$$

where $\nu = \Delta - 1$, and R is a cut-off scale regularizing the AdS$_3$ volume. In going to the second line we have used the result of [21] (their eq. (3.8)) for the spectral zeta function. Combining (3.4) with (3.6), and using the regulated $\text{vol}(AdS_3) = -2\ell^3\pi\log R$, we arrive at [10][3]

$$\delta c(\Delta, s) = -\frac{1}{2}(-1)^{2s}\nu(\nu^2 - 3s^2) \qquad (\nu = \Delta - 1). \tag{3.7}$$

We may also translate the AdS$_3$ labels (Δ, s) to the equivalent CFT weights h and $\bar{h}$, so that

$$\delta c(h, \bar{h}) = -\frac{1}{2}(-1)^{2s}\nu(\nu^2 - 3s^2) \qquad (\nu = h + \bar{h} - 1, \quad s = h - \bar{h}). \tag{3.8}$$

Finally, for *massless* bulk fields which have $\bar{h}$ (resp. h) equal to zero, a further ghost contribution should be included so that the combined contribution to $\delta c_{\text{massless}}$ is $\delta c(h, 0) - \delta c(h, 1)$ (resp. $\delta c(0, \bar{h}) - \delta c(1, \bar{h})$) [22].

3.2 The Casimir energy

Turning now to the Casimir energy, we take a 2D CFT point of view and start with the partition function

$$Z(q, \bar{q}) = \text{Tr}\,(-1)^{F_L + F_R}q^{L_0}\bar{q}^{\bar{L}_0}, \tag{3.9}$$

where L_0 and $\bar{L}_0$ are left and right Virasoro generators and the total fermion number can be replaced with the spin of the representation as $(-1)^{F_L + F_R} = (-1)^{2s} = (-1)^{2(L_0 - \bar{L}_0)}$. For free fields we can then formally define

$$E_c = \frac{1}{2}\text{Tr}(-1)^F(L_0 + \bar{L}_0) = \frac{1}{2}q\frac{d}{dq}Z(q, \bar{q} = q)\Big|_{q \to 1}. \tag{3.10}$$

The idea expressed by the formula is very basic, a generalization of the ground state energy $\frac{1}{2}\hbar\omega$ for the harmonic oscillator. However, as always in quantum field theory, we must address divergences. We find it convenient to introduce a regulator through the substitution $q = e^{-\epsilon}$ which renders (3.10) a meromorphic function of ϵ. We subsequently take the limit $q \to 1$ by retaining only the constant term in the Laurent expansion around $\epsilon = 0$. As a concise shorthand for this procedure we write:

$$E_c = -\frac{1}{2}\lim_{\epsilon \to 0}\frac{d}{d\epsilon}Z(e^{-\epsilon}, e^{-\epsilon})\Big|^{\text{finite}}. \tag{3.11}$$

This prescription for extracting the Casimir energy is equivalent to that defined via the spectral zeta function. Indeed, following [10], one defines

$$E_c = \frac{1}{2}(-1)^{2s}\sum_n (h_n + \bar{h}_n) = \frac{1}{2}(-1)^{2s}\zeta(-1), \tag{3.12}$$

---

[3]Note that our δc is denoted in [10] by $c_{\text{AdS}_3}^+$.

where h_n and $\bar{h}_n$ are eigenvalues of L_0 and $\bar{L}_0$ and

$$\zeta(z) = \sum_n \frac{1}{(h_n + \bar{h}_n)^z} = \frac{1}{\Gamma(z)} \int_0^\infty d\beta \, \beta^{z-1} Z(e^{-\beta}, e^{-\beta}). \tag{3.13}$$

One can refine (3.10) by taking independent q and $\bar{q}$ derivatives of the partition function. In this way, we may define the "left" and "right" contributions to the Casimir energy

$$E_c^L = -\frac{1}{2} \lim_{\epsilon \to 0} \left(\frac{d}{d\epsilon} Z(e^{-\epsilon}, \bar{q}) \right)_{\bar{q}=e^{-\epsilon}} \Big|^{\text{finite}}, \tag{3.14}$$

$$E_c^R = -\frac{1}{2} \lim_{\epsilon \to 0} \left(\frac{d}{d\epsilon} Z(q, e^{-\epsilon}) \right)_{q=e^{-\epsilon}} \Big|^{\text{finite}}. \tag{3.15}$$

In these two formulae the large round brackets serve to stress that the finite substitutions $\bar{q} \to e^{-\epsilon}$ and $q \to e^{-\epsilon}$ are performed after differentiation with respect to ϵ but before extracting the constant part of the expressions from the Laurent expansion in ϵ around $\epsilon = 0$. The prescriptions for E_c^L and E_c^R in (3.14) and (3.15) can also be extracted from the Casimir energy computed from a suitably refined zeta function. However, they do not themselves correspond precisely to the Casimir energy and are instead given by particular derivatives of the refined Casimir energy. Therefore, while it is tempting to relate E_c^L and E_c^R to the left and right-moving central charges, it is not clear that such an identification is justified. However, it follows straightforwardly from the chain rule that the sum

$$E_c = E_c^L + E_c^R, \tag{3.16}$$

reproduces the physical Casimir energy.

Until now, we have not yet specified the one-particle partition function $Z(q, \bar{q})$. For a bulk field dual to an operator with weights h and $\bar{h}$, it takes the simple form

$$Z_{h,\bar{h}}(q, \bar{q}) = \frac{q^h \bar{q}^{\bar{h}}}{(1-q)(1-\bar{q})}, \tag{3.17}$$

where the denominator arises from the $\text{SL}(2, \mathbb{R}) \times \text{SL}(2, \mathbb{R})$ descendents of the highest weight state. Performing the manipulations above, we find

$$E_c^L(h, \bar{h}) = -\frac{1}{48}(-1)^{2(h-\bar{h})} \left(1 - 10h + 18h^2 - 8h^3 + 12h\bar{h} - 12h^2\bar{h} - 6\bar{h}^2 + 4\bar{h}^3 \right),$$

$$E_c^R(h, \bar{h}) = -\frac{1}{48}(-1)^{2(h-\bar{h})} \left(1 - 10\bar{h} + 18\bar{h}^2 - 8\bar{h}^3 + 12h\bar{h} - 12h\bar{h}^2 - 6h^2 + 4h^3 \right). \tag{3.18}$$

Adding these together and replacing $h = \frac{1}{2}(\nu + s + 1)$ and $\bar{h} = \frac{1}{2}(\nu + 1 - s)$ then gives [10]

$$E_c(h, \bar{h}) = \frac{1}{24}(-1)^{2s}\nu(2\nu^2 - 1) \qquad (\nu = h + \bar{h} - 1, \quad s = h - \bar{h}). \tag{3.19}$$

As noted in [10], the central charge expression (3.8) and the Casimir energy (3.19) in general do not obey the two-dimensional CFT relation $E_c = -c/12$. This is perhaps an unusual aspect of $\text{AdS}_3/\text{CFT}_2$ holography, and we will explore this connection in more detail below. To do so, we find it convenient to define the would-be central charge

$$\delta\tilde{c} \equiv -12E_c = -12(E_c^L + E_c^R), \tag{3.20}$$

obtained from the Casimir energy. Below, we examine the relation between these two notions of the central charge for theories with varying amounts of supersymmetry.

$\mathcal{N}=(0,2)$	long ($\bar{h}>|\bar{r}|$)	short ($\bar{h}=|\bar{r}|$)
E_c^L	$-\frac{1}{8}\bar{h}$	$-\frac{1}{48}(1-6h+6h^2+3\bar{h}-6\bar{h}^2)$
E_c^R	$-\frac{1}{16}(1-2h-4\bar{h})$	$-\frac{1}{32}(1-8\bar{h}+8\bar{h}^2-2h+8h\bar{h})$
$\delta\tilde{c}$	$\frac{3}{4}(1-2h-2\bar{h})$	$\frac{1}{8}(5+12h^2+12\bar{h}^2-18h-18\bar{h}+24h\bar{h})$
δc	$\frac{3}{4}(1-2h+2\bar{h})$	$\frac{1}{8}(5+12h^2-12\bar{h}^2-18h-6\bar{h}+24h\bar{h})$

Table 1. The left and right Casimir energies, E_c^L and E_c^R, would-be central charge $\delta\tilde{c}$ and central charge δc for $\mathcal{N}=(0,2)$ multiplets. Note that $\delta\tilde{c}\neq\delta c$ for both short and long multiplets. All entries should be multiplied by the spin-dependent factor $(-1)^{2(h-\bar{h})}$.

3.3 Holography for various amounts of supersymmetry

The finite-dimensional subgroup $SO(2,2)$ of the two-dimensional conformal group splits into left and right components, $SU(1,1)\times SU(1,1)$. This can be extended by including varying amounts of supersymmetry on the left and the right, independently. We now turn to a few important cases.

3.3.1 $\mathcal{N}=(0,2)$

The $\mathcal{N}=2$ superconformal algebra corresponds to $SU(1,1|1)$, and extends the conformal algebra formed by L_0, $L_{\pm 1}$ with the supercurrents $G_{\pm 1/2}^{\pm}$ and a $U(1)$ current J_0. Lowest weight representations are labeled by the weight h and charge r, and unitary representations exist for $h\geq|r|$. Such representations fall into two categories, namely long for $h>|r|$ and short for $h=|r|$. The latter may be classified as either chiral for $h=r$ or anti-chiral for $h=-r$. The content of these representations are given by

$$\text{long:} \qquad |h,r\rangle \oplus |h+\tfrac{1}{2},r+\tfrac{1}{2}\rangle \oplus |h+\tfrac{1}{2},r-\tfrac{1}{2}\rangle \oplus |h+1,r\rangle,$$

$$\text{chiral:} \qquad |h,h\rangle \oplus |h+\tfrac{1}{2},h-\tfrac{1}{2}\rangle,$$

$$\text{antichiral:} \qquad |h,-h\rangle \oplus |h+\tfrac{1}{2},\tfrac{1}{2}-h\rangle. \tag{3.21}$$

For the $\mathcal{N}=(0,2)$ case, we take a single irreducible representation $|h\rangle$ on the left and tensor it with $|\bar{h},\bar{r}\rangle$ on the right. The resulting representations are then classified by the right-moving superalgebra, and the result for the Casimir energies and central charge are shown in table 1. As shown in the table, the would-be central charge $\delta\tilde{c}$ computed from the Casimir energy does not agree with the holographic calculation of δc.

3.3.2 $\mathcal{N}=(2,2)$

Although $\mathcal{N}=(0,2)$ supersymmetry is insufficient to give agreement between $\delta\tilde{c}$ and δc, it turns out that a match is obtained for $\mathcal{N}=(2,2)$ superconformal theories. Here, since both sides are supersymmetric, we may tensor together either long or short multiplets on both sides. This gives rise to the four possibilities shown in table 2. Importantly, the long-long representations do not contribute to either the Casimir energy nor the central charge. Moreover, E_c^L only receives contributions from shortened representations on the right,

$\mathcal{N} = (2,2)$	long-long	long-short	short-long	short-short
E_c^L	0	$-\frac{1}{16}$	0	$-\frac{1}{32}(1-4h)$
E_c^R	0	0	$-\frac{1}{16}$	$-\frac{1}{32}(1-4\bar{h})$
$\delta\tilde{c}$	0	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}(1-2h-2\bar{h})$
δc	0	$\frac{3}{4}$	$\frac{3}{4}$	$\frac{3}{4}(1-2h-2\bar{h})$

Table 2. The left and right Casimir energies, E_c^L and E_c^R, would-be central charge $\delta\tilde{c}$ and central charge δc for $\mathcal{N} = (2,2)$ multiplets. All multiplets have $\delta\tilde{c} = \delta c$. All entries should be multiplied by the spin-dependent factor $(-1)^{2(h-\bar{h})}$.

while E_c^R only receives contributions from shortened representations on the left. The fact that short representations suffice for certain computations underpins our ability to employ two-dimensional superconformal indices for efficiently packaging those computations below.

3.3.3 Small $\mathcal{N} = (0,4)$

We now turn to theories with $\mathcal{N} = 4$ supersymmetry. Here we have to make a distinction between the "small" and the "large" $\mathcal{N} = 4$. The small algebra contains $PSU(1,1|2)$ as its finite dimensional subalgebra and the large $\mathcal{A}_\gamma$ contains $D(2,1|\alpha)$.

We start with the small $\mathcal{N} = 4$ algebra, with the finite dimensional subalgebra generated by L_0, $L_{\pm 1}$, SU(2) currents J_0^i and supercurrents $G_{\pm 1/2}^a$, $\tilde{G}_{\pm 1/2}^a$ transforming as doublets under SU(2). In addition, the central charge is related to the level k of the affine SU(2) via $c = 6k$.

Lowest weight representations of the small $\mathcal{N} = 4$ algebra are built from a state $|h,j\rangle$, where j now labels the SU(2) representation. Once again, unitary representations exist for $h \geq j$, with saturation of the inequality corresponding to shortened representations. Complete representations for the small $\mathcal{N} = (0,4)$ case are then obtained by tensoring a representation $|h\rangle$ on the left with an $\mathcal{N} = 4$ representation $|\bar{h},\bar{j}\rangle$ on the right. The resulting representations are short for $\bar{h} = \bar{j}$ and long for $\bar{h} > \bar{j}$.

The resulting Casimir energies and central charges are given in table 3. We find that the small $\mathcal{N} = (0,4)$ supersymmetry is sufficient for the would-be central charge derived from the Casimir energy $\delta\tilde{c}$ to coincide with the holographic central charge δc.

3.3.4 Small $\mathcal{N} = (4,4)$

Turning to the small $\mathcal{N} = (4,4)$ case, we can build representations by tensoring together either long or short $\mathcal{N} = 4$ representations on the left and on the right. Since the Casimir energy and central charge vanishes identically whenever there is an $\mathcal{N} = 4$ long representation, whether on the left or the right, only the short-short case will contribute non-trivially. The results are summarized in table 4. The short-short result can also be obtained by decomposing the multiplet into $\mathcal{N} = (2,2)$ representations and using the results shown in table 2. Once again, we find that there is sufficient supersymmetry that $\delta\tilde{c} = \delta c$. This is the result previously reported in [10].

small $\mathcal{N} = (0,4)$	long $(\bar{h} > \bar{j})$	short $(\bar{h} = \bar{j})$
E_c^L	0	$-\frac{1}{24}(1 - 6h + 6h^2)$
E_c^R	0	$\frac{1}{8}\bar{h}(1 - 2h)$
$\delta\tilde{c}$	0	$\frac{1}{2}(1 + 6h^2 + 6h\bar{h} - 6h - 3\bar{h})$
δc	0	$\frac{1}{2}(1 + 6h^2 + 6h\bar{h} - 6h - 3\bar{h})$

Table 3. The left and right Casimir energies, E_c^L and E_c^R, would-be central charge $\delta\tilde{c}$ and central charge δc for small $\mathcal{N} = (0,4)$ multiplets labeled by $|h; \bar{h}, \bar{j}\rangle$. All multiplets have $\delta\tilde{c} = \delta c$. All entries should be multiplied by the spin-dependent factor $(-1)^{2(h-\bar{h})}$.

small $\mathcal{N} = (4,4)$	long-long	long-short	short-long	short-short
E_c^L	0	0	0	$\frac{1}{4}h$
E_c^R	0	0	0	$\frac{1}{4}\bar{h}$
$\delta\tilde{c}$	0	0	0	$-3(h + \bar{h})$
δc	0	0	0	$-3(h + \bar{h})$

Table 4. The left and right Casimir energies, E_c^L and E_c^R, would-be central charge $\delta\tilde{c}$ and central charge δc for small $\mathcal{N} = (4,4)$ multiplets. All multiplets have $\delta\tilde{c} = \delta c$. A ll entries should be multiplied by the spin-dependent factor $(-1)^{2(h-\bar{h})}$.

3.3.5 Large $\mathcal{N} = (0,4)$

We now consider the large $\mathcal{N} = 4$ cases. The large $\mathcal{N} = 4$ superalgebra contains, in addition to Virasoro, two affine SU(2)'s and a U(1) algebra. The fermionic generators include dimension-3/2 supercurrents and dimension-1/2 fields, both transforming as $(2,1) + (1,2)$ under the two SU(2)'s. The two affine SU(2)'s have levels k^+ and k^-, and the central charge is given by $c = 6k^+k^-/(k^+ + k^-)$. It is customary to define the parameters

$$\gamma = \frac{k^-}{k^+ + k^-}, \qquad \alpha = \frac{\gamma}{1 - \gamma} = \frac{k^-}{k^+}. \tag{3.22}$$

For a holographic point of view, we are mostly interested in the finite subalgebra $D(2,1|\alpha)$ with bosonic component $\mathrm{SL}(2;\mathbb{R}) \times \mathrm{SU}(2) \times \mathrm{SU}(2)$. It admits unitary representations whenever [23]

$$h \geq \gamma j^+ + (1 - \gamma)j^-, \tag{3.23}$$

Once again, representations split into long and short, with the latter saturating this bound. Using (3.18) and (3.8), we then arrive at the results shown in table 5. We see that generally the large $\mathcal{N} = (0,4)$ is insufficient to ensure $\delta\tilde{c} = \delta c$ even though it has four supersymmetries. However, the equality applies for some representations, notably those with $\bar{j}^+ = \bar{j}^-$.

large $\mathcal{N}=(0,4)$	long	short
E_c^L	0	$-\frac{1}{24}[1-6h+6h^2+6\gamma(1-\gamma)(\bar{j}^+-\bar{j}^-)^2]$
E_c^R	0	$-\frac{1}{8}(\bar{j}^+-\bar{j}^-)[(1-2\gamma)(1-2h)-4\gamma(1-\gamma)(\bar{j}^+-\bar{j}^-)]$
$\delta\tilde{c}$	0	$\frac{1}{2}[1+6h^2-6h+3(\bar{j}^+-\bar{j}^-)((1-2\gamma)(1-2h)-2\gamma(1-\gamma)(\bar{j}^+-\bar{j}^-))]$
δc	0	$\frac{1}{2}[1+6h^2-6h+3(\bar{j}^+-\bar{j}^-)((1-2\gamma)(1-2h)+2\gamma(1-\gamma)(\bar{j}^+-\bar{j}^-))]$

Table 5. The left and right Casimir energies, E_c^L and E_c^R, would-be central charge $\delta\tilde{c}$ and central charge δc for large $\mathcal{N}=(0,4)$ multiplets labeled by $|h;\bar{h},\bar{j}^+,\bar{j}^-\rangle$. We find $\delta\tilde{c}\neq\delta c$ except for special cases such as the representations $\bar{j}^+=\bar{j}^-$ or the limits $\gamma\to 0,1$. All entries should be multiplied by the spin-dependent factor $(-1)^{2(h-\bar{h})}$.

large $\mathcal{N}=(4,4)$	long-long	long-short	short-long	short-short
E_c^L	0	0	0	$-\frac{1}{4}(1-2\gamma)(j^+-j^-)$
E_c^R	0	0	0	$-\frac{1}{4}(1-2\bar{\gamma})(\bar{j}^+-\bar{j}^-)$
$\delta\tilde{c}$	0	0	0	$3[(1-2\gamma)(j^+-j^-)+(1-2\bar{\gamma})(\bar{j}^+-\bar{j}^-)]$
δc	0	0	0	$3[(1-2\gamma)(j^+-j^-)+(1-2\bar{\gamma})(\bar{j}^+-\bar{j}^-)]$

Table 6. The left and right Casimir energies, E_c^L and E_c^R, would-be central charge $\delta\tilde{c}$ and central charge δc for large $\mathcal{N}=(4,4)$ multiplets. All multiplets have $\delta\tilde{c}=\delta c$. All entries should be multiplied by the spin-dependent factor $(-1)^{2(h-\bar{h})}$.

3.3.6 Large $\mathcal{N}=(4,4)$

Finally, we examine the large $\mathcal{N}=(4,4)$ case, or more precisely representations of $D(2,1|\alpha)\times D(2,1|\bar{\alpha})$. Again, we tensor together either long or short representations on the left and the right. The results summarized in table 6 show that in this case $\delta\tilde{c}=\delta c$.

Although we allow for different parameters on the left and the right in the discussion here, the case we have in mind is string theory on $\text{AdS}_3\times S^3\times S^3\times S^1$ which has $\alpha=\bar{\alpha}$, equivalent to $\gamma=\bar{\gamma}$. Furthermore, the BPS spectrum of Type II supergravity on $\text{AdS}_3\times S^3\times S^3\times S^1$ only contains states with $j^+=j^-$ and $\bar{j}^+=\bar{j}^-$ [24, 25]. In this special case all states in the Kaluza-Klein reduction therefore give vanishing contributions to E_c^L, E_c^R and δc.

3.4 How much supersymmetry is sufficient?

As we have seen, the holographic dual to the CFT relation $\delta E_c=-\delta c/12$ (equivalent to $\delta\tilde{c}=\delta c$ in the notation used in this section) is not generally upheld on a state by state basis in the bulk. In particular, it fails for bulk states with no supersymmetry at all and it similarly fails for $\mathcal{N}=(0,2)$ multiplets.

On the other hand, it holds for all theories with four or more supercharges, with the exception of large $\mathcal{N}=(0,4)$ supersymmetry where restriction to representations with $\bar{j}^+=\bar{j}^-$ is needed. The restriction needed in the latter case may or may not be significant; it appears to be satisfied for all known models with large $\mathcal{N}=4$ supersymmetry in the bulk.

This indicates that the standard Casimir energy may not be a useful tool for computing subleading central charges in $\mathrm{AdS}_3/\mathrm{CFT}_2$ duals with too little supersymmetry.

Of course, even in theories with fewer supercharges, we still expect $\delta E_c = -\delta c/12$ to be valid once we sum over the complete bulk spectrum. However, the mechanism for this equality is far from obvious in the cases where it does not follow from supersymmetry applied to individual multiplets.

4 Supersymmetric Casimir energy in AdS$_3$

In this section we propose an improved method for computing quantum corrections to the bulk central charge in $\mathrm{AdS}_3/\mathrm{CFT}_2$ holography. The linchpin is a supersymmetric version of the Casimir energy due to bulk fluctuations. This supersymmetric Casimir energy arises naturally in the context of partition functions that preserve some amount of supersymmetry which are closely related to the superconformal index.

As we will see, in the context of chiral $\mathcal{N} = (0, 2)$ or $\mathcal{N} = (0, 4)$ theories this approach will provide a useful tool in computing the left-moving subleading central charge δc_L. In theories where both chiralities preserve two or more supercharges we are able to compute quantum corrections to both the left and right-moving central charges in a manifestly supersymmetric fashion.

4.1 The superconformal index and the supersymmetric Casimir energy

In 2D CFTs with $\mathcal{N} = (0, 2)$ supersymmetry the superconformal index is defined as

$$\mathcal{I}_R(q) = \mathrm{Tr}(-1)^{2(L_0 - \bar{L}_0)} q^{L_0} \bar{q}^{\bar{L}_0 - \bar{J}_0}, \tag{4.1}$$

where the trace is over all states in the theory on the cylinder $S^1 \times \mathbb{R}$. The index only receives contributions from states satisfying $\bar{L}_0 = \bar{J}_0$. These only appear in shortened (chiral) multiplets with respect to a particular supercharge in the superconformal algebra. The index is thus independent of $\bar{q}$ and depends only on the BPS spectrum of the theory. We use a subscript R to emphasize that the index localizes on multiplets that are shortened with respect to one of the right moving supercharges.

One can similarly define an anti-chiral index that localizes on states that are shortened with respect to the other supercharge. Also, in theories with a left-moving supersymmetry one can straightforwardly define analogous left-handed indices that localize onto shortened states of the left-moving algebra.

The index also has an interpretation as the partition function given by the Euclidean path-integral for the theory on $S^1 \times S^1$, with fermions satisfying periodic boundary conditions around the temporal circle. The trace and the partition function representations of the index are related by an overall factor

$$Z_R(q) = e^{-\beta E_{SUSY}^L} \mathcal{I}_R(q). \tag{4.2}$$

Here $q = e^{-\beta}$, where β is defined by the ratio of the length of the Euclidean time circle to the length of the spatial circle. We will often refer to β as an inverse temperature even though

we have given fermions periodic boundary conditions. The quantity E^L_{SUSY} in the exponent of the prefactor is the supersymmetric Casimir energy. We use a superscript L in order to emphasize that in free field theory it is essentially a sum of eigenvalues of the left-moving L_0 over all the states killed by the right-moving supercharge (cf. eq. (4.12) below). Assuming a unique ground state and an appropriate gap above it so that the index satisfies $\mathcal{I}_R(q = 0) = 1$, then E^L_{SUSY} can be extracted from the $\beta \to \infty$ limit of the partition function.

The supersymmetric Casimir energy has a universal structure determined by the anomaly polynomial of the SCFT. In particular, the Casimir energy determined from the right-handed partition function through (4.2) is given simply by the left-moving central charge [16]

$$E^L_{\text{SUSY}} = -\frac{c_L}{24}. \tag{4.3}$$

In theories that admit a holographic dual at large c the central charge decomposes into a sum of a leading contribution derived from the classical supergravity action and a subleading contribution that is due to one-loop determinants of fields in the bulk. Similarly, the supersymmetric Casimir energy can be decomposed as

$$E^L_{SUSY} = E^L_{(0)} + E^L_{(1)}. \tag{4.4}$$

The first term $E^L_{(0)}$ is a classical contribution from the bulk on-shell action evaluated on the AdS_3 background with an appropriate supersymmetric regularization prescription [26, 27].[4] The second term $E^L_{(1)}$ arises from one-loop contributions to the supersymmetric bulk partition function. In notation that is a natural generalization of the previous section one then has the relations

$$E^L_{(0)} = -\frac{c^{(0)}_L}{24} \tag{4.5}$$

and

$$E^L_{(1)} = -\frac{\delta c_L}{24}. \tag{4.6}$$

Our focus is on the computation of $E^L_{(1)}$ and will use (4.6) to relate it to δc_L.

In order to extract δc_L we will utilize a particular property of the index at large N. Rewriting (4.2) as

$$\mathcal{I}_R(q) = e^{\beta E^L_{SUSY}} Z_R(q), \tag{4.7}$$

and evaluating $Z_R(q)$ holographically for large c limit we see that, since the large-c index does not scale with c, the leading behavior at large c arising from $E^L_{(0)}$ should cancel against the corresponding contribution to the partition function at large c. We can then simplify this to[5]

$$\mathcal{I}^{\text{large-}c}_R(q) = e^{\beta E^L_{(1)}} Z^{\text{one-loop}}_R(q), \tag{4.8}$$

---

[4]While references [26, 27] only consider the cases of AdS$_4$ and AdS$_5$, we expect that similar results also hold for AdS$_3$.

[5]Our estimates all refer to the leading saddle point at large c and quantum fluctuations around it. Additional contributions from subleading saddles can be neglected here because their suppression is of order e^{-c}.

where $\mathcal{I}_R^{\text{large-}c}(q)$ is the $c \to \infty$ limit of the index and $Z_R^{\text{one-loop}}(q)$ refers to the contribution to the partition function from one-loop determinants of all bulk fields. At one-loop, the contributions to $Z_R(q)$ correspond to free fields in the bulk which have an interpretation as generalized free fields in the CFT.

In a high-temperature (small-β) expansion, the logarithm of the partition function for free fields has no term linear in β [28–30]. Because of its generalized free field interpretation, we expect that $\ln Z_R^{\text{one-loop}}$ similarly has no term linear in β in its small-β expansion. We can therefore compute the quantum correction $E_{(1)}^L$ to the supersymmetric Casimir energy from the index at large c and then infer the corresponding quantum correction δc_L from (4.6). We substantiate this procedure in appendix B, by demonstrating that the linear-in-β term of $\ln \mathcal{I}_R^{\text{large-}c}$ can be reduced to a sum over left-moving energies of free single-particle states (annihilated by the right-handed supercharge), and so can naturally be identified with $E_{(1)}^L$ in the bulk.

4.2 The single-particle index

In theories with a holographic description the index at large c is naturally expressed in terms of contributions from single-trace operators in the theory. For theories with $\mathcal{N} = (0, 2)$ we express it as

$$\mathcal{I}_R^{s.t.}(q) = \text{Tr}_{s.t.}(-1)^{2(L_0 - \bar{L}_0)} q^{L_0} \bar{q}^{\bar{L}_0 - \bar{J}_0}. \tag{4.9}$$

Each single-trace operator in the CFT is dual to a supergravity field, so in the bulk the trace can be thought of as being over single-particle states and we can replace "s.t." above with "s.p." While we will not use it in the following, we note for completeness that the full index (4.1) at large c is constructed from the single-particle result by taking the plethystic exponential

$$\mathcal{I}_R^{\text{large-}c}(q) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k} \mathcal{I}_R^{s.p.}(q^k)\right), \tag{4.10}$$

which takes the multi-particle contributions into account .

The supersymmetric Casimir energy can be computed from the index in the same way that the standard Casimir energy was extracted from the partition function in (3.11), viz.

$$E_{(1)}^L = -\frac{1}{2} \lim_{\beta \to 0} \frac{d}{d\beta} \mathcal{I}_R^{s.p.}(q = e^{-\beta})\Big|^{\text{finite}}. \tag{4.11}$$

We show in appendix B that this relation is equivalent to computing

$$E_{(1)}^L = \sum (-1)^{2s} \frac{1}{2} h_n \tag{4.12}$$

using zeta-function regularization. It therefore has a natural interpretation as a Casimir energy (of the supersymmetric states) in the bulk, as expected.

Combining the supersymmetric Casimir energy (4.11) with (4.6) we have our main result

$$\delta c_L = 12 \lim_{\beta \to 0} \frac{d}{d\beta} \mathcal{I}_R^{s.p.}(q = e^{-\beta})\Big|^{\text{finite}}. \tag{4.13}$$

This provides an algorithm for computing the quantum corrections to the left-moving central charge in $\mathcal{N} = (0,2)$ supersymmetric theories with a holographic dual.

To find explicit formulae, we organize bulk states into multiplets of the $\mathcal{N} = (0,2)$ algebra, which come in two types: short and long. Short chiral multiplets satisfy $\bar{L}_0 = \bar{J}_0$ and contribute to the index as

$$\mathcal{I}^{s.p.}_{R,h,\bar{h}}(q) = (-1)^{2(h-\bar{h})}\frac{q^h}{1-q}, \tag{4.14}$$

where h and $\bar{h}$ are the left and right-moving weights of the operator and the denominator arises from descendant contributions. Importantly, short anti-chiral multiplets (with $\bar{L}_0 = -\bar{J}_0$) and long multiplets (with $\bar{L}_0 > |\bar{J}_0|$) give vanishing contributions to (4.9).

Inserting the single-particle index (4.14) for a chiral $\mathcal{N} = (0,2)$ multiplet into (4.11), we find

$$E^L_{(1)}(h,\bar{h}) = -\frac{1}{24}(-1)^{2(h-\bar{h})}(1 - 6h + 6h^2), \tag{4.15}$$

which gives

$$\delta c_L(h,\bar{h}) = (-1)^{2(h-\bar{h})}(1 - 6h + 6h^2). \tag{4.16}$$

While for $\mathcal{N} = (0,2)$ theories we do not yet have a direct way of corroborating this result, when we apply it and its corresponding right-moving counterpart in theories with both left and right-moving supercharges we can compare with the Weyl anomaly results for δc. As we will see, the result in (4.16) corresponds to the sum of contributions to δc_L from a chiral and the corresponding CP conjugate anti-chiral multiplet.

4.3 Quantum corrections from one-loop Chern-Simons levels

In an effort to support the veracity of our formula (4.16) for the central charge δc_L in theories with only $\mathcal{N} = (0,2)$ supersymmetry it is useful to compute the right-moving central charge δc_R. This necessarily involves another method.

The central charge is related to the level k_R of the right-moving $U(1)_R$ current by

$$\delta c_R = -3\delta k_R, \tag{4.17}$$

where δk_R is the bulk one-loop contribution to coefficient of the Chern-Simons term

$$\delta\mathcal{L} \propto \frac{1}{4\pi}\delta k_R A_R \wedge dA_R. \tag{4.18}$$

Holographic anomaly matching gives the $F_R \wedge F_R$ term in the $U(1)_R$ anomaly. The quantum correction δk_R can be computed from one-loop vacuum polarization diagrams in three-dimensional flat space, extracting the contribution to the Chern-Simons term in the bulk. The topological nature of the Chern-Simons term implies that the flat space result is valid also in curved space. Explicit computations presented in appendix C give[6]

$$\delta k_R(s,\bar{r}) = -(-1)^{2s} 2s\,\bar{r}^2, \tag{4.19}$$

---

[6]We derive these results for spins $s = \frac{1}{2}, 1$ and $\frac{3}{2}$. While we do not consider spins greater than $\frac{3}{2}$, the simple extrapolation used here gives results consistent with expectations. It would be interesting to verify it by an explicit calculation. See also [32] for an independent computation of these Chern-Simons levels using an index theorem.

for a field of spin $s = h - \bar{h}$ and right-moving U(1)$_R$-charge $\bar{r}$. For an analogous calculation of one-loop Chern-Simons terms in five dimensions, see [31].

Summing the result (4.19) over the states in an $\mathcal{N} = (0,2)$ multiplet we find

$$\delta k_R^{\text{short}}(h, \bar{h}) = -\frac{1}{4}(-1)^{2(h-\bar{h})}(1 - 4\bar{h}^2 - 2h - 2\bar{h} + 8h\bar{h}), \tag{4.20}$$

$$\delta k_R^{\text{long}}(h, \bar{h}) = -\frac{1}{2}(-1)^{2(h-\bar{h})}(1 - 2h + 2\bar{h}). \tag{4.21}$$

and then (4.17) gives the corresponding values for δc_R. Combining these results with our formula for δc_L, taking the degeneracies arising from chiral and anti-chiral multiplets into account, we arrive at values for the total central charge

$$\delta c = \frac{1}{2}(\delta c_L + \delta c_R), \tag{4.22}$$

for both short and long multiplets. The values for the Weyl anomaly δc obtained this way agree with the results in table 1, summed over a chiral multiplet and its CP conjugate anti-chiral multiplet. Importantly, we see also that $(0,2)$ supersymmetry is enough to ensure that long supermultiplets in the bulk do not contribute to δc_L. These observations provide evidence supporting our formula (4.11) computing $E_{(1)}^L$ from the right-handed index.

However, note that there is no a priori obvious way to regularize a sum of Chern-Simons levels over a Kaluza-Klein tower (without recourse to the higher-dimensional embedding as in [32]) as in the case of quantities computed directly from the index, such as δc_L. Therefore, utilizing the results in (4.20) and (4.21) to compute such a KK sum should be done with care.

4.4 Higher amounts of supersymmetry

In the remainder of this section we present results for cases with higher amounts of supersymmetry. We will see that the case of $\mathcal{N} = (0,4)$ is similar to $\mathcal{N} = (0,2)$ discussed above and that the story simplifies when there is both a left-moving and a right-moving supersymmetry.

4.4.1 Small $\mathcal{N} = (0,4)$

In the case of $\mathcal{N} = (0,4)$ supersymmetry, the index of a short multiplet is given by the same expression (4.14) as in the $\mathcal{N} = (0,2)$ case. This means that the result for δc_L in (4.16) is also valid for short multiplets with $\mathcal{N} = (0,4)$ supersymmetry. Indeed, it matches exactly with the result from E_c^L in table 3, upon identifying E_c^L in section 3 with the $E_{(1)}^L$ defined in this section. This indicates that, for theories with at least four "small" supercharges, the somewhat ad hoc prescription pursued in section 3 is equivalent to the more principled method advanced in the current section, at least at the level of individual multiplets.

Note that for $\mathcal{N} = (0,4)$ we do not need to sum the results from table 3 over chiral and anti-chiral sectors as we did in the $\mathcal{N} = (0,2)$ case. This is because a short multiplet of $\mathcal{N} = (0,4)$ contains an entire SU(2) multiplet as its highest weight state. In particular, for each state in the highest weight representation the corresponding CP conjugate state is also included and one does not need to add the two together as in the case with $\mathcal{N} = (0,2)$.

Like in the $\mathcal{N} = (0, 2)$ case, we can check that our result for δc_L is consistent with the Weyl anomaly δc found in section 3 and presented in table 3. For this comparison we compute the Chern-Simons levels for $(0, 4)$ multiplets and find

$$\delta k_R^{\text{short}}(h, \bar{h}) = (-1)^{2(h - \bar{h})} \bar{h}(1 - 2h), \tag{4.23}$$

$$\delta k_R^{\text{long}}(h, \bar{h}) = 0, \tag{4.24}$$

with the corresponding central charges again $\delta c_R = -3\delta k_R$. Just as δc_L found here was consistent with E_c^L in table 3, the result for $\delta k_R^{\text{short}}$ is consistent with E_c^R in that table. With $\mathcal{N} = (0, 4)$ supersymmetry the long multiplets offer no further check because, unlike the $\mathcal{N} = (0, 2)$ case, the contribution from long multiplets to δk_R vanishes.

4.4.2 Large $\mathcal{N} = (0, 4)$

With large $\mathcal{N} = (0, 4)$ supersymmetry, the index of a short multiplet is again given by the same expression (4.14) as in the $\mathcal{N} = (0, 2)$ case, and hence the result for δc_L in (4.16) is also valid for short multiplets with large $\mathcal{N} = (0, 4)$ supersymmetry.

Interestingly, this result does not match with the one found using the ordinary Casimir energy E_c^L and presented in table 5, except for representations with $\bar{j}^+ = \bar{j}^-$ (or $\gamma = 0, 1$). Moreover, the discrepancies occur precisely in the situations where we had already identified problems with the CFT$_2$ relation $\delta E_c = -\delta c/12$. It is hard to escape the conclusion that the principled method using supersymmetric Casimir energy and a relation to an index is correct, while the prescription in section 3 is unreliable.

In this situation it is particularly important that we can check that the result for δc_L is consistent with the quantum correction δc to the holographic Weyl anomaly given in table 5, by performing an independent computation in Chern-Simons theory. The result for the Chern-Simons levels of the $U(1)_R$ reported in (A.12) are

$$\delta k_R^{\text{short}}(h, \bar{h}, \bar{j}_+, \bar{j}_-) = (-1)^{2(h - \bar{h})}(\bar{j}_- - \bar{j}_+)((1 - 2\gamma)(1 - 2h) - 2\gamma(1 - \gamma)(\bar{j}_+ - \bar{j}_-)), \tag{4.25}$$

$$\delta k_R^{\text{long}}(h, \bar{h}, \bar{j}_+, \bar{j}_-) = 0, \tag{4.26}$$

and again corresponds to central charge $\delta c_R = -3\delta k_R$. We find agreement with the central charge δc but generally not with the (presumably erroneous) result $\delta\tilde{c}$ that was deduced from the ordinary Casimir energy E_c^L.

As in the small $\mathcal{N} = (0, 4)$ case, with large $\mathcal{N} = (0, 4)$ the contribution from long multiplets to δk_R vanishes.[7]

4.4.3 $\mathcal{N} = (2, 2)$

When there is supersymmetry on both the left and right we can utilize the existence of a right-moving index $\mathcal{I}_R$ as well as a left-moving index $\mathcal{I}_L$ to compute both δc_L and δc_R.

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[7]This computation proceeds most easily if one first computes the sum over the $SU(2) \times SU(2)$ states in a representation of $SL(2, R) \times SU(2)^2$, giving

$$\sum_{(\bar{j}^+, \bar{j}^-)} \bar{r}^2 = \frac{1}{3}(1 + 2\bar{j}^+)(1 + 2\bar{j}^-)[\gamma^2 \bar{j}^+(1 + \bar{j}^+) + (1 - \gamma)^2 \bar{j}^-(1 + \bar{j}^-)].$$

$\mathcal{N} = (2, 2)$	long-long	long-short	short-long	short-short
δc_L	0	3	0	$\frac{3}{2}(1 - 4h)$
δc_R	0	0	3	$\frac{3}{2}(1 - 4\bar{h})$
δc	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{3}{2}(1 - 2h - 2\bar{h})$

Table 7. The left and right central charges δc_L and δc_R as derived from the SUSY Casimir energy and the central charge δc for $\mathcal{N} = (2, 2)$ multiplets. The line δc agrees with a sum of the results from table 2 over a chiral plus its CP conjugate anti-chiral multiplet. All entries should be multiplied by the spin-dependent factor $(-1)^{2(h - \bar{h})}$.

small $\mathcal{N} = (4, 4)$	long-long	long-short	short-long	short-short
δc_L	0	0	0	$-12h$
δc_R	0	0	0	$-12\bar{h}$
δc	0	0	0	$-6(h + \bar{h})$

Table 8. The left and right central charges δc_L and δc_R as derived from the SUSY Casimir energy and the central charge δc for $\mathcal{N} = (4, 4)$ multiplets. The line δc agrees with the result from table 4. All entries should be multiplied by the spin-dependent factor $(-1)^{2(h - \bar{h})}$.

large $\mathcal{N} = (4, 4)$	long-long	long-short	short-long	short-short
δc_L	0	0	0	$6[(1 - 2\gamma)(j^+ - j^-)]$
δc_R	0	0	0	$6[(1 - 2\bar{\gamma})(\bar{j}^+ - \bar{j}^-)]$
δc	0	0	0	$3[(1 - 2\gamma)(j^+ - j^-) + (1 - 2\bar{\gamma})(\bar{j}^+ - \bar{j}^-)]$

Table 9. The left and right central charges δc_L and δc_R as derived from the SUSY Casimir energy and the central charge δc for large $\mathcal{N} = (4, 4)$ multiplets. The line δc corresponds to the result from table 6. All entries should be multiplied by the spin-dependent factor $(-1)^{2(h - \bar{h})}$.

The results for δc_L and δc_R are presented in table 7. We find independent agreements with the chiral Casimir energies E_c^L and E_c^R reported in table 2, after summing the latter over a chiral multiplet plus its CP conjugate anti-chiral multiplet. This implies agreement also for the holographic Weyl anomaly

4.4.4 Small and large $\mathcal{N} = (4, 4)$

With four supercharges on both the left and the right, we only find non-zero contributions to δc_L and δc_R from short-short multiplets. The results for the small $\mathcal{N} = (4, 4)$ are summarized in table 8 and those for the large $\mathcal{N} = (4, 4)$ are similarly given in table 9. Both are completely consistent with the results from section 3, reported in table 4 and table 6, respectively.

5 Quantum corrected central charge for specific dualities

In this section we apply our formula for the quantum correction to the bulk central charge to full-fledged AdS_3/CFT_2 dualities in string theory. In these examples the AdS_3 matter content is qualitatively more complicated than the individual multiplets considered in the previous sections. We must consider infinite KK towers, corresponding to supergravity fields in higher dimensions. The resulting divergences present challenges that are addressed correctly by our formula, at least in the examples we consider.

We first study cases where the full central charge c is known from the boundary description; the challenge would then be to reproduce the $O(1)$ term in the central charge, denoted δc, from the bulk description.

In the last two subsections we move on to less-understood dualities where we will make predictions for the $O(1)$ piece of the central charges of the yet-to-be-discovered finite-c boundary CFTs.

5.1 $AdS_3 \times S^3 \times K3$

Type IIB theory on $AdS_3 \times S^3 \times K3$ has small $\mathcal{N} = (4, 4)$ supersymmetry. The dual CFT is a $(K3)^{Q_1 Q_5 + 1}/S(Q_1 Q_5 + 1)$ superconformal sigma model [33]. The seed theory with target space $K3$ has central charge $c_L = c_R = 6$ and the full symmetric orbifold has central charge $c_L = c_R = 6(Q_1 Q_5 + 1)$. The expansion parameter around the classical saddle point is $\frac{G_{\mathrm{AdS_3}}}{\ell_{\mathrm{AdS_3}}} \propto 1/Q_1 Q_5$ [34]. The CFT_2 therefore has quantum corrections $\delta c_L = \delta c_R = 6$.

The input to the AdS_3 computation of δc is the supersymmetric index. The KK towers comprising the supergravity spectrum was computed by de Boer [35, 36] who presented it in various forms including the one-particle index[8]

$$\mathcal{I}_R^{\mathrm{s.p.}}(q) = \mathcal{I}_L^{\mathrm{s.p.}}(q) = 1 + \frac{44\sqrt{q} + 28q}{1 - q}. \tag{5.1}$$

Applying the master formula (1.1) and its left-moving analogue we arrive at

$$\delta c_L = \delta c_R = -\frac{864}{\beta^2} + 6 + O(\beta) \quad (\text{as } \beta \to 0). \tag{5.2}$$

The divergence in the $\beta \to 0$ limit is due to the infinite sum over towers of Kaluza-Klein modes. Regularizing the divergence by expanding around $\beta = 0$ and keeping the finite piece we obtain the desired result $\delta c_L = \delta c_R = 6$. Our regularization is manifestly supersymmetric and similar to the one employed successfully in AdS_5/CFT_4 [7, 9]. The divergence could also be tamed more traditionally via a zeta-function regularization [10].

5.2 $AdS_3 \times S^3 \times T^4$

This case is also a duality with small $\mathcal{N} = (4, 4)$ supersymmetry.

The CFT_2 description is a σ-model on the target space $\tilde{T}^4 \times (T^4)^{Q_1 Q_5}/S(Q_1 Q_5)$. The decoupled $\tilde{T}^4$ in the target space corresponds to the quantum Higgs branch of the D1-D5 gauge theory [37]. From the viewpoint of the $U(Q_1)$ gauge theory which flows to the

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[8]This is the function $s(1, q, 1)$ in eq. (5.10) of [36] except that, since we are considering the "one"-particle index, we do not include the $+1$ vacuum contribution included in the reference.

sigma model in the IR (see e.g. [38]), the $\tilde{T}^4$ represents an $\mathcal{N} = (4,4)$ hypermultiplet decoupled from the adjoint representation (see e.g. section 4 of [39]). The $\tilde{T}^4$ corresponds in the gravity picture to two pairs of $(1/2,0)_s + (0,1/2)_s$ singletons living on the boundary of AdS$_3$. These two singletons are equivalent to the $\tilde{T}^4$ sigma model on the boundary. Therefore, as far as the $\tilde{T}^4$ factor is concerned, bulk and boundary theory are identical and their central charges match trivially: $c_R = c_L = 6$.[9]

The more interesting part of the CFT$_2$ is the $(T^4)^{Q_1 Q_5}/S(Q_1 Q_5)$ orbifold. The central charges are $c_L = c_R = 6 Q_1 Q_5$. Since the expansion parameter again is $\frac{G_{\text{AdS}_3}}{\ell_{\text{AdS}_3}} \propto 1/Q_1 Q_5$ [34] the central charges of the CFT$_2$ are not corrected by quantum effects $\delta c_L = \delta c_R = 0$.

The supersymmetric index for the excitations in the AdS$_3$ is largely trivial, due to the high symmetry of T^4. Once the $(1/2,0)_s + (0,1/2)_s$ singletons are removed from the bulk spectrum the indices turn out to be[10]

$$\mathcal{I}_R^{\text{s.p.}}(q) = \mathcal{I}_L^{\text{s.p.}}(q) = -1. \tag{5.3}$$

The prescription (1.1) thus gives the bulk quantum correction $\delta c_L = \delta c_R = 0$, in agreement with the boundary dual.

Collecting the two parts of the analysis we find the quantum corrections $\delta c_L = \delta c_R = 6$ for both bulk and boundary theory, with the nontrivial contributions from singletons and the overall $\tilde{T}^4$, respectively. BMT achieved this agreement in [10] using different methods.

5.3 AdS$_3 \times S^3 \times S^3 \times S^1$

This case is a large $\mathcal{N} = (4,4)$ duality. The dual CFT is the $Q_1 Q_5^+$-fold symmetric orbifold of a 2D CFT seed theory referred to as the $\mathcal{S}_\kappa$ theory [41] (and also [42, 43] for earlier works). For comparison with other references note that we take $\kappa \equiv \frac{Q_5^-}{Q_5^+} - 1$. The central charges of this CFT$_2$ are given by [41]

$$c_L = c_R = \frac{6 Q_1 Q_5^- Q_5^+}{Q_5^- + Q_5^+}. \tag{5.4}$$

In the semi-classical gravity regime, the D-brane charges Q_1 and Q_5^+, as well as the D5-flux Q_5^-, are simultaneously taken to infinity as $Q_1 \propto Q_5^+ \propto Q_5^- \to \infty$. Moreover, there is a constraint that $\frac{Q_5^-}{Q_5^+} \in \mathbb{Z}^{>1}$. However, the expansion parameter is $\frac{G_{\text{AdS}_3}}{\ell_{\text{AdS}_3}} \propto 1/Q_1 Q_5^+$ [42] so the leading quantum correction vanishes $\delta c_L = \delta c_R = 0$.

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[9]This situation is analogous to the one-loop Weyl anomaly matching for the U(N) $\mathcal{N} = 4$ SYM and its bulk dual containing a "doubleton" living on the AdS$_5$ boundary. Once the Weyl anomaly matching of the decoupled U(1) of the boundary and the doubleton of the bulk is established (essentially through their equivalence), it remains to show that the SU(N) $\mathcal{N} = 4$ SYM and the bulk theory without the doubleton have identical Weyl anomaly.

[10]Some authors add a $+1$ vacuum contribution to this index, so it vanishes (cf. [40]). We avoid that in order to consider a "one"-particle index. The plethystic exponentiation of (5.3) yields a vanishing multi-particle index, as expected.

In order to compute $\delta c_{L,R}$ from the bulk we start from the BPS spectrum of KK supergravity in this case [24, 25]

$$\bigoplus_{j \in \frac{1}{2}\mathbb{Z}^{\geq 0}}^{\infty} \left([j,j]_s + \left[j + \frac{1}{2}, j + \frac{1}{2} \right]_s \right) \otimes \left([j,j]_s + \left[j + \frac{1}{2}, j + \frac{1}{2} \right]_s \right), \, . \tag{5.5}$$

The notation $[j^+, j^-]_s$ indicates a short$_{j^+,j^-}$ representation of $D(2,1|\alpha)$. See the appendix for the details of the multiplet structure.

It is straightforward to check that each KK level j in the spectrum (5.5) gives a vanishing contribution to the supergravity one-particle index. Since the $j = 0$ level includes also the vacuum state whose contribution to the index we should not add, the total supergravity single-particle index comes out

$$\mathcal{I}_R^{\text{s.p.}}(q) = \mathcal{I}_L^{\text{s.p.}}(q) = -1. \tag{5.6}$$

Then (1.1) establishes the $0 = 0$ match, very similarly to the $\mathcal{N} = (4,4)$ duality of the $(T^4)^{Q_1 Q_5}/S(Q_1 Q_5)$ sigma-model.

In this case, instead of going through the index, we could observe that both on the left and on the right, the short multiplets have $j^+ = j^-$. However, subsection 3.3.6 established that such short×short multiplets do not contribute to $\delta c_{L,R}$. The $0 = 0$ agreement between quantum corrections in CFT_2 and in AdS_3 therefore follows effortlessly.

5.4 M-theory on $\text{AdS}_3 \times S^2 \times CY_3$

These compactifications are interesting because they describe large classes of 4D black holes [44]. Moreover, the duality has $\mathcal{N} = (0,4)$ supersymmetry and is the only chiral example we study in this paper.

The boundary CFT is a sigma model with left-moving central charge [44]

$$c_L = P^3 + c_2 \cdot P - 3, \tag{5.7}$$

where P is a very ample divisor in CY_3 and P^3 is the triple self-intersection number of the divisor. We have subtracted the $+3$ contribution of the three translational zero-modes of the underlying M5-brane from the result of [44]. This subtraction is analogous to removing the decoupled $\tilde{T}^4$ from the boundary target space in the duality of subsection 5.2.

Because the expansion parameter is $\frac{G_{\text{AdS}_3}}{\ell_{\text{AdS}_3}} \propto 1/P^3$ [45], we have $\delta c_L = -3$. The term that is linear in P is due to higher-derivative corrections in bulk [19], rather then one-loop effects; so it is not our concern in this work.

Since we have supersymmetry on the right, in the bulk we can compute the right-handed index of the supergravity theory, and extract δc_L from the index. To do so, we need the single-particle index of the KK supergravity. It is given in eq. (6.12) of [46]) as

$$\mathcal{I}_R^{\text{s.p.}}(q) = 2(h^{1,1} - h^{1,2})\frac{q}{(1-q)^2} - 3\frac{q}{1-q}, \tag{5.8}$$

where $h^{i,j}$ are the hodge numbers of the CY_3. Applying our master formula (1.1) we get

$$\delta c_L = \frac{-48(h^{1,1} - h^{1,2})}{\beta^3} + \frac{36}{\beta^2} - 3, \tag{5.9}$$

with the finite piece (as $\beta \to 0$) exactly reproducing the expected boundary result.

Replacing the CY_3 with T^6 or $K3 \times T^2$ gives more subtle examples of $\text{AdS}_3/\text{CFT}_2$. We leave the holographic study of quantum corrections in these examples to the future.

5.5   AdS$_3 \times (S^3 \times T^4)/G$ and AdS$_3 \times (S^3 \times K3)/\mathbb{Z}_2$

These cases are proposed $\mathcal{N} = (2,2)$ dualities [47]. The dual CFTs are not yet known at finite central charge. Therefore instead of presenting matchings, in this section we will be making predictions.

There are eight dualities in this class: seven dualities involving orbifolds of $S^3 \times T^4$, and one duality involving a $\mathbb{Z}_2$ orbifold of $S^3 \times K3$. In all the eight cases the orbifold group G rotates the S^3 such that a great circle S^1 remains fixed, but the action on the M_4 (i.e. T^4 or $K3$) is free so that the internal geometry $(S^3 \times M_4)/G$ is smooth. In the first seven cases T^4/G is a Hyperelliptic Surface (HS), while in the eighth case $K3/\mathbb{Z}_2$ is the Enriques surface (ES). The proposed infinite-c dual CFTs are infinite-fold symmetric product orbifolds with seed target spaces HS or ES, respectively.

For the seven HS cases the bulk one-particle indices are[11]

$$\mathcal{I}_R^{\text{s.p.}}(q) = \mathcal{I}_L^{\text{s.p.}}(q) = -1, \tag{5.10}$$

just like the case of AdS$_3 \times S^3 \times T^4$ without the extra singletons, or the case of AdS$_3 \times S^3 \times S^3 \times S^1$. Hence our prescription (1.1) gives the quantum correction

$$\delta c_L^{HS} = \delta c_R^{HS} = 0, \tag{5.11}$$

from the bulk side. Of course, this is disregarding possible bulk singletons that would be dual to decoupled factors in the boundary sigma model.

For the ES case, the bulk one-particle indices are nontrivial

$$\mathcal{I}_R^{\text{s.p.}}(q) = \mathcal{I}_L^{\text{s.p.}}(q) = \frac{22\sqrt{q} + 14q}{1 - q} \ . \tag{5.12}$$

where again we removed a $+1$ vacuum contribution from the result reported in [47]. Applying our master formula (1.1) we get

$$\delta c_L^{ES} = \delta c_R^{ES} = -\frac{432}{\beta^2} + 3, \tag{5.13}$$

with the finite part (as $\beta \to 0$) giving our prediction for the boundary theory: $\delta c_L^{ES} = \delta c_R^{ES} = 3$.

5.6   AdS$_3 \times (S^3 \times T^4)/D_n$

As in the previous subsection, these cases are proposed $\mathcal{N} = (2,2)$ dualities [48]. The dual CFTs are not yet known at finite central charge, so in this subsection we will again be making predictions rather than presenting agreements.

There are a total of eight distinct cases. In the notation of [48], there are two for the D_1 group (the $D_1^{(1)}$ and $D_1^{(2)}$ cases), and six more for $D_2^{(1)}$, $D_2^{(2)}$, $D_3^{(1)}$, $D_3^{(2)}$, D_4, and D_6. The internal geometry $(S^3 \times T^4)/D_n$ is singular in all the cases, so the bulk spectrum is not just KK modes from supergravity in higher dimensions, it also has a part coming from

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[11]Reference [47] reports zero because it adds a $+1$ contribution from the vacuum.

the twisted sector of IIB string theory. The proposed infinite-c dual CFTs are infinite-fold symmetric product orbifolds with seed target space T^4/D_n.

For the two dualities with D_1 group, the bulk one-particle indices are simply -1, just like the case of $\text{AdS}_3 \times S^3 \times T^4$ without the extra singletons, the case of $\text{AdS}_3 \times S^3 \times S^3 \times S^1$, or the seven HS examples. Therefore, just as in those cases (1.1) gives

$$\delta c_L^{D_1} = \delta c_R^{D_1} = 0, \tag{5.14}$$

for the quantum correction computed from the bulk side. Of course, this is assuming no extra bulk singletons that would be dual to possible decoupled factors in the boundary sigma model.

For the remaining six dualities the superconformal index for single particle excitations in the bulk are nontrivial. It is given by eq. (6.33) of [41] as

$$\mathcal{I}_R^{\text{s.p.}}(q) = \frac{4(1+a)q^{1/4} + 22\sqrt{q} + 4(1+a)q^{3/4} + (18+4a)q}{1-q}, \tag{5.15}$$

after setting the parameter in that work z to zero, and again removing the $+1$ vacuum contribution included there. The parameter a takes different values $\{1, -1/2, -1, -1, 1/4, -1/2\}$ for the six distinct bulk geometries that we are considering. Our prescription (1.1) for the quantum correction to the central charge gives

$$\delta c_L = \delta c_R = -\frac{144(4+a)}{\beta^2} + 6\left(1 + \frac{a}{2}\right). \tag{5.16}$$

The finite piece, evaluated for the appropriate values of a, gives our predictions

$$\delta c_L = \delta c_R = 6(1 + a/2) = \{9, 9/2, 3, 3, 27/4, 9/2\}. \tag{5.17}$$

for the six boundary theories that are based on orbifold groups $D_2^{(1)}$, $D_2^{(2)}$, $D_3^{(1)}$, $D_3^{(2)}$, D_4, D_6.

6 Summary and outlook

The main result (1.1) of this article is an efficient prescription for the holographic computation of quantum corrections to central charges in the supersymmetric $\text{AdS}_3/\text{CFT}_2$ correspondence. Our one-loop (i.e. $\mathcal{O}(G^0_{\text{AdS}_3})$) formula follows from the relation between the central charges and the supersymmetric Casimir energies, as extracted from the high-temperature behavior of the bulk one-particle superconformal indices.

For theories with non-chiral supersymmetry where the left- and right-moving central charges are equal, our results agree with those of BMT [10], at the level of individual bulk multiplets. However, when considering complete Kaluza-Klein towers of states our method is advantageous because our regularization of the infinite KK sums is completely unambiguous and manifestly supersymmetric.

We also leveraged the notion of supersymmetric Casimir energy to motivate our formula even in cases with chiral supersymmetry. With "small" $\mathcal{N} = (0, 4)$ SUSY our results

again coincide with those of BMT [10] at the level of individual bulk multiplets, but not necessarily for entire KK towers. For multiplets of $\mathcal{N} = (0, 2)$ or even multiplets of "large" $\mathcal{N} = (0, 4)$ (for representations with $\bar{j}^+ \neq \bar{j}^-$) we find disagreement.

Although it is possible that agreement would be restored after summing over the whole bulk spectrum, it is not obvious that this can be realized without a supersymmetric prescription for the sum. It would therefore be interesting to test our formula (1.1) for the KK spectra in instances of $\mathcal{N} = (0, 2)$ and $\mathcal{N} = (0, 4)$ AdS$_3$/CFT$_2$, such as those in [49–51].

Among the concrete outcomes of the present work were the predictions in subsection 5.5 for the quantum corrections to central charges in the dualities recently discovered by Eberhardt [47]. Initial investigations seem to confirm our predictions, and reveal novel aspects of AdS$_3$/CFT$_2$ at the quantum level [52].

A more ambitious question left for future work is whether a version of our main result (1.1) can be found that applies at finite-N. Such a formula would relate the exact (left-handed) central charge of any 2D SCFT to its full (right-handed) superconformal index. A simple finite-N version would be

$$c_L \overset{?}{=} 12 \lim_{\beta \to 0} \frac{\mathrm{d}}{\mathrm{d}\beta} \mathrm{PL}\big(\mathcal{I}^R(q = e^{-\beta})\big), \tag{6.1}$$

where PL refers to the plethystic logarithm and $\mathcal{I}^R$ is the full right-handed SCFT index. A simple calculation shows that this proposal implies that the term that is linear in β in the asymptotic small-β expansion of $\ln \mathcal{I}^R$ is equal to $-\beta c_L / 24$.

Similar finite-N prescriptions were conjectured for 4d SCFTs in [9], but were later found to be invalid for SO(3) SQCD with two flavors [53]. This example shows that such finite-N formulas do not necessarily apply when the Rains function of the 4d SCFT has flat directions [53]. Such flat directions appear in turn related to unlifted Coulomb branches in the crossed channel — i.e. on $R^3 \times S^1$ [53].

Analogously, we expect that (6.1) may be violated when certain gap conditions on the spectrum of the 2D SCFT are not met. With appropriate gap conditions, on the other hand, one can establish the proposal (6.1). For example, for $\mathcal{N} = (2, 2)$ SCFTs with a gap in the spectrum above $L_0 = J_0$ and with finite degeneracies (so that the "low-temperature" asymptotics of the index is dominated by the contribution of the states with $L_0 = J_0$), it is a simple exercise to show that the modular properties of the elliptic genus imply (6.1). It is an interesting prospect that, ultimately, such gap conditions might have connections with the sparsity criteria (such as [54, 55]) for a 2D SCFT having a holographic dual in the first place.

Acknowledgments

A Some 2D superconformal representation theory

In this appendix, we present a brief overview of superconformal representations used in the body of the paper. Our starting point is the $\mathcal{N} = 2$ superconformal algebra, which can be further extended to yield the small and large $\mathcal{N} = 4$ algebras. The two-dimensional conformal algebra decomposes into left- and right-moving sectors, and for the most part we will focus on a single copy. However, we also report the contribution of individual representations to the right-handed superconformal index

$$\mathcal{I}_R(q) = \mathrm{Tr}(-1)^{2L_0 - 2\bar{L}_0} q^{L_0} \bar{q}^{\bar{L}_0 - \bar{J}_0}. \tag{A.1}$$

When doing so, we will consider theories with equal amounts of supersymmetry on both sides. Note that this index assumes at least $\mathcal{N} = (0,2)$ supersymmetry, with L_0 and J_0 defined by the $\mathcal{N} = 2$ algebra given below. The contributions to $\tilde{\mathcal{I}}_R(q) = \mathrm{Tr}(-1)^{2L_0 - 2\bar{L}_0} q^{L_0} \bar{q}^{\bar{L}_0 + \bar{J}_0}$ and the left-handed indices (assuming supersymmetry on the left) are obtained similarly.

A.1 The $\mathcal{N} = 2$ superconformal algebra

The $\mathcal{N} = 2$ superconformal algebra corresponds to $SU(1,1|1)$, and it has a bosonic subgroup $SU(1,1) \times U(1)$. The global part of the algebra is given by L_0, $L_{\pm 1}$, $G^+_{\pm 1/2}$, $G^-_{\pm 1/2}$, J_0, with corresponding (anti-)commutation relations

$$
\begin{aligned}
[L_0, L_{\pm 1}] &= \mp L_{\pm 1}, & [L_1, L_{-1}] &= 2L_0, \\
[L_0, G^+_{\pm 1/2}] &= \mp \frac{1}{2} G^+_{\pm 1/2}, & [L_0, G^-_{\pm 1/2}] &= \mp \frac{1}{2} G^-_{\pm 1/2}, \\
[L_1, G^{\pm}_{-1/2}] &= G^{\pm}_{1/2}, & [L_{-1}, G^{\pm}_{1/2}] &= -G^{\pm}_{-1/2}, \\
\{G^+_{\pm 1/2}, G^-_{\pm 1/2}\} &= 2L_{\pm 1}, & \{G^+_{\pm 1/2}, G^-_{\mp 1/2}\} &= 2(L_0 \pm J_0), \\
[J_0, G^{\pm}_{1/2}] &= \pm \frac{1}{2} G^{\pm}_{1/2}, & [J_0, G^{\pm}_{-1/2}] &= \pm \frac{1}{2} G^{\pm}_{-1/2}.
\end{aligned}
\tag{A.2}
$$

Lowest weight representations are built from the state $|h, r\rangle$ where h and r are the L_0 and J_0 eigenvalues, respectively. The full representation is then obtained by acting with a combination of the creation operators L_{-1}, $G^+_{-1/2}$ and $G^-_{-1/2}$.

Representations of the $\mathcal{N} = 2$ superconformal algebra. Unitary irreducible representations exist for $h \geq |r|$. The case $h > |r|$ gives rise to an $\mathcal{N} = 2$ long representation built from four $\mathcal{N} = 0$ representations

$$|h, r\rangle, \quad |h + \tfrac{1}{2}, r + \tfrac{1}{2}\rangle = G^+_{-1/2}|h, r\rangle, \quad |h + \tfrac{1}{2}, r - \tfrac{1}{2}\rangle = G^-_{-1/2}|h, r\rangle, \quad |h+1, r\rangle = G^+_{-1/2} G^-_{-1/2}|h, r\rangle. \tag{A.3}$$

When $h = |r|$, we end up with a shortened representation. The chiral representation corresponds to starting with $|h, h\rangle$, in which case $G^+_{-1/2}$ creates a null state. As a result, the chiral multiplet is generated by L_{-1} and $G^-_{-1/2}$. The $\mathcal{N} = 0$ decomposition is

$$|h, h\rangle, \qquad |h + \tfrac{1}{2}, h - \tfrac{1}{2}\rangle = G^-_{-1/2}|h, h\rangle. \tag{A.4}$$

Finally, the antichiral multiplet is built from

$$|h, -h\rangle, \qquad |h + \tfrac{1}{2}, \tfrac{1}{2} - h\rangle = G^+_{-1/2}|h, -h\rangle. \tag{A.5}$$

Contribution to the index. Focusing on the right-handed index (A.1), it is easy to see that long and antichiral representations on the right do not contribute. Hence only chiral multiplets on the right contribute, giving a factor $(-1)^{-2\bar{h}}$ for the $|\bar{h}, \bar{h}\rangle$ multiplet. As a result, we have

$$\mathcal{I}_R(q)[\text{any} \times \text{chiral}_{\bar{h}}] = \text{Tr}_L(-1)^{2(L_0 - \bar{h})} q^{L_0}, \tag{A.6}$$

where the trace is taken over the left-moving representation. For $\mathcal{N} = (2, 2)$ supersymmetry, the only non-vanishing contributions are then

$$\mathcal{I}_R(q)[\text{long}_{h,r} \times \text{chiral}_{\bar{h}}] = (-1)^{2(h-\bar{h})} \frac{(1 - \sqrt{q})^2 q^h}{1 - q},$$

$$\mathcal{I}_R(q)[\text{short}_h \times \text{chiral}_{\bar{h}}] = (-1)^{2(h-\bar{h})} \frac{(1 - \sqrt{q}) q^h}{1 - q}, \tag{A.7}$$

where the short representation on the left can be either chiral or antichiral. It is now straightforward to check that the prescription (1.1) acting on $\mathcal{I}_R(q)$ successfully reproduces the result of table 7.

A.2 The small $\mathcal{N} = 4$ superconformal algebra

The $\mathcal{N} = 4$ superconformal algebra corresponds to PSU(1, 1|2), with bosonic subgroup SU(1, 1)×SU(2). The global part of the algebra is given by L_0, $L_{\pm 1}$, $G^a_{\pm 1/2}$, $\tilde{G}^a_{\pm 1/2}$ and J^i_0. Lowest weight representations are built from the state $|h, j, m = j\rangle$, where h is the L_0 eigenvalue and j, m are the SU(2) quantum numbers. This state is annihilated by L_1, $G^a_{1/2}$, $\tilde{G}^a_{1/2}$, J^i_1 and J^+_0. Descendants are built using the creation operators L_{-1}, $G^a_{-1/2}$, $\tilde{G}^a_{-1/2}$ as well as the SU(2) generators J^i_0.

Representations of the small $\mathcal{N} = 4$ superconformal algebra. Representations come in two forms, long (massive, with $h > j$) and short (massless, with $h = j$). The long representations are generated by acting with all four supercharge creation operators $G^a_{-1/2}$ and $\tilde{G}^a_{-1/2}$, transforming as $\mathbf{2} + \bar{\mathbf{2}}$ of SU(2). A generic long representation is built from 16 $\mathcal{N} = 0$ representations, all transforming as representations of SU(2)

$$\begin{aligned}
&|h, j\rangle, \\
&2|h + \tfrac{1}{2}, j + \tfrac{1}{2}\rangle, \qquad 2|h + \tfrac{1}{2}, j - \tfrac{1}{2}\rangle, \\
&|h + 1, j + 1\rangle, \qquad 4|h + 1, j\rangle, \qquad |h + 1, j - 1\rangle, \\
&2|h + \tfrac{3}{2}, j + \tfrac{1}{2}\rangle, \qquad 2|h + \tfrac{3}{2}, j - \tfrac{1}{2}\rangle, \\
&|h + 2, j\rangle.
\end{aligned} \tag{A.8}$$

These representations decompose into a number of long $\mathcal{N} = 2$ multiplets, and hence do not contribute to the index.

A short representation of the $\mathcal{N} = 4$ superconformal algebra has $h = j$. Starting from $|j, j, m = j\rangle$, we can see that $G^2_{-1/2}$ and $\tilde{G}^1_{-1/2}$ generate null states. We thus use $G^1_{-1/2}$ and $\tilde{G}^2_{-1/2}$ to generate descendants. The short multiplet contains

$$|j, j\rangle, \qquad 2|j + \frac{1}{2}, j - \frac{1}{2}\rangle, \qquad |j + 1, j - 1\rangle. \tag{A.9}$$

In particular, these short representations decompose into one chiral, one antichiral, and a number of long $\mathcal{N} = 2$ multiplets. Therefore if such an $\mathcal{N} = 4$ short representation is present on the right sector, it contributes to $\mathcal{I}_R$ just as its constituent $\mathcal{N} = 2$ chiral multiplet does, namely $(-1)^{-2\bar{h}}$.

Contribution to the index. The right-handed index only receives contributions from shortened representations on the right

$$\mathcal{I}_R(q)[\text{any} \times \text{short}_{\bar{h}}] = \text{Tr}_L (-1)^{2(L_0 - \bar{h})} q^{L_0}. \tag{A.10}$$

For the non-chiral $\mathcal{N} = (4, 4)$ algebra, we have

$$\mathcal{I}_R(q)[\text{long}_{h,r} \times \text{short}_{\bar{h}}] = (-1)^{2(h - \bar{h})} \frac{(2h + 1)(1 - \sqrt{q})^3 q^h}{1 + \sqrt{q}},$$

$$\mathcal{I}_R(q)[\text{short}_h \times \text{short}_{\bar{h}}] = (-1)^{2(h - \bar{h})} \frac{(1 + \sqrt{q} + 2h(1 - \sqrt{q})) q^h}{1 + \sqrt{q}}. \tag{A.11}$$

The $\text{short}_h \times \text{short}_{\bar{h}}$ representations of the $\mathcal{N} = (4, 4)$ algebra are sometimes denoted as $(h, \bar{h})_s$. It is straightforward to check that the prescription (1.1) acting on the above successfully reproduces the result of table 8.

A.3 The large $\mathcal{N} = 4$ superconformal algebra

The bulk supergravity symmetry group for the duality involving the large $\mathcal{N} = (4, 4)$ SCA is $D(2, 1|\alpha) \times D(2, 1|\alpha)$. Therefore we focus our discussion on a single copy of $D(2, 1|\alpha)$. The bosonic subgroup of $D(2, 1|\alpha)$ is $\text{SL}(2, R) \times \text{SU}(2) \times \text{SU}(2)$. Therefore three quantum numbers h, j^+, j^- will be used to label its lowest weight representations. Note that the large $\mathcal{N} = 4$ algebra has an extra U(1), but only the singlet sector of that U(1) is of interest to us. Hence the corresponding quantum number never appears in our discussion.

For comparison with the better known $\mathcal{N} = 2$ algebra, note that $D(2, 1|\alpha)$ has an $\mathcal{N} = 2$ subgroup, whose $\text{SL}(2, R)$ part coincides with that of $D(2, 1|\alpha)$, and whose U(1) part is given by the combination

$$J^{\mathcal{N}=2} = \gamma J^+_z + (1 - \gamma) J^-_z, \tag{A.12}$$

of the third-component generators of the two SU(2)'s inside $D(2, 1|\alpha)$, where $\gamma = \alpha/(1 + \alpha)$.

Representations of $D(2, 1|\alpha)$. Unitary representations of $D(2, 1|\alpha)$ satisfy the BPS inequality

$$h \geq \gamma j^+ + (1 - \gamma) j^-, \tag{A.13}$$

$$\begin{aligned}
&|h, j^+, j^-\rangle, \\
&|h+\tfrac{1}{2}, j^+ +\tfrac{1}{2}, j^- +\tfrac{1}{2}\rangle, \quad |h+\tfrac{1}{2}, j^+ +\tfrac{1}{2}, j^- -\tfrac{1}{2}\rangle, \quad |h+\tfrac{1}{2}, j^+ -\tfrac{1}{2}, j^- +\tfrac{1}{2}\rangle, \quad |h+\tfrac{1}{2}, j^+ -\tfrac{1}{2}, j^- -\tfrac{1}{2}\rangle, \\
&|h+1, j^+ +1, j^-\rangle, |h+1, j^+, j^- +1\rangle, 2|h+1, j^+, j^-\rangle, |h+1, j^+ -1, j^-\rangle, |h+1, j^+, j^- -1\rangle, \\
&|h+\tfrac{3}{2}, j^+ +\tfrac{1}{2}, j^- +\tfrac{1}{2}\rangle, \quad |h+\tfrac{3}{2}, j^+ +\tfrac{1}{2}, j^- -\tfrac{1}{2}\rangle, \quad |h+\tfrac{3}{2} j^+ -\tfrac{1}{2}, j^- +\tfrac{1}{2}\rangle, \quad |h+\tfrac{3}{2}, j^+ -\tfrac{1}{2}, j^- -\tfrac{1}{2}\rangle, \\
&|h+2, j^+, j^-\rangle.
\end{aligned}$$

Table 10. The long representations of $D(2,1|\alpha)$, where $h > \gamma j^+ + (1-\gamma)j^-$.

$$\begin{aligned}
&|h, j^+, j^-\rangle, \\
&|h+\tfrac{1}{2}, j^+ +\tfrac{1}{2}, j^- -\tfrac{1}{2}\rangle, \quad |h+\tfrac{1}{2}, j^+ -\tfrac{1}{2}, j^- +\tfrac{1}{2}\rangle, \quad |h+\tfrac{1}{2}, j^+ -\tfrac{1}{2}, j^- -\tfrac{1}{2}\rangle, \\
&|h+1, j^+, j^-\rangle, \qquad\qquad |h+1, j^+ -1, j^-\rangle, \qquad |h+1, j^+, j^- -1\rangle, \\
&|h+\tfrac{3}{2}, j^+ -\tfrac{1}{2}, j^- -\tfrac{1}{2}\rangle.
\end{aligned}$$

Table 11. The short representations of $D(2,1|\alpha)$, where $h = \gamma j^+ + (1-\gamma)j^-$.

with short representations saturating the bound. The long representations of $D(2,1|\alpha)$ are shown in table 10, and the short representations are shown in table 11. Note that only short representations with $j^+ = j^- = j$ appear in the spectrum of string theory on $\text{AdS}_3 \times S^3 \times S^3 \times S^1$ [24, 25]. For such representations, the BPS relation becomes $h = j$.

Contribution to the index. Long representations of $D(2,1|\alpha)$ can be decomposed into a number of long $\mathcal{N} = 2$ multiplets, and hence do not contribute to the right-handed superconformal index. On the other hand, short representations of $D(2,1|\alpha)$ can yield short $\mathcal{N} = 2$ multiplets, and hence can contribute to the index. To see how this arises, we compare the $D(2,1|\alpha)$ shortening condition, $h = \gamma j^+ + (1-\gamma)j^-$, with the $\mathcal{N} = 2$ shortening condition, $h = |r|$. Using the relation (A.12) between the $\mathcal{N} = 2$ current and the $D(2,1|\alpha)$ currents then demonstrates that a short $D(2,1|\alpha)$ representation contains one chiral and one anti-chiral $\mathcal{N} = 2$ multiplet, along with a set of long $\mathcal{N} = 2$ multiplets. The chiral (anti-chiral) multiplet corresponds to the state within $|h, j^+, j^-\rangle$ where the third components of the two SU(2)'s are both maximal (minimal).

As a result, the contribution of a short $D(2,1|\alpha)$ multiplet on the right to $\mathcal{I}_R(q)$ takes the form

$$\mathcal{I}_R(q)[\text{any} \times \text{short}_{\bar{j}^+, \bar{j}^-}] = \text{Tr}_L(-1)^{2(L_0 - \bar{h})} q^{L_0}. \tag{A.14}$$

The contribution of $D(2,1|\alpha) \times D(2,1|\alpha)$ representations to the right-handed index can then be determined to be

$$\mathcal{I}_R(q)[\text{long}_{h,j^+,j^-} \times \text{short}_{\bar{j}^+, \bar{j}^-}] = \frac{(-1)^{2(h-\bar{h})}(2j^+ +1)(2j^- +1)(1-\sqrt{q})^4 q^h}{1-q},$$

$$\mathcal{I}_R(q)[\text{short}_{j^+, j^-} \times \text{short}_{\bar{j}^+, \bar{j}^-}] \tag{A.15}$$

$$= \frac{(-1)^{2(h-\bar{h})}(1 + 2(j^+ + j^-)(1-\sqrt{q}) + 4j^+ j^-(1-\sqrt{q})^2 + \sqrt{q})q^h}{1 + \sqrt{q}}.$$

Once again, we can verify that (1.1) acting the index successfully reproduces the result of table 9.

B Zeta function representation of the large-c index and the supersymmetric Casimir energy

In the main text we have focused on the relation between the supersymmetric Casimir energy and the single-particle supersymmetric index. The single-particle index is advantageous for practical computations but the full multi-particle index would perhaps be conceptually more satisfying. Happily, the relation between the two is particularly simple in the large-c limit. In this appendix we show how the supersymmetric Casimir energy as given in (4.12) naturally appears as a linear-in-β term in the natural logarithm of the full multi-particle large-c index. Additionally, we show how this term is related to the standard definition of the Casimir energy in terms of a regulated sum over energy eigenvalues which appears as the zero temperature contribution to the free energy. We stress that we are not presenting a CFT_2 discussion in this appendix, but studying the AdS_3 side.

B.1 Supersymmetric Casimir energy from the index

Starting from the large-c single-particle index in (4.9), the full index is given by enumerating all multi-particle states which arise as products of single-particle states. This is accomplished by taking the Plethystic exponential as in (4.10), which we reproduce here

$$\mathcal{I}_R^{\text{large-c}}(q) = \exp\left(\sum_{k=1}^{\infty} \frac{1}{k}\mathcal{I}_R^{s.p.}(q^k)\right). \tag{B.1}$$

To simplify the notation we will omit the large-c superscript in the following.

For concreteness, consider an $\mathcal{N} = (0,2)$ chiral multiplet (cases with higher amounts of supersymmetry can be treated similarly). The single-particle index (4.14) of such a chiral multiplet is

$$\mathcal{I}_{R,h,\bar{h}}^{s.p.}(q) = (-1)^{2(h-\bar{h})}\frac{q^h}{1-q} = (-1)^{2(h-\bar{h})}\sum_{n=0}^{\infty} q^{h+n}. \tag{B.2}$$

Inserting this into (B.1), the natural logarithm of the large-c index is given by the series

$$\ln \mathcal{I}_{R,h,\bar{h}}(e^{-\beta}) = (-1)^{2(h-\bar{h})}\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{1}{k}e^{-k\beta(h+n)}, \tag{B.3}$$

where we have set $q = e^{-\beta}$.

In order to extract the small-β asymptotics of this expression, we follow Cardy [56] and write the exponential as the inverse Mellin transform of the Gamma function

$$e^{-x} = \frac{1}{2\pi i}\int_{\mathcal{C}} x^{-s}\Gamma(s)\,ds, \tag{B.4}$$

where the contour runs to the right of and parallel to the imaginary axis. Inserting this into the logarithm of the index we have

$$\ln \mathcal{I}_{R,h,\bar{h}}(e^{-\beta}) = (-1)^{2(h-\bar{h})}\frac{1}{2\pi i}\int_{\mathcal{C}}\beta^{-s}\zeta(s+1)\Gamma(s)\sum_{n=0}^{\infty}(h+n)^{-s}\,ds, \tag{B.5}$$

where $\zeta(s+1)$ comes from the sum over k in (B.3). The contour is now to the right of $\mathrm{Re}\,s = 1$ where the integrand is convergent. This contour can be closed by deforming the upper ($\mathrm{Im}\,s > 0$) and lower ($\mathrm{Im}\,s < 0$) segments up and down towards the negative real axis from above and below. Cauchy's theorem then gives the small-β expansion

$$\ln \mathcal{I}_{R,h,\bar{h}}(e^{-\beta}) = (-1)^{2(h-\bar{h})} \left(\frac{1}{\beta} \frac{\pi^2}{6} + \zeta'(h; 0) + \beta \frac{1}{2} \zeta(h; -1) + \cdots \right), \qquad \text{(B.6)}$$

where

$$\zeta(h; s) = \sum_{n=0}^{\infty} (h + n)^{-s}, \qquad \text{(B.7)}$$

is a Hurwitz zeta function. The dots in (B.6) refer to higher powers in β that arise from the infinite set of poles in the Γ-function at $s = -n$ for integer $n > 1$. These terms are subleading for small β.

We are particularly interested in the term that is linear in β. It is due to the pole of the integrand in (B.5) at $s = -1$ which arises from the Γ-function. To find its coefficient we have used $\mathrm{Res}(\Gamma, -1) = -1$ as well as $\zeta(0) = -1/2$.[12] We now recognize that the coefficient of β in (B.6) is nothing but the sum (4.12) over the left-handed quantum zero-energies of the supersymmetric AdS$_3$ modes in the multiplet, and thus it gives $E_{(1)}^L(h, \bar{h})$ as claimed. Of course, the Hurwitz zeta function that appears in this term is elementary so we can evaluate the coefficient of β in (B.6) explicitly and find

$$\frac{1}{2}(-1)^{2(h-\bar{h})} \zeta(h; -1) = -\frac{1}{24}(-1)^{2(h-\bar{h})}(1 - 6h + 6h^2) \ . \qquad \text{(B.8)}$$

This matches our result in (4.15) that was derived from the single-particle index using the prescription (4.11).

That the coefficient of β in the small-β expansion of $\ln \mathcal{I}_R$ should be $E_{(1)}^L$ follows from our prescription (4.11) as well; the reader might convince themselves of this using (B.1), or more carefully using the relation between the small-β asymptotics of the single-particle and multi-particle indices presented in appendix D of [57].

To summarize this appendix so far, we have shown that

- the supersymmetric Casimir energy $E_{(1)}^L$ that we obtained from a single-particle-index regularization (4.11) of the expression (4.12), also arises as the coefficient of β in the small-β asymptotic expansion of logarithm of the bulk multi-particle index;

- at the level of individual bulk multiplets, the index prescription (4.11) and the zeta-function regularization (B.8) amount to the same result for the expression (4.12).

---

[12]To get the other terms shown in (B.6) we have used $\mathrm{Res}(\Gamma, 0) = 1$ and $\zeta(2) = \pi^2/6$ for the β^{-1} term. For the β^0 term, we note that the Laurent expansion of $\zeta(s+1)\Gamma(s) = 1/s^2 + \mathcal{O}(s^0)$ near $s = 0$, which gives rise to the derivative $\zeta'(0; h) \equiv \frac{\partial}{\partial s} \zeta(s; h)|_{s=0}$ appearing in the residue of the pole at this point.

B.2 Supersymmetric Casimir energy as a sum over normal mode energies

We now want to show that the SUSY Casimir energy in (B.8) corresponds to a sum over *supersymmetric energies* of states in the bulk chiral multiplet.

To see this note that a field in global AdS$_3$ with conformal weight $(h, \bar{h})$ has quantized normal mode energies [58]

$$E_{n,l}(h, \bar{h}) = \frac{2n + |l| + h + \bar{h}}{\ell_3}, \qquad n = 0, 1, 2, \ldots, \ l = 0, \pm 1, \pm 2, \ldots, \tag{B.9}$$

where l is the "angular" momentum around the spatial circle, n is a radial mode number and ℓ_3 is the AdS radius. We will set $\ell_3 = 1$ in the following.

In the supersymmetric context that we consider, these energies are modified due to the presence of a constant background U(1)$_R$ gauge field that is required in order for the bulk partition function in global AdS$_3$ to correspond to the supersymmetric partition function (or elliptic genus) [46].[13] This constant background U(1)$_R$ gauge field is a flat connection so it simply shifts the energies by an amount equal to the R-charge such that the supersymmetric energies become

$$E_{n,l}^{SUSY}(h, \bar{h}; j) = 2n + |l| + h + \bar{h} - \bar{r}, \tag{B.10}$$

where n and l have the same ranges as in (B.9) and $\bar{r}$ is the U(1)$_R$ charge of the field.

Each entry of the oscillator spectrum (B.10) contributes the usual vacuum energy $\frac{1}{2}\hbar\omega$. The supersymmetric Casimir energy follows as a renormalized sum over these contributions so using zeta-function regularization we find:

$$E^{SUSY}(h, \bar{h}; \bar{r}) := \frac{1}{2}(-1)^{2(h-\bar{h})} \sum_{\kappa=0}^{\infty} (\kappa + 1)(\kappa + h + \bar{h} - \bar{r})^{-s} \Big|_{s=-1}. \tag{B.11}$$

We have inserted an extra $(-1)^{2(h-\bar{h})}$ in order to count bosons with a plus sign and fermions with a minus sign. Also, in (B.11) we have replaced the double sum on n and l with a single sum that takes into account the appropriate degeneracy of each energy level.

For the $\mathcal{N} = (0, 2)$ chiral multiplet in (A.4), the U(1)$_R$ charge is related to the weight by $\bar{r} = \bar{h}$, so we have

$$E^{ch}(h, \bar{h}; \bar{r} = \bar{h}) = \frac{1}{2}(-1)^{2(h-\bar{h})} \sum_{\kappa=0}^{\infty} \left[(\kappa + 1)(\kappa + h)^{-s} - (\kappa + 1)(\kappa + h + 1)^{-s} \right]_{s=-1}. \tag{B.12}$$

After shifting the summation index in the second term, this can be written

$$E^{ch}(h, \bar{h}) = \frac{1}{2}(-1)^{2(h-\bar{h})} \sum_{\kappa=0}^{\infty} (\kappa + h)^{-s} \Big|_{s=-1}, \tag{B.13}$$

which is equal to the SUSY Casimir energy defined from the index in (B.6).

On the other hand, one can check that for the long $\mathcal{N} = (0, 2)$ multiplet in (A.3) the sum over the supersymmetric energies vanishes due to cancellation between bosons and fermions.

---

[13]See also [26, 27] for the corresponding statements in AdS$_5$.

C Computation of one-loop Chern-Simons terms

In this appendix we collect some details of the computation of the Chern-Simons levels claimed in (4.18) of section 4, for spins less than two. It amounts to computing gauge boson vacuum polarization diagrams in flat space with charged fields running in the loop and extracting the parity-odd term that survives in the low-energy limit. We proceed directly to the vacuum polarization computation and only then providing the full context. The main result is given in equation (4.19), where the spin s is given from the results below by identifying $\operatorname{sign}(m) = \frac{s}{|s|}$ [59].

The generation of Chern-Simons couplings in the three-dimensional effective action from one-loop vacuum polarization diagrams due to spin-1/2 fermions was first found in [60, 61], which we generalize to fields of spin-1 and spin-3/2 and conjecture an extrapolation to higher spins. The analysis of this section follows the methods and regularization of [31] where the analogous calculation in five dimensions is performed.

C.1 Details on the vacuum polarization diagrams

The integrand of the one-loop diagrams we will consider are all of the form

$$\Sigma^{\mu\nu}(p) = \int \frac{d^3 k}{(2\pi)^3} \frac{\mathbb{N}^{\mu\nu}(p, k; m)}{(k^2 + m^2)((k-p)^2 + m^2)}, \tag{C.1}$$

where p^μ is the external photon momentum and k^μ is the loop momentum.

Here we will go through some of the details in extracting the relevant terms in the numerator factors. In particular, since we are interested in extracting the parity odd term we can restrict to the terms in the numerator of the form

$$\mathbb{N}^{\mu\nu}(p, k; m) = f(k^2; m)\epsilon^{\mu\nu\rho}p_\rho. \tag{C.2}$$

In particular, we can discard terms with higher powers of p as well as those that do not include the Levi-Civita tensor.

C.1.1 Spin-1/2

For fermions, the relevant parity odd terms arise from traces of γ matrices. The spin-1/2 propagator is

$$\Delta^{1/2}(p) = \frac{-\not{p} + im}{p^2 + m^2} \tag{C.3}$$

and the interaction vertex is simply

$$V_{1/2}^\mu = q\gamma^\mu. \tag{C.4}$$

From these, we construct the numerator for a spin-1/2 field in the loop to be

$$\begin{aligned}
\mathbb{N}_{1/2}^{\mu\nu}(p, k; m) &= -\operatorname{Tr}\left[(-q\gamma^\mu)(-\not{k} + im)(-q\gamma^\nu)(-(\not{k} - \not{p}) + m)\right] \\
&\simeq q^2\left[-\operatorname{Tr}(\gamma^\mu\gamma^\rho\gamma^\nu)(imk_\rho) - \operatorname{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho)(im(k_\rho - p_\rho))\right] \\
&\simeq 2imq^2\left[-\epsilon^{\mu\rho\nu}k_\rho - \epsilon^{\mu\nu\rho}(k_\rho - p_\rho)\right] \\
&\simeq 2imq^2\epsilon^{\mu\nu\rho}p_\rho. \tag{C.5}
\end{aligned}$$

Putting this all together we find the following parity-odd contribution to the effective action

$$
\begin{aligned}
\Sigma_{1/2}^{\mu\nu} e_\mu e_\nu &= (-) \int \frac{d^3k}{(2\pi)^3} \operatorname{Tr}\left((-q\gamma^\mu) \frac{-\slashed{k}+im}{k^2+m^2}(-q\gamma^\nu)\frac{-(\slashed{k}-\slashed{p})+im}{(k-p)^2+m^2} \right) \\
&\simeq -2imq^2 p_\rho e_\mu e_\nu \epsilon^{\rho\mu\nu} \int \frac{d^3k}{(2\pi)^3} \frac{1}{(k^2+m^2)((k-p)^2+m^2)} \\
&\simeq -\frac{i}{2\pi^2}\frac{m}{|m|}q^2 \Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2},\Lambda^{-2}\right) p_\rho e_\mu e_\nu \epsilon^{\rho\mu\nu} \\
&\simeq \frac{i}{4\pi}\operatorname{sign}(m)q^2 p_\rho e_\mu e_\nu \epsilon^{\rho\mu\nu}.
\end{aligned} \tag{C.6}
$$

C.1.2 Spin-1

For bosons, the parity odd term arises from explicit ϵ tensors in both the propagator and the vertex. For spin-1, these are

$$
\begin{aligned}
\Delta_{\mu\nu}^1(p) &= -i\left(m\eta_{\mu\nu} + \frac{1}{m}p_\mu p_\nu - i\epsilon_{\mu\nu\gamma}p^\gamma \right), \\
V_1^{\nu\mu\lambda} &= -q\epsilon^{\nu\mu\lambda}.
\end{aligned} \tag{C.7}
$$

These give the numerator

$$
\begin{aligned}
\mathbb{N}_1^{\mu\nu}(p,k;m) &= (-q\epsilon^{\lambda\mu\sigma})\Delta_{\lambda\rho}^1(k)(-q\epsilon^{\rho\nu\gamma})\Delta_{\gamma\sigma}^1(k-p), \\
&\simeq -imq^2\epsilon^{\mu\nu\sigma}p_\sigma\left(1 - \frac{k^2}{m^2}\right).
\end{aligned} \tag{C.8}
$$

From which we compute

$$
\begin{aligned}
\Sigma_1^{\mu\nu} e_\mu e_\nu &= (+) \int \frac{d^3k}{(2\pi)^3} \frac{(-q\epsilon^{\lambda\mu\sigma})\Delta_{\lambda\rho}^1(k)(-q\epsilon^{\rho\nu\gamma})\Delta_{\gamma\sigma}^1(k-p)}{(k^2+m^2)((k-p)^2+m^2)} \\
&\simeq -imq^2 p_\rho e_\mu e_\nu \epsilon^{\rho\mu\nu} \int \frac{d^3k}{(2\pi)^3}\left(1 - \frac{k^2}{m^2}\right)\frac{1}{(k^2+m^2)((k-p)^2+m^2)} \\
&\simeq -\frac{i}{4\pi^2}\frac{m}{|m|}q^2\left(\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2},\Lambda^{-2}\right) - \Gamma\left(\frac{5}{2}\right)\Gamma\left(-\frac{1}{2},\Lambda^{-2}\right)\right) p_\rho e_\mu e_\nu \epsilon^{\rho\mu\nu} \\
&\simeq \frac{i}{2\pi}\left(\frac{3\Lambda}{4\sqrt{\pi}} - 1\right)\operatorname{sign}(m)q^2 p_\rho e_\mu e_\nu \epsilon^{\rho\mu\nu}.
\end{aligned} \tag{C.9}
$$

C.1.3 Spin-3/2

For massive spin-3/2 fields in the loop, we have

$$
\begin{aligned}
\Delta_{\mu\nu}^{3/2}(p) &= \left(\eta_{\mu\nu} + \frac{p_\mu p_\nu}{m^2}\right)(-\slashed{p}+im) + \frac{1}{2}\left(\gamma_\mu - \frac{ip_\mu}{m}\right)(-\slashed{p}-im)\left(\gamma_\nu - \frac{ip_\nu}{m}\right), \\
V^{\nu\mu\lambda} &= -q\gamma^{\nu\mu\lambda}.
\end{aligned} \tag{C.10}
$$

These give a numerator of

$$
\mathbb{N}_{3/2}^{\mu\nu}(p,k;m) = -\operatorname{Tr}\left[(-q\epsilon^{\lambda\mu\sigma})\Delta_{\lambda\rho}^{\frac{3}{2}}(k)(-q\epsilon^{\rho\nu\gamma})\Delta_{\gamma\sigma}^{\frac{3}{2}}(k-p) \right]. \tag{C.11}
$$

To simplify this, schematically write

$$\Delta^{\frac{3}{2}}_{\mu\nu}(p) = A_{\mu\nu}(p) + B_{\mu\nu}(p), \tag{C.12}$$

where $A_{\mu\nu}(p)$ refers to the first term in (C.10) and $B_{\mu\nu}(p)$ to the second. The numerator can then be written

$$\mathbb{N}^{\mu\nu}_{3/2}(p, k; m) = -q^2 \epsilon^{\lambda\mu\sigma}\epsilon^{\rho\nu\gamma}\text{Tr}\left[\left(A_{\lambda\rho}(k) + B_{\lambda\rho}(k)\right)\left(A_{\gamma\sigma}(k-p) + B_{\gamma\sigma}(k-p)\right)\right]. \tag{C.13}$$

Evaluating the various terms in the product independently we find

$$\epsilon^{\lambda\mu\sigma}\epsilon^{\rho\nu\gamma}\text{Tr}\left[A_{\lambda\rho}(k)A_{\gamma\sigma}(k-p)\right] \simeq 0,$$

$$\epsilon^{\lambda\mu\sigma}\epsilon^{\rho\nu\gamma}\text{Tr}\left[A_{\lambda\rho}(k)B_{\gamma\sigma}(k-p)\right] \simeq -im\epsilon^{\mu\nu\rho}p_\rho\left(1 + \frac{k^2}{m^2}\right),$$

$$\epsilon^{\lambda\mu\sigma}\epsilon^{\rho\nu\gamma}\text{Tr}\left[B_{\lambda\rho}(k)A_{\gamma\sigma}(k-p)\right] \simeq -im\epsilon^{\mu\nu\rho}p_\rho\left(1 + 3\frac{k^2}{m^2}\right),$$

$$\epsilon^{\lambda\mu\sigma}\epsilon^{\rho\nu\gamma}\text{Tr}\left[B_{\lambda\rho}(k)B_{\gamma\sigma}(k-p)\right] \simeq 2im\epsilon^{\mu\nu\rho}p_\rho\left(1 + \frac{k^2}{m^2}\right). \tag{C.14}$$

Adding everything together we find the numerator is given by

$$\mathbb{N}^{\mu\nu}_{3/2}(p, k; m) = -2imq^2 k^2 \epsilon^{\mu\nu\rho}p_\rho. \tag{C.15}$$

Putting it altogether, we find the effective parity odd term

$$\begin{aligned}
\Sigma^{\mu\nu}_{3/2}e_\mu e_\nu &= (-)\int \frac{d^3k}{(2\pi)^3}\frac{\text{Tr}\left[(-q\epsilon^{\delta\mu\rho})\Delta^{\frac{3}{2}}_{\delta\sigma}(k)(-q\epsilon^{\lambda\nu\sigma})\Delta^{\frac{3}{2}}_{\lambda\rho}(k-p)\right]}{(k^2+m^2)((k-p)^2+m^2)}\\
&\simeq -2imq^2 p_\rho e_\mu e_\nu \epsilon^{\rho\mu\nu}\int \frac{d^3k}{(2\pi)^3}\frac{k^2}{m^2}\frac{1}{(k^2+m^2)((k-p)^2+m^2)}\\
&\simeq -\frac{i}{2\pi^2}\frac{m}{|m|}q^2\Gamma\left(\frac{5}{2}\right)\Gamma\left(-\frac{1}{2}, \Lambda^{-2}\right)p_\rho e_\mu e_\nu \epsilon^{\rho\mu\nu}\\
&\simeq -\frac{i}{2\pi}\left(\frac{3\Lambda}{2\sqrt{\pi}} - \frac{3}{2}\right)\text{sign}(m)q^2 p_\rho e_\mu e_\nu \epsilon^{\rho\mu\nu}.
\end{aligned} \tag{C.16}$$

C.1.4 Extrapolation to arbitrary spin

The results above all have a finite term given by

$$\Sigma^{\mu\nu}_{|s|}e_\mu e_\nu = -(-1)^{2s}\frac{i}{2\pi}s\, q^2 p_\rho e_\mu e_\nu \epsilon^{\rho\mu\nu}, \tag{C.17}$$

where s is the spin of the field appearing in the loop, which is related to the mass by $m = \frac{s}{|s|}$. Given the chiral nature of the massive higher spin fields in three-dimensions [59], it is reasonable to suspect that charged massive higher spin fields will continue to follow this pattern. This suspicion is corroborated by the results in section 4. It would be very interesting to verify this conjectured structure with an explicit calculation, but we will leave this for future work.

C.2 Momentum integrals

In order to compute the momentum integrals we have utilized the Schwinger parametrization of the integrand. Here are some expressions relevant for this.

Since we are only interested in terms in the effective action of the form $p_\rho e_\mu e_\nu \epsilon^{\rho\mu\nu}$, we can restrict to the terms from the loop integrals which are independent of the external momentum p.

One can rewrite the denominator as

$$\frac{1}{(k^2 + m^2)((k-p)^2 + m^2)} = \int_0^\infty du \int_0^\infty dv \; e^{-(u+v)(\ell^2 + \bar{m}^2)}, \tag{C.18}$$

where we have defined the shifted momentum ℓ and the shifted mass $\bar{m}^2$ as

$$\ell = k - \frac{v}{u+v}p,$$
$$\bar{m}^2 = m^2 + \frac{uv}{(u+v)^2}p^2. \tag{C.19}$$

Since we are interested in the p-independent part of the integral, we drop the p dependence of these.

The integral over the momentum can now be computed by evaluating the Gaussian integral

$$\int \frac{d^3\ell}{(2\pi)^3} \ell^{2n} e^{-\frac{x\ell^2}{m^2}} = \frac{|m|^{2n+3}}{4\pi^2} \frac{\Gamma\left(n + \frac{3}{2}\right)}{(u+v)^{n+3/2}}. \tag{C.20}$$

Putting this all together we have

$$\int \frac{d^3k}{(2\pi)^3} \frac{k^{2n}}{(k^2 + m^2)((k-p)^2 + m^2)} = \frac{1}{2\pi^2} \frac{m^{2n}}{|m|} \Gamma\left(n + \frac{3}{2}\right) \int du\, dv\; \Theta\left(u + v - \Lambda^{-2}\right) \frac{e^{-(u+v)}}{(u+v)^{n+3/2}}$$
$$= \frac{1}{4\pi^2} \frac{m^{2n}}{|m|} \Gamma\left(n + \frac{3}{2}\right) \Gamma\left(-n + \frac{1}{2}, \Lambda^{-2}\right). \tag{C.21}$$

In the last line we used

$$\int_0^\infty du \int_0^\infty dv\; \Theta\left(u + v - \Lambda^{-2}\right) \frac{e^{-(u+v)}}{(u+v)^\alpha} = \Gamma\left(2 - \alpha, \Lambda^{-2}\right). \tag{C.22}$$

The parameter Λ is inserted as a UV cut-off. In this way the Schwinger variables act as a heat kernel regulator of the loop integral and Λ provides a UV cut-off for integration over the heat variables u and v. One can then perform the integral and expand in large-Λ. The regularization procedure is to simply discard the infinite terms proportional to positive powers of Λ.

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