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U.U.D.M. Project Report 2019:6

A study of an approximate equation for sharp fronts in the generalized Surface Quasi-Geostrophic Equations in the cylinder

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Examensarbete i matematik, 30 hp
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Februari 2019

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Abstract

The Surface Quasi-Geostrophic Equation is a pseudo-differential equation that arises in Fluid Dynamics. It admits piecewise constant solutions, whose fronts (lines of discontinuity) follow their own pseudo-differential set of equations. A cubic model for these fronts was introduced in [5] by J. Hunter and J. Shu. This master thesis explores whether this toy model admits soliton solutions, using both numerical simulation and theoretical tools.

Acknowledgments

I would like to thank my supervisor Jordi-Lluís Figueras for his help and guidance, without which this thesis would have never been possible.

Chapter 1

Introduction

1.1 The physical setting

The *Surface Quasi-Geostrophic Equation* (1.5) is a pseudo-differential equation that originates in the field of geophysics. For an in-depth reference of all the involved concepts see [1]; we will give a quick overview in this section.

In *fluid mechanics*, focus is given to physical quantities that move along a fluid medium. Let $\mathbf{u}$ be the velocity of such a medium: given a physical quantity θ that flows with $\mathbf{u}$, its time evolution is more easily described through the *material derivative*

$$\frac{D\theta}{Dt} = \frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta.$$

When studying atmospherical phenomena dominated by the Earth's spin, it is sometimes good enough to ignore any vertical motion of the air and describe its winds as a 2-dimensional flow on the surface of a sphere. Using polar coordinates $(x, y) \in \mathbb{R}^2$, where x represents longitude and y latitude, the wind velocity field $\mathbf{u}$ can be described as a $\mathbb{R}^2 \to \mathbb{R}^2$ function with components (u, v). This coordinate system is degenerate at the poles, but works well in the mid-latitudes where interesting phenomena like the *jet streams* happen. Under certain approximations, $\mathbf{u}$ can be taken as the *perpendicular gradient* $\nabla^\perp := (-\partial_y, \partial_x)$ of a *stream function* ψ:

$$\mathbf{u} = \nabla^\perp \psi = (-\psi_y, \psi_x).$$

That is, $\mathbf{u}$ flows along the contour lines of ψ. Two interesting cases arise here: First, under the full 3-dimensional setting, the *vorticity* $\boldsymbol{\omega}$ of the flow $\mathbf{u}$ is defined as its rotational $\boldsymbol{\omega} = \nabla \times \mathbf{u}$. When reduced to a 2-dimensional setting, this gets reduced to the z-component:

$$\omega = u_y - v_x = -\psi_{yy} - \psi_{xx} = -\Delta \psi.$$

In some cases, one can assume ω to be conserved, that is,

$$\frac{D\omega}{Dt} = 0, \tag{1.1a}$$

$$\omega = -\Delta \psi. \tag{1.1b}$$

Equations (1.1) are known as the *stream-vorticity formulation of the incompressible Euler Equations*, as written using Physics notation. Assuming the Laplacian operator can be inverted (that is, the choice of ψ is restricted enough that we can uniquely determine it given a valid ω) and using a more mathematically inclined notation, this becomes:

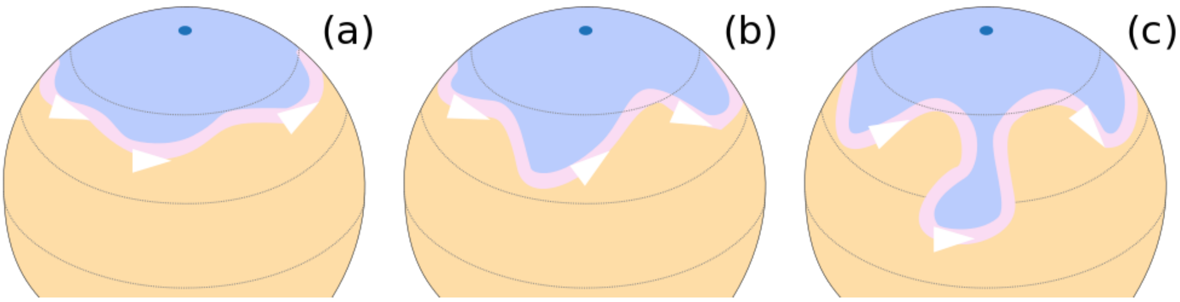

Figure 1.1: Jet streams are self-sustaining stripes of high velocity wind in the surface of a planet, caused by the interaction between pressure differences and the Coriolis force. This figure shows the Northern Hemisphere jet stream (pink) on Earth, fueled by the temperature difference between cold polar air (blue) and warmer air (orange). It shows meanders (Rossby Waves) developing (a), (b); then finally detaching a "drop" of cold air (c). (CC) Wikimedia Commons.

Definition 1.1 (Euler Equations)**.** The *Euler Equations* are the following set of equations, where $\omega : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2$:

$$\omega_t + \mathbf{u} \cdot \nabla \omega = 0, \tag{1.2a}$$

$$\mathbf{u} = \nabla^{\perp} (-\Delta)^{-1} \omega. \tag{1.2b}$$

Second, in geophysics a useful quantity is the (quasi-geostrophic) *potential vorticity* θ, which again can be described as a real-valued function on the sphere. Under a set of approximations which are collectively described as *quasi-geostrophic flow*, valid for mid-latitudes, θ is conserved ($D\theta/Dt = 0$), and a peculiar relationship appears between the Fourier transforms of θ and ψ. Formally considering ψ as a plane wave $\psi(\mathbf{x}) = \hat{\psi}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$ for some $\mathbf{k} \in \mathbb{R}^2$ yields θ as a plane wave $\theta(\mathbf{x}) = \hat{\theta}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}}$, with the defining relationship

$$\hat{\psi}(\mathbf{k}) = -\frac{1}{|\mathbf{k}|} \hat{\theta}(\mathbf{k}). \tag{1.3}$$

This is briefly explained in [2] and [3]. It establishes an algebraic relation between the Fourier transforms $\hat{\psi}$ and $\hat{\theta}$, which a priori lets us write a conservation law without ψ, simiilar to the substitution we did for (1.2). To describe this law, which we will find to be very similar to (1.2), we need to discuss the *fractional Laplacian*.

1.2 Fractional Laplacian and Fourier multipliers

1.2.1 General case

Given $f \in L^1(\mathbb{R}^n) \cap C^1(\mathbb{R}^n)$ and $1 \leq j \leq n$, assume $\partial_{x_j} f \in L^1(\mathbb{R}^n)$, and further assume that $\mathcal{F}[f]$ and $\mathcal{F}[\partial_{x_j} f]$ are integrable functions. Then

$$\partial_{x_j} f(\mathbf{x}) = \partial_{x_j} \int_{\mathbb{R}^n} \mathcal{F}[f](\mathbf{k}) e^{i\mathbf{x}\cdot\mathbf{k}} d\mathbf{k} = \int_{\mathbb{R}^n} \mathcal{F}[f](\mathbf{k}) (ik_j) e^{i\mathbf{x}\cdot\mathbf{k}} d\mathbf{k} \qquad \forall \mathbf{x} \in \mathbb{R}^n,$$

$$\partial_{x_j} f(\mathbf{x}) = \int_{\mathbb{R}^n} \mathcal{F}[\partial_{x_j} f](\mathbf{k}) e^{i\mathbf{x}\cdot\mathbf{k}} d\mathbf{k} \qquad \forall \mathbf{x} \in \mathbb{R}^n.$$

The uniqueness of the Fourier transform in $L^1(\mathbb{R}^n)$ then implies

$$\mathcal{F}[\partial_{x_j} f] = ik_j \mathcal{F}[f].$$

Similar operations reveal similar relations. Using multiindices of the form $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$, and writing derivatives as $D^\alpha = (i\partial_{x_1})^{\alpha_1} \cdots (i\partial_{x_n})^{\alpha_n}$, any linear differential operator of order n with constant coefficients can be written as

$$P = \sum_{|\alpha| \leq n} a_\alpha D^\alpha, \qquad a_\alpha \in \mathbb{C},$$

where $|a| = \sum_{k=1}^n a_k$. Using similar reasoning as above, one can then prove that if, all involved functions are regular enough,

$$\mathcal{F}[Pf](\mathbf{k}) = p(\mathbf{k})\mathcal{F}[f](\mathbf{k}),$$

where $p : \mathbb{R}^n \to \mathbb{C}$ is the complex polynomial given by

$$p(\mathbf{k}) = \sum_{|\alpha| \leq n} a_\alpha \mathbf{k}^\alpha,$$

and $\mathbf{k}^\alpha := k_1^{\alpha_1} \cdots k_n^{\alpha^n}$. This means many differential operators can be treated through the Fourier transform as multiplicative functions. As an important particular example, the Laplacian becomes

$$\mathcal{F}[\Delta f](\mathbf{k}) = -|\mathbf{k}|^2 \mathcal{F}[f](\mathbf{k}).$$

If we ask f to be a zero-mean function, $\mathcal{F}[f](\mathbf{0}) = 0$, we can invert this formula: if $g = \Delta f$ and both functions are regular enough, we can uniquely determine f from g through

$$\mathcal{F}[f](\mathbf{k}) = -\frac{1}{|\mathbf{k}|^2}\mathcal{F}[g](\mathbf{k}) \quad \forall \mathbf{k} \neq \mathbf{0}, \qquad \mathcal{F}[f](0) = 0.$$

This algebraic relation allows us to define the *inverse Laplacian* $(-\Delta)^{-1}$, and more generally a family of operators which do not correspond to classical differential equations, the *fractional Laplacian* $(-\Delta)^s$, $s \in \mathbb{R}$, through the relation

$$\mathcal{F}[(-\Delta)^s f](\mathbf{k}) = |\mathbf{k}|^{2s}\mathcal{F}[f](\mathbf{k}).$$

To be specific, one needs the right hand side of this equation to be an element of $L^1(\mathbb{R}^n)$, so that the formula can be reversed:

Definition 1.2 (Fractional Laplacian)**.** Given $s \in \mathbb{R}$, let A be the set of all functions $f \in L^1(\mathbb{R}^n)$ such that $|\mathbf{k}|^{2s}\mathcal{F}[f](\mathbf{k}) \in L^1(\mathbb{R}^n)$, and let $m(\mathbf{k}) := |\mathbf{k}|^{2s}$. The *fractional Laplacian* $(-\Delta)^s$ is defined as the operator $A \to L^\infty(\mathbb{R}^n)$ given by

$$(-\Delta)^s f = \mathcal{F}^{-1}\left(m\mathcal{F}[f]\right).$$

This construction can be further generalized to a much wider class of operators. As described in [4] and outlined in Appendix A, the Fourier transform can be defined on functions $f : G \to \mathbb{C}$ for any *locally compact Abelian group* (LCA group) G, which is associated to a dual LCA group Γ (unique up to isomorphism), and $\hat{f}$ is a function from Γ to $\mathbb{C}$. With this in mind,

Definition 1.3 (Fourier multiplier)**.** Given a LCA group G, denote its dual as Γ. Given a function $m : \Gamma \to \mathbb{C}$, define the set A_m as

$$A_m := \{f \in L^1(G) \ : \ m \cdot \mathcal{F}[f] \in L^1(\Gamma)\}.$$

The *Fourier multiplier* $T_m : A_m \to L^\infty(G)$ is the operator

$$T_m f = \mathcal{F}^{-1}(m\mathcal{F}[f]),$$

and m is called the *symbol* of T_m. When no confusion is caused, T_m is just called a *multiplier*.

Examples 1.1.

1. Given $a \in \mathbb{C}$, the operator $f \mapsto af$ is a multiplier with symbol $m(\mathbf{k}) = a$.
2. For $G = \mathbb{R}$ or $G = \mathbb{T}$, the derivative operator $f \mapsto f'$ is a multiplier with symbol $m(k) = ik$.
3. For $G = \mathbb{R}^n$ and $1 \leq j \leq n$, the partial derivative operator ∂_{x_j} is a multiplier with symbol $m(\mathbf{k}) = ik_j$.
4. The Laplacian Δ is a multiplier with symbol $m(\mathbf{k}) = -|\mathbf{k}|^2$.
5. Given $s \in \mathbb{R}$, the fractional Laplacian $(-\Delta)^s$ is a multiplier with symbol $m(\mathbf{k}) = |\mathbf{k}|^{2s}$.
6. The operator $|\partial_x| := (-\Delta)^{1/2}$ is a multiplier with symbol $m(\mathbf{k}) = |\mathbf{k}|$.
7. The operator $\log|\partial_x|$ is defined as the multiplier over $\mathbb{T}$ with symbol $m(0) = 0$, $m(k) = \log|k|$ for $k \neq 0$.

If m never vanishes, T_m can be easily inverted:

$$\hat{g} = m\hat{f} \quad \Leftrightarrow \quad \hat{f} = \frac{1}{m}\hat{g}. \tag{1.4}$$

One must be careful with the domain of definition, since g might not be in $L^1(G)$. To formalize this:

Lemma 1.1 (Fourier multiplier invertibility)**.** *Let $m : \Gamma \to \mathbb{C}^*$. Let $B_m = T_m^{-1}(L^1(G) \cap A_{1/m})$, $C_m = T_m(B_m)$. Then $T_{\frac{1}{m}}|_{C_m}$ is the inverse of $T_m|_{B_m}$.*

Proof. By definition, $C_m \subseteq L^1(G)$, and if $g \in C_m$ there is $f \in B_m$ such that $g = \mathcal{F}^{-1}(m\mathcal{F}[f])$. Since $m\mathcal{F}[f] \in L^1(\Gamma)$, the Fourier inversion theorem applies, so $\mathcal{F}[g] = m\mathcal{F}[f]$. Also by definition, $C_m \subseteq A_{1/m}$, so $(1/m)\mathcal{F}[g] = \mathcal{F}[f]$ is integrable and we can apply the Fourier inversion theorem. Thus, $T_{\frac{1}{m}} g = \mathcal{F}^{-1}(\mathcal{F}[f]) = f$. □

1.2.2 The particular case $L^2(\mathbb{T})$

In the case $G = \mathbb{T}$ ($\Gamma = \mathbb{Z}$), restricting Fourier multipliers to $T_m : L^2(\mathbb{T}) \to L^2(\mathbb{T})$ lets us use the Hilbert space structure of $L^2(\mathbb{T})$ to our advantadge. In particular, it lets us separate neatly the kernel of an operator from its orthogonal complement, and use the relation (1.4) to characterize any multiplier.

Lemma 1.2 (Fourier multipliers in $L^2(\mathbb{T})$)**.** *Given $m : \mathbb{Z} \to \mathbb{C}$, consider the restricted set*

$$D_m := \{f \in L^2(\mathbb{T}) \ : \ m \cdot \mathcal{F}[f] \in L^1(\mathbb{Z})\}.$$

Then $D_m \subseteq A_m$, and $T_m(D_m) \subseteq L^2(\mathbb{T})$. When dealing with $L^2(\mathbb{T})$ we will assume Fourier multipliers to be restricted to $T_m : D_m \to L^2(\mathbb{T})$.

Proof. Since $\mathbb{T}$ has finite measure, $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$, proving $D_m \subseteq A_m$. Since $\mathbb{Z}$ is discrete, the opposite relation holds: $L^1(\mathbb{Z}) \subset L^2(Z)$, so if $f \in D_m$ then $m \cdot \mathcal{F}[f] \in L^2(\mathbb{Z})$. As pointed out in Appendix A, this implies $\mathcal{F}^{-1}(m \cdot \mathcal{F}[f]) \in L^2(\mathbb{T})$. □

Lemma 1.3 (Fourier multiplier kernels and inverses)**.** *Let $m : \mathbb{Z} \to \mathbb{C}$. Denote*

$$Z := \{k \in \mathbb{Z} \ : \ m(k) = 0\} \subseteq \mathbb{Z}.$$

Then $\ker T_m = \langle \phi_k \rangle_{k \in Z}$. Furthermore, $T_m|_{(\ker T_m)^\perp}$ is a bijection into $T_m(D_m)$, with inverse

$$T_m^{-1} g = \sum_{k \notin Z} \frac{1}{m(k)} \hat{g}_k \phi_k, \qquad g \in T_m(D_m).$$

Finally, $T_m(D_m) \subseteq \langle \phi_k \rangle_{k \notin Z}$, and $\{\phi_k\}_{k \notin Z} \subset T_m(D_m)$.

Proof.

1. Clearly $\langle\phi_k\rangle_{k\in Z} \subseteq \ker T_m$, since $T_m\phi_k = 0$ for all $k \in Z$ (in particular, $\langle\phi_k\rangle_{k\in Z} \subseteq D_m$). Consider now $f \in D_m$ such that $T_m f = 0$. It follows that $m(k)\hat{f}_k = 0$ for all $k \in \mathbb{Z}$, so $\hat{f}_k$ can only be nonzero if $k \in Z$, proving $\ker T_m \subseteq \langle\phi_k\rangle_{k\in Z}$.

2. Given $g \in T_m(D_m)$, it follows that $g \in L^2(\mathbb{T})$ and $\mathcal{F}[g] = m\mathcal{F}[f]$ for some $f \in D_m$, so $f \in L^2(\mathbb{Z})$ and $\mathcal{F}[f] \in L^2(\mathbb{Z})$. Consider now the projection of f into $(\ker T_m)^\perp$, $f^\perp$. It follows that $f = f^\perp + f^\parallel$, with $f^\parallel \in \ker T_m$, so $T_m f^\perp = g$. Consider now the function $\omega(k) = 1/m(k)$ for $k \notin Z$, $\omega(k) = 0$ for $k \in Z$. Then $\omega\mathcal{F}[g] = \mathcal{F}[f^\perp]$, and by the Fourier inversion theorem

$$f^\perp = \mathcal{F}^{-1}(\omega\mathcal{F}[g]) = \sum_{k\notin Z} \frac{1}{m(k)}\hat{g}_k\phi_k.$$

   Assume now $h \in (\ker T_m)^\perp$ also follows $T_m h = g$. It follows that $f^\perp - h \in (\ker T_m)^\perp$, but also $T_m(f^\perp - h) = 0$, so $f^\perp = h$, proving $f^\perp$ is unique in $(\ker T_m)^\perp$.

3. Since $\{\phi_k\}_{k\notin Z}$ is a basis of $L^2(\mathbb{T})$, $T_m(D_m) \subseteq \langle\phi_k\rangle_{k\notin Z}$ is a direct consequence of $\ker T_m = \langle\phi_k\rangle_{k\in Z}$. Furthermore, for $k \notin Z$, $T_m(\phi_k/m(k)) = \phi_k$.

□

Note that we cannot say in general that $T_m(D_m) = \langle\phi_k\rangle_{k\notin Z}$, since T_m might not be defined in the whole $L^2(\mathbb{T})$ space. However, this is still a good rule of thumb when reasoning about such operators. Introducing the notation $R(T)$ for the range of an arbitrary operator T, let us explore some important examples.

Examples 1.2.

1. $\ker\partial_x = \langle\phi_0\rangle$. The inverse operator is

$$\partial_x^{-1}g = \sum_{k\neq 0} -\frac{i}{k}\hat{g}_k\phi_k, \qquad g \in R(\partial_x),$$

   which corresponds to the zero-mean integral of f. Unlike ∂_x, ∂_x^{-1} has bounded norm: $\|\partial_x^{-1}\phi_k\| \leq 1$ for $k \neq 0$, so $\|\partial_x^{-1}g\| \leq \|g\|$. It can thus be extended by continuity to a bounded linear operator $\partial_x^{-1} : \langle\phi_0\rangle^\perp \to \langle\phi_0\rangle^\perp$ with unit norm. Further setting $\partial_x^{-1}\phi_0 = 0$, it can be taken as bounded linear operator $L^2(\mathbb{T}) \to L^2(\mathbb{T})$ with unit norm.

2. $\ker\log|\partial_x| = \langle\phi_{-1}, \phi_0, \phi_{+1}\rangle$. The inverse operator is

$$\log|\partial_x|^{-1}g = \sum_{|k|\geq 2} (\log|k|)^{-1}\hat{g}_k\phi_k, \qquad g \in R(\log|\partial_x|).$$

   Again, $\log|\partial_x|^{-1}$ has bounded norm, so it can be extended by continuity to a bounded linear operator in $\langle\phi_{-1}, \phi_0, \phi_{+1}\rangle^\perp$ with norm $(\log 2)^{-1} \simeq 1.44$. Again, it can be further extended to $L^2(\mathbb{T}) \to L^2(\mathbb{T})$ while conserving its norm by setting $\log|\partial_x|^{-1}\phi_k = 0$, $|k| \leq 1$.

1.3 The Surface Quasi-Geostrophic Equation and a front model

Back to the physical problem at hand, the fractional Laplacion notation gives us a specific meaning of what $(-\Delta)^{-1}$ means in Equation (1.2). Furthermore, it lets us rewrite Equation (1.3) to reveal its common form in mathematics:

Definition 1.4 (Surface Quasi-Geostrophic Equation)**.** The following set of equations is known as the *Surface Quasi-Geostrophic Equation* (SQG), where $\theta : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\boxed{\begin{aligned} \theta_t + \mathbf{u} \cdot \nabla\theta = 0, \\ \mathbf{u} = \nabla^{\perp}(-\Delta)^{-1/2}\theta. \end{aligned}} \qquad \begin{aligned} &(1.5a) \\ &(1.5b) \end{aligned}$$

This equation is the focus of [12], where the formation of discontinuous fronts is studied. When written in mathematical terms, Equations (1.2) and (1.5) become particular cases of a one-parameter family of pseudo-differential equations:

Definition 1.5 (Generalized Surface Quasi-Geostrophic equation)**.** The following set of equations is known as the *generalized Surface Quasi-Geostrophic equation* (gSQG), where $\theta : \mathbb{R}^2 \to \mathbb{R}$ and $\mathbf{u} : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\boxed{\begin{aligned} \theta_t + \mathbf{u} \cdot \nabla\theta = 0, \\ \mathbf{u} = \nabla^{\perp}(-\Delta)^{-\alpha/2}\theta, \\ 0 < \alpha \leq 2. \end{aligned}} \qquad \begin{aligned} &(1.6a) \\ &(1.6b) \\ &(1.6c) \end{aligned}$$

The case $\alpha = 1$ corresponds to the SQG Equation and the case $\alpha = 2$ corresponds to the Euler Equations. An often-studied property of this family of equations is the existence of weak solutions that only take two possible values (see, for example, [13]):

$$\theta(\mathbf{x}, t) = \begin{cases} \theta_+ & \mathbf{x} \in \Omega(t), \\ \theta_- & \mathbf{x} \notin \Omega(t), \end{cases}$$

for $\theta_+, \theta_- \in \mathbb{R}$ and $\Omega(t)$ a domain whose boundary evolves over time. If the boundary is regular enough, one can model $\partial\Omega(t)$ as a function that follows a differential equation derived from the original gSQG Equation. Using vocabulary from Physics, this function is often called a *front*. In a recent paper [5], John Hunter and Jingyang Shu study fronts which are periodic perturbations of $\partial\Omega(0) = \{y = 0\}$. In particular, they assume $\partial\Omega(t)$ can be parameterized as $\partial\Omega(t) = (x, \varphi(x, t))$, with $\varphi(x + 2\pi, t) = \varphi(x, t)$. Using a cubic approximation for small amplitudes, they reach the following dynamic equation for the front:

$$\varphi_t + \frac{1}{2}\partial_x \left\{ \varphi^2 \mathbf{A}\varphi - \varphi \mathbf{A}\varphi^2 + \frac{1}{3}\mathbf{A}\varphi^3 \right\} + \mathbf{L}\varphi = 0, \tag{1.7}$$

where $\mathbf{A}$ and $\mathbf{L}$ are operators that depend on the value of α. In the SQG case ($\alpha = 1$), $\mathbf{L} = -2\log|\partial_x|$ and $\mathbf{A} = \partial_x^2 \log|\partial_x|$. This brings us to the main matter of interest in this thesis: expanding (1.7) for $\alpha = 1$ we obtain the equation

$$\boxed{\varphi_t + \frac{1}{2}\partial_x \left\{ \varphi^2 \log|\partial_x|\varphi_{xx} - \varphi \log|\partial_x|(\varphi^2)_{xx} + \frac{1}{3}\log|\partial_x|(\varphi^3)_{xx} \right\} = 2\log|\partial_x|\varphi_x.} \tag{1.8}$$

Hunter and Shu proved in a later paper [6] that this equation is weakly well-posed, but many questions remain open. The goal of this thesis is trying to provide an answer for one of these questions:

Does Equation (1.8) allow for time-periodic solutions?

The interest lies in the fact that the SQG Equation is known to have time-periodic solutions, but it is unclear if this new model captures this property. The rest of this thesis explains our investigations into this subject.

Chapter 2

Numerical simulation

As a first step to study Equation (1.8), we replicated the numerical results in [5]. Once this proved successful, we explored the use of Newton's method for Banach spaces to find periodic soliton solutions. This method showed the existence of possible candidate solutions, and inspired the theoretical approach explored in the next chapter. Before discussing these facts, we need to describe the theory behind numerical simulation using the Fourier transform. The approach towards Fourier transforms in this thesis, as exposed in appendix A, is mostly based on [4]. Other computational Fourier concepts are covered from an engineer's perspective in [7].

2.1 Fourier pseudo-spectral methods

2.1.1 Theory

For each time point $t \in \mathbb{R}$, the functions we deal with are elements of $L^2(\mathbb{T})$, so their Fourier transforms are elements of $L^2(\mathbb{Z})$ (see appendix A for more details) and they can be described using the Fourier basis $\{\phi_k\}_{k\in\mathbb{Z}}$. This is a very convenient setting for theoretical work, but a finite dimensional representation is needed to apply numerical simulations in a computer. The scheme followed by all the tools we used consists in sampling the original function at N equidistant points, so all operations can be taken in $\mathbb{Z}/N\mathbb{Z}$ instead: given $f(x) \in L^2(\mathbb{T})$, we can associate it to a function $f_N \in L^2(\mathbb{Z}/N\mathbb{Z})$ through the relation

$$f_N(n) := f\left(2\pi \frac{n}{N}\right), \quad n \in \mathbb{Z}/N\mathbb{Z}.$$

We can think of this as the action of an operator:

Definition 2.1 (sampling operator)**.** Let $N \in \mathbb{N}$. The *sampling operator* $S_N : L^2(\mathbb{T}) \to L^2(\mathbb{Z}/N\mathbb{Z})$ is defined as

$$(S_N f)(n) := f_N(n) = f(2\pi n/N), \qquad \forall f \in L^2(\mathbb{T}), \quad \forall n \in \mathbb{Z}/N\mathbb{Z}.$$

This opens an important question: how should we represent the actions of the operators ∂_x and $\log|\partial_x|$ over f_N? The solution is applicable to all Fourier multipliers. Given a function $f \in L^2(\mathbb{T})$ and a symbol $m : \mathbb{Z} \to \mathbb{C}$,

$$T_m f(x) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} m(k)\hat{f}(k)e^{2\pi ikx} = \sum_{k\in\mathbb{Z}} m(k)\hat{f}(k)\phi_k(x),$$

and we can introduce an equivalent basis for $\mathbb{Z}/N\mathbb{Z}$:

Definition 2.2 (Fourier basis of $L^2(\mathbb{Z}/N\mathbb{Z})$)**.** The *Fourier basis* of $L^2(\mathbb{Z}/N\mathbb{Z})$ is the collection $\{\phi_K^N\}_{K\in\mathbb{Z}/N\mathbb{Z}} \subset L^2(\mathbb{Z}/N\mathbb{Z})$ defined by

$$\phi_K^N(n) = \frac{1}{\sqrt{N}} e^{2\pi i k n/N}, \qquad \forall n \in \mathbb{Z}/N\mathbb{Z},$$

where $k \in K$ is any representative of K.

Since $e^{2\pi ikn/N}$ is N-periodic in k, the choice of representative $k \in K$ does not matter here. Furthermore, the Fourier inversion theorem in $\mathbb{Z}/N\mathbb{Z}$ implies that $\{\phi_K^N\}_{K\in\mathbb{Z}/N\mathbb{Z}}$ is indeed a basis for $L^2(\mathbb{Z}/N\mathbb{Z})$. Any one choice of representative $k \in K$ allows us to apply any $L^2(\mathbb{T})$ Fourier multiplier to f_N through the formula

$$(T_m f_N)(n) = \sum_{K\in\mathbb{Z}/N\mathbb{Z}} m(k) \hat{f}_N(K) \phi_K^N(n), \qquad k \in K,$$

but there is an infinite number of choices of representatives and each one yields a different action of T_m over f_N. However, there is a common property to the operators we are interested in: $\|T_m \phi_k\|_{L^2(\mathbb{T})}$ is an increasing function of $|k|$. This means that the L^2-norm minimizing choice of representatives is $\{-N/2, \ldots, +N/2\}$ if N is odd, and either $\{-N/2+1, \ldots, +N/2\}$ or $\{-N/2, \ldots, +N/2-1\}$ if N is even. This leads to the following definition:

Definition 2.3 (Action of Fourier multipliers over sampled functions)**.** Let $N \in \mathbb{N}$. The *norm-minimizing choice of representatives* $\mathcal{K}_N : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}$ is defined as

$$\mathcal{K}_N(K) = \begin{cases} K \cap \{-N/2, \ldots, +N/2\}, & N \text{ odd}, \\ K \cap \{-N/2, \ldots, +N/2-1\}, & N \text{ even}. \end{cases}$$

Let $m : \mathbb{Z} \to \mathbb{C}$, and let T_m be a multiplier with symbol m. The action $T_m : L^2(\mathbb{Z}/N\mathbb{Z}) \to L^2(\mathbb{Z}/N\mathbb{Z})$ is defined as

$$(T_m f)(n) = \sum_{K\in\mathbb{Z}/N\mathbb{Z}} m(\mathcal{K}_N(K)) \hat{f}(K) \phi_K^N(n), \qquad \forall f \in L^2(\mathbb{Z}/N\mathbb{Z}), \quad \forall n \in \mathbb{Z}/N\mathbb{Z}.$$

2.1.2 Practice

In practical terms, this means most of the computer's work can be done representing any solution $\varphi(x,t)$ of Equation (1.8) as a collection of timesteps $\varphi^t(x) := \varphi(x,t)$, with each timestep containing N complex samples, or equivalently $2N$ floating point values, representing the frequencies $\hat{\varphi}_N^t$. If these complex values are stored in an array indexed $k = 0, \ldots, N-1$, ∂_x can be calculated as

$$\mathcal{F}[\partial_x \varphi_N^t](k) = i \left((k \mod N) - \frac{N}{2} \right) \hat{\varphi}_N^t(k),$$

and similarly for other relevant operators, like $\log|\partial_x|$ and ∂_x^2. The time evolution can then be calculated through a forward integration method like those in the Runge-Kutta family.

As a general principle, the precision of the simulation improves as N grows, and this particular problem seems to require especially high values – as a reference, Hunter and Shu use $N = 2^{14}$ and $N = 2^{15}$ in [5]. This makes it vital to keep the time complexity of any involved algorithms as low as possible. Since we represent the function as its Fourier transform, linear combinations and the application of Fourier multipliers are $O(N)$ operations. However, function products in the base space become convolutions in the dual space, which have order $O(N^2)$. Similarly, a naive implementation of the Fourier transform or its inverse is again an $O(N^2)$ operation. These two factors become, a priori, the worse bottlenecks in our simulation.

Luckily, for highly divisible values of N there are strategies that can be used to reduce the transform and inverse transform times to $O(N \log N)$, collectively known as the *Fast Fourier Transform* methods (FFT). This also allows base space products (or rather, dual space convolutions) to be realised in $O(N \log N)$ time: first, apply the inverse Fourier transform using FFT, second, perform the function product in base space as an $O(N)$ operation, third, apply the Fourier transform using FFT again.

Another important computational problem is that of numerical stability. Although the exact spectral form of Equation (1.8) is complicated, a general notion is that the k-th frequency is multiplied at every step by $k^3 \log k$, ignoring convolutions (every term is differentiated three times and gets $\log|\partial_x|$ applied once). Furthermore, the convolutions expand the quickly growing error from the higher frequencies to the lower frequencies.This means that the evolution equations are stiff, and any numerical errors are expected to grow exponentially as the number of iterations increases. This can be attenuated by choosing smooth initial conditions, since the Fourier transform of a $C^\infty(\mathbb{T})$ function is rapidly decaying (see appendix A). However, the choice of initial conditions is independent of the number of iterations, so it does not scale well with increasing values of N. A filtering step is needed at every step of the iteration process.

Filtering consists of convoluting the original function with a (base space) filter function, or equivalently multiplying the Fourier transform by a (dual space) filter function, so as to reduce unwanted artifacts in the simulation. Two common filters are the *sharp spectral filter* and the *exponential filter*:

Definition 2.4 (sharp spectral filter)**.** Let $M \in \mathbb{N}$. The *sharp spectral filter* with cutoff M is the Fourier multiplier with symbol

$$m(k) = \begin{cases} 1, & |k| \leq M, \\ 0, & |k| > M. \end{cases}$$

Definition 2.5 (exponential filter)**.** Let $\alpha, \beta > 0$. The *exponential filter* with parameters α, β is the Fourier multiplier with symbol

$$m(k) = \exp\left(-\alpha|k|^\beta\right).$$

These filters act on $L^2(\mathbb{Z}/N\mathbb{Z})$ as described in Definition 2.3. A general rule of thumb is to discard the higher third of frequencies. For the sharp spectral filter, this means choosing $M = N/3$. For the exponential filter, [14] suggests using

$$m(k) = \exp\left(-36\left(\frac{2|k|}{N}\right)^{19}\right).$$

Another reason why filtering is needed is avoiding aliasing phenomena: the choice of representatives $\mathcal{K}_N$ effectively embeds $L^2(\mathbb{Z}/N\mathbb{Z})$ into $L^2(\mathbb{T})$ as the trigonometric polynomials of degree at most $N/2$. However, the product in $L^2(\mathbb{T})$ of two degree-$N/2$ trigonometric polynomials is a degree-N polynomial, whereas the product in $L^2(\mathbb{Z}/N\mathbb{Z})$ of two functions stays in $L^2(\mathbb{Z}/N\mathbb{Z})$ and gets embedded as an at most degree-($N/2$) polynomial. Hence, the high frequency information of an $L^2(\mathbb{T})$ product becomes mixed with the lower frequencies when this product is approximated in $L^2(\mathbb{Z}/N\mathbb{Z})$. There are various ad-hoc ways to attenuate this problem; the theoretically simplest is to use a sharp spectral filter with cutoff $N/2$ to avoid any frequency overlap.

2.2 Replicating the results in Hunter-Shu

In [5], Hunter and Shu discuss the numerical simulation of Equation (1.8) with two sets of initial conditions:

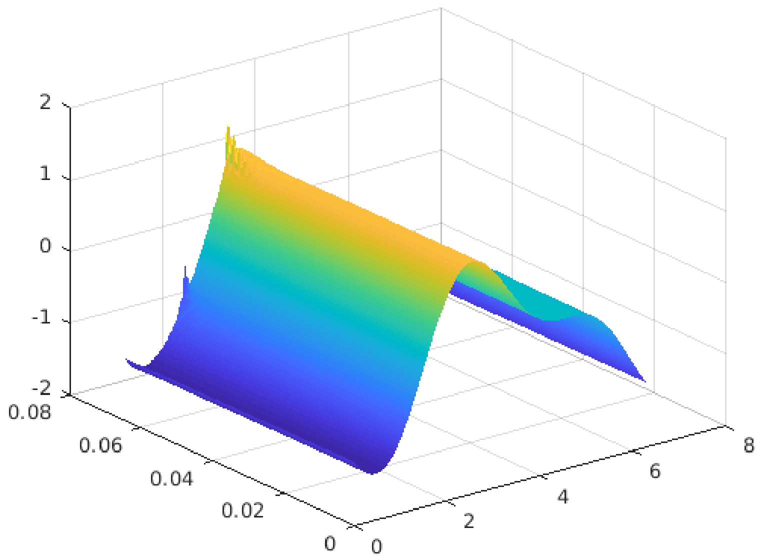

Figure 2.1: Our own simulation with $N = 2^{10}$ for the initial condition (2.1).

$$\varphi_0(x) = \cos(x+\pi) + \frac{1}{2}\cos[2(x+\pi+2\pi^2)], \tag{2.1}$$

$$\varphi_0(x) = \operatorname{sech}^2\left(\frac{5(x-\pi)}{2}\right). \tag{2.2}$$

When we set out to replicate the numerical simulations in [5], our first approach consisted in programming the simulation directly in C. For this purpose, we used a commonplace FFT library, called FFTW [8], that implements $O(N \log N)$ Fourier transform and inverse Fourier transform algorithms for highly divisible N, as discussed previously. Following Hunter and Shu, we restricted ourselves to powers of two to obtain high performance. The time iteration was realised using the standard 4th-order Runge-Kutta method (RK4) with a fixed time step. This involves rearranging Equation (1.8) as

$$\hat{\varphi}_t = \mathcal{F}[T\varphi], \qquad T\varphi := 2\log|\partial_x|\varphi_x - \frac{1}{2}\partial_x\left\{\varphi^2\log|\partial_x|\varphi_{xx} - \varphi\log|\partial_x|(\varphi^2)_{xx} + \frac{1}{3}\log|\partial_x|(\varphi^3)_{xx}\right\}, \tag{2.3}$$

and calculating $\mathcal{F}[T\varphi]$ for various values of $\hat{\varphi}$. In the customary Runge-Kutta notation of solving $\dot{y} = f(y)$, this corresponds to $y = \hat{\varphi}$, $f(y) = \mathcal{F}[T\varphi]$.

We tested the initial conditions (2.1) and (2.2), but also a simple sine $f(x) = \sin(x)$, the discontinuous sawtooth function $f(x) = x$, and low-degree trigonometric polynomials. Similarly, the code was tested not only against Equation (2.3), but also against two commonplace equations: the Transport Equation $\varphi_t = \varphi_x$ and the Heat Equation $\varphi_t = \varphi_{xx}$. The simpler cases highlighted how unstable spectral numerical simulations are, and the need for filtering. As a consequence, we tried three options: no filtering, applying an $M = N/3$ sharp spectral filter at the end of the $\mathcal{F}[T\varphi]$ calculations, and applying the previously mentioned exponential filter at the end of the $\mathcal{F}[T\varphi]$ calculations.

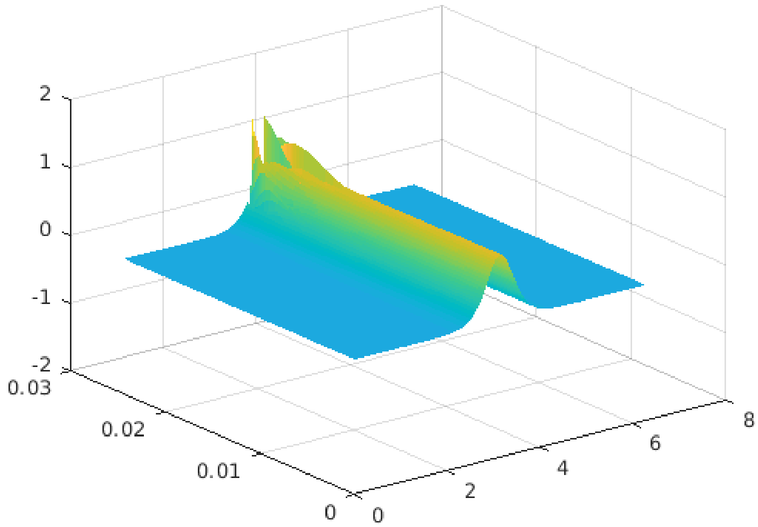

Figure 2.2: Our own simulation with $N = 2^{10}$ for the initial condition (2.2).

Using RK4 with filtering and smooth initial conditions proved stable enough to correctly simulate the Transport and Heat Equations, but insufficient for Equation (2.3). This led to a second approach: using MatLab and its built-in routines. The achievable performance in MatLab is necessarily lower than pure C programming, but it allows for easier and faster iteration of the code. It also includes adaptive time step integration methods whose manual implementation would be beyond the scope of this thesis, and easy to use FFT methods (that internally use the FFTW library, according to MatLab's documentation). We implemented integration through Matlab's **ode45** method, which uses the *Runge–Kutta–Fehlberg method* for adaptive timestep integration (see [9]).

The new approach was met with mixed success: the adaptive time step integration allowed for more stable results in the Heat and Transport Equations with lower values of N, but was still insufficient for Equation (2.3). This led us to contact Dr. Hunter, who suggested two modifications: using a *viscosity approximation method*, and applying an $M = N/2$ sharp cutoff filter before every single function product to avoid aliasing artifacts. It was only after these modifications that satisfactory results were achieved. Figures 2.1 and 2.2 show the results we obtained for the initial conditions (2.1) and (2.2), respectively. In them, oscillatory phenomena similar to those described in the original paper can be observed, although considering the high numerical instability we cannot determine if this is a real property of the underlying equation or a property of the simulation.

2.3 A Newton method

After we succeeded in replicating the numerical results in [5], we moved on to explore numerically the possibility of periodic solutions in (1.8). This process can be simplified by considering *soliton* solutions, that is, constant shape solutions. This allows us to reduce the problem to functions $f \in L^2(\mathbb{T})$, represent them by their Fourier transforms, and apply Newton's method.

2.3.1 Soliton solutions

For the context of this thesis, we define soliton solutions as follows:

Definition 2.6 (soliton)**.** A map $\varphi : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ is called a *soliton* if there exist $f : \mathbb{T} \to \mathbb{R}$, $a \in \mathbb{R}$ such that

$$\varphi(x,t) = f(x + at).$$

Assuming the solution is a soliton, Equation (1.8) can be written as

$$af' + \frac{1}{2}\partial_x \left\{ f^2 \log|\partial_x| f'' - f \log|\partial_x|(f^2)'' + \frac{1}{3}\log|\partial_x|(f^3)'' \right\} = 2\log|\partial_x| f', \qquad f : \mathbb{T} \to \mathbb{R}, \quad a \in \mathbb{R}.$$

Similarly to Equation (2.3), this can be written in operator form:

$$\boxed{\begin{aligned} &F_a f = 0, \\ &F_a f := af' - 2\log|\partial_x| f' + \frac{1}{2}\partial_x \left\{ f^2 \log|\partial_x| f'' - f \log|\partial_x|(f^2)'' + \frac{1}{3}\log|\partial_x|(f^3)'' \right\}. \end{aligned}} \qquad \begin{aligned} (2.4\text{a}) \\ (2.4\text{b}) \end{aligned}$$

Definition 2.7 (Hunter-Shu soliton operator)**.** Let $A \subseteq L^2(\mathbb{T})$ be the set of functions $f \in L^2(\mathbb{T})$ such that

$$\exists g \in L^2(\mathbb{T}), \quad g = \frac{1}{2}\partial_x \left\{ f^2 \log|\partial_x| f'' - f \log|\partial_x|(f^2)'' + \frac{1}{3}\log|\partial_x|(f^3)'' \right\},$$

where all terms in the equation must be valid expressions (in particular, f must be in the domain of $\log|\partial_x|\partial_x$). The operator $F : \mathbb{R} \times A \to L^2(\mathbb{T})$ is defined as $F(a, f) := F_a f$, as defined in Equation (2.4b).

Lemma 2.1 (Properties of the Hunter-Shu soliton operator)**.**

1. *Constant functions are a solution to Equation* (2.4)*.*
2. *For each $a \in \mathbb{R}$, the operator $F_a := F(a, \cdot)$ is unbounded.*
3. *The domain of $\log|\partial_x|$ includes $C^1(\mathbb{T})$, and $\|\log|\partial_x| f\| \le \|\partial_x f\|$ for all $f \in C^1(\mathbb{T})$.*
4. *$C^\infty(\mathbb{T}) \subseteq A$.*
5. *A is dense in $L^2(\mathbb{T})$.*

Proof.

1. This is a simple check on Equation (2.4b).
2. Consider the sequence $\|F_a \phi_k\|_{L^2}$, for $k \in \mathbb{N}$:

$$\begin{aligned} F_a \phi_k &= ik(a - 2\log|k|)\phi_k + \frac{1}{2}\partial_x \left\{ \phi_k^2 \log|\partial_x|(\phi_k)'' - \phi_k \log|\partial_x|(\phi_k^2)'' + \frac{1}{3}\log|\partial_x|(\phi_k^3)'' \right\} = \\ &= ik(a - 2\log|k|)\phi_k + \frac{1}{2}\partial_x \left(\frac{4\log 4 - 3\log 3}{2\pi} k^2 \phi_{3k} \right) = \\ &= ik(a - 2\log|k|)\phi_k + i\frac{3(4\log 4 - 3\log 3)}{4\pi} k^3 \phi_{3k}, \end{aligned}$$

and since ϕ_k and ϕ_{3k} are orthogonal with unit norm, this means $\|F_a \phi_k\| \ge |k|^3 C$, where $C := 3(4\log 4 - 3\log 3)/4\pi$.

3. Given $n \in \mathbb{N}$, $\log n < n$, so

$$\|\log|\partial_x|\phi_k\| = \log|k| < |k| = \|\partial_x \phi_k\|, \quad |k| \geq 1,$$

and $\log|\partial_x|\phi_0 = \partial_x \phi_0 = 0$. Therefore, $\|\log|\partial_x|f\| \leq \|\partial_x f\|$ whenever $\partial_x f \in L^2(\mathbb{T})$. Furthermore, C^1 functions are in the local integrable sets L^p_{loc} for all $1 \leq p \leq \infty$, and $\mathbb{T}$ has finite measure, so $C^1(\mathbb{T}) \subset L^p(\mathbb{T})$. In particular, $C^1(\mathbb{T}) \subset L^2(\mathbb{T})$.

4. This is a direct consequence of the previous point.

5. This is a direct consequence of the previous point and the fact that $C^\infty(\mathbb{T})$ is dense in $L^2(\mathbb{T})$.

□

$F_a f$ can further be subdivided into a linear part $L_a f'$ and a nonlinear part Nf:

$$F_a f = L_a f' + Nf,$$
$$L_a f := (a - 2\log|\partial_x|)f,$$
$$Nf := \frac{1}{2}\partial_x \left\{ f^2 \log|\partial_x| f'' - f \log|\partial_x|(f^2)'' + \frac{1}{3}\log|\partial_x|(f^3)'' \right\}.$$

Lemma 2.2 (Properties of L_a and N)**.**

1. *For each $a \in \mathbb{R}$, L_a is an unbounded linear operator.*

2. *Given $n \in \mathbb{N}$, $\ker L_{2\log n} = \langle \phi_{-n}, \phi_{+n} \rangle$. For other values of $a \in \mathbb{R}$, $\ker L_a = \{0\}$.*

3. *N is cubic, in the sense that $N(\lambda f) = \lambda^3 Nf$.*

4. *N is unbounded.*

5. *Both L_a and N are defined in the dense subset $A \subset L^2(\mathbb{T})$.*

Proof.

1. This follows from the fact that $\log|\partial_x|$ is an unbounded linear operator.

2. This can be obtained by direct calculation:

$$L_a f = 0 \qquad \Leftrightarrow \qquad af = 2\log|\partial_x|f \qquad \Leftrightarrow \qquad a\hat{f}_k = 2\log|k|\hat{f}_k \quad \forall |k| \geq 1,$$

so that either $\hat{f}_k = 0$ or $a = 2\log|k|$. In the case $k = 0$ this means $\hat{f}_0 = 0$, independently of $a \in \mathbb{R}$.

3. This follows from the definition of N.

4. The same proof used for the unboundedness of F_a applies in this case.

5. The definition of A implies both these facts.

□

2.3.2 Newton's method: theory

Newton's method was originally designed for univariate functions and is usually taught for multivariate functions, but it can be generalized to Banach spaces. However, this requires a notion of differentiation for functions acting on those spaces – in our case, in the space of operators from $L^2(\mathbb{T})$ to itself. It turns out the notion of a derivative in $\mathbb{R}^n$ extends naturally to this domain:

Definition 2.8 (Fréchet derivative)**.** Let A and B be Banach spaces. Let $U \subseteq A$ be an open set, and let $F : U \to B$. F is said to be *Fréchet differentiable* at $x \in U$ if there exists a bounded linear operator $DF(x) : A \to B$ such that

$$\lim_{h \to 0} \frac{\|F(x+h) - F(x) - DF(x)h\|_B}{\|h\|_A} = 0. \tag{2.5}$$

$DF(x)$ is called the *Fréchet derivative* of F at x. If F is Fréchet differentiable at every point in U, the map $DF : U \to \mathcal{L}(A, B)$ mapping x to $DF(x)$ is called the *Fréchet derivative* of F.

Lemma 2.3 (Properties of the Fréchet derivative)**.**

1. *If it exists, $DF(x)$ is unique. This extends to DF.*
2. *The map $F \mapsto DF$ is linear.*
3. *In the case $A = \mathbb{R}^n$, $B = \mathbb{R}^m$, this corresponds with the usual notion of a multivariate derivative.*

We shall skip the proof, since it is exactly the same covered in any multivariate course. The important point to keep in mind when working with spaces of infinite dimension is that $DF(x)$ must be *bounded*. As described in [10], this allows us to define Newton's method in such spaces:

Definition 2.9 (Newton's method)**.** Let A, B be Banach spaces, $U \subseteq A$ an open set and $F : U \to B$ a Fréchet differentiable map with derivative DF. *Newton's method* consists in calculating the sequence $\{X_n\}_{n=0}^{\infty} \subseteq U$, for arbitrary $X_0 \in U$, and X_{n+1} implicitly defined by

$$DF(X_n)(X_{n+1} - X_n) = -F(X_n). \tag{2.6}$$

If $DF(x)$ is invertible for each $x \in U$, this can be rewritten in explicit form as

$$X_{n+1} = X_n - DF(X_n)^{-1}F(X_n). \tag{2.7}$$

If $DF(x)$ is not generally invertible, a strategy must be provided to find a valid X_{n+1} at each step. A possible strategy is finding the *minimum-norm least-squares* solution, that is, finding the set of approximate solutions that minimize $\|DF(X_n)(X_{n+1} - X_n) + F(X_n)\|^2$ and then choosing the minimum-norm solution among those.

In general, Newton's method is more effective if the initial value X_0 is close to a root. Since we are interested in nontrivial solutions, we may consider bifurcations from a known constant solution. Considering the use of the operators ∂_x and $\log|\partial_x|$ in F_a, we cannot expect $DF(x)$ to be invertible in general. However, as pointed out in section 1.2.2, we might have a better chance inverting those operators if we restrict our solution to the space of zero-mean functions. This strongly suggests investigating bifurcations from $f_0 = 0$. Let us start from this point and try to calculate the Fréchet derivative of F_a. It is worthwhile to introduce the notation

$$[F, G] := FG - GF, \qquad Tf := \log|\partial_x| f''.$$

We follow the same scheme we would use in multivariate calculus: given $f_0, h \in L^2(\mathbb{T})$, we calculate the increment $\Delta F_a(f_0)h = F_a(f_0 + h) - F_a f_0$, and then separate its linear and nonlinear parts. In this case,

$$\Delta F_a(f_0)h = ah' - 2\log|\partial_x|h' + \frac{1}{2}\partial_x\left\{f_0^2 Th + 2f_0[h, Tf_0] - [h, Tf_0^2]\right\} + O(h^2).$$

Thus, the linear part is simply

$$DF_a(f_0)h = ah' - 2\log|\partial_x|h' + \frac{1}{2}\partial_x\left\{f_0^2 Th + 2f_0[h, Tf_0] - [h, Tf_0^2]\right\}, \tag{2.8}$$

and Equation (2.5) is mantained. However, this operator is not bounded for arbitrary f_0, independently of the value of $a \in \mathbb{R}$. Consider the case $f_0 = 0$:

$$DF_a(0)h = ah' - 2\log|\partial_x|h' = L_a h',$$

which we know to be unbounded. This means F_a is *not* Fréchet differentiable in $f_0 = 0$. However, the derivative is invertible in $\langle\phi_{-1}, \phi_0, \phi_{+1}\rangle^\perp$, and its inverse is bounded. Therefore, Equation (2.7) is well-defined, and we can explore the results of Newton's method.

Of course, we cannot choose $X_0 = 0$, since then $\{X_n\}_{n=0}^\infty$ would stay fixed in the trivial solution. By the implicit function theorem, bifurcations can only happen for values of $a \in \mathbb{R}$ such that $DF_a(0)$ is singular. Since we consider zero-mean functions, ∂_x is invertible, and this means L_a must be singular. As discussed in Lemma 2.2, this can only happen if $a = 2\log n$ for some $n \in \mathbb{N}$. Let us choose a fixed $n \in \mathbb{N}$ and consider the possible bifurcation from $a_0 := 2\log n$ as a one-parameter family of functions f_ε for $\varepsilon > 0$, where

$$F_{a_0+\varepsilon} f_\varepsilon = 0.$$

Consider then a series expansion of f_ε by the parameter ε, that is,

$$f_\varepsilon = 0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots = \sum_{n=1}^\infty f_n \varepsilon^n, \qquad f_n : \mathbb{T} \to \mathbb{R} \quad \forall n \in \mathbb{N}.$$

This suggests a first-order approximation in terms of $\varepsilon > 0$ of the form $f_\varepsilon \simeq \varepsilon f_1$. Furthermore, since $F_{a_0+\varepsilon} = F_{a_0} + \varepsilon\partial_x$,

$$\partial_\varepsilon(F_{a_0+\varepsilon} f_\varepsilon) = F_{a_0}\partial_\varepsilon f_\varepsilon + \partial_x f_\varepsilon + \varepsilon\partial_\varepsilon\partial_x f_\varepsilon = F_{a_0}(f_1 + 2\varepsilon f_2 + \cdots) + f'_\varepsilon + \varepsilon(f'_1 + 2\varepsilon f'_2 + \cdots),$$

so

$$\partial_\varepsilon(F_{a_0+\varepsilon} f_\varepsilon)|_{\varepsilon=0} = F_{a_0} f_1 + f'_0 = F_{a_0} f_1,$$

but since $F_{a_0+\varepsilon} f_\varepsilon = 0$, this means $F_{a_0} f_1 = 0$. For f_1 real this implies $f_1 = \alpha\phi_{+n} + \overline{\alpha}\phi_{-n}$, $\alpha \in \mathbb{C}$. This suggests using the Newton method with the set of initial values

$$\hat{f}_k = 0, \quad |k| \neq n, \qquad \hat{f}_{+n} = \varepsilon\alpha, \qquad \hat{f}_{-n} = \varepsilon\overline{\alpha},$$

for a suitable choice of $\alpha \in \mathbb{C}$, $\varepsilon > 0$.

2.3.3 Newton's method: practice

With all of this in mind, we wrote a script in MatLab that simulates the infinite-dimensional Newton's method in a finite-dimensional representation. As we did before, we represented $f \in L^2(\mathbb{T})$ as the $L^2(\mathbb{Z}/N\mathbb{Z})$ element $\mathcal{F}[S_N f]$, with N a suitably large power of two. We calculated $DF_a(X_n)$ as an $N \times N$ complex matrix following Equation (2.8), and solved Equation (2.6) using MatLab's **lsqminnorm** least-squares norm minimizing algorithm.

There are several parameters we had to choose for this: the value $n \in \mathbb{N}$ that determines $a_0 := 2\log n$ and $f_1 \in \langle\phi_{-n}, \phi_{+n}\rangle$, the value $\varepsilon > 0$, and the value $\alpha \in \mathbb{C}$. We expressed α in polar form as $\alpha = Ae^{i\theta}$.

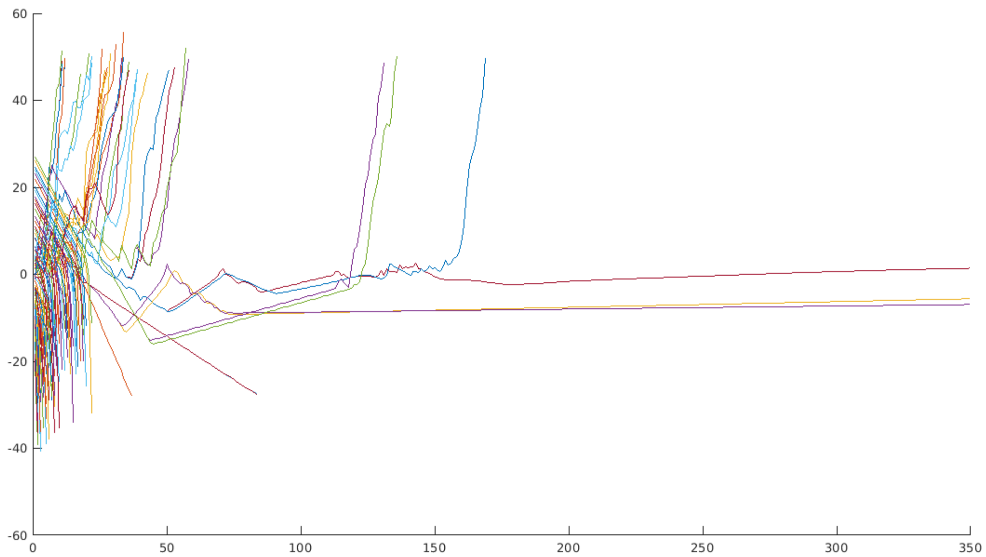

Figure 2.3: l^∞ norm of successive iterations of $\hat{f}$, according to Newton's method. Each line corresponds to a different set of initial values. Horizontal axis: number of iterations. Vertical axis: $\log \|\hat{f}\|_{l^\infty}$.

We set the script to repeat the simulation for each possible combination of values, with each parameter chosen from the following possibilities:

$$\begin{aligned} &n \in \{2, 3, 4, 5, 8, 11\}, \\ &\varepsilon \in \{10^{-3}, 10^{-2}, 10^{-1}, 10^{0}, 10^{1}, 10^{2}\}, \\ &A \in \{10^{-1}, 10^{0}, 10^{1}\}, \\ &\theta \in \{0, \pi/2, \pi, 3\pi/2\}. \end{aligned}$$

The script runs a maximum of 10^3 iterations for each combination of values and uses the maximum absolute value of the stored frequencies ($\|\hat{f}_N\|_{L^\infty(\mathbb{Z}/N\mathbb{Z})}$, in representation of $\|\hat{f}\|_{l^\infty}$) to discard sequences that are considered divergent (the calculated value surpasses 10^{20}) or zero (the calculated value falls under 10^{-7}). It records as successes the results of any combination for which the maximum absolute value stored for $F_a f$, $\|\mathcal{F}[F_a f]\|_{L^\infty(\mathbb{Z}/N\mathbb{Z})}$, becomes smaller than 10^{-12} before any other end conditions are met.

For the first version of this script, the results obtained were negative. However, the theoretical tools described in this chapter suggested a theoretical approach that is discussed in the next chapter. This theoretical approach has been so far inconclusive, but does shed clarity on the subject.

After later revision of the code, some implementation problems were found. This prompted a revision of the script, and yielded the final version described in this section. Figure 2.3 shows the maximum norm of X_n for different sets of initial values. As can be seen, most combinations either quickly converge to the trivial solution $X_n = 0$ or diverge, but a few solutions stay stable. Upon further investigation, some of these functions seem meaningless, but others suggest real solutions – and are remarkably similar in structure to what is described in the next chapter. Figures 2.4, 2.6 and 2.9 showcase some cases that were detected as successful, both promising ones and false positives.

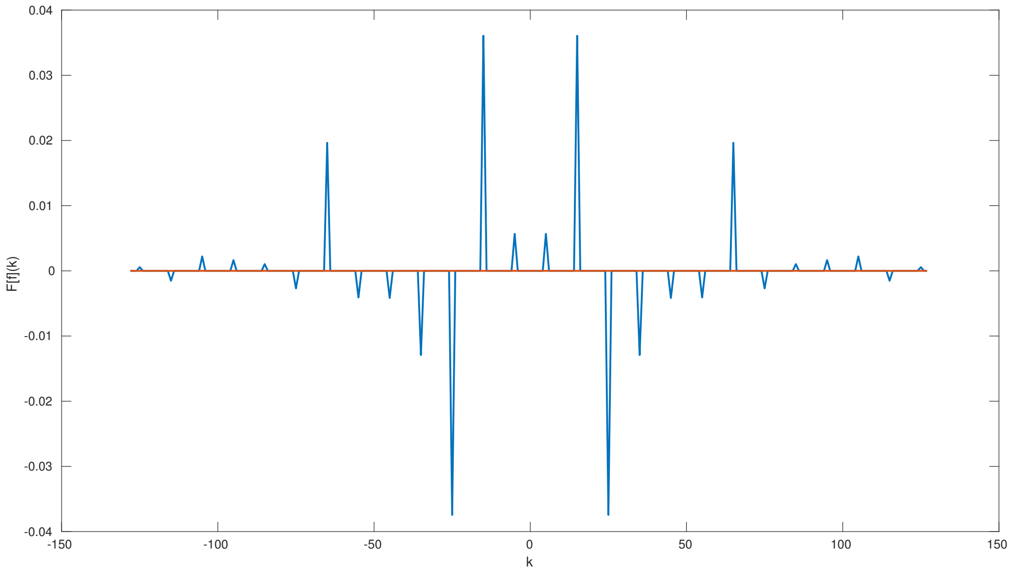

Figure 2.4: Real (blue) and imaginary (orange) parts of $\hat{f}_k$ for the Newton method run with values $n = 5$, $\varepsilon = 10^{-1}$, $A = 10$, $\theta = 0$, which was detected as successful. This is a purely real-valued successful case, showing symmetry with respect to k.

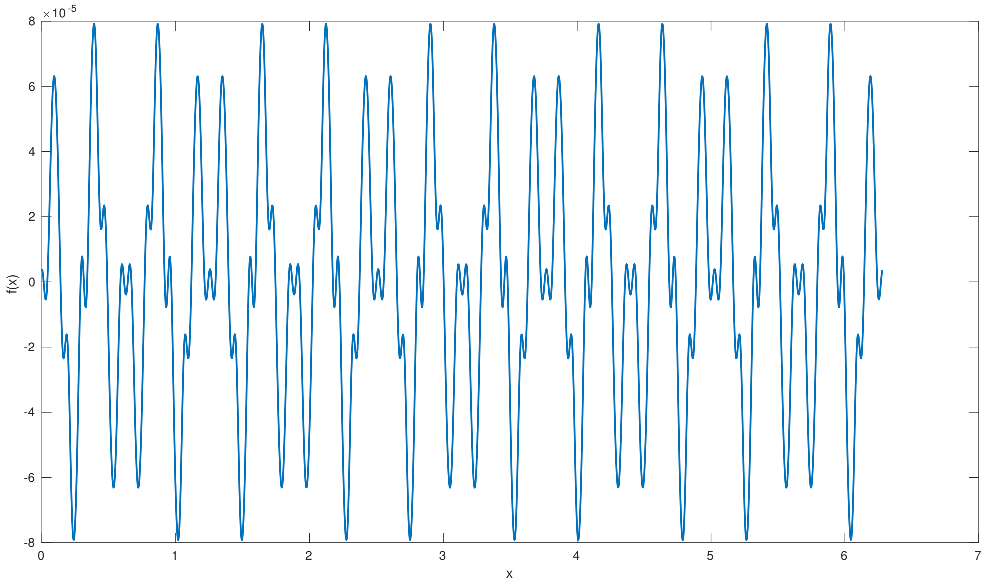

Figure 2.5: Spatial form $f(x)$ for the Newton method run with values $n = 5$, $\varepsilon = 10^{-1}$, $A = 10$, $\theta = 0$.

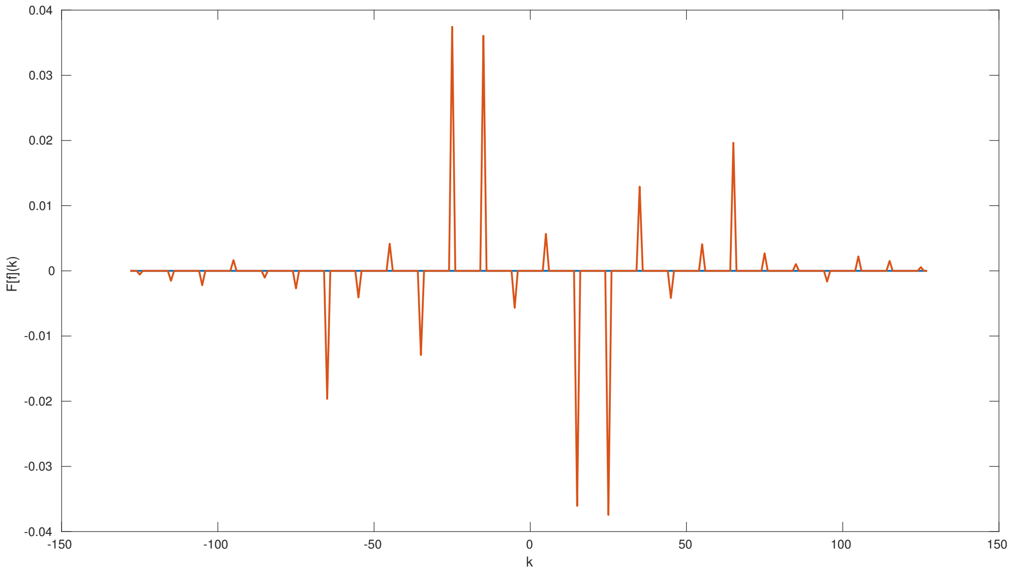

Figure 2.6: Real (blue) and imaginary (orange) parts of $\hat{f}_k$ for the Newton method run with values $n = 5$, $\varepsilon = 10^{-1}$, $A = 10$, $\theta = \pi/2$, which was detected as successful. This is a purely imaginary-valued successful case, showing anti-symmetry with respect to k.

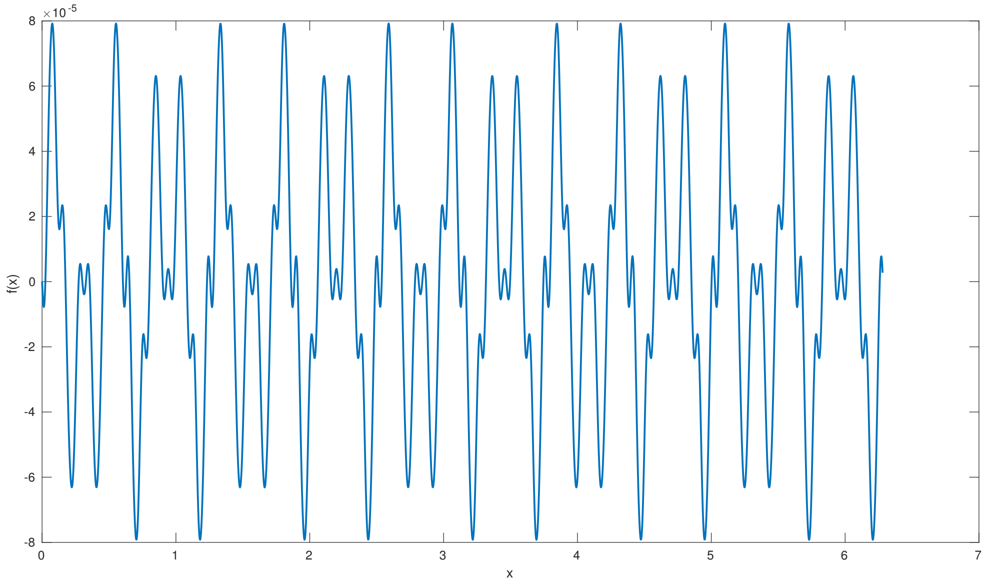

Figure 2.7: Spatial form $f(x)$ for the Newton method run with values $n = 5$, $\varepsilon = 10^{-1}$, $A = 10$, $\theta = \pi/2$.

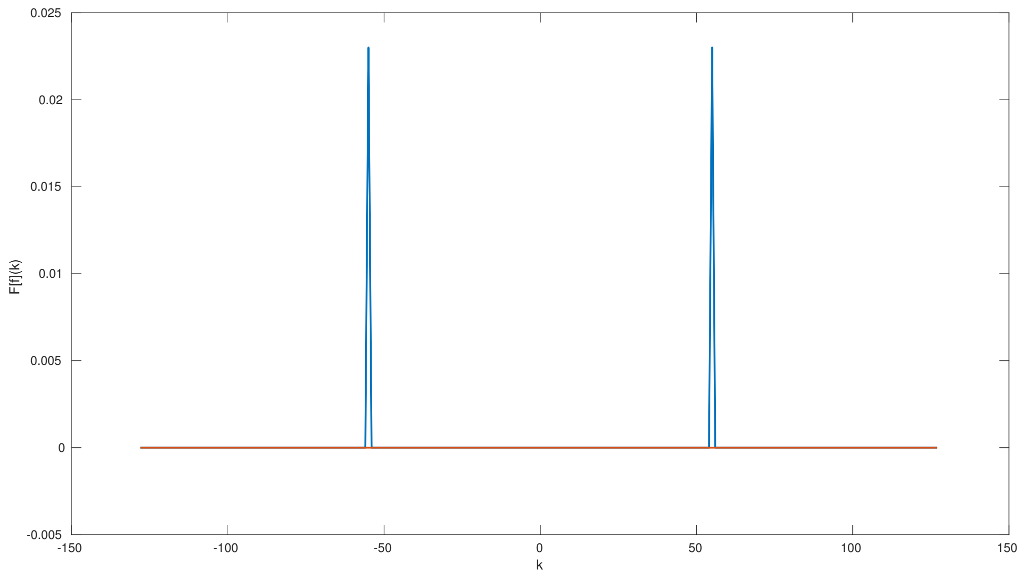

Figure 2.8: Real (blue) and imaginary (orange) parts of $\hat{f}_k$ for the Newton method run with values $n = 11$, $\varepsilon = 10^0$, $A = 10$, $\theta = 0$, which was detected as successful. This is a purely real-valued dubious case. Notice the scale of the peaks is less than 1/40.

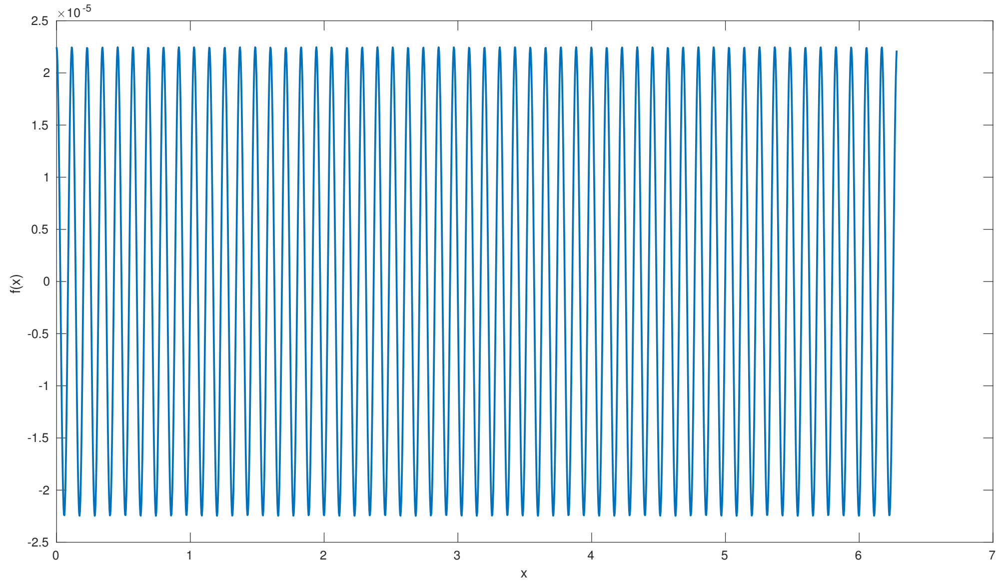

Figure 2.9: Spatial form $f(x)$ for the Newton method run with values $n = 11$, $\varepsilon = 10^0$, $A = 10$, $\theta = 0$.

Chapter 3

A theoretical approach

This chapter contains a theoretical approach to the existence of periodic solutions for the Soliton Equation (2.4). It is heavily inspired by section 2.3 and reuses all the theoretical background that we have exposed so far. Section 3.1 describes how a bifurcation f_ε from the trivial solution $f_0 = 0$, parameterized by $\varepsilon > 0$, can be expanded into a formal series. The terms in this series are defined recursively and provide a formal solution. Section 3.2 explores the convergence of this series using a majorant method for a simplified version of the equation (3.2.1) and for the full case (3.2.2); it is shown that this approach fails.

3.1 An assimptotic expansion

Following the steps of section 2.3, we want to find a one-parameter family of solutions f_ε, with $\varepsilon > 0$ small enough. Let us start from the most evident case, $f_0 = 0$ and $a_0 = 2\log 2$, and this time consider a_ε as a function of ε too. Thus, we are looking for a mapping

$$\varepsilon \mapsto (a_\varepsilon, f_\varepsilon) \in \mathbb{R} \times L^2(\mathbb{T}), \qquad (a_0, f_0) = (2\log 2, 0), \qquad F_{a_\varepsilon} f_\varepsilon = 0.$$

Assuming a_ε and f_ε are regular enough, they can be simultaneously expanded as series on ε:

$$\begin{cases} f_\varepsilon = 0 + \varepsilon f_1 + \varepsilon^2 f_2 + \cdots = \sum_{n=1}^\infty f_n \varepsilon^n, & f_n : \mathbb{T} \to \mathbb{R}, \\ a_\varepsilon = a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots = \sum_{n=0}^\infty a_n \varepsilon^n, & a_n \in \mathbb{R}. \end{cases} \tag{3.1}$$

We are interested in finding f_n that are regular enough and have zero mean, so the operators ∂_x, $\log|\partial_x|$ and their combinations are well defined and are invertible (this would imply $f_n \in L^2(\mathbb{T})$ and $f_n \in \langle \phi_{-1}, \phi_0, \phi_{+1} \rangle^\perp$). In this scenario, Equation (3.1) lets us expand the linear part of F_{a_ε}, $L_{a_\varepsilon} f'_\varepsilon$, as a series on ε, $\{a\}_{n=0}^\infty$ and $\{f'_n\}_{n=1}^\infty$:

$$L_{a_\varepsilon} f'_\varepsilon = (a_\varepsilon - \log|\partial_x|)\partial_x f_\varepsilon = \left(\sum_{n=0}^\infty a_n \varepsilon^n - \log|\partial_x| \right) \partial_x \left(\sum_{n=1}^\infty f_n \varepsilon^n \right) =$$
$$= \sum_{n=1}^\infty \left(\sum_{m=1}^n a_{n-m} f'_m - \log|\partial_x| f'_n \right) \varepsilon^n = \sum_{n=1}^\infty \left(\sum_{m=1}^{n-1} a_{n-m} f'_m + L_{a_0} f'_n \right) \varepsilon^n.$$

A similar calculation can be done on Nf_ε, but it is more involved. For now, let us simply write

$$Nf_\varepsilon = \sum_{n=1}^\infty \varepsilon^n N_n f_\varepsilon,$$

and note that, since N is cubic, $N_n f_\varepsilon$ can be described by $\{f_1, \dots, f_{n-2}\}$. In particular, $N_1 f_\varepsilon = N_2 f_\varepsilon = 0$. Putting it all together, we can write

$$F_{a_\varepsilon} f_\varepsilon = \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n-1} a_{n-m} f_m' + L_{a_0} f_n' + N_n f_\varepsilon \right) \varepsilon^n,$$

and since we expect $F_{a_\varepsilon} f_\varepsilon = 0$ for all values of ε, this implies

$$\boxed{0 = \sum_{m=1}^{n-1} a_{n-m} f_m' + L_{a_0} f_n' + N_n f_\varepsilon, \qquad \forall n \in \mathbb{N}.} \tag{3.2}$$

This set of equations lets us give a formal recursive definition of a_{n-1} and f_n', assuming the values $\{a_0, \dots, a_{n-2}\}$ and $\{f_1, \dots, f_{n-1}\}$ are known. To deduce these relations, we need to consider the kernel $Z := \ker L_{a_0} = \langle \phi_{-2}, \phi_{+2} \rangle$, and the projectors $\mathbb{P}_Z : L^2(\mathbb{T}) \to Z$ and $\mathbb{P}_{Z^\perp} : L^2(\mathbb{T}) \to Z^\perp$:

$$\begin{aligned} \mathbb{P}_Z f &= \sum_{|k|=2} \hat{f}_k \phi_k = \hat{f}_{-2} \phi_{-2} + \hat{f}_{+2} \phi_{+2}, \\ \mathbb{P}_{Z^\perp} f &= \sum_{|k| \neq 2} \hat{f}_k \phi_k. \end{aligned}$$

Since ∂_x, $\log|\partial_x|$, $\mathbb{P}_Z$, $\mathbb{P}_{Z^\perp}$ and their linear combinations (including L_{a_0}) are diagonal operators in the Fourier basis, they commute with each other whenever their joint action is well-defined. With this in mind, the cases $n = 1, 2$ provide initial conditions:

$$\begin{aligned} n &= 1 & &\Rightarrow & L_{a_0} f_1' &= 0 & &\Rightarrow & f_1' &\in Z, \\ n &= 2 & &\Rightarrow & L_{a_0} f_2' &= -a_1 f_1' & &\Rightarrow & \mathbb{P}_{Z^\perp} L_{a_0} f_2' &= 0. \end{aligned}$$

Using commutation, $L_{a_0} \mathbb{P}_{Z^\perp} f_2' = 0$. Separating f_2',

$$L_{a_0} f_2' = L_{a_0} (\mathbb{P}_Z f_2' + \mathbb{P}_{Z^\perp} f_2') = 0 + L_{a_0} \mathbb{P}_{Z^\perp} f_2' = 0,$$

so $f_2' \in Z$ and $a_1 = 0$. A priori there are no restrictions on the precise values of f_1' and f_2' within Z, but we will treat both cases differently: let us choose f_1' freely as any real-valued function, so that $f_1' = \overline{\alpha} \phi_{-2} + \alpha \phi_{+2}$ for some nonzero $\alpha \in \mathbb{C}^*$, and let us take $f_2' = 0$. Assuming f_n to have zero mean, this uniquely determines f_1 and f_2 as

$$f_1 = \partial_x^{-1} f_1' = \frac{i}{2} \overline{\alpha} \phi_{-2} - \frac{i}{2} \alpha \phi_{+2}, \qquad f_2 = \partial_x^{-1} f_2' = 0.$$

The fact that $\mathbb{P}_Z f_n'$ has been undetermined so far turns out to be a recurring pattern which will allow us to obtain formulas for f_n' and a_n. For this, consider the operator $(L_{a_0}|_{Z^\perp})^{-1} : R(L_{a_0}) \to Z^\perp$. This operator has norm $(2[\log 3 - \log 2])^{-1} \simeq 1.23$, so we can consider its extension by continuity, the bounded linear operator $G : Z^\perp \to Z^\perp$. With this tool, we can provide the following result:

Lemma 3.1 (formulas for f_n', a_n)**.** *Let $\alpha \in \mathbb{C}^*$. There are a unique sequence $\{a_n\}_{n=0}^{\infty} \subset \mathbb{C}$ and a unique sequence $\{f_n\}_{n=1}^{\infty} \subset L^2(\mathbb{T}) \cap C^\infty(\mathbb{T})$ of zero mean functions that follow the Equation Set* (3.2)*, and such that*

$$a_0 = 2\log 2, \qquad f_1' = \overline{\alpha} \phi_{-2} + \alpha \phi_{+2}, \qquad f_2 = 0, \qquad f_n \in Z^\perp \quad \forall n \geq 2.$$

Furthermore, each f_n' is determined by

$$\boxed{f_n' = -G \left(\sum_{m=2}^{n-1} a_{n-m} f_m' + \mathbb{P}_{Z^\perp} N_n f_\varepsilon \right), \quad \forall n \geq 3,} \tag{3.3}$$

and each a_n is determined by

$$\boxed{a_n = -\frac{\langle N_{n+1} f_\varepsilon, \phi_2\rangle}{\langle f_1', \phi_2\rangle} \qquad \forall n \in \mathbb{N}.} \tag{3.4}$$

Proof. The cases $n = 1, 2$ for f_n, and $n = 0$ for a_n, are already given by the statement of the lemma. This already provides solutions for the cases $n = 1, 2$ in (3.2), and determines f_1 and f_2 using ∂_x^{-1}. For $n \geq 3$, assume $f_m' \in Z^\perp \cap C^\infty(\mathbb{T})$ for all $2 \leq m < n$ and there are valid $\{a_1, \ldots, a_{n-2}\}$ that solve (3.2). Projecting Equation (3.2) into $Z^\perp$ and using commutation,

$$0 = \sum_{m=1}^{n-1} a_{n-m} \mathbb{P}_{Z^\perp} f_m' + L_{a_0} \mathbb{P}_{Z^\perp} f_n' + \mathbb{P}_{Z^\perp} N_n f_\varepsilon = \sum_{m=2}^{n-1} a_{n-m} f_m' + L_{a_0} \mathbb{P}_{Z^\perp} f_n' + \mathbb{P}_{Z^\perp} N_n f_\varepsilon.$$

This can be rearranged as

$$L_{a_0} \mathbb{P}_{Z^\perp} f_n' = -\sum_{m=2}^{n-1} a_{n-m} f_m' - \mathbb{P}_{Z^\perp} N_n f_\varepsilon,$$

with the right hand side an element of $Z^\perp$. We can then use G to obtain

$$\mathbb{P}_{Z^\perp} f_n' = -G\left(\sum_{m=2}^{n-1} a_{n-m} f_m' + \mathbb{P}_{Z^\perp} N_n f_\varepsilon\right), \quad \forall n \geq 3.$$

Assuming $f_n' \in Z^\perp$ yields Equation (3.3), and gives a unique definition for f_n:

$$f_n := -\partial_x^{-1} G\left(\sum_{m=2}^{n-1} a_{n-m} f_m' + \mathbb{P}_{Z^\perp} N_n f_\varepsilon\right) \in Z^\perp.$$

The assumption $f_m \in C^\infty(\mathbb{T})$ and the fact that both G and ∂_x^{-1} are bounded guarantee that $f_n \in C^\infty(\mathbb{T})$. Projecting Equation (3.2) into Z now yields

$$a_{n-1} f_1' = -\mathbb{P}_Z N_n f_\varepsilon.$$

A priori this does not guarantee the existence of a valid a_{n-1}: we have no guarantee that f_1' and $\mathbb{P}_Z N_n f_\varepsilon$ are proportional. However, both functions are real-valued elements of Z, so they must be of the form $\overline{\beta}\phi_{-2} + \beta\phi_{+2}$. This means they *are* proportional, and a_{n-1} can be defined by the inner product

$$a_{n-1} \langle f_1', \phi_2\rangle = -\langle N_n f_\varepsilon, \phi_2\rangle.$$

Since $\langle f_1', \phi_2\rangle = \alpha \neq 0$, this implies Equation (3.4) and shows the equation is well-defined. □

In other words, we have found a mapping $\alpha \in \mathbb{C} \mapsto (\{a_n\}_{n=0}^\infty, \{f_n\}_{n=1}^\infty)$, such that if the series (3.1) converge, then $\varepsilon \mapsto (a_\varepsilon, f_\varepsilon)$ is a one-parameter family of solutions to $F_{a_\varepsilon} f_\varepsilon = 0$.

3.2 Bounding the sequences

For any particular mapping $\alpha \in \mathbb{C} \mapsto (\{a_n\}_{n=0}^\infty, \{f_n\}_{n=1}^\infty)$ to define a valid converging series, we need the sequences $|a_n|$ and $\|f_n\|$ to grow at most exponentially. This section delves first in a simple model where $N f_\varepsilon$ is substituted by the simplest possible cubic operator, $N f_\varepsilon = f_\varepsilon^3$ (Section 3.2.1), and tackles later the full form of $N f_\varepsilon$ (Section 3.2.2). The method fails in both accounts, subtly in the simple case and dramatically in the full form.

3.2.1 A simpler case

Consider the simplified equation

$$\tilde{F}_a f = 0, \qquad \tilde{F}_a f = L_a f + \tilde{N} f, \qquad \tilde{N} f = f^3,$$

which still has $f_0 = 0$ as a trivial solution and $a = 2\log n$ as the points of singularity for $D\tilde{F}_a$, including $a_0 = 2\log 2$. Assume as before that $\varepsilon \mapsto (a_\varepsilon, f_\varepsilon)$ is a one-parameter family of solutions to $\tilde{F}_{a_\varepsilon} f_\varepsilon = 0$, with $0 \mapsto (2\log 2, 0)$. Further assume that the expansion (3.1) can be performed, and the functions f_n are smooth with zero mean. This allows us to give an explicit form for $\tilde{N}_n f_\varepsilon$:

$$\tilde{N} f_\varepsilon = \left(\sum_{n=1}^{\infty} f_n \varepsilon^n \right)^3 = \sum_{n=3}^{\infty} \sum_{m=1}^{n-2} \sum_{l=1}^{n-m-1} f_m f_l f_{n-m-l} \varepsilon^n,$$

so we can identify

$$\tilde{N} f_\varepsilon = \sum_{n=3}^{\infty} \tilde{N}_n f_\varepsilon, \qquad \tilde{N}_n f_\varepsilon := \sum_{m=1}^{n-2} \sum_{l=1}^{n-m-1} f_m f_l f_{n-m-l} \quad \forall n \geq 3.$$

It is often more convenient to write

$$\tilde{N}_n f = \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma \geq 1}} f_\alpha f_\beta f_\gamma, \qquad \forall n \geq 3. \tag{3.5}$$

Of course, this can be extended to $n \in \mathbb{N}$ by taking $\tilde{N}_1 f_\varepsilon = \tilde{N}_2 f_\varepsilon = 0$. Now, the conclusions from the previous section were all taken without giving an explicit form to $N_n f_\varepsilon$; they just assumed it depends on $\{f_1, \ldots, f_{n-2}\}$, and therefore they still aply. In particular, Lemma 3.1 is valid, so we can take $f_2 = f_2' = 0$ and

$$a_1 = -\frac{\langle \tilde{N}_2 f_\varepsilon, \phi_2 \rangle}{\langle f_1', \phi_2 \rangle} = 0.$$

The tuple of zeros $(a_1 = 0, f_2 = 0, \tilde{N}_2 f_\varepsilon = 0)$ starts a notable sequence:

Lemma 3.2 (zeros of a_n, f_n)**.** *Let $\{a_n\}_{n=0}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ be sequences as in Lemma 3.1, for the simplified problem $\tilde{F}_{a_\varepsilon} f_\varepsilon = 0$. Then*

$$\tilde{N}_{2n} f_\varepsilon = 0, \quad a_{2n-1} = 0, \quad f_{2n} = 0, \qquad \forall n \in \mathbb{N}.$$

Proof. The case $n = 1$ has been already proven in the above discussion. Set now $n \geq 2$ and assume that this is true for all $m < n$.

1. If $\alpha, \beta, \gamma \in \mathbb{N}$ are all odd, then $\alpha + \beta + \gamma$ is odd too. Therefore, if $\alpha + \beta + \gamma = 2n$, at least one of them is an even integer in the range $\{1, \ldots, n-2\}$. Without loss of generality, assume it is α. The induction hypothesis then implies $f_\alpha = 0$, so $f_\alpha f_\beta f_\gamma = 0$. Thus, Equation (3.5) implies $\tilde{N}_{2n} f_\varepsilon = 0$.

2. Equation (3.4) then immediately implies $a_{2n-1} = 0$.

3. Equation (3.3) is reduced by $\tilde{N}_{2n} f_\varepsilon = 0$ to

$$f_{2n}' = -G \left(\sum_{m=2}^{2n-1} a_{2n-m} f_m' \right).$$

   If m is even, the induction hypothesis implies $f_m' = 0$. If m is odd, $2n - m$ is an odd number no bigger than $2n-3$, so the induction hypothesis implies $a_{2n-m} = 0$. In any case, $a_{2n-m} f_m' = 0$, and so $f_{2n}' = 0$. Since f_{2n} is zero-mean, this implies $f_{2n} = 0$.

□

Suppose $\{X_n\}_{n=1}^{\infty}$, $\{Y_n\}_{n=1}^{\infty}$ and $\{Z_n\}_{n=1}^{\infty}$ are sequences of bounds for $\tilde{N}_n f_\varepsilon$, a_n and f_n', respectively:

$$\|\tilde{N}_n f_\varepsilon\| \le X_n, \quad |a_n| \le Y_n, \quad \|f_n'\| \le Z_n \qquad \forall n \in \mathbb{N}. \tag{3.6}$$

We would like to use the existing relations between these sequences to prove the bounds can be chosen with at most an exponential rate of growth. For this goal, let us use some comfort notation:

$$K := \|G\| \simeq 1.23, \qquad A := |\langle f_1', \phi_2\rangle| > 0, \qquad B := \sqrt{2} \simeq 1.44. \tag{3.7}$$

These three constants, with the help of Equations (3.3), (3.4) and (3.5), let us construct a minimal set of bounds by induction:

Lemma 3.3 (bounds X_n, Y_n, Z_n)**.** *Let $\{a_n\}_{n=0}^{\infty}$ and $\{f_n\}_{n=1}^{\infty}$ be sequences as in Lemma 3.1, for the simplified problem $\tilde{F}_{a_\varepsilon} f_\varepsilon = 0$, and let $A, B, K > 0$ be the constants defined in* (3.7)*. There are three sequences of bounds $\{X_n\}_{n=1}^{\infty}, \{Y_n\}_{n=0}^{\infty}, \{Z_n\}_{n=1}^{\infty} \subset \mathbb{R}_{>0}$, which follow the relations* (3.6)*, defined by the initial values*

$$X_1 = 0, \qquad Y_0 = 2\log 2, \qquad Z_1 = AB,$$
$$X_2 = 0, \qquad Y_1 = 0, \qquad Z_2 = 0,$$

and which follow the recurrence formulas

$$X_n = \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma\ge 1}} Z_\alpha Z_\beta Z_\gamma \qquad \forall n \ge 3, \tag{3.8a}$$

$$Y_n = A^{-1} X_{n+1}, \qquad n \in \mathbb{N}, \tag{3.8b}$$

$$Z_n = K\left(\sum_{m=2}^{n-1} A^{-1} X_{n-m+1} Z_m + X_n\right), \qquad \forall n \ge 2. \tag{3.8c}$$

Proof. In Equations (3.8), X_n depends only on $\{Z_1, \dots, Z_{n-2}\}$, and Z_n depends only on $\{Z_1, \dots, Z_{n-1}\}$ and $\{X_1, \dots, X_n\}$. This means both sequences are uniquely determined by their recurrence relation, and $\{Y_n\}_{n=0}^{\infty}$ can be determined from $\{X_n\}_{n=1}^{\infty}$. Thus, we must only prove that each value is a bound of its respective sequence. The cases $n = 1, 2$ are already covered in the statement of the lemma. For $n \ge 3$, assume the bounds hold for $m < n$. Equation (3.5) then implies

$$\|\tilde{N}_n f_\varepsilon\| \le \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma\ge 1}} \|f_\alpha\|\|f_\beta\|\|f_\gamma\| \le \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma\ge 1}} Z_\alpha Z_\beta Z_\gamma,$$

since for zero-mean functions $\|f\| \le \|f'\|$, so X_n is an upper bound of $\|\tilde{N}_n f_\varepsilon\|$. Equation (3.4) then implies

$$|a_n| = \frac{|\langle \tilde{N}_n f_\varepsilon, \phi_2\rangle|}{|\langle f_1', \phi_2\rangle|} = A^{-1}|\langle \tilde{N}_n f_\varepsilon, \phi_2\rangle| \le A^{-1}\|\tilde{N}_n f_\varepsilon\|\|\phi_2\| = A^{-1}\|\tilde{N}_n f_\varepsilon\| \le A^{-1} X_{n+1},$$

due to the Cauchy-Schwartz inequality and the fact that ϕ_2 is a unit vector. Again, this implies Y_n is an upper bound of $|a_n|$. Finally, Equation (3.3) implies

$$\|f_n'\| \le K\left\|\sum_{m=2}^{n-1} a_{n-m} f_m' + \tilde{N}_n f_\varepsilon\right\| \le K\left(\sum_{m=2}^{n-1} |a_{n-m}|\|f_m'\| + \|\tilde{N}_n f_\varepsilon\|\right) \le K\left(\sum_{m=2}^{n-1} Y_{n-m} Z_m + X_n\right),$$

which means Z_n is a bound for $\|f_n'\|$. □

We would like to provide a simple relation between the three sequences of bounds, so that an easy check would suffice. We can clearly ignore Y_n, but X_n and Z_n have a more involved interaction.

Lemma 3.4. *For all $n \in \mathbb{N}$, $Z_n \geq KX_n$, and $X_{n+2} \geq A^2B^2Z_n$. In particular, $X_n \leq Z_n \leq (AB)^{-2}X_{n+2}$, and $X_{2n+3} \geq K^n(AB)^{2n+3}$ for all $n \in \mathbb{N}$.*

Proof.

1. $Z_n \geq KX_n$ is trivially true for $n = 1, 2$. For $n \geq 3$, Equation (3.8c) implies $Z_n \geq KX_n$.
2. Equation (3.8a) implies $X_{n+2} \geq Z_n Z_1^2$, and $Z_1 = AB$.
3. Since $K > 1$, $X_n \leq Z_n \leq (AB)^{-2}X_{n+2}$ is a direct consequence of the two previous points.
4. Using the previous points, $X_5 \geq A^2B^2Z_3 \geq KA^2B^2X_3 \geq KA^4B^4Z_1 = KA^5B^5$. Assuming this is true for $m < n$, the same process yields $X_{2n+3} \geq KA^2B^2X_{2(n-1)+3}$. The result follows by induction.

□

This proves X_n and Z_n have the same rate of growth and grow *at least* exponentially, even though the freedom of choice in A lets us make this a *shrinking* lower bound. If we could find such simple upper bounds we would complete our proof, but that does not seem possible. The sums in Equations (3.8a) and (3.8c) grow in terms as $n \to \infty$, so any naive attempts at upper bounds fail. As an example, assume $Z_m = CD^m$ for all $m < n$, for some constants $C, D \in \mathbb{R}_{>0}$. We would like to find values such that $Z_n = CD^n$, but this is a contradiction:

$$X_n = C^3D^n \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma\geq 1}} 1 = C^3D^n \frac{(n-1)(n-2)}{2},$$

which implies

$$Z_n \geq KX_n = KC^3D^n \frac{(n-1)(n-2)}{2},$$

so if we assume $Z_n = CD^n$ we obtain

$$2C^{-2}K^{-1} \geq (n-1)(n-2),$$

which cannot be true for a fixed $C > 0$ and all $n \in \mathbb{N}$. Therefore, a more delicate approach is needed. Luckily, some patterns can be glimpsed by calculating the first few terms of the bounding sequences:

$$\begin{aligned}
&X_1 = 0, && Z_1 = AB,\\
&X_2 = 0, && Z_2 = 0,\\
&X_3 = Z_1^3 = A^3B^3, && Z_3 = KA^3B^3,\\
&X_4 = 0, && Z_4 = 0,\\
&X_5 = 3Z_1^2Z_3 = 3KA^5B^5, && Z_5 = K^2A^5B^5(B+3),\\
&X_6 = 0, && Z_6 = 0,\\
&X_7 = 3Z_1^2Z_5 + 3Z_1Z_3^2 = 3K^2A^7B^7(B+4), && Z_7 = K^3A^7B^7(B^2+9B+12),\\
&X_8 = 0, && Z_8 = 0,\\
&X_9 = 3Z_1^2Z_7 + 6Z_1Z_3Z_5 + Z_3^3 = K^3A^9B^9(3B^2+33B+55), && Z_9 = K^4A^9B^9(B^3+18B^2+36B+55).
\end{aligned}$$

Here we can clearly observe the sequence of zeros we predicted, and more clear patterns emerge: all bounds have an exponential term $(AB)^n$, a slower exponential term of approximate form $K^{n/2}$, and a polynomial term on B. The polynomial term can be defined as a pair of sequences P_n, Q_n that are defined recursively:

Lemma 3.5. *Let $\{X_n\}_{n=1}^\infty, \{Y_n\}_{n=0}^\infty, \{Z_n\}_{n=1}^\infty \subset \mathbb{R}_{>0}$ be as in Lemma 3.3. Then there exist two sequences $\{P_n\}_{n=1}^\infty, \{Q_n\}_{n=1}^\infty \subset \mathbb{R}_{>0}$, defined by the initial values*

$$P_1 = 0, \quad P_2 = 0, \quad P_3 = 1, \quad Q_1 = 1, \quad Q_2 = 0,$$

and the recurrence relations for $n \geq 3$,

$$P_n = \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma\geq 1}} Q_\alpha Q_\beta Q_\gamma, \tag{3.9a}$$

$$Q_n = P_n + \sum_{m=2}^{n-1} B P_{n-m+1} Q_m. \tag{3.9b}$$

Such that the following equations hold for $n \geq 3$:

$$X_n = P_n K^{(n-3)/2} A^n B^n, \qquad Y_n = P_{n+1} K^{(n-2)/2} A^n B^{n+1}, \qquad Z_n = Q_n K^{(n-1)/2} A^n B^n. \tag{3.10}$$

Proof. As was the case in Lemma 3.3, the recurrence relations and initial values uniquely determine $\{P_n\}_{n=1}^\infty$ and $\{Q_n\}_{n=1}^\infty$, so we only have to prove the equalities (3.10). Again, we can ignore the equality for Y_n, since it is derived from $Y_n = A^{-1} X_{n+1}$. For X_n and Z_n, consider the initial cases

$$\begin{aligned}
n = 1: &\quad P_1 = 0, &\quad X_1 = 0 = P_1 K^{-1} AB, &\quad Q_1 = 1, &\quad Z_1 = AB = Q_1 K^0 AB,\\
n = 2: &\quad P_2 = 0, &\quad X_2 = 0 = P_2 K^{-1/2} A^2 B^2, &\quad Q_2 = 0, &\quad Z_2 = 0 = Q_2 K^{1/2} A^2 B^2,\\
n = 3: &\quad P_3 = 1, &\quad X_3 = A^3 B^3 = P_3 K^0 A^3 B^3, &\quad Q_3 = 1, &\quad Z_3 = K A^3 B^3 = Q_3 K^1 A^3 B^3.
\end{aligned}$$

That is, the equalities (3.10) hold for $1 \leq n \leq 3$, even when it would appear absurd. For $n \geq 4$, assume the equalities (3.10) also hold for $m < n$. Equation (3.8a) then immediately implies Equation (3.9a), and Equation (3.8c) implies Equation (3.9b). □

Lemma 3.6. *Let $\{X_n\}_{n=1}^\infty, \{Y_n\}_{n=0}^\infty, \{Z_n\}_{n=1}^\infty, \{P_n\}_{n=1}^\infty, \{Q_n\}_{n=1}^\infty \subset \mathbb{R}_{>0}$ be as in Lemma 3.5. Then*

1. *$X_{2n} = 0$, $Y_{2n-1} = 0$, $Z_{2n} = 0$, for all $n \in \mathbb{N}$.*
2. *$P_{2n} = 0$, $Q_{2n} = 0$, for all $n \in \mathbb{N}$.*
3. *$P_n \leq Q_n \leq P_{n+2}$, for all $n \in \mathbb{N}$.*

Proof.

1. The same pairing argument as in Lemma 3.2 applies.
2. Same as above.
3. Equation (3.9b) implies $P_n \leq Q_n$, and Equation (3.9a) implies $Q_n = Q_n Q_1^2 \leq P_n$.

□

We have simplified the sequences we work with, eliminated any obvious exponentially growing factors and found an interlocking relation $P_n \leq Q_n \leq P_{n+2}$. However, P_n and Q_n are still defined in terms of growing sums, so naive upper bounds are still doomed to failure. We turn thus to *generating functions* for a solution. Consider the formal power series

$$P(x) = \sum_{n=1}^{\infty} P_n x^n, \qquad Q(x) = \sum_{n=1}^{\infty} Q_n x^n. \tag{3.11}$$

These allow us to convert the recurrence relations (3.9) into algebraic equations.

Lemma 3.7. *Let* $\{P_n\}_{n=1}^{\infty}, \{Q_n\}_{n=1}^{\infty} \subset \mathbb{R}_{>0}$ *be as in Lemma 3.5, and let* P, Q *be as in* (3.11)*. Then*

1. $P = Q^3$.
2. $Q = P(BQ/x - (B-1))$.
3. $BQ^4 - (B-1)Q^3 x - xQ = 0$.

Proof.

1. Using Equation (3.9a),

$$(Q(x))^3 = \left(\sum_{n=1}^{\infty} Q_n x^n\right)^3 = \sum_{n=3}^{\infty} \left(\sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma \geq 1}} Q_\alpha Q_\beta Q_\gamma \right) x^n = \sum_{n=3}^{\infty} P_n x^n = P(x).$$

2. Using Equation (3.9b) and $P_1 = 0$, $Q_1 = 1$,

$$\begin{aligned}
Q(x) = \sum_{n=1}^{\infty} Q_n x^n &= \sum_{n=1}^{\infty} \left(P_n + \sum_{m=2}^{n-1} B P_{n-m+1} Q_m \right) x^n = \\
&= \sum_{n=1}^{\infty} \left((1-B)P_n + \sum_{m=1}^{n-1} B P_{n-m+1} Q_m \right) x^n = \\
&= (1-B)P(x) + B \sum_{n=1}^{\infty} \left(\sum_{m=1}^{n-1} P_{n-m+1} Q_m \right) x^n = \\
&= (1-B)P(x) + B \left(\sum_{n=1}^{\infty} P_{n+1} x^n \right) \left(\sum_{n=1}^{\infty} Q_n x^n \right) = \\
&= (1-B)P(x) + B \frac{P(x)}{x} Q(x).
\end{aligned}$$

3. This follows from the two previous formulas.

□

We have found Q is a solution to a quartic equation. We would like to deduce from this that its series form has a nonzero radius of convergence, and therefore the coefficients Q_n grow at most exponentially. Sadly, the opposite turns out to be true.

Lemma 3.8. *Let* $\{P_n\}_{n=1}^{\infty}, \{Q_n\}_{n=1}^{\infty} \subset \mathbb{R}_{>0}$ *be as in Lemma 3.5. Then* $P_n, Q_n \notin O(e^n)$.

Proof. The cuartic equation $BQ^4 - (B-1)Q^3x - xQ = 0$ admits a trivial solution $Q(x) = 0$, which does not correspond to the formal series $Q(x) = \sum_{n=1}^{\infty} Q_n x^n$, since $Q_1 = 1$, $Q_2 = 0$, and therefore $Q(x) = x + \sum_{n=3}^{\infty} Q_n x^n$, with $Q_n >= 0$ for all $n \in \mathbb{N}$. Removing that, we obtain the cubic equation

$$BQ^3 - (B-1)Q^2x - x = 0. \tag{3.12}$$

If Q converges, it can be differentiated as

$$Q'(x) = \sum_{n=1}^{\infty} nQ_n x^{n-1} = 1 + \sum_{n=3} nQ_n x^{n-1}.$$

This leads to $Q'(0) = 1$, and we can differentiate Equation (3.13).

$$3BQ^2Q' - 2(B-1)QQ'x - (B-1)Q^2 - 1 = 0. \tag{3.13}$$

Since $Q(x) = 0$ and $Q'(0) = 1$, this leads by substitution to $-1 = 0$, a contradiction. Therefore, the assumption that Q converges in a neighborhood of $(Q, x) = (0, 0)$ must be false. This implies Q_n grows at a faster-than-exponential rate. Since $Q_n \leq P_{n+2} \leq Q_{n+2}$ for all $n \in \mathbb{N}$, the same is true for P_n. □

Lemma 3.9. *The bounds $\{X_n\}_{n=1}^{\infty}$, $\{Y_n\}_{n=1}^{\infty}$, $\{Z_n\}_{n=1}^{\infty}$, as defined in Lemma 3.3, are not $O(e^n)$. Therefore, convergence of the series expansion* (3.1) *is not guaranteed using this method.*

Proof. This is a direct consequence of Lemma 3.8 and Equations (3.10). □

The bounds X_n, Y_n, Z_n cannot be made tighter without explicitly calculating terms of each generation. Therefore, this method fails in the simple case.

3.2.2 The full nonlinear term

We have seen the asymptotic expansion method fails in the simple case, since we cannot find bounds that grow slowly enough. This problem turns out to be much more dramatic considering the original equation. In that case,

$$Nf = \frac{1}{2}\partial_x \left\{ f^2 \log|\partial_x| f'' - f \log|\partial_x| (f^2)'' + \frac{1}{3}\log|\partial_x|(f^3)'' \right\}.$$

Again, substituting $f_\varepsilon = \sum_{n=1}^{\infty} f_n \varepsilon^n$ we can find $N_n f_\varepsilon$:

$$f_\varepsilon^2 \log|\partial_x| f_\varepsilon'' = \left(\sum_{n=1}^{\infty} f_n\varepsilon^n\right)^2 \log|\partial_x| \left(\sum_{n=1}^{\infty} f_n\varepsilon^n\right)'' = \sum_{n=1}^{\infty}\left(\sum_{\substack{\alpha+\beta+\gamma=n\\ \alpha,\beta,\gamma\geq 1}} f_\alpha f_\beta \log|\partial_x| f_\gamma''\right)\varepsilon^n,$$

and by similar calculations on $f_\varepsilon \log|\partial_x|(f_\varepsilon^2)''$ and $\log|\partial_x|(f_\varepsilon^3)''$ we arrive at

$$N_n f_\varepsilon = \frac{1}{2}\partial_x \sum_{\substack{\alpha+\beta+\gamma=n\\ \alpha,\beta,\gamma\geq 1}} \left(f_\alpha f_\beta \log|\partial_x| f_\gamma'' - f_\alpha \log|\partial_x|(f_\beta f_\gamma)'' + \frac{1}{3}\log|\partial_x|(f_\alpha f_\beta f_\gamma)'' \right), \quad \forall n \geq 3, \tag{3.14}$$

and again $N_1 f_\varepsilon = 0$, $N_2 f_\varepsilon = 0$. From these results, Lemma 3.2 easily translates to this case:

$$N_{2n} f_\varepsilon = 0, \quad a_{2n-1} = 0, \quad f_{2n} = 0, \qquad \forall n \in \mathbb{N}.$$

We would like to generalize the bounds from Lemma 3.3 into this case. This implies bounding expressions of the form $\log|\partial_x| f''$, which a priori is a contradiction: this is an unbounded action on f. However, this is alleviated by the fact that the functions f_n are trigonometric polynomials of degree at most $2n$. Proving this requires some theoretical work.

Definition 3.1 (trigonometric polynomial)**.** For the purpose of this thesis, a *trigonometric polynomial* of degree $n \in \mathbb{N}$ is any element of $\langle \phi_{-n}, \ldots, \phi_{+n} \rangle \backslash \langle \phi_{-(n-1)}, \ldots, \phi_{+(n-1)} \rangle$, that is, any function of the form

$$f(x) = \sum_{k=-n}^{+n} \alpha_k e^{ikx}, \qquad x \in \mathbb{T}, \quad \alpha_k \in \mathbb{C},$$

with either $\alpha_{-n} \neq 0$ or $\alpha_{+n} \neq 0$.

Lemma 3.10. *Let T_m be a Fourier multiplier with symbol $m : \mathbb{T} \to \mathbb{C}$, and let f be a trigonometric polynomial of degree n. Then $T_m f$ is a trigonometric polynomial of degree $\leq n$.*

Proof. Fourier multipliers are diagonal operators on the Fourier basis, so they map $\langle \phi_{-n}, \ldots, \phi_{+n} \rangle$ to $\langle \phi_{-n}, \ldots, \phi_{+n} \rangle$. In particular, $T_m f$ is a trigonometric polynomial of degree at most n. □

Lemma 3.11. *Let $\{f_n\}_{n=1}^{\infty}$ be as in Lemma 3.1. For each $n \in \mathbb{N}$, f_n is a trigonometric polynomial of degree $\leq 2n$.*

Proof. f_1 is defined as a linear combination of ϕ_{-2} and ϕ_{+2}, so it is a trigonometric polynomial of degree 2. Let now $n \geq 2$ and assume this is true for all $m < n$. Since ∂_x, $\log|\partial_x|$, G and their linear combinations are Fourier multipliers, each term in the sum in Equation (3.14) is a trigonometric polynomial of degree $\leq 2n$, so $N_n f_\varepsilon$ is a trigonometric polynomial of degree $\leq 2n$. A similar logic applies to the operator G and Equation (3.3), so f_n is a trigonometric polynomial of degree $\leq 2n$. □

This means we can bound the action of ∂_x on f_n by $2n$, and we can bound the action of $\log|\partial_x| f_n''$ on f_n by $4n^2 \log(2n)$. This leads to the following global bound for $N_n f_\varepsilon$:

Lemma 3.12. *Let $\{f_n\}_{n=1}^{\infty}$ be as in Lemma 3.1, and $N_n f_\varepsilon$ as in Equation* (3.14)*. The following bounds hold:*

$$\|N_n f_\varepsilon\| \leq \frac{28}{3} n^3 \log(2n) \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma \geq 1}} \|f_\alpha f_\beta f_\gamma\|, \qquad \forall n \in \mathbb{N}.$$

Proof. This is trivially true for $n = 1, 2$, since both coefficients are zero. Suppose now $n \geq 3$. Since $\|\partial_x \phi_k\| = |k|$, the restriction of ∂_x to n-th degree trigonometric polynomials has norm n. Similarly, the restriction of $\log|\partial_x|$ to n-th degree polynomials has norm $\log n$. Since, as we discussed, each term in Equation (3.14) is a $2n$-th degree trigonometric polynomial,

$$\begin{aligned}
\|N_n f_\varepsilon\| &= \left\| \frac{1}{2} \partial_x \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma \geq 1}} \left(f_\alpha f_\beta \log|\partial_x| f_\gamma'' - f_\alpha \log|\partial_x| (f_\beta f_\gamma)'' + \frac{1}{3} \log|\partial_x| (f_\alpha f_\beta f_\gamma)'' \right) \right\| \leq \\
&\leq n \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma \geq 1}} \left\| f_\alpha f_\beta \log|\partial_x| f_\gamma'' - f_\alpha \log|\partial_x| (f_\beta f_\gamma)'' + \frac{1}{3} \log|\partial_x| (f_\alpha f_\beta f_\gamma)'' \right\| \leq \\
&\leq n \cdot (4n^2 \log(2n)) \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma \geq 1}} \left(\|f_\alpha\| \|f_\beta\| \|f_\gamma\| + \|f_\alpha\| \|f_\beta\| \|f_\gamma\| + \frac{1}{3} \|f_\alpha\| \|f_\beta\| \|f_\gamma\| \right) = \\
&= 4n^3 \log(2n) \cdot \frac{7}{3} \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma \geq 1}} \|f_\alpha f_\beta f_\gamma\| = \frac{28}{3} n^3 \log(2n) \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma \geq 1}} \|f_\alpha f_\beta f_\gamma\|.
\end{aligned}$$

□

Again, this allows us to introduce a set of "minimal" bounds for a_n, f_n and $N_n f_\varepsilon$, as we did in Lemma 3.3.

Lemma 3.13. *Let $\{a_n\}_{n=0}^\infty$ and $\{f_n\}_{n=1}^\infty$ be sequences as in Lemma 3.1, for the full problem $F_{a_\varepsilon} f_\varepsilon = 0$, and let $A, B, K > 0$ be the constants defined in (3.7). There are three sequences of bounds $\{X_n\}_{n=1}^\infty$, $\{Y_n\}_{n=0}^\infty$, $\{Z_n\}_{n=1}^\infty \subset \mathbb{R}_{>0}$, defined by the initial values*

$$X_1 = 0, \qquad Y_0 = 2\log 2, \qquad Z_1 = AB,$$
$$X_2 = 0, \qquad Y_1 = 0, \qquad Z_2 = 0,$$

which follow the bounds

$$\|N_n f_\varepsilon\| \le X_n, \quad |a_n| \le Y_n, \quad \|f_n\| \le Z_n,$$

and which follow the recurrence formulas

$$X_n = \frac{28}{3} n^3 \log(2n) \sum_{\substack{\alpha+\beta+\gamma=n \\ \alpha,\beta,\gamma\ge 1}} Z_\alpha Z_\beta Z_\gamma \qquad \forall n \ge 3, \tag{3.15a}$$

$$Y_n = A^{-1} X_{n+1}, \qquad n \in \mathbb{N}, \tag{3.15b}$$

$$Z_n = K\left(\sum_{m=2}^{n-1} A^{-1} X_{n-m+1} Z_m + X_n\right), \qquad \forall n \ge 2. \tag{3.15c}$$

Proof. The proof is essentially the same as for Lemma 3.3. □

Sadly, even this lean set of bounds is not enough to prove convergence in the full case: X_n has order $O(n!)$, independently of the initial condition $A := |\langle f_1', \phi_2\rangle|$. The next lemma proves this:

Lemma 3.14. *Let $\{X_n\}_{n=1}^\infty$ be as in Lemma 3.13. Then,*

$$X_{2n+1} \ge \left(\frac{28}{3}\right)^n (AB)^{2n+1} \prod_{m=1}^n (2m+1)^3 \log(4m+2), \qquad \forall n \in \mathbb{N}. \tag{3.16}$$

In particular, $X_n \ge A^n n!$ for $n \ge 3$ odd. Therefore, $X_n \notin O(e^n)$.

Proof. Since all terms in Equation (3.15c) are positive and $K > 1$, $Z_n \ge X_n$. Furthermore, Equation (3.15a) implies

$$X_n \ge \frac{28}{3} n^3 \log(2n) A^2 B^2 Z_{n-2}, \qquad n \ge 3.$$

For $n = 1$,

$$X_3 \ge \frac{28}{3} 3^3 \log(6) A^3 B^3 = \left(\frac{28}{3}\right)^1 (AB)^3 \prod_{m=1}^1 (2m+1)^3 \log(4m+2).$$

Let $n \ge 2$ and assume inequality (3.16) holds for $m < n$. Then,

$$\begin{aligned} X_{2n+1} &\ge \frac{28}{3}(2n+1)^3 \log(2(2n+1)) A^2 B^2 Z_{2(n-1)+1} \ge \frac{28}{3}(2n+1)^3 \log(2(2n+1)) A^2 B^2 X_{2(n-1)+1} = \\ &= \frac{28}{3}(2n+1)^3 \log(2(2n+1)) \left(\frac{28}{3}\right)^{n-1} (AB)^{2n-1} \prod_{m=1}^{n-1} (2m+1)^3 \log(4m+2) = \\ &= \left(\frac{28}{3}\right)^n (AB)^{2n+1} \prod_{m=1}^n (2m+1)^3 \log(4m+2). \end{aligned}$$

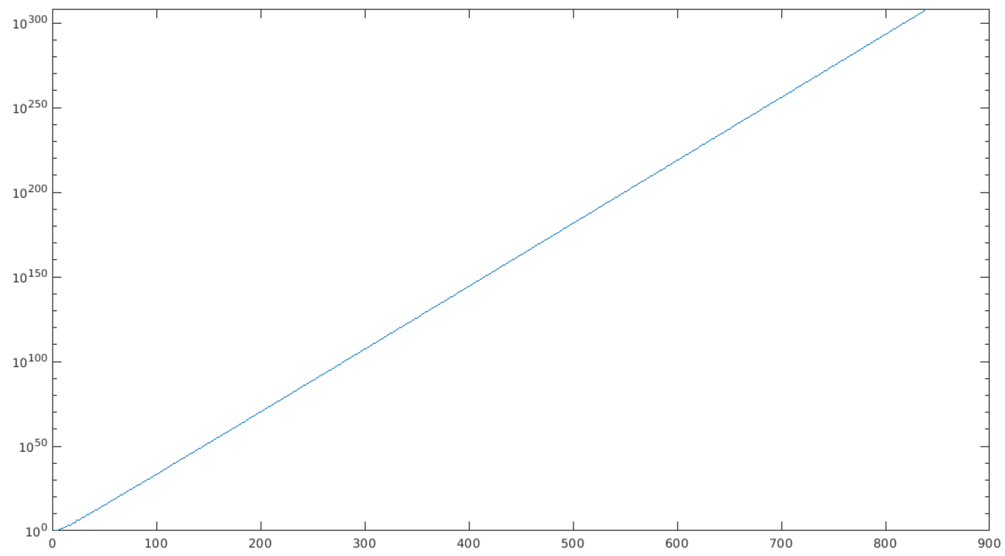

Figure 3.1: Simulated value of the bounds Z_n in the simplified equation $\tilde{F}_{a_\varepsilon} f_\varepsilon = 0$, with initial value $A = 1$, for $n \leq 2^{10}$, in logarithmic scale. Strong suggestion of exponential growth. The simulation overflows before $n = 900$.

In particular, $(2m+1)^3 \geq (2m+1)^2 \geq 2m(2m+1)$, so $\prod_{m=1}^{n}(2m+1)^3 \geq (2n+1)!$. Since all other terms except A^n are guaranteed to be greater than 1, it follows that $X_{2n+1} \geq A^{2n+1}(2n+1)!$, $n \in \mathbb{N}$. This is equivalent to $X_n \geq A^n n!$ for $n \geq 3$ odd, and has a faster-than-exponential rate of growth, independently of the value of $A > 0$. □

As Lemma 3.14 shows, the bounding strategies used for the simple case $\tilde{N}_n f = f^3$ fail in the full equation. No matter our initial choice of $\alpha \in \mathbb{C}^*$, X_n has faster-than-exponential growth, and thus it cannot be used to prove convergence of the series (3.1).

3.3 Numerical simulations

To help in the visualization of $\{f_n\}_{n=1}^{\infty}$ and its bounds $\{Z_n\}_{n=1}^{\infty}$, we wrote some MatLab scripts that calculate the first terms of both sequences, both for the simple and full equations.

3.3.1 Simple case

Figures 3.1, 3.2 and 3.3 show different possible behaviours of the bounds Z_n in the simple case $\tilde{F}_{a_\varepsilon} f_\varepsilon = 0$, depending on the initial value $A := |\langle f_1', \phi_2 \rangle|$. As we would expect from exponential growth, there is a critical value for A such that $Z_n \to \infty$ for greater values of A, and $Z_n \to 0$ for smaller values. However, we have proven this to be misleading, since Z_n has faster-than-exponential growth.

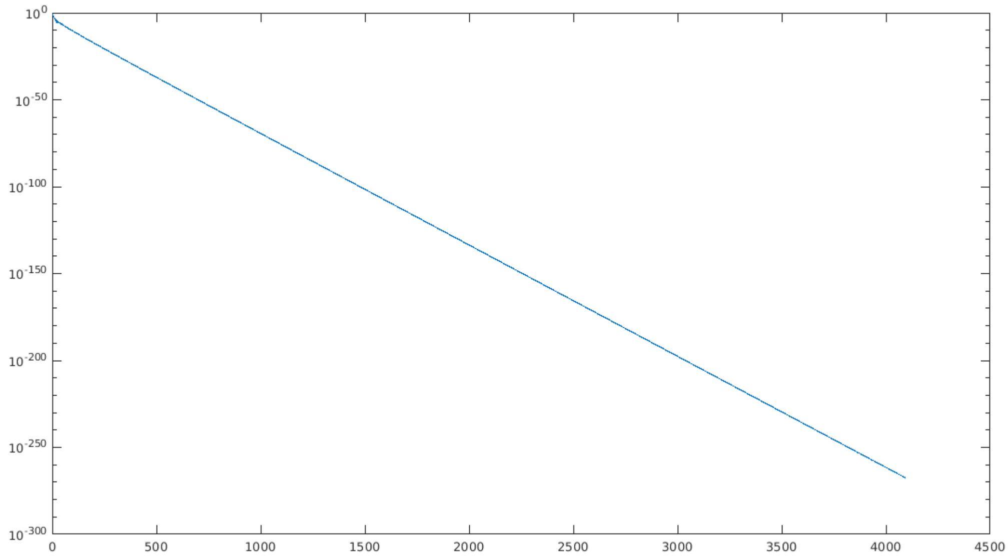

Figure 3.2: Simulated value of the bounds Z_n in the simplified equation $\tilde{F}_{a_\varepsilon} f_\varepsilon = 0$, with initial value $A = 0.01$, for $n \leq 2^{12}$, in logarithmic scale. Strong suggestion of descent into zero.

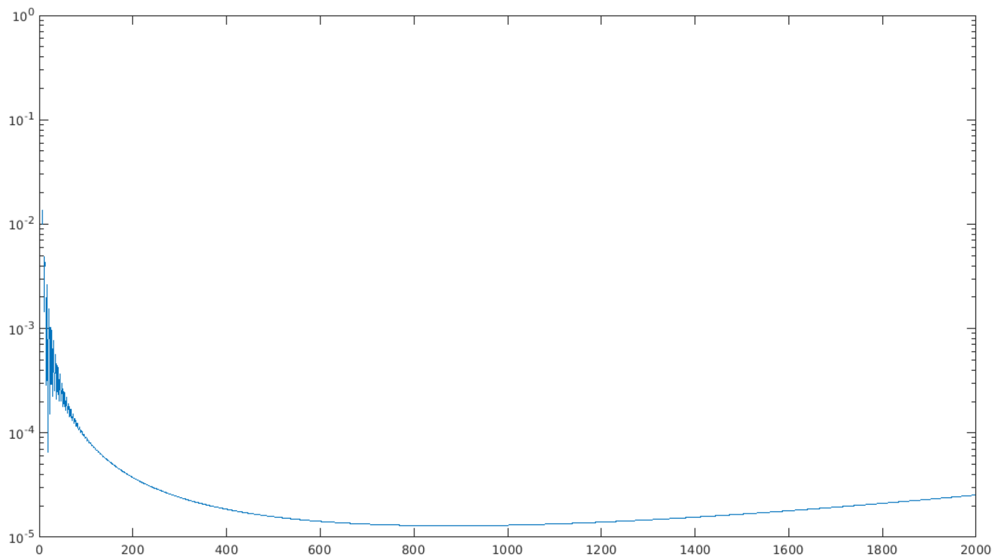

Figure 3.3: Simulated value of the bounds Z_n in the simplified equation $\tilde{F}_{a_\varepsilon} f_\varepsilon = 0$, with initial value $A = 0.1415$, for $n \leq 2000$, in logarithmic scale. A preliminary search suggests there is a turning point for A within 10^{-3} of this value.

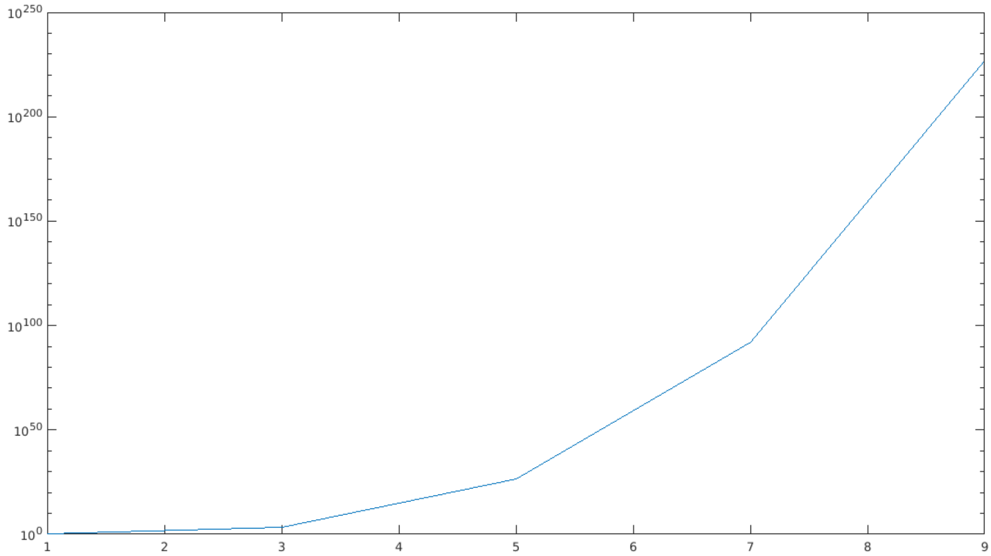

Figure 3.4: Simulated value of the bounds Z_n in the full equation $F_{a_\varepsilon} f_\varepsilon = 0$, with initial value $A = 1$, in logarithmic scale. As we proved, it shows faster-than exponential growth. The simulation overflows at $n = 10$.

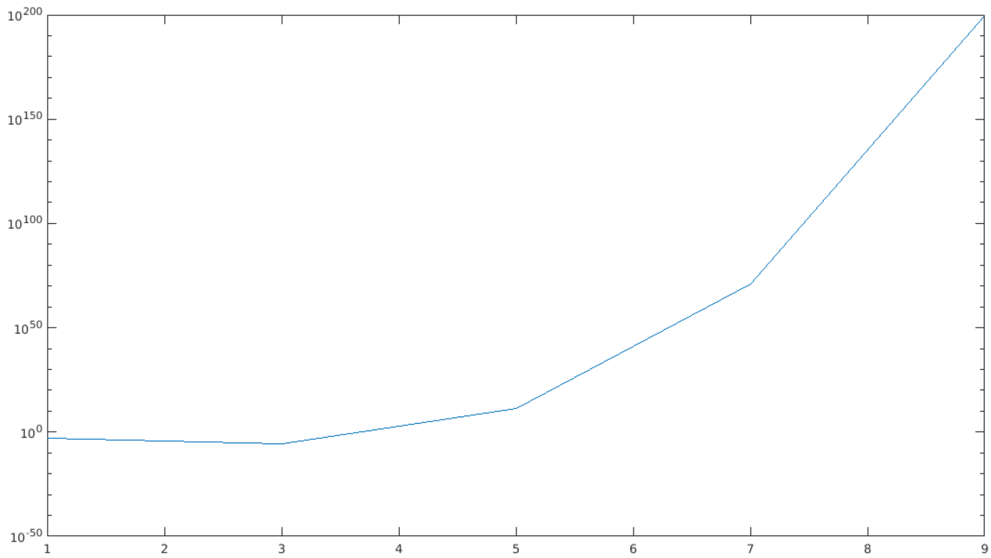

Figure 3.5: Simulated value of the bounds Z_n in the full equation $F_{a_\varepsilon} f_\varepsilon = 0$, with initial value $A = 10^{-3}$, in logarithmic scale. Again, this shows the bounds are insufficient, even for small values of A. The simulation overflows at $n = 10$.

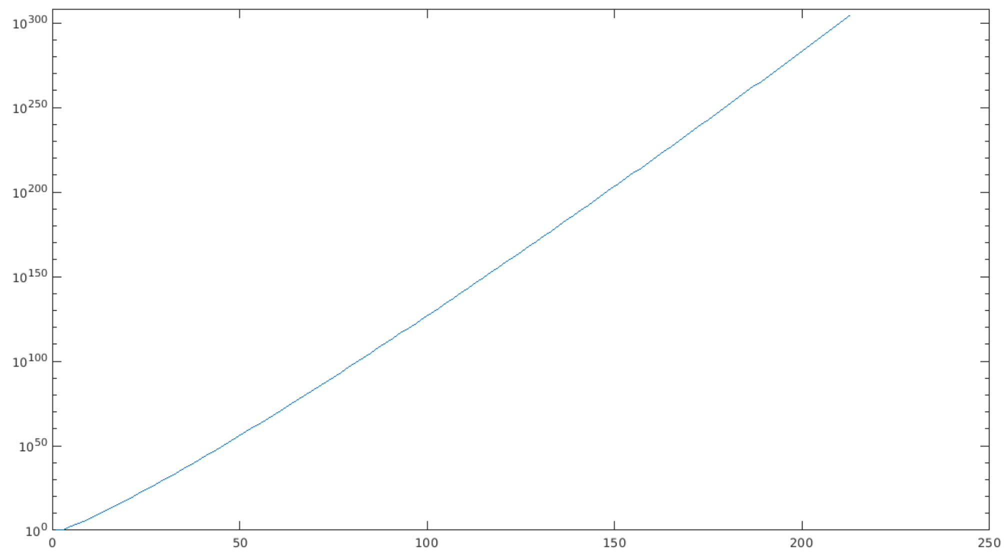

Figure 3.6: $\|\hat{f}_n\|_{l^\infty}$ in the full equation $F_{a_\varepsilon} f_\varepsilon = 0$, with initial value $A = 10$, in logarithmic scale. The rate of growth might be exponential.

3.3.2 Full case

Figures 3.4 and 3.5 show the factorially growing behaviour of Z_n in the full case $F_{a_\varepsilon} f_\varepsilon = 0$, independent of the initial value A. Figures 3.6 and 3.7 show the maximum absolute value of any calculated frequency for each f_n, where the recursive relations (3.3) and (3.4) were used explicitly. This is of course much more computationally intensive, so the number of elements is necessarily much lower.

3.4 Next steps

Since majorant methods fail, a different approach is needed. Biffurcation Theory offers an interesting lead in the form of the Crandall-Rabinowitz Theorem [15], which offers conditions under which a nontrivial bifurcation of $F_a f = 0$ happens. The theorem requires F_a to have a Fréchet differential, which does not happen in the largest possible definition set. However, it can be proven that the restrictions

$$\partial_x, \log|\partial_x| : C^{r+1}(\mathbb{T}) \to C^r(\mathbb{T})$$

are bounded linear operators, from which it follows that the restriction $F_a : C^{r+4}(\mathbb{T}) \to C^r(\mathbb{T})$ is a continuous operator with bounded Fréchet derivative. Proving all conditions for Crandall-Rabinowitz are met would prove the existence of a non-trivial periodic soliton solution to the Hunter-Shu Equation.

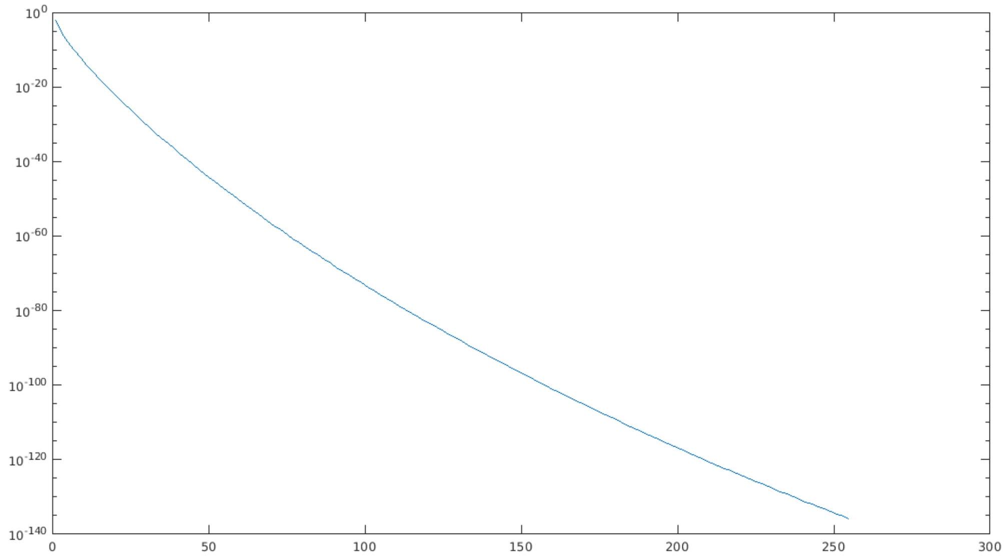

Figure 3.7: $\|\hat{f}_n\|_{l^\infty}$ in the full equation $F_{a_\varepsilon} f_\varepsilon = 0$, with initial value $A = 10^{-2}$, in logarithmic scale. The rate of growth is ambiguous, but clearly more subdued than in Figure 3.5.

Appendix A

Fourier transforms

There are several conventions for the Fourier transform and series, and different spaces it can be applied to. What follows is a brief theoretical background that unifies all these cases, and further derivation into the specific cases that are needed in the main text. The information shown here is mostly derived from [4], where proper proofs are given; this is merely a summary.

A.1 Theoretical background

Let G be a locally compact, Hausdorff abelian topological group (LCA group). Then we can equip it with a well-behaved measure called the *Haar measure*:

Theorem A.1 (Haar measure)**.** *There exists a regular, non-negative, nontrivial measure μ, defined on the Borel sets over G, which is translation invariant. That is, for all $E \in \mathcal{B}(G)$ and $x \in G$,*

$$\mu(x + E) = \mu(E).$$

Such a measure is called a Haar measure *over G. Furthermore, μ is unique up to a multiplicative constant: if μ' is another Haar measure over G, there exists $C > 0$ such that $\mu' = C\mu$.*

The Lebesgue measure over $\mathbb{R}^n$, restricted to $\mathcal{B}(\mathbb{R}^n)$, and the counting measure over $\mathbb{Z}^n$, are Haar measures. The counting measure is also a Haar measure over the finite abelian groups $\mathbb{Z}/n\mathbb{Z}$. In the following, when we write $L^p(G)$ we assume the functions to be complex-valued, and the Haar measure is implied.

Definition A.1 (convolution)**.** Given two functions $f, g : G \to \mathbb{C}$, their *convolution* $f * g$ is defined as

$$(f * g)(x) = \int_G f(x - y)g(y)dy,$$

whenever this integral is defined.

Theorem A.2 (convolution properties)**.**

1. *If $1 \leq p < q \leq \infty$, $1/p + 1/q = 1$, $f \in L^p(G)$ and $g \in L^q(G)$, then $f * g \in L^\infty(G)$.*
2. *If $f, g \in L^1(G)$, then $f * g \in L^1(G)$.*

Definition A.2 (character)**.** A function $\gamma : G \to \mathbb{C}$ is called a *character* if $|\gamma(x)| = 1$ for all $x \in G$ and $\gamma(x + y) = \gamma(x)\gamma(y)$ for all $x, y \in G$.

It follows from the definition that if γ is a character and $x \in G$, then $\gamma(0) = 1$, $\gamma(-x) = \gamma(x)^{-1} = \overline{\gamma(x)}$. We will sometimes write $(x, \gamma) := \gamma(x)$.

Theorem A.3 (dual group definition)**.** *The set Γ of continuous characters over G form a group under the operation $(\gamma_1 + \gamma_2)(x) := \gamma_1(x)\gamma_2(x)$. This is called the* dual group *of G.*

Theorem A.4 (Fourier transform)**.** *Given $f \in L^1(G)$, there is a function $\hat{f} \in L^\infty(\Gamma)$ defined by*

$$\hat{f}(\gamma) = \int_G f(x)(-x, \gamma) d\mu.$$

We call $\hat{f}$ the Fourier transform *of f and we define the* Fourier operator *over $L^1(G)$ as the operator*

$$\begin{array}{rccc} \mathcal{F}: & L^1(G) & \rightarrow & L^\infty(\Gamma) \\ & f & \mapsto & \hat{f}. \end{array}$$

Theorem A.5 (Fourier properties)**.**

1. *$\mathcal{F}$ is a bounded $\mathbb{C}$-linear operator.*
2. *$\mathcal{F}[\overline{f}](\gamma) = \overline{\mathcal{F}[f](-\gamma)}$.*
3. *In particular, if f is real valued and $\hat{f} := \mathcal{F}[f]$, then $\hat{f}$ is complex-conjugate: $\hat{f}(-\gamma) = \overline{\hat{f}(\gamma)}$.*
4. (convolution theorem) *$\mathcal{F}[f * g] = \mathcal{F}[f] \cdot \mathcal{F}[g]$.*

Theorem A.6 (dual group properties)**.**

1. *Γ is an LCA group.*
2. *The dual of Γ is isomorphic to the base group G. We shall identify it with G from this point.*
3. (Fourier inversion theorem) *Given a choice of Haar measure μ for G, there is a unique choice of Haar measure η for Γ such that for any $f \in L^1(G)$ such that $\hat{f} := \mathcal{F}[f] \in L^1(\Gamma)$, the following identity holds:*
$$f(x) = \int_\Gamma \hat{f}(\gamma)(x, \gamma) d\eta.$$
This means that $\mathcal{F}$ is invertible if restricted to $\mathcal{F}^{-1}(L^1(\Gamma))$, and its inverse is equivalent to the above integral.

The inverse of the Fourier transform can be extended by its integral formula to a greater domain:

Theorem A.7 (inverse Fourier transform)**.** *Given $\hat{f} \in L^1(\Gamma)$, there is a function $f \in L^\infty(G)$ defined by*

$$f(x) = \int_\Gamma \hat{f}(\gamma)(x, \gamma) d\eta.$$

We call f the inverse Fourier transform *of $\hat{f}$ and we define the* inverse Fourier operator *over $L^1(\Gamma)$ as the operator*

$$\begin{array}{rccc} \mathcal{F}^{-1}: & L^1(\Gamma) & \rightarrow & L^\infty(G) \\ & \hat{f} & \mapsto & f. \end{array}$$

Furthermore, the inverse Fourier operator is a bounded linear operator.

A.2 Application to common cases

In this section we shall see what specific formulas and relations derive from the previous theoretical section. The choice of a specific Haar measure affects how the normalization constants are spread between the forward and backward Fourier transforms; we will choose to spread them equally for reasons that will become apparent when we discuss $L^2(\mathbb{T})$.

Theorem A.8.

1. *The Haar measure in $\mathbb{R}^n$ is the Lebesgue measure. Its dual space is $\mathbb{R}^n$. The forward Fourier transform for $f \in L^1(\mathbb{R}^n)$ is*

$$\mathcal{F}[f](\mathbf{k}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} f(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}, \qquad \forall \mathbf{k} \in \mathbb{R}^n,$$

and the inverse Fourier transform for $\hat{f} \in L^1(\mathbb{R}^n)$ is

$$\mathcal{F}^{-1}[\hat{f}](\mathbf{x}) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} \hat{f}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{k}, \qquad \forall \mathbf{x} \in \mathbb{R}^n.$$

2. *The Haar measure in $\mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}$ is the Lebesgue measure. Its dual space is $\mathbb{Z}$, whose Haar measure is the counting measure. The forward Fourier transform for $f \in L^1(\mathbb{T})$ is*

$$\mathcal{F}[f](k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{T}} f(x) e^{-ikx} dx, \qquad \forall k \in \mathbb{Z},$$

and the inverse Fourier transform for $\hat{f} \in L^1(\mathbb{Z})$ is

$$\mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \hat{f}(k) e^{ikx}, \qquad \forall x \in \mathbb{T}.$$

3. *The Haar measure in $\mathbb{Z}/n\mathbb{Z}$ is the counting measure. Its dual space is $\mathbb{Z}/n\mathbb{Z}$. The forward Fourier transform for $f \in L^1(\mathbb{Z}/n\mathbb{Z})$ is*

$$\mathcal{F}[f](k) = \frac{1}{\sqrt{n}} \sum_{m\in\mathbb{Z}/n\mathbb{Z}} f(m) e^{-2\pi i \frac{km}{n}}, \qquad \forall k \in \mathbb{Z}/n\mathbb{Z},$$

and the inverse Fourier transform for $\hat{f} \in L^1(\mathbb{Z}/n\mathbb{Z})$ is

$$\mathcal{F}^{-1}[\hat{f}](m) = \frac{1}{\sqrt{n}} \sum_{k\in\mathbb{Z}/n\mathbb{Z}} \hat{f}(k) e^{2\pi i \frac{km}{n}}, \qquad \forall m \in \mathbb{Z}/n\mathbb{Z}.$$

In the equations for $\mathbb{Z}/n\mathbb{Z}$, we take $e^{2\pi i \frac{km}{n}}$ and $e^{-2\pi i \frac{km}{n}}$ to act upon any representatives of $k, m \in \mathbb{Z}/n\mathbb{Z}$. Since e^{ix} is 2π-periodic on x, the value is independent of the choice of representative. Furthermore, all finite valued functions are in $L^p(\mathbb{Z}/n\mathbb{Z})$ for all $1 \leq p \leq \infty$, since it is a finite set of finite measure. In particular, the Fourier transform and inverse Fourier transform are defined over all functions in $\mathbb{Z}/n\mathbb{Z}$ and are mutual inverses in the usual operator sense.

A.3 The Hilbert space $L^2(\mathbb{T})$

Since $\mathbb{T}$ has finite Haar measure, $L^2(\mathbb{T}) \subset L^1(\mathbb{T})$ and so the Fourier transform is defined in $L^2(\mathbb{T})$. Furthermore, $L^2(\mathbb{T})$ is a Hilbert space, and admits a countable basis. This all comes together with the *Fourier basis*:

Definition A.3 (Fourier basis)**.** The *Fourier basis* of $L^2(\mathbb{T})$ is the sequence $\{\phi_k\}_{k\in\mathbb{Z}} \subset L^2(\mathbb{T})$ defined by

$$\phi_k(x) = \frac{1}{\sqrt{2\pi}} e^{ikx}.$$

Theorem A.9 (Fourier basis properties)**.**

1. *The Fourier basis is an orthonormal basis for $L^2(\mathbb{T})$.*
2. *$\langle f, \phi_k \rangle = \mathcal{F}[f](k)$, for all $f \in L^2(\mathbb{T})$.*
3. *$\mathcal{F}[f] \in L^2(\mathbb{Z})$ for all $f \in L^2(\mathbb{T})$.*
4. *$\mathcal{F}^{-1}[\hat{f}] \in L^2(\mathbb{T})$ for all $\hat{f} \in L^2(\mathbb{Z})$.*
5. *The Fourier inversion theorem holds: $\mathcal{F}^{-1}(\mathcal{F}[f]) = f$ for all $f \in L^2(\mathbb{T})$.*

For commodity, given $f \in L^2(\mathbb{T})$ we write $\hat{f}_k := \mathcal{F}[f](k)$. A characteristic property of the Fourier transform of smooth functions in $L^2(\mathbb{T})$ is *rapid decay*: the sequence $\{f_k\}_{k\in\mathbb{Z}}$ goes exponentially fast to zero as $|k| \to \infty$. To be more exact,

Definition A.4 (rapid decay)**.**

1. Let $\{a_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a sequence of complex numbers. $\{a_n\}_{n\in\mathbb{N}}$ is said to have *rapid decay* if

$$\sup_{n\in\mathbb{N}} |a_n| n^N < \infty, \qquad \forall N \in \mathbb{N}.$$

2. By extension, let $\{b_k\}_{k\in\mathbb{Z}} \subset \mathbb{C}$ be a sequence of complex numbers indexed by the integers. $\{b_k\}_{k\in\mathbb{Z}}$ is said to have *rapid decay* if

$$\sup_{k\in\mathbb{Z}} |a_n| |k|^N < \infty, \qquad \forall N \in \mathbb{N}.$$

Lemma A.10 (smooth functions and rapid decay)**.** *Let $f \in C^\infty(\mathbb{T})$. Then $\{\hat{f}_k\}_{k\in\mathbb{Z}}$ has rapid decay. Conversely, if $\{\hat{f}_k\}_{k\in\mathbb{Z}}$ has rapid decay, then $f \in C^\infty(\mathbb{T})$ almost everywhere.*

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