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Searching for self-injective planar quivers with potential

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Abstract

To every quiver with potential (QP), one may associate an associative algebra called the Jacobian algebra. In this article, we study QPs whose potential is induced from a plane embedding of the quiver (in which case the QP is said to be planar) and whose Jacobian algebra is self-injective (in which case the QP is said to be self-injective). Motivation to study self-injective QPs comes from their relationship with 2-representation-finite algebras: basic 2-representation-finite algebras are precisely truncated Jacobian algebras of self-injective QPs [6]. Some families and sporadic examples of self-injective planar QPs appear in the literature [6, 7], but whether there exist other, still unknown self-injective planar QPs is an open problem.

We focus our attention on planar QPs, for which self-injectivity can be determined algorithmically, and search for self-injective planar QPs by means of computer assistance. Algorithms for working with these QPs (and bound quivers in greater generality) are presented, and implementations in C$^\sharp$ of the algorithms are provided. Using the provided implementations, we find fourteen self-injective planar QPs not previously appearing in the literature. Some of these new QPs fit into four infinite families of planar QPs, and we are led to the conjectures that all QPs of these families are self-injective.

Contents

Acknowledgments

I would like to express my gratitude to my advisor, Martin Herschend, for the countless discussions had, explanations given, and ideas and suggestions proposed, from the inception to the conclusion of this work.

1 Algebras and their modules

In this section, we give basic definitions and results for algebras and modules over an algebra, much of which will be essential for the coming sections. Most definitions and results appear in one form or another in chapter I of [1].

Throughout this article, K denotes an algebraically closed field.

In this section, whenever I is not explicitly said to denote an ideal, I denotes an arbitrary index set.

1.1 Algebras

Definition 1.1 A *K-algebra* is a vector space A over K equipped with a multiplication of vectors

$$\cdot\colon A \times A \to A$$

that is bilinear. The algebra is said to be *associative* if the multiplication is associative. The algebra is said to be *unital* (or *with unity*) if there is an identity element with respect to multiplication, i.e., an element $1_A \in A$ such that $1_A a = a = a 1_A$ for every $a \in A$. □

Definition 1.2 Let A be a K-algebra. A *right (left) ideal* of A is a K-subspace $I \subseteq A$ that absorbs multiplication from the right (left), i.e., for every $x \in I$ and $a \in A$, we have $xa \in I$ ($ax \in I$). A *two-sided ideal* of A is a K-subspace $I \subseteq A$ that is a right and left ideal. □

In this article, most ideals will be two-sided and we will write just "ideal" for "two-sided ideal".

Ideals can be multiplied in the following sense.

Definition 1.3 Let A be a unital K-algebra and $I, J \subseteq A$ be two ideals. We then define the *product* of I and J as the ideal

$$IJ := \left\{ \sum_{i=1}^{n} x_i y_i \;\middle|\; n \in \mathbb{Z}_{\geq 0},\ \forall i \in \{1, \ldots, n\}\colon x_i \in I,\ y_i \in J \right\} \subseteq A,$$

which is the ideal generated by $\{xy \mid x \in I,\ y \in J\}$. We also define the nth *power* of I as

$$I^n := \begin{cases} A & \text{if } n = 0 \\ I^{n-1} I & \text{if } n \geq 1. \end{cases}$$ □

Definition 1.4 Let A be a K-algebra and I be an ideal of A. The *quotient algebra A/I* is the algebra obtained by endowing the quotient space A/I with bilinear multiplication given by

$$(a + I)(b + I) := ab + I$$

for all $a, b \in A$. □

Proposition 1.5 *Let A be a K-algebra and I be an ideal of A. If A is associative, then A/I is associative. If A is unital with unity 1_A, then A/I is unital with unity $1_A + I$.* □

1.2 Modules

Throughout this article, "module" means "right module".

Definition 1.6 Let A be an associative unital K-algebra. A *(right) module over* A is a K-vector space M equipped with a (right) action

$$M \times A \to M$$
$$(m, a) \mapsto ma$$

by A which is compatible with the vector space structure on M: for every $m, n \in M$, $a, b \in A$, and $\lambda \in K$, we require that

1. $m1_A = m$
2. $m(a + b) = ma + mb$
3. $m(a \cdot b) = (ma)b$
4. $m(\lambda a) = \lambda(ma)$
5. $(m + n)a = ma + na$
6. $(\lambda m)a = \lambda(ma)$ □

For the remainder of this section, let A be an associative unital K-algebra that is finite-dimensional (as a vector space).

1.2.1 Morphisms of modules

Definition 1.7 A *morphism* of A-modules is a K-linear map $f : M \to N$ between A-modules that respects the A-action on M and N, i.e., such that $f(ma) = f(m)a$ for every $m \in M$ and $a \in A$. If f is injective, then f is a *monomorphism.* If f is surjective, then f is an *epimorphism.* If f is bijective, then f is an *isomorpism* and M and N are said to be *isomorphic*, denoted by $M \cong N$. If $M = N$, then f is an *endomorphism.* □

Notation 1.8 Let M and N be A-modules. We write $\mathrm{Hom}_A(M, N)$ for the K-vector space

$$\mathrm{Hom}_A(M, N) := \{f : M \to N \mid f \text{ is an } A\text{-module morphism}\}$$

of all A-module morphisms from M to N with valuewise addition and scalar multiplication. We write $\mathrm{End}_A(M)$ for the associative unital K-algebra

$$\mathrm{End}_A(M) := \{f : M \to M \mid f \text{ is an } A\text{-module morphism}\}$$

of all A-module endomorphisms on M with valuewise addition, valuewise scalar multiplication, and function composition as bilinear multiplication. □

Definition 1.9 Let M be an A-module. An A*-submodule of* M is a K-subspace U of M that is closed under the A-action, i.e., $xa \in U$ for every $x \in U$ and $a \in A$. We associate to the submodule $U \subseteq M$ the *inclusion morphism* $\iota : U \to M$ defined by $\iota(x) = x$ for every $x \in U$. □

Definition 1.10 Let M be an A-module and $U \subseteq M$ be a submodule. The *quotient module* M/U is the A-module obtained by endowing the quotient space M/U with the A-action defined by $(m+U)a = ma+U$ for every $m \in M$ and $a \in A$. We associate to the quotient module M/U the *projection morphism* $\pi : M \to M/U$ defined by $\pi(m) = m + U$ for every $m \in M$. □

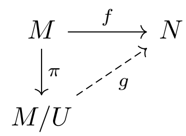
Definition 1.11 Let $f : M \to N$ be a morphism of A-modules. The *kernel* of f is the A-submodule

$$\ker f := \{m \in M \mid f(m) = 0\} \subseteq M.$$

The *image* of f is the A-submodule

$$\mathrm{im}\, f := \{f(m) \mid m \in M\} \subseteq N.$$ □

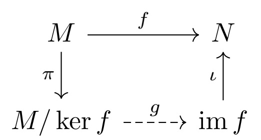
Proposition 1.12 *Let $f\colon M \to N$ be a morphism of A-modules and let $U \subseteq \ker f$ be a submodule of the kernel. Then there is a unique morphism $g\colon M/U \to N$ such that $f = g \circ \pi$, where $\pi\colon M \to M/U$ denotes the projection morphism. In terms of diagrams, the following diagram should commute:*



Explicitly, this g is given by

$$g(m + U) = f(m).$$ □

Proposition 1.13 (First isomorphism theorem for modules) *Let $f\colon M \to N$ be a morphism of A-modules. Then there is a unique morphism $g\colon M/\ker f \to \operatorname{im} f$ such that $f = \iota \circ g \circ \pi$, where $\iota\colon \operatorname{im} f \to N$ and $\pi\colon M \to M/\ker f$ denote the inclusion morphism and projection morphism, respectively. In terms of diagrams, the following diagram should commute:*



Explicitly, this g is given by

$$g(m + \ker f) = f(m).$$ □

1.2.2 Direct sum decompositions and idempotents

Definition 1.14 Let M be an A-module. For every (possibly infinite) family $\{M_i \mid i \in I\}$ of submodules of M, we define its *(internal) sum* as the A-submodule

$$\sum_{i \in I} M_i := \left\{ \sum_{i \in I} m_i \;\middle|\; (\forall i \in I\colon m_i \in M_i) \text{ and } m_i = 0 \text{ for all but finitely many } i \in I \right\} \subseteq M.$$ □

If every $m \in \sum_{i \in I} M_i$ can be written as a finite sum $\sum_{i \in I} m_i$ (with $m_i \in M_i$ for every $i \in I$) in a unique way, the sum is said to be *direct*, which we may emphasize by writing $\bigoplus_{i \in I} M_i$ instead of $\sum_{i \in I} M_i$.

Definition 1.15 An A-module M is said to be *indecomposable* if $M \neq \{0\}$ and there are no two nonzero submodules $U, V \subseteq M$ such that $M = U \oplus V$.

A direct sum decomposition $\bigoplus_{i \in I} M_i$ of M is said to be a *decomposition into indecomposables* if every direct summand M_i is indecomposable. □

Every associative unital K-algebra can be viewed as a module over itself.

Definition 1.16 The *regular* right A-module, denoted by A_A, has underlying K-vector space A and right A-action given by the multiplication in the algebra. □

Proposition 1.17 *A_A admits a decomposition into indecomposables.* □

By multiplying the algebra from the left by any fixed element, we get a submodule of the regular module. This is particularly useful for left-multiplication by an idempotent.

Proposition 1.18 *Let $x \in A$ be an arbitrary element. Then*

$$xA := \{x \cdot a \mid a \in A\}$$

is a submodule of A_A with $x \in xA$. □

Direct sum decompositions of the module A_A correspond intimately to idempotents of A with certain properties that we now define.

Definition 1.19 An element $e \in A$ is said to be an *idempotent* if $e^2 = e$. Two idempotents $e_1, e_2 \in A$ are said to be *orthogonal* if $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$. A set $\{e_1, \dots, e_n\}$ of idempotents of A is said to be *complete* if $e_1 + \dots + e_n = 1_A$. An idempotent $e \in A$ is said to be *primitive* if it cannot be written as a sum $e = e_1 + e_2$ of two nonzero orthogonal idempotents $e_1, e_2 \in A$. □

Proposition 1.20 *Let $\{e_1, \dots, e_n\}$ be a complete set of pairwise orthogonal idempotents of A. Then*

$$A_A = e_1 A \oplus \dots \oplus e_n A.$$

Conversely, suppose that $A_A = M_1 \oplus \dots \oplus M_n$ for some submodules $M_1, \dots, M_n \subseteq A_A$. Let $e_1 \in M_1, \dots, e_n \in M_n$ denote the unique elements such that

$$1_A = e_1 + \dots + e_n.$$

Then $\{e_1, \dots, e_n\}$ is a complete set of pairwise orthogonal idempotents.

The above provide mutually inverse one-to-one correspondences between complete sets of pairwise orthogonal idempotents and direct sum decompositions of A_A. □

Proposition 1.21 *Let $e \in A$ be an idempotent. Then eA is indecomposable if and only if e is nonzero and primitive.* □

Corollary 1.22 *The one-to-one correspondence between complete sets of pairwise orthogonal idempotents and direct sum decompositions of Proposition 1.20 restricts to a one-to-one correspondence between complete sets of* nonzero primitive *orthogonal idempotents of A and direct sum decompositions of A_A* into indecomposables*.* □

Direct sum decompositions of an arbitrary A-module M correspond intimately to idempotents of $\operatorname{End}_A(M)$.

Proposition 1.23 *Let M be an A-module. If $e \in \operatorname{End}_A(M)$ is an idempotent, then $M = \operatorname{im} e \oplus \ker e$. Conversely, if $M = M_1 \oplus M_2$, then $\iota_1 \circ \pi_1 \in \operatorname{End}_A(M)$ (where $\pi_1 \colon M \to M_1$ and $\iota_1 \colon M_1 \to M$ denote the projection and inclusion morphisms) is an idempotent. These are mutually inverse one-to-one correspondences between direct sum decompositions of M into two submodules and idempotents of* $\operatorname{End}_A(M)$*.* □

In the same vein as indecomposability of a submodule but stronger is the notion of simpleness of a submodule.

Definition 1.24 An A-module M is said to be *simple* if $M \neq \{0\}$ and its only submodules are $\{0\}$ and M. □

Proposition 1.25 (Schur's lemma) *Let $f \colon M \to N$ be a nonzero morphism of A-modules. If M is simple, then f is a monomorphism. If N is simple, then f is an epimorphism. If both M and N are simple, then f is an isomorphism.* □

Definition 1.26 Let M be an A-module. The *socle* of an A-module M is the submodule generated by all the simple submodules of M:

$$\operatorname{soc} M := \sum_{\substack{S \subseteq M \\ S \text{ simple}}} S.$$ □

1.2.3 The direct product and direct sum

Definition 1.27 Let $\{M_i \mid i \in I\}$ be a possibly infinite family of A-modules.

The *direct product* of $\{M_i \mid i \in I\}$ is the A-module

$$\prod_{i \in I} M_i := \{(m_i)_{i \in I} \mid \forall i \in I \colon m_i \in M_i\}$$

with operations given by

$$(m_i)_{i\in I} + (n_i)_{i\in I} := (m_i + n_i)_{i\in I}$$
$$(m_i)_{i\in I}\, a := (m_i a)_{i\in I}.$$

In other words, the direct product is the Cartesian product with componentwise operations. We associate to this direct product a family $\{\pi_j \colon \prod_{i\in I} M_i \to M_j \mid j \in I\}$ of *projection morphisms* defined by

$$\pi_j((m_i)_{i\in I}) = m_j.$$

The *direct sum* of $\{M_i \mid i \in I\}$ is the A-submodule

$$\begin{aligned}\bigoplus_{i\in I} M_i &:= \{(m_i)_{i\in I} \mid (\forall i \in I \colon m_i \in M_i) \text{ and } m_i = 0 \text{ for all but finitely many } i \in I\} \\ &\subseteq \prod_{i\in I} M_i.\end{aligned}$$

We associate to this direct sum a family $\{\iota_j \colon M_j \to \bigoplus_{i\in I} M_i \mid j \in I\}$ of *inclusion morphisms* defined by

$$\iota_j(m_j) = (m_i)_{i\in I},$$

where

$$m_i := \begin{cases} m_j & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$ □

The direct product and direct sum are characterized (up to isomorphism) by the following dual universal properties.

Proposition 1.28 (Universal property of the direct product and direct sum) *Let $\{N_i \mid i \in I\}$ be a family of A-modules and $N = \prod_{i\in I} N_i$. For every A-module M and family $f_i \colon M \to N_i$ of A-module morphisms, there is a unique morphism $f \colon M \to N$ such that $f_i = \pi_i \circ f$ for every $i \in I$.*

Let $\{M_i \mid i \in I\}$ be a family of A-modules and $M = \bigoplus_{i\in I} M_i$. For every A-module N and family $g_i \colon M_i \to N$ of A-module morphisms, there is a unique morphism $g \colon M \to N$ such that $g_i = g \circ \iota_i$ for every $i \in I$. □

1.2.4 Free modules

Definition 1.29 Let M be an A-module and $X := \{x_i \mid i \in I\} \subseteq M$ be a possibly infinite subset. X is said to be A-*linearly independent* if for every $m \in M$, there is at most one family of elements $a_i \in A$ (all but finitely many zero) such that

$$m = \sum_{i\in I} x_i a_i.$$

X is said to *generate M over A* if for every $m \in M$, there is at least one family of elements $a_i \in A$ (all but finitely many zero) such that

$$m = \sum_{i\in I} x_i a_i.$$

X is said to be an A-*basis* of M and M is said to be *free (on X)* if X is linearly independent and generates M over A.

An A-module M is said to be *finitely generated* if it admits a finite generating set. □

Proposition 1.30 *An A-module M is free if and only if it is isomorphic to a direct sum of copies of A_A, i.e.,*

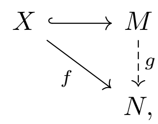
$$M \cong \bigoplus_{i \in I} A_A.$$

for some possibly infinite index set I. Moreover, if $f\colon \bigoplus_{i\in I} A_A \to M$ is an isomorphism, then M is free on the set

$$\{(f \circ \iota_i)(1_A) \mid i \in I\} \subseteq M. \qquad \square$$

Corollary 1.31 *The regular module A_A is free on $\{1_A\}$.* $\square$

Proposition 1.32 (Universal property of the free module) *Let M be an A-module with A-basis $X = \{x_i \mid i \in I\} \subseteq M$ (for some index set I), and let N be an A-module. Then for every function $f\colon X \to N$, there exists a unique A-module morphism $g\colon M \to N$ that extends f to all of M. In terms of diagrams, the following diagram should commute:*



where $X \hookrightarrow M$ denotes the inclusion map. Moreover, this g is given by

$$g\Big(\sum_{i\in I} x_i a_i\Big) = \sum_{i \in I} f(x_i) a_i.$$

for every family of $a_i \in A$ (all but finitely many equal to zero). $\square$

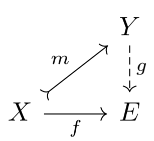
Proposition 1.33 *An A-module M is finitely generated if and only if it is finite-dimensional as a vector space.* $\square$

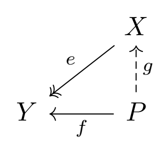
1.2.5 Injective and projective modules

Definition 1.34 An A-module E is said to be *injective* if for every monomorphism $m\colon X \to Y$ and morphism $f\colon X \to E$ there exists a morphism $g\colon Y \to E$ such that $f = g \circ m$.

An A-module P is said to be *projective* if for every epimorphism $e\colon X \to Y$ and morphism $f\colon P \to Y$ there exists a morphism $g\colon P \to X$ such that $f = e \circ g$.

In terms of diagrams, the following diagrams should commute:





In terms of Hom functors, the contravariant functor $\mathrm{Hom}(-, E)$ should map monomorphisms to epimorphisms and the covariant functor $\mathrm{Hom}(P, -)$ should map epimorphisms to epimorphisms. $\square$

Proposition 1.35 *Let $\{E_i \mid i \in I\}$ be a possibly infinite family of A-modules and let E be its direct product $E = \prod_{i\in I} E_i$. Then E is injective if and only if every E_i is injective.*

Let $\{P_i \mid i \in I\}$ be a possibly infinite family of A-modules and let P be its direct sum $P = \bigoplus_{i\in I} P_i$. Then P is projective if and only if every P_i is projective. $\square$

Proof. The implications from injectivity and projectivity of the factors and the summands, respectively, hold in great category-theoretical generality. We prove this implication for injective modules and note that the corresponding implication for projective modules can be shown in a dual manner.

Suppose that every E_i is injective. Given a monomorphism $m\colon X \to Y$ and a morphism $f\colon X \to E$, set $f_i := \pi_i \circ f\colon X \to E_i$ and invoke injectivity of E_i with m and f_i to get $g_i\colon Y \to E_i$ such that $f_i = g_i \circ m$. By the universal property of the direct product, the family of morphism g_i induces a morphism $g\colon Y \to E$, unique with the property that $g_i = \pi_i \circ g$ for every $i \in I$. Observe that this g satisfies the identity $f = g \circ m$ required for injectivity of E, because $\pi_i \circ g \circ m = g_i \circ m = f_i = \pi_i \circ f$ for every $i \in I$. Thus E is injective.

For the converse, we suppose that E is injective and show that E_i is injective for a fixed $i \in I$. Given a monomorphism $m\colon X \to Y$ and a morphism $f_i\colon X \to E_i$, set $f := \iota_i \circ f_i\colon X \to E$ so that $\pi_i \circ f = \pi_i \circ \iota_i \circ f_i = f_i$. By the injectivity of E, there is a $g\colon Y \to E$ such that $f = g \circ m$. By setting $g_i := \pi_i \circ g$, we have $f_i = \pi_i \circ f = \pi_i \circ g \circ m = g_i \circ m$. Thus E_i is injective.

For the corresponding implication for projective modules, suppose that P is projective and show that P_i is projective for a fixed $i \in I$. Given an epimorphism $e\colon X \to Y$ and a morphism $f_i\colon P_i \to Y$, set $f := f_i \circ \pi_i\colon P \to Y$ so that $f \circ \iota_i = f_i \circ \pi_i \circ \iota_i = f_i$. By the projectivity of P, there is a $g\colon P \to X$ such that $f = e \circ g$. By setting $g_i := g \circ \iota_i$, we have $f_i = f \circ \iota_i = e \circ g \circ \iota_i = e \circ g_i$. Thus P_i is projective. □

Corollary 1.36 *A finite direct sum $E = \bigoplus_{i=1}^n E_i$ is injective if and only if every summand E_i is injective.* □

Proof. The finite direct sum $E = \bigoplus_{i=1}^n E_i$ coincides with the direct product $E = \prod_{i=1}^n E_i$. □

Lemma 1.37 *Every free A-module is projective.* □

Proof. Let F be a free module with A-basis $\{m_i \mid i \in I\}$ (for some index set I). Let $f\colon F \to Y$ be a morphism and $e\colon X \twoheadrightarrow Y$ be an epimorphism, as in Definition 1.34 but with F in place of P. For every $i \in I$, let $x_i \in X$ be a preimage of $f(m_i) \in Y$ under e, and define $g\colon F \to X$ via the universal property of the free module by $g(m_i) = x_i$. For an arbitrary element $m = \sum_{i\in I} m_i a_i \in F$, we then have

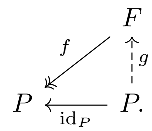
$$\begin{aligned}(e \circ g)(m) &= e(g(m))\\ &= e\Big(g\Big(\sum_{i\in I} m_i a_i\Big)\Big)\\ &= e\Big(\sum_{i\in I} g(m_i) a_i\Big)\\ &= \sum_{i\in I} e\big(g(m_i)\big) a_i\\ &= \sum_{i\in I} e(x_i) a_i\\ &= \sum_{i\in I} f(m_i) a_i\\ &= f\Big(\sum_{i\in I} m_i a_i\Big)\\ &= f(m),\end{aligned}$$

which shows that $e \circ g = f$ and hence that F is projective. □

The content of the following proposition is that the projective modules are precisely the direct summands of the free modules.

Proposition 1.38 *Let P be an A-module. Then P is projective if and only if there is an A-module P' such that $P \oplus P'$ is free.* □

Proof. Assume that P is projective. Let F be a free module with an epimorphism $f\colon F \twoheadrightarrow P$, for instance by taking F as a free module on all of P and f as the morphism induced by the identity map $P \to P$ via the universal property of the free module. By the definition of projectivity, the identity morphism $\mathrm{id}_P\colon P \to P$ factors through f as $\mathrm{id}_P = f \circ g$ for some $g\colon P \to F$ (which is necessarily injective), as depicted in the following diagram:



It follows that $(g \circ f) \in \operatorname{End}_A(F)$ is an idempotent. By Proposition 1.23, F decomposes as

$$F = \operatorname{im}(g \circ f) \oplus \ker(g \circ f).$$

Because f is surjective and g is injective, we have

$$\begin{aligned} F &= \operatorname{im}(g \circ f) \oplus \ker(g \circ f) \\ &= \operatorname{im} g \oplus \ker(g \circ f) \\ &\cong P \oplus \ker(g \circ f), \end{aligned}$$

which shows that P is a direct summand of a free module.

For the converse, assume that there is a module P' such that $P \oplus P'$ is free. Then $P \oplus P'$ is projective by Lemma 1.37, and the direct summand P is projective by Proposition 1.35. □

Definition 1.39 [1, p. 171] An associative unital K-algebra A is said to be *self-injective* if it is finite-dimensional and its regular module A_A is injective. □

Remark 1.40 The given definition of self-injectivity appears to depend on the arbitrarily made choice of considering right A-modules rather than left A-modules. However, under the assumption of finite-dimensionality, it can be shown that the regular left module is injective if and only if the regular right module is. □

Proposition 1.41 *Let $A_A = e_1 A \oplus \cdots \oplus e_n A$ be a direct sum decomposition of A_A. Then A is self-injective if and only if $e_i A$ is injective for every $i = 1, \ldots, n$.* □

Proof. This follows immediately from Definition 1.39 and Corollary 1.36 □

1.2.6 Ideals acting on a module

Notation 1.42 Let M be a (right) A-module and I be a (right) ideal of A. We write MI for the submodule

$$MI := \left\{ \sum_{i=1}^{n} m_i x_i \;\middle|\; n \in \mathbb{Z}_{\geq 0},\ \forall i \in \{1, \ldots, n\} \colon m_i \in M,\ x_i \in I \right\} \subseteq M \qquad \square$$

The Jacobson radical and Nakayama's lemma, to be defined and stated next, give rise to a characterization of the socle of a module (see Proposition 2.21) over a type of algebra to be studied in Section 2. The socles of certain modules turn out to contain all the information needed to determine whether the algebra is self-injective (see Theorem 2.22), and the characterization allows for an algorithm to work with the socles.

Definition 1.43 [1, p. 4] The *Jacobson radical* of A is the intersection of all maximal right ideals of A and is denoted by $\operatorname{rad} A$. □

Proposition 1.44 [1, p. 4] $\operatorname{rad} A$ *is a two-sided ideal of* A. □

Proposition 1.45 (Nakayama's lemma) [1, p. 7] *Let M be a nonzero finitely generated A-module. Then,*

$$M \operatorname{rad} A \subsetneq M. \qquad \square$$

2 Quivers and their path algebras

In this section, we develop the theory for quivers, their path algebras, and quotients of the path algebra by an admissible ideal. A characterization of self-injectivity of such quotients in terms of the socles of the indecomposable projective modules is given. We finish the section by studying quivers for which these socles can be computed algorithmically. The characterization of self-injectivity in terms of the socles thus provides an algorithmic means by which to determine self-injectivity.

Most definitions and results appear in chapter II of [1].

2.1 Quivers and their path algebras

A quiver is a directed multigraph with loops allowed. We will associate to every quiver an algebra called the path algebra of the quiver. One viewpoint of quivers is thus as a very concrete tool for constructing algebras.

All the definitions and results of this subsection are found in section II.1 of [1].

Definition 2.1 A *quiver* Q is a quadruple $Q = (Q_0, Q_1, s, t)$ consisting of a set Q_0 of *vertices*, a set Q_1 of *arrows* between two vertices, and functions $s, t\colon Q_1 \to Q_0$ sending each arrow $a \in Q_1$ to its *source vertex* $s(a) \in Q_0$ and *target vertex* $t(a) \in Q_0$, respectively. Q is said to be *finite* if Q_0 and Q_1 are finite sets. □

We will represent quivers graphically by drawing the vertices of the quiver in the plane and for each arrow of the quiver an arrow from its source vertex to its target vertex. For instance, the diagram

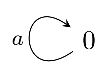
$$1 \xrightarrow{a} 2$$

is understood to represent the quiver $Q = (\{1, 2\}, \{a\}, s, t)$ with $s, t\colon \{a\} \to \{1, 2\}$ defined by $s(a) = 1$ and $t(a) = 2$. The names of the vertices and arrows in a quiver are often of no importance, and we will instead draw the quiver as

$$\bullet \longrightarrow \bullet,$$

which still captures the "shape" of the quiver.

Example 2.2 All of the diagrams below represent well-defined quivers.



$$0 \underset{b}{\overset{a}{\rightrightarrows}} 1$$

$$0 \underset{b}{\overset{a}{\rightleftarrows}} 1$$

$$0 \qquad 1$$

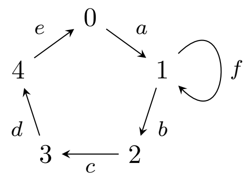
□

Definition 2.3 A *path* p from i to j of length $l \geq 1$ in a quiver $Q = (Q_0, Q_1, s, t)$ is a triple $(i \mid a_1, \ldots, a_l \mid j)$ consisting of a source vertex i, denoted by $s(p)$, a target vertex j, denoted by $t(p)$, and a finite sequence of (possibly repeated) arrows $a_1, \ldots, a_l$ such that $s(a_1) = i$, $t(a_k) = s(a_{k+1})$ for every k such that $1 \leq k < l$, and $t(a_l) = j$. For every vertex i, there is also a path $\varepsilon_i := (i \mid\mid i)$ of length 0 called the *stationary path* at i.

For a quiver Q and $l \geq 0$, we denote by Q_l the set of all paths in Q of length l. Note that the paths of length 0 may be identified with the vertices and that the paths of length 1 may be identified with the arrows, so that the overloaded notation Q_0 and Q_1 is no concern. Note also that the overloaded notation s and t for arrows and paths agree under this identification.

A *cycle* of length l (or l-*cycle*) is a path $(i \mid a_1, \ldots, a_l \mid i)$ of length $l \geq 1$ where the source and target coincide. A *loop* is a cycle of length 1. A quiver that contains no cycles is said to be *acyclic*. □

Example 2.4 Consider the quiver



.

$(0 \mid a, f, b \mid 2)$ is a path from 0 to 2, $(1 \mid f \mid 1)$ is a loop, $(0 \mid a, b, c, d, e \mid 0)$ is a 5-cycle starting and ending at 0, and $(0 \mid a, f, b, c, d, e \mid 0)$ is a 6-cycle starting and ending at 0. □

Definition 2.5 Let Q be a quiver. The *path algebra* KQ of Q is the K-algebra defined as the free K-vector space on all paths in Q, with bilinear multiplication given on the basis elements by

$$(i \mid a_1, \ldots, a_l \mid j_1) \cdot (j_2 \mid b_1, \ldots, b_m \mid k) = \begin{cases} (i \mid a_1, \ldots, a_l, b_1, \ldots, b_m \mid k) & \text{if } j_1 = j_2 \\ 0 & \text{if } j_1 \neq j_2 \end{cases}$$

and extended bilinearly to all of KQ.

In other words, the product of two basis elements is their concatenation or the zero element, depending on whether the paths can be concatenated. □

Remark 2.6 That the multiplication in the path algebra really is associative is shown in Proposition 2.8. It follows that a non-stationary path $(i \mid a_1, \ldots, a_l \mid j)$ may be expressed more succinctly as the product $a_1 \cdots a_l$ of the constituent arrows viewed as elements of the path algebra. □

Example 2.7 Let Q be the quiver

$$1 \xrightarrow{a} 2 \; .$$

The paths in Q are $\varepsilon_1, \varepsilon_2, a$, and the multiplication table for the basis elements in KQ is

	ε_1	ε_2	a
ε_1	ε_1	0	a
ε_2	0	ε_2	0
a	0	a	0

.

This algebra is seen to be isomorphic to the algebra of upper triangular 2×2 matrices

$$\begin{bmatrix} K & K \\ 0 & K \end{bmatrix} = \left\{ \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \;\middle|\; x, y, z \in K \right\}$$

via the isomorphism given on the basis elements by

$$\begin{aligned} \varepsilon_1 &\mapsto \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \\ \varepsilon_2 &\mapsto \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \\ a &\mapsto \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right). \end{aligned}$$

□

Proposition 2.8 *Let Q be a quiver, and consider the path algebra KQ of Q.*

1. *KQ is associative.*
2. *KQ is unital if and only if Q_0 is finite.*

3. *KQ is finite-dimensional if and only if Q is finite and acyclic.* □

Proof. 1. Associativity on the basis elements follows by the associativity of concatenation: consider the two products $(p_1 \cdot p_2) \cdot p_3$ and $p_1 \cdot (p_2 \cdot p_3)$ of paths p_1, p_2, p_3 with sources i_1, i_2, i_3 and targets j_1, j_2, j_3, respectively. If the paths can be concatenated, i.e., $j_1 = i_2$ and $j_2 = i_3$, then both products are nonzero and coincide. Otherwise, both products are zero. Associativity for arbitrary elements of the path algebra follows by the bilinearity of the multiplication.

2. One readily verifies that if Q_0 is finite, then $\sum_{i \in Q_0} \varepsilon_i$ is a unity. If Q_0 is infinite, consider an arbitrary element $x \in KQ$. Write it as a finite linear combination $x = \sum_{k=1}^{m} \lambda_k p_k$ of paths. The set of target vertices $T = \{t(p_k) \mid k = 1, \ldots, m\} \subseteq Q_0$ is finite, so $Q_0 \setminus T \neq \varnothing$. Let $i \in Q_0 \setminus T$ and consider any path with source i, for instance ε_i. Then $x \cdot \varepsilon_i = 0 \neq \varepsilon_i$, showing that x is not a unity. Seeing as x was arbitrary, the algebra has no unity.

3. KQ is finite-dimensional if and only if there are only finitely many paths in Q. If Q is infinite, then there are infinitely many paths of length 0 or 1. If Q has a cycle c, then $c, c^2, c^3, \ldots$ is an infinite sequence of distinct paths. In both cases, Q has infinitely many paths and KQ is hence infinite-dimensional. If, on the other hand, Q is finite and acyclic, then there are finitely many stationary paths, and each of the finitely many arrows can occur at most once in a path, so the non-stationary paths are also finitely many. That is, Q has finitely many paths and is hence finite-dimensional. □

2.2 Quotients of the path algebra

The path algebra is generally too large to be of interest to us; by Proposition 2.8, the path algebra of a quiver with a cycle is infinite-dimensional, and we will want a finite-dimensional algebra when studying self-injectivity. By taking a quotient of the path algebra of a finite quiver by a so-called admissible ideal, we obtain a well-behaved algebra that is associative, unital, and finite-dimensional. Thus, all of Section 1 applies to such quotient algebras.

Most definitions and results of this subsection are found in section II.2 of [1]. The Nakayama permutation is treated in more generality than given here in [4].

Definition 2.9 Let Q be a finite quiver. The *arrow ideal* of Q is the two-sided ideal R_Q of KQ generated by all arrows (paths of length 1):

$$R_Q := \langle a \mid a \in Q_1 \rangle.$$ □

Definition 2.10 Let Q be a finite quiver. A two-sided ideal I of KQ is said to be *admissible* if

$$R_Q^m \subseteq I \subseteq R_Q^2$$

for some integer $m \geq 2$. Intuitively, I contains only paths of length at least 2 and I contains all paths of length m or more.

If I is an admissible ideal of KQ, then the pair (Q, I) is called a *bound quiver* and the quotient algebra KQ/I is said to be the *algebra of* (Q, I). □

Remark 2.11 With the point of view of quivers as a tool with which to construct algebras, it is of interest to note that every so-called basic and connected finite-dimensional associative K-algebra with unity arises (up to isomorphism) as the algebra of a bound quiver. This relies on the assumption that K is algebraically closed and is studied in detail in section II.3 of [1]. □

For the remainder of this section, let (Q, I) be a bound quiver and $A = KQ/I$ be its algebra.

Proposition 2.12 [1, p. 56] $\dim_K A < +\infty$. □

Proposition 2.13 [1, p. 55] *The following is a complete set of nonzero primitive orthogonal idempotents for A:*

$$\{\varepsilon_i + I \mid i \in Q_0\}.$$ □

Notation 2.14 For every $i \in Q_0$, we set $e_i := \varepsilon_i + I \in A$. □

Proposition 2.15 [1, p. 57] $\operatorname{rad} A = R_Q/I$. □

By Proposition 2.12, A is self-injective if and only if A_A is injective. In order to give a more concrete characterization of self-injectivity of A (namely, Theorem 2.22), we need to define two A-modules for every vertex of Q.

Definition 2.16 For every $i \in Q_0$, we define the *indecomposable projective A-module corresponding to i* as

$$P(i) := e_i A$$

and the *simple A-module corresponding to i* as

$$S(i) := \frac{P(i)}{P(i)\operatorname{rad} A}.$$ □

Remark 2.17 Concretely, $P(i)$ is simply the span of all equivalence classes of paths starting at i:

$$P(i) = \operatorname{span}_K\{p + I \mid p \text{ is a path in } Q \text{ with } s(p) = i\}.$$ □

Proposition 2.18 *For every $i \in Q_0$,*

1. *$P(i)$ is an indecomposable projective A-module, and*
2. *$S(i)$ is one-dimensional as a vector space and is in particular a simple A-module. Moreover, the A-module structure on $S(i)$ is given by*

$$x(p+I) = \begin{cases} x & \text{for } p = \varepsilon_i \\ 0 & \text{for } p \neq \varepsilon_i \end{cases}$$

for every $x \in S(i)$ and path p in Q. □

Proof.

1. By Proposition 2.13 and Corollary 1.22, we have the decomposition into indecomposables

$$A_A = \bigoplus_{i \in Q_0} P(i).$$

By Proposition 1.35, every direct summand $P(i)$ is projective.

2. We obtain a K-generating set for $P(i)$ from the set of all paths in Q, which generates KQ, as follows:

$$\begin{aligned} \{p \mid p \text{ is a path in } Q\} \text{ generates } KQ &\implies \{p + I \mid p \text{ is a path in } Q\} \text{ generates } A = KQ/I \\ &\implies \{e_i(p+I) \mid p \text{ is a path in } Q\} \text{ generates } P(i) = e_i A. \end{aligned}$$

For every path p in Q, we have

$$\begin{aligned} e_i(p+I) &= (\varepsilon_i + I)(p + I) \\ &= \varepsilon_i p + I \\ &= \begin{cases} p + I & \text{if } s(p) = i \\ 0 + I & \text{if } s(p) \neq i, \end{cases} \end{aligned}$$

so we obtain the smaller generating set

$$\{p + I \mid p \text{ is a path in } Q \text{ with } s(p) = i\}$$

for $P(i)$. From this, we obtain the following generating set for $S(i)$:

$$\{(p+I) + P(i)\operatorname{rad} A \mid p \text{ is a path in } Q \text{ with } s(p) = i\}.$$

For every non-stationary path p with $s(p) = i$, we have $(p + I) \in R_Q/I = \operatorname{rad} A$ and hence

$$(p + I) = (\varepsilon_i p + I) = (\varepsilon_i + I)(p + I) \in P(i) \operatorname{rad} A,$$

so that $(p + I) + P(i) \operatorname{rad} A = 0 + P(i) \operatorname{rad} A$. This shows that $S(i)$ is generated by the single element $(\varepsilon_i + I) + P(i) \operatorname{rad} A = e_i + P(i) \operatorname{rad} A$ and is of dimension at most 1 as a vector space.

By Proposition 1.45, $P(i) \operatorname{rad} A \subsetneq P(i)$, so that $S(i) \neq \{0\}$. Thus $S(i)$ is one-dimensional as a vector space.

As for the A-module structure on $S(i)$, we have the following for the generator and an arbitrary path p in Q:

$$\begin{aligned}\big((\varepsilon_i + I) + P(i) \operatorname{rad} A\big)(p + I) &= (\varepsilon_i p + I) + P(i) \operatorname{rad} A \\ &= \begin{cases} (p + I) + P(i) \operatorname{rad} A & \text{if } s(p) = i \\ 0 + P(i) \operatorname{rad} A & \text{if } s(p) \neq i \end{cases} \\ &= \begin{cases} (\varepsilon_i + I) + P(i) \operatorname{rad} A & \text{for } p = \varepsilon_i \\ 0 + P(i) \operatorname{rad} A & \text{for } p \neq \varepsilon_i. \end{cases}\end{aligned}$$

By homogeneity of the A-action, this extends to all elements of $S(i)$. ☐

Self-injectivity of the algebra of a bound quiver may be reduced to injectivity of the indecomposable projective modules.

Proposition 2.19 *A is self-injective if and only if $P(i)$ is injective for every $i \in Q_0$.* ☐

Proof. This is an immediate consequence of Propositions 1.41, 2.12 and 2.13 and the definition of $P(i)$. ☐

The content of the following proposition is that the simple modules in Definition 2.16 is a complete set of pairwise non-isomorphic simple A-modules.

Proposition 2.20 *Every simple A-module S is isomorphic to $S(i)$ for some unique $i \in Q_0$.* ☐

Proof. Let S be a simple A-module. We first show the uniqueness of $i \in Q_0$ with $S \cong S(i)$, by showing that $S(i) \ncong S(j)$ for distinct $i, j \in Q_0$. This follows immediately from Proposition 2.18: we have that e_i acts on one hand as the identity map on $S(i)$ and on the other hand as the zero map on $S(j)$, and the modules in question are nonzero.

Now we show the existence of $i \in Q_0$ with $S \cong S(i)$. Consider the regular module A_A. By Corollary 1.31 and Proposition 1.32, we may take any nonzero $x \in S$ and extend the function $f\colon \{1_A\} \to S$, $1_A \mapsto x$ to an A-module morphism $g\colon A_A \to S$ that is nonzero. By the simpleness of S and Proposition 1.25, we have that g is an epimorphism and hence that

$$\dim_K S \leq \dim_K A_A < +\infty.$$

By Proposition 1.33, S is finitely generated and we may thus use Proposition 1.45 to conclude that

$$S \operatorname{rad} A = \{0\}.$$

Moreover, we have by Proposition 1.28 that $g_i := g \circ \iota_i\colon P(i) \to S$ is nonzero for some $i \in Q_0$. From $S \operatorname{rad} A = \{0\}$, we get $g_i(P(i) \operatorname{rad} A) = \{0\}$: an arbitrary element of $P(i) \operatorname{rad} A$ is of the form $\sum_{i=1}^n x_i a_i$ for some $n \in \mathbb{Z}_{\geq 0}$, $x_i \in P(i)$, and $a_i \in \operatorname{rad} A$, and its image under g_i is

$$g_i\Big(\sum_{i=1}^n x_i a_i\Big) = \sum_{i=1}^n g_i(x_i) a_i,$$

which is an element of $S \operatorname{rad} A = \{0\}$. By Proposition 1.12, there is thus a morphism h such that the following diagram commutes:

$$\begin{array}{ccc} P(i) & \xrightarrow{g_i} & S \\ \downarrow{\scriptstyle \pi} & \nearrow{\scriptstyle h} & \\ \frac{P(i)}{P(i) \operatorname{rad} A} & & \end{array}$$

Because $g_i \neq 0$, we have $h \neq 0$. Note moreover that $\frac{P(i)}{P(i) \operatorname{rad} A} = S(i)$, which is simple by Proposition 2.18. Thus, $h\colon S(i) \to S$ is a nonzero morphism of simple modules, and Proposition 1.25 says that h is an isomorphism. ☐

Proposition 2.21 *For every A-module M, we have*

$$\begin{aligned} \operatorname{soc} M &= \{x \in M \mid x \operatorname{rad} A = \{0\}\} \\ &= \{x \in M \mid \forall a \in Q_1 \colon x(a + I) = 0\}, \end{aligned}$$

where $x \operatorname{rad} A = \{xa \mid a \in \operatorname{rad} A\}$*.* ☐

Proof. Let $U = \{x \in M \mid x \operatorname{rad} A = \{0\}\}$. One readily verifies that U is a submodule of M.

To show the inclusion $U \supseteq \operatorname{soc} M$, it suffices to show that $U \supseteq S$ for every simple submodule $S \subseteq M$. Thus, let $S \subseteq M$ be a simple submodule. By Proposition 2.20, S is finite-dimensional, so Proposition 1.45 and the simpleness of S gives $S \operatorname{rad} A = \{0\}$ and hence $x \operatorname{rad} A = \{0\}$ for every $x \in S$. This shows that $S \subseteq U$ and hence that $\operatorname{soc} M \subseteq U$.

To show the inclusion $U \subseteq \operatorname{soc} M$, we show that every $x \in U$ is in the sum of some simple submodules of M. Thus, let $x \in U$ be an arbitrary element. We may express x as

$$\begin{aligned} x &= x 1_A \\ &= x \sum_{i \in Q_0} e_i \\ &= \sum_{i \in Q_0} x e_i. \end{aligned}$$

Now we claim that for every $i \in Q_0$, the K-linear span of xe_i is a submodule of M. That is, we claim that $\operatorname{span}_K(xe_i)$ is closed under the A-action. Because the set of all paths in Q generates KQ, the set of all equivalence classes of the form $p + I$ for p a path in Q generates $KQ/I = A$. By the linearity of the A-action in both slots, it thus suffices to verify for an arbitrary $i \in Q_0$ and every path p in Q that

$$(xe_i)(p + I) = x(\varepsilon_i p + I) \in \operatorname{span}_K(xe_i).$$

If $s(p) \neq i$, we have

$$\begin{aligned} x(\varepsilon_i p + I) &= x(0 + I) \\ &= 0 \\ &\in \operatorname{span}_K(xe_i). \end{aligned}$$

Otherwise, we have $\varepsilon_i p = p$ and want to show

$$x(p + I) \in \operatorname{span}_K(xe_i).$$

We consider the two cases that p is stationary and non-stationary, respectively. For $p = \varepsilon_i$ stationary, we have

$$\begin{aligned} x(p + I) &= x(\varepsilon_i + I) \\ &= xe_i \\ &\in \operatorname{span}_K(xe_i). \end{aligned}$$

For p non-stationary, we have $p \in R_Q$ and hence $p + I \in R_Q/I = \operatorname{rad} A$. It follows from $x \in U$ that

$$\begin{aligned} x(p+I) &= 0 \\ &\in \operatorname{span}_K(xe_i). \end{aligned}$$

We have now shown that $\operatorname{span}_K(xe_i)$ is an A-submodule of M. Its dimension as a vector space is at most 1, so it is either the zero module or a simple module. Because this holds for every $i \in Q_0$, we can now write

$$\begin{aligned} x &= \sum_{i \in Q_0} xe_i \\ &= \sum_{\substack{i \in Q_0 \\ xe_i \neq 0}} xe_i \\ &\in \sum_{\substack{i \in Q_0 \\ xe_i \neq 0}} \operatorname{span}_K(xe_i), \end{aligned}$$

where the right-hand side is a sum of simple submodules of M. This shows that

$$\begin{aligned} U &\subseteq \sum_{x \in U} \sum_{\substack{i \in Q_0 \\ xe_i \neq 0}} \operatorname{span}_K(xe_i) \\ &\subseteq \sum_{\substack{S \subseteq M \\ S \text{ simple}}} S \\ &= \operatorname{soc} M, \end{aligned}$$

which finishes the proof that $U = M$.

It follows readily from Proposition 2.15 that

$$U = \{x \in M \mid \forall a \in Q_1 \colon x(a+I) = 0\},$$

which finishes the proof of the proposition. □

Theorem 2.22 *[4] Let (Q, I) be a bound quiver and $A = KQ/I$. Then A is self-injective if and only if there exists a permutation $\sigma \colon Q_0 \to Q_0$ with the property that for every $i \in Q_0$,*

$$\operatorname{soc} P(i) \cong S(\sigma(i)).$$

□

Definition 2.23 Let (Q, I) be a bound quiver with KQ/I self-injective. The permutation $\sigma \colon Q_0 \to Q_0$ in Theorem 2.22 is called the *Nakayama permutation* for (Q, I). □

Remark 2.24 By Proposition 2.20, there is at most one Nakayama permutation (and in fact at most one *map* $\sigma \colon Q_0 \to Q_0$ satisfying $\operatorname{soc} P(i) \cong S(\sigma(i))$ for every $i \in Q_0$) for (Q, I). This justifies referring to σ as *the* Nakayama permutation. □

2.3 Semimonomial bound quivers and weak cancellativity

In this subsection, we define the notions of semimonomial bound quivers and weak cancellativity at a vertex of a bound quiver. These notions are important because they ultimately allow for a computer to determine the socles $\operatorname{soc} P(i)$ of the indecomposable projective modules and hence determine whether the bound quiver algebra is self-injective via Theorem 2.22.

More precisely, let (Q, I) a bound quiver and $i \in Q_0$. Under the assumption of semimonomiality and weak cancellativity at i, the socle $\operatorname{soc} P(i)$ can be characterized in terms of maximal nonzero equivalence classes of individual paths starting at i (see Proposition 2.31). Moreover, under the assumption of semimonomiality, the restriction of the equivalence relation on KQ given by I to individual paths of KQ can be thought of as transformations of paths according to two types of moves: annihilating a path and replacing part of the path by another path (with the same source and target). This allows for an algorithmic computation of a basis for $\operatorname{soc} P(i)$.

Definition 2.25 [5, p. 5] Let (Q, I) be a bound quiver. We say that I is *semimonomial* and (Q, I) is *semimonomial* if I is generated by elements of the form p and $p - q$ for p, q paths in Q. □

The algebras of semimonomial bound quivers admit very nice bases as per the following proposition.

Proposition 2.26 [5, p. 6] *Let (Q, I) be a bound quiver. If (Q, I) is semimonomial, then the set*

$$B := \left\{ p + I \;\middle|\; \begin{matrix} p \text{ is a path in } Q \\ p + I \neq 0 + I \end{matrix} \right\}$$

is a multiplicative K-basis of KQ/I. □

Proof. Multiplicativity is immediate: $(p + I)(q + I) = pq + I$, where either $pq + I = 0 + I$ or $pq + I \neq 0 + I$ for pq a path in Q.

That B has K-span all of KQ/I is also readily shown:

$$\begin{aligned} \{p \mid p \text{ is a path in } Q\} \text{ spans } KQ &\implies \{p + I \mid p \text{ is a path in } Q\} \text{ spans } KQ/I \\ &\implies \left\{ p + I \;\middle|\; \begin{matrix} p \text{ is a path in } Q \\ p + I \neq 0 + I \end{matrix} \right\} = B \text{ spans } KQ/I. \end{aligned}$$

For K-linear independence of B, we will follow the proof in [5] and give an equivalent expression for I from which linear independence follows readily. The setup is as follows: let

$$P = \left\{ p \;\middle|\; \begin{matrix} p \text{ is a path in } Q \\ p + I \neq 0 + I \end{matrix} \right\},$$

$\sim$ denote the equivalence relation on P defined by

$$p \sim q \iff p + I = q + I,$$

$P/{\sim}$ denote the set of all equivalence classes, and

$$I' = \left\{ \sum_{p \text{ a path in } Q} \lambda_p p \;\middle|\; \sum_{p \in A} \lambda_p = 0 \text{ for every } A \in P/{\sim} \right\}.$$

Lemma $I = I'$ □

Proof. For the inclusion $I \subseteq I'$, it suffices to prove that I' is an ideal of KQ containing a set of generators of I. By the assumption that I is semimonomial, we may take the generators to be of the form p_i and $p_j - q_j$, for p_i, p_j, q_j paths in Q such that $p_j + I \neq 0 + I$ and $q_j + I \neq 0 + I$.

For every generator p_i, the singleton sum $\sum \lambda_p p = p_i$ has all scalars λ_p for $p \in P$ equal to zero, seeing as $p_i \in I \implies p_i \notin P$. Thus we have for every $A \in P/{\sim}$ that

$$\sum_{p \in A} \lambda_p = \sum_{p \in A} 0 = 0,$$

which shows that $p_i \in I'$.

For every generator $p_j - q_j$, note that

$$\begin{aligned} p_j - q_j \in I &\implies p_j + I = q_j + I \\ &\implies p_j \sim q_j. \end{aligned}$$

The “doubleton” sum $\sum \lambda_p p = p_j - q_j$ therefore has

$$\sum_{p \in A} \lambda_p = 1 - 1 = 0$$

for the equivalence class A with $p_j, q_j \in A$ and

$$\sum_{p \in A'} \lambda_p = \sum_{p \in A'} 0 = 0$$

for all the other equivalence classes. This shows that $p_j - q_j \in I'$, and we conclude that I' contains a set of generators of I.

The fact that I' is a K-linear subspace of KQ follows by noting that it is the kernel of the linear map

$$\begin{aligned} KQ &\to K(P/{\sim}) \\ \sum_{p \text{ a path in } Q} \lambda_p p &\mapsto \sum_{A \in P/\sim} \Big(\sum_{p \in A} \lambda_p \Big) A, \end{aligned}$$

where $K(P/{\sim})$ denotes the space of all formal K-linear combinations of elements in $P/{\sim}$.

To show that I' is closed under multiplication from both sides by arbitrary elements in KQ, note that it suffices, by bilinearity of the multiplication, to show that I' is closed under multiplication from both sides by paths. We verify that the latter is the case for multiplication from the right.

Let $\sum \lambda_p p \in I'$ be an arbitrary element and q be an arbitrary path in Q (viewed as an element of KQ). Their product is

$$\Big(\sum \lambda_p p \Big) q = \sum \lambda_p (pq),$$

and we will write μ_r for the coefficient of a path r in the above linear combination. Note that every path r can be written as $r = pq$ for at most one path p, so that we have

$$\mu_r = \begin{cases} \mu_{pq} = \lambda_p & \text{if } r = pq \text{ for some } p \\ 0 & \text{otherwise.} \end{cases}$$

To show that $\big(\sum \lambda_p p \big) q \in I'$, we need to show that every equivalence class $A \in P/{\sim}$ has

$$\sum_{r \in A} \mu_r = 0.$$

Fix an arbitrary equivalence class A and decompose it into two parts based on whether the paths end with q or not:

$$\begin{aligned} A' &:= \left\{ pq \;\middle|\; \begin{array}{c} p \text{ is a path in } Q \\ pq \in A \end{array} \right\} \\ A'' &:= A \setminus A'. \end{aligned}$$

For every $r \in A''$, we have $\mu_r = 0$ and hence

$$\sum_{r \in A} \mu_r = \sum_{r \in A'} \mu_r.$$

Write C' for the paths in A' with the trailing q removed:

$$C' := \left\{ p \,\middle|\, \begin{array}{c} p \text{ is a path in } Q \\ pq \in A \end{array} \right\}.$$

Then observe that $C' \subseteq P$, seeing as

$$\begin{aligned} p \notin P &\implies p + I = 0 + I \\ &\implies p \in I \\ &\implies pq \in I \\ &\implies pq + I = 0 + I \\ &\implies pq \notin P \\ &\implies pq \notin A' \\ &\implies p \notin C', \end{aligned}$$

and that C' contains entire equivalence classes of P in the sense that $p \in C' \implies [p]_\sim \subseteq C'$ (where $[p]_\sim$ denotes the equivalence class of p under $\sim$), seeing as if $p \in C'$ then $pq \in A'$ and

$$\begin{aligned} p \sim p' &\implies p + I = p' + I \\ &\implies pq + I = p'q + I \\ &\implies pq \sim p'q \\ &\implies p'q \in A' \\ &\implies p' \in C'. \end{aligned}$$

It follows that C' is a union of equivalence classes under $\sim$, namely

$$C' = \bigcup_{\substack{C \in P/\sim \\ C \subseteq C'}} C.$$

Using the assumption that $\sum \lambda_p p \in I'$, i.e., that $\sum_{p \in C} \lambda_p = 0$ for every $C \in P/\sim$, we then have

$$\begin{aligned} \sum_{r \in A} \mu_r &= \sum_{r \in A'} \mu_r \\ &= \sum_{pq \in A'} \mu_{pq} \\ &= \sum_{p \in C'} \mu_{pq} \\ &= \sum_{p \in C'} \lambda_p \\ &= \sum_{\substack{C \in P/\sim \\ C \subseteq C'}} \sum_{p \in C} \lambda_p \\ &= \sum_{\substack{C \in P/\sim \\ C \subseteq C'}} 0 \\ &= 0. \end{aligned}$$

Thus, we have shown that I' is closed under multiplication by arbitrary elements from the right. The proof for multiplication from the left is similar. With that said, we have shown that I' is a two-sided ideal of KQ containing a set of generators of I, from which $I \subseteq I'$ follows.

For the reverse inclusion $I \supseteq I'$, we proceed by induction over the number, k say, of nonzero scalars λ_p in the sum $\sum \lambda_p p$ for an element in I'. The case $k = 0$ is immediate:

$$k = 0 \implies \sum \lambda_p p = 0 \in I,$$

and in the case $k = 1$, we have $\sum \lambda_p p = \lambda_{p_0} p_0$ for some path p_0 and scalar $\lambda_{p_0} \neq 0$. The condition for membership in I' rules out the possibility that $p_0 \in P$, which is to say that $p_0 \in I$ and hence $\lambda_{p_0} p_0 \in I$.

Now suppose that every element of I' with at most k nonzero scalars are also members of I and consider an element $\sum \lambda_p p \in I'$ with $k + 1$ nonzero scalars λ_p.

If at least one of the paths, p_0 say, corresponding to nonzero scalars is in I, then its term, $\lambda_{p_0} p_0$, can safely be removed from the sum without violating the condition for membership in I'. We then have

$$\sum \lambda_p p = \sum_{p \neq p_0} \lambda_p p + \lambda_{p_0} p_0,$$

where the sum in the right-hand side has k terms and is hence an element of I by the induction hypothesis, and $\lambda_{p_0} p_0$ is also an element of I. Thus, the entire right-hand side is in I, which is what we wanted to show.

Otherwise, all the paths corresponding to nonzero scalars are in P. If all the paths belong to the same equivalence class $A \in P/{\sim}$, then by fixing an arbitrary $p_0 \in A$, we have

$$\begin{aligned}
\Big(\sum_{p \text{ a path in } Q} \lambda_p p\Big) + I &= \sum_{p \text{ a path in } Q} \lambda_p (p + I) \\
&= \sum_{p \in A} \lambda_p (p + I) \\
&= \sum_{p \in A} \lambda_p (p_0 + I) \\
&= \Big(\sum_{p \in A} \lambda_p\Big)(p_0 + I) \\
&= 0(p_0 + I) \\
&= 0 + I.
\end{aligned}$$

This is to say that $\sum \lambda_p p \in I$, which is what we wanted to show.

Otherwise, we pick an arbitrary equivalence class A containing at least one path with nonzero scalar and write

$$\sum \lambda_p p = \sum_{p \in A} \lambda_p p + \sum_{p \notin A} \lambda_p p,$$

where each of the two sums in the right-hand side belongs to I'. The first term contains (by definition of A) at least one nonzero scalar and, because A does not contain all paths with nonzero scalar, so does the second term. It follows that neither of the terms contains strictly more than k elements. The induction hypothesis then says that both terms are in I, so that the entire right-hand side is in I, which is what we wanted to show. This finishes the proof that $I = I'$. □

For K-linear independence of B, take an arbitrary K-linear combination of B. If we fix an arbitrary transversal T of B consisting of paths, we can write the linear combination as

$$\sum_{p \in T} \lambda_p (p + I)$$

for some $\lambda_p \in K$. Now suppose that the linear combination is zero. We then have

$$\Big(\sum_{p\in T} \lambda_p p\Big) + I = \sum_{p\in T} \lambda_p (p + I)$$
$$= 0 + I,$$

which is to say that

$$\sum_{p\in T} \lambda_p p \in I.$$

By the lemma, we have

$$\sum_{p\in T} \lambda_p p \in I',$$

so that for every $A \in P/{\sim}$,

$$\sum_{p\in T\cap A} \lambda_p = 0.$$

Because T is a transversal of B, there is at most one element in every $T \cap A$. It follows that $\lambda_p = 0$ for every $A \in P/{\sim}$ and $p \in T \cap A$ and hence that $\lambda_p = 0$ for every $p \in T$, which is to say that the linear combination we started with is trivial. This shows that B is linearly independent and finishes the proof that B is a basis of KQ/I. □

The equivalence relation $\sim$ in the proof of Proposition 2.26 is readily extended to the collection of all paths in Q and the zero element of KQ. In Section 4, we will need to be able to algorithmically compute the equivalence classes under this extended relation. In order to do so, we introduce two types of "moves" on the paths in Q, which transform a path into another path or the zero element of KQ, and express in Proposition 2.28 the equivalence relation in terms of these moves.

Definition 2.27 Let (Q, I) be a semimonomial bound quiver and p be a path in Q. If there is a monomial generator p' of I that appears in p, i.e., with

$$p = xp'y$$

for some paths x, y in Q, we define an *annihilation move* for p that transforms p into 0. For every non-monomial generator $p' - q'$ of I with p' appearing in p, i.e., with

$$p = xp'y$$

for some paths x, y in Q, we define a *replacement move* for p that transforms $p = xp'y$ into $xq'y$. Similarly, if q' appears in p as $p = xq'y$, we define the replacement move for p that transforms $p = xq'y$ into $xp'y$.

If there is a finite sequence of moves transforming a path p into a path q or the zero element (where the empty sequence is understood to transform p into p itself), we write

$$p \rightsquigarrow q$$

and

$$p \rightsquigarrow 0,$$

respectively. □

Proposition 2.28 *Let (Q, I) be a semimonomial bound quiver and $\sim$ denote the equivalence relation on*

$$\{p \mid p \text{ is a path in } Q\} \cup \{0\} \subseteq KQ$$

defined by

$$x \sim y \iff x + I = y + I.$$

We then have

1. *for every path p in Q,*

$$p \sim 0 \iff p \rightsquigarrow 0,$$

and

2. *for all paths p, q in Q with $p, q \nsim 0$,*

$$p \sim q \iff p \rightsquigarrow q.$$

□

Proof. We show the implications from right to left by showing that annihilation moves exist only for zero-equivalent paths and that replacement moves transform a path into an equivalent path.

If p is a path with an annihilation move, then $p = xp'y$ for some monomial generator p' of I and paths x, y, and we have

$$p' + I = 0 + I \implies p + I = (x + I)(p' + I)(y + I) = 0 + I,$$

so that $p \sim 0$.

If p is a path with a replacement move transforming p into a path q, then $p = xp'y$ and $q = xq'y$ for some non-monomial generator $p' - q'$ of I and paths x, y, and we have

$$p' + I = q' + I \implies p + I = (x + I)(p' + I)(y + I) = (x + I)(q' + I)(y + I) = q + I,$$

so that $p \sim q$.

The implications from right to left now follow: if $p \rightsquigarrow q$, then there is a sequence of replacement moves transforming $p =: p_1$ into $q =: p_n$, and we have

$$\begin{aligned} p = p_1 \rightsquigarrow p_2 \rightsquigarrow \cdots \rightsquigarrow p_n = q &\implies p = p_1 \sim p_2 \sim \cdots \sim p_n = q \\ &\implies p \sim q. \end{aligned}$$

If $p \rightsquigarrow 0$, then there is a sequence of replacement moves transforming p into some path p' and an annihilation move transforming p' into 0, and we have

$$\begin{aligned} (p \rightsquigarrow p') \wedge (p' \rightsquigarrow 0) &\implies (p \sim p') \wedge (p' \sim 0) \\ &\implies p \sim 0. \end{aligned}$$

For the implication $p \sim 0 \implies p \rightsquigarrow 0$, we assume that p is a path with $p + I = 0 + I$, i.e., $p \in I$, and write it as

$$\begin{aligned} p &= \sum_{r \in R} \lambda_r x_r p_r y_r + \sum_{s \in S} \lambda_s x_s (p_s - q_s) y_s \\ &= \sum_{r \in R} \lambda_r x_r p_r y_r + \sum_{s \in S} \lambda_s (x_s p_s y_s - x_s q_s y_s), \end{aligned}$$

where R, S are some index sets, $\lambda_r, \lambda_s \in K$, x_r, y_r, x_s, y_s are paths in Q, p_r are monomial generators of I, and $p_s - q_s$ are non-monomial generators of I. Because p appears in the left-hand side, p must appear in the right-hand side. If p appears in the sum over R, as $x_r p_r y_r$, then the monomial generator p_r gives rise to an annihilation move for p, so that

$$p \rightsquigarrow 0.$$

Otherwise, p appears in the sum over S, and we look for a sequence of replacement moves that transform p into a path appearing in the sum over R. Towards defining the set P of paths appearing in the sum over S that are "reachable from p via the sum over S", we set

$$P_0 := \{p\}$$

and recursively define P_{k+1} for $k \geq 0$ as

$$P_{k+1} := P_k \cup \left\{ q \;\middle|\; \begin{array}{c} q \text{ is a path in } Q \\ \exists s \in S\colon (x_s p_s y_s \in P_k \wedge q = x_s q_s y_s) \vee (x_s q_s y_s \in P_k \wedge q = x_s p_s y_s) \end{array} \right\}.$$

In words, P_{k+1} is the set obtained from P_k by locating the terms in the sum over S with precisely one of its two paths appearing in P_k and adding the other path to P_{k+1}. After at most $|S|$ recursion steps, we will run out of terms in the sum over S with precisely one path in P_k. That is, every term in the sum over S will have either both or neither of its two paths in the set

$$P := P_{|S|}.$$

We note, by induction, that for every $k \geq 0$ and $q \in P_k$, there is a sequence of replacement moves transforming p into q, which for $k = |S|$ is to say that

$$q \in P \implies p \rightsquigarrow q.$$

Let $S' \subseteq S$ denote the set of indices of the terms with both paths in P:

$$S' := \{s \in S \mid x_s p_s y_s, x_s q_s y_s \in P\},$$

and consider the sum

$$\sum_{s \in S'} \lambda_s (x_s p_s y_s - x_s q_s y_s),$$

which we may write as

$$\sum_{s \in S'} \lambda_s (x_s p_s y_s - x_s q_s y_s) = \sum_{t \in T} \lambda_t p_t, \tag{1}$$

for some index set T, $\lambda_t \in K$, and distinct paths p_t in Q. Note that

$$\sum_{t \in T} \lambda_t = 0,$$

because every term in the left-hand side of Eq. (1) has coefficients adding up to 0. Consider the identity

$$p = \sum_{r \in R} \lambda_r x_r p_r y_r + \sum_{s \in S'} \lambda_s (x_s p_s y_s - x_s q_s y_s) + \sum_{s \in S \setminus S'} \lambda_s (x_s p_s y_s - x_s q_s y_s). \tag{2}$$

The coefficient, λ_{t_0} say, for p in the sum over S' must be 1, because the coefficient for p in the left-hand side is 1 and p does not appear in the sum over R or $S \setminus S'$. Therefore, there is a path $p_{t_1} \in P \setminus \{p\}$ with coefficient $\lambda_{t_1} \neq 0$ in the sum over S'. This path p_{t_1} does not appear in the left-hand side of Eq. (2) or in the sum over $S \setminus S'$ in the right-hand side, so p_{t_1} must appear in the sum over R with coefficient $-\lambda_{t_1} \neq 0$, which gives an annihilation move $p_{t_1} \rightsquigarrow 0$. Because $p_{t_1} \in P$, we have

$$p \rightsquigarrow p_{t_1} \rightsquigarrow 0,$$

which finishes the proof that $p \sim 0$ implies $p \rightsquigarrow 0$.

For the implication $p \sim q \implies p \rightsquigarrow q$, we assume that p, q are paths with $p + I = q + I \neq 0 + I$, i.e., that $p - q \in I$ but $p, q \notin I$, and proceed much as in the proof that $p \sim 0 \implies p \rightsquigarrow 0$. We write

$$\begin{aligned} p - q &= \sum_{r \in R} \lambda_r x_r p_r y_r + \sum_{s \in S} \lambda_s x_s (p_s - q_s) y_s \\ &= \sum_{r \in R} \lambda_r x_r p_r y_r + \sum_{s \in S} \lambda_s (x_s p_s y_s - x_s q_s y_s), \end{aligned}$$

and note that p cannot appear in the sum over R in the right-hand side, because $p \nsim 0$. We define the set P of paths appearing in the sum over S that are "reachable from p via the sum over S" and the set S' of indices of terms with both paths in P as before, and we consider the same sum as before,

$$\sum_{s \in S'} \lambda_s (x_s p_s y_s - x_s q_s y_s),$$

which we write as

$$\sum_{s \in S'} \lambda_s (x_s p_s y_s - x_s q_s y_s) = \sum_{t \in T} \lambda_t p_t.$$

We again have

$$\sum_{t \in T} \lambda_t = 0$$

and consider the identity

$$p - q = \sum_{r \in R} \lambda_r x_r p_r y_r + \sum_{s \in S'} \lambda_s (x_s p_s y_s - x_s q_s y_s) + \sum_{s \in S \setminus S'} \lambda_s (x_s p_s y_s - x_s q_s y_s).$$

The coefficient, λ_{t_0} say, for p in the sum over S' in the right-hand side is 1, because the coefficient for p in the left-hand side is 1 and p does not appear in the sums over R or $S \setminus S'$. We conclude that there is a path $p_{t_1} \in P \setminus \{p\}$ with coefficient $\lambda_{t_1} \neq 0$ in the sum over S'. This path p_{t_1} cannot appear in the sum over R in the right-hand side, because we would then have

$$p \sim p_{t_1} \sim 0,$$

and p_{t_1} does not appear in the sum over $S \setminus S'$, so its coefficient in the entire right-hand side is λ_{t_1}. Thus, p_{t_1} must appear with coefficient $\lambda_{t_1} \neq 0$ in the left-hand side. Because p_{t_1} is distinct from p, we must have $p_{t_1} = q$. From $p_{t_1} \in P$, we conclude that

$$p \rightsquigarrow p_{t_1} = q,$$

which finishes the proof that $p \sim q$ implies $p \rightsquigarrow q$. □

As a corollary of Proposition 2.26, we have a concrete basis for every indecomposable projective module.

Corollary 2.29 *Let (Q, I) be a bound quiver and $i \in Q_0$. Then the following set is a K-basis of $P(i)$:*

$$B_i := \left\{ p + I \;\middle|\; \begin{array}{c} p \text{ is a path in } Q \\ s(p) = i \\ p + I \neq 0 + I \end{array} \right\}.$$ □

Proof. Note that

$$B = \bigsqcup_{j \in Q_0} B_j$$

and that $B_j \subseteq P(j)$ for every $j \in Q_0$, because every $p + I \in B_j$ has $s(p) = j$. It follows from

$$KQ/I = \bigoplus_{j \in Q_0} P(j)$$

that B_j is a basis of $P(j)$ for every $j \in Q_0$. □

Definition 2.30 Let (Q, I) be a bound quiver and $i \in Q_0$. We say that (Q, I) is *weakly cancellative at i* if for every path p in Q starting at i, we have either

$$\forall a \in Q_1 \colon pa + I = 0 + I$$

or

$$\exists a \in Q_1 \colon \big((pa + I \neq 0 + I) \wedge (\forall q \colon pa + I = qa + I \implies p + I = q + I) \big),$$

where q is quantified over all paths in Q. □

For semimonomial bound quivers, the socle $\operatorname{soc} P(i)$ at a vertex i at which the bound quiver is weakly cancellative admits a very concrete basis, expressed in terms of maximal nonzero equivalence classes of paths in Q.

Proposition 2.31 *Let (Q, I) be a semimonomial bound quiver and $i \in Q_0$. If (Q, I) is weakly cancellative at i, then the following set is a K-basis of* $\operatorname{soc} P(i)$*:*

$$\left\{ p + I \;\middle|\; \begin{array}{c} p \text{ is a path in } Q \\ s(p) = i \\ p + I \neq 0 + I \\ \forall a \in Q_1 \colon pa + I = 0 + I \end{array} \right\} \qquad \square$$

Proof. Let C_i denote the tentative basis:

$$C_i = \left\{ p + I \;\middle|\; \begin{array}{c} p \text{ is a path in } Q \\ s(p) = i \\ p + I \neq 0 + I \\ \forall a \in Q_1 \colon pa + I = 0 + I \end{array} \right\}$$

First, we show that $C_i \subseteq \operatorname{soc} P(i)$, using Proposition 2.21. We have

$$\begin{aligned} C_i &= \left\{ p + I \;\middle|\; \begin{array}{c} p \text{ is a path in } Q \\ s(p) = i \\ p + I \neq 0 + I \\ \forall a \in Q_1 \colon pa + I = 0 + I \end{array} \right\} \\ &\subseteq \left\{ p + I \;\middle|\; \begin{array}{c} p \text{ is a path in } Q \\ s(p) = i \end{array} \right\} \\ &\subseteq P(i). \end{aligned}$$

For every $p + I \in C_i$ and $a \in Q_1$, the condition that $pa + I = 0 + I$ is equivalent to $(p + I)(a + I) = 0 + I$. Proposition 2.21 thus lets us conclude that

$$\begin{aligned} C_i &\subseteq \{x \in P(i) \mid \forall a \in Q_1 \colon x(a + I) = 0\} \\ &= \operatorname{soc} P(i). \end{aligned}$$

Next, we note that C_i is K-linearly independent. This is an immediate consequence of the fact that C_i is a subset of the basis of $P(i)$ given by Corollary 2.29.

Finally, we show that C_i has K-linear span all of $\operatorname{soc} P(i)$, which relies on the assumption that (Q, I) is weakly cancellative at i. Let $x \in \operatorname{soc} P(i) \subseteq P(i)$ be an arbitrary element of the socle, and recall the K-basis B_i of $P(i)$ given by Corollary 2.29:

$$B_i = \left\{ p + I \;\middle|\; \begin{array}{c} p \text{ is a path in } Q \\ s(p) = i \\ p + I \neq 0 + I \end{array} \right\}.$$

Express x in this basis B_i as

$$x = \sum_k \lambda_k (p_k + I),$$

with all $\lambda_k \neq 0$. We want to show that every $p_k + I$ is an element of not just B_i but of the subset $C_i \subseteq B_i$. In other words, we want to show that every $p_k + I$ satisfies

$$\forall a \in Q_1 \colon p_k a + I = 0 + I.$$

Assume towards a contradiction that this is not the case, i.e., that there is a k_0 and an $a_0 \in Q_1$ with

$$p_{k_0} a_0 + I \neq 0 + I.$$

By weak cancellativity at i, we can take a_0 to satisfy

$$\forall q \colon p_{k_0} + I \neq q + I \implies p_{k_0} a_0 + I \neq q a_0 + I.$$

For paths $q = p_k$ with $k \neq k_0$ (which satisfy $p_{k_0} + I \neq p_k + I$), this means that

$$p_{k_0} a_0 + I \neq p_k a_0 + I.$$

Because $x \in \operatorname{soc} P(i)$ and by Proposition 2.21, we have

$$\begin{aligned} 0 + I &= x(a_0 + I) \\ &= \Big(\sum_k \lambda_k (p_k + I) \Big)(a_0 + I) \\ &= \sum_k \lambda_k \big((p_k + I)(a_0 + I) \big) \\ &= \sum_k \lambda_k (p_k a_0 + I), \end{aligned}$$

which lets us write

$$\begin{aligned} -\lambda_{k_0}(p_{k_0} a_0 + I) &= \sum_{k \neq k_0} \lambda_k (p_k a_0 + I) \\ &= \sum_{\substack{k \neq k_0 \\ p_k a_0 + I \neq 0 + I}} \lambda_k (p_k a_0 + I). \end{aligned}$$

The rightmost side is a linear combination of elements of B_i distinct from $p_{k_0} a_0 + I \in B_i$, while the leftmost side is a nonzero scalar multiple of $p_{k_0} a_0 + I$. This is impossible by K-linear independence of B_i. We thus conclude that we for every k have

$$\forall a \in Q_1 \colon p_k a + I = 0 + I,$$

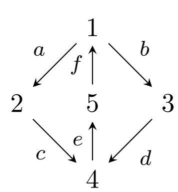
so that

$$x = \sum_k \lambda_k (p_k + I)$$

is a K-linear combination of elements of C_i. Because $x \in \operatorname{soc} P(i)$ was arbitrary, we have shown that C_i spans $\operatorname{soc} P(i)$. □

The following example shows that the assumption of weak cancellativity in Proposition 2.31 cannot be omitted.

Example 2.32 Let Q denote the quiver



and I denote the ideal $\langle bde - ace, cef, def \rangle$.

This I is a semimonomial and admissible ideal. However, (Q, I) is not weakly cancellative at 1, because ac is a path that is killed when right-multiplied by every arrow except for e and e fails to distinguish $ac + I$ from $bd + I$ in the sense that $ace + I = bde + I$, despite $ac + I$ and $bd + I$ being distinct.

There is only one maximal nonzero equivalence class of paths starting at $i = 1$ in Q, namely $ace + I = bde + I$, so we have

$$\begin{aligned} C_1 &:= \left\{ p + I \;\middle|\; \begin{array}{c} p \text{ is a path in } Q \\ s(p) = 1 \\ p + I \neq 0 + I \\ \forall a' \in Q_1 \colon pa' + I = 0 + I \end{array} \right\} \\ &= \{ace + I\} \end{aligned}$$

However, the equivalence class $x := (bd - ac) + I$ is annihilated by every arrow in Q, and we thus have $x \in \operatorname{soc} P(1) \setminus \operatorname{span}_K C_1$. In particular, $\operatorname{soc} P(1) \neq \operatorname{span}_K C_1$. □

For a bound quiver that is semimonomial and weakly cancellative at every vertex, we may use Proposition 2.31 to give a very concrete characterization of the Nakayama permutation, namely in terms of maximal nonzero equivalence classes of paths. This is the result that allows us to algorithmically find the Nakayama permutation of such a bound quiver, if it exists, and hence determine if the bound quiver algebra is self-injective, as described in Section 4.1.

Proposition 2.33 *Let (Q, I) be a semimonomial bound quiver that is weakly cancellative at every $i \in Q_0$. For every $i \in Q_0$, denote by C_i the basis from Proposition 2.31:*

$$C_i := \left\{ p + I \;\middle|\; \begin{array}{c} p \text{ is a path in } Q \\ s(p) = i \\ p + I \neq 0 + I \\ \forall a \in Q_1 \colon pa + I = 0 + I \end{array} \right\}.$$

If every C_i is singleton, say $C_i = \{p_i + I\}$ for some path p_i, and the map $\sigma \colon Q_0 \to Q_0$ defined by

$$\sigma(i) = t(p_i)$$

is a permutation, then σ is a Nakayama permutation for (Q, I). Otherwise, there is no Nakayama permutation for (Q, I). □

Proof. First, we note that every C_i being singleton is a necessary condition for the existence of a Nakayama permutation, seeing as the simple modules are one-dimensional by Proposition 2.18.

Under the assumption that every C_i is singleton (and writing $C_i = \{p_i + I\}$), we then show that

$$\operatorname{soc} P(i) \cong S(t(p_i))$$

for every $i \in Q_0$. For an arbitrary $i \in Q_0$, the assumption is that $\operatorname{soc} P(i)$ is one-dimensional. Hence, $\operatorname{soc} P(i)$ is simple, and by Proposition 2.20 we must have

$$\operatorname{soc} P(i) \cong S(j)$$

for some $j \in Q_0$. Note that $e_{t(p_i)}$ acts as the identity map on $\operatorname{soc} P(i) = \operatorname{span}_K(p_i + I)$. By Proposition 2.18, the only $j \in Q_0$ for which $e_{t(p_i)}$ acts as the identity map on $S(j)$ is $j = t(p_i)$, so we have

$$\operatorname{soc} P(i) \cong S(t(p_i)),$$

and this holds for every $i \in Q_0$. This shows that σ in Proposition 2.33 is well-defined and satisfies

$$\operatorname{soc} P(i) \cong S(\sigma(i))$$

for every $i \in Q_0$. If σ is a permutation, it follows that σ is a Nakayama permutation for (Q, I). If σ is not a permutation, then σ is not a Nakayama permutation for (Q, I), and we have by Remark 2.24 that there is no Nakayama permutation for (Q, I). □

3 Quivers with potential

In this section, we define quivers with potential (QPs) and focus our attention on quotients KQ/I of the path algebra where the ideal I is induced by the potential on Q (the Jacobian ideal of the QP). In the case that this quotient KQ/I is self-injective, the QP is said to be self-injective. Motivation to study self-injective QPs can be found in their relationship with so-called "2-representation-finite algebras"; see Theorem 3.11 in [6].

We give a sufficient condition for when the Jacobian ideal of a QP is semimonomial in Proposition 3.9, which may be viewed as a step towards applying Proposition 2.33 to QPs. This step turns out to be sufficient for performing a computer search for self-injective QPs using algorithms described in Section 4.

The section concludes with an account of all the self-injective planar QPs known to the author, in particular fourteen QPs not previously appearing in the literature, which were found using the algorithms from Section 4.

3.1 Quivers with potential and their Jacobian algebras

The definitions in this section are concrete versions of the more abstract and general definitions in [7].

Definition 3.1 Let Q be a quiver. Consider the set Q_{cyc} of all cycles in Q and the equivalence relation $\sim$ on Q_{cyc} defined by

$$(i \mid a_1, \dots, a_m \mid i) \sim (j \mid b_1, \dots, b_n \mid j)$$

if and only if

$$m = n \text{ and } \exists k \in \{1, \dots, n\} \colon (b_1, \dots, b_n) = (a_k, \dots, a_n, a_1, \dots, a_{k-1}).$$

That is, two cycles are equivalent if one of them can be cyclically permuted to the other.

A *detached cycle* in Q is an element of $Q_{\text{cyc}}/{\sim}$. We denote by $[a_1 \cdots a_m]$ the equivalence class of a cycle $a_1 \cdots a_m$. □

Definition 3.2 Let Q be a finite quiver. A *potential* on Q is a finite K-linear combination W of detached cycles in Q. To avoid certain non-admissibility of the Jacobian ideal (see Definition 3.6), we will assume that the detached cycles in W all have length at least 3. A *quiver with potential* (QP) is a pair (Q, W) where Q is a quiver and W is a potential on Q. □

Example 3.3 Consider the quiver of Example 2.4. Then $abcde$, $bcdea$, and f are distinct cycles. As detached cycles, however, the first two cycles are equal: $[abcde] = [bcdea]$.

$W = [bcdea] - [f^3]$ is an example of a potential on Q. □

Detached cycles and potentials can be "differentiated", which is a notion necessary to define the Jacobian ideal.

Definition 3.4 Let Q be a quiver. For every $a \in Q_1$ and detached cycle $[a_1 \cdots a_l]$, the *cyclic derivative* with respect to a of $[a_1 \cdots a_l]$ is an element of KQ defined as

$$\partial_a([a_1 \cdots a_l]) := \sum_{a_i = a} a_{i+1} \cdots a_l a_1 \cdots a_{i-1}.$$

For $l = 1$ and $a_1 = a$, the right-hand side is understood as a singleton sum with the stationary path at $s(a_1)$ as its only term.

Cyclic differentiation of a detached cycle extends by linearity to cyclic differentiation of a potential W. □

While perhaps not of much importance to us, it is worth noting that the notion of cyclic differentiation has several properties that justifies the terminology, such as a Leibniz rule and a chain rule [3, p. 11]. In the following example, we see that the cyclic derivative of a "monomial" is what one would expect from calculus.

Example 3.5 Let Q be a quiver with a loop $x \in Q_1$. Write x^n for the cycle $x \cdots x$ of length $n \geq 1$, or more explicitly, $x_1 \cdots x_n$ with $x_i = x$ for every $i = 1, \ldots, n$. We then have

$$\begin{aligned} \partial_x([x^n]) &= \sum_{x_i = x} x_{i+1} \cdots x_n x_1 \cdots x_{i-1} \\ &= \sum_{i=1}^{n} x_{i+1} \cdots x_n x_1 \cdots x_{i-1} \\ &= \sum_{i=1}^{n} x^{n-1} \\ &= n x^{n-1} \end{aligned}$$

(where x^{n-1} for $n - 1 = 0$ is understood as the stationary path at $s(x)$). □

Definition 3.6 Let (Q, W) be a QP. The *Jacobian ideal* of (Q, W) is the two-sided ideal

$$\langle \partial_a(W) \mid a \in Q_1 \rangle \subseteq KQ$$

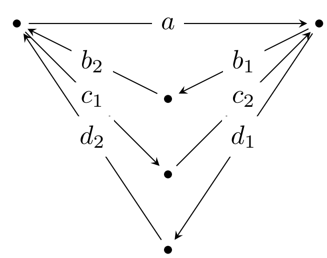
of KQ generated by the cyclic derivatives of W with respect to each arrow $a \in Q_1$. The *Jacobian algebra* of (Q, W) is the quotient

$$P(Q, W) := \frac{KQ}{\langle \partial_a(W) \mid a \in Q_1 \rangle},$$

of the path algebra by the Jacobian ideal. □

The following example shows that the Jacobian ideal of a QP is not necessarily admissible (not even when the QP is planar; see Definition 3.12).

Example 3.7 Let (Q, W) be the QP with quiver



and potential

$$W = [ab_1b_2] - [b_1b_2c_1c_2] + [c_1c_2d_1d_2],$$

and let I denote the Jacobian ideal of (Q, W). For every $n \in \mathbb{Z}_{>0}$, we have that $(ad_1d_2)^n$ is a path of length $3n$. Moreover, because every term of every generator of I contains one of the arrows b_1, b_2, c_1, c_2 not present in $(ad_1d_2)^n$, we have $(ad_1d_2)^n \notin I$. Thus, the Jacobian ideal I of (Q, W) is not admissible. □

Definition 3.8 Let (Q, W) be a QP with admissible Jacobian ideal.

(Q, W) is said to be *self-injective* if its Jacobian algebra $P(Q, W)$ is self-injective, and the *Nakayama permutation* of (Q, W) is the Nakayama permutation of the underlying bound quiver (Q, I), where I denotes the Jacobian ideal of (Q, W).

(Q, W) is said to be *weakly cancellative at* $i \in Q_0$ if the underlying bound quiver (Q, I) is weakly cancellative at i. □

In order to make use of Proposition 2.33 to draw conclusions about QPs, it would be useful to have conditions under which the Jacobian ideal of a QP is admissible and semimonomial. We now give a simple sufficient condition for semimonomiality.

Proposition 3.9 *Let (Q, W) be a QP. If*

1. *every cycle in W has coefficient -1 or $+1$,*
2. *every arrow $a \in Q_1$ appears in at most one positive cycle and in at most one negative cycle, and*
3. *every arrow $a \in Q_1$ appears at most once in every cycle in W,*

then the Jacobian ideal of (Q, W) is semimonomial. □

Proof. Fix an arrow $a \in Q_1$ and a path p occurring in the cyclic derivative $\partial_a(W)$. The key observation is that the occurrences of p in $\partial_a(W)$ are all due to the detached cycle $[pa]$; no other detached cycle contributes to the occurrences of p.

With the third assumption, that every arrow occurs at most once in every cycle, we have that p appears in $\partial_a(W)$ with the same scalar $\lambda_{[pa]} \in K$ as $[pa]$ appears with in W (without the third assumption, the detached cycle $[pa]$ could have "repetitions" and thus give rise to several occurrences of p in $\partial_a(W)$, which combine to a single term with scalar equal to an integer multiple of $\lambda_{[pa]}$). By the first assumption, this scalar is -1 or $+1$.

Keeping a fixed, the second assumption, that a appears in at most one positive cycle and in at most one negative cycle, gives rise to four possible cases:

1. If a does not appear in any positive cycle or in any negative cycle, then $\partial_a(W) = 0$.
2. If a appears in one positive cycle, $[pa]$, and in no negative cycle, then $\partial_a(W) = p$
3. If a appears in no positive cycle and in one negative cycle, $[qa]$, then $\partial_a(W) = -q$
4. If a appears in one positive cycle, $[pa]$, and in one negative cycle, $[qa]$, then $\partial_a(W) = p - q$

In the third case, we may replace the generator $-q$ by q. We thus see that the first case contributes with no generator, the second and third cases contribute with a monomial generator p and q, respectively, and the fourth case contributes with a non-monomial generator $p - q$ to the Jacobian ideal. Letting a vary, we get a semimonomial Jacobian ideal. □

A natural way in which QPs with semimonomial Jacobian ideals arise is from certain quivers embedded in the plane.

Definition 3.10 Let Q be a quiver without loops and 2-cycles. By an *embedding* of Q in the plane, we mean an injective map $e : Q_0 \to \mathbb{R}^2$ such that the collection $\{l_a \mid a \in Q_1\}$, where l_a denotes the open line segment from $e(s(a))$ to $e(t(a))$, satisfies

- $e(i) \notin l_a$ for every $i \in Q_0$ and $a \in Q_1$ and
- $l_a \cap l_b = \varnothing$ for all distinct $a, b \in Q_1$.

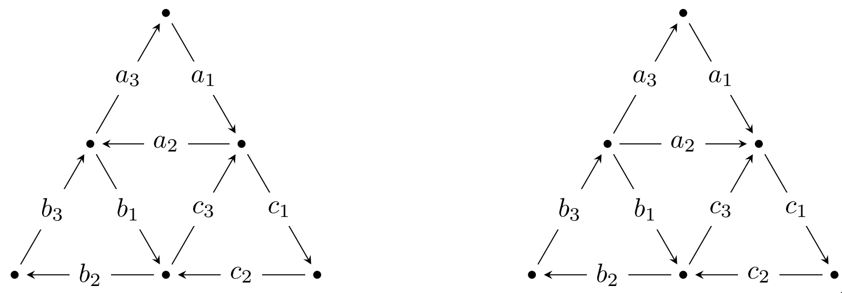
The pair (Q, e) is called a *plane quiver*, and its *image* is the set

$$e(Q) := \{e(i) \mid i \in Q_0\} \cup \bigcup_{a \in Q_1} l_a \subseteq \mathbb{R}^2.$$ □

Definition 3.11 Let (Q, e) be a plane quiver. A *face* of (Q, e) is a bounded connected component of $\mathbb{R}^2 \setminus e(Q)$ which is an open polygon. □

Definition 3.12 Let (Q, e) be a plane quiver such that every bounded connected component of $\mathbb{R}^2 \setminus e(Q)$ is a face and the arrows bounding every face are cyclically oriented. The linear combination W of the bounding cycles (as detached cycles) of all the faces of (Q, e) with coefficient $+1$ for clockwise oriented cycles and -1 for counterclockwise oriented cycles is called the *potential induced by* (Q, e). The QP (Q, W) is called the *planar QP induced by* (Q, e), and a QP that is induced by some plane quiver is said to be a *planar QP*. □

Example 3.13 Consider the two plane quivers



Both quivers have the property that all the bounded connected components of the complement of the image are faces. The quiver on the left-hand side has the bounding arrows of every face cyclically oriented, and its induced potential is

$$W = [a_1a_2a_3] + [b_1b_2b_3] + [c_1c_2c_3] - [a_2b_1c_3].$$

The quiver on the right-hand side has two faces with bounding arrows that are not cyclically oriented and so does not have an induced potential. □

Example 3.14 The QP in Example 3.7 is planar. □

Proposition 3.15 *Every planar QP has semimonomial Jacobian ideal.* □

Proof. The proof idea is to use Proposition 3.9. Let (Q, W) be a planar QP and (Q, e) be a plane quiver inducing the QP.

The first condition of Proposition 3.9 holds by definition of the induced potential.

For the second condition of Proposition 3.9, we let $a \in Q_1$ be an arbitrary arrow and consider the directed open line segment l_a from $e(s(a))$ to $e(t(a))$. Immediately to the left of l_a, there will be a connected component of $\mathbb{R}^2 \setminus e(Q)$, and immediately to the right of l_a, there will be a connected component of $\mathbb{R}^2 \setminus e(Q)$. These components are each either the unbounded component or a face (i.e., an open polygon). In the latter case, l_a is one of the sides of the face. This gives rise to three possible cases that we consider separately: both components are faces, precisely one of the components is a face, and neither of the components is a face.

If both components are faces, then a appears in two cycles of the potential (once for each face). The face on the right-hand side of l_a has clockwise orientation, and the face on the left-hand side of l_a has counterclockwise orientation. Thus, a appears in one cycle of positive sign and one cycle of negative sign in W.

If precisely one of the components is a face, then a appears in one cycle in W.

If neither of the components is a face, then a does not appear in any cycle in W.

In all three cases, we see that the second condition of Proposition 3.9 holds.

Finally, the third condition of Proposition 3.9 holds because every cycle in W corresponds to a face of (Q, e) and every arrow of a cycle in W corresponds to a side of the corresponding face. Because every side of a polygon appears only once as we trace its boundary, every arrow $a \in Q_1$ appears at most once in every cycle in W. □

3.2 Known self-injective planar quivers with potential

In this subsection, an account of all the self-injective planar QPs known to the author is given. The previously known self-injective planar QPs are covered by the articles [6] and [7]. Not covered by these articles are fourteen self-injective planar QPs, which were found by means of a computer search as described in Section 6.

Ten of these belong to one of four different infinite families of planar QPs that we shall call “odd flowers”, “even flowers of type 1”, “even flowers of type 2”, and “pointed flowers”. These families appear to consist solely of self-injective planar QPs.

There are two sources of self-injective planar QPs in the literature: [6] and [7]. In [6], three infinite families of self-injective planar QPs are given that we call “triangles”, “squares”, and “n-gons”. In [7], it is shown that every symmetric (k, n)-Postnikov diagram gives rise to a self-injective planar QPs, where k, n are positive integers satisfying $1 \leq k \leq n-1$. A necessary and sufficient condition on k and n in order for there to exist a symmetric (k, n)-Postnikov diagram is given in [8].

In addition to these sources, [6] give a notion of mutation of planar QPs called planar mutation, which preserves planarity. They further show that if the planar QP is self-injective, then planar mutation along an orbit of the Nakayama permutation subject to the condition that there are no arrows between the vertices of the orbit preserves also self-injectivity. This type of mutation is called iterated planar mutation. For brevity, we say that two self-injective planar QPs that are related by a sequence of such iterated planar mutations are *mutation-equivalent*.

Every self-injective planar QP thus induces an entire class of mutation-equivalent planar QPs that are also self-injective. By definition, iterated planar mutation preserves the number of vertices of the QP, N say. It follows from Theorem 4.2 in [6] that also the order of the Nakayama permutation, S say, is preserved by iterated planar mutation. Thus, if two self-injective planar QPs have different N or different S, then they are not mutation-equivalent. If a self-injective planar QP has (N, S) different from that of all self-injective planar QPs given in [6] and [7], then it is necessarily “new” in that it is not mutation-equivalent to any previously known self-injective planar QP.

The values of (N, S) for the self-injective planar QPs from [6] are as follows. The triangles have

$$N = \frac{n(n+1)}{2}, \qquad S = 3,$$

the squares have

$$N = n^2, \qquad S = 2,$$

and the n-gons have

$$N = n, \qquad S = \begin{cases} n & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

The value of (N, S) of a self-injective planar QP induced by a symmetric (k, n)-Postnikov diagram is determined entirely by k and n as follows:

$$N = k(n-k) - n + 1, \qquad S = \frac{n}{\gcd(n, k)},$$

and the criterion for (k, n) in order for there to exist at least one symmetric (k, n)-Postnikov diagram (given in [8]) is

$$k \equiv -1, 0, 1 \quad \left(\bmod \frac{n}{\gcd(n, k)}\right).$$

3.2.1 Odd flowers

The first four new self-injective planar QPs are shown in Figs. 2 to 5. They have an obvious generalization to a planar QP with $2m+1$ vertices in the center polygon and $m+1$ “layers” of vertices, which we may denote by OddFlower(m), for $m \geq 2$. In words, for every $m \geq 2$, we write OddFlower(m) for the QP constructed from a self-injective cobweb QP with $2m+1$ vertices in the center polygon as defined in [7] by gluing on a pentagon to each of the $2(2m+1)$ arrows of the boundary and letting neighboring pentagons share one arrow. For $m = 1$, a reasonable definition of OddFlower(m) is given by Fig. 1, which is mutation-equivalent to the triangle on 15 vertices and hence previously known to be self-injective.

The following proposition has been shown by means of computer assistance.

Proposition 3.16 *For $1 \leq m \leq 5$, the planar QP* OddFlower(m) *is self-injective with Nakayama permutation given by clockwise rotation by $\frac{2\pi \cdot m}{2m+1}$.* □

We are led to make the conjecture that this holds for every m:

Conjecture 3.17 *For every $m \geq 1$, the planar QP* OddFlower(m) *is self-injective with Nakayama permutation given by clockwise rotation by $\frac{2\pi \cdot m}{2m+1}$.* □

It turns out that for every $m \geq 2$ for which Conjecture 3.17 holds, OddFlower(m) is not mutation-equivalent to any previously known self-injective planar QP. In particular, we conclude that OddFlower(2), OddFlower(3), OddFlower(4), and OddFlower(5) are new self-injective planar QPs.

Proposition 3.18 *Let $m \geq 2$. If* OddFlower(m) *is self-injective with Nakayama permutation given by clockwise rotation by $\frac{2\pi \cdot m}{2m+1}$, then* OddFlower($m$) *is not mutation-equivalent to any previously known self-injective planar QP.* □

Proof. The number of vertices of OddFlower(m) is $N := (2m+1)(2m+3)$, and the order of the Nakayama permutation is $S := 2m+1$, seeing as

$$\begin{aligned}\gcd(m, 2m+1) &= \gcd(m, 2m+1-2m) \\ &= \gcd(m, 1) \\ &= 1.\end{aligned}$$

We have $S \geq 5$, so OddFlower(m) is not mutation-equivalent to any triangle or square. Moreover, OddFlower(m) is not mutation-equivalent to any n-gon, because an n-gon on $N = (2m+1)(2m+3)$ vertices has Nakayama permutation of order $n = (2m+1)(2m+3) > 2m+1 = S$. This rules out all self-injective planar QPs in [6].

To rule out also the self-injective planar QPs arising from symmetric (k, n)-Postnikov diagrams, we assume that there is such a diagram that induces a QP on N vertices with Nakayama permutation of order S and derive a contradiction. That is, we assume that k and n are integers satisfying the following conditions:

$$\begin{cases} 1 \leq k \leq n-1 & (3) \\ N = k(n-k) - n + 1 & (4) \\ S = \dfrac{n}{\gcd(n,k)} & (5) \\ k \equiv -1, 0, 1 \pmod{S}. & (6) \end{cases}$$

We first consider Eq. (4) modulo 2 and deduce that k and n are both even. We have $N = (2m+1)(2m+3)$ odd, so the right-hand side must be odd and hence $k(n-k) - n$ must be even. If n were odd, then $k(n-k)$ would have to be odd, which is impossible because the factors would have different parity, so n is even. Then $k(n-k)$ must be even, which is the case only when k is even, seeing as the factors of $k(n-k)$ have the same parity.

We now consider Eq. (4) modulo S to find, without loss of generality, that $k \equiv 1 \pmod{S}$. By Eq. (5), we have $n \equiv 0 \pmod{S}$, so that modding out by S in Eq. (4) gives

$$N \equiv -k^2 + 1 \pmod{S}.$$

On the other hand, we have $N = (2m+1)(2m+3) = S(S+2)$, so that

$$1 - k^2 \equiv 0 \pmod{S}$$

and hence, by Eq. (6),

$$k \equiv -1, 1 \pmod{S}.$$

We may without loss of generality assume that

$$k \equiv 1 \pmod{S};$$

otherwise, replace k by $n - k$.

Now write

$$k = qS + 1 \tag{7}$$

for some $q \geq 0$, where q is odd because k is even, and

$$n = dS \tag{8}$$

for $d = \gcd(n, k)$. We then have

$$\begin{aligned} N &= k(n-k) - n + 1 \\ &= (qS+1)(dS - (qS+1)) - dS + 1 \\ &= qdS^2 - q^2S^2 - qS + dS - qS - 1 - dS + 1 \\ &= qdS^2 - q^2S^2 - 2qS \\ &= S \cdot q(S(d-q) - 2). \end{aligned}$$

Because $N = S(S+2)$, we conclude that

$$S + 2 = q \cdot (S(d-q) - 2) \tag{9}$$

and hence in particular

$$q \leq S + 2.$$

Modding out Eq. (9) by S, we find that

$$2 \equiv -2q \pmod{S},$$

and because S is odd, we equivalently get

$$q \equiv -1 \pmod{S}.$$

Because $q \geq 0$, we get

$$q \geq S - 1$$

and because q is odd, we in fact get

$$q \geq 2S - 1.$$

We have thus shown that

$$\begin{aligned} (S+2) + (S-3) &= 2S - 1 \\ &\leq q \\ &\leq S + 2. \end{aligned} \tag{10}$$

Given that $S \geq 5$, we have $S - 3 > 0$ and hence a contradiction. We conclude that $\mathrm{OddFlower}(m)$ does not arise up to mutation equivalence from a symmetric Postnikov diagram.

For completeness, we may note that the QP does not arise from a symmetric (k,n)-Postnikov diagram in the case $m = 1$ ($S = 3$) either. Instead of an immediate contradiction from Eq. (10), we get

$$q = 5$$

so that

$$\begin{aligned} S(d-q) - 2 &= 3(d-q) - 2 \\ &= 1 \end{aligned}$$

by Eq. (9) and hence

$$d = q + 1 = 6.$$

Plugging these values into Eqs. (7) and (8) gives

$$\begin{aligned} k &= qS + 1 = 5 \cdot 3 + 1 = 16 \\ n &= dS = 6 \cdot 3 = 18. \end{aligned}$$

But these values have greatest common divisor

$$\begin{aligned} \gcd(n,k) &= \gcd(16,18) \\ &= 2 \cdot \gcd(8,9) \\ &= 2, \end{aligned}$$

which contradicts $\gcd(n,k) = d = 6$. □

Corollary 3.19 *For $2 \le m \le 5$, $\mathrm{OddFlower}(m)$ is a self-injective planar QP that is not mutation-equivalent to any previously known self-injective planar QP.* □

3.2.2 Even flowers of type 1

The first five QPs of the second infinite family of QPs are shown in Figs. 6 to 10. We may parametrize the QPs of the family by the number $m \ge 2$ of "layers" of vertices, which is half the number of vertices in the center polygon, and denote the corresponding QP by $\mathrm{EvenFlower}_1(m)$.

The following proposition has been shown by means of computer assistance.

Proposition 3.20 *For $2 \le m \le 6$, the planar QP $\mathrm{EvenFlower}_1(m)$ is self-injective with Nakayama permutation given by clockwise rotation by*

$$\begin{cases} \frac{2\pi \cdot (m-1)}{2m} & \text{if } m \text{ is even} \\ \pi & \text{if } m \text{ is odd.} \end{cases}$$ □

We are led to make the conjecture that this holds for every m:

Conjecture 3.21 *For every $m \ge 2$, the planar QP $\mathrm{EvenFlower}_1(m)$ is self-injective with Nakayama permutation given by clockwise rotation by*

$$\begin{cases} \frac{2\pi \cdot (m-1)}{2m} & \text{if } m \text{ is even} \\ \pi & \text{if } m \text{ is odd.} \end{cases}$$ □

It turns out that for every even $m \ge 4$ for which Conjecture 3.21 holds, $\mathrm{EvenFlower}_1(m)$ is not mutation-equivalent to any previously known self-injective planar QP. In particular, we conclude that $\mathrm{EvenFlower}_1(4)$ and $\mathrm{EvenFlower}_1(6)$ are new self-injective planar QPs. For $m = 2$, we have that $\mathrm{EvenFlower}_1(m)$ arises up to iterated planar mutation from a symmetric Postnikov diagram.

Proposition 3.22 *Let $m \geq 4$ be even. If* $\text{EvenFlower}_1(m)$ *is self-injective with Nakayama permutation given by clockwise rotation by* $\frac{2\pi\cdot(m-1)}{2m}$*, then* $\text{EvenFlower}_1(m)$ *is not mutation-equivalent to any previously known self-injective planar QP.* □

Proof. The number of vertices of $\text{EvenFlower}_1(m)$ is $N := (2m)^2$, and the order of the Nakayama permutation is $S := 2m$, seeing as m is even, and so

$$\begin{aligned}\gcd(m-1, 2m) &= \gcd(m-1, 2m-2(m-1)) \\ &= \gcd(m-1, 2) \\ &= 1.\end{aligned}$$

We first rule out the self-injective planar QPs in [6], which we can do for every even $m \geq 2$. For such m, we have $S = 2m \geq 4$, which rules out all triangles and squares. The N-gon has Nakayama permutation of order

$$\begin{aligned}\frac{N}{2} &= \frac{4m^2}{2} \\ &= 2m^2 \\ &> 2m \\ &= S,\end{aligned}$$

which rules out the last type of self-injective planar QPs in [6].

To see that $\text{EvenFlower}_1(m)$ does not arise up to iterated planar mutation from a symmetric (k, n)-Postnikov diagram, we assume that there is such a diagram that induces a QP on N vertices with Nakayama permutation of order S and derive a contradiction. That is, we assume that k and n are integers satisfying the conditions in Eqs. (3) to (6) given in Section 3.2.1.

By considering Eq. (4) modulo S, we get

$$N \equiv -k^2 + 1 \pmod{S}.$$

On the other hand, $N = (2m)^2 = S^2$ is a multiple of S, so that

$$1 - k^2 \equiv 0 \pmod{S}$$

and hence, by Eq. (6),

$$k \equiv -1, 1 \pmod{S}.$$

We may without loss of generality assume that

$$k \equiv 1 \pmod{S};$$

otherwise, replace k by $n - k$. Thus, we can write

$$k = qS + 1$$

for some $q \geq 0$. For brevity, also write

$$d = \gcd(n, k).$$

We then have

$$\begin{aligned}N &= k(n-k) - n + 1 \\ &= (qS+1)(dS - (qS+1)) - dS + 1 \\ &= qdS^2 - q^2S^2 - qS + dS - qS - 1 - dS + 1 \\ &= qdS^2 - q^2S^2 - 2qS \\ &= S \cdot q(S(d-q) - 2).\end{aligned}$$

Because $N = S^2 = S \cdot S$, we conclude that

$$S = q \cdot (S(d - q) - 2). \tag{11}$$

Considering Eq. (11) modulo S, we find that

$$0 \equiv -2q \pmod{S},$$

and because S is even, we equivalently get

$$q \equiv 0 \quad \left(\bmod\ \frac{S}{2}\right).$$

Because both factors in the right-hand side of Eq. (11) must be strictly positive, we get

$$q \geq \frac{S}{2}$$

and

$$d - q \geq 1,$$

which implies that

$$S(d - q) - 2 \geq S - 2.$$

For $S \geq 8$ (i.e., $m \geq 4$), we get

$$\begin{aligned} S &= q \cdot (S(d - q) - 2) \\ &\geq \frac{S}{2} \cdot (S - 2) \\ &> \frac{S}{2} \cdot 2 \\ &= S, \end{aligned}$$

which is absurd. We conclude that $\text{EvenFlower}_1(m)$ does not arise up to mutation equivalence from a symmetric Postnikov diagram.

For completeness, we may note that $\text{EvenFlower}_1(2)$ does arise up to iterated planar mutation from a $(3, 12)$-Postnikov diagram, as shown in Figure 19 in [7]. □

Corollary 3.23 $\text{EvenFlower}_1(4)$ *and* $\text{EvenFlower}_1(6)$ *are self-injective planar QPs that are not mutation-equivalent to any previously known self-injective planar QP.* □

Assuming the veracity of Conjecture 3.21, the number of vertices and the order of the Nakayama permutation for $\text{EvenFlower}_1(m)$ with $m \geq 3$ odd are $N = (2m)^2$ and $S = 2$. These values agree with the squares in [6], and one might ponder the question whether $\text{EvenFlower}_1(m)$ is mutation-equivalent to a square for every odd $m \geq 3$. This is indeed the case for $m = 3$, as has been verified by hand. It seems reasonable to expect that it should be the case for every odd $m \geq 3$.

Proposition 3.24 $\text{EvenFlower}_1(3)$ *is mutation-equivalent to the square on* $6^2 = 36$ *vertices.* □

Conjecture 3.25 $\text{EvenFlower}_1(m)$ *is mutation-equivalent to the square on* $(2m)^2$ *vertices for every odd* $m \geq 3$. □

3.2.3 Even flowers of type 2

The first five QPs of the third infinite family of QPs are shown in Figs. 11 to 15. We may parametrize the QPs of the family by the number $m \geq 2$ of "layers" of vertices, which is half the number of vertices in the center polygon, and denote the corresponding QP by $\mathrm{EvenFlower}_2(m)$.

The following proposition has been shown by means of computer assistance.

Proposition 3.26 *For $2 \leq m \leq 6$, the planar QP $\mathrm{EvenFlower}_2(m)$ is self-injective with Nakayama permutation given by clockwise rotation by*

$$\begin{cases} \pi & \text{if } m \text{ is even} \\ \frac{2\pi\cdot(m-1)}{2m} & \text{if } m \text{ is odd.} \end{cases} \qquad \square$$

We are led to make the conjecture that this holds for every m:

Conjecture 3.27 *For every $m \geq 2$, the planar QP $\mathrm{EvenFlower}_2(m)$ is self-injective with Nakayama permutation given by clockwise rotation by*

$$\begin{cases} \pi & \text{if } m \text{ is even} \\ \frac{2\pi\cdot(m-1)}{2m} & \text{if } m \text{ is odd.} \end{cases} \qquad \square$$

Remark 3.28 At first glance, the even flowers of type 2 might appear to be the same as or at least mutation-equivalent to the even flowers of type 1, and Conjecture 3.27 might appear to be a restatement of Conjecture 3.21. Note, however, that the clauses for the Nakayama permutation are swapped: the angle of rotation for the Nakayama permutation is π in Conjecture 3.21 for m odd and in Conjecture 3.27 for m even. This ensures that $\mathrm{EvenFlower}_1(m)$ and $\mathrm{EvenFlower}_2(m)$ are not mutation-equivalent for any $m \geq 2$ (assuming that the conjectures hold).

In hindsight, a more unified definition of the even flower families would be that obtained by swapping $\mathrm{EvenFlower}_1(m)$ and $\mathrm{EvenFlower}_2(m)$ for m odd. Then, Conjecture 3.21 would have the Nakayama permutation given by rotation by $\frac{2\pi\cdot(m-1)}{2m}$ regardless of the parity of m and similarly for Conjecture 3.27 with rotation by π, Propositions 3.22 and 3.29 would say that $\mathrm{EvenFlower}_1(m)$ is a new self-injective planar QP for every $m \geq 4$ regardless of the parity of m, and Conjectures 3.25 and 3.32 would say that $\mathrm{EvenFlower}_2(m)$ is mutation-equivalent to a square for every $m \geq 2$ regardless of the parity of m. $\square$

It turns out that for every odd $m \geq 5$ for which Conjecture 3.27 holds, $\mathrm{EvenFlower}_2(m)$ is not mutation-equivalent to any previously known self-injective planar QP. In particular, we conclude that $\mathrm{EvenFlower}_2(5)$ is a new self-injective planar QP. For $m = 3$, we have that $\mathrm{EvenFlower}_2(m)$ is mutation-equivalent to the triangle on 36 vertices.

Proposition 3.29 *Let $m \geq 5$ be odd. If $\mathrm{EvenFlower}_2(m)$ is self-injective with Nakayama permutation given by rotation by $\frac{2\pi\cdot(m-1)}{2m}$, then $\mathrm{EvenFlower}_2(m)$ is not mutation-equivalent to any previously known self-injective planar QP.* $\square$

Proof. The number of vertices of $\mathrm{EvenFlower}_2(m)$ is $N := (2m)^2$, and the order of the Nakayama permutation is $S := m$, seeing as m is odd and so

$$\begin{aligned} \gcd(m-1, 2m) &= \gcd(m-1, 2m - 2(m-1)) \\ &= \gcd(m-1, 2) \\ &= 2. \end{aligned}$$

We first rule out the self-injective planar QPs in [6]. We have $S = m \geq 5$, which rules out all triangles and squares. The N-gon has Nakayama permutation of order

$$\begin{aligned}\frac{N}{2} &= \frac{4m^2}{2}\\ &= 2m^2\\ &> m\\ &= S,\end{aligned}$$

which rules out the last type of self-injective planar QPs in [6].

To see that $\mathrm{EvenFlower}_2(m)$ does not arise up to iterated planar mutation from a symmetric (k,n)-Postnikov diagram, we assume that there is such a diagram that induces a QP on N vertices with Nakayama permutation of order S and derive a contradiction. That is, we assume that k and n are integers satisfying the conditions in Eqs. (3) to (6) given in Section 3.2.1.

By considering Eq. (4) modulo S, we get

$$N \equiv -k^2 + 1 \pmod{S}.$$

On the other hand, $N = (2m)^2 = (2S)^2 = 4S^2$ is a multiple of S, so that

$$1 - k^2 \equiv 0 \pmod{S}$$

and hence, by Eq. (6),

$$k \equiv -1, 1 \pmod{S}.$$

We may without loss of generality assume that

$$k \equiv 1 \pmod{S};$$

otherwise, replace k by $n-k$. Thus, we can write

$$k = qS + 1$$

for some $q \geq 0$. For brevity, also write

$$d = \gcd(n,k).$$

We then have

$$\begin{aligned}N &= k(n-k) - n + 1\\ &= (qS+1)(dS - (qS+1)) - dS + 1\\ &= qdS^2 - q^2S^2 - qS + dS - qS - 1 - dS + 1\\ &= qdS^2 - q^2S^2 - 2qS\\ &= S \cdot q(S(d-q) - 2).\end{aligned}$$

Because $N = 4S^2 = S \cdot 4S$, we conclude that

$$q \cdot (S(d-q) - 2) = 4S. \tag{12}$$

Considering Eq. (12) modulo S, we find that

$$0 \equiv -2q \pmod{S},$$

and because S is odd, we equivalently get

$$q \equiv 0 \pmod{S}.$$

Because both factors in the left-hand side of Eq. (12) must be strictly positive, we have

$$d - q \geq 1,$$

so that

$$S(d-q) - 2 \geq S - 2.$$

It follows that

$$\begin{aligned} 4S &= q \cdot (S(d-q) - 2) \\ &\geq q(S-2) \\ &= qS - 2q. \end{aligned}$$

We cannot have $q \geq 7$, because that together with the assumption that $S = m \geq 5$ would imply that

$$\begin{aligned} 4S &\geq qS - 2q \\ &= 4S + (q-4)S - 2(q-4) - 8 \\ &= 4S + (S-2)(q-4) - 8 \\ &\geq 4S + 3(q-4) - 8 \\ &\geq 4S + 3 \cdot 3 - 8 \\ &> 4S, \end{aligned}$$

which is absurd. Thus, $q \leq 6$. Given that q is a multiple of $S \geq 5$, which is odd, the only possibility is that $S = q = 5$. Plugging these values into Eq. (12), we get

$$5 \cdot (5(d-5) - 2) = 4 \cdot 5$$

and hence

$$5(d-5) - 2 = 4,$$

which we see is impossible by considering residue classes modulo 5. We conclude that $\mathrm{EvenFlower}_2(m)$ does not arise up to mutation equivalence from a symmetric Postnikov diagram. □

Corollary 3.30 $\mathrm{EvenFlower}_2(5)$ *is a self-injective planar QP that is not mutation-equivalent to any previously known self-injective planar QP.* □

Assuming the veracity of Conjecture 3.27, the number of vertices and the order of the Nakayama permutation for $\mathrm{EvenFlower}_2(m)$ with $m \geq 2$ even are $N = (2m)^2$ and $S = 2$. These values agree with the squares in [6], and one might ponder the question whether $\mathrm{EvenFlower}_2(m)$ is mutation-equivalent to a square for every even $m \geq 2$. This is indeed the case for $m = 2$ and $m = 4$, as has been verified by hand. It seems reasonable to expect that it should be the case for every even $m \geq 2$.

Proposition 3.31 $\mathrm{EvenFlower}_2(2)$ *and* $\mathrm{EvenFlower}_2(4)$ *are mutation-equivalent to the squares on* $4^2 = 16$ *and* $8^2 = 64$ *vertices, respectively.* □

Conjecture 3.32 $\mathrm{EvenFlower}_2(m)$ *is mutation-equivalent to the square on* $(2m)^2$ *vertices for every even* $m \geq 2$. □

3.2.4 Pointed flowers

The first four QPs of the fourth infinite family of QPs are shown in Figs. 16 to 19. We may parametrize the QPs of the family by the number $m \geq 1$ of "layers" of vertices (ignoring the center vertex), and denote the corresponding QP by $\mathrm{PointedFlower}(m)$.

The following proposition has been shown by means of computer assistance.

Proposition 3.33 *For* $1 \leq m \leq 4$*, the planar QP* PointedFlower(m) *is self-injective with Nakayama permutation given by clockwise rotation by*

$$\begin{cases} \frac{2\pi\cdot(m+1)}{2m+1} & \text{if } m \text{ is odd} \\ \frac{2\pi\cdot m}{2m+1} & \text{if } m \text{ is even.} \end{cases} \qquad \square$$

We are led to make the conjecture that this holds for every m:

Conjecture 3.34 *For every* $m \geq 1$*, the planar QP* PointedFlower(m) *is self-injective with Nakayama permutation given by clockwise rotation by*

$$\begin{cases} \frac{2\pi\cdot(m+1)}{2m+1} & \text{if } m \text{ is odd} \\ \frac{2\pi\cdot m}{2m+1} & \text{if } m \text{ is even.} \end{cases} \qquad \square$$

For $m = 1$, we have that PointedFlower(m) is mutation-equivalent to the triangle on 10 vertices, which is previously known to be self-injective. However, it turns out that for every $m \geq 2$ for which Conjecture 3.34 holds, PointedFlower(m) is not mutation-equivalent to any previously known self-injective planar QP. In particular, we conclude that PointedFlower(2), PointedFlower(3), and PointedFlower(4) are new self-injective planar QPs.

Proposition 3.35 *Let* $m \geq 2$*. If* PointedFlower(m) *is self-injective with Nakayama permutation given by clockwise rotation by*

$$\begin{cases} \frac{2\pi\cdot(m+1)}{2m+1} & \text{if } m \text{ is odd} \\ \frac{2\pi\cdot m}{2m+1} & \text{if } m \text{ is even,} \end{cases}$$

then PointedFlower(m) *is not mutation-equivalent to any previously known self-injective planar QP.* $\square$

Proof. The number of vertices of PointedFlower(m) is $N := (2m+1)^2 + 1$, and the order of the Nakayama permutation is $S := 2m + 1$, seeing as

$$\begin{aligned} \gcd(m, 2m+1) &= \gcd(m, 2m+1-2m) \\ &= \gcd(m, 1) \\ &= 1 \end{aligned}$$

and

$$\begin{aligned} \gcd(m+1, 2m+1) &= \gcd(m, 2m+1-2(m+1)) \\ &= \gcd(m, -1) \\ &= 1. \end{aligned}$$

We have $S \geq 5$, so PointedFlower(m) is not mutation-equivalent to any triangle or square. Moreover, PointedFlower(m) is not mutation-equivalent to any n-gon, because an n-gon on $N = (2m+1)^2 + 1$ vertices has Nakayama permutation of order

$$\begin{aligned} n &= \frac{(2m+1)^2 + 1}{2} \\ &> 2m + 1 \\ &= S. \end{aligned}$$

This rules out all self-injective planar QPs in [6].

To rule out also the self-injective planar QPs arising from symmetric (k, n)-Postnikov diagrams, we assume that there is such a diagram that induces a QP on N vertices with Nakayama permutation of order S and derive a contradiction. That is, we assume that k and n are integers satisfying the conditions in Eqs. (3)

to (6) given in Section 3.2.1. We may without loss of generality assume that $k \leq \frac{n}{2}$ (replace k by $n-k$ otherwise).

Consider the residue class of N modulo S. On one hand, we have

$$\begin{aligned} N &= (2m+1)^2 + 1 \\ &= S^2 + 1 \\ &\equiv 1 \pmod{S}. \end{aligned}$$

On the other hand, we have by Eqs. (4) and (5)

$$\begin{aligned} N &= k(n-k) + n - 1 \\ &\equiv -k^2 + 1 \pmod{S}. \end{aligned}$$

By Eq. (6), we conclude that

$$k \equiv 0 \pmod{S}.$$

Thus, both k and n are multiples of S, which gives

$$\begin{aligned} S \mid k, n &\implies S \mid \gcd(k, n) \\ &\implies S^2 \mid S \cdot \gcd(k, n) = n. \end{aligned}$$

We have now shown that

$$\begin{aligned} k &\in \{S, 2S, \ldots\} \\ n &\in \{S^2, 2S^2, \ldots\}. \end{aligned}$$

Note that the expression $k(n-k) - n + 1$ is increasing in k (for $k \leq \frac{n}{2}$) and n separately, so that the least value is attained for $k = S$ and $n = S^2$. This gives the inequality

$$\begin{aligned} N &= k(n-k) - n + 1 \\ &\geq S(S^2 - S) - S^2 + 1 \\ &= S^2(S-1) - S^2 + 1 \\ &= S^2(S-2) + 1. \end{aligned}$$

For $S \geq 5$, we get

$$\begin{aligned} N &\geq S^2(S-2) + 1 \\ &> S^2 + 1, \end{aligned}$$

which contradicts the fact that $N = S^2 + 1$. We conclude that PointedFlower(m) does not arise up to mutation equivalence from a symmetric Postnikov diagram.

For completeness, we may note that the QP does arise from a symmetric $(3, 9)$-Postnikov diagram in the case $m = 1$ ($S = 3$), as shown in Figure 3 in [7]. □

Corollary 3.36 *For $2 \leq m \leq 4$,* PointedFlower(m) *is a self-injective planar QP that is not mutation-equivalent to any previously known self-injective planar QP.* □

3.2.5 Sporadic examples

The remaining four new self-injective planar QPs are shown in Figs. 20 to 23. Their values of (N, S) are $(40, 4)$, $(51, 5)$, $(65, 4)$, and $(85, 5)$, respectively.

The following proposition was shown by means of computer assistance.

Proposition 3.37

1. *The planar QP shown in Fig. 20 is self-injective with Nakayama permutation given by clockwise rotation by* $\frac{\pi}{2}$.
2. *The planar QP shown in Fig. 21 is self-injective with Nakayama permutation given by clockwise rotation by* $\frac{4\pi}{5}$.
3. *The planar QP shown in Fig. 22 is self-injective with Nakayama permutation given by clockwise rotation by* $\frac{\pi}{2}$
4. *The planar QP shown in Fig. 23 is self-injective with Nakayama permutation given by clockwise rotation by* $\frac{2\pi}{5}$. □

Proposition 3.38 *The planar QPs shown in Figs. 20 to 23 are not mutation-equivalent to any previously known self-injective planar QPs.* □

Proof. Consider the QP shown in Fig. 20. We first rule out the self-injective planar QPs in [6]. We have $S = 4 > 3$, so the QP cannot be mutation-equivalent to a triangle or a square. The 40-gon has Nakayama permutation of order $\frac{40}{2} = 20 > 4 = S$, which rules out the n-gons.

Next, we rule out the self-injective planar QPs arising from symmetric (k, n)-Postnikov diagrams, namely by assuming that the QP does arise up to mutation equivalence from a symmetric (k, n)-Postnikov diagram and deriving a contradiction. That is, we assume that k and n are integers satisfying the conditions in Eqs. (3) to (6) given in Section 3.2.1.

By considering Eq. (4) modulo $S = 4$, we get

$$N \equiv -k^2 + 1 \pmod{S}.$$

On the other hand, $N = 40 = 10 \cdot 4 = 10S$ is a multiple of S, so that

$$1 - k^2 \equiv 0 \pmod{S}$$

and hence, by Eq. (6),

$$k \equiv -1, 1 \pmod{S}.$$

We may without loss of generality assume that

$$k \equiv 1 \pmod{S};$$

otherwise, replace k by $n - k$. Thus, we can write

$$k = qS + 1$$

for some $q \geq 0$. For brevity, also write

$$d = \gcd(n, k).$$

We then have

$$\begin{aligned}
N &= k(n-k) - n + 1 \\
&= (qS+1)(dS - (qS+1)) - dS + 1 \\
&= qdS^2 - q^2S^2 - qS + dS - qS - 1 - dS + 1 \\
&= qdS^2 - q^2S^2 - 2qS \\
&= S \cdot q(S(d-q) - 2) \\
&= 4 \cdot q(4(d-q) - 2)
\end{aligned}$$

Because $N = 40$, we conclude that

$$q \cdot (4(d-q) - 2) = 10.$$

Considering both sides modulo 4, we find that q is odd and hence the only possibilities are

$$q = 1, 5.$$

The first possibility, $q = 1$, is actually an impossibility, because $q = 1$ implies $d = 4$, which gives

$$\begin{aligned} k &= qS + 1 = 5 \\ n &= dS = 16. \end{aligned}$$

This value of (k, n) fails to satisfy Eq. (5). Similarly, we have that $q = 5$ implies $d = 6$, which gives

$$\begin{aligned} k &= qS + 1 = 21 \\ n &= dS = 24. \end{aligned}$$

This value of (k, n) also fails to satisfy Eq. (5). We conclude that the planar QP shown in Fig. 20 does not arise up to mutation equivalence from a symmetric (k, n)-Postnikov diagram.

Consider the QP shown in Fig. 21. We first rule out the self-injective planar QPs in [6]. We have $S = 5 > 3$, so the QP cannot be mutation-equivalent to a triangle or a square. The 51-gon has Nakayama permutation of order $51 > 5 = S$, which rules out the n-gons.

Next, we rule out the self-injective planar QPs arising from symmetric (k, n)-Postnikov diagrams, namely by assuming that the QP does arise up to mutation equivalence from a symmetric (k, n)-Postnikov diagram and deriving a contradiction. That is, we assume that k and n are integers satisfying the conditions in Eqs. (3) to (6) given in Section 3.2.1. We may without loss of generality assume that $k \leq \frac{n}{2}$ (replace k by $n - k$ otherwise).

Consider the residue class of N modulo $S = 5$. On one hand, we have

$$\begin{aligned} N &= 51 \\ &\equiv 1 \pmod{S}. \end{aligned}$$

On the other hand, we have by Eqs. (4) and (5)

$$\begin{aligned} N &= k(n-k) + n - 1 \\ &\equiv -k^2 + 1 \pmod{S}. \end{aligned}$$

By Eq. (6), we conclude that

$$k \equiv 0 \pmod{S}.$$

Thus, both k and n are multiples of S, which gives

$$\begin{aligned} S \mid k, n &\implies S \mid \gcd(k, n) \\ &\implies S^2 \mid S \cdot \gcd(k, n) = n. \end{aligned}$$

We have now shown that

$$\begin{aligned} k &\in \{S, 2S, \ldots\} \\ n &\in \{S^2, 2S^2, \ldots\}. \end{aligned}$$

Note that $k(n-k)-n+1$ is increasing in k (for $k \leq \frac{n}{2}$) and n separately, so that the least value is attained for $k = S$ and $n = S^2$. This gives the inequality

$$\begin{aligned} N &= k(n-k)-n+1 \\ &\geq S(S^2-S)-S^2+1 \\ &= S^2(S-1)-S^2+1 \\ &= S^2(S-2)+1 \\ &= 25 \cdot 3 + 1 \\ &= 76, \end{aligned}$$

which contradicts the fact that $N = 51$. We conclude that the planar QP shown in Fig. 21 does not arise up to mutation equivalence from a symmetric (k, n)-Postnikov diagram.

Consider the QP shown in Fig. 22. We first rule out the self-injective planar QPs in [6]. We have $S = 4 > 3$, so the QP cannot be mutation-equivalent to a triangle or a square. The 65-gon has Nakayama permutation of order $65 > 4 = S$, which rules out the n-gons.

Next, we rule out the self-injective planar QPs arising from symmetric (k, n)-Postnikov diagrams, namely by assuming that the QP does arise up to mutation equivalence from a symmetric (k, n)-Postnikov diagram and deriving a contradiction. That is, we assume that k and n are integers satisfying the conditions in Eqs. (3) to (6) given in Section 3.2.1. We may without loss of generality assume that $k \leq \frac{n}{2}$ (replace k by $n-k$ otherwise).

Consider the residue class of N modulo $S = 4$. On one hand, we have

$$\begin{aligned} N &= 65 \\ &\equiv 1 \pmod{S}. \end{aligned}$$

On the other hand, we have by Eqs. (4) and (5)

$$\begin{aligned} N &= k(n-k)+n-1 \\ &\equiv -k^2+1 \pmod{S}. \end{aligned}$$

By Eq. (6), we conclude that

$$k \equiv 0 \pmod{S}.$$

Thus, both k and n are multiples of S, which gives

$$\begin{aligned} S \mid k, n &\implies S \mid \gcd(k, n) \\ &\implies S^2 \mid S \cdot \gcd(k, n) = n. \end{aligned}$$

We have now shown that

$$\begin{aligned} k &\in \{S, 2S, \ldots\} \\ n &\in \{S^2, 2S^2, \ldots\}. \end{aligned}$$

We cannot have $n = S^2 = 16$, because then

$$\begin{aligned} N &= k(n-k)-n+1 \\ &\leq \frac{n^2}{4}-n+1 \\ &= 64-16+1 \\ &= 49. \end{aligned}$$

Note that the expression $k(n-k)-n+1$ is increasing in k (for $k \leq \frac{n}{2}$) and n separately. Therefore, $n \geq 2S^2$ implies that

$$\begin{aligned} N &= k(n-k)-n+1 \\ &\geq S(2S^2-S)-2S^2+1 \\ &= 4(32-4)-32+1 \\ &= 81, \end{aligned}$$

which is also impossible. We conclude that the planar QP shown in Fig. 22 does not arise up to mutation equivalence from a symmetric (k, n)-Postnikov diagram.

Consider the QP shown in Fig. 23. We first rule out the self-injective planar QPs in [6]. We have $S = 5 > 3$, so the QP cannot be mutation-equivalent to a triangle or a square. The 85-gon has Nakayama permutation of order $85 > 5 = S$, which rules out the n-gons.

Next, we rule out the self-injective planar QPs arising from symmetric (k, n)-Postnikov diagrams, namely by assuming that the QP does arise up to mutation equivalence from a symmetric (k, n)-Postnikov diagram and deriving a contradiction. That is, we assume that k and n are integers satisfying the conditions in Eqs. (3) to (6) given in Section 3.2.1.

By considering Eq. (4) modulo $S = 5$, we get

$$N \equiv -k^2+1 \pmod{S}.$$

On the other hand, $N = 85 = 17 \cdot 5 = 17S$ is a multiple of S, so that

$$1-k^2 \equiv 0 \pmod{S}$$

and hence, by Eq. (6),

$$k \equiv -1, 1 \pmod{S}.$$

We may without loss of generality assume that

$$k \equiv 1 \pmod{S};$$

otherwise, replace k by $n-k$. Thus, we can write

$$k = qS+1$$

for some $q \geq 0$. For brevity, also write

$$d = \gcd(n, k).$$

We then have

$$\begin{aligned} N &= k(n-k)-n+1 \\ &= (qS+1)(dS-(qS+1))-dS+1 \\ &= qdS^2-q^2S^2-qS+dS-qS-1-dS+1 \\ &= qdS^2-q^2S^2-2qS \\ &= S \cdot q(S(d-q)-2) \\ &= 5 \cdot q(5(d-q)-2) \end{aligned}$$

Because $N = 85$, we conclude that

$$q \cdot (5(d-q)-2) = 17.$$

The right-hand side is prime, so the second factor of the left-hand side must be either 1 or 17. We see that this is impossible by considering its residue class modulo 5. We conclude that the planar QP shown in Fig. 23 does not arise up to mutation equivalence from a symmetric (k, n)-Postnikov diagram. □

4 Algorithms

In this section, we give an algorithm for determining whether the algebra of a semimonomial bound quiver is self-injective (and in the positive case also outputting the Nakayama permutation) and in particular whether the Jacobian algebra of a QP is self-injective in the case that the Jacobian ideal is semimonomial. We also outline an algorithm for generating planar QPs that seem likely to be self-injective.

Combining the algorithms for generating and analyzing planar QPs, we are able to perform a computer search for self-injective planar QPs.

4.1 Determining self-injectivity

In this subsection, we give an algorithm for determining whether a semimonomial bound quiver is weakly cancellative at every vertex or not and, in the weakly cancellative case, whether the bound quiver algebra is self-injective. In the self-injective case, the algorithm also outputs the Nakayama permutation of the bound quiver algebra.

In order to use the algorithm in practice, we would have to make sure that the ideal is admissible. This is important in order for the algorithm to terminate and provide a correct result. Determining whether an ideal is admissible seems difficult, so instead, we make a simple modification to the algorithm to terminate when it encounters a path of length exceeding some given bound. This makes it feasible to run the algorithm on a larger class of quivers equipped with a semimonomial ideal, and the following property of the algorithm is retained: if the algorithm indicates that the algebra is self-injective, then the algebra is for certain self-injective. In other words, the modified algorithm can prove self-injectivity but not always disprove it.

The algorithm is applicable to QPs with semimonomial Jacobian ideal and thus, in particular (by Proposition 3.15), to planar QPs.

The algorithms that are presented assume that the quivers have no parallel arrows (i.e., no two arrows with equal sources and equal targets), in which case every arrow a can be identified with the pair $(s(a), t(a))$ of its source and target vertices. It is the author's belief that the algorithms would generalize painlessly to quivers with parallel arrows. Representing the arrows of the quivers with adjacency lists is convenient for the algorithms. Finally, refer to the implementations of the algorithms (see Section 5) for more details.

4.1.1 Semimonomial bound quivers

By Proposition 2.33 and Theorem 2.22, the maximal nonzero equivalence classes of paths in a semimonomial bound quiver (Q, I) that is weakly cancellative at every vertex contain all the information needed to conclude whether the bound quiver algebra KQ/I is self-injective or not. An algorithm for computing the maximal nonzero equivalence classes of paths starting at a given vertex $i \in Q_0$ (a basis of $\operatorname{soc} P(i)$, if (Q, I) is weakly cancellative at i) is therefore important.

It does not seem to be easy to tell without careful analysis whether a semimonomial bound quiver is weakly cancellative at a vertex, so instead of requiring the input bound quiver to be weakly cancellative at every vertex, we devise the algorithm to determine whether the bound quiver is weakly cancellative at every vertex.

We arrive at an algorithm specification as follows: the input is a semimonomial bound quiver (Q, I) and a vertex $i \in Q_0$. The output is a collection of all maximal nonzero equivalence classes of paths starting at i and, depending on whether or not there is a path that contradicts weak cancellativity of (Q, I) at i, either "weakly cancellative at i" or "not weakly cancellative at i".

Pseudocode for an algorithm that matches this specification is given in Algorithm 1. Before giving an explanation of the pseudocode, we describe roughly how the algorithm works on a higher level. The algorithm performs a graph search in the tree of all paths in Q starting at the given vertex $i \in Q_0$. For every path p that is explored, the equivalence class $p + I$ is computed, in the sense that all its representative paths are determined (using Proposition 2.28). If $p + I = 0 + I$, then the children of p (paths that extend p by one arrow) are not considered in the search. The algorithm thus explores every path p starting at i with a nonzero equivalence class $p + I$. By checking which equivalence classes have no path that extends to a path with nonzero equivalence class, the algorithm records the maximal nonzero equivalence classes during the search. After the search, the nonzero equivalence classes are used to determine whether weak cancellativity at i fails. The algorithm outputs the maximal nonzero equivalence classes and a value indicating whether (Q, I) is weakly cancellative at i and then terminates.

Algorithm 1 Computing a basis of the socle at a given vertex

```
function ComputeSocleBasis(Q, I, i)
    b, E ← PerformSearch(Q, I, i)
    if WeakCancellativityFails(E) then
        return b, "not weakly cancellative at i"
    else
        return b, "weakly cancellative at i"
    end if
end function

function PerformSearch(Q, I, i)
    b ← Set()
    E ← DisjointSets()
    E.MakeSet(0)
    E.MakeSet(ε_i)
    q ← Queue(ε_i)

    while ¬q.IsEmpty do
        p ← q.Dequeue()
        m ← true
        for v in Q.AdjacencyList[p.LastVertex] do
            p′ ← p.Append(v)
            if DetermineEquivalenceClass(Q, I, p′, E) = "nonzero" then
                m ← false
                q.Enqueue(p′)
            end if
        end for
        if m and ¬b.Contains(E.FindSet(p)) then
            b.Add(E.FindSet(p))
        end if
    end while

    return b, E
end function

function WeakCancellativityFails(E)
    d ← Dictionary()
    for e in E do
        if e.Contains(0) or e.Contains(ε_i) then
            continue
        end if
```

```
        P ← Dictionary()
        for q in e do
            a ← q.LastArrow
            p ← E.FindSet(q.Parent)
            if d.ContainsKey(p) and d[p] = "has distinguishing arrow" then
                continue
            end if
            if ¬P.ContainsKey(a) then
                P[a] ← p
                d[p] ← "has distinguishing arrow so far"
            else if p ≠ P[a] then
                d[p] ← "has no distinguishing arrow so far"
                d[P[a]] ← "has no distinguishing arrow so far"
            end if
        end for
        for p in P.Values do
            if d[p] = "has distinguishing arrow so far" then
                d[p] ← "has distinguishing arrow"
            end if
        end for
    end for
    return d.Values.Contains("has no distinguishing arrow so far")
end function

function DetermineEquivalenceClass(Q, I, p, E)
    if p.IsDiscoveredInEquivalenceClassSearch then
        if E.FindSet(p) = E.FindSet(0) then
            return "zero"
        else
            return "nonzero"
        end if
    end if
    return ComputeEquivalenceClass(Q, I, p, E)
end function

function ComputeEquivalenceClass(Q, I, p, E)
    p.IsDiscoveredInEquivalenceClassSearch ← true
    s ← Queue(p)
    while ¬s.IsEmpty do
        p ← s.Dequeue()
        for all subpaths p′ of p do
            if p′ is a generator of I then
                E.Union(p, 0)
                return "zero"
            end if
            for all q′ such that p′ − q′ is a generator of I do
                q ← (p with p′ replaced by q′)
                if E.FindSet(q) = E.FindSet(0) then
                    E.Union(p, 0)
                    return "zero"
                end if
```

```
86:             if ¬q.IsDiscoveredInEquivalenceClassSearch then
87:                 q.IsDiscoveredInEquivalenceClassSearch ← true
88:                 E.Union(p, q)
89:                 s.Enqueue(q)
90:             end if
91:         end for
92:     end for
93:   end while
94:   return "nonzero"
95: end function
```

Throughout the pseudocode, E is a variable containing an instance of the disjoint-set data structure. This is a data structure designed to represent a collection of disjoint sets (or an equivalence relation) and is optimized for three operations: MakeSet(x), FindSet(x), and Union(x, y). The operation MakeSet(x) takes a new element x and inserts the singleton set $\{x\}$ into the disjoint-set collection. The operation FindSet(x) takes an element x and returns a canonical representative for the set containing x, where "canonical" means that if x and y belong to the same set, then FindSet(x) = FindSet(y). The operation Union(x, y) takes two elements x, y belonging to different sets and unites the two sets. More details on this data structure can be found in [2].

We now provide an explanation of the pseudocode in Algorithm 1. The ComputeSocleBasis function is straightforward: perform the search to find a tentative basis b of $\operatorname{soc} P(i)$ (or more precisely, a transversal of such a basis) and obtain an instance E of the disjoint-set data structure containing all the paths in Q starting at i that are not zero-equivalent (as well as some paths that are zero-equivalent) partitioned into equivalence classes according to I. Then check if the equivalence classes contradict weak cancellativity at i of (Q, I). If there is a contradiction, return b and "not weakly cancellative at i". Otherwise, the tentative basis b is in fact a basis (by Proposition 2.31) and we return b and "weakly cancellative at i".

The PerformSearch function performs a graph search in the tree of all paths in Q starting at i, terminating after all the paths with nonzero equivalence class has been explored (which eventually occurs, because I is admissible).

It should be noted that the paths that appear in the PerformSearch, DetermineEquivalenceClass, and ComputeEquivalenceClass functions have an attribute (IsDiscoveredInEquivalenceClassSearch) and so are not *just* paths of arrows. Between the lines, we are storing and retrieving the paths in and from a search tree that contains for each path the IsDiscoveredInEquivalenceClassSearch attribute.

The variable b holds a collection of (representatives of) tentative basis elements of $\operatorname{soc} P(i)$, the variable E holds an instance of the disjoint-set data structure containing all the paths encountered (including a "zero path" corresponding to the zero equivalence class $0+I$) partitioned into equivalence classes, and the variable q holds a queue of paths that have been discovered but not explored in the tree search. An invariant for q is that it contains only paths whose equivalence class has been computed and is nonzero. The variable p holds the path being explored in the current iteration of the graph search, and the variable m holds a boolean value indicating whether p is maximal with respect to the "is a prefix of" relation among all paths with nonzero equivalence class.

Lines 10 to 14 initialize variables for the graph search: start with an empty set of basis elements b, an instance E of the disjoint-set data structure containing a singleton representing the zero equivalence class and a singleton for the equivalence class of the stationary path ε_i, and a queue q containing only the stationary path ε_i. The while loop spanning Lines 15 to 28 performs the graph search. In the loop body, a path p whose equivalence class is computed and nonzero is dequeued for exploration. The equivalence classes of all its immediate "extension paths" ($p' := pa$ for $a \in Q_1$) are determined in Lines 18 to 24. For every extension path with nonzero equivalence class, enqueue it for later exploration. When Line 25 is reached, m will be **true** if and only if p has no extension path with nonzero equivalence class, i.e., if and only if p is maximal.

Lines 25 to 27 add p to the tentative basis if p is maximal and an equivalent path has not already been added. Eventually, we will run out of paths with nonzero equivalence class, the queue will be emptied, and we will return the tentative basis b and the collection of equivalence classes E.

The idea behind the WeakCancellativityFails function is as follows: for every nonzero and non-stationary equivalence class[1] (to play the role of $pa + I$ in the second condition of weak cancellativity), write every path in the equivalence class as pa for p a path (the "parent" path, say) and $a \in Q_1$ and record the parent p (up to equivalence) of every path pa for which there is no other path $p'a$ in the equivalence class $pa + I$ with non-equivalent parent p' (i.e., with $p + I \neq p' + I$) but that ends with the same arrow a. These recorded parents p are "distinguished" by the corresponding arrow a in the sense that they satisfy the second condition for weak cancellativity via the arrow a. If a path p appears as a parent for some of the nonzero equivalence classes and the path p is not distinguished by its corresponding arrow for any of the equivalence classes, then p contradicts weak cancellativity at i: there is an arrow a such that $pa + I \neq 0 + I$ (so that p fails the first condition for weak cancellativity) and there is no arrow satisfying the second condition for weak cancellativity for p.

The variable d keeps track of the parents that have been encountered and whether a distinguishing arrow has been found for them. In every iteration of the outer loop (Lines 33 to 57), the variable P is assigned a dictionary that maps an arrow a to the canonical representative p of the parent of the first path in the equivalence class that ends with a. This is used on Lines 44 to 50 to determine if the arrow a of a parent p distinguishes p or not. Note that these lines need not and should not be executed if we have already (in an earlier iteration of the outer for loop) found a distinguishing arrow for the parent p, as made sure by Lines 41 to 43. The second inner for loop (Lines 52 to 56) iterates over the values of P, which necessarily contain all parents p (up to equivalence) with $d[p] =$ "has distinguishing arrow so far", and records as having a distinguishing arrow all parents p that were encountered, previously had no distinguishing arrow, and was found to have a distinguishing arrow. When Line 58 is reached, d will contain a value for the canonical representative p of every nonzero equivalence class that is not maximal nonzero, which corresponds precisely to the paths that fail to satisfy the first condition of weak cancellativity. In other words, the paths that appear as keys in d are precisely those that dictate whether weak cancellativity at the starting vertex i fails. Moreover, the value $d[p]$ for a path p is "has distinguishing arrow" precisely if there is an arrow a that satisfies the second condition of weak cancellativity for p and "has no distinguishing arrow so far" otherwise. Thus, we return **true**, indicating that weak cancellativity at i fails, if and only if some value in d is "has no distinguishing arrow so far".

With the interpretation of IsDiscoveredInEquivalenceClassSearch as recording whether the equivalence class has been computed using ComputeEquivalenceClass (which is indeed what the IsDiscoveredInEquivalenceClassSearch attribute records outside of calls to ComputeEquivalenceClass), the DetermineEquivalenceClass function is straightforward: if the equivalence class of p has already been computed, check whether it is the zero class and return accordingly. Otherwise, compute the equivalence class with a call to ComputeEquivalenceClass and just pass on the "zero" or "nonzero" return value.

The ComputeEquivalenceClass is given in less detail than the other functions. Essentially, it performs a graph search in the graph of all paths equivalent to p, terminating either when the zero path is encountered or when the equivalence class is exhausted, and returns "zero" or "nonzero" depending on whether p is zero-equivalent or not. The variable s holds a queue of paths discovered but not yet explored and is initialized to contain only the argument p. In the while loop, the variable p plays the role of the path currently being explored. The for loop starting on Line 75 iterates over all contiguous (non-stationary, say) subpaths p'

[1] [2] do not mention iterating over the sets in a disjoint-set data structure. However, using a disjoint-set forest implementation with the modification alluded to in exercise 21.3-4 in [2], we may efficiently iterate over the set of any given representative. If we add another boolean attribute, indicating whether the set has already been iterated over, we may efficiently (in time linear in the number of elements of the forest) iterate over all the sets in the disjoint-set forest exactly once by iterating over all the elements of the forest and iterating over the set containing the element if and only if the new attribute indicates that we have not yet iterated over the set.

of p (e.g., if $p = abc$, then all the paths a, b, c, ab, bc, abc are considered). To be specific, the subpaths are considered to know their start and end index (e.g., $p = abab$ has two distinct subpaths here denoted by ab). This distinction matters when performing surgery on the paths to replace one subpath with another. If tbe subpath p' is a generator of I, then $p' + I = 0 + I$ so that $p + I = 0 + I$. Hence we unite the equivalence class of p with the zero class and return "zero" on Lines 77 and 78. For every path q' such that $p' - q'$ is a generator of I, we have $p' + I = q' + I$ so that p is equivalent to $q := (p$ with the subpath p' replaced by $q')$. For this q, there are three cases to consider:

1. q has been found to be zero equivalent in a previous call to ComputeEquivalenceClass,
2. q has already been discovered in the current call to ComputeEquivalenceClass, and
3. q has not ever been discovered in ComputeEquivalenceClass.

The first case is possible because all the previous calls to ComputeEquivalenceClass may have terminated upon encountering the zero path before discovering p (the current argument to ComputeEquivalenceClass). The other two cases are straightforward. Note that there is no fourth case, that q has been discovered in a previous call to ComputeEquivalenceClass and found to be not zero equivalent: if q were not zero equivalent and discovered in a previous call to ComputeEquivalenceClass, then the entire equivalence class of q (including p) would have been computed. Thus, p would have its IsDiscoveredInEquivalenceClassSearch set to **true**, so that the if statement on Line 61 would prevent the current call to ComputeEquivalenceClass from DetermineEquivalenceClass from ever having been made.

In the first case, we conclude that p is zero equivalent, so we combine the equivalence class of p with the zero class and return "zero" on Lines 83 and 84. In the second case, there is nothing to do; p and q already belong to the same set in E, the attribute q.IsDiscoveredInEquivalenceClassSearch is already **true** (by assumption), and we should not enqueue q. In the third case, we discover q by updating its IsDiscoveredInEquivalenceClassSearch attribute, combining the equivalence classes of p and q, and enqueuing q on Lines 87 to 89.

Given the ComputeSocleBasis function, which computes a transversal, D_i say, of the set C_i in Proposition 2.33, it is now straightforward to concoct an algorithm for determining whether a semimonomial bound quiver is weakly cancellative at every vertex or not and, in the weakly cancellative case, whether the bound quiver algebra is self-injective, and in the self-injective case also outputting the Nakayama permutation: for every $i \in Q_0$, compute D_i using ComputeSocleBasis. If weak cancellativity is indicated to fail at any i, output "not weakly cancellative" and terminate. If some D_i is not singleton, output "weakly cancellative but not self-injective" and terminate. Otherwise, denote by p_i the unique path in D_i for every $i \in Q_0$ and by σ the map $\sigma \colon Q_0 \to Q_0$ in Proposition 2.33. If σ is not a permutation, output "weakly cancellative but not self-injective". Otherwise, output "weakly cancellative and self-injective" and the Nakayama permutation σ. Then terminate.

Detailed pseudocode for essentially the same algorithm is given in Algorithm 2. The set J is used to detect non-injectivity of the tentative Nakayama permutation σ.

Algorithm 2 Computing the Nakayama permutation, if it exists

```
function ComputeNakayamaPermutation(Q, I)
    J ← Set()
    σ ← Dictionary()
    for i in Q_0 do
        b, c ← ComputeSocleBasis(Q, I, i)
        if c = "not weakly cancellative at i" then
            return "not weakly cancellative"
        end if
```

```
9:          if b.Count > 1 then
10:             return "possibly weakly cancellative but not self-injective"
11:         end if
12:         j ← b.FirstElement.LastVertex
13:         if j ∈ J then
14:             return "possibly weakly cancellative but not self-injective"
15:         end if
16:         J.Add(j)
17:         σ[i] ← j
18:     end for
19:     return "weakly cancellative and self-injective", σ
20: end function
```

4.1.2 Addressing non-admissibility

The algorithms given in Section 4.1.1 assume that the ideal I in addition to being semimonomial is admissible. In particular, they assume that the ideal satisfies $R_Q^m \subseteq I$ for some $m \geq 2$. If the ComputeNakayamaPermutation function given in Algorithm 2 is executed on a semimonomial ideal I without this lower bound, the execution might not terminate.

A simple way to ensure termination is to modify the involved functions to return "probably not admissible" whenever a path of length exceeding some fixed bound, M say, is encountered. Concretely, we may modify all the functions given in Algorithms 1 and 2 except for WeakCancellativityFails to also take this bound M as an argument and have the functions just pass on the bound until it reaches ComputeEquivalenceClass.

Between Line 74 and Line 75 in ComputeEquivalenceClass in Algorithm 1, we may insert an if statement to return "probably not admissible" if p has length greater than M. Then make sure that this return value is bubbled up properly: the return statement on Line 68 in DetermineEquivalenceClass in Algorithm 1 does not need to be updated. On Line 20 in PerformSearch in Algorithm 1, also compare the return value to "probably not admissible" and return "probably not admissible" accordingly. On Line 2 in ComputeSocleBasis in Algorithm 1, handle the return value "probably not admissible" by returning "probably not admissible". Finally, between Line 5 and Line 6 in ComputeNakayamaPermutation in Algorithm 2, add an if statement comparing the return value to "probably not admissible" and return "probably not admissible" if they match.

The algorithm obtained by making this modification has the property that it may output "probably not admissible" even if (Q, I) is admissible (and even weakly cancellative at every vertex and self-injective), if we were to choose the bound M too small. In other words, the algorithm may produce false negatives. On the other hand, if the algorithm outputs "weakly cancellative and self-injective", then (Q, I) is guaranteed to be bound, be weakly cancellative at every vertex, and have a self-injective algebra. Thus, the algorithm cannot produce false positives. This latter property together with the fact that there is a large enough M for every bound quiver such that the algorithm correctly classifies the bound quiver as admissible makes the algorithm useful for searching for bound quivers with self-injective algebra in practice.

4.1.3 Quivers with potential

By Proposition 3.9, we obtain from the algorithms presented in Sections 4.1.1 and 4.1.2 an algorithm to compute the Nakayama permutation, if it exists, of a QP whose potential has all scalars in $\{-1, +1\}$, every arrow appearing in at most one cycle with scalar -1 and in at most one cycle with scalar $+1$, and every arrow appearing at most once in every cycle in the potential. More precisely, the output of the algorithm is one of "probably not admissible", "not weakly cancellative", "weakly cancellative but not self-injective", and "weakly cancellative and self-injective", with the guarantee that if the output is "weakly cancellative and self-injective" then the QP induces a bound quiver that is weakly cancellative at every vertex and the QP is self-injective. In this latter case, the Nakayama permutation of the QP is also computed.

4.2 Generating QPs

Many of the known self-injective planar QPs (see Section 3.2) are induced by certain plane quivers that we shall call "layered quivers", by which we mean roughly that the quivers have vertices that may be partitioned into disjoint "layers" and every face but the center face resides between two consecutive layers. Given that many self-injective planar QPs are induced by layered quivers but most plane quivers are not layered quivers, layered quivers seem like an interesting class of quivers from the point of view of self-injectivity.

In this subsection, we describe how layered quivers may be encoded as a collection of natural numbers. Conversely, from the collection of numbers of a layered quiver, we may construct the planar QP induced by the layered quiver. By generating suitable collections of natural numbers, we thus obtain an algorithm for generating planar QPs induced by layered quivers.

4.2.1 Layered quivers

Definition 4.1 A *layered quiver* is a plane quiver (Q, e) such that all bounded connected components of $\mathbb{R}^2 \setminus e(Q)$ are faces bounded by cycles and whose vertices can be partitioned into a finite sequence $L_1, \ldots, L_m \subseteq Q_0$ of subsets called *layers* in such a way that the underlying graph of Q satisfies the following conditions:

- For every $k \in \{1, \ldots, m\}$, we have $|L_k| \geq 3$.
- For every $k \in \{1, \ldots, m-1\}$, we have

$$e(L_k) \subseteq \operatorname{conv}\big(e(L_{k+1})\big) \subseteq \mathbb{R}^2,$$

  i.e., every layer is embedded into the convex hull of the next layer.
- For every layer L_k, the induced subgraph of L_k is a cycle graph.
- All edges are between vertices of the same layer L_k or vertices of consecutive layers L_k, L_{k+1}.

We will call L_m the *boundary layer* and the other layers *interior layers*. We will call an arrow between consecutive layers a *vertical* arrow. □

Being interested in self-injectivity, we may restrict our attention to quivers whose underlying graph is connected. Accordingly, we will tacitly assume that every layered quiver considered has arrows between the consecutive layers.

Example 4.2 The quiver in Fig. 24 is a layered quiver with three layers:

$$\begin{aligned} L_1 &= \{1, \ldots, 4\} \\ L_2 &= \{5, \ldots, 10\} \\ L_3 &= \{11, \ldots, 21\}. \end{aligned}$$

□

Example 4.3 The odd and even flowers shown in Figs. 1 to 15 are layered quivers. □

Given a layered quiver and an order on the vertices of an interior layer L_k, we may define an order on the vertices of the next layer L_{k+1} as follows: find the first vertex $i \in L_k$ with a vertical arrow into L_{k+1}, and define the first vertex of L_{k+1} to be the target $t(a)$ of the "first" arrow a from i to L_{k+1}. By the "first" arrow, we mean either the arrow b that bounds the same face as the arrow between i and its predecessor i' in L_k (if the arrow between i' and i is oriented from i' to i) or the vertical arrow arrow b' from i bounding the same face as b (if the arrow between i' and i is oriented from i to i'). Then define the order on L_{k+1} as that obtained by following the cycle of the underlying graph in the clockwise direction starting from $t(a)$.

For a layered quiver with a designated first vertex of the first layer, this gives rise to an order on the vertices of every layer in the quiver. This also gives rise to an order on the vertical arrows between two fixed consecutive layers L_k and L_{k+1}, by taking as the first arrow the arrow denoted by a above, as the second arrow the other vertical arrow (b say) of the clockwise oriented face bounded by a, as the third arrow the other vertical arrow (c say) of the counterclockwise oriented face bounded by b, and so on.

Example 4.4 Consider the layered quiver from Example 4.2 where we take 1 to be the first vertex of the first layer. The induced order on the vertices of every layer then coincides with the usual order on the natural numbers, with a_1 being the first arrow from L_1 to L_2 and b_1 being the first arrow from L_2 to L_3. The arrows a_2 and a_3 are the second and third, respectively, vertical arrow between L_1 and L_2. The arrows b_2 and b_3 are the second and third, respectively, vertical arrow between L_2 and L_3. □

Given a layered quiver with the vertices of each layer ordered, we can associate to every interior layer L_k a finite sequence $s^{(k)}$ of non-negative integers whose terms are the number of arrows in L_{k+1} between the endpoints in L_{k+1} of two consecutive vertical arrows between L_k and L_{k+1}.

Example 4.5 Consider the layered quiver from Example 4.4. The sequence $s^{(1)}$ for the first layer is

$$s^{(1)} = 1, 0, 1, 2, 1, 1$$

with the first term $s_1^{(1)}$ signifying that the face bounded by a_1 and a_2 has $s_1^{(1)} = 1$ arrow in L_2, the second term $s_2^{(1)}$ signifying that the face bounded by a_2 and a_3 has $s_2^{(1)} = 0$ arrows in L_2, and so on. The sequence $s^{(2)}$ for the second layer is

$$s^{(2)} = 1, 0, 1, 1, 1, 0, 1, 1, 1, 1, 1, 0, 1, 1.$$

□

Given a layered quiver with the vertices of each layer ordered, we can associate to every vertex $i \in L_k$ of an interior layer a non-negative integer $f(i)$ defined by

$$f(i) = \left\lfloor \frac{1}{2} \cdot \left| \left\{ a \in Q_1 \;\middle|\; \begin{matrix} (s(a) = i \wedge t(a) \in L_{k+1}) \\ \vee (t(a) = i \wedge s(a) \in L_{k+1}) \end{matrix} \right\} \right| \right\rfloor .$$

In words, $f(i)$ is the number of pairs of vertical arrows between i and L_{k+1} with attention paid to the possibility of there being an odd number of arrows between i and L_{k+1} (which happens precisely when there is an odd number of arrows between L_{k-1} and i). We may also define $f(i)$ to be 0 for every vertex i in the boundary layer, so that we can think of f as a function $f\colon Q_0 \to \mathbb{Z}_{\geq 0}$.

Example 4.6 Consider the layered quiver from Example 4.4. The values for f are

i	1	2	3	4	5	6	7	8	9	10
$f(i)$	1	1	0	1	1	2	1	1	0	0

(and $f(i) = 0$ for $11 \leq i \leq 21$). □

Let (Q, e) be a layered quiver whose first layer (which is a cycle) is clockwise oriented. Given only the number of layers, m say, and their sizes, $|L_1|, \ldots, |L_m|$, the sequences $s^{(1)}, \ldots, s^{(m-1)}$, and the function f as defined above for (Q, e), we can construct a layered quiver (Q', e') (that might possibly require curved line segments for some of the arrows) that induces the same planar QP (up to a relabeling of the vertices) as (Q, e). An implementation of the construction of the planar QP from this data is given in Section 5.

Let us fix the number of layers, m say, and their sizes, $|L_1|, \ldots, |L_m|$. In order for a collection of sequences $s^{(1)}, \ldots, s^{(m-1)}$ of non-negative integers and a function $f\colon Q_0 \to \mathbb{Z}_{\geq 0}$ to correspond to those of a layered quiver with the prescribed number of layers and layer sizes, some conditions must be satisfied: for every $k \in \{1, \ldots, m-1\}$, the number of terms in $s^{(k)}$ (which is to be the number of vertical arrows between L_k

and L_{k+1}) needs to be

$$\sum_{i\in L_k} 2f(i) + \begin{cases} 1 & \text{if } \left| \left\{ a \in Q_1 \;\middle|\; \begin{array}{l} (s(a) = i \wedge t(a) \in L_{k-1}) \\ \vee (t(a) = i \wedge s(a) \in L_{k-1}) \end{array} \right\} \right| \text{ is odd} \\ 0 & \text{otherwise,} \end{cases}$$

and the sum of the terms in $s^{(k)}$ needs to be $|L_{k+1}|$. Moreover, in order to avoid 2-cycles, some of the terms must be strictly positive.

By choosing sequences $s^{(1)}, \dots, s^{(m-1)}$ and functions $f\colon Q_0 \to \mathbb{Z}_{\geq 0}$ that satisfy these conditions and constructing the corresponding planar QP, we obtain a procedure for generating planar QPs induced by layered quivers. Given in Section 5 is an implementation of a procedure that generates the planar QPs induced by all layered quivers (with clockwise oriented first layer) with a prescribed number of layers, layer sizes, and numbers of vertical arrows between consecutive layers.

5 Implementations

The author has implemented the algorithms given in Section 4 and made a user interface for the algorithm outlined in Section 4.1.3 in the form of a quiver editor, in which the user can check whether a quiver drawn in the plane induces a planar QP and whether this planar QP is weakly cancellative at every vertex and self-injective. These implementations are publically available, as both source code and binary files, under the MIT license in a GitHub repository at `https://github.com/Samuel-Q/SelfInjectiveQuiversWithPotential/`.

In some more detail, important pieces of the codebase include:

- the implementation of the algorithm given in Section 4.1.1 for determining whether a semimonomial bound quiver is weakly cancellative at every vertex and has self-injective algebra (with support for dealing with non-admissible ideals according to Section 4.1.2),
- the implementation of the corresponding algorithm for QPs with semimonomial Jacobian ideal given in Section 4.1.3,
- construction of (the planar QP induced by) a layered quiver from a collection of integers as outlined in Section 4.2.1, and more generally adding layers to any planar QP (that is not necessarily layered),
- exhaustive generation of layered quivers as outlined in Section 4.2.1, and more generally exhaustive generation of planar QPs obtained by adding layers to a fixed planar QP, and
- the quiver editor.

The entire codebase is written in C$^\sharp$ (version 7.3) and targets .NET Framework version 4.7.1.

6 Conclusion

Using the implementations in Section 5, several computer searches for self-injective planar QPs with rotational symmetry have been performed, including among the following classes of QPs:

1. layered quivers with 2–4 layers of nondecreasing sizes and a cycle of length 3–12 at the center,
2. layered quivers but with a single vertex instead of a cycle at the center, with 2–5 layers of nondecreasing sizes and symmetry when rotated by $\frac{2\pi}{k}$ for $3 \leq k \leq 12$, and
3. planar QPs obtained by adding a layer of vertices to some fixed planar QP (selected manually).

These searches revealed fourteen self-injective planar QPs (presented in Section 3.2) that do not appear in the literature, some of which fit into families of planar QPs that we have conjectured consist solely of self-injective QPs.

7 Figures

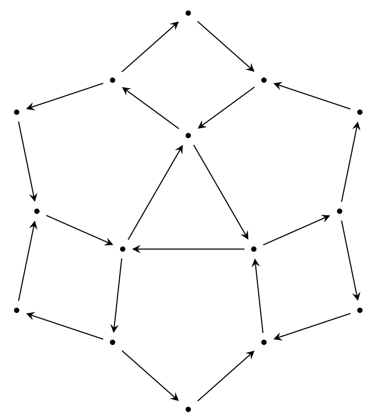


Figure 1: OddFlower(1), which is a previously known self-injective QP, with $(N, S) = (15, 3)$. It is the first QP in the first infinite family of QPs conjectured to consist solely of self-injective QPs.

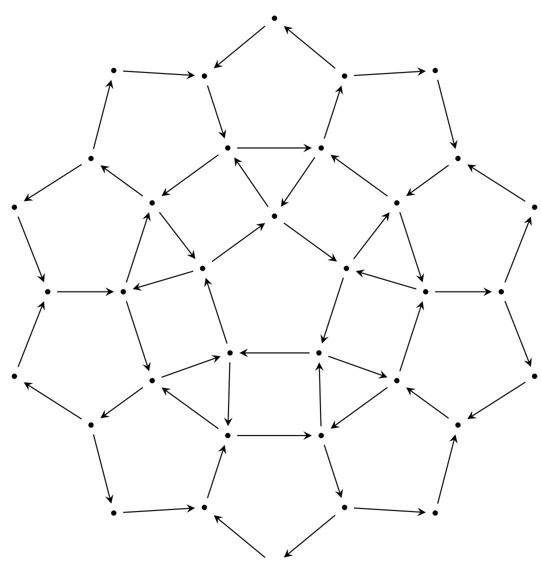


Figure 2: OddFlower(2), which is the first of the fourteen new self-injective QPs, with $(N, S) = (35, 5)$. It is the second QP in the first infinite family of QPs conjectured to consist solely of self-injective QPs.

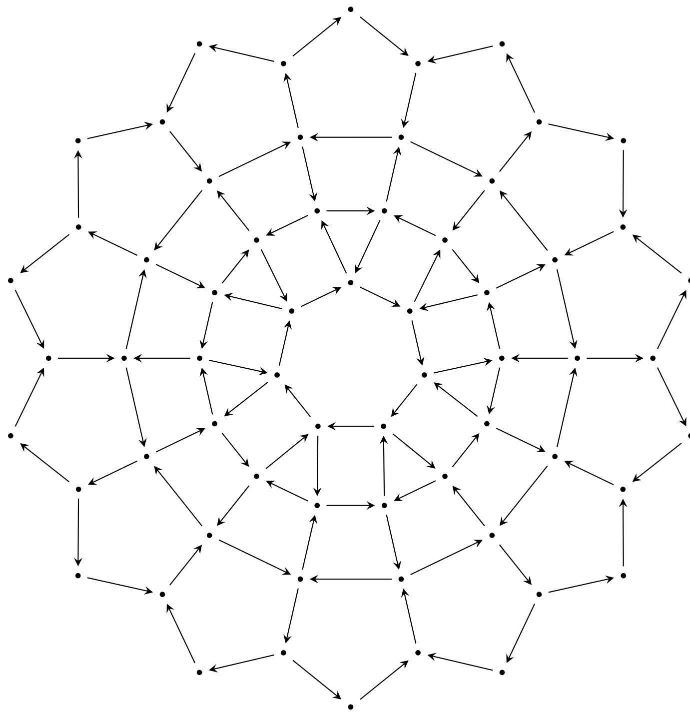


Figure 3: OddFlower(3), which is the second of the fourteen new self-injective QPs, with $(N, S) = (63, 7)$. It is the third QP in the first infinite family of QPs conjectured to consist solely of self-injective QPs.

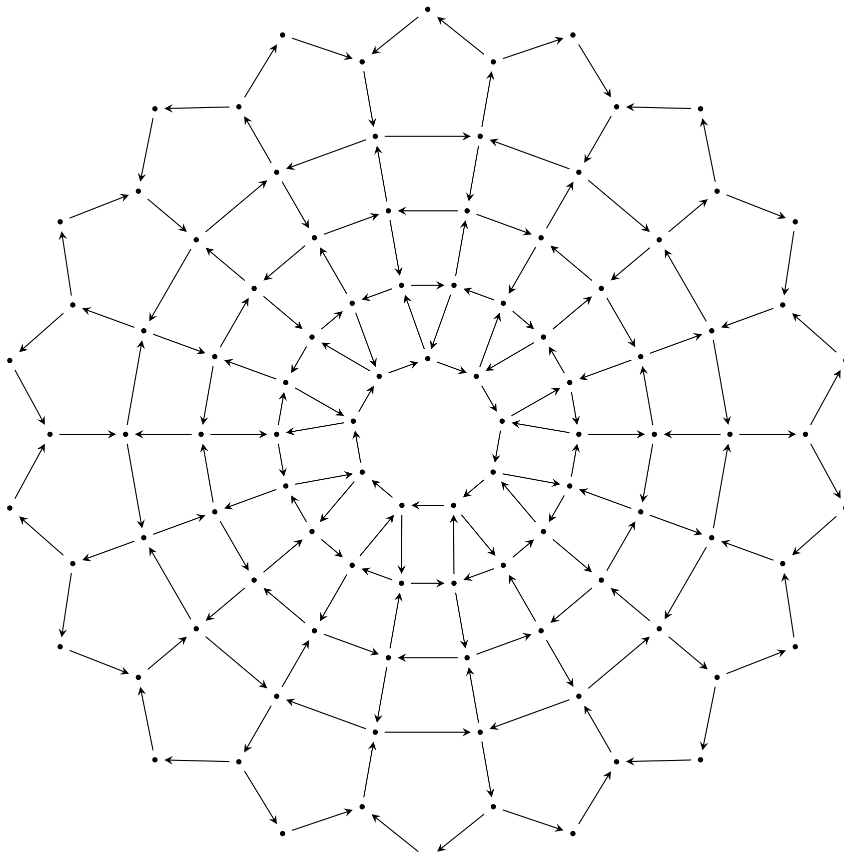


Figure 4: OddFlower(4), which is the third of the fourteen new self-injective QPs, with $(N, S) = (99, 9)$. It is the fourth QP in the first infinite family of QPs conjectured to consist solely of self-injective QPs.

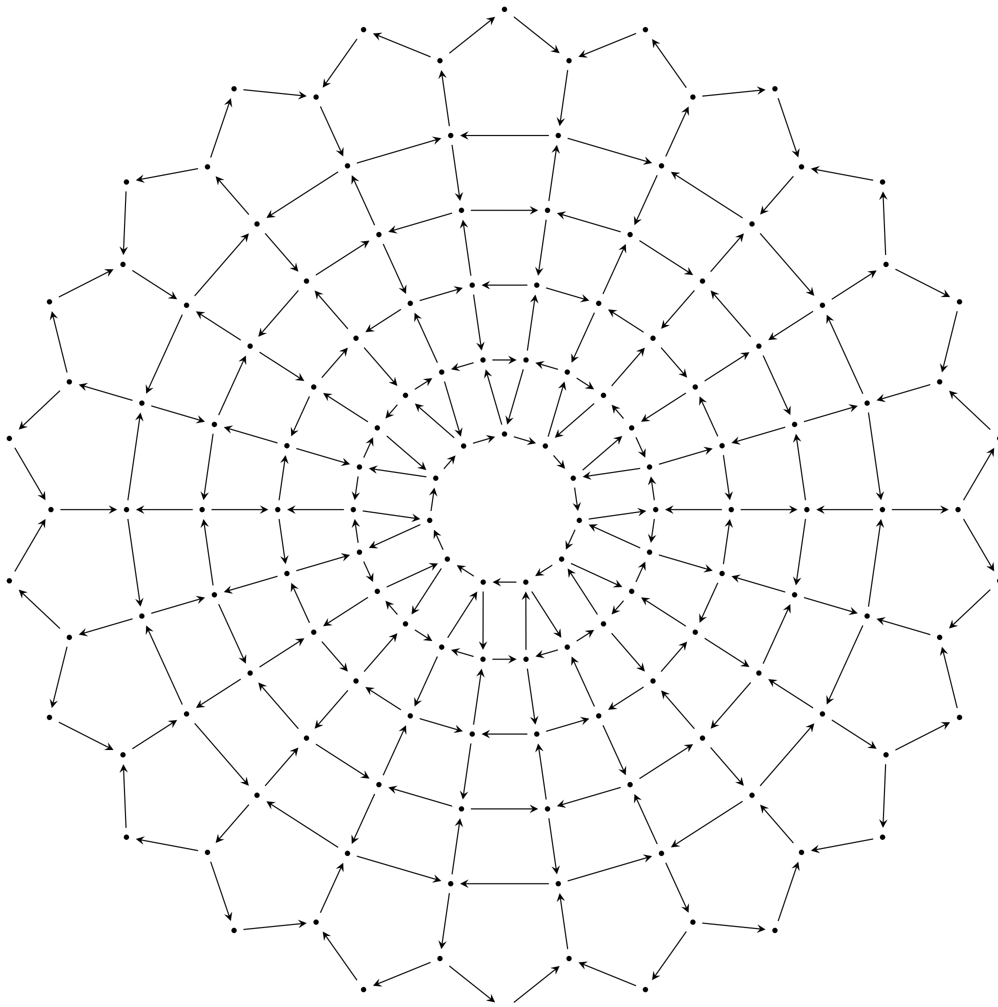


Figure 5: OddFlower(5), which is the fourth of the fourteen new self-injective QPs, with $(N, S) = (143, 11)$. It is the fifth QP in the first infinite family of QPs conjectured to consist solely of self-injective QPs.

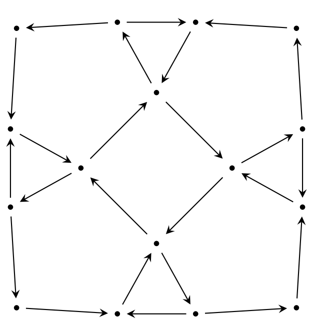


Figure 6: $\text{EvenFlower}_1(2)$, which is a previously known self-injective QP, with $(N, S) = (16, 4)$. It is the first QP in the second infinite family of QPs conjectured to consist solely of self-injective QPs.

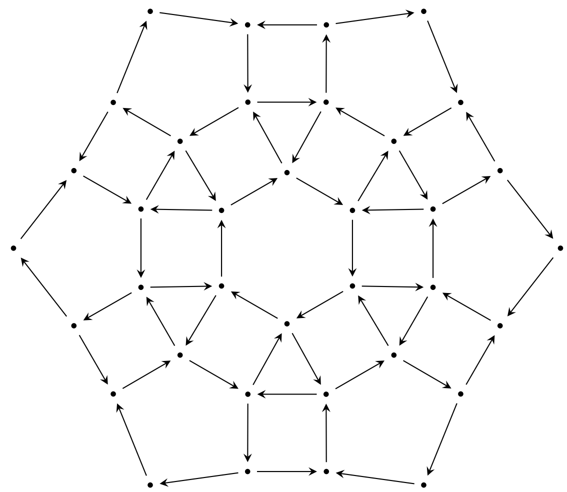


Figure 7: $\text{EvenFlower}_1(3)$, which is a previously known self-injective QP, with $(N, S) = (36, 2)$. It is the second QP in the second infinite family of QPs conjectured to consist solely of self-injective QPs.

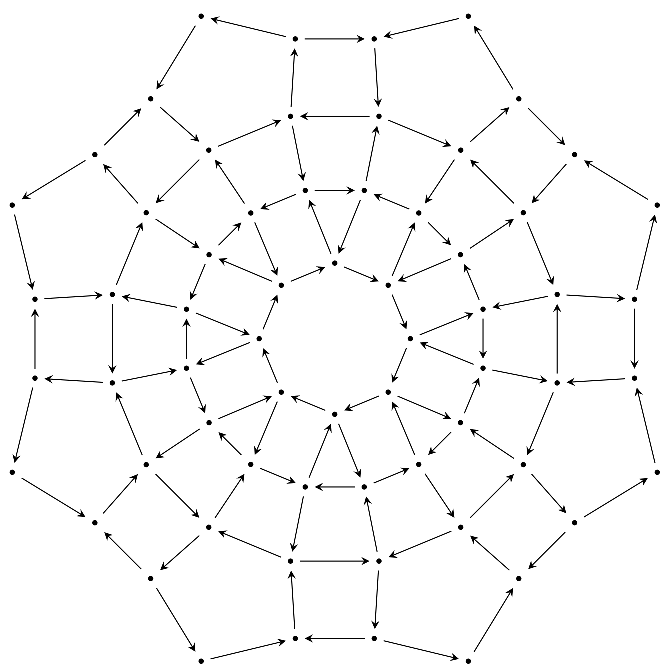


Figure 8: $\text{EvenFlower}_1(4)$, which is the fifth of the fourteen new self-injective QPs, with $(N, S) = (64, 8)$. It is the third QP in the second infinite family of QPs conjectured to consist solely of self-injective QPs.

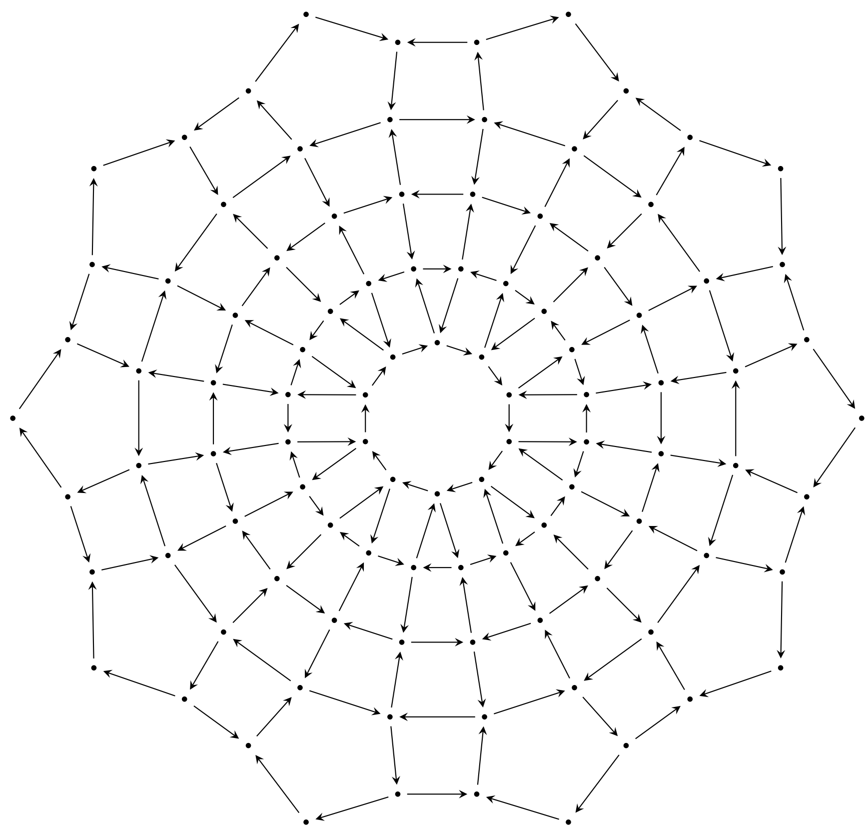


Figure 9: $\text{EvenFlower}_1(5)$, which is a self-injective QP with $(N, S) = (100, 2)$, that is conjectured to be mutation-equivalent to a previously known self-injective QP. It is the fourth QP in the second infinite family of QPs conjectured to consist solely of self-injective QPs.

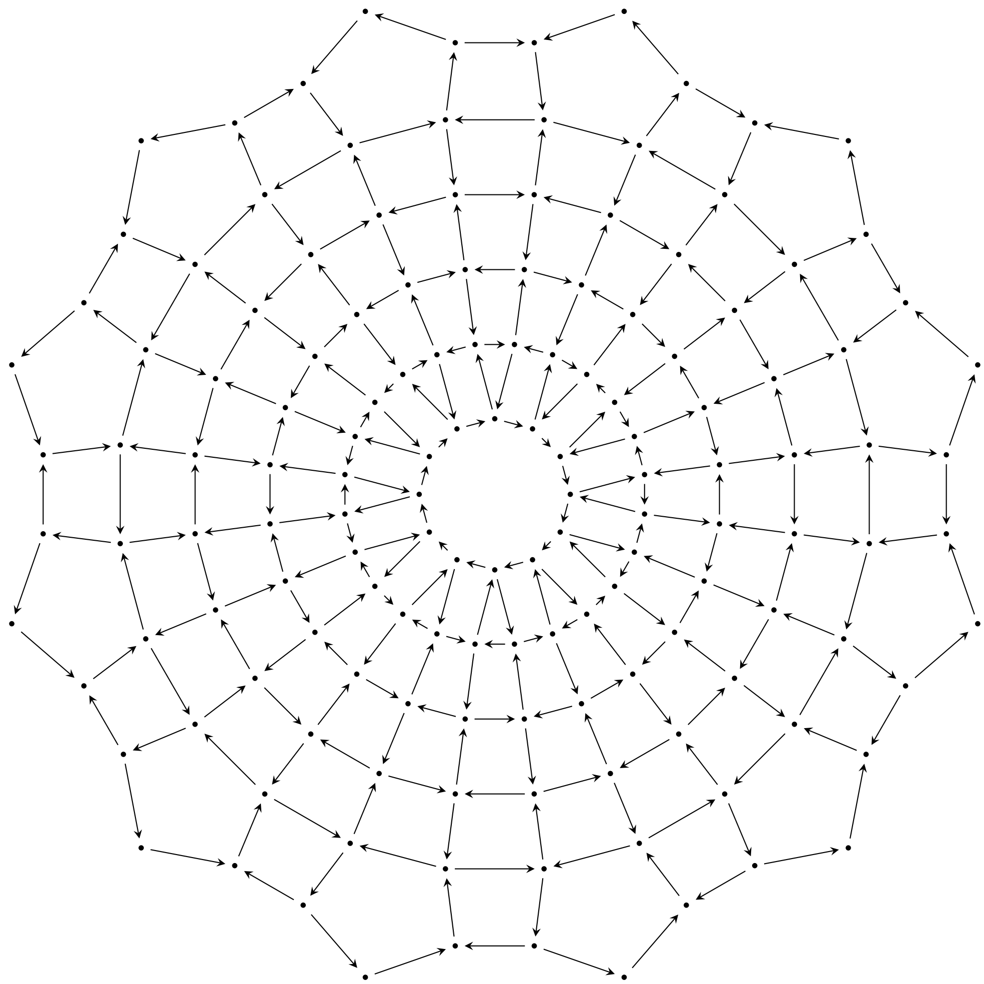


Figure 10: EvenFlower$_1$(6), which is the sixth of the fourteen new self-injective QPs, with $(N, S) = (144, 12)$. It is the fifth QP in the second infinite family of QPs conjectured to consist solely of self-injective QPs.

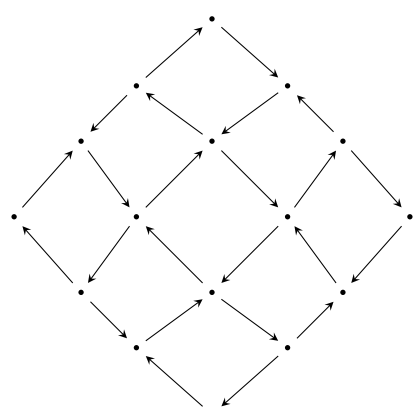


Figure 11: EvenFlower$_2$(2), which is a previously known self-injective QP, with $(N, S) = (16, 2)$. It is the first QP in the third infinite family of QPs conjectured to consist solely of self-injective QPs.

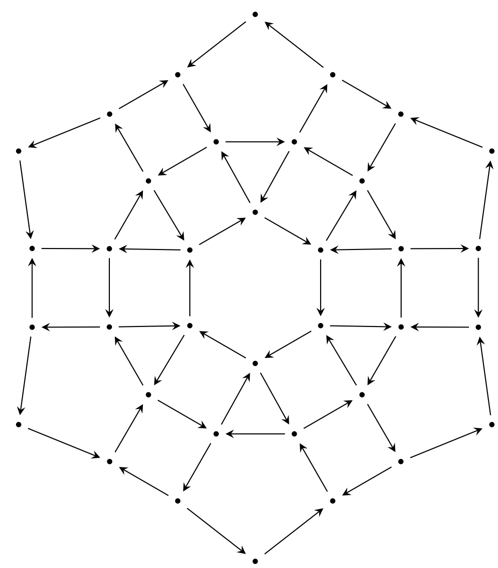


Figure 12: EvenFlower$_2$(3), which is a previously known self-injective QP, with $(N, S) = (36, 3)$. It is the second QP in the third infinite family of QPs conjectured to consist solely of self-injective QPs.

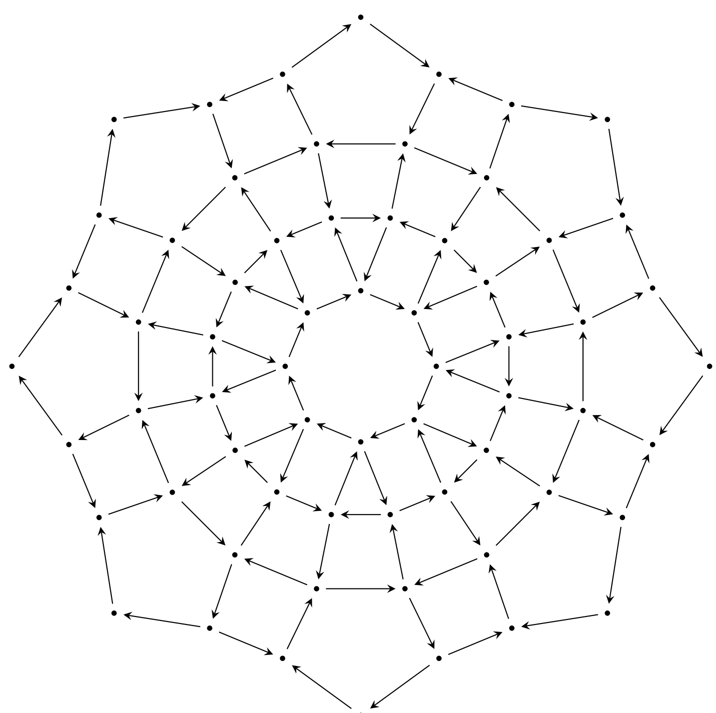


Figure 13: EvenFlower$_2$(4), which is a previously known self-injective QP, with $(N, S) = (64, 2)$. It is the third QP in the third infinite family of QPs conjectured to consist solely of self-injective QPs.

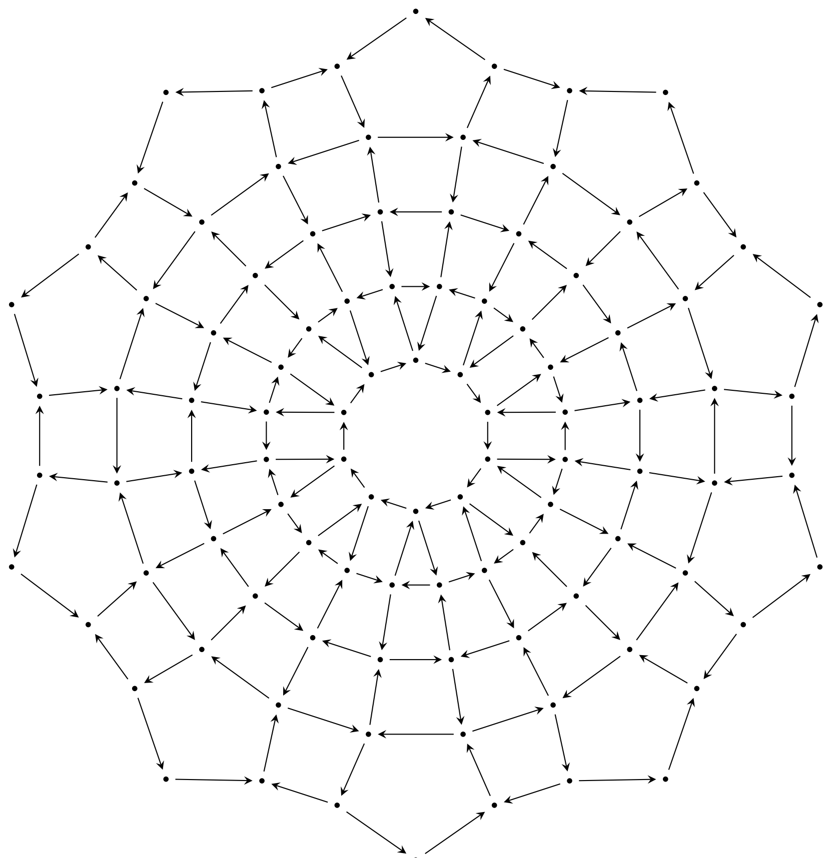


Figure 14: $\text{EvenFlower}_2(5)$, which is the seventh of the fourteen new self-injective QPs, with $(N, S) = (100, 5)$. It is the fourth QP in the third infinite family of QPs conjectured to consist solely of self-injective QPs.

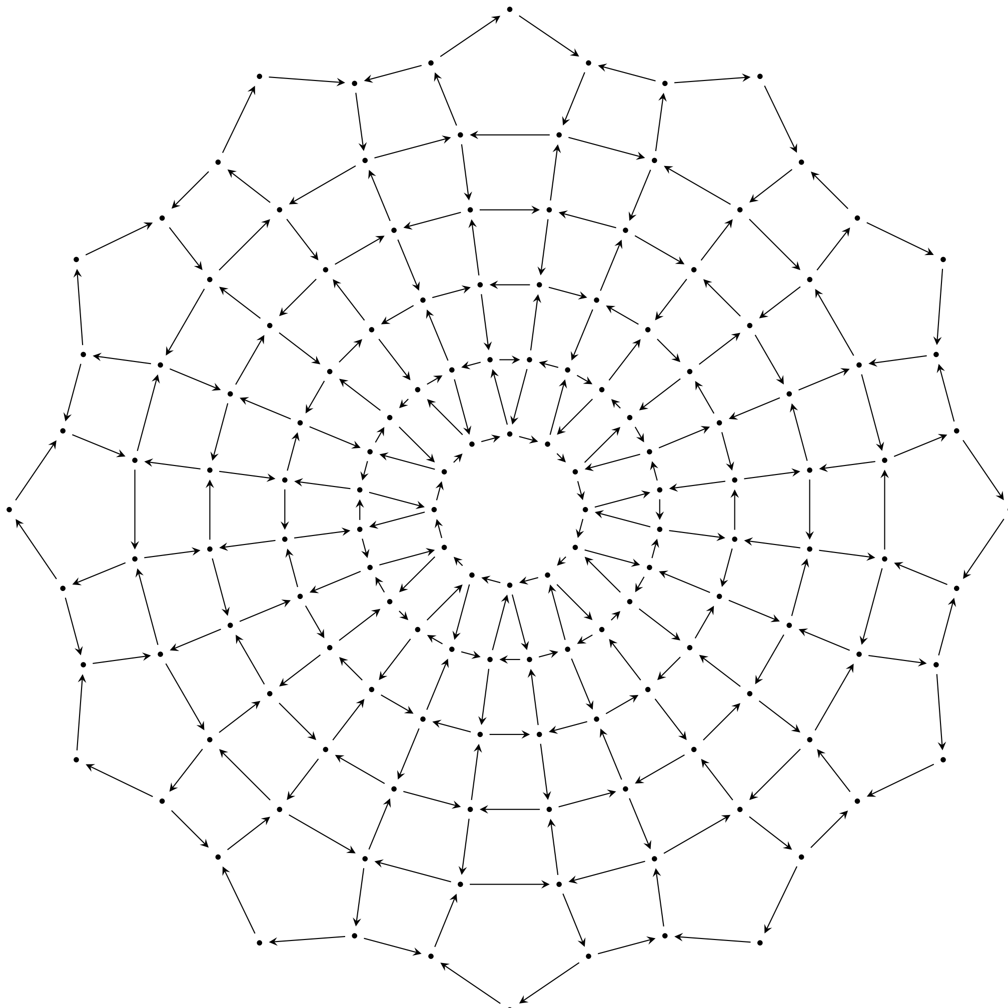


Figure 15: $\text{EvenFlower}_2(6)$, which is a self-injective QP with $(N, S) = (144, 2)$, that is conjectured to be mutation-equivalent to a previously known self-injective QP. It is the fifth QP in the third infinite family of QPs conjectured to consist solely of self-injective QPs.

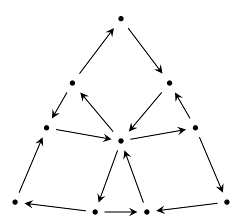


Figure 16: PointedFlower(1), which is a previously known self-injective QP, with $(N, S) = (10, 3)$. It is the first QP in the fourth infinite family of QPs conjectured to consist solely of self-injective QPs.

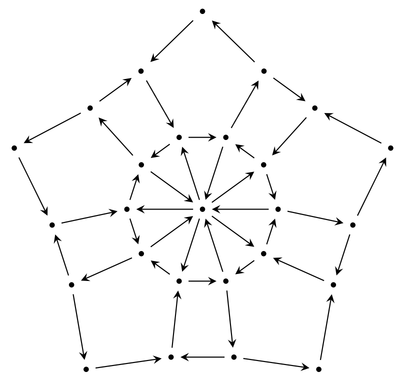


Figure 17: PointedFlower(2), which is the eighth of the fourteen new self-injective QPs, with $(N, S) = (26, 5)$. It is the second QP in the fourth infinite family of QPs conjectured to consist solely of self-injective QPs.

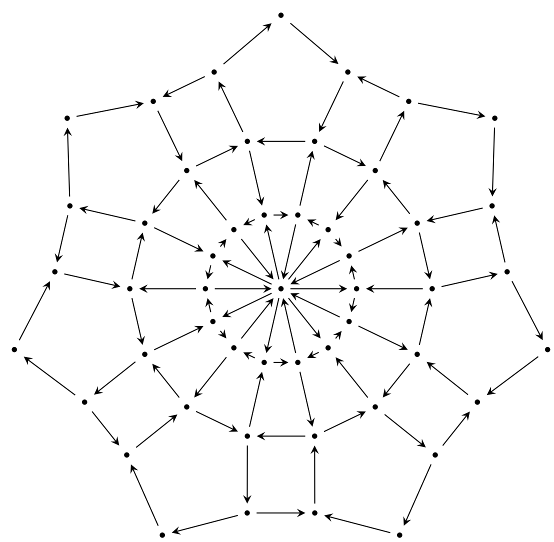


Figure 18: PointedFlower(3), which is the ninth of the fourteen new self-injective QPs, with $(N, S) = (50, 7)$. It is the third QP in the fourth infinite family of QPs conjectured to consist solely of self-injective QPs.

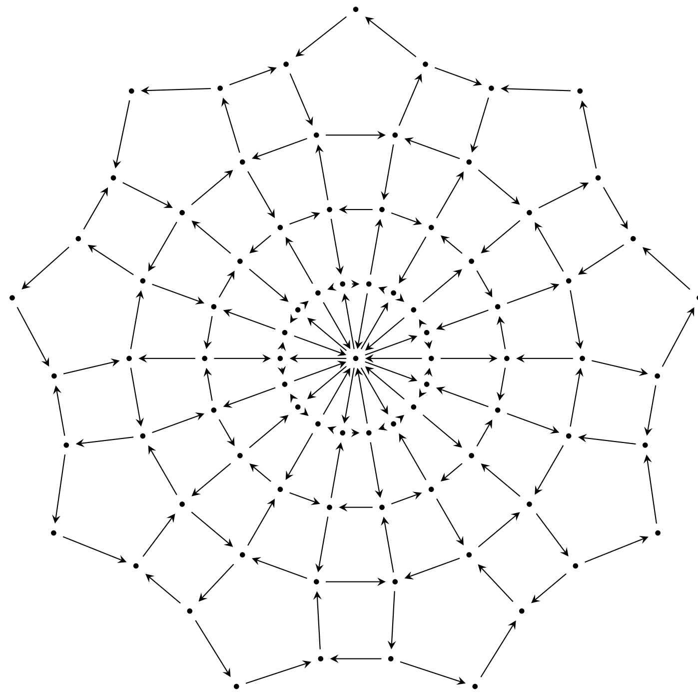


Figure 19: PointedFlower(4), which is the tenth of the fourteen new self-injective QPs, with $(N, S) = (82, 9)$. It is the fourth QP in the fourth infinite family of QPs conjectured to consist solely of self-injective QPs.

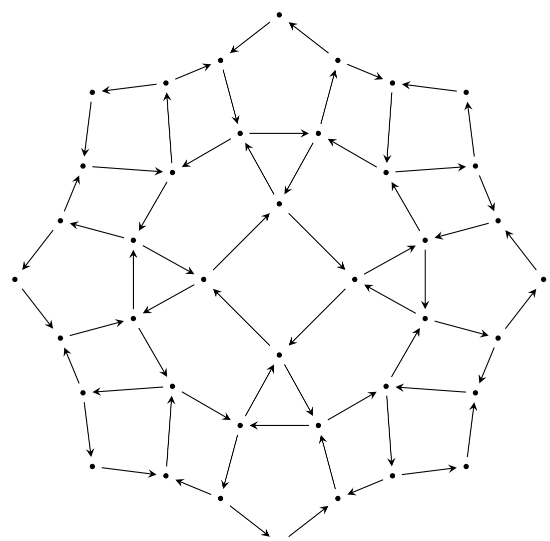


Figure 20: The eleventh of the fourteen new self-injective QPs, with $(N, S) = (40, 4)$.

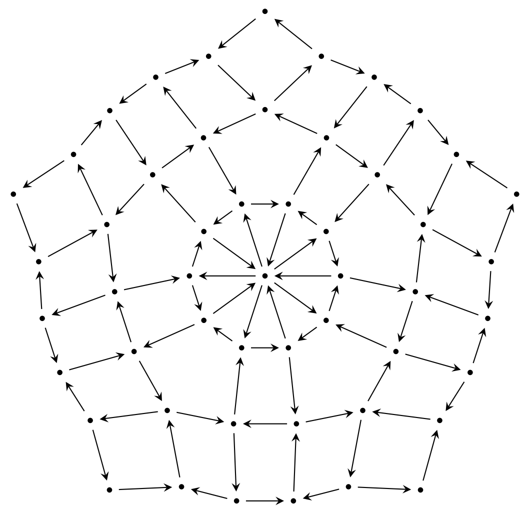


Figure 21: The twelfth of the fourteen new self-injective QPs, with $(N, S) = (51, 5)$.

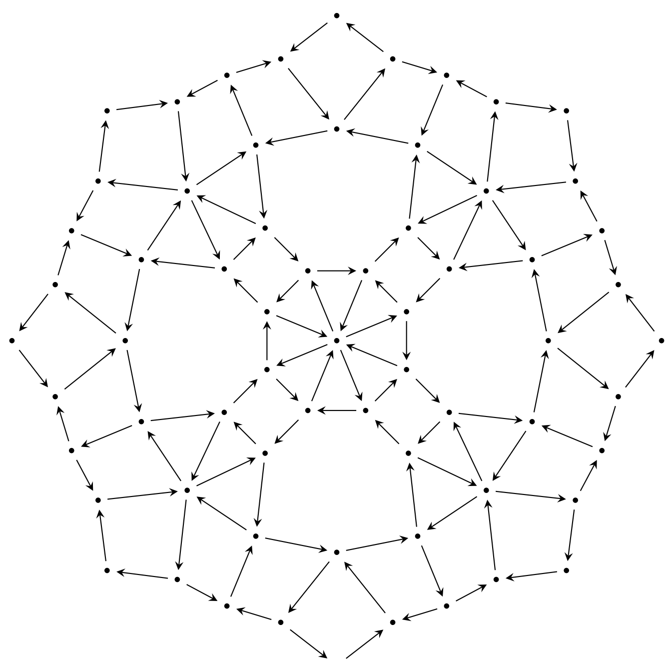


Figure 22: The thirteenth of the fourteen new self-injective QPs, with $(N, S) = (65, 4)$.

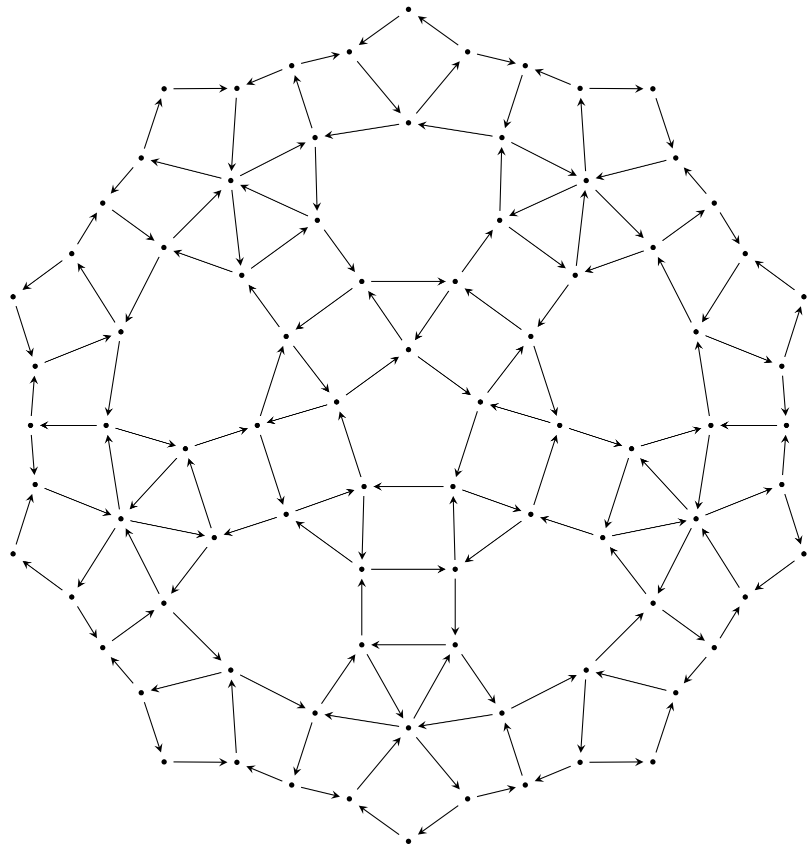


Figure 23: The fourteenth of the fourteen new self-injective QPs, with $(N, S) = (85, 5)$.

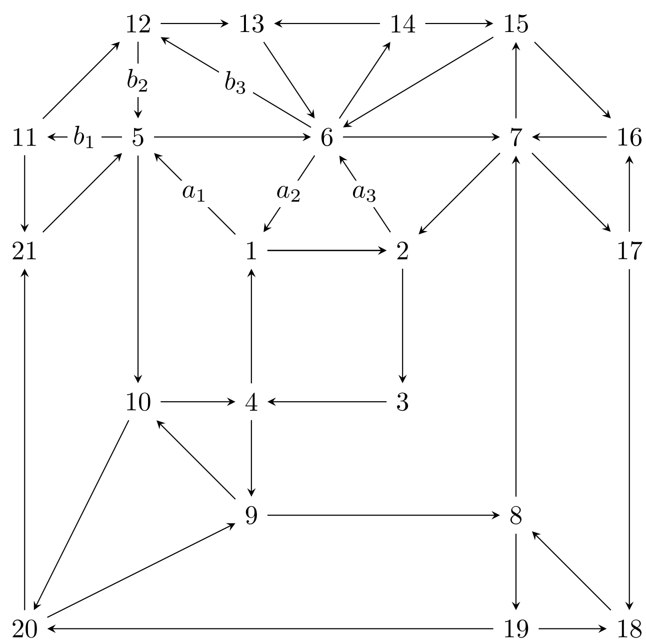


Figure 24: A layered quiver.

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