



Cohomology of the toric arrangement associated with A_n

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Abstract. We compute the total cohomology of the complement of the toric arrangement associated with the root system A_n as a representation of the corresponding Weyl group via fixed point theory of a “twisted” action of the group. We also provide several proofs of an explicit formula for the Poincaré polynomial of the complement of the toric arrangement associated with A_n .

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1. Introduction

An arrangement \mathcal{A} is a finite set of closed subvarieties of a variety X . A finite group Γ of automorphisms of X , which fixes \mathcal{A} as a set, acts on the complement $X_{\mathcal{A}}$ of \mathcal{A} . The group Γ , therefore, also acts on the de Rham cohomology groups $H^i(X_{\mathcal{A}})$ which in this way become Γ -representations. It is an interesting, but often hard, problem to determine these representations. A somewhat easier, but still interesting, problem is to determine the total cohomology $H^*(X_{\mathcal{A}})$ as a representation of Γ .

In Sect. 2 we consider this problem and, generalizing ideas of Felder and Veselov [9], we develop a method for computing $H^*(X_{\mathcal{A}})$ via fixed point theory, provided that the cohomology groups $H^i(X_{\mathcal{A}})$ have sufficiently nice mixed Hodge structure. It is known that many important classes of arrangements are of this type and in Sect. 4 we apply our method in the case of a toric arrangement associated with the root system A_n to compute the total cohomology as a representation of the Weyl group of A_n .

Theorem 1.1. *Let W_{A_n} be the Weyl group of the root system A_n . Then the total cohomology of the complement X_{A_n} of the toric arrangement associated with A_n is the W_{A_n} -representation*

$$H^*(X_{A_n}) = \text{Reg}_{W_{A_n}} + n \cdot \text{Ind}_{\langle s \rangle}^{W_{A_n}}(\text{Triv}_{\langle s \rangle}),$$

where $\text{Reg}_{W_{A_n}}$ is the regular representation of W_{A_n} and $\text{Ind}_{\langle s \rangle}^{W_{A_n}}(\text{Triv}_{\langle s \rangle})$ denotes the representation of W_{A_n} induced up from the trivial representation of the subgroup generated by the simple reflection $s = (12)$.

In Sect. 5 we continue our study of the complement of the toric arrangement associated with A_n , but we now forget about the action of the Weyl group and focus on the Poincaré polynomial, i.e. we study the dimensions of the individual cohomology groups.

Theorem 1.2. *The Poincaré polynomial of the complement X_{A_n} of the toric arrangement associated with A_n is given by*

$$P(X_{A_n}, t) = \prod_{i=1}^n (1 + (i+1) \cdot t).$$

1.1. Previous work

Motivated by questions related to the braid group, the study of the cohomology of arrangement complements was initiated by Arnol'd in [2] where he computed the cohomology ring of the complement of the hyperplane arrangement associated with A_n . In particular, he found the formula

$$\prod_{i=1}^n (1 + i \cdot t). \tag{1.1}$$

for the Poincaré polynomial. This formula was later generalized to general root systems by Orlik and Solomon [21]. Theorem 1.2 is thus a toric analogue of Arnol'd's formula. It is certainly well known to experts (it is in fact implicit already in the work of Arnol'd), but the explicit statement seems to be missing from the literature. The purpose of Sect. 5 is thus to record the result as well as to discuss its proof from various points of view.

The study of Arnol'd was continued in a slightly different direction by Brieskorn [5] who described the action of the Weyl group of A_n on the cohomology ring of the complement of the associated hyperplane arrangement. These results were later improved and extended by Lehrer [15] and Lehrer and Solomon [16]. The study of the cohomology of complements of toric arrangements was initiated by Looijenga in [17] as part of his computation of the cohomology of the moduli space \mathcal{M}_3 of smooth curves of genus three (it should be mentioned that this paper contains some mistakes which later have been corrected by Getzler and Looijenga himself in [12] and by De Concini and Procesi in [7]). Theorem 1.1 is a toric analogue of the works of Brieskorn [5], Lehrer [15] and Lehrer–Solomon [16]. It should be pointed out that it can be derived from results of Getzler [13] (who was investigating moduli spaces of rational curves), as well as from results of Gaiffi [11], Mathieu [19] and Robinson and Whitehouse [22] (who all were more directly interested in arrangements). Relevant formulas (from the point of view of representation stability) can also be found in the work of Hersh and Reiner [14]. Our main contribution is thus the new, fixed point theoretic proof. It also seems to

us that our method is the most promising for generalizations, e.g. to make computations for other root systems of classical type.

2. Arrangements

Unless otherwise specified, we shall always work over the complex numbers.

Definition 2.1. Let X be a variety. An *arrangement* \mathcal{A} in X is a finite set $\{A_i\}_{i \in I}$ of closed subvarieties of X , where I is a finite index set.

Given an arrangement \mathcal{A} in a variety X one may define its *cycle*

$$D_{\mathcal{A}} = \bigcup_{i \in I} A_i \subset X,$$

and its *open complement*

$$X_{\mathcal{A}} = X \setminus D_{\mathcal{A}}.$$

The variety $X_{\mathcal{A}}$, or rather its de Rham cohomology groups $H^i(X_{\mathcal{A}})$, will be our main object of study. We shall always consider cohomology with rational coefficients.

Let Γ be a finite group of automorphisms of X that stabilizes \mathcal{A} as a set. The action of Γ induces actions on $X_{\mathcal{A}}$ and on the cohomology groups $H^i(X_{\mathcal{A}})$. Each individual cohomology group $H^i(X_{\mathcal{A}})$ thus becomes a Γ -representation and, therefore, so does the total cohomology $H^*(X_{\mathcal{A}})$. We shall now explain a method to determine $H^*(X_{\mathcal{A}})$ in a large class of interesting cases, including arrangements of hyperplanes and toric arrangements, generalizing ideas of Felder and Veselov [9].

2.1. The total cohomology

Let \mathcal{A} be an arrangement in a variety X and let Γ be a finite group of automorphisms of X that fixes \mathcal{A} as a set. The group Γ will then act on the individual cohomology groups of $X_{\mathcal{A}}$ and thus on the total cohomology

$$H^*(X_{\mathcal{A}}) := \bigoplus_{i \geq 0} H^i(X_{\mathcal{A}}).$$

The value of the *total character* at $g \in \Gamma$ is defined as

$$P(X_{\mathcal{A}})(g) := \sum_{i \geq 0} \text{Tr}(g, H^i(X_{\mathcal{A}})),$$

and the *Lefschetz number* of $g \in \Gamma$ is defined as

$$L(X_{\mathcal{A}})(g) := \sum_{i \geq 0} (-1)^i \cdot \text{Tr}(g, H^i(X_{\mathcal{A}})).$$

Let $X_{\mathcal{A}}^g$ denote the fixed point locus of $g \in \Gamma$. Lefschetz fixed point theorem, see [6], then states that the Euler characteristic $E(X_{\mathcal{A}}^g)$ of $X_{\mathcal{A}}^g$ equals the Lefschetz number of g , i.e.

$$E(X_{\mathcal{A}}^g) = L(X_{\mathcal{A}})(g).$$

We now specialize to the case when each cohomology group $H^i(X_{\mathcal{A}})$ has mixed Hodge structure of the form

$$H^i(X_{\mathcal{A}}) = \bigoplus_{j \equiv i \pmod 2} W_{j,j} H^i(X_{\mathcal{A}}),$$

where $W_{j,j} H^i(X_{\mathcal{A}})$ denotes the part of $H^i(X_{\mathcal{A}})$ of Tate type (j, j) and where each element of \mathcal{A} is fixed by complex conjugation. It is known, through work of Brieskorn [5] and Looijenga [17], that if \mathcal{A} is a hyperplane arrangement or a toric arrangement, then $H^i(X_{\mathcal{A}})$ has Tate type (i, i) so many interesting examples are of this type (see [8] for a more complete discussion). We define an action of $\Gamma \times \mathbb{Z}_2$ on X by letting $(g, 0) \in \Gamma \times \mathbb{Z}_2$ act as $g \in \Gamma$ and $(\text{id}, 1) \in \Gamma \times \mathbb{Z}_2$ act by complex conjugation. Since \mathcal{A} is fixed under conjugation, this gives an action on $X_{\mathcal{A}}$. We write \bar{g} to denote the element $(g, 1) \in \Gamma \times \mathbb{Z}_2$.

Since $H^i(X_{\mathcal{A}})$ only has parts with Tate type congruent to (i, i) modulo 2, complex conjugation acts as $(-1)^i$ on $H^i(X_{\mathcal{A}})$. We thus have

$$\begin{aligned} L(X_{\mathcal{A}})(\bar{g}) &= \sum_{i \geq 0} (-1)^i \text{Tr}(\bar{g}, H^i(X_{\mathcal{A}})) \\ &= \sum_{i \geq 0} (-1)^i \cdot (-1)^i \cdot \text{Tr}(g, H^i(X_{\mathcal{A}})) \\ &= P(X_{\mathcal{A}})(g). \end{aligned}$$

Since $L(X_{\mathcal{A}})(\bar{g}) = E(X_{\mathcal{A}}^{\bar{g}})$, we have proved the following lemma:

Lemma 2.2. *Let X be a smooth variety and let \mathcal{A} be an arrangement in X which is fixed by complex conjugation and such that $H^i(X_{\mathcal{A}})$ only has parts of Tate type congruent to (i, i) modulo 2. Let Γ be a finite group which acts on X as automorphisms and which fixes \mathcal{A} as a set. Then*

$$P(X_{\mathcal{A}})(g) = E(X_{\mathcal{A}}^{\bar{g}}).$$

3. Toric arrangements associated with root systems

Classically, arrangements of hyperplanes have been given most attention. However, in the past two decades an increasing number of authors have considered also toric arrangements and they have been studied from the point of view of geometry, topology, algebra and combinatorics.

Definition 3.1. Let X be an n -torus. An arrangement \mathcal{A} in X is called a *toric arrangement* if each element of \mathcal{A} is a subtorus.

Remark 3.2. We shall only be interested in the case where each subtorus in the arrangement has codimension one, i.e. where the arrangement is divisorial, and write “toric arrangement” to mean “divisorial toric arrangement”.

Let Φ be a root system, let $\Delta = \{\beta_1, \dots, \beta_n\}$ be a set of simple roots and let Φ^+ be the set of positive roots of Φ with respect to Δ . We think of Φ as a set of vectors in some real Euclidean vector space V and we let M be the \mathbb{Z} -linear span of Φ . Thus, M is a free \mathbb{Z} -module of finite rank n .

Define $X = \text{Hom}(M, \mathbb{C}^*) \cong (\mathbb{C}^*)^n$. The Weyl group W_Φ of Φ acts on X from the right by precomposition, i.e.

$$(\chi \cdot g)(v) = \chi(g \cdot v).$$

For each $\alpha \in \Phi$ we define

$$A_\alpha = \{\chi \in X \mid \chi(\alpha) = 1\}.$$

We thus obtain an arrangement of hypertori in X

$$\mathcal{A}_\Phi = \{A_\alpha\}_{\alpha \in \Phi}.$$

To avoid cluttered notation we shall write X_Φ instead of the more cumbersome $X_{\mathcal{A}_\Phi}$.

Let $\chi \in X$. We introduce the notation $\chi(\beta_i) = z_i$ for the simple roots β_i , $i = 1, \dots, n$. The coordinate ring of X is then

$$\mathbb{C}[X] = \mathbb{C}[z_1, \dots, z_n, z_1^{-1}, \dots, z_n^{-1}].$$

If α is a root, there are integers m_1, \dots, m_n such that

$$\alpha = m_1 \cdot \beta_1 + \dots + m_n \cdot \beta_n.$$

With this notation we have that $\chi(\alpha) = 1$ if and only if

$$z_1^{m_1} z_2^{m_2} \dots z_n^{m_n} = 1.$$

4. Root systems of type A

The root system A_n is most naturally viewed in an n -dimensional subspace of \mathbb{R}^{n+1} . Denote the i th coordinate vector of \mathbb{R}^{n+1} by e_i . The roots Φ can then be chosen to be

$$\alpha_{i,j} = e_i - e_j, \quad i \neq j.$$

A choice of positive roots is

$$\alpha_{i,j} = e_i - e_j, \quad i < j,$$

and the simple roots with respect to this choice of positive roots are

$$\beta_i = e_i - e_{i+1}, \quad i = 1, \dots, n.$$

The Weyl group of group A_n is isomorphic to the symmetric group S_{n+1} and an element of S_{n+1} acts on an element in $M = \mathbb{Z}\langle\Phi\rangle$ by permuting the indices of the coordinate vectors in \mathbb{R}^{n+1} .

4.1. The total character

In this section we shall compute the value of the total character at any element $g \in W_{A_n}$. This will determine the total cohomology $H^*(X_{A_n})$ as an W_{A_n} -representation. Although we have not pursued this, similar methods should allow the computation of $H^*(X_\Phi)$ also in the case of root systems of type B_n , C_n and D_n .

Lemma 4.1. *Let W_{A_n} be the Weyl group of A_n and suppose that $g \in W_{A_n}$ has a cycle of length greater than two. Then $X_{A_n}^g$ is empty.*

Proof. The statement only depends on the conjugacy class of g so suppose that g contains the cycle $(1, 2, \dots, s)$, where $s \geq 3$. We then have

$$\begin{aligned} g \cdot \beta_1 &= e_2 - e_3 = \beta_2, \\ g \cdot \beta_2 &= e_3 - e_4 = \beta_3, \\ &\vdots \\ g \cdot \beta_{s-2} &= e_{s-1} - e_s = \beta_{s-1}, \\ g \cdot \beta_{s-1} &= e_s - e_1 = -(\beta_1 + \dots + \beta_{s-1}). \end{aligned}$$

If $\bar{g} \cdot \chi = \chi$ we must have

$$z_i = \bar{z}_{i+1} \quad \text{for } i = 1, \dots, s - 2, \tag{1}$$

$$z_{s-1} = \bar{z}_1^{-1} \bar{z}_2^{-1} \dots \bar{z}_{s-1}^{-1}. \tag{2}$$

We insert (1) into (2) and take absolute values to obtain $|z_1|^s = 1$. We thus see that $|z_1| = 1$. Since we have $z_2 = \bar{z}_1$ it follows that

$$\chi(\alpha_{1,3}) = \chi(\beta_1 + \beta_2) = z_1 \cdot z_2 = z_1 \cdot \bar{z}_1 = |z_1|^2 = 1.$$

Thus, χ lies in $X_{\alpha_{13}}$ so $X_{\bar{\Phi}}$ is empty. □

If we apply Lemma 2.2 and Lemma 4.1 we obtain the following corollary:

Corollary 4.2. *If g is an element of the Weyl group of A_n such that $g^2 \neq \text{id}$, then*

$$P(X_{A_n})(g) = 0.$$

We thus know the total character of all elements in the Weyl group of A_n of order greater than 2. We shall, therefore, turn our attention to the involutions.

Lemma 4.3. *If g is an element of the Weyl group of A_n of order 2 which is not a reflection, then*

$$E(X_{A_n}^{\bar{g}}) = P(X_{A_n})(g) = 0.$$

Proof. Let $k > 1$ and consider the element $g = (1, 2)(3, 4) \dots (2k - 1, 2k)$. We define a new basis for M :

$$\begin{aligned} \gamma_i &= \beta_i + \beta_{i+1}, & i &= 1, \dots, 2k - 2, \\ \gamma_j &= \beta_j & j &= 2k - 1, \dots, n \end{aligned}$$

Then

$$g \cdot \gamma_{2i-1} = \gamma_{2i}, \quad \text{and} \quad g \cdot \gamma_{2i} = \gamma_{2i-1},$$

for $i = 1, \dots, k - 1$,

$$g \cdot \gamma_{2k-1} = -\gamma_{2k-1}, \quad g \cdot \gamma_{2k} = \gamma_{2k-1} + \gamma_{2k},$$

and $g \cdot \gamma_i = \gamma_i$ for $i > 2k$. If we put $\chi(\gamma_i) = t_i$, then $X_{A_n}^{\bar{g}} \subseteq X_{A_n}$ is given by the equations

$$\begin{aligned} t_{2i-1} &= \bar{t}_{2i} & i &= 1, \dots, k-1, \\ t_{2k-1} &= \bar{t}_{2k-1}^{-1}, \\ t_{2k} &= \bar{t}_{2k-1} \cdot \bar{t}_{2k}, \\ t_i &= \bar{t}_i, & i &= 2k+1, \dots, n. \end{aligned}$$

Thus, the points of $X_{A_n}^{\bar{g}}$ have the following form:

$$(t_1, \bar{t}_1, t_2, \bar{t}_2, \dots, t_{k-1}, \bar{t}_{k-1}, s, s^{-1/2} \cdot r, t_{2k}, \dots, t_n),$$

where $s \in S^1 \setminus \{1\} \subset \mathbb{C}$, $r \in \mathbb{R}^+$ and $t_i \in \mathbb{R}$ for $i = 2k, \dots, n$.

We can now see that each connected component of $X_{A_n}^{\bar{g}}$ is homeomorphic to $\mathcal{C}_{k-1}(\mathbb{C} \setminus \{0, 1\}) \times (0, 1) \times \mathbb{R}^{n-2k+1}$, where $\mathcal{C}_{k-1}(\mathbb{C} \setminus \{0, 1\})$ is the configuration space of $k-1$ points in the twice punctured complex plane. This space is in turn homotopic to the configuration space $\mathcal{C}_{k+1}(\mathbb{C})$ of $k+1$ points in the complex plane. The space $\mathcal{C}_{k+1}(\mathbb{C})$ is known to have Euler characteristic zero for $k \geq 1$. □

Remark 4.4. Note that it is essential that we use ordinary cohomology in the above proof, since compactly supported cohomology is not homotopy invariant. However, since X_{Φ} satisfies Poincaré duality, the corresponding result follows also in the compactly supported case.

We now turn to the reflections:

Lemma 4.5. *If g is a reflection in the Weyl group of A_n , then*

$$E(X_{A_n}^{\bar{g}}) = P(X_{A_n})(g) = n!.$$

Proof. Let $g = (1, 2)$. We then have

$$\begin{aligned} g \cdot \beta_1 &= -\beta_1, \\ g \cdot \beta_2 &= \beta_1 + \beta_2, \\ g \cdot \beta_i &= \beta_i, & i &> 2. \end{aligned}$$

This gives the equations

$$\begin{aligned} z_1 &= \bar{z}_1^{-1}, \\ z_2 &= \bar{z}_1 \cdot \bar{z}_2, \\ z_i &= \bar{z}_i, & i &> 2. \end{aligned}$$

Thus $z_1 \in S^1 \setminus \{1\}$, z_2 is not real and satisfies $z_2 = \bar{z}_1 \cdot \bar{z}_2$ so we choose z_2 from a space isomorphic to \mathbb{R}^* . Hence, $X_{A_n}^{\bar{g}} \cong [0, 1] \times \mathbb{R}^* \times Y$, where Y is the space where the last $n-2$ coordinates z_3, \dots, z_n takes their values.

These coordinates satisfy $z_i = \bar{z}_i$, i.e. they are real. We begin by choosing z_3 . Since $\chi(e_i - e_j) \neq 0, 1$ we need $z_3 \neq 0, 1$. We thus choose z_3 from $\mathbb{R} \setminus \{0, 1\}$. We then choose z_4 in $\mathbb{R} \setminus \{0, 1, \frac{1}{z_3}\}$, z_5 in $\mathbb{R} \setminus \{0, 1, \frac{1}{z_4}, \frac{1}{z_3 \cdot z_4}\}$ and so on. In the i th step we have i components to choose from. Thus, Y consists of

$$3 \cdot 4 \cdots n = \frac{n!}{2}$$

components, each isomorphic to \mathbb{R}^{n-2} . Hence, $E(Y) = \frac{n!}{2}$ and it follows that

$$E(T_{\mathbb{F}}^{\bar{g}}) = E([0, 1]) \cdot E(\mathbb{R}^*) \cdot E(Y) = n!.$$

□

It remains to compute the value of the total character at the identity element.

Lemma 4.6. $E(X_{A_n}^{\text{id}}) = P(X_{A_n})(\text{id}) = \frac{(n+2)!}{2}$

Proof. The proof is a calculation similar to that in the proof of Lemma 4.5. We note that the equations for $X_{A_n}^{\text{id}}$ are $z_i = \bar{z}_i, i = 1, \dots, n$ so the computation of $E(X_{A_n}^{\text{id}})$ is essentially the same as that for $E(Y)$ above. The difference is that we have n steps and in the i th step we have $i + 1$ choices. This gives the result. □

Lemmas 4.1, 4.3, 4.5 and 4.6 together determine the character of W_{A_n} on $H^*(X_{A_n})$. The representation $\text{Ind}_{\langle s \rangle}^{W_{A_n}}(\text{Triv}_{\langle s \rangle})$ takes value $\frac{(n+1)!}{2}$ on the identity element, $2(n - 1)!$ on transpositions and is zero elsewhere. Since the character of $H^*(X_{A_n})$ takes the value $(n + 2)!/2$ on the identity, $n!$ on transpositions and is zero elsewhere, we see that Theorem 1.1 holds.

Remark 4.7. The corresponding calculation for the affine hyperplane case was first computed by Lehrer in [15] and later by Felder and Veselov in [9]. In the hyperplane case, the total cohomology turned out to be $2 \text{Ind}_{\langle s \rangle}^{W_{A_n}}(\text{Triv}_{\langle s \rangle})$, where s is a transposition, i.e. the cohomology is twice the representation induced up from the trivial representation of the subgroup generated by a transposition. Thus, the representation $\text{Ind}_{\langle s \rangle}^{W_{A_n}}(\text{Triv}_{\langle s \rangle})$ accounts for most of the cohomology also in the hyperplane case.

5. The Poincaré polynomial

In this section we shall see that the Poincaré polynomial of the complement of the toric arrangement associated with A_n satisfies the formula given in Theorem 1.2. We give several proofs of this result. Before giving the first proof we remark that setting $t = 1$ in Theorem 1.2 gives another proof of Lemma 4.6.

Proof 1. The key observation in the first proof is that X_{A_n} is isomorphic to the moduli space $\mathcal{M}_{0,n+3}$ of smooth rational curves marked with $n + 3$ points. To see this, note that

$$X_{A_n} = \{(x_0, x_1, \dots, x_n) \in (\mathbb{C}^*)^{n+1} / \mathbb{C}^* | x_i \neq x_j\}.$$

We thus have that

$$(x_0, x_1, \dots, x_n) \mapsto [x_1/x_0, \dots, x_n/x_0, 0, 1, \infty] \in (\mathbb{P}^1)^{n+3}$$

gives an isomorphism $X_{A_n} \rightarrow \mathcal{M}_{0,n+3}$.

We count the number of points of $\mathcal{M}_{0,n+3}$ over a finite field \mathbb{F}_q with q elements by choosing $n + 3$ distinct points on \mathbb{P}^1 and dividing the result

by the order of $\text{PGL}(2, \mathbb{F}_q)$. We have $|\mathbb{P}^1(\mathbb{F}_q)| = q + 1$ and $|\text{PGL}(2, \mathbb{F}_q)| = (q + 1)q(q - 1)$. This gives

$$|\mathcal{M}_{0,n+3}(\mathbb{F}_q)| = \frac{\prod_{i=1}^{n+3} (q + 1 - (i - 1))}{(q + 1)q(q - 1)} = \prod_{i=1}^n (q - (i + 1)).$$

Call the above polynomial $p(q)$. By results of Dimca and Lehrer [8] we obtain the Poincaré polynomial of $\mathcal{M}_{0,n+3}$ as $(-t)^n \cdot p(-1/t)$. \square

The Poincaré polynomial of $\mathcal{M}_{0,n}$ is well known and the above proof is not new, although it seems to be missing in the literature. The advantage of the above proof is that it easily extends to make equivariant computations. Neither this is new and we refer the interested reader to Getzler [13] for the results (although his methods are quite different).

In the above proof, we computed something quite different from what we originally were interested in (the number of points of $\mathcal{M}_{0,n+3}$ over \mathbb{F}_q) and arrived at the desired result via a change of variables. This will also be the case in the second proof which has a more combinatorial flavour.

Proof 2. Let $M_{A_n}(x, y)$ denote the arithmetic Tutte polynomial corresponding to the toric arrangement associated with A_n . By Theorem 5.11 of [20] we have

$$P(X_{A_n}, t) = t^n M_{A_n} \left(\frac{2t + 1}{t}, 0 \right).$$

We introduce the new variables $X = (x - 1)(y - 1)$ and $Y = y$ and define $\psi_{A_n}(X, Y) = (-1)^n M_{A_n}(x, y)$. We then have

$$\psi_{A_n} \left(-\frac{t + 1}{t}, 0 \right) = \left(-\frac{1}{t} \right)^n P(X_{A_n}, t).$$

Let

$$F(x, y) = \sum_{n \geq 0} \frac{x^n y^{\binom{n}{2}}}{n!}$$

and define the generating function

$$\Psi_A(X, Y, Z) = 1 + X \cdot \sum_{n \geq 1} \psi_{A_{n-1}}(X, Y) \frac{Z^n}{n!}.$$

Then

$$\Psi_A \left(-\frac{t + 1}{t}, 0, Z \right) = 1 + \left(-\frac{t + 1}{t} \right) \cdot \sum_{n \geq 1} \left(-\frac{1}{t} \right)^{n-1} P(X_{A_{n-1}}, t) \frac{Z^n}{n!}. \tag{5.1}$$

By Theorem 1.14 of [1] we have $\Psi_A(X, Y, Z) = F(Z, Y)^X$. We thus get

$$\begin{aligned}
 \Psi_A \left(-\frac{t+1}{t}, 0, Z \right) &= F(Z, 0)^{-\frac{t+1}{t}} \\
 &= (1 + Z)^{-\frac{t+1}{t}} \\
 &= \sum_{n \geq 0} \binom{-\frac{t+1}{t}}{n} Z^n \\
 &= \sum_{n \geq 0} \frac{\left(-\frac{t+1}{t}\right) \left(-\frac{t+1}{t} - 1\right) \cdots \left(-\frac{t+1}{t} - n + 1\right)}{n!} Z^n \\
 &= 1 + \left(-\frac{t+1}{t}\right) \cdot \sum_{n \geq 1} \left(-\frac{1}{t}\right)^{n-1} \cdot \prod_{i=1}^{n-1} ((i+1)t + 1) \frac{Z^n}{n!}.
 \end{aligned}$$

We now get the result by comparing the above expression with Eq. 5.1. □

The advantage of the above proof is that it extends to other root systems of classical type. However, it does not seem to work well equivariantly.

To give the third and final proof we need some new terminology. Given an arrangement $\mathcal{A} = \{A_i\}_{i \in I}$ in a variety X we define the intersection poset of \mathcal{A} as the set

$$\mathcal{L}(\mathcal{A}) = \{\cap_{j \in J} A_j \mid J \subseteq I\}$$

ordered by reverse inclusion.

Let Φ be a root system in \mathbb{R}^n and let $M = \mathbb{Z}\langle\Phi\rangle$. We have defined a toric arrangement $\mathcal{A}_\Phi = \{A_\alpha\}_{\alpha \in \Phi}$ in $X = \text{Hom}(M, \mathbb{C}^*)$ associated with Φ by setting

$$A_\alpha = \{\chi \in X \mid \chi(\alpha) = 1\}$$

for all $\alpha \in \Phi$. Similarly, we define a hyperplane arrangement $\mathcal{B}_\Phi = \{B_\alpha\}_{\alpha \in \Phi}$ in $Y = \text{Hom}(M, \mathbb{C})$ by setting

$$B_\alpha = \{\phi \in Y \mid \phi(\alpha) = 0\}.$$

The arrangements \mathcal{A}_Φ and \mathcal{B}_Φ and their posets are related but to see how, we shall change the perspective slightly.

Let $V = M \otimes \mathbb{C}$. Fleischmann and Janiszczak [10] define the poset $\mathcal{P}(\Phi)$ as the poset of linear spans in V of subsets of Φ , ordered by inclusion, and show that the posets $\mathcal{L}(\mathcal{B}_\Phi)$ and $\mathcal{P}(\Phi)$ are isomorphic. We follow them and define $\mathcal{R}(\Phi)$ as the poset of submodules of M spanned by elements of Φ , ordered by inclusion. In very much the same way we have that the posets $\mathcal{L}(\mathcal{A}_\Phi)$ and $\mathcal{R}(\Phi)$ are isomorphic.

There is a surjective order preserving map $\rho : \mathcal{R}(\Phi) \rightarrow \mathcal{P}(\Phi)$ sending a submodule $N \subset M$ to $N \otimes \mathbb{C}$. If we represent elements in $\mathcal{R}(\Phi)$ and $\mathcal{P}(\Phi)$ by their echelon basis matrices, ρ sends a matrix C to the matrix obtained from C by dividing each row by the greatest common divisor of its entries, i.e. sending a matrix to its saturation. We thus see that ρ is an isomorphism if and only if every module in $\mathcal{R}(\Phi)$ is saturated. For $\Phi = A_n$ this is indeed the case. The pivotal observation for proving this is the following lemma, which

can be proven via a simple induction argument on the number of rows using Gaussian elimination.

Lemma 5.1. *Let C be a binary matrix of full rank such that the 1's in each row of C are consecutive. Then each pivot element in the row reduced echelon matrix (over \mathbb{Z}) obtained from C is 1.*

If one expresses the positive roots of A_n in terms of the simple roots β_i , each root is a vector of zeros and ones with all ones consecutive. It thus follows from Lemma 5.1 that the modules in $\mathcal{R}(A_n)$ are saturated and, therefore, we have that the map $\rho : \mathcal{R}(A_n) \rightarrow \mathcal{P}(A_n)$ is an isomorphism of posets.

Remark 5.2. After the appearance of the first preprint of this paper, Bibby [4] showed the stronger result that the intersection poset of a hyperplane arrangement, a toric arrangement and of an abelian arrangement associated with A_n is isomorphic to the partition lattice.

The above observation is one key ingredient in our third proof of Theorem 1.2. Another key ingredient is the following theorem of MacMeikan:

Theorem 5.3 (MacMeikan [18]). *Let $\mathcal{A} = \{A_i\}_{i \in I}$ be a toric arrangement or an arrangement of hyperplanes. Then*

$$P(X_{\mathcal{A}}, t) = \sum_{Z \in \mathcal{L}(\mathcal{A})} \mu(Z)(-t)^{\text{cd}(Z)} P(Z, t),$$

where $\text{cd}(Z)$ denotes the codimension of Z in X and μ denotes the Möbius function of $\mathcal{L}(\mathcal{A})$.

MacMeikan's result is in fact quite a bit stronger but this version is enough for our purposes. Note that if \mathcal{A} is a toric arrangement, then each $Z \in \mathcal{L}(\mathcal{A})$ is a disjoint union of tori and $P(Z, t) = c_Z(1 + t)^{\dim(Z)}$ where c_Z is the number of components of Z . Note also that if \mathcal{A} is a hyperplane arrangement, then each $Z \in \mathcal{L}(\mathcal{A})$ is an affine space and, therefore, $P(Z, t) = 1$ and Theorem 5.3 thus reduces to the Orlik–Solomon formula in this case.

Proof 3. Let $\mu_{\mathcal{P}}$ denote the Möbius function of $\mathcal{P}(\Phi)$. Theorem 5.3 and Eq. (1.1) tell us that

$$\sum_{V \in \mathcal{P}(\Phi)} \mu_{\mathcal{P}}(V) \cdot (-t)^{\dim(V)} = \prod_{i=1}^n (1 + i \cdot t).$$

By equating the coefficients of t^r we get

$$\begin{aligned} \sum_{\substack{V \in \mathcal{P}(\Phi) \\ \dim(V)=r}} \mu_{\mathcal{P}}(V)(-1)^r &= \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=r}} \prod_{i \in I} i \\ &= e_r(1, \dots, n), \end{aligned}$$

where e_r denotes the r th elementary symmetric polynomial. We saw above that the map $\rho : \mathcal{R}(\Phi) \rightarrow \mathcal{P}(\Phi)$ is an isomorphism of posets. Hence, if $\mu_{\mathcal{R}}$ denotes the Möbius function of $\mathcal{R}(\Phi)$ we have that

$$\mu_{\mathcal{R}}(N) = \mu_{\mathcal{P}}(\rho(N)).$$

It thus follows that

$$\sum_{\substack{N \in \mathcal{R}(\Phi) \\ \text{rk}(N)=r}} \mu_{\mathcal{R}}(N)(-1)^r = e_r(1, \dots, n). \tag{5.2}$$

If we apply Theorem 5.3 to X_{Φ} , we obtain

$$\begin{aligned} P(X_{\Phi}, t) &= \sum_{N \in \mathcal{R}(\Phi)} \mu_{\mathcal{R}}(N)(-t)^{\text{rk}(N)} \cdot (1+t)^{n-\text{rk}(N)} \\ &= \sum_{r=0}^n t^r \cdot (1+t)^{n-r} \sum_{\text{rk}(N)=r} \mu_{\mathcal{R}}(N) \cdot (-1)^r. \end{aligned}$$

If we use Eq. (5.2) we now see that the coefficient of t^k in $P(X_{\Phi}, t)$ is

$$\sum_{j=0}^k \binom{n-j}{k-j} \cdot e_j(1, \dots, n).$$

The coefficient of t^k in $\prod_{i=1}^n (1 + (i+1) \cdot t)$ is

$$\begin{aligned} \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=k}} (i_1 + 1) \cdots (i_k + 1) &= \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=k}} \sum_{j=0}^k e_j(i_1, \dots, i_k) \\ &= \sum_{j=0}^k \sum_{\substack{I \subseteq \{1, \dots, n\} \\ |I|=k}} e_j(i_1, \dots, i_k) \\ &= \sum_{j=0}^k \binom{n-j}{k-j} \cdot e_j(1, \dots, n). \end{aligned}$$

This proves the claim. □

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