



Superconformal indices on $S^1 \times (S^5/\mathbb{Z}_p)$

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Abstract

We obtain generating functions associated to the abelian superconformal indices for 6d $(1, 0)$ tensor and hypermultiplets on $S^1 \times (S^5/\mathbb{Z}_p)$. We extract the superconformal indices and their high and low temperature behaviors. We consider round and generically squashed S^5 in turn. We show that the unsquashed limit of the superconformal indices is smooth. We examine S-duality in the large p limit that acts by exchanging the Hopf circle with the temporal circle.

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1. Introduction

The partition function for 5d SYM was first computed perturbatively (i.e. by suppressing instanton contributions) on a round S^5 in [4], [5]. Later this result was extended to the instanton sector in [7], [8] by regularizing with generic squashing parameters a, b, c subject to the relation $a + b + c = 0$. For abelian gauge group, the 5d partition function including the instanton sector, was matched with the 6d superconformal index on $S^1 \times S^5$ in [8], providing supporting evidence for the M5/D4 correspondence [2], [3].

It remains difficult to generalize the abelian tests of the M5/D4 correspondence to nonabelian gauge groups (except for the case of infinite rank gauge group where we may use the AdS/CFT

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correspondence) since that requires a definition of the nonabelian M5 brane theory itself.¹ But we can also try to generalize the test to other geometries while keeping the gauge group abelian. As a first step in that direction, in this paper we will generalize the computation of the abelian 6d superconformal index to $S^1 \times (S^5/\mathbb{Z}_p)$. Keeping $(2, 0)$ supersymmetry of the M5 brane leads to a very restricted class of possible 6d geometries. But these lens space geometries belong to that class. The lens spaces have non-trivial topology that is detected by the Ray-Singer torsion. In [15] we found a mismatch (a factor that is related to the Ray-Singer torsion) between the 6d index and the 5d partition function, for the maximally topologically twisted 5d maximally supersymmetric Yang-Mills. This mismatch was traced to the nontrivial circle reduction of the selfdual two-form from 6d to 5d. It is then natural to ask whether one can find the same type of mismatch also between the 6d $(1, 0)$ theories on $\mathbb{R} \times (S^5/\mathbb{Z}_p)$ and 5d SYM on S^5/\mathbb{Z}_p .

The abelian 6d superconformal index on $S^1 \times (S^5/\mathbb{Z}_p)$ has been computed previously in [10]. We reproduce that result in eq. (5.5). We then use this result to extract its high temperature expansion, which is what we should get when we compute the partition function for 5d SYM theory on S^5/\mathbb{Z}_p . On the other hand, in [10] the low temperature expansion of this index was found to match with the partition function of 5d SYM theory on $\mathbb{R} \times \mathbb{C}P^2$.

The abelian M5 brane superconformal index on $S^1 \times S^5$ was first obtained in [1] by using radial quantization. We reproduced and generalized this in [13] by using Hamiltonian quantization. Two types of squashings were considered, squashing of the fiber and of the base manifold respectively. These squashings were also independently found in [12]. Three squashing parameters A, B, C are introduced associated to the three Cartan rotations of $SO(6)$, which is the isometry group of S^5 . The trace parameter

$$h = \frac{1}{3}(A + B + C)$$

squashes the Hopf fiber of S^5 . The three traceless squashing parameters

$$a = A - h$$

$$b = B - h$$

$$c = C - h$$

which are subject to the constraint $a + b + c = 0$, squashes the base manifold $\mathbb{C}P^2$. We consider a mass deformed M5 brane theory that we will call $(2, 0)^*$ theory. Mass deformation breaks supersymmetry by half and the $(2, 0)$ tensor multiplet splits into one $(1, 0)$ tensor multiplet and one $(1, 0)$ hypermultiplet with a mass parameter m_H . The R-symmetry is $SU(2)_R$ and the flavor symmetry is $SU(2)_F$. We have the following commuting charges: three Cartans j_i of $SO(6)$, one Cartan $R_1 + R_2$ of $SU(2)_R$, one Cartan $R_1 - R_2$ of $SU(2)_F$. We can associate one chemical potential to each of these Cartan generators while preserving one supercharge $Q_{j_1 j_2 j_3}^{R_1 R_2} = Q_{---}$ with the $SO(6)$ Cartan charges $j_i = -1/2$, R charges $R_1 = R_2 = -1/2$ and scaling dimension $\Delta = 1/2$. The following operators commute with Q_{---} ,

$$\mathcal{O}_1 = \Delta + \frac{1}{2}(R_1 + R_2)$$

¹ However, some recent tests have been made in this direction. The half-BPS index has been computed for A_k 5d SYM by localization in [10] and corresponding expressions for the D_k and E_k gauge groups have been conjectured in [11]. These half-BPS indices have been recently deconstructed for the A_k and the D_k series from corresponding 4d quiver theories in [16], [17]. I thank A. Bourget for bringing these references to my attention.

$$\mathcal{O}_2 = R_1 - R_2$$

$$\mathcal{O}_3 = j_1 - j_2$$

$$\mathcal{O}_4 = j_2 - j_3$$

$$\mathcal{O}_5 = j_1 + j_2 + j_3 - 2R_1 - R_2$$

In [1], [8], [7] the charge \mathcal{O}_5 that is associated with squashing of the Hopf fiber was not included. The existence of \mathcal{O}_5 was first noted in [6], [10], although there again only four independent charges were again included in the 6d superconformal index. But we can have five independent mutually commuting charges that commute with $Q_{\overline{1}\overline{2}\overline{3}}$. In particular the generator \mathcal{O}_5 is crucial in this paper when we consider the theory on a lens space. We assign chemical potentials to each of these charges and define the superconformal index as

$$I(\beta, \omega, a, b, c, m_H) = \text{tr}(-1)^F \omega^{\mathcal{O}_5} e^{-\beta(\mathcal{O}_1 + m_H \mathcal{O}_2 + a j_1 + b j_2 + c j_3)}$$

Generating functions associated to this index are [12], [13]

$$f_{\text{tensor}}(\beta, \omega, a, b, c) = \frac{e^{-3\beta} \omega^3 - e^{-2\beta} \omega^2 (e^{\beta a} + e^{\beta b} + e^{\beta c})}{(1 - \omega e^{-\beta(1+a)})(1 - \omega e^{-\beta(1+b)})(1 - \omega e^{-\beta(1+c)})} \tag{1.1}$$

$$f_{\text{hyper}}(\beta, \omega, a, b, c, m_H) = \frac{e^{-\frac{3}{2}\beta} (\omega e^{\beta m_H} + \omega^2 e^{-\beta m_H})}{(1 - \omega e^{-\beta(1+a)})(1 - \omega e^{-\beta(1+b)})(1 - \omega e^{-\beta(1+c)})} \tag{1.2}$$

for the tensor and hypermultiplets respectively. Here m_H is a parameter that determines the hypermultiplet mass and we will put the radius of S^5 to be $r = 1$.

We will obtain the abelian indices on $\mathbb{R} \times (S^5/\mathbb{Z}_p)$ by Fourier transforming with respect to the chemical potential ω . We do this first for the special case $a = b = c = 0$ in section 4, and later in full generality in section 5 and obtain the results presented in eqs. (5.9) and (5.11). We dualize the corresponding indices and obtain their high temperature expansions by using either one of three different dualization methods: zeta function regularization and the Abel-Plana formula method following [9] for the unsquashed case $a = b = c = 0$, and the plethystic dualization method following [8] for the squashed case with generic a, b, c . We then show that the unsquashed limit $a, b, c \rightarrow 0$ is both well-defined (independent of how we take the limit) and smooth. In the limit, we recover the previously obtained results for the unsquashed case with $a = b = c = 0$. We summarize these three dualization methods in three appendices. Finally, in section 6 we test an asymptotic S-duality conjecture [14] in the limit of large p .

2. Supersymmetry enhancement

We define the M5 brane generating function

$$f_{M5} = f_{\text{tensor}} + f_{\text{hyper}}$$

But for generic mass parameter m_H this is a mass deformed version of the M5 brane. It has the same field content as the M5 brane, but not enough supersymmetry to make this a theory of a single $(2, 0)$ tensor multiplet. We call this a $(2, 0)^*$ theory. The preserved supercharges can be easily counted for each new chemical potential we insert into the index. Supercharges that preserve \mathcal{O}_1 have charges such that $R_1 + R_2 = -1$, hence they are $Q_{\overline{j_1 \overline{j_2 j_3}}}$. There are 8 such supercharges. This is the amount of supersymmetry that we have in the $(2, 0)^*$ theory for generic m_H since these supercharges also commute with \mathcal{O}_2 whose chemical potential is m_H . But let us

now consider the linear combination $\mathcal{O}_1 + m_H \mathcal{O}_2$ that appears in the index. The condition for this to commute with the supercharges, is that

$$\frac{1}{2} + \left(\frac{1}{2} + m_H\right) R_1 + \left(\frac{1}{2} - m_H\right) R_2 = 0$$

For generic values of m_H the only solution is $R_1 = R_2 = -1/2$. But for $m_H = 1/2$ we only need $R_1 = -1/2$ while R_2 can be either of $\pm 1/2$. This means we have enhancement of supersymmetry from 8 to 16 supercharges at $m_H = 1/2$. A similar enhancement of supersymmetry happens at $m_H = -1/2$ where instead R_1 can be either of $\pm 1/2$ and $R_2 = -1/2$. Let us now turn on \mathcal{O}_5 . Then for $m_H = 1/2$ we find the four preserved supercharges, $Q_{---}^{--}, Q_{+--}^{+-}, Q_{-+-}^{+-}, Q_{--+}^{-+}$ and for $m_H = -1/2$ we find the four preserved supercharges $Q_{---}^{--}, Q_{+--}^{+-}, Q_{+--}^{+-}, Q_{--+}^{-+}$. Had we instead defined \mathcal{O}_5 the symmetric way as

$$\mathcal{O}_5^{sym} = j_1 + j_2 + j_3 - \frac{3}{2}(R_1 + R_2)$$

we would only get one preserved supercharge Q_{---}^{--} when we turn on the chemical potential for \mathcal{O}_5^{sym} . We would like to preserve as much supersymmetry as possible for each new chemical potential that we turn on, and therefore we will not define \mathcal{O}_5 in the symmetric way.

If we pick $m_H = 1/2$ and $a = b = c = 0$, then we find a simplification also when we turn on the chemical potential ω (corresponding to \mathcal{O}_5). The generating function becomes

$$f_{M5}(\beta, \omega) = \frac{\omega e^{-\beta}}{1 - \omega e^{-\beta}}$$

This simplification would not occur had we instead used \mathcal{O}_5^{sym} , but also we do not get this simplification if we take $m_H = -1/2$ despite then again we have four supercharges. Instead, the simplification at $m_H = -1/2$ occurs if we replace \mathcal{O}_5 with $\mathcal{O}_5^{\vee} = j_1 + j_2 + j_3 - R_1 - 2R_2$. We normalized \mathcal{O}_5 such that $e^{2\pi i \mathcal{O}_5} = 1$ on bosonic states. This gives the quantization condition $\mathcal{O}_5 \in \mathbb{Z}$. By lensing, we will identify the rotation by the angle $2\pi/p$ with the identity operator, and this gives the quantization condition $\mathcal{O}_5 \in p\mathbb{Z}$. To project out all the integer modes except multiples of p , we put $\eta = e^{2\pi i/p}$ and introduce the projection operator

$$\frac{1}{p} \sum_{\ell=0}^{p-1} \eta^\ell = \sum_{\ell \in \mathbb{Z}} \delta_{n, \ell p}$$

Thus we get the generating function on $\mathbb{R} \times (S^5/\mathbb{Z}_p)$ as

$$f_{M5}(\beta, p) = \frac{1}{p} \sum_{\ell=0}^{p-1} f_{M5}(\beta, \eta^\ell)$$

Lensing by using the generator \mathcal{O}_5 leads to the lens space $L(p; 1, 1)$. For the lens space $S^5/\mathbb{Z}_p = L(p; q_1, q_2)$ we shall replace \mathcal{O}_5 with the operator

$$\mathcal{O}_{q_1, q_2} = q_1 j_1 + q_2 j_2 + j_3 - (q_1 + q_2) R_1 - R_2$$

Then for $m_H = 1/2$ we have generically only two conserved supercharges $Q_{---}^{--}, Q_{--+}^{-+}$, and for $m_H = -1/2$ we preserve $Q_{---}^{--}, Q_{+--}^{+-}$. It is easy to compute the generating function on $\mathbb{R} \times L(p; 1, 1)$ for $m_H = 1/2$. The result is

$$f_{M5}(\beta, p) = \frac{e^{-p\beta}}{1 - e^{-p\beta}}$$

Lensing for the (1, 0) tensor multiplet does not depend on the choice of m_H nor on the detailed definition of \mathcal{O}_5 , whether we use $\mathcal{O}_5, \mathcal{O}_5^\vee, \mathcal{O}_5^{sym}$ or some other combination. This is simply because all fields in the (1, 0) tensor multiplet are neutral with respect to the R-charges R_1, R_2 . But for the hypermultiplet, lensing will depend on the choice of \mathcal{O}_5 . Does this mean that lensing leads to an ambiguous result for the hypermultiplet? We will now explain that the answer is either yes or no depending on the amount of supersymmetry that we like to preserve. When we put a theory on a curved manifold, we always need to specify the amount of supersymmetry that we like to preserve. At the point where $a = b = c = 0$ and $m_H = 1/2$, the preserved supercharges may tell us how we shall choose \mathcal{O}_5 so that there is no ambiguity in the lensing procedure. In this paper we will consider the (2, 0)* theory that has four preserved supercharges $Q_{----}^-, Q_{+---}^+, Q_{-+-+}^+, Q_{--++}^-$ at the point specified by $a = b = c = 0$ and $m_H = 1/2$. This specification uniquely tells us that we shall use the operator \mathcal{O}_5 to define the superconformal index to squash the Hopf fiber, rather than \mathcal{O}_5^\vee or some other combination. If we would like to preserve less than four supercharges at this particular point, then we will find a certain ambiguity in how to pick \mathcal{O}_5 . But this is a common situation. When we lower the amount of supersymmetry we always find more freedom in how to set up a corresponding theory that preserves that amount of supersymmetry.

3. From generating function to superconformal index

If we have a spectrum of discrete energy levels, then this may be encoded in a generating function

$$f(\beta) = \sum_n d_n e^{-\beta E_n}$$

where E_n are the energy levels with degeneracies d_n and β is the inverse temperature. (To get the generating function of an index, we count degeneracies of fermionic fields with a minus sign.) The index is an infinite product

$$I(\beta) = \prod_{\ell \in \mathbb{Z}} \prod_n (a_\ell^2 + E_n^2)^{-d_n/2}$$

where

$$a_\ell = \frac{2\pi \ell}{\beta}$$

The product is divergent and needs to be regularized. We may use zeta function regularization and define [9]

$$\zeta(s) = \mu^{2s} \sum_n \sum_{\ell \in \mathbb{Z}} d_n (a_\ell^2 + E_n^2)^{-s} \tag{3.1}$$

where μ is an energy scale that we need to insert in order for $\zeta(s)$ to be dimensionless. Then the index will be given by

$$I(\beta) = e^{-\beta F(\beta)} = e^{\frac{1}{2}\zeta'(0)} \tag{3.2}$$

In [9] it was shown by a direct computation that (3.2) can be rewritten as a low temperature expansion

$$I(\beta) = e^{-\beta E} \prod_n (1 - e^{-\beta E_n})^{-d_n} \quad (3.3)$$

where E is the Casimir energy. The nice feature with the derivation in [9], is that the Casimir energy factor appears automatically and does not need to be multiplied by hand. If we take the logarithm, we get

$$\begin{aligned} \log I(\beta) &= -\beta E - \sum_n d_n \log(1 - e^{-\beta E_n}) \\ &= -\beta E + \sum_n \sum_{k=1}^{\infty} \frac{1}{k} d_n e^{-k\beta E_n} \end{aligned}$$

If we exchange the sums, the result can be expressed as the plethystic sum of the generating function

$$\log I(\beta) = -\beta E + \sum_{k=1}^{\infty} \frac{1}{k} f(k\beta)$$

The plethystic sum may involve a divergent sum in certain applications. Then, even if this divergent sum is canceled out in the dualization process that takes us from the low temperature expansion (3.3) to a high temperature expansion, we may nevertheless miss out some logarithmic terms (or prefactors of the superconformal index). Another drawback is that the plethystic method can only be applied to generating functions $f(\beta)$ that are antisymmetric, $f(\beta) = -f(-\beta)$. The advantage with the plethystic method is that it leads to simpler computations and in particular it is easy to extract the Stefan-Boltzmann terms from the generating function, which is much more involved to do by using the other methods.

4. Turning off squashing parameters

In this section we put the squashing parameters to zero, $a = b = c = 0$, and obtain the generating functions on $S^1 \times (S^5/\mathbb{Z}_p)$ by Fourier transforming with respect to the chemical potential ω . We also obtain the high temperature expansions of the corresponding indices by using zeta function dualization method, the Abel-Plana dualization method, and the plethystic dualization method. We outline each of these dualization methods in three appendices.

4.1. The (1, 0) tensor multiplet index

The generating function for the (1, 0) tensor multiplet on $\mathbb{R} \times S^5$ is given by

$$f_{\text{tensor}}(\beta, \omega) = \frac{\omega^3 e^{-3\beta} - 3\omega^2 e^{-2\beta}}{(1 - \omega e^{-\beta})^3}$$

where we just turn on one chemical potential $\omega = e^{-\beta h}$ that corresponds to a rotation along the Hopf fiber (and h was introduced in the Introduction). By noting that

$$\frac{1}{(1 - e^{-\beta})^3} = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} e^{-\beta n}$$

we can write this as

$$f_{tensor}(\beta, \omega) = \sum_{n=1}^{\infty} (1 - n^2) \omega^n e^{-\beta n}$$

To obtain the generating function on the lens space $S^5/\mathbb{Z}_p = L(p; 1, 1)$, we sum over all ω that are taken as the p different p -th roots of unity. This sum amounts to projecting onto the lens space where we are identifying

$$\psi \sim \psi + \frac{2\pi}{p}$$

where ψ is the 2π ranged Hopf fiber coordinate on S^5 . We then use the identity

$$\frac{1}{p} \sum_{\ell=0}^{p-1} e^{2\pi i \ell n/p} = \sum_{q \in \mathbb{Z}} \delta_{n,pq}$$

and get

$$f_{tensor}(\beta, p) = \sum_{n=1}^{\infty} (1 - n^2 p^2) e^{-\beta pn}$$

which we evaluate to

$$f_{tensor}(\beta, p) = \frac{e^{-p\beta}}{1 - e^{-p\beta}} - \frac{p^2}{4} \frac{\cosh \frac{p\beta}{2}}{\left(\sinh \frac{p\beta}{2}\right)^3} \tag{4.1}$$

The small- β' expansion is

$$f_{tensor}(\beta', p) = -\frac{2}{p\beta'^3} + \frac{1}{p\beta'} - \frac{1}{2} + \frac{p^3 + 10p}{120} \beta' + \mathcal{O}(\beta'^2) \tag{4.2}$$

The low temperature (or large β) behavior of the full index is governed by the Casimir energy E ,

$$I_{tensor}(\beta, p) \sim e^{-\beta E}$$

The Casimir energy can be read off from the small- β' expansion of the generating function, by interpreting β' as a small regulator, and after subtracting the divergent terms proportional to β'^{-3} and β'^{-1} , thus defining a renormalized f_{ren} . The Casimir energy is then computed as

$$E = -\frac{1}{2} \frac{d}{d\beta'} f_{tensor,ren}(\beta', p) \Big|_{\beta'=0}$$

We get

$$E_{tensor} = -\frac{p^3 + 10p}{240}$$

If we use zeta function regularization, we will compute the Casimir energy as follows. We begin by defining

$$\begin{aligned} E(s) &= \frac{1}{2} \sum_{n=1}^{\infty} (1 - p^2 n^2) (pn)^{1-s} \\ &= \frac{1}{2} p^{1-2s} \left(\zeta(s-1) - p^2 \zeta(s-3) \right) \end{aligned}$$

and then

$$\begin{aligned} E(0) &= \frac{1}{2} p \left(\zeta(-1) - p^2 \zeta(-3) \right) \\ &= -\frac{p^3 + 10p}{240} \end{aligned}$$

is the Casimir energy. The advantage with this latter method is that we do not need to worry about how to remove the singular terms proportional to β'^{-3} and β'^{-1} .

The zeta function that is associated to the generating function $f_{tensor}(\beta, p)$ is

$$\zeta(s) = \mu^{2s} \sum_{\ell \in \mathbb{Z}} \sum_{n=1}^{\infty} (1 - n^2 p^2) (n^2 p^2 + a_{\ell}^2)^{-s} \quad (4.3)$$

We decompose $\zeta(s) = \zeta_{\ell=0}(s) + \sum_{\ell \neq 0} \zeta_{\ell \neq 0}(s)$ and rewrite

$$\begin{aligned} \zeta_{\ell=0}(s) &= \mu^{2s} \left(p^{-2s} \zeta(2s) - p^{2-2s} \zeta(2s-2) \right) \\ \zeta_{\ell \neq 0}(s) &= \mu^{2s} \sum_{n=1}^{\infty} \left((1 + a_{\ell}^2) (n^2 p^2 + a_{\ell}^2)^{-s} - (n^2 p^2 + a_{\ell}^2)^{1-s} \right) \end{aligned}$$

Then by using $\zeta(0) = -\frac{1}{2}$, $\zeta(-2) = 0$ and $\zeta'(0) = -\frac{1}{2} \log(2\pi)$, $\zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}$ we get

$$\zeta'_{\ell=0}(0) = -\log \frac{\mu}{p} - \log(2\pi) + p^2 \frac{\zeta(3)}{2\pi^2}$$

For $\ell \neq 0$ we get²

$$\zeta'_{\ell \neq 0, n=0}(0) = -\frac{2\pi^4}{45 p \beta^3} + \frac{\pi^2}{3 p \beta} + \log(\mu \beta)$$

By also including the sectors with $n \neq 0$, we get the result

$$\begin{aligned} -\beta F_{tensor} &= -\frac{\pi^4}{45 p \beta^3} + \frac{\pi^2}{6 p \beta} + \frac{1}{2} \log \frac{p \beta}{2\pi} + p^2 \frac{\zeta(3)}{4\pi^2} \\ &\quad + \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{p^2}{n^3} \left(\frac{1}{2\pi^2} + \frac{2\ell n}{p \beta} + \frac{n^2}{p^2} + \left(\frac{2\pi \ell n}{p \beta} \right)^2 \right) e^{-\frac{4\pi^2 \ell n}{p \beta}} \end{aligned} \quad (4.4)$$

In particular, we see that the dependence on the scale μ cancels out. We computed the contribution from terms with $n \neq 0$ by using the zeta function dualization method that we outline in appendix A, following [9].

4.2. The (1, 0) hypermultiplet index

If we pick the hypermultiplet mass parameter $m_H = 1/2$, then we have the following generating function

² We have a sum over $n = 1, 2, \dots$ that we wish to dualize. To this end, we add the term that we would get by taking $n = 0$. We also add the terms with n replaced by $-n$, which is possible since only n^2 appears in the summand. The details are in appendix A.

$$f_{hyper}(\beta) = \frac{e^{-2\beta} + e^{-\beta}}{(1 - e^{-\beta})^3}$$

We may use

$$\frac{1}{(1 - e^{-\beta})^3} = \sum_{n=0}^{\infty} \frac{1}{2} (n + 1)(n + 2)e^{-\beta n}$$

to bring this into the form

$$f_{hyper}(\beta) = \sum_{n=1}^{\infty} n^2 e^{-\beta n}$$

Now let us take $m_H = 1/2 + \epsilon$. Then the generating function is

$$f_{hyper}(\beta) = \frac{1}{2} \sum_{n=0}^{\infty} (n - 1)n e^{-\beta(n+\epsilon)} + \frac{1}{2} \sum_{n=0}^{\infty} (n + 1)n e^{-\beta(n-\epsilon)}$$

We will now use the Abel-Plana method to obtain the high temperature expansion. We outline this method in the appendix. Associated to this is the function

$$f(z) = \frac{1}{2} (z - 1)z \log(1 - e^{-\beta(z+\epsilon)}) + \frac{1}{2} (z + 1)z \log(1 - e^{-\beta(z-\epsilon)})$$

Now we Taylor expand this function in ϵ

$$f(z) = z^2 \log(1 - e^{-\beta z}) - \beta \epsilon \frac{ze^{-\beta z}}{1 - e^{-\beta z}} - \frac{1}{2} \beta^2 \epsilon^2 \frac{z^2 e^{-\beta z}}{(1 - e^{-\beta z})^2} + \mathcal{O}(\epsilon^3)$$

We have

$$\int_0^{\infty} dx f(x) = -\frac{\pi^4}{45\beta^3} - \epsilon \frac{\pi^2}{6\beta} - \epsilon^2 \frac{\pi^2}{6\beta} + \mathcal{O}(\epsilon^3)$$

which is consistent with

$$\frac{\pi^4}{45\beta^3} + \frac{12m_H^2 - 3\pi^2}{12\beta}$$

by taking $m_H = 1/2 + \epsilon$ and expanding in ϵ . A rather curious fact is that the series terminates at order ϵ^2 . But the integrals at order ϵ^n for $n > 2$ all diverges, although by summing them all up, we shall of course expect a finite result, and actually the contribution from all those divergent integrals should sum up to zero.

It is easy to get the Casimir energy from the generating function. We get

$$f(\beta', m_H) = \frac{2}{\beta'^3} + \frac{4m_H^2 - 1}{4\beta'} + \frac{1}{960} (17 - 120m_H^2 + 80m_H^4)\beta' + \mathcal{O}(\beta'^2)$$

Then we see that the Casimir energy is

$$E = \frac{1}{240} + \epsilon \frac{1}{24} - \epsilon^3 \frac{1}{12} - \epsilon^4 \frac{1}{24}$$

as an exact expression in ϵ .

4.2.1. Refinement

If we take $m_H = 1/2 + \epsilon$, then we have

$$f_{hyper} = \frac{1}{2} \sum_{n=0}^{\infty} (n+1)(n+2) \left(e^{-\beta(n+2+\epsilon)} \omega^{n+2} + e^{-\beta(n+1-\epsilon)} \omega^{n+1} \right)$$

Now we can lens this expression, and get

$$\begin{aligned} f_{hyper} &= \frac{1}{2} \sum_{q=0}^{\infty} (pq-1)pq e^{-\beta(pq+\epsilon)} + \frac{1}{2} \sum_{q=0}^{\infty} (pq+1)pq e^{-\beta(pq-\epsilon)} \\ &= \frac{p^2 \cosh(\beta\epsilon) \cosh \frac{p\beta}{2} + p \sinh(\beta\epsilon) \sinh \frac{p\beta}{2}}{4 \left(\sinh \frac{p\beta}{2} \right)^3} \end{aligned} \quad (4.5)$$

From this we can extract the Casimir energy

$$E_{hyper} = \frac{p^3}{240} + \epsilon \frac{p}{24} - \epsilon^3 \frac{1}{12p} - \epsilon^4 \frac{1}{24p}$$

For the high temperature expansion, we define

$$f(z, \epsilon) = \frac{1}{2} \left((pz-1)pz \log \left(1 - e^{-\beta(pz+\epsilon)} \right) + (pz+1)pz \log \left(1 - e^{-\beta(pz-\epsilon)} \right) \right)$$

that we Taylor expand in ϵ and then we are able to compute its integral to the first few orders, with the result

$$\int_0^{\infty} dx f(x, \epsilon) = -\frac{\pi^4}{45p\beta^3} - \epsilon \frac{\pi^2}{6p\beta} - \epsilon^2 \frac{\pi^2}{6p\beta} + \mathcal{O}(\epsilon^3)$$

Let us move on to the other term in the Abel-Plana formula

$$i \int_0^{\infty} dx \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1}$$

We will content ourselves to compute up to linear order in ϵ , and thus we only need to work with f expanded to first order

$$f(z, \epsilon) = p^2 z^2 \log(1 - e^{-p\beta z}) + \epsilon p\beta \frac{z e^{-p\beta z}}{1 - e^{-p\beta z}}$$

To this order, we get

$$\begin{aligned} f(ix, \epsilon) - f(-ix, \epsilon) &= p^2 x^2 \log \left(-e^{ip\beta x} \right) + ipx\beta\epsilon \\ &= p^2 x^2 i (p\beta x - 2\pi n - \pi) + ipx\beta\epsilon \end{aligned}$$

where

$$\int_0^{\infty} dx = \sum_{n=0}^{\infty} \int_{2\pi n}^{2\pi(n+1)} dx$$

Thus we like to compute (and here we changed the sign of $f(z)$)

$$\sum_{n=0}^{\infty} \int_{2\pi n/(p\beta)}^{2\pi(n+1)/(p\beta)} dx \frac{p^2 x^2 (p\beta x - 2\pi n - \pi) + px\beta\epsilon}{e^{2\pi x} - 1}$$

We report the following partial results,

$$\int_0^{\infty} dx \frac{x^3}{e^{2\pi x} - 1} = \frac{1}{240}$$

$$\int_0^{\infty} dx \frac{x^2}{e^{2\pi x} - 1} = \frac{\zeta(3)}{4\pi^3}$$

$$\int_0^{\infty} dx \frac{x}{e^{2\pi x} - 1} = \frac{1}{24}$$

We may expand the denominator as

$$\frac{e^{-2\pi x}}{1 - e^{-2\pi x}} = \sum_{k=1}^{\infty} e^{-2\pi kx}$$

and consider the integrals

$$\int dx x^2 e^{-2\pi kx} = -\frac{1}{4\pi^3 k^3} (1 + 2\pi kx + 2\pi^2 k^2 x^2) e^{-2\pi kx}$$

We get

$$\frac{p^3 + 10\epsilon p}{240} \beta - p^2 \frac{\zeta(3)}{4\pi^2}$$

plus the double sum

$$\begin{aligned} & \frac{p}{2\pi^2} \sum_{k,n=1}^{\infty} \frac{n}{k^3} \left[(1 + 2\pi kx + 2\pi^2 k^2 x^2) e^{-2\pi kx} \right]_{2\pi n/(p\beta)}^{2\pi(n+1)/(p\beta)} \\ &= - \sum_{k,n=1}^{\infty} \frac{p^2}{k^3} \left(\frac{1}{2\pi^2} + \frac{2kn}{p\beta} + \left(\frac{2\pi kn}{p\beta} \right)^2 \right) e^{-\frac{4\pi^2 kn}{p\beta}} \end{aligned}$$

Upon adding the Casimir energy term $-\beta E_{hyper}$ we find a cancellation. Summarizing, we have got

$$\begin{aligned} -\beta F_{hyper} &= \frac{\pi^4}{45 p\beta^3} + \epsilon \frac{\pi^2}{6 p\beta} + \epsilon^2 \frac{\pi^2}{6 p\beta} - p^2 \frac{\zeta(3)}{4\pi^2} \\ &\quad - \sum_{k,n=1}^{\infty} \frac{p^2}{k^3} \left(\frac{1}{2\pi^2} + \frac{2kn}{p\beta} + \left(\frac{2\pi kn}{p\beta} \right)^2 \right) e^{-\frac{4\pi^2 kn}{p\beta}} \end{aligned}$$

If we add the contributions of the tensor and the hypermultiplets, we get

$$-\beta (F_{tensor} + F_{hyper}) = \frac{\pi^2(1 + \epsilon + \epsilon^2)}{6p\beta} + \frac{1}{2} \log \frac{p\beta}{2\pi} + \sum_{n,k=1}^{\infty} \frac{1}{n} e^{-\frac{4\pi^2 kn}{p\beta}}$$

If we put $\epsilon = 0$, we may express this result as

$$I(\beta) = \sqrt{\frac{p\beta}{2\pi}} e^{\frac{\pi^2}{6p\beta}} \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} \frac{e^{-\frac{4\pi^2 n}{p\beta}}}{1 - e^{-\frac{4\pi^2 n}{p\beta}}}\right)$$

This result was also obtained in [8] by using the known modular property of the Dedekind eta function.

We can also easily perform the sum over n . For the hypermultiplet and if we put $\epsilon = 0$, we then get

$$-\beta F_{hyper} = \frac{\pi^4}{45p\beta^3} - \sum_{k=1}^{\infty} \left[\frac{p^2}{4\pi^2 k^3} \frac{\cosh \frac{2\pi^2 k}{p\beta}}{\sinh \frac{2\pi^2 k}{p\beta}} + \frac{p}{2k^2 \beta} \frac{1}{\left(\sinh \frac{2\pi^2 k}{p\beta}\right)^2} + \frac{\pi^2}{k\beta^2} \frac{\cosh \frac{2\pi^2 k}{p\beta}}{\left(\sinh \frac{2\pi^2 k}{p\beta}\right)^3} \right]$$

4.2.2. The same result from the plethystic exponent

Let us return to the generating function (4.5) and take $\epsilon = 0$ for simplicity,

$$\begin{aligned} f_{hyper}(\beta, p) &= \frac{p^2}{4} \frac{\cosh \frac{p\beta}{2}}{\left(\sinh \frac{p\beta}{2}\right)^3} \\ &= \frac{2}{p\beta^3} - \frac{p^3}{120}\beta + \mathcal{O}(\beta^2) \end{aligned} \tag{4.6}$$

We can use this to compute the index by taking the plethystic exponent,

$$\begin{aligned} \log \tilde{I}_{hyper}(\beta, p) &= \sum_{n=1}^{\infty} \frac{1}{n} f_{hyper}(n\beta, p) \\ &= \sum_{n=-\infty}^{\infty} \int_{\epsilon}^{\infty} \frac{ds}{s} f_{hyper}(s\beta, p) e^{2\pi ins} \end{aligned}$$

The correct index is given by $I_{hyper} = e^{-\beta E} \tilde{I}_{hyper}$. For the second identity to hold, we need to pick up the points $s = n$ for $n = 1, 2, \dots$ from

$$\sum_{n \in \mathbb{Z}} e^{2\pi ins} = \sum_{n \in \mathbb{Z}} \delta(s - n)$$

To this end, we shall take lower integration bound such that $0 < \epsilon < \beta$.

Dualization is now performed as follows. First we regularize as follows

$$f(s) = f_{sing}(s) + f_{reg}(s)$$

where $f_{reg}(s) = f(s) - f_{sing}(s)$ and $f_{sing}(s)$ involves terms of the form s^{-n} for $n > 0$ such that $f_{reg}(0)$ is finite. Next we can compute the contribution from $f_{sing}(s)$. We have the small- s expansion

$$f_{hyper}(s\beta, p) = \frac{2}{p\beta^3 s^3} - \frac{p^3 \beta s}{120} + \mathcal{O}(s^3)$$

The first term in this expansion is the singular term, which gives rise to the Stefan-Boltzmann term

$$\sum_{n=-\infty}^{\infty} \int_{\epsilon}^{\infty} ds \frac{2}{p\beta^3 s^4} e^{2\pi i n s} = \frac{2}{p\beta^3} \zeta(4) = \frac{\pi^4}{45 p\beta^3}$$

The remaining piece is

$$\sum_{n=1-\infty}^{\infty} \int \frac{ds}{s} f_{reg}(\beta s) e^{2\pi i n s} + \frac{1}{2} \int_{-\infty}^{\infty} \frac{ds}{s} f_{reg}(\beta s) - \frac{1}{2} \frac{f_{reg}(s)}{s} \Big|_{s=0}$$

There are triple poles at

$$s = \frac{2\pi i k}{p\beta}, \quad k \in \mathbb{Z}$$

and by encircling those triple poles that lie in the upper half plane and picking up the residues, we get

$$\sum_{n=1-\infty}^{\infty} \int \frac{ds}{s} f_{reg}(\beta s) e^{2\pi i n s} = - \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{p^2}{k^3} \left(\frac{1}{2\pi^2} + \frac{2kn}{p\beta} + \left(\frac{2\pi kn}{p\beta} \right)^2 \right) e^{-\frac{4\pi^2 kn}{p\beta}} \tag{4.7}$$

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{ds}{s} f_{reg}(\beta s) = -\frac{p^2}{4\pi^2} \zeta(3) \tag{4.8}$$

By a general argument that we prove in the appendix, we have that

$$-\frac{1}{2} \frac{f_{reg}(s)}{s} \Big|_{s=0}$$

is canceled by the Casimir energy.

By comparing the two computations, and by looking at the term in (4.8), we see that what was referred to as the topological subleading term in [9] corresponds to what was referred to as the perturbative contribution in [8].

5. Turning on squashing parameters

We will now consider a more complicated situation with generic squashing parameters turned on. For this case, the plethystic method is easy to use, whereas the other two methods, the generalized zeta function and the Abel-Plana formula, become difficult to use. The generating functions for general squashing parameters a, b, c and generic lensing parameter p , are quite complicated. However, already from the small β expansion of these generating functions, we can extract the

high temperature and low temperature asymptotic behavior of the corresponding indices, so that is where we will start.

We begin by extracting the Stefan-Boltzmann terms with squashing. Given a generating function $f(\beta)$, the Stefan-Boltzmann terms are obtained by computing the following quantity,

$$\sum_{n \in \mathbb{Z}} \int_{\epsilon}^{\infty} \frac{ds}{s} e^{\frac{2\pi i n s}{\beta}} f_{sing}(s)$$

For the explicit computations, we will only need the following results,

$$\sum_{n \in \mathbb{Z}} \int_{\epsilon}^{\infty} \frac{ds}{s} e^{\frac{2\pi i n s}{\beta}} \frac{1}{s^n} = \frac{\zeta(n+1)}{\beta^n}$$

and in this paper we will encounter

$$\zeta(4) = \frac{\pi^4}{90}$$

$$\zeta(2) = \frac{\pi^2}{6}$$

5.1. The tensor multiplet

The refined generating function for the tensor multiplet is given by (1.1). We begin by writing a series expansion for

$$\frac{1}{(1 - \omega e^{-\beta(1+a)})(1 - \omega e^{-\beta(1+b)})(1 - \omega e^{-\beta(1+c)})} = - \sum_{n=0}^{\infty} \omega^n e^{-\beta n} f_n(a, b, c) \tag{5.1}$$

where

$$f_n(a, b, c) = \frac{e^{-\beta a(n+1)}(e^{\beta b} - e^{\beta c}) + e^{-\beta b(n+1)}(e^{\beta c} - e^{\beta a}) + e^{-\beta c(n+1)}(e^{\beta a} - e^{\beta b})}{(1 - e^{-\beta(a-b)})(1 - e^{-\beta(b-c)})(1 - e^{-\beta(c-a)})}$$

We now need to multiply this by the numerator

$$\omega^3 e^{-3\beta} - \omega^2 e^{-2\beta} (e^{\beta a} + e^{\beta b} + e^{\beta c})$$

We then get two terms, $\omega^{n+3}(\dots) + \omega^{n+2}(\dots)$. The trick is to shift the sum for the first term to bring both terms into the same form $\omega^{n+2}(\dots + \dots)$, and then replace $n + 2$ by mp where $m = 1, 2, \dots$ for $p > 1$. (If $p = 1$, then we shall take $m = 2, 3, \dots$) For $p > 1$ we get

$$f_{tensor}(p, \beta, a, b, c) = - \sum_{m=1}^{\infty} e^{-\beta pm} \left(f_{pm-3}(\beta, a, b, c) - (e^{\beta a} + e^{\beta b} + e^{\beta c}) f_{pm-2}(\beta, a, b, c) \right)$$

This is a geometric sum, which we evaluate to

$$f_{tensor}(p, \beta, a, b, c) = - \frac{1 + e^{\beta(b-c)}}{(1 - e^{-\beta(a-b)})(1 - e^{-\beta(c-a)})} \frac{1}{1 - e^{p\beta(1+a)}} + \text{cycl} \tag{5.2}$$

We note that

$$f_{tensor}(p, \beta, a, b, c) + f_{tensor}(p, -\beta, a, b, c) = -1$$

This enables us to write this in the manifestly antisymmetric form by adding $-1/2$,

$$f_{tensor}(p, \beta, a, b, c) = \left(\frac{\cosh \frac{\beta}{2}(b-c)}{4 \sinh \frac{\beta}{2}(a-b) \sinh \frac{\beta}{2}(a-c)} \frac{\cosh \frac{\beta}{2}(1+a)}{\sinh \frac{\beta}{2}(1+a)} + \text{cycl} \right) - \frac{1}{2}$$

The small β expansion reads

$$f_{tensor}(p, \beta, a, b, c) = -\frac{2}{Np\beta^3} + \frac{6 + 5(ab + bc + ca)}{6Np\beta} - \frac{1}{2} + \frac{p^3 + 10p}{120}\beta - \frac{a^4 + b^4 + c^4 - 36abc}{240Np}\beta + \mathcal{O}(\beta^2)$$

where $N := (1+a)(1+b)(1+c)$ and is a generalization of (4.2) to the squashed case. From the singular terms we extract the Stefan-Boltzmann terms

$$\beta F_{tensor} = \frac{\pi^4}{45Np\beta^3} - \frac{(6 + 5(ab + bc + ca))\pi^2}{36Np\beta}$$

and from the linear term we extract the Casimir energy

$$E_{tensor} = -\frac{p^3 + 10p}{240} + \frac{a^4 + b^4 + c^4 - 36abc}{480Np}$$

5.2. The hypermultiplet

We pick $m_H = 1/2 + \epsilon$ and put $t = e^{\beta\epsilon}$ where we have the following refined generating function,

$$f_{hyper}(\beta, \omega, t) = \frac{e^{-2\beta}\omega^2 t^{-1} + e^{-\beta}\omega t}{(1 - \omega e^{-\beta(1+a)})(1 - \omega e^{-\beta(1+b)})(1 - \omega e^{-\beta(1+c)})}$$

We again use the series expansion (5.1) for the denominator and write

$$\begin{aligned} f_{hyper}(\beta, \omega) &= -\sum_{n=0}^{\infty} \omega^n e^{-\beta n} (e^{-2\beta}\omega^2 t^{-1} + e^{-\beta}\omega t) f_n(a, b, c) \\ &= -\sum_{n=-1}^{\infty} \omega^{n+2} e^{-\beta(n+2)} (f_n t + f_{n+1} t^{-1}) \end{aligned}$$

Summing over ω picks out $n + 2 = mp$ for $m = 1, 2, \dots$ We get

$$f_{hyper}(\beta, p, a, b, c, t) = \sum_{m=1}^{\infty} e^{-\beta mp} \left(t f_{mp-1} + t^{-1} f_{mp-2} \right)$$

The sum can be evaluated with the result

$$f_{hyper}(\beta, p, a, b, c, t) = \frac{te^{-\beta c} + t^{-1}e^{\beta b}}{(1 - e^{-\beta(a-b)})(1 - e^{-\beta(c-a)})} \frac{1}{1 - e^{\beta(1+a)}} + \text{cycl} \tag{5.3}$$

By noting that

$$f_{\text{hyper}}(p, \beta, a, b, c, t) + f_{\text{hyper}}(p, -\beta, a, b, c, 1/t) = 0$$

we can write this as

$$f_{\text{hyper}}(p, \beta, a, b, c, t) = \frac{\cosh \frac{\beta}{2}(b+c+2\epsilon)}{4 \sinh \frac{\beta}{2}(a-b) \sinh \frac{\beta}{2}(a-c)} \frac{\cosh \frac{p\beta}{2}(1+a)}{\sinh \frac{p\beta}{2}(1+a)} + \text{cycl}$$

The small β expansion reads

$$f_{\text{hyper}} = \frac{2}{Np\beta^3} + \frac{ab+bc+ca+6(\epsilon+\epsilon^2)}{6Np\beta} + \frac{M\beta}{120Np} + \mathcal{O}(\epsilon^2, \beta^2) \tag{5.4}$$

where

$$M = \frac{1}{2}(a^4 + b^4 + c^4) + 2abc - Np^4 + (20(ab+bc+ca) + 10abc - 10Np^2)\epsilon + 10(ab+bc+ca)\epsilon^2 + 20\epsilon^3 + 10\epsilon^4$$

which is a generalization of (4.6) to the squashed case. From the singular terms we extract the Stefan-Boltzmann terms

$$-\beta F_{\text{hyper}} = \frac{\pi^4}{45Np\beta^3} + \frac{(ab+bc+ca+6(\epsilon+\epsilon^2))\pi^2}{36Np\beta}$$

and from the linear term we extract the Casimir energy

$$E_{\text{hyper}} = \frac{p^3}{240} + \frac{p\epsilon}{24} - \frac{\epsilon^3}{12Np} - \frac{\epsilon^4}{24Np} - \frac{a^4 + b^4 + c^4 + 4abc}{480Np} - \frac{2(ab+bc+ca) + abc}{24Np}\epsilon - \frac{ab+bc+ca}{24Np}\epsilon^2$$

5.3. Summing the contributions

If we sum the contributions from tensor and hypermultiplets,

$$f_{M5}(\beta, p) = f_{\text{tensor}}(\beta, p) + f_{\text{hyper}}(\beta, p)$$

we get the result

$$f_{M5} = \frac{\sinh \frac{\beta(\epsilon-c)}{2} \sinh \frac{\beta(\epsilon-b)}{2}}{\sinh \frac{\beta(a-b)}{2} \sinh \frac{\beta(c-a)}{2}} \frac{1}{1 - e^{p\beta(1+a)}} + \text{cycl} \tag{5.5}$$

This result agrees precisely with eq. (4.5) in [10]. At $\epsilon = 0$, the small β expansion reads

$$f_{M5} = \frac{1+ab+ac+bc}{Np\beta} - \frac{1}{2} + \frac{1}{12}p\beta + \frac{abc}{6Np}\beta + \mathcal{O}(\beta^2)$$

From the divergent term we obtain the Stefan-Boltzmann term

$$\log I_{SB} = \frac{1 + ab + bc + ca}{Np} \frac{\pi^2}{6\beta}$$

and from the linear term we obtain the Casimir energy,

$$E_{M5} = -\frac{p}{24} - \frac{abc}{12Np}$$

5.4. More on the exact results

The exact expressions for generating functions that we have obtained in (5.2) and (5.3) have not yet been written in the fully reduced form, by which we mean the following. If we write the sum of the three cyclic permutations on a common denominator (for $i = \{tensor, hyper\}$ respectively)

$$f_i = \frac{P_i}{(1 - e^{\beta(b-a)})(1 - e^{\beta(c-b)})(1 - e^{\beta(a-c)})(1 - e^{p\beta(1+a)})(1 - e^{p\beta(1+b)})(1 - e^{p\beta(1+c)})}$$

then this can be always further reduced to the form

$$f_i = \frac{P_{i, reduced}}{(1 - e^{p\beta(1+a)})(1 - e^{p\beta(1+b)})(1 - e^{p\beta(1+c)})}$$

by which we mean that the poles associated to the vanishing of $1 - e^{\beta(a-b)}$ and any of its cyclic permutations are all removable poles. We have tested this up to large values of p and seen that this cancellation of poles always happens so we conjecture this always happens for all values of p , but we have no proof. We will here obtain a general formula for $P_{i, reduced}$, which again will be a conjecture. For notational simplicity, we put

$$\begin{aligned} u &= e^{\beta a} \\ v &= e^{\beta b} \\ w &= e^{\beta c} \end{aligned}$$

which are subject to the constraint $uvw = 1$, and we put

$$x = e^{\beta}$$

In this notation, we have

$$\begin{aligned} f_{hyper}(p) &= \frac{1/w + v}{(1 - v/u)(1 - u/w)(1 - x^p u^p)} + \text{cycl} \\ f_{tensor}(p) &= -\frac{1 + v/w}{(1 - v/u)(1 - u/w)(1 - x^p u^p)} + \text{cycl} \end{aligned}$$

We rewrite these in the form

$$f_i = \frac{g_i}{1 - x^p u^p} + \text{cycl}$$

where

$$g_{hyper} = -\frac{u+1}{(u-v)(u-w)}$$

$$g_{tensor} = \frac{u(v+w)}{(u-v)(u-w)}$$

By adding the three cyclic terms, we get

$$f_i = \frac{a_i - x^p b_i + x^{2p} c_i}{(1-x^p u^p)(1-x^p v^p)(1-x^p w^p)}$$

where

$$a_i = g_i + \text{cycl}$$

$$b_i = (v^p + w^p) g_i + \text{cycl}$$

$$c_i = \frac{1}{u^p} g_i + \text{cycl}$$

Explicitly we find

$$a_{hyper} = 0$$

$$a_{tensor} = -1$$

For the other two terms, we have a rather complicated dependence on p . For the first few values of p we find that

$$b_{hyper}(1) = 1$$

$$b_{hyper}(2) = 1 + u + v + w$$

$$b_{hyper}(3) = u + v + w + u^2 + v^2 + w^2 + uv + uw + vw$$

$$b_{tensor}(1) = -u - v - w$$

$$b_{tensor}(2) = -u^2 - v^2 - w^2 - uv - uw - vw$$

$$b_{tensor}(3) = -u^3 - v^3 - w^3 - u^2v - uv^2 - u^2w - v^2w - uw^2 - vw^2 - 2uvw$$

$$c_{hyper}(1) = -1$$

$$c_{hyper}(2) = -1 - \frac{1}{u} - \frac{1}{v} - \frac{1}{w}$$

$$c_{hyper}(3) = -\frac{1}{u} - \frac{1}{v} - \frac{1}{w} - \frac{1}{u^2} - \frac{1}{v^2} - \frac{1}{w^2} - \frac{1}{uv} - \frac{1}{vw} - \frac{1}{wu}$$

$$c_{tensor}(1) = 0$$

$$c_{tensor}(2) = u + v + w$$

$$c_{tensor}(3) = 2 + \frac{u}{v} + \frac{v}{u} + \frac{u}{w} + \frac{w}{u} + \frac{v}{w} + \frac{w}{v}$$

We have the relations

$$c_{hyper}(p, u, v, w) = -b_{hyper}(p, 1/u, 1/v, 1/w)$$

$$c_{tensor}(p, u, v, w) = -b_{tensor}(p, 1/u, 1/v, 1/w) - \frac{1}{u^p} - \frac{1}{v^p} - \frac{1}{w^p}$$

These relations determine c_i once we know b_i . We also have the relation

$$b_{tensor}(p) + b_{hyper}(p+1) - b_{hyper}(p) + b_{hyper}(p-1) = 0$$

that we can use to determine b_{tensor} once we know b_{hyper} . Our task has been reduced to determine b_{hyper} , before we have used the relation $uvw = 1$. Let us now switch to a short notation. If $b_{hyper}(2) = 1 + u + v + w$, then we will write this as $p = 2: (100), (010), (001), (000)$ where (100) represents the term u and so on. We will also suppress all terms that are obtained by trivial permutations, so instead of writing out (100), (010), (001), we will just write (100). This way we get for the first few values of p the following results,³

$$\begin{aligned}
 p = 1 & : [(000)] \\
 p = 2 & : [(100)], [(000)] \\
 p = 3 & : [(200), (110)], [(100)] \\
 p = 4 & : [(300), (210), (111)], [(200), (110)] \\
 p = 5 & : [(400), (310), (220), (211)], [(300), (210), (111)] \\
 p = 6 & : [(500), (410), (320), (221), (311)], [(400), (310), (220), (211)]
 \end{aligned}$$

where we have grouped the elements into two classes. From this, we see the following pattern

$$p = p : [(p - 1, 0, 0), (p - 2, 1, 0) \dots], [(p - 2), (p - 3, 1, 0) \dots]$$

where the first class of elements are all those elements whose entries sum up to $p - 1$, and the second class are all those elements whose entries sum up to $p - 2$. We have now in principle completed the computation, although the result has not been presented in an explicit way. This situation can be improved by restricting to $(u, v, w) = (u, 1/u, 1)$ where we get

$$\begin{aligned}
 b_{hyper}(p, u, 1/u, 1) & = (p - 1) (u + u^{-1}) + (p - 2) (u^2 + u^{-2}) + \dots \\
 & \quad + (u^{p-1} + u^{-(p-1)}) + p \\
 c_{hyper}(p, u, 1/u, 1) & = -b_{hyper}(p, u, 1/u, 1) \\
 b_{tensor}(p, u, 1/u, 1) & = -(p - 1) (u + u^{-1}) - (p - 2) (u^2 + u^{-2}) - \dots \\
 & \quad - (u^{p-1} + u^{-(p-1)}) - p - (u^p + u^{-p}) \\
 c_{tensor}(p, u, 1/u, 1) & = (p - 1) (u + u^{-1}) + (p - 2) (u^2 + u^{-2}) + \dots \\
 & \quad + (u^{p-1} + u^{-(p-1)}) + p - 1
 \end{aligned}$$

For the sum we have a closed form,

$$b_{hyper}(p, u, 1/u, 1) = \left(\frac{u^{p/2} - u^{-p/2}}{u^{1/2} - u^{-1/2}} \right)^2 \tag{5.6}$$

Using this result, we get

$$f_{hyper}(p, u, 1/u, 1) = - \left(\frac{u^{p/2} - u^{-p/2}}{u^{1/2} - u^{-1/2}} \right)^2 \frac{x^p + x^{2p}}{(1 - x^p u^p)(1 - x^p v^p)(1 - x^p w^p)} \Bigg|_{v=1/u, w=1}$$

³ We carried out this computation up to $p = 6$ by using Mathematica.

$$f_{tensor}(p, u, 1/u, 1) = \left(\frac{u^{p/2} - u^{-p/2}}{u^{1/2} - u^{-1/2}} \right)^2 \frac{x^p + x^{2p}}{(1 - x^p u^p)(1 - x^p v^p)(1 - x^p w^p)} \Bigg|_{v=1/u, w=1} + \frac{-1 + (u^p + u^{-p})x^p - x^{2p}}{(1 - x^p u^p)(1 - x^p v^p)(1 - x^p w^p)} \Bigg|_{v=1/u, w=1}$$

The second term in the second line can be simplified by noting that

$$(1 - x^p u^p)(1 - x^p u^{-p}) = 1 - x^p(u^p + u^{-p}) + x^{2p}$$

Then, by adding the two contributions, we get the result,

$$f_{M5}(p, u, 1/u, 1) = \frac{1}{-1 + x^p}$$

We notice that surprisingly this M5 brane generating function does not depend on the squashing parameter a .

To get a nontrivial dependence on squashing parameters for the M5 brane generating function at $m_H = 1/2$, we shall consider generic a, b, c . Then we shall return to our result above. To streamline the notation, we define

$$s_q(u, v, w) = \sum_{r+s+t=q} u^r v^s w^t$$

and

$$Q_{q_1, q_2, \dots}(u, v, w) = s_{q_1}(u, v, w) + s_{q_2}(u, v, w) + \dots$$

Then we have

$$b_{hyper}(p, u, v, w) = Q_{p-1, p-2}(u, v, w) \tag{5.7}$$

Let us further define the set

$$Q_{q_1, q_2, \dots} = \{r, s, t | r + s + t = q_1\} \cup \{r, s, t | r + s + t = q_2\} \cup \dots$$

We then conjecture the following general expression,

$$f_{hyper}(p, \beta, a, b, c) = \sum_{r+s+t \in Q_{p-1, p-2}} \frac{\cosh \left[\frac{p\beta}{2} + \beta(ar + bs + ct) \right]}{4 \sinh \frac{p\beta}{2}(1+a) \sinh \frac{p\beta}{2}(1+b) \sinh \frac{p\beta}{2}(1+c)} \tag{5.8}$$

As consistency checks, we note that if we take the limit $u, v, w \rightarrow 1$, then the sum (5.7) reduces to

$$b_{hyper}(p, 1, 1, 1) = \frac{p(p+1)}{2} + \frac{(p-1)p}{2} = p^2$$

and if we put $(u, v, w) = (u, 1/u, 1)$ in (5.8), then it reduces to (5.6).

Let us move on to the tensor multiplet. We clearly seem to have

$$b_{tensor}(p, u, v, w) = -Q_{p, p-3}(u, v, w)$$

As checks, we see that in special cases, this reduces to the previous results,

$$b_{tensor}(p, 1, 1, 1) = -p^2 - 2$$

$$b_{tensor}(p, u, 1/u, 1) = -\left(\frac{u^{p/2} - u^{-p/2}}{u^{1/2} - u^{-1/2}}\right)^2 - (u^p + u^{-p})$$

We are then ready to conjecture the general result for the generating function,

$$f_{tensor}(p, x, u, v, w) = \frac{-1 + x^p Q_{p,p-3} + x^{2p} (Q_{p,p-3}^\vee - u^{-p} - v^{-p})}{(1 - x^p u^p)(1 - x^p v^p)(1 - x^p w^p)}$$

where we define

$$Q_{p,p-3}^\vee(\beta) = Q_{p,p-3}(-\beta)$$

We can take out a simple term from this and write the rest in a manifestly antisymmetric form,

$$f_{tensor} = -\frac{1}{1 - x^p w^p} \frac{x^{\frac{p}{2}} (Q_{p,p-3}^\vee - u^{-p} - v^{-p}) + x^{-\frac{p}{2}} (Q_{p,p-3} - u^p - v^p)}{\left((xu)^{\frac{p}{2}} - (xu)^{-\frac{p}{2}}\right) \left((xv)^{\frac{p}{2}} - (xv)^{-\frac{p}{2}}\right) \left((xw)^{\frac{p}{2}} - (xw)^{-\frac{p}{2}}\right)}$$

which can also be written as

$$f_{tensor} = \frac{e^{-p\beta(1+c)}}{1 - e^{-p\beta(1+c)}} + \frac{\cosh \frac{p\beta(1-2a)}{2} + \cosh \frac{p\beta(1-2b)}{2} - \sum_{r,s,t \in Q_{p,p-3}} \cosh \frac{p\beta(1-\frac{2}{p}(ra+sb+tc))}{2}}{4 \sinh \frac{p\beta(1+a)}{2} \sinh \frac{p\beta(1+b)}{2} \sinh \frac{p\beta(1+c)}{2}} \quad (5.9)$$

It is easy to see that we reproduce the previously known result if we put $p = 1$. For $p = 1$ we have

$$\sum_{r,s,t \in Q_{p,p-3}} \cosh \frac{p\beta}{2} \left(1 - \frac{2}{p}(ra + sb + tc)\right) \Big|_{p=1}$$

$$= \cosh \frac{\beta(1-2a)}{2} + \cosh \frac{\beta(1-2b)}{2} + \cosh \frac{\beta(1-2c)}{2}$$

and so we get

$$f_{tensor} = \frac{e^{-\beta(1+c)}}{1 - e^{-\beta(1+c)}} - \frac{\cosh \frac{\beta(1-2c)}{2}}{4 \sinh \frac{\beta(1+a)}{2} \sinh \frac{\beta(1+b)}{2} \sinh \frac{\beta(1+c)}{2}} \quad (5.10)$$

Let us now return to the hypermultiplet. To understand how to generalize to general m_H , it is sufficient to just look at say the case with $p = 2$ for which we get

$$f_{hyper}(2, x, u, v, w, t) = \frac{(t^{-1} + t(u + v + w))x^{-1} + (t + t^{-1}(u^{-1} + v^{-1} + w^{-1}))x}{(ux - u^{-1}x^{-1})(vx - v^{-1}x^{-1})(wx - w^{-1}x^{-1})}$$

If we express this in the form

$$f_{hyper}(2, x, u, v, w, t) = \sum_{r,s,t \in \mathcal{Q}_0} \frac{t^{-1}u^r v^s w^t + tu^{-r} v^{-s} w^{-t}}{(ux - u^{-1}x^{-1})(vx - v^{-1}x^{-1})(wx - w^{-1}x^{-1})} + \sum_{r,s,t \in \mathcal{Q}_1} \frac{t^{-1}u^r v^s w^t + tu^{-r} v^{-s} w^{-t}}{(ux - u^{-1}x^{-1})(vx - v^{-1}x^{-1})(wx - w^{-1}x^{-1})}$$

then it seems clear that this should generalize as

$$f_{hyper}(p, \beta, a, b, c, \epsilon) = \sum_{r,s,t \in \mathcal{Q}_{p-2}} \frac{\cosh\left[\frac{p\beta}{2} - \beta(ar + bs + ct) - \beta\epsilon\right]}{4 \sinh \frac{p\beta}{2}(1+a) \sinh \frac{p\beta}{2}(1+b) \sinh \frac{p\beta}{2}(1+c)} + \sum_{r,s,t \in \mathcal{Q}_{p-1}} \frac{\cosh\left[\frac{p\beta}{2} - \beta(ar + bs + ct) + \beta\epsilon\right]}{4 \sinh \frac{p\beta}{2}(1+a) \sinh \frac{p\beta}{2}(1+b) \sinh \frac{p\beta}{2}(1+c)} \tag{5.11}$$

As two consistency checks for our conjectured formula (5.11), we first notice that the small β expansion agrees with (5.4) for arbitrary p , and second, if we put $a = b = c = 0$ and keep p arbitrary, then we reproduce (4.5) to all orders in β .

5.5. Dualization using the plethystic method

Having obtained the generating functions in their fully reduced forms, we are now ready to dualize these generating functions using the plethystic method as we outline in the appendix following [8]. We have already extracted the Stefan-Boltzmann terms. Let us move on to compute the integrals

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{ds}{s} f_{i,reg}(s)$$

for i running over hyper and tensor multiplets. If there are only simple poles, the instanton contribution is computed in a similar way,

$$\sum_{k=1-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{ds}{s} f_{i,reg}(s) e^{2\pi i ks/\beta}$$

and if simple poles are located at (5.12), then the sum over k becomes a geometric series,

$$\sum_{k=1}^{\infty} e^{-\frac{4\pi^2 kn}{p\beta(1+a)}} = \frac{e^{-\frac{4\pi^2 n}{p\beta(1+a)}}}{1 - e^{-\frac{4\pi^2 n}{p\beta(1+a)}}}$$

5.5.1. The hypermultiplet

We note that this integral does not depend on β so it should be part of the perturbative contribution from the 5d viewpoint. The advantage with turning on generic a, b, c parameters is that there now will appear only contributions from simple poles located at

$$\begin{aligned}
 s &= \frac{2\pi in}{p(1+a)} \\
 s &= \frac{2\pi in}{p(1+b)} \\
 s &= \frac{2\pi in}{p(1+c)}
 \end{aligned}
 \tag{5.12}$$

and by closing the contour in the upper halfplane, we will pick up contributions only from those poles with $n = 1, 2, 3, \dots$. We expand around a pole,

$$\sinh \frac{ps(1+a)}{2} = (-1)^n \frac{p(1+a)}{2} \left(s - \frac{2\pi in}{p(1+a)} \right) + \dots$$

and get the corresponding residue

$$\begin{aligned}
 H_n(a, b, c) &= - \sum_{\mathcal{Q}_{p-2}} \frac{1}{2n} \frac{\cos \frac{\pi n}{1+a} \left(2+a - \frac{2}{p}(ar+bs+ct) - \frac{2\epsilon}{p} \right)}{\sin \frac{\pi n(b-a)}{1+a} \sin \frac{\pi n(c-a)}{1+a}} \\
 &\quad - \sum_{\mathcal{Q}_{p-1}} \frac{1}{2n} \frac{\cos \frac{\pi n}{1+a} \left(2+a - \frac{2}{p}(ar+bs+ct) + \frac{2\epsilon}{p} \right)}{\sin \frac{\pi n(b-a)}{1+a} \sin \frac{\pi n(c-a)}{1+a}}
 \end{aligned}$$

after some computation. The contribution from the other two poles can be obtained by cyclic permutations of a, b, c . Thus we have obtained

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{ds}{s} f_{hyper,reg}(s) = \sum_{n=1}^{\infty} (H_n(a, b, c) + H_n(b, c, a) + H_n(c, a, b))$$

If we take $p = 1$ we get only the contribution from $r = s = t = 0$ from the sum in the second line,

$$H_n(a, b, c) = - \frac{1}{2n} \frac{\cos \frac{\pi n}{1+a} (2+a+2\epsilon)}{\sin \frac{\pi n(b-a)}{1+a} \sin \frac{\pi n(c-a)}{1+a}}$$

which then leads to a result that is in a good agreement with eq. (2.67) in [8].

To take the unsquashed limit, we first assume that $\epsilon = 0$ and define

$$\tilde{H}_n(\lambda, a, b, c) = H_n(\lambda a, \lambda b, \lambda c) + H_n(\lambda b, \lambda c, \lambda a) + H_n(\lambda c, \lambda a, \lambda b)$$

and then Taylor expand

$$\tilde{H}_n(\lambda, a, b, c) = -\frac{1}{2n^3\pi^2} + \frac{2abc}{15}n\pi^2\lambda^3 + \mathcal{O}(\lambda^4)$$

and we get

$$\begin{aligned}
 &\frac{1}{2} \sum_{n=1}^{\infty} (H_n(a, b, c) + H_n(b, c, a) + H_n(c, a, b)) \\
 &= -\frac{p^2}{4\pi^2} \zeta(3) + \frac{p^2}{15} abc \zeta(-1) + \mathcal{O}(|a, b, c|^4)
 \end{aligned}$$

where the factor p^2 comes from the sum $\sum_{\mathcal{Q}_{p-1,p-2}} 1 = p^2$. This result shows that the unsquashed limit is smooth and we reproduce our previous result (4.8), if we interpret the sum by means of zeta function regularization.

If we keep ϵ nonzero, then there will be a correction to this result on the form

$$\tilde{H}_n(\lambda, a, b, c) = -\frac{1}{2n^3\pi^2} + \mathcal{O}(\epsilon^3, \lambda^4)$$

which is still consistent with our previous result that we computed up to order $\mathcal{O}(\epsilon^2)$.

Let us move on to the instanton contribution. We then consider the sum

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} (H_{n,k}(a, b, c) + H_{n,k}(b, c, a) + H_{n,k}(c, a, b))$$

where we define

$$H_{n,k}(a, b, c) = H_n(a, b, c) e^{-\frac{4\pi^2 nk}{p\beta(1+a)}}$$

To take the unsquashed limit, we define

$$\tilde{K}_{n,k}(\lambda; a, b, c) = K_{n,k}(\lambda a, \lambda b, \lambda c) + K_{n,k}(\lambda b, \lambda c, \lambda a) + K_{n,k}(\lambda c, \lambda a, \lambda b)$$

and expand \tilde{K} up zeroth order in λ ,

$$\tilde{K}_{n,k}(\lambda; a, b, c) = \left(\frac{2k}{\beta n^2 p} + \frac{1}{2\pi^3 n^3} + \frac{4\pi^2 k^2}{p^2 \beta^2 n} \right) e^{-\frac{4\pi^2 kn}{p\beta}} + \mathcal{O}(\lambda)$$

We notice that no singular terms appear in this expansion, and that the finite term is independent of a, b, c , which means that the unsquashed limit is well-defined and does not depend on how we let a, b, c approach to zero as long as $a + b + c = 0$. This result then leads to a perfect agreement with our previous result in eq. (4.7). Hence also for the instanton contribution, the unsquashed limit is smooth.

5.5.2. The tensor multiplet

For the tensor multiplet, the cyclic symmetry in a, b, c is hidden once we separate out the first term in (5.9). We will now dualize the terms in the second line in (5.9) which are antisymmetric under $\beta \rightarrow -\beta$ so that we can apply the plethystic dualization method on these terms alone.⁴ We get the following residues at a given n when we compute the integral $\int ds f_{tensor,reg}(s)/s$ from the first two terms on the second line of eq. (5.9),

$$\begin{aligned} & \frac{\cos \frac{\pi n 3a}{1+a} + \cos \frac{\pi n(a+2b)}{1+a}}{2n \sin \frac{\pi n(b-a)}{1+a} \sin \frac{\pi n(c-a)}{1+a}} \\ & \frac{\cos \frac{\pi n(b+2a)}{1+b} + \cos \frac{\pi n 3b}{1+b}}{2n \sin \frac{\pi n(a-b)}{1+b} \sin \frac{\pi n(c-b)}{1+b}} \\ & \frac{\cos \frac{\pi n(c+2a)}{1+c} + \cos \frac{\pi n(c+2b)}{1+c}}{2n \sin \frac{\pi n(a-c)}{1+c} \sin \frac{\pi n(b-c)}{1+c}} \end{aligned} \tag{5.13}$$

⁴ Later we will take back this statement, due to regularization issues.

These are cyclic permutations up to the following terms

$$-\frac{\cos \frac{\pi n 3b}{1+b} - \cos \frac{\pi n(b+2c)}{1+b}}{2n \sin \frac{\pi n(a-b)}{1+b} \sin \frac{\pi n(c-b)}{1+b}} = \frac{1}{n}$$

and

$$-\frac{\cos \frac{\pi n 3a}{1+a} - \cos \frac{\pi n(a+2c)}{1+a}}{2n \sin \frac{\pi n(b-a)}{1+a} \sin \frac{\pi n(c-a)}{1+a}} = \frac{1}{n}$$

where we have used the trigonometric identity $2 \sin A \sin B = \cos(A - B) - \cos(A + B)$. The last term on the second line in eq. (5.9) contributes something that has already cyclic permutation symmetry,

$$\sum \frac{\cos \frac{\pi n(a+2(ra+sb+tc)/p)}{1+a}}{2n \sin \frac{\pi n(b-a)}{1+a} \sin \frac{\pi n(c-a)}{1+a}} + \text{cycl}$$

Summing all the contributions coming from the second line in eq. (5.9) we get the result

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{ds}{s} f_{\text{tensor,reg,2nd line}}(s) = \sum_{n=1}^{\infty} \frac{1}{2n} (t_n(a, b, c) + t_n(b, c, a) + t_n(c, a, b) + 2)$$

where

$$t_n(a, b, c) = -\frac{\cos \frac{\pi n(b-c)}{1+a}}{\sin \frac{\pi n(b-a)}{1+a} \sin \frac{\pi n(c-a)}{1+a}} + \sum_{r,s,t \in Q_{p,p-3}} \frac{\cos \frac{\pi n(a+2(ra+sb+tc)/p)}{1+a}}{2 \sin \frac{\pi n(b-a)}{1+a} \sin \frac{\pi n(c-a)}{1+a}}$$

For $p = 1$ this reduces to

$$t_n(a, b, c) = \frac{1}{2} T_n(a, b, c) - 1$$

where

$$T_n(a, b, c) = \frac{\cos \frac{\pi n(b-c)}{1+a}}{2 \sin \frac{\pi n(b-a)}{1+a} \sin \frac{\pi n(c-a)}{1+a}}$$

and we get

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{ds}{s} f_{\text{tensor,reg}}(s) = \sum_{n=1}^{\infty} \frac{1}{2n} (T_n(a, b, c) + T_n(b, c, a) + T_n(c, a, b) - 1) \tag{5.14}$$

This is in good agreement with eq. (2.66) in [8]. Let us expand around $a = b = c = 0$ up to cubic order. For simplicity let us take $p = 1$. Thus we define

$$\tilde{T}_n(\lambda, a, b, c) = T_n(\lambda a, \lambda b, \lambda c) + T_n(\lambda b, \lambda c, \lambda a) + T_n(\lambda c, \lambda a, \lambda b)$$

The small λ -expansion reads

$$\tilde{T}_n(\lambda, a, b, c) = \frac{1}{4\pi^3} \frac{1}{n^3} + \frac{1}{2n} - \frac{3\pi^2}{5} abc n \lambda^3 + \mathcal{O}(\lambda^4)$$

The problematic term $1/(2n)$ cancels against $-1/(2n)$. Then after carrying out the summation over n using zeta function regularization, we get

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{ds}{s} f_{tensor,reg}(s) = \frac{1}{4\pi^3} \zeta(3) - \frac{3\pi^2}{5} abc \zeta(-1) + \mathcal{O}(|a, b, c|^4)$$

Up to cubic order, this expression is completely symmetric in a, b, c and there is no reason to expect this symmetry will be broken at higher orders. But there is obviously a problem here since if we dualize the first simple term in (5.10), then we get a term proportional to

$$\frac{1}{2} \log(\beta(1+c)) \tag{5.15}$$

which breaks the permutation symmetry among a, b, c . To restore it by adding the contribution that we get from dualizing the second term, the second term can not lead to a result that is completely symmetric in a, b, c [8]. We may not be allowed to dualize the second term by the plethystic method, since we can not dualize the first simple term in (5.10) by the plethystic method. We demonstrate this fact at the end of appendix C. To use the plethystic method to dualize the first term, we need to regularize a divergent plethystic sum $\sum 1/n$ which can not be regularized using the zeta function. We may for instance multiply the generating function by a gaussian regulator $e^{-\epsilon\beta^2}$ (which is symmetric in β) and then at the end take $\epsilon \rightarrow 0$. This effectively places a cutoff at $n \sim N_\epsilon \sim 1/\sqrt{\epsilon}$ for the sum over n . The same regularization should then be used throughout, hence to the whole expression in (5.10). Now, if we apply this regularization to the second term as well, it will amount to a multiplication of each of the three terms at the three lines in eq. (5.13) by their corresponding regulator factor $e^{-\epsilon s^2}$ where s is evaluated at the three poles in (5.12) for each line respectively. The correction is proportional to ϵ for the zeroth order term in λ , and so it goes to zero as we take ϵ to zero. But interesting regularization effects can show up at cubic order where we are regularizing a divergent sum $\sum n$. This will then contain a divergent piece $\sum_{n=1}^{N_\epsilon} n \sim N_\epsilon^2 \sim 1/\epsilon$, which is canceled against the order ϵ correction to (5.13), which is asymmetric in a, b, c . A detailed such computation would involve the error function and so it would be quite involved. Let us here content ourselves with noting that in the limit $a, b, c \rightarrow 0$ we reproduce the correct result, which agrees with our previous results that we obtained by rigorous methods. When a, b, c is away from zero, we may not have got the entirely correct result by the plethystic dualization method, but the error should be well-confined and small as long as λ is small. The error we made is only concerning the perturbative part. No instanton sum is affected by this issue, since the instanton sums are convergent.

Let us move on to the instanton sum. We define

$$T_{n,k}(a, b, c) = T_n(a, b, c) e^{-\frac{4\pi^2 nk}{\beta(1+a)}}$$

and

$$\tilde{T}_{n,k}(\lambda, a, b, c) = T_{n,k}(\lambda a, \lambda b, \lambda c) + T_{n,k}(\lambda b, \lambda c, \lambda a) + T_{n,k}(\lambda c, \lambda a, \lambda b)$$

This has the small λ -expansion

$$\tilde{T}_{n,k}(\lambda, a, b, c) = \left(\frac{k}{\beta n^2} + \frac{1}{2n} + \frac{1}{4\pi^2 n^3} + \frac{2\pi^2 k}{\beta^2 n} \right) e^{-\frac{4\pi^2 kn}{\beta}} + \mathcal{O}(\lambda^2)$$

By finally adding the contribution coming from the first term in (5.9), which gives the contribution

$$\frac{\pi^2}{6\beta} + \frac{1}{2} \log \frac{\beta}{2\pi}$$

we reduce to our previous result (4.4) in the unsquashed limit.⁵

6. Asymptotic S-duality

In [14] it was argued that for the geometry $S^1_\beta \times (S^5_r/\mathbb{Z}_p)$ there appears an emergent rectangular T^2 spanned by S^1_β and the Hopf fiber of S^5 , in the limit when p becomes very large. It was then argued that there would be an S-duality associated with this emergent T^2 . The radius of the temporal S^1_β is β , while the radius of the Hopf fiber is $2\pi r/p$ where r is the radius of S^5 . In order to exchange these two circles, it is convenient to follow [14] and put $\beta = 2\pi r/k$ for some integer k . The S-dual geometry will then correspond to $S^1_{\beta_D} \times (S^5_r/\mathbb{Z}_k)$ with $\beta_D = 2\pi r/p$. For a T^2 to emerge on both sides of the duality, we need to assume that both p and k are very large integer numbers. For the duality to relate high and low temperature behaviors, we need to in addition assume that $k \ll p$. Then $\beta_D \ll \beta$ which means that the S-dual geometry corresponds to the high temperature side of the duality.

We will now test whether asymptotic S-duality holds, which we can do since we know both the low temperature and the high temperature behaviors of the logarithm our indices (free energies). The high temperature behavior of the free energy is governed by the Stefan-Boltzmann terms, in which we shall put $\beta = 2\pi r/k$. The low temperature behavior is governed by the Casimir energy computed on S^5/\mathbb{Z}_k multiplied by $\beta_D = 2\pi r/p$.

6.1. Accidental asymptotic S-duality for the index

We begin with listing our results for the Stefan-Boltzmann terms

$$\beta F_{tensor} = \frac{\pi^4}{45Np\beta^3} - \frac{(6 + 5(ab + bc + ca))\pi^2}{36Np\beta}$$

$$\beta F_{hyper} = -\frac{\pi^4}{45Np\beta^3} - \frac{(ab + bc + ca + 6(\epsilon + \epsilon^2))\pi^2}{36Np\beta}$$

and for the Casimir energies

$$E_{tensor} = -\frac{p^3}{240} - \frac{p}{24} + \frac{a^4 + b^4 + c^4 - 36abc}{480Np}$$

$$E_{hyper} = \frac{p^3}{240} + \frac{p\epsilon}{24} - \frac{\epsilon^3}{12Np} - \frac{\epsilon^4}{24Np}$$

$$-\frac{a^4 + b^4 + c^4 + 4abc}{480Np}$$

⁵ For the comparison with (4.4) we should remember to multiply \tilde{T}_k by 2 since we defined this out of T that is in the perturbative part where we have the factor of 1/2 multiplying the integral $\int ds f_{reg}/s$, while there is no such factor 1/2 for the corresponding integral for the instanton contributions.

$$-\frac{2(ab+bc+ca)+abc}{24Np}\epsilon$$

$$-\frac{ab+bc+ca}{24Np}\epsilon^2$$

We are now ready to test asymptotic S-duality. In the Stefan-Boltzmann terms we put $\beta = 2\pi r/k$ and get

$$\beta F_{tensor} = \frac{\pi k^3}{360p} - \frac{(6+5(ab+bc+ca))\pi k}{72Np}$$

$$\beta F_{hyper} = -\frac{\pi k^3}{360p} - \frac{(ab+bc+ca)\pi k}{72Np}$$

and for the Casimir energies we replace p by k and multiply by $\beta_D = 2\pi r/p$ to get

$$\beta_D E_{tensor} = -\frac{\pi k^3}{120p} - \frac{\pi k}{12p} + \frac{\pi(a^4+b^4+c^4-36abc)}{240Nkp}$$

$$\beta_D E_{hyper} = \frac{\pi k^3}{120p} + \frac{\pi k\epsilon}{12p} - \frac{\pi\epsilon^3}{6Nkp} - \frac{\pi\epsilon^4}{12Nkp}$$

$$-\frac{\pi(a^4+b^4+c^4+4abc)}{240Nkp}$$

$$-\frac{\pi(2(ab+bc+ca)+abc)\epsilon}{12Nkp}$$

$$-\frac{\pi(ab+bc+ca)\epsilon^2}{12Nkp}$$

S-duality for (1, 0) supermultiplets would hold if we had $\beta F_i = \beta_D E_i$ for $i = \{tensor, hyper\}$. Clearly we do not have such an S-duality. Things improve if we consider (2, 0) theory for which we put $\epsilon = 0$. Then we have for the sum

$$\beta F_{M5} = -\frac{\pi k}{12Np} - \frac{(ab+bc+ca)\pi k}{12Np}$$

$$= -\frac{\pi k}{12p} + \frac{abc\pi k}{12Np}$$

where in the second step we used $N = 1 + ab + bc + ca + abc$. We also have

$$\beta_D E_{M5} = -\frac{\pi k}{12p} - \frac{abc}{6Nkp}$$

Thus we have $\beta F_{M5} = \beta_D E_{M5}$ if and only if $abc = 0$. Moreover, if we put $a = b = c = 0$ and keep ϵ arbitrary, then we have

$$\beta F_{M5} = -\frac{\pi k}{12p} - \epsilon \frac{\pi k}{12p} - \epsilon^2 \frac{\pi k}{12p}$$

$$\beta_D E_{M5} = -\frac{\pi k}{12p} + \epsilon \frac{\pi k}{12p} - \epsilon^3 \frac{\pi}{6kp} - \epsilon^4 \frac{\pi}{12kp}$$

Hence only when $\epsilon = 0$ and $abc = 0$ can we have asymptotic S-duality. We believe that this asymptotic S-duality that we see here is rather accidental, and a result of two competing effects.

On the one hand we have an increased amount of supersymmetry at $\epsilon = 0$ and $a = b = c = 0$. On the other hand we have with increased amount of supersymmetry also further cancellation of leading powers that lowers the leading power from T^3 down to T in the Stefan-Boltzmann terms in the large T limit, where $T = 1/\beta$ is the temperature. The cancellation of leading power appears to make asymptotic S-duality less likely to hold, but then increased supersymmetry apparently compensates for that so that we can see asymptotic S-duality nevertheless.

But asymptotic S-duality was expected to hold by a very general argument in [14], and thus we would not expect to only see this duality by some accident. We have found that in general there is no such asymptotic S-duality for indices, other than for a rather accidental choice of parameters. We could then ask ourselves why this is so. We believe that the answer is due to the fact that in 5d the generic leading term in the Stefan-Boltzmann law should generically grow like T^5 for large T and asymptotic S-duality is expected only for this leading term. But for the supersymmetric indices, we have no such high power leading term as T^5 due to supersymmetric cancellation. The asymptotic S-duality only holds in the very large p and k limits and thus is expected to be seen only for the T^5 term in the Stefan-Boltzmann law, which grows like k^5 when we put $\beta \sim 1/k$. To see those terms, we may instead consider the contribution to the index coming from each individual field before the cancellation has taken place. Or we may consider the partition function rather than the index. Indeed, here we will see asymptotic S-duality that seems to be generic, rather than accidental.

6.2. Generic asymptotic S-duality for individual fields

We will now demonstrate asymptotic S duality at leading order T^5 for each individual field in the (1, 0) tensor multiplet.

For the fields in the tensor multiplet, the scalar field (S), the tensor gauge field (T) and the Weyl fermions (F), we have on a round S^5 the following refined degeneracies [13]

$$d_n^S(\omega) = \sum_{m=0}^n d_{m,n-m} \omega^{2m-n}$$

$$d_n^T(\omega) = \sum_{m=0}^{n-1} \left(d_{m,n-m-1} \omega^{2m-n-2} + d_{m,n-m} \omega^{2m-n} + d_{m,n-m+1} \omega^{2m-n+2} \right)$$

$$d_n^F(\omega) = \sum_{m=0}^n \left(d_{m,n-m} \omega^{2m-n-3/2} + d_{m,n-m+1} \omega^{2m-n+1/2} \right)$$

where

$$d_{p,q} = \frac{1}{2} (p+1)(q+1)(p+q+2)$$

Although we have the relation $4d_n^S = d_{n-1}^F + d_n^F$ for the unrefined degeneracies, this relation does not extend to the refined case.

We define

$$D_n^S(\omega) = d_{n-2}^S(\omega)$$

$$D_n^T(\omega) = d_{n-2}^T(\omega)$$

$$D_n^F(\omega) = d_{n-3}^F(\omega) \omega^{-3/2} + d_{n-2}^F(\omega) \omega^{3/2}$$

and then we have the refined generating functions

$$f^i(\beta, \omega) = \sum_{n=0}^{\infty} D_n^i(\omega) e^{-\beta n}$$

for $i = S, T, F$ and where one may check that we can extend the sum all the way down to $n = 0$ since for $n = 0, 1$ there is no nonzero contribution to the sum. Although there are no nice and simple explicit expressions for these refined dimensions, we are able to repackage these refined dimensions into manageable closed form expressions for the refined generating functions,

$$f^S(\beta, \omega) = \frac{e^{-2\beta} - e^{-4\beta}}{(1 - \omega e^{-\beta})^3 (1 - \omega^{-1} e^{-\beta})^3}$$

$$f^T(\beta, \omega) = \frac{e^{-3\beta}(\omega^{-3} + 3\omega^{-1} + 6\omega) - 3e^{-4\beta}(\omega^{-2} + \omega^2 + 3) + 3e^{-5\beta}(\omega^{-1} + \omega) - e^{-6\beta}}{(1 - \omega e^{-\beta})^3 (1 - \omega^{-1} e^{-\beta})^3}$$

$$f^F(\beta, \omega) = \frac{e^{-2\beta}(1 + 3\omega^2) + e^{-3\beta}(\omega^{-3} + 3\omega^{-1} - 3\omega - \omega^3) - e^{-4\beta}(1 + 3\omega^{-2})}{(1 - \omega e^{-\beta})^3 (1 - \omega^{-1} e^{-\beta})^3}$$

We get a simplification when we compute the refined tensor multiplet generating function,

$$f_{tensor} = f^S + f^T - f^F = \frac{e^{-3\beta}\omega^3 - 3e^{-2\beta}\omega^2}{(1 - \omega e^{-\beta})^3}$$

but we will not consider this object here, but rather the contributions from the individual fields. To obtain the corresponding generating functions on lens space $L(p; 1, 1)$, we expand the denominator in an infinite series

$$f^i(\beta, \omega) = \frac{\sum_{\lambda} \omega^{\lambda} f_{\lambda}^i(\beta)}{(1 - \omega e^{-\beta})^3 (1 - \omega^{-1} e^{-\beta})^3}$$

$$= \sum_{\lambda} f_{\lambda}^i(\beta) \sum_{n,m=0}^{\infty} \frac{1}{4} (n+1)(n+2)(m+1)(m+2) e^{-\beta(n+m)} \omega^{n-m+\lambda}$$

and then we sum over ω running over all the p distinct p -th roots of unity that will put $n - m + \lambda = pq$. Let us assume that p is sufficiently large, such that

$$\lambda < p$$

Let us furthermore restrict ourselves to the case that $\lambda \geq 0$. Since $n = m + pq - \lambda \geq 0$, we see that for $q \geq 1$ there will be no further restriction on m coming from requiring that $m \geq \lambda - pq$ since by our assumptions we will have $\lambda - pq < 0$. Hence part of our sum will consist of

$$\sum_{q=1}^{\infty} \sum_{m=0}^{\infty} f(m + pq - \lambda, m)$$

Let us next assume that $q \leq -1$. We then bring this into $q \geq 1$ by first exchanging m and n assuming that the summand has this exchange symmetry, and next replacing λ by $\lambda' = -\lambda$ which is negative. We then need to analyze the case when $-p < \lambda' \leq 0$. Here we find no restrictions at all, so we have the contribution

$$\sum_{q=1}^{\infty} \sum_{m=0}^{\infty} f(m + pq + \lambda, m)$$

Now only remains the case when $q = 0$. Then $n = m - \lambda$. If $\lambda \geq 0$, then we have the contribution

$$\sum_{n=0}^{\infty} f(n, n + \lambda) = \sum_{m=0}^{\infty} f(m + \lambda, m)$$

If λ is negative, we have the contribution

$$\sum_{m=0}^{\infty} f(m - \lambda, m)$$

Summing all contributions, we get a quantity that we call S_{λ} ,

$$S_{\lambda} = \sum_{q=1}^{\infty} \sum_{m=0}^{\infty} (f(m + pq - \lambda, m) + f(m + pq + \lambda, m)) + \sum_{m=0}^{\infty} f(m + |\lambda|, m)$$

where we shall take

$$f(n, m) = \frac{1}{4}(n + 1)(n + 2)(m + 1)(m + 2)e^{-\beta(n+m)}$$

We then get the lensed indices as follows,

$$\begin{aligned} f^S &= (e^{-2\beta} - e^{-4\beta})S_0 \\ f^T &= e^{-3\beta}(S_3 + 9S_1) - e^{-4\beta}3(2S_2 + 3S_0) + e^{-5\beta}6S_1 - e^{-6\beta}S_0 \\ f^F &= (e^{-2\beta} - e^{-4\beta})(S_0 + 3S_2) \end{aligned}$$

Unlensed indices are reproduced by taking $p = 1$ and are

$$\begin{aligned} f^S &= \frac{e^{-2\beta} + e^{-3\beta}}{(1 - e^{-\beta})^5} \\ f^T &= \frac{e^{-5\beta} - 5e^{-4\beta} + 10e^{-3\beta}}{(1 - e^{-\beta})^5} \\ f^F &= 4 \frac{e^{-2\beta} + e^{-3\beta}}{(1 - e^{-\beta})^5} \end{aligned}$$

and in total

$$f^S + f^T - f^F = \frac{e^{-3\beta} - 3e^{-2\beta}}{(1 - e^{-\beta})^3}$$

Lensing gives

$$f^S + f^T - f^F = \frac{e^{-3p\beta} - (2 + p^2)e^{-2p\beta} + (1 - p^2)e^{-p\beta}}{(1 - e^{-p\beta})^3}$$

but the expressions for the individual contributions are quite lengthy. Let us therefore only present their small β expansions,

$$f^S = \frac{2}{p\beta^5} - \frac{1}{6p\beta^3} + \frac{-160 + 168p^2 + 21p^4 + 2p^6}{30240p}\beta$$

$$f^T = \frac{6}{p\beta^5} - \frac{5}{2p\beta^3} + \frac{1}{p\beta} - \frac{1}{2} + \frac{-832 + 1848p^2 - 63p^4 + 2p^6}{10080p}\beta$$

$$f^F = \frac{8}{p\beta^5} - \frac{2}{3p\beta^3} + \frac{-664 + 798p^2 - 105p^4 + 2p^6}{7560p}\beta$$

The leading order Stefan-Boltzmann terms are associated with

$$\zeta(6) = \frac{\pi^6}{945}$$

and are given by

$$\beta F_S = -\frac{2}{945} \frac{\pi^6 r^5}{p\beta^5}$$

$$\beta F_T = -\frac{2}{315} \frac{\pi^6 r^5}{p\beta^5}$$

$$\beta F_F = -\frac{8}{945} \frac{\pi^6 r^5}{p\beta^5}$$

The Casimir energy contributions in the large p limit are

$$E^S = -\frac{p^5}{30240r}$$

$$E^T = -\frac{p^5}{10080r}$$

$$E^F = -\frac{p^5}{7560r}$$

In the Stefan-Boltzmann behavior in the high temperature limit, we put $\beta = 2\pi r/k$, to get

$$\beta F^S = -\frac{2\pi k^5}{30240p}$$

$$\beta F^T = -\frac{2\pi k^5}{10080p}$$

$$\beta F^F = -\frac{2\pi k^5}{7560p}$$

We would now like to reproduce this from the low temperature and the Casimir energy. We then take $\beta_D = 2\pi r/p$ and consider $L(k; 1, 1)$ and get

$$\beta_D E^S = -\frac{2\pi k^5}{30240p}$$

$$\beta_D E^T = -\frac{2\pi k^5}{10080p}$$

$$\beta_D E^F = -\frac{2\pi k^5}{7560p}$$

We thus have got a complete agreement

$$\beta F^i = \beta_D E^i$$

for all the individual fields in the tensor multiplet and so we have an asymptotic S-duality at order T^5 that appears to be generic. We also can see that at subleading orders, we have no such agreement, just as was to be expected.

7. Discussion

We have found that the unsquashing limit $a, b, c \rightarrow 0$ is smooth. In the 5d localization computation, the unsquashed case is difficult to analyze since the instanton particles will spread out as we take $a = b = c = 0$ [8]. As long as the squashing parameters are not exactly zero, then no matter how small they are, once we fix their values and then take the localization limit, these instanton particles become localized at three fixed points on $\mathbb{C}P^2$, in the localization limit. But we may worry that the unsquashed limit is singular, or that the limit is discontinuous and that we get a different result when we put $a = b = c = 0$ compared to what we get by taking the limit $a, b, c \rightarrow 0$. Indeed such a discontinuity is natural to expect because we need a different computation when $a = b = c = 0$ in the 5d theory. It is therefore a nontrivial result to have found that the limit $a, b, c \rightarrow 0$ is actually smooth. It would be interesting to see if this smooth behavior extends to nonabelian gauge groups.

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Appendix A. The zeta function dualization method

We follow [9] and decompose the zeta function (3.1) into two parts,

$$\zeta(s) = \zeta_{\ell=0} + 2\zeta_{\ell>0}(s)$$

where we notice that $\zeta_{\ell<0} = \zeta_{\ell>0}$, which explains the factor 2. By a rewriting of the zeta function, we encounter coefficients

$$C_\lambda(a_\ell^2) = \sum_\sigma C_{\lambda,n} a_\ell^{2\sigma}$$

that are polynomials in a_ℓ^2 , where

$$a_\ell = \frac{2\pi\ell}{\beta}$$

Let us illustrate how such coefficient polynomials arise by a simple example. We consider the tensor multiplet zeta function (4.3) that (for $p = 1$) involves terms of the form

$$\begin{aligned} (n^2 - 1)(n^2 + a_\ell^2)^{-s} &= (n^2 + a_\ell^2 - a_\ell^2 - 1)(n^2 + a_\ell^2)^{-s} \\ &= (n^2 + a_\ell^2)^{1-s} - (a_\ell^2 + 1)(n^2 + a_\ell^2)^{-s} \end{aligned}$$

In this example, we have the coefficient polynomials

$$C_0(a_\ell^2) = -1 - a_\ell^2$$

$$C_1(a_\ell^2) = 1$$

In a more general situation, these coefficient polynomials are defined through the following expansions of the zeta function components above

$$\begin{aligned} \zeta_{\ell=0}(s) &= (\mu r)^{2s} \sum_{\lambda} \sum_{n=1}^{\infty} C_{\lambda,0}(np)^{2\lambda-2s} \\ &= (\mu r)^{2s} \sum_{\lambda} C_{\lambda,0} p^{2\lambda-2s} \zeta(2s - 2\lambda) \end{aligned}$$

and

$$\zeta_{\ell>0}(s) = (\mu r)^{2s} \sum_{\lambda} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} C_{\lambda}(a_\ell^2) (a_\ell^2 + (np)^2)^{\lambda-s}$$

respectively. We then apply the Mellin transform that puts $a_\ell^2 + (np)^2$ in the exponent,

$$(a_\ell^2 + (np)^2)^{\lambda-s} = \frac{1}{\Gamma(s - \lambda)} \int_0^\infty \frac{dt}{t} t^{s-\lambda} e^{-(a_\ell^2+n^2 p^2)t}$$

We now wish to dualize the sum with respect to n . To this end, we extend the sum over $n = 1, 2, \dots$ to also include $n = 0, -1, -2, \dots$, which we can do since n^2 is even. But then we have to remember to subtract the term with $n = 0$ again, and also divide the result by 2. Once we have got a sum over $n \in \mathbb{Z}$, we can apply the Poisson resummation formula to that sum,

$$\sum_{n \in \mathbb{Z}} e^{-tn^2} = \sqrt{\frac{\pi}{t}} \sum_{n_D \in \mathbb{Z}} e^{-\frac{\pi^2 n_D^2}{t}}$$

For the terms with $n_D \neq 0$ we then use the following integral formula for the modified Bessel function

$$\int_0^\infty \frac{dt}{t} t^v e^{-\frac{a}{t}-bt} = 2 \left(\frac{a}{b}\right)^{v/2} K_v(2\sqrt{ab})$$

while the integral we get for $n_D = 0$ is gives a Gamma function. Then we must, as we have said, also subtract the term with $n = 0$ (and then divide the whole thing by 2). This way, we end up with the following result,

$$\zeta_{\ell>0}(s) = \zeta_{\ell>0,n=0}(s) + \zeta_{\ell>0,n \neq 0}(s)$$

where

$$\begin{aligned} \zeta_{\ell>0,n=0}(s) &= \frac{\sqrt{\pi}}{2p} \mu^{2s} \sum_{\lambda,\sigma} C_{\lambda,\sigma} \frac{\Gamma(s - \lambda - 1/2)}{\Gamma(s - \lambda)} \left(\frac{2\pi}{\beta}\right)^{1+2\lambda+2\sigma-2s} \zeta(2s - 2\sigma - 2\lambda - 1) \\ &\quad - \frac{1}{2} \mu^{2s} \sum_{\lambda,\sigma} C_{\lambda,\sigma} \left(\frac{2\pi}{\beta}\right)^{2\lambda+2n-2\sigma} \zeta(2s - 2\sigma - 2\lambda) \end{aligned}$$

and

$$\zeta_{\ell>0, n\neq 0}(s) = \frac{\sqrt{\pi}}{p} \mu^{2s} \sum_{\lambda} \sum_{\ell=1}^{\infty} \sum_{n=1}^{\infty} \frac{C_{\lambda}}{\Gamma(s-\lambda)} \left(\frac{n\beta}{2p\ell}\right)^{s-\lambda-1/2} K_{s-\lambda-1/2}\left(\frac{4\pi^2 n\ell}{p\beta}\right)$$

Explicit forms of the Bessel functions that we will encounter are

$$K_{\pm 1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$$

$$K_{\pm 3/2}(x) = \sqrt{\frac{\pi}{2x}} \left(1 + \frac{1}{x}\right) e^{-x}$$

To get the index, we need to compute the derivative of the zeta function at $s = 0$. We have

$$\zeta'_{\ell=0}(0) = \sum_{\lambda} C_{\lambda,0} p^{2\lambda} 2\zeta'(-2\lambda)$$

For $\zeta'_{\ell>0}(0)$ we first bring up an overall factor of s by using $1/\Gamma(s) = s/\Gamma(s+1)$ and $1/\Gamma(s-1) = s(s-1)/\Gamma(s+1)$. We then obtain the zeta function on the form $\zeta(s) = s\tilde{\zeta}(s)$ and the derivative is then simply given by $\zeta'(0) = \tilde{\zeta}(0)$ since in all our examples $\tilde{\zeta}(s)$ will be regular at $s = 0$. Thus to compute the derivative, we never need to actually compute any derivative.

It would be more natural to apply Poisson resummation with respect to the sum over ℓ that is already over \mathbb{Z} . But then we would get the low temperature expansion (3.3) as was shown in [9]. That computation is very elegant. In particular the Casimir energy factor drops out automatically without any need to consider normal ordering and zero point energies.

Appendix B. The Abel-Plana dualization method

If we write the log of the index as a low temperature expansion

$$\log I = -\beta E + \sum_n d_n \log(1 - e^{-\beta E_n})$$

then we may compute the sum over n using the Abel-Plana integral formula. The analytic function that we need to consider in this application is given by

$$f(z) = d(z) \log(1 - e^{-\beta E(z)})$$

where $d(n) = d_n$ and $E(n) = E_n$ for $n = 0, 1, 2, \dots$, and analytically continued away from these integer values. If we compute the sum over n by the Abel-Plana integral formula, then we automatically turn this sum into a high temperature expansion.

Let us now present the Abel-Plana integration formula. A sum over n may (under certain conditions) be computed by a contour integral,

$$\sum_{n=0}^{\infty} f(n) = \oint_C dz \frac{f(-z)}{e^{-2\pi iz} - 1}$$

Here C is a counter clockwise contour surrounding the positive real axis, including the origin. Next we assume that f is analytic in the positive halfplane and behave nicely at infinity, to rotate the contour to the imaginary axis. Then the right-hand side becomes

$$\frac{1}{2} f(0) + i \int_{-\infty}^{\infty} dx \frac{f(ix)}{e^{2\pi x} - 1}$$

where we add $f(0)/2$ because the integral contour that goes through the pole at $z = 0$ picks up the other half of that same residue, to make up $f(0)/2 + f(0)/2 = f(0)$ in total. We separate the integral domain into two pieces $[-\infty, \infty] = [-\infty, 0] \cup [0, \infty]$ and rewrite the former integral

$$\begin{aligned} & i \int_{-\infty}^0 dx \frac{f(ix)}{e^{2\pi x} - 1} \\ &= i \int_0^{\infty} dx \frac{f(-ix)}{e^{-2\pi x} - 1} \\ &= -i \int_0^{\infty} dx \frac{f(-ix)}{e^{2\pi x} - 1} e^{2\pi x} \\ &= -i \int_0^{\infty} dx f(-ix) - i \int_0^{\infty} dx \frac{f(-ix)}{e^{2\pi x} - 1} \end{aligned}$$

The first integral is along the imaginary axis, but by assumption our function f is well-behaved at infinity and analytic in the positive real halfplane, and the integral can be Wick rotated to the positive real axis,

$$- \int_0^{\infty} dx f(x) + i \int_0^{-\infty} dx f(ix) = 0$$

Adding up, we then have

$$\sum_{n=0}^{\infty} f(n) = \frac{1}{2} f(0) + \int_0^{\infty} dx f(x) + i \int_0^{\infty} dx \frac{f(ix) - f(-ix)}{e^{2\pi x} - 1}$$

which is the Abel-Plana formula.

It is not a priori clear to us why the application of the Abel-Plana formula turns the low temperature expansion into a high temperature expansion, but it works this way in all explicit examples that we have encountered.

Appendix C. The plethystic dualization method

Here we describe the plethystic dualization method that was used in [8] to compute the index $I(\beta) = e^{-\beta E} \tilde{I}_{SB} \tilde{I}_{reg}$ where E is the Casimir energy. We separate the generating function into a singular and a regular part at $\beta = 0$. The singular part goes into the Stefan-Boltzmann factor \tilde{I}_{SB} as shown in the main text. The regular part is treated as follows,

$$\ln \tilde{I}_{reg}(\beta) = \sum_{n \in \mathbb{Z}} \int_{\epsilon}^{\infty} \frac{ds}{s} e^{\frac{2\pi i n s}{\beta}} f_{reg}(s)$$

We shall assume that $f_{reg}(s) = -f_{reg}(-s)$ and so $f_{reg}(s)/s$ does not have a simple pole at $s = 0$. We put

$$\mu(n, s) = \frac{ds}{s} e^{\frac{2\pi i n s}{\beta}} f_{reg}(s)$$

where we notice that $\mu(-n, s) = \mu(n, -s)$. We then consider the following rewritings

$$\begin{aligned} \sum_{n \in \mathbb{Z}_\epsilon} \int_\epsilon^\infty \mu(n, ns) &= \sum_{n=1}^\infty \int_\epsilon^\infty \mu(n, s) + \sum_{n=1}^\infty \int_\epsilon^\infty \mu(-n, s) + \int_\epsilon^\infty \mu(0, s) \\ &= \sum_{n=1}^\infty \left(\int_{-\infty}^{-\epsilon} \mu(n, s) + \int_\epsilon^\infty \mu(n, s) \right) + \frac{1}{2} \left(\int_{-\infty}^{-\epsilon} \mu(0, s) + \int_\epsilon^\infty \mu(0, s) \right) \\ &= \sum_{n=1}^\infty \int_{\mathbb{R}} \mu(n, s) + \frac{1}{2} \int_{\mathbb{R}} \mu(0, s) - \sum_{n=1-\epsilon}^\infty \int_\epsilon^\epsilon \mu(n, s) - \frac{1}{2} \int_\epsilon^\epsilon \mu(0, s) \end{aligned}$$

We next look at the third and fourth terms,

$$\begin{aligned} - \sum_{n=1-\epsilon}^\infty \int_\epsilon^\epsilon \mu(n, s) - \frac{1}{2} \int_{-\epsilon}^\epsilon \mu(0, s) &= -\frac{1}{2} \sum_{n \in \mathbb{Z}_{-\epsilon}} \int_\epsilon^\epsilon \mu(n, s) \\ &= -\frac{1}{2} \sum_{n \in \mathbb{Z}_{-\epsilon}} \int_\epsilon^\epsilon \frac{ds}{s} e^{\frac{2\pi i n s}{\beta}} f_{reg}(s) \\ &= -\frac{\beta}{2} \left[\frac{1}{s} f_{reg}(s) \right] \Big|_{s=0} \end{aligned}$$

We thus have three terms to compute,

$$\ln \tilde{I}_{reg} = A + B + C$$

where

$$\begin{aligned} A &= \sum_{n=1}^\infty \int_{\mathbb{R}} \frac{ds}{s} e^{\frac{2\pi i n s}{\beta}} f_{reg}(s) \\ B &= \frac{1}{2} \int_{\mathbb{R}} \frac{ds}{s} f_{reg}(s) \\ C &= -\frac{\beta}{2} \left[\frac{1}{s} f_{reg}(s) \right] \Big|_{s=0} \end{aligned}$$

By noting that $f_{reg}(s) = -f_{reg}(-s)$ implies $f_{reg}(0) = 0$, we can write

$$C = -\frac{\beta}{2} \lim_{s \rightarrow 0} \left[\frac{f_{reg}(s) - f_{reg}(0)}{s} \right] = -\frac{\beta}{2} \lim_{s \rightarrow 0} \partial_s f_{reg}(s)$$

We now see that

$$C = \beta E$$

where E is the Casimir energy

$$E = -\frac{1}{2} \partial_\beta \text{freg}(\beta)|_{\beta=0}$$

Then e^C cancels against $e^{-\beta E}$.

Let us try to illustrate this method by dualizing the Dedekind eta function. We start by rewriting its corresponding generating function as

$$f(\beta) = \frac{e^{-\beta}}{1 - e^{-\beta}} = \frac{\cosh \frac{\beta}{2}}{2 \sinh \frac{\beta}{2}} - \frac{1}{2}$$

Since the first term is antisymmetric, we can apply the plethystic dualization method on this term. The plethystic sum is given by

$$\sum_{n=1}^{\infty} \frac{1}{n} f(n\beta) = \sum_{n=1}^{\infty} \left(\frac{1}{2n} \frac{\cosh \frac{n\beta}{2}}{\sinh \frac{n\beta}{2}} - \frac{1}{2n} \right) \quad (\text{C.1})$$

We note that although the whole expression is convergent, being equal to $-\sum_{n=1}^{\infty} \log(1 - e^{-n\beta})$, the sum $\sum_{n=1}^{\infty} -\frac{1}{2n}$ is divergent and has to be regularized if we shall be able to separate the two terms. But let us ignore this, and just apply the plethystic dualization method on the first term. From the singular piece, we get the Stefan-Boltzmann term

$$\sum_{n \in \mathbb{Z}} \int_{\epsilon}^{\infty} \frac{ds}{s} e^{2\pi i n s / \beta} \frac{1}{s} = \frac{\pi^2}{6\beta}$$

and from the regular piece we get the perturbative and nonperturbative contributions

$$\begin{aligned} \frac{1}{2} \int_{-\infty}^{\infty} \frac{ds}{s} \frac{\cosh \frac{s}{2}}{\sinh \frac{s}{2}} &= \sum_{n=1}^{\infty} \frac{1}{2n} \\ \int_{-\infty}^{\infty} \frac{ds}{s} \frac{\cosh \frac{s}{2}}{\sinh \frac{s}{2}} e^{-4\pi^2 k n / \beta} &= \sum_{n=1}^{\infty} \frac{1}{n} e^{-4\pi^2 k n / \beta} \end{aligned}$$

respectively. The perturbative contribution cancels against the second term in the plethystic sum (C.1), leaving us with

$$\sum_{n=1}^{\infty} \frac{1}{n} f(n\beta) = \frac{\pi^2}{6\beta} + \sum_{k,n=1}^{\infty} \frac{1}{n} e^{-\frac{4\pi^2 k n}{\beta}}$$

The correct answer should have in addition a log-term $\frac{1}{2} \log \beta$, which we are missing.

To make this computation rigorous, we may regularize the divergences by replacing $f(\beta)$ with $f(\beta)e^{-\epsilon\beta^2}$ and then at the end take $\epsilon \rightarrow 0$. We could alternatively use the zeta function dualization method and start with $\zeta(s) = \mu^{2s} \sum_{n \in \mathbb{N}} \sum_{\ell \in \mathbb{Z}} (a_\ell^2 + n^2)^{-s}$ and obtain the high temperature expansion by dualizing with respect to n . We expect that the result will not depend on which regularization we use.

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