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The Laplacian in its different guises

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Introduction

The Laplacian

$$\Delta u := \nabla \cdot \nabla u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$$

is a scalar operator which gives the divergence of the gradient of a scalar field. It is prevalent in the famous Dirichlet problem, whose importance cannot be overstated. It entails finding the solution u to the problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega. \end{cases}$$

The Dirichlet problem is of fundamental importance in mathematics and its applications, and the efforts to solve the problem has led to many revolutionary ideas and important advances in mathematics.

Harmonic functions in the complex plane give rise to conformal maps, which are important in this context as they are used to map the domain of the Dirichlet problem into the unit circle in order to solve it there, and then map it back without loosing the validity of the solutions.

The Dirichlet problem also models stochastic processes. This is seen by discretizing a planar domain into an ϵ-grid and considering movement up, down, left, and right along the grid. Assuming that the movement is chosen at random in each step with a boundary $\partial\Omega$ such that it is constituted by the disjoint sets Γ_1 and Γ_2 we obtain the Dirichlet problem

$$-\Delta u = 0, \quad u = 1 \text{ on } \Gamma_1, \quad u = 0 \text{ on } \Gamma_2,$$

where u describes the probability of hitting Γ_1 the first time the boundary $\partial\Omega$ is hit.

Another way of viewing the Laplacian is as the Euler-Lagrange equation to the Dirichlet integral

$$D(u) = \int_\Omega |\nabla u|^2 dx$$

which means that the solutions to the Laplacian are the minimizers of the Dirichlet energy.

The p-Laplacian, and, in the limit, the ∞-Laplacian, are generalizations of the Laplacian, which, in the case of the p-Laplacian, is given by

$$\Delta_p u := \operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0,$$

where we see that indeed Δ_2 equals Δ, and Δ_∞ comes by letting $p \to \infty$. This gives us

$$\Delta_\infty u = \frac{1}{|\nabla u|^2} \sum_{i,j=1}^{n} u_{x_i} u_{x_j} u_{x_i x_j}$$

as will be shown later. In the case of the p-Laplacian in the complex plane we obtain quasiconformal maps, which have a bounded distortion from their conformal counterparts. Instead of a random walk a game theoretic connection with or without drift is deduced.

Chapter 1

$p = 2$: The Laplacian

1.1 In the Dirichlet problem

As stated in the introduction, the Laplacian is defined as

$$\Delta u := \nabla \cdot \nabla u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

We will now use this to solve a version of the so called Dirichlet problem. We follow [5] here.

How would one try to solve the problem where it is given that $\Delta u = 0$ inside a rectangle with side lengths a and b, and that $u(0, x) = f_1(x)$ and $u(x, b) = f_2(x)$ on the horisontal sides, and $u(0, y) = g_1(x)$ and $u(a, y) = g_2(x)$ on the vertical sides of that rectangle? Such a problem is called a Dirichlet problem on a rectangular domain. Here, we follow the discussion in [5].

The answer is to use separation of variables. This requires homogeneous boundary conditions, which we can obtain by breaking down the domain into parts A, B, C, and D which are homogeneous enough, and then using the superposition principle to obtain a solution for the entire domain. The latter uses the fact that equation is linear. Here, the domains would be sufficiently homogeneous if at most one side was inhomogeneous as will be seen later.

Part A in this problem formulation is that we have $\Delta u = 0$ in the subdomain, and that we have $u(0, y) = u(x, b) = u(a, y) = 0$ and $u(x, 0) = f_1(x)$ on the boundary. The method for part C is similar.

We use that the separation of variables $u(x, y) = X(x)Y(y)$ gives us

$$\Delta u = u_{xx} + u_{yy} = X''(x)Y(y) + X(x)Y''(y) = 0,$$

and

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \pm\lambda^2$$

by rearrangement. This is then split into the cases of λ^2 and $-\lambda^2$. For λ^2 we have that

$$\begin{cases} X'' + \lambda^2 X = 0 \\ Y'' - \lambda^2 Y = 0 \end{cases}$$

which implies that

$$\begin{cases} X = A\cos\lambda x + B\sin\lambda x \\ Y = C\cosh\lambda y + D\sinh\lambda y. \end{cases}$$

This is useful for parts A and C. For $-\lambda^2$ we have that

$$\begin{cases} X'' - \lambda^2 X = 0 \\ Y'' + \lambda^2 Y = 0 \end{cases}$$

which implies that

$$\begin{cases} X = A\cosh\lambda x + B\sinh\lambda x \\ Y = C\cos\lambda y + D\sin\lambda y, \end{cases}$$

which is useful for parts B and D.

Here, $X(0) = 0 \Rightarrow A = 0$, and $X(a) = 0 \Rightarrow B\sin\lambda a = 0$, which means that $X_n(x) = \sin\lambda_n x$ where $\lambda_n = \frac{n\pi}{a}$ for $n = 1, 2, \dots$, and $u(x, b) = X(x)Y(y) = 0 \Rightarrow Y(b) = 0$, which means that $C\cosh\lambda b + D\sinh\lambda b = 0 \Rightarrow C = -D\tanh\lambda b$. Now, we have that

$$\begin{aligned} Y(y) &= C\cosh\lambda y + D\sinh\lambda y \\ &= -D\tanh\lambda b\cosh\lambda y + D\sinh\lambda y \\ &= D\frac{\cosh\lambda b\sinh\lambda y - \sinh\lambda b\cosh\lambda y}{\cosh\lambda b} \\ &= \frac{D}{\cosh\lambda b}\sinh\lambda(y-b) = E\sinh\lambda(y-b) \end{aligned}$$

where we defined

$$E := \frac{D}{\cosh\lambda b}.$$

We can build in the boundary condition $y(b) = 0$ by letting $Y_n(y) = E\sinh\lambda_n(y-b)$.

We now have that the functions

$$u_n(x, y) = X(x)Y(y) = \sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}(y-b)$$

satisfy the boundary conditions of the simplified problem.

In order to piece these subdomains together and fulfill the boundary condition $u(x, 0) = f_1(x)$ we need to superimpose all of these solutions. Here

$$u(x, y) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a}\sinh\frac{n\pi}{a}(y-b)$$

gives us

$$f_1(x) = u(x, 0) = \sum_{n=1}^{\infty}(-B_n\sinh\frac{n\pi b}{a})\sin\frac{n\pi x}{a}$$

where

$$-B_n\sinh\frac{n\pi b}{a} = \frac{2}{a}\int_0^a f_1(x)\sin\frac{n\pi x}{a}dx.$$

Consequently

$$u(x, y) = \sum_{n=1}^{\infty} B_n\sinh\frac{n\pi}{a}(y-b)\sin\frac{n\pi x}{a}$$

where

$$B_n = -\frac{2}{a \sinh \frac{n\pi b}{a}} \sin_0^a f_1(x) \sin \frac{n\pi x}{a} dx.$$

Part B in this problem formulation is that we have $\Delta u = 0$ in the subdomain, and that we have $u(x, 0) = u(0, y) = u(x, b) = 0$ and $u(b, 0) = g_2(y)$ on the boundary. The method for part D is similar.

We use separation of variables again. Again, $u(x, y) = X(x)Y(y)$ gives us

$$\Delta u = u_{xx} + u_{yy} = X''(x)Y(y) + X(x)Y''(y) = 0,$$

and

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = \pm\lambda^2$$

by rearrangement.

We want the function $Y(y)$ to have the behavior of sines and cosines since we have homogeneous boundary conditions at $y = 0$ and $y = b$. Thus, we choose the constant as λ^2. We have that

$$\begin{cases} X'' - \lambda^2 X = 0 \\ Y'' + \lambda^2 Y = 0 \end{cases}$$

which implies that

$$\begin{cases} X = A \cosh \lambda x + B \sinh \lambda x \\ Y = C \cos \lambda y + D \sin \lambda y, \end{cases}$$

as seen before.

Now, we have that $u(x, 0) = 0$, which implies that $Y(0) = 0$, which gives us that $C = 0$, and $u(x, b) = 0$, which implies that $Y(b) = 0$, which gives us that $B \sin \lambda b = 0$. Consequently, $Y_n(y) = \sin \lambda_n y$, where $\lambda_n = \frac{n\pi}{b}$ for $n = 1, 2, \ldots$, and $u(0, y) = 0$, which implies that $X(0) = 0$. Namely, $A = 0$.

Hence, $X_n(x) = B \sinh \lambda_n x$, which givs us that

$$u_n(x, y) = X(x)Y(y) = \sin \frac{n\pi y}{b} \sinh \frac{n\pi x}{b}$$

satisfies the homogeneous boundary conditions. Again, in order to piece these subdomains together and fulfills the boundary condition $u(a, y) = g_2(y)$ we need to superimpose all of these solutions. Here,

$$u(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{b} \sinh \frac{n\pi y}{b}$$

gives us

$$g_2(x) = u(a, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi a}{b} \sinh \frac{n\pi y}{b},$$

where

$$C_n \sinh \frac{n\pi a}{b} = \frac{2}{b} \int_0^b g_2(x) \sin \frac{n\pi y}{b} dy.$$

Hence

$$C_n = \frac{2}{b \sinh \frac{n\pi a}{b}} \int_0^b g_2(x) \sin \frac{n\pi y}{b} dy$$

and

$$u(x, y) = \sum_{n=1}^{\infty} c_n \sinh \left(\frac{n\pi x}{b}\right) \sin \left(\frac{n\pi y}{b}\right).$$

1.2 In vector calculus

The discussion here comes from [8]. In order to define a number of properties of a scalar field we first have to define what it means to differentiate a vector field. The gradient of a scalar field $f(x, y, z)$ is given by

$$\operatorname{grad} f(x,y,z) = \nabla f(x,y,z) = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k}.$$

The divergence of a vector field is given by

$$\operatorname{div} \boldsymbol{F}(x,y,z) = \nabla \cdot \boldsymbol{F}(x,y,z) = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

where F_1, F_2, and F_3 denote the components of $\boldsymbol{F}(x, y, z)$. Its connection to the Laplacian is given by $\Delta f = \operatorname{div} \operatorname{grad} f$. These notions are easily extended to arbitrary dimensions as considered later on.

An important result for the divergence of a vector field is that the divergence of the vector field inside a domain determines the net flow through its surface. This result is called the divergence theorem, which we will now state and prove as done in [12].

Throughout this thesis we will use Ω to denote a domain, $\partial\Omega$ its surface, and $B_r(x)$ a ball centered at x with radius r, but in the following theorem and its proof we will deviate this convention. We will instead denote of the domain by D and its surface by S, and the volume of D will be denoted by V.

A domain in the plane $\mathbb{R}^2$ which can be bounded by a pair of vertical lines $x = a$ and $x = b$ is called ***y-simple***. The definition of ***x-simple*** is obtained by simply changing places of x and y. A domain which is a union of finitely many non-overlapping subdomains, that are both x-simple and y-simple, is called ***regular***.

The case definitions in $\mathbb{R}^3$ is similar. A domain in $\mathbb{R}^3$ is called x-simple if it is bounded by a piecewise smooth surface S, and if every straight line parallel to the x-axis which passes through an interior point of D meets S at exactly two points. The definitions for y and z-simple are analogous. In $\mathbb{R}^3$ a domain is called regular if it is a union of finitely many, non-overlapping subdomains, which are each $x-$, $y-$, and z-simple.

Theorem 1.1. *Let $D \subset \mathbb{R}^3$ be a regular domain whose boundary S is an oriented and closed surface with unit normal field $\hat{N}$ pointing out of D. If $\boldsymbol{F}$ is a smooth vector field defined on D, then*

$$\iiint_D \operatorname{div} \boldsymbol{F} dV = \oiint_S \boldsymbol{F} \cdot \hat{N} dS \tag{1.1}$$

Proof. It is sufficient to consider a subdomain D which is regular as D itself is regular. This is seen by considering a domain D and surface S which are divided into the parts D_1 and D_2 with surfaces S_1 and S_2 by the surface S^* slicing D in half, which results in a union of abutting domains.

Here, S^* is a part of the boundary of both D_1 and D_2, but their exterior normals $\hat{N}_1$ and $\hat{N}_2$ point in opposite directions of S^*. If (1.5) holds for both subdomains we get

$$\iiint_{D_1} \operatorname{div} \boldsymbol{F} dV = \oiint_{S_1 \cup S^*} \boldsymbol{F} \cdot \hat{N}_1 dS$$
$$\iiint_{D_2} \operatorname{div} \boldsymbol{F} dV = \oiint_{S_2 \cup S^*} \boldsymbol{F} \cdot \hat{N}_2 dS$$

and then, adding them together, we get

$$\begin{aligned}\iiint_D \operatorname{div} \boldsymbol{F} dV = \iiint_{D_1 \cup D_2} \operatorname{div} \boldsymbol{F} dV &= \iiint_{D_1} \operatorname{div} \boldsymbol{F} dV + \iiint_{D_2} \operatorname{div} \boldsymbol{F} dV \\ &= \oiint_{S_1 \cup S^*} \boldsymbol{F} \cdot \hat{N}_1 dS + \oiint_{S_2 \cup S^*} \boldsymbol{F} \cdot \hat{N}_2 dS = \oiint_S \boldsymbol{F} \cdot \hat{N} dS\end{aligned}$$

since the contributions from S^* cancel out because of $\hat{N}_2 = -\hat{N}_1$ on that surface.

Thus, we can from now on assume, without loss of generality, that D is regular. Since it's z-simple we know that it lies between the graph of two functions $f(x,y)$ and $g(x,y)$ defined in a region R in $\mathbb{R}^2$. This means that if (x,y,v) is in D, then (x,y) is in R, and $f(x,y) \leq z \leq g(x,y)$. The third term in the lefthand side equals

$$\iiint_D \frac{\partial F_3}{\partial z} dV = \iint_R dxdy \int_{f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z} dz = \iint_R (F_3(x,y,g(x,y)) - F_3(x,y,f(x,y)))dxdy. \tag{1.2}$$

The third term in the righthand side can be written as

$$\oiint_S F_3(x,y,z)\boldsymbol{k} \cdot \hat{\boldsymbol{N}} dS = \left(\iint_{\text{Top}} + \iint_{\text{Bottom}} + \iint_{\text{Side}} \right) F_3(x,y,z)\boldsymbol{k} \cdot \hat{\boldsymbol{N}} dS.$$

The top and the bottom of the domain are given by $z = g(x,y)$ and $z = f(x,y)$, respectively. Along the sides we have $\boldsymbol{k} \cdot \hat{\boldsymbol{N}} = 0$, which means that its contribution is zero. The top vector area element is given by

$$\hat{\boldsymbol{N}} dS = \left(-\frac{\partial g}{\partial x}\boldsymbol{i} - \frac{\partial g}{\partial y}\boldsymbol{j} + \boldsymbol{k} \right) dxdy$$

as the top is given by $z = g(x,y)$, and the bottom vector area element is given by

$$\hat{\boldsymbol{N}} dS = -\left(-\frac{\partial f}{\partial x}\boldsymbol{i} - \frac{\partial f}{\partial y}\boldsymbol{j} + \boldsymbol{k} \right) dxdy$$

as their normals have opposite orientation. Hence, we have

$$\iint_{\text{Top}} F_3(x,y,z)\boldsymbol{k} \cdot \hat{\boldsymbol{N}} dS = \iint_R F_3(x,y,g(x,y))dxdy$$

and

$$\iint_{\text{Bottom}} F_3(x,y,z)\boldsymbol{k} \cdot \hat{\boldsymbol{N}} dS = -\iint_R F_3(x,y,f(x,y))dxdy.$$

Hence

$$\begin{aligned}
\oiint_S F_3(x,y,z)\boldsymbol{k} \cdot \hat{\boldsymbol{N}} dS &= \left(\iint_{\text{Top}} + \iint_{\text{Bottom}} + \iint_{\text{Side}} \right) F_3(x,y,z)\boldsymbol{k} \cdot \hat{\boldsymbol{N}} dS \\
&= \iint_R F_3(x,y,g(x,y))dxdy - \iint_R F_3(x,y,f(x,y))dxdy \\
&= \iint_R F_3(x,y,g(x,y)) - F_3(x,y,f(x,y))dxdy \\
&= \iint_R dxdy \int_{f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z} dz \\
&= \iiint_D \frac{\partial F_3}{\partial z} dV
\end{aligned}$$

where we used (1.2). Likewise

$$\iiint_D \frac{\partial F_1}{\partial x} dV = \oiint_S F_1 \boldsymbol{i} \cdot \hat{\boldsymbol{N}} dS$$

and

$$\iiint_D \frac{\partial F_2}{\partial y} dV = \oiint_S F_2 \boldsymbol{j} \cdot \hat{\boldsymbol{N}} dS$$

as D is also $x-$ and y-simple. Together, we now have

$$\iiint_D \operatorname{div} \boldsymbol{F} dV = \iiint_D \frac{\partial F_1}{\partial x} dV + \iiint_D \frac{\partial F_2}{\partial y} dV + \iiint_D \frac{\partial F_3}{\partial z} dV = \oiint_S \boldsymbol{F} \cdot \hat{\boldsymbol{N}} dS$$

which is the desired result. □

1.3 As a minimizer

We start off by proving Taylor's theorem as it is a fundamental tool throughout this discussion, and it will appear many times throughout this thesis. In order to do this we will state and prove Rolle's theorem, which in itself requires the extreme value theorem. We follow the discussion of [8].

Theorem 1.2. *If f has a local extremum at c and f is differentiable at c, then $f'(c) = 0$*

Proof. Assume for definitiveness that f has a local maximum at c. Then $-f$ has a local minimum as $f(c) \geq f(x)$ for all $x \in [a,b]\backslash\{c\}$ by definition of a local maximum means that $-f(c) \leq -f(x)$ for all $x \in [a,b]\backslash\{c\}$, which is the definition of a local minimum.

Now

$$\frac{f(x) - f(c)}{x - c} \geq 0$$

if $x \in [a,b]$, and $x < c$, and

$$\frac{f(x) - f(c)}{x - c} \leq 0$$

if $x \in [a,b]$, and $x > c$. This means that the left-hand derivative is greater than or equal to zero, and the right-hand derivative is less than or equal to zero. Thus, it's equal to zero. □

Theorem 1.3. *Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there exists a point $c \in (a,b)$ such that $f'(c) = 0$.*

Proof. The extreme value theorem tells us that there exist $x_m, x_M \in [a,b]$ such that $f(x_m) \leq f(x) \leq f(x_M)$ for all $x \in [a,b]$. If $f(x_m) = f(x_M)$, then f is a constant function, which means that the result follows trivially. If instead $f(x_m) \neq f(x_M)$, then either $x_m \in (a,b)$ or $x_M \in (a,b)$, which means that the result follows from the extreme value theorem as this means that the function has a local extremum on $[a,b]$. □

Now we proceed to state and prove Taylor's theorem.

Theorem 1.4. *Let f be an $n+1$-differentiable function on an open interval containing the points a and x. Then*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \tag{1.3}$$

for some c between a and x.

Proof. We start by assuming that $a < x$ for definitiveness, and defining a function g on $[a,x]$ such that

$$g(t) = \sum_{k=0}^{n} \frac{f^{(k)}(t)}{k!}(x-t)^k + \alpha\frac{(x-t)^{n+1}}{(n+1)!} - f(x), \tag{1.4}$$

where we choose α so that $g(a) = 0$. Clearly, g is continuous on $[a,x]$ and it is also differentiable on (a,x). By Rolle's theorem there exists a $c \in (a,x)$ such that $g'(c) = 0$ since $g(x) = g(a)$, as is seen by inspection. The calculations

$$\frac{d}{dt}\frac{f^{(k)}(t)}{k!}(x-t)^k = \begin{cases} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \frac{f^{(k)}(t)}{(k-1)!}(x-t)^{k-1} & \text{if } k \geq 1 \\ f'(t) & \text{if } k = 0 \end{cases}$$

show that

$$\begin{aligned} g'(t) &= \sum_{k=0}^{n} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \sum_{k=0}^{n-1} \frac{f^{(k+1)}(t)}{k!}(x-t)^k - \alpha\frac{(x-t)^n}{n!} \\ &= \frac{f^{(n+1)}(t)}{n!}(x-t)^n - \alpha\frac{(x-t)^n}{n!}. \end{aligned}$$

In particular, we have

$$g'(c) = \frac{f^{(n+1)}(c)}{n!}(x-c)^n - \alpha\frac{(x-c)^n}{n!} = 0,$$

which gives us that $\alpha = f^{(n+1)}(c)$. Now, (1.4) give us

$$g(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k + \alpha\frac{(x-a)^{n+1}}{(n+1)!} - f(x) = 0,$$

which is (1.3). □

The remainder of this section we show that the Laplacian also is known as the Euler-Lagrange equation for the Dirichlet integral

$$D(u) = \int_\Omega |\nabla u|^2 dx.$$

We roughly follow the line of reasoning of [14]. The reasoning here is analogue to when we want to find where in its domain a function attains its extremum. In that case one checks where the derivative is zero to see where it changes its sign.

Here, one checks the sign of the so called first variation. The so called extremal is a function for which the first variation is zero, similar to how an extremum is a point where the derivative of the function is zero. In the case of finding extrema in the domain of a function we manipulate the function itself. In order to find the functions which give extrema we consider the functional

$$J(y) = \int_{x_0}^{x_1} f(x, y, y')dx. \tag{1.5}$$

The first variation

$$\delta J(\eta, y) = \int_{x_0}^{x_1} \left(\eta\frac{\partial f}{\partial x} + \eta'\frac{\partial f}{\partial y'}\right) dx \tag{1.6}$$

comes from the following computation

$$\begin{aligned}
J(\hat{y}) - J(y) &= \int_{x_0}^{x_1} f(x, \hat{y}, \hat{y}')dx - \int_{x_0}^{x_1} f(x, y, y')dx \\
&= \int_{x_0}^{x_1} \left\{\left(f(x, y, y') + \epsilon\left\{\eta\frac{\partial f}{\partial x} + \eta'\frac{\partial f}{\partial y'}\right\} + O(\epsilon^2)\right) - f(x, y, y')\right\} dx \\
&= \epsilon\int_{x_0}^{x_1} \left(\eta\frac{\partial f}{\partial x} + \eta'\frac{\partial f}{\partial y'}\right) dx + O(\epsilon^2) \\
&= \delta J(\eta, y) + O(\epsilon^2),
\end{aligned}$$

which in turn comes from considering the Taylor expansion

$$\begin{aligned}
f(x, \hat{y}, \hat{y}') &= f(x, y + \epsilon\eta, y' + \epsilon\eta') \\
&= f(x, y, y') + \epsilon\left\{\eta\frac{\partial f}{\partial x} + \eta'\frac{\partial f}{\partial y'}\right\} + O(\epsilon^2)
\end{aligned}$$

of a perturbation $\hat{y} = y + \epsilon\eta$ where $\epsilon > 0$ and $\eta \in H$ for

$$H = \{\eta \in C^2[x_0, x_1] : \eta(x_0) = \eta(x_1) = 0\}.$$

This means that $\hat{y}$ has the same values at its boundary x_0 and x_1 as y does. The choice of y is that it has to attain specified boundary values $y(x_0)$ and $y(x_1)$. This means that $y \in S$, where

$$S = \{y \in C^2[x_0, x_1] : y(x_0) = y_0 \wedge y(x_1) = y_1\}. \tag{1.7}$$

We want to minimize the functional in (1.5). It tells us whether we have a minimum or a maximum by observing the sign of $J(\hat{y}) - J(y)$ for the perturbation of the extremal y. If $J(\hat{y}) - J(y) \geq 0$ for all $\hat{y}$ such that $\|\hat{y} - y\| < \epsilon$, then we have that $J(\hat{y}) \geq J(y)$, which means that it is a minimum. The argument in the case of a maximum is similar. This is completely analogous to the definition of extrema for functions.

We observe that J has local minimum $y \in S$ if and only if it has a local maximum for $-J$. This follows from the simple computation

$$(-J(\hat{y})) - (-J(y)) = -(J(\hat{y}) - J(y)) \leq 0 \Leftrightarrow J(\hat{y}) - J(y) \geq 0.$$

The first variation is the key for swiftly evaluating this quantity. However, it can be simplified using integration by parts as follows

$$\int_{x_0}^{x_1} \eta' \frac{\partial f}{\partial y'} dx = \eta \frac{\partial f}{\partial y'}\Big|_{x_0}^{x_1} - \int_{x_0}^{x_1} \eta \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx = -\int_{x_0}^{x_1} \eta \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) dx,$$

where it is used that $\eta(x_0) = \eta(x_1) = 0$.

Remember that integration by parts simply comes from the product rule. Namely, that

$$\frac{d}{dx}(f(x)g(x)) = f'(x)g(x) + f(x)g'(x),$$

and

$$\int_a^b \frac{d}{dx} f(x)g(x)dx = f(x)g(x)\Big|_a^b = \int_a^b f'(x)g(x)dx + \int_a^b f(x)g'(x)dx \Leftrightarrow$$
$$\int f(x)g'(x)dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)dx$$

where it's required that $f(x)$ and $g(x)$ are differentiable. This means that the first variation as seen in (1.6) can be written as

$$\int_{x_0}^{x_1} \eta \left\{\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)\right\} dx, \tag{1.8}$$

which gives us

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = \frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - \frac{\partial^2 f}{\partial y \partial y'} y' - \frac{\partial^2 f}{\partial y' \partial y''} y''.$$

This means that

$$E(x) = \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right)$$

is continuous for any fixed $y \in C^2[x_0, x_1]$ given that f has at least two continuous derivatives. The analogue of the finite-dimensional case in (1.8) is the inner product condition

$$\langle \eta, E \rangle = \int_{x_0}^{x_1} \eta(x)E(x)dx = 0$$

as E and η are elements of the Hilbert space $L^2[x_0, x_1]$ since $\eta \in H$ and $E(x)$ is continuous on $[x_0, x_1]$. As in the finite-dimensional case, it can be shown that this condition is equivalent with $E = 0$.

Lemma 1.1. *Let α and β be real numbers such that $\alpha < \beta$. Then there exists a function $\nu \in C^2(\mathbb{R})$ such that $\nu(x)$ for all $x \in (\alpha, \beta)$ and $\nu(x) = 0$ for all $x \in \mathbb{R}\backslash(\alpha, \beta)$.*

Proof. Let

$$\nu(x) = \begin{cases} (x-\alpha)^3(\beta - x)^3 & \text{if } x \in (\alpha, \beta) \\ 0 & \text{otherwise.} \end{cases}$$

It should be clear that this function fulfills the criteria, except possibly for continuity of the derivatives at $x = \alpha$ and $x = \beta$. Here

$$\lim_{x \searrow \alpha} \frac{\nu(x) - \nu(\alpha)}{x - \alpha} = \lim_{x \searrow \alpha} \frac{(x-\alpha)^3(\beta - x)^3 - 0}{x - \alpha} = \lim_{x \searrow \alpha} (x-\alpha)^2(\beta - x)^3,$$

and

$$\lim_{x \nearrow \alpha} \frac{\nu(x) - \nu(\alpha)}{x - \alpha} = \lim_{x \nearrow \alpha} \frac{0 - 0}{x - \alpha} = 0.$$

Hence, $\nu'(\alpha) = 0$. Similarly

$$\begin{aligned} \lim_{x \searrow \alpha} \frac{\nu'(x) - \nu'(\alpha)}{x - \alpha} &= \lim_{x \searrow \alpha} \frac{3(x-\alpha)^2(\beta - x)^2(\beta + \alpha - 2x) - 0}{x - \alpha} \\ &= \lim_{x \searrow \alpha} 3(x - \alpha)(\beta - x)^2(\beta + \alpha - 2x) = 0, \end{aligned}$$

and

$$\lim_{x \nearrow \alpha} \frac{\nu'(x) - \nu'(\alpha)}{x - \alpha} = \lim_{x \nearrow \alpha} \frac{0 - 0}{x - \alpha} = 0.$$

Thus, $\nu''(\alpha) = 0$. The argument to show that $\nu''(\beta) = 0$ is similar.

Consequently, we know that the second derivative is

$$\nu''(x) = \begin{cases} 6(x-\alpha)(\beta - x)\{(x-\alpha)^2 + (\beta - x)^2 - 3(x-\alpha)(\beta - x)\} & \text{if } x \in (\alpha, \beta) \\ 0 & \text{otherwise,} \end{cases}$$

and that

$$\lim_{x \to \alpha} \nu''(x) = \nu''(\alpha) = 0$$

and

$$\lim_{x \to \beta} \nu''(x) = \nu''(\beta) = 0,$$

which means that $\nu \in C^2(\mathbb{R})$. ∎

Lemma 1.2. *Suppose that $\langle \eta, g \rangle = 0$ for all $n \in H$. If $g : [x_0, x_1] \to \mathbb{R}$ is a continuous function, then $g = 0$ on the interval $[x_0, x_1]$.*

Proof. Assume towards a contradiction that $g \neq 0$ for some $c \in [x_0, x_1]$. We can assume without loss of generality that $g(c) > 0$, and, consequently, by continuity, that $c \in (x_0, x_1)$. Furthermore, continuity of g on $[x_0, x_1]$ gives us that there are numbers α and β such that $x_0 < \alpha < c < \beta < x_1$ and $g(x) > 0$ for $x \in (\alpha, \beta)$.

Now, Lemma 1.1 implies that there exists a function $\nu \in C^2[x_0, x_1]$ such that $\nu(x) > 0$ for all $x \in (\alpha, \beta)$ and $\nu(x) = 0$ otherwise. Consequently, $\nu \in H$, and

$$\langle \nu, g \rangle = \int_{x_0}^{x_1} \nu(x) g(x) dx = \int_{\alpha}^{\beta} \nu(x) g(x) dx > 0,$$

which contradicts the assumption that $\langle \eta, g \rangle = 0$ for all $\eta \in H$. Thus, $g = 0$ on (x_0, x_1), and, by continuity, on $[x_0, x_1]$. ∎

Now we proceed to the theorem that has been proved throughout this section.

Theorem 1.5. *Let $J : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form*

$$J(y) = \int_{x_0}^{x_1} f(x, y(x), y'(x))dx$$

where f has continuous partial derivatives of second order with respect to x, y and y', $x_0, x_1 \in \mathbb{R}$ such that $x_0 < x_1$, and $y_0, y_1 \in S$ with define as in (1.7). If $y \in S$ is an extremal for J, then

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial y}{\partial y} = 0 \tag{1.9}$$

for all $x \in [x_0, x_1]$.

To see that the Laplacian is the Euler-Lagrange equation of the Dirichlet energy $|y'|^2$ we have that

$$\begin{aligned}\delta J(\eta, y) &= \int_{x_0}^{x_1} \left(\eta \frac{\partial f}{\partial y} + \eta' \frac{\partial f}{\partial y'}\right) dx \\ &= \int_{x_0}^{x_1} \eta' 2|y'| \frac{y'}{|y'|} dx \\ &= 2\int_{x_0}^{x_1} \eta' y' dx.\end{aligned}$$

The n-dimensional analogue with n independent variables is to instead consider minimization of the twice differentiable Lagrangian $L(x, u, \nabla u)$ over a regular bounded domain Ω with a smooth boundary $\partial\Omega$. Recall that a domain is called regular if it is union of finitely many non-overlapping subdomains which are simple in its respective dimensions.

The only difference here is to consider the gradient instead of only one derivative, and a domain instead of an interval. The methods will stay the same, but with slight adjustments to account for these changes.

The problem is to find the minimum of $I(u)$ subject to the boundary conditions $u\Big|_{\partial\Omega} = u_0$ where

$$I(u) = \int_\Omega L(x, u, \nabla u)dx.$$

To derive the Euler-Lagrange equation we consider a variation by δu and the difference

$$\delta I = I(u + \delta u) - I(u).$$

We assume that the perturbation δu lays within an ϵ-neighborhood of the point x for all x, that it is twice differentiable and small. This means that the norm of the gradient goes to zero as $\epsilon \to 0$. Namely

$$\delta u(x + z) = 0,\ \forall z : |z| > \epsilon,\ \text{and}\ \forall x : |\nabla(\delta u)| < C\epsilon.$$

This is important we can linearize the perturbed Lagrangian when the perturbation and its gradient are both infinitesimal and twice differentiable. Here

$$\begin{aligned}L(x, \hat{u}, \nabla\hat{u}) &= L(x, u + \delta u, \nabla(u + \delta u)) \\ &= L(x, u, \nabla) + \frac{\partial L(x, u, \nabla u)}{\partial u}\delta u + \frac{\partial L(x, u, \nabla u)}{\partial \nabla u}\delta\nabla u + o(\|\delta u\|, \|\nabla\delta u\|),\end{aligned}$$

where $\frac{\partial L(x,u,\nabla u)}{\partial \nabla u}$ denotes the vector of the partial derivatives of L with respect to the partial derivatives of u. Namely

$$\frac{\partial L(x, u, \nabla u)}{\partial \nabla u} = \left(\frac{\partial L(x, u, \nabla u)}{\partial\left(\frac{\partial u}{\partial x_1}\right)}, \frac{\partial L(x, u, \nabla u)}{\partial\left(\frac{\partial u}{\partial x_2}\right)}, \ldots, \frac{\partial L(x, u, \nabla u)}{\partial\nabla\left(\frac{\partial u}{\partial x_n}\right)}\right).$$

The little O notation $f(\epsilon) = o(g(\epsilon))$ means that

$$\left|\frac{f(\epsilon)}{g(\epsilon)}\right| < c \tag{1.10}$$

for some constant $c \in \mathbb{R}$. Now we have the expression

$$\begin{aligned}
I(\hat{u}) - I(u) &= I(u + \delta u) - I(u) \\
&= \int_\Omega \left(L + \frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial \nabla u}\delta \nabla u \right) dx + o(\|\delta u\|, \|\nabla \delta u\|) - \int_\Omega L dx \\
&= \int_\Omega \left(\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial \nabla u}\delta \nabla u \right) dx + o(\|\delta u\|, \|\nabla \delta u\|) \\
&= \delta I(u) + o(\|\delta u\|, \|\nabla \delta u\|)
\end{aligned}$$

for the Lagrangian, where $\delta I(u)$ is its version of the first variation. Similarly, we use integration by parts to be rid ourselves of the term $\frac{\partial L}{\partial \nabla u}\delta \nabla u$. Here

$$\int_\Omega \left(\frac{\partial L}{\partial \nabla u}\nabla(\delta u) \right) dx = -\int_\Omega \delta u \left(\nabla \frac{\partial L}{\partial \nabla u} \right) dx + \int_{\partial\Omega} \delta u \left(\frac{\partial L}{\partial \nabla u} n \right) ds,$$

where we used that $\delta \nabla u = \nabla(\delta u)$ due to linearity. Now

$$\begin{aligned}
I(\hat{u}) - I(u) &= \int_\Omega \left(\frac{\partial L}{\partial u}\delta u + \frac{\partial L}{\partial \nabla u}\delta \nabla u \right) dx + o(\|\delta u\|, \|\nabla \delta u\|) \\
&= \int_\Omega \left(\frac{\partial L}{\partial u} - \nabla \frac{\partial L}{\partial \nabla u} \right) \delta u dx + \int_{\partial\Omega} \delta u \left(\frac{\partial L}{\partial \nabla u} n \right) ds + o(\|\delta u\|, \|\nabla \delta u\|).
\end{aligned}$$

The coefficients here are given specific names. The coefficient δu in the first integral is called the ***variational derivative in*** Ω, and is given by

$$S_L(u) = \frac{\partial L}{\partial u} - \nabla \left(\frac{\partial L}{\partial \nabla u} \right).$$

The coefficient by δu in the boundary integral is called the ***variational derivative on the boundary of*** Ω, which is given by

$$S_L^\partial(u, n) = \frac{\partial L}{\partial \nabla u} n.$$

Hence, we can represent $I(\hat{u}) - I(u)$ as

$$I(\hat{u}) - I(u) = \int_\Omega S_L(u)\delta u dx + \int_{\partial\Omega} S_L^\partial(u, n)\delta u ds.$$

The fact that $I(\hat{u}) - I(u) \geq 0$ and that the variation of $\hat{u}$ in the domain Ω is arbitrary leads us to the Euler-Lagrange equation

$$\begin{cases} S_L(u) & = 0 \text{ in } \Omega \\ S_L^\partial(u, n)\delta u & = 0 \text{ on } \partial\Omega. \end{cases}$$

1.4 In the complex plane

The Cauchy-Riemann equations

The discussion here follows [4]. The Cauchy-Riemann equations are

$$\begin{aligned}
u_x &= v_y \\
u_y &= -v_x,
\end{aligned}$$

which follow straight from the requirement that

$$if_x = f_y.$$

This comes from the fact that

$$if_x = i(u_x + iv_x) = iu_x - v_x = u_y + iv_y = f_y$$

requires that $\text{Im}(if_x) = \text{Im}(f_y)$ and $\text{Re}(if_x) = \text{Re}(f_y)$, where

$$\text{Im}(if_x) = u_x = v_y = \text{Im}(f_y)$$

and

$$\text{Re}(if_x) = -v_x = u_y = \text{Re}(f_y).$$

The Cauchy-Riemann equations mean that the level curves of u and v are orthogonal. That is

$$\langle \nabla u, \nabla v \rangle = 0.$$

This is seen by the following computation.

$$\langle \nabla u, \nabla v \rangle = u_x v_x + u_y v_y = v_y v_x - v_x v_y = 0.$$

Furthermore

$$|\nabla u|^2 = |\nabla v|^2,$$

which follows from

$$|\nabla u|^2 = u_x^2 + u_y^2 = v_y^2 + (-v_x)^2 = |\nabla v|^2.$$

The Cauchy-Riemann equations also imply harmonicity. That is, that $\Delta u = 0$. This follows from

$$\Delta u = u_{xx} + u_{yy} = (u_x)_x + (u_y)_y = (v_y)_x + (-v_x)_y = v_{yx} - v_{xy} = 0.$$

A way to see that the Cauchy-Riemann equations need to be true for a complex derivative to exist is to look at what happens when the limits for the Laplacian of x and y change place. The definition of the complex derivative is

$$f'(z_0) := \lim_{\Delta z \to 0} \frac{f(z_0 + \nabla z) - f(z_0)}{\Delta z},$$

where $z = x + iy$. Here, Δz can approach zero either along the real or imaginary axis. This follows from the fact that it can be written as $\Delta z = \Delta x + i\Delta y$. Write f as $f = u + iv$, and $z_0 = x_0 + iy_0$. Hence

$$f(z_0) = u(x_0, y_0) + iv(x_0, y_0)$$

and

$$f(z_0 + \Delta z) = u(x_0 + \Delta x, y_0 + \Delta y) + iv(x_0 + \Delta x, y_0 + \Delta y).$$

Consequently

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta u(x_0, y_0) + i\Delta v(x_0, y_0)}{\Delta x + i\Delta y},$$

where

$$\Delta u(x_0, y_0) = u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)$$
$$\Delta v(x_0, y_0) = v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0).$$

Let $\Delta z = \Delta x$. Then

$$\begin{aligned} f'(z) &= \lim_{\Delta x \to 0} \frac{\Delta u(x_0, y_0) + i\Delta v(x_0, y_0)}{\Delta x} \\ &= u_x \Big|_{(x_0,y_0)} + i v_x \Big|_{(x_0,y_0)} = f_x. \end{aligned}$$

Let $\Delta z = i\Delta y$. Then

$$\begin{aligned} f'(z) &= \lim_{\Delta y \to 0} \frac{\Delta u(x_0, y_0) + i\Delta v(x_0, y_0)}{i\Delta y} \\ &= \lim_{\Delta y \to 0} (-i) \frac{\Delta u(x_0, y_0) + i\Delta v(x_0, y_0)}{(-i)i\Delta y} \\ &= -i u_y \Big|_{(x_0,y_0)} + v_y \Big|_{(x_0,y_0)} = \frac{1}{i} f_y. \end{aligned}$$

These are the same if and only if

$$\begin{aligned} u_x &= v_y \\ u_y &= -v_x. \end{aligned}$$

Conformal maps

A map that preserves angles, is injective, and differentiable, is called **conformal**. That a function f fulfills the Cauchy-Riemann equations is equivalent with that function being a conformal map. This follows from the fact that a map is conformal if it is a rotation with a rescaling. Geometrically, this means that

$$f = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Hence, a map f that is conformal fulfills

$$J(f) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

where $u = u(x, y)$ and $v = v(x, y)$. Consequently

$$J(f) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = r \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \Leftrightarrow \begin{cases} u_x &= r\cos\theta \\ u_y &= -r\sin\theta \\ v_x &= r\sin\theta \\ v_y &= r\cos\theta \end{cases} \Rightarrow \begin{cases} u_x &= r\cos\theta = v_y \\ u_y &= -r\sin\theta = -v_x. \end{cases}$$

Moreover

$$J(f) = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} v_y & -v_x \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} b & -a \\ a & b \end{bmatrix},$$

where v_x and v_y were denoted as a and b respectively in the last step. Similarly

$$J(f) = \begin{bmatrix} u_x & u_y \\ v_x & v_y \end{bmatrix} = \begin{bmatrix} u_x & u_y \\ -u_y & u_y \end{bmatrix} = \begin{bmatrix} b & -a \\ a & b \end{bmatrix}$$

where u_y and u_x were denoted as a and b respectively in that last step. This matrices are exactly the ones that appear when multiplying a complex number with another, which is exacthly what the complex derivative is. To see this, consider the multiplication of $x + iy$ by $a + ib$ by the map

$$x + iy \mapsto (a + ib)(x + iy) = ax + iay + ibx - by = ax - by + i(bx + ay).$$

This corresponds to multiplying a vector $(x, y) \in \mathbb{R}^2$ by the matrix

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

since

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = (ax - by) + i(bx + ay).$$

All this means is that things are stretched and rotated the same in both the real and imaginary direction. The section about quasiconformal maps will deal with the case when these stretchings are not the same. That

$$\det(J(f)) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} v_y & -v_x \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} u_x & u_y \\ -u_y & u_x \end{vmatrix} = v_x^2 + v_y^2 = u_x^2 + u_y^2 \geq 0$$

means that the map is orientation preserving whenever the derivative is non-zero. It will be shown that this is also is true for quasiconformal maps.

1.5 As the mean value property

Before we proceed to the proof of the mean value property we need to introduce the notation

$$\fint_{B_r(x)} u dx := \frac{1}{|B_r(x)|} \int u dx.$$

Here, we follow [3].

Theorem 1.6. *Assume that $u \in C^2(\Omega)$ is harmonic on an open set Ω and $B_r(x) \subset \Omega$. Then*

$$u(x) = \fint_{B_r(x)} u dx \quad \text{and} \quad u(x) = \fint_{\partial B_r(x)} u dS.$$

Proof. If $u \in C^2(\Omega)$ and $B_r(x) \subset \Omega$, then

$$\begin{aligned} \int_{B_r(x)} \Delta u dx &= \int_{\partial B_r(x)} \frac{\partial u}{\partial \nu} dS(z) \\ &= \{y \in \partial B_1 : z = ry \Rightarrow dS(z) = r^{n-1} d(S(y))\} \\ &= r^{n-1} \int_{\partial B_1(0)} \frac{\partial u}{\partial r}(x + ry) dS(y) \\ &= r^{n-1} \frac{\partial}{\partial r} \left[\int_{\partial B_1(0)} u(x + ry) dS(y) \right] \end{aligned}$$

is implied by the divergence theorem. Let σ_N denote the area of $\partial B_r(x)$. Divison by $\sigma_N r^n$ yields

$$\fint_{B_r(x)} \Delta u dx = \frac{n}{r} \frac{\partial}{\partial r} \left[\fint_{\partial B_r(x)} u dS \right]$$

It follows that if u is harmonic, then its mean value over a sphere centered at x is independent of r since

$$\Delta u(x) = 0 \Rightarrow \int_{B_r(x)} \Delta u dx = 0.$$

The mean value property for sphere follows as the integral for $r \to 0$ is equal to $u(x)$. The mean value property for balls follows from the mean value property for spheres by radial integration. □

1.6 Viscosity solutions

To conclude the proof of the following section we need to introduce the concept of viscosity solutions. To this end we need to define what a proper function is. Here, we follow [11]. Given an equation $F = 0$ we will require F to satisfy a fundamental monotonicity condition

$$F(x, r, p, X) \leq F(x, s, p, Y) \quad \text{whenever} \quad r \leq s \quad \text{and} \quad Y \leq X. \tag{1.11}$$

where $r, s \in \mathbb{R}$, $x, p \in \mathbb{R}^n$, and $X, Y \in S(n)$. Here, $S(n)$ denote symmetric $n \times n$ matrices. This can be split into the conditions

$$F(x, r, p, X) \leq F(x, s, p, X) \quad \text{whenever} \quad r \leq s \tag{1.12}$$

and

$$F(x, r, p, X) \leq F(x, r, p, Y) \quad \text{whenever} \quad X \leq Y, \tag{1.13}$$

where an equation fulfilling the condition given by (1.13) is referred to as ***degenerate elliptic***. An equation which fulfills both (1.12) and (1.13), and, consequently, (1.11), is referred to as a ***proper*** equation.

We assume that a solution u to equation is twice differentiable in $\mathbb{R}^n$, and that it fulfills

$$F(x, u(x), Du(x), D^2u(x)) \leq 0$$

for all $x \in \mathbb{R}^n$. Given a test function φ, which also is twice differentiable in $\mathbb{R}^n$, and a point $\hat{x} \in \mathbb{R}^n$ where $u - \varphi$ has a local maximum, we have that $Du(\hat{x}) = D\varphi(\hat{x})$ and $D^2u(\hat{x}) \leq D^2\varphi(\hat{x})$. This can be seen as the test function touching it from above. Degenerate ellipticity gives us

$$F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq F(\hat{x}, u(\hat{x}), Du(\hat{x}), D^2u(\hat{x}) \leq 0.$$

That the extremes of this inequality are independent of the derivatives of u means that an arbitrary function u is a subsolution of $F = 0$ if it fulfills the inequality

$$F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x}) \leq 0,$$

whenever $u - \varphi$ is a local maximum at $\hat{x}$ for a twice differentiable test function φ.

That $u - \varphi$ has a local maximum at $\hat{x}$ means that we have $u(x) - \varphi(x) \leq u(\hat{x}) - \varphi(\hat{x})$ for all $x \in B_\epsilon(\hat{x})$. This gives us $u(x) \leq u(\hat{x}) - \varphi(\hat{x}) + \varphi(x)$, which by gives us

$$u(x) \leq u(\hat{x}) + \langle D\varphi(\hat{x}), x - \hat{x}\rangle + \frac{1}{2}\langle D^2\varphi(\hat{x})(x - \hat{x}), (x - \hat{x})\rangle + o(|x - \hat{x}|^2) \quad \text{as} \quad x \to \hat{x} \tag{1.14}$$

by performing a Taylor expansion at $\hat{x}$. If (1.14) holds for some $(D\varphi(\hat{x}), D^2\varphi(\hat{x})) \in \mathbb{R}^n \times S(n)$ and u is twice differentiable at $\hat{x}$, then $D\varphi(\hat{x}) = Du(\hat{x})$ and $D^2u(\hat{x}) \leq D^2\varphi(\hat{x})$. Hence, if u is a classical solution to $F \leq 0$, then it follows that $F(\hat{x}, u(\hat{x}), D\varphi(\hat{x}), D^2\varphi(\hat{x})) \leq 0$ when (1.14) holds. The argument for a supersolution is given by considering by reversal of the inequalities. That is, considering a test function which touches the solution from below, which gives a minimum of $u - \varphi$ instead.

This discussion may even be expanded to the case when the solution is not differentiable. It is also based on test functions, but we follow a slightly different approach which is still based on (1.14).

We define the ***superjet*** $J^{2,+}_{\mathcal{O}}$ of $u : \mathcal{O} \to \mathbb{R}$ as the map $J^{2,+}_{\mathcal{O}} u : \mathcal{O} \to \mathbb{R}$, where that $\mathcal{O}$ is a locally compact set such that $\hat{x} \in \mathcal{O}$ and (1.14) holds for $x \in \mathcal{O}$ such that $x \to \hat{x}$ means that $(D\varphi(\hat{x}), D^2\varphi(\hat{x})) \in J^{2,+}_{\mathcal{O}} u$.

The analogue discussion for ***subjets*** is given by reversing the inequality in (1.14). These are denoted as $J^{2,-}_{\mathcal{O}} u$, or, equivalently, as $J^{2,+}u$. They relate to the superjets by the relationship $J^{2,-}_{\mathcal{O}} u(x) = -J^{2,+}_{\mathcal{O}}(-u)(x)$.

We are now ready to formally define viscosity subsolutions, supersolutions, and solutions.

Definition 1.1. *Let F satisfy* (1.11), *and $\mathcal{O} \subset \mathbb{R}^n$. A **viscosity solution** of $F = 0$ on $\mathcal{O}$ is an upper semicontinuous function $u : \mathcal{O} \to \mathbb{R}$ such that*

$$F(x, u(x), D\varphi(\hat{x}), D^2\varphi(\hat{x}) \le 0 \quad \textit{for all} \quad x \in \mathcal{O} \quad \textit{and} \quad (D\varphi(\hat{x}), D^2\varphi(\hat{x})) \in J^{2,+}_{\mathcal{O}} u(x).$$

*Similarly, a **viscosity subsolution** of $F = 0$ on $\mathcal{O}$ is a lower semicontinuous function $u : \mathcal{O} \to \mathbb{R}$ such that*

$$F(x, u(x), D\varphi(\hat{x}), D^2\varphi(\hat{x}) \ge 0 \quad \textit{for all} \quad x \in \mathcal{O} \quad \textit{and} \quad (D\varphi(\hat{x}), D^2\varphi(\hat{x})) \in J^{2,-}_{\mathcal{O}} u(x).$$

*If u is both a viscosity subsolution and viscosity supersolution of $F = 0$ in $\mathcal{O}$, then it is called a **viscosity solution** of $F = 0$ on $\mathcal{O}$.*

1.7 In stochastic processes

The following discussion comes from [13]. Consider a smooth and bounded domain $\Omega \subset \mathbb{R}^2$ in the plane which has a boundary $\partial\Omega$ divided into the parts Γ_1 and Γ_2 such that $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$. The behavior of moving at random starting at $(x, y) \in \Omega \backslash \partial\Omega$ is called a **random walk**.

Consider the following question: What is the probability $u(x_0, y_0)$ that you hit the boundary at Γ_1 the first time that you hit the boundary if you are moving at random starting at the point $(x_0, y_0) \in \Omega\backslash\partial\Omega$?

Answering the question is facilitated by considering a discretized version of it where only movements up, down, left, and right in a fixed step size of length $\epsilon > 0$ are considered. That is, movement to $(x + \epsilon, y)$, $(x - \epsilon, y)$, $(x, y + \epsilon)$, or $(x, y - \epsilon)$ from the point $(x, y) \in \Omega$. Each direction is chosen with equal probability. Hence, the probability for a direction to be chosen is $1/4$.

Let $u_\epsilon(x, y)$ denote the probability of hitting the boundary at the part $\Gamma_1 + B_\delta(0)$ the first time that the enlarged boundary $\partial\Omega + B_\delta(0)$ is hit when moving the lattice of size $\epsilon > 0$ of the discretized version of the walk. Here, we have chosen to consider some sufficiently large neighborhood $B_\delta(0)$ of the boundary as the boundary does not necessarily have to lie on the lattice. Note that $B_\delta(x) = x + B_\delta(0)$ for $x \in \partial\Omega$.

Now, conditional expectations yields

$$u_\epsilon(x, y) = \frac{1}{4} u_\epsilon(x + \epsilon, y) + \frac{1}{4} u_\epsilon(x - \epsilon, y) + \frac{1}{4} u_\epsilon(x, y + \epsilon) + \frac{1}{4} u_\epsilon(x, y - \epsilon).$$

Hence

$$0 = \{u_\epsilon(x + \epsilon, y) - 2u_\epsilon(x, y) + u_\epsilon(x - \epsilon, y)\} + \{u_\epsilon(x, y + \epsilon) - 2u_\epsilon(x, y) + u_\epsilon(x, y - \epsilon)\} \tag{1.15}$$

Assume that u_ϵ converges uniformly to a function u in $\bar{\Omega}$ as $\epsilon \to 0$. Intuitively, one can see this by considering that $0 \le u_\epsilon \le 1$ for all $\epsilon > 0$ and all $(x, y) \in \Omega$.

Let ϕ be a smooth function which touches u from below at $(x_0, y_0) \in \Omega$. Thus $u - \phi$ has a strict minimum at $(x_0, y_0) \in \Omega$. Due to uniform convergence of u_ϵ to u there are points (x_ϵ, y_ϵ) such that

$$(u_\epsilon - \phi)(x_\epsilon, y_\epsilon) \le (u_\epsilon - \phi)(x, y) + o(\epsilon^2) \quad \text{for} \quad (x, y) \in \Omega \tag{1.16}$$

and

$$(x_\epsilon, y_\epsilon) \to (x_0, y_0) \quad \text{as} \quad \epsilon \to 0.$$

The following result is obtained by simply rearranging (1.16).

$$u_\epsilon(x,y) - u_\epsilon(x_\epsilon, y_\epsilon) \geq \phi(x,y) - \phi(x_\epsilon, y_\epsilon) + o(\epsilon^2) \quad \text{for} \quad (x,y) \in \Omega$$

Using this and (1.15) the following is obtained.

$$0 \geq \{\phi(x_\epsilon + \epsilon, y_\epsilon) - 2\phi(x_\epsilon, y_\epsilon) + \phi(x_\epsilon - \epsilon, y_\epsilon)\} + \{\phi(x_\epsilon, y_\epsilon + \epsilon) - 2\phi(x_\epsilon, y_\epsilon) + \phi(x_\epsilon, y_\epsilon - \epsilon)\} \tag{1.17}$$

Moreover

$$\{\phi(x_\epsilon + \epsilon, y_\epsilon) - 2\phi(x_\epsilon, y_\epsilon) + \phi(x_\epsilon - \epsilon, y_\epsilon)\} = \epsilon^2 \frac{\partial^2 \phi}{\partial x^2}(x_\epsilon, y_\epsilon) + o(\epsilon^2) \tag{1.18}$$

$$\{\phi(x_\epsilon, y_\epsilon + \epsilon) - 2\phi(x_\epsilon, y_\epsilon) + \phi(x_\epsilon, y_\epsilon - \epsilon)\} = \epsilon^2 \frac{\partial^2 \phi}{\partial y^2}(x_\epsilon, y_\epsilon) + o(\epsilon^2), \tag{1.19}$$

follows from cancellation of the first order terms in the Taylor expansion of $\phi(x,y)$. Hence

$$0 \geq \frac{\partial^2 \phi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \phi}{\partial y^2}(x_0, y_0),$$

by substituing in (1.18) and (1.19) into (1.17), and dividing by ϵ^2 and taking the limit $\epsilon \to 0$.

Thus, we have now shown that if a smooth function ϕ touches u from below at a point (x_0, y_0), then the derivatives of ϕ must satisfy

$$0 \geq \frac{\partial^2 \phi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \phi}{\partial y^2}(x_0, y_0).$$

An analoguous argument can be made, where ψ is considered as a smooth function which touches u from above at $(x_0, y_0) \in \Omega$, which gives the reverse inequality. Then, we have shown that if a smooth function ψ touches u from above at a point $(x_0, y_0) \in \Omega$, then the derivatives of ψ must satisfy

$$0 \leq \frac{\partial^2 \phi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \phi}{\partial y^2}(x_0, y_0).$$

A solution to a PDE is called a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution of that PDE. Thus

$$\Delta u = \frac{\partial^2 \phi}{\partial x^2}(x_0, y_0) + \frac{\partial^2 \phi}{\partial y^2}(x_0, y_0) = 0.$$

This means that the uniform limit of the sequence of solutions to the discretized, and therefore approximated, problems u_ϵ, is the unique viscosity solution u to the boundary value problem

$$-\Delta u = 0, \quad u = 1 \text{ on } \Gamma_1, \quad u = 0 \text{ on } \Gamma_2.$$

The boundary value conditions follow naturally from the fact that $u_\epsilon := 1$ in the neighborhood $B_\epsilon(0)$ of Γ_1, and $u_\epsilon := 0$ in the neighborhood $B_\epsilon(0)$ of Γ_2. This was achieved by only assuming that u_ϵ was uniformly convergent, which can be proven rigorously.

Chapter 2

$2 < p < \infty$ and $p = \infty$: The p-Laplacian and ∞-Laplacian

2.1 Deduction of the p-Laplacian and ∞-Laplacian

The the discussion in this section is entirely from [10]. If, instead of a square, the considered exponent is some p in $2 < p < \infty$, then

$$D(u) = \int_{\Omega} |\nabla u|^p dx. \tag{2.1}$$

The corresponding Euler-Lagrange equation is

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

as the first variation of the nonlinear analogue $|u'|^p$ of the one dimensional linear Dirichlet energy $|u'|^2$ is

$$\begin{aligned}
\delta J(\eta, u) &= \int_{x_0}^{x_1} \left(\eta \frac{\partial f}{\partial u} + \eta' \frac{\partial f}{\partial u'} \right) dx \\
&= \int_{x_0}^{x_1} \eta' p |u'|^{p-1} \frac{u'}{|u'|} dx \\
&= \int_{x_0}^{x_1} \eta' p |u'|^{p-2} u' dx,
\end{aligned}$$

where $f = |u'|^p$, and we used that

$$\frac{\partial f}{\partial u} = 0$$

as $f(x, u, u')$ lacks explicit dependence of u, and that

$$\frac{\partial |u|}{\partial u} = \frac{u}{|u|}.$$

Furthermore, in the n-dimensional case we have the gradient instead of the one-dimensional derivative, and we integrate over an n-dimensional domain Ω instead of a one-dimensional domain $[x_0, x_1]$. Here

$$\begin{aligned}
\delta D(\eta, u) &= \int_\Omega \langle \nabla\eta, |\nabla u|^{p-2}\nabla u\rangle dx \\
&= \int_\Omega \sum_{i=1}^n \frac{\partial \eta}{\partial x_i}|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}dx \\
&= \sum_{i=1}^n \left[\left(\eta|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right)\Big|_{\partial\Omega} - \int_\Omega \eta\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}dx\right)\right] \\
&= -\int_\Omega \eta\nabla\cdot(|\nabla u|^{p-2}\nabla u)dx = \int_\Omega \eta\Delta_p u dx = 0 \Leftrightarrow \Delta_p u = 0,
\end{aligned}$$

which means that the p-Laplacian is the Euler-Lagrange equation for the p-Dirichlet problem. Moreover

$$\delta D(\eta, u) = \int_\Omega \langle \nabla\eta, p|\nabla u'|^{p-2}\nabla u\rangle dx.$$

The p-Laplace operator is defined as

$$\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$$

where we have that

$$\Delta_p u = |\nabla u|^{p-4}\left\{|\nabla u|^2\Delta u + (p-2)\sum_{i,j=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial u}{\partial x_i\partial x_j}\right\} \tag{2.2}$$

which follows from the computation

$$\begin{aligned}
\frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right) &= |\nabla u|^{p-2}\left(\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial u}{\partial x_i}\frac{\partial}{\partial x_i}|\nabla u|^{p-2}\right) \\
&= |\nabla u|^{p-2}\frac{\partial^2 u}{\partial x_i^2} + \frac{\partial u}{\partial x_i}(p-2)|\nabla u|^{p-3}\frac{\partial}{\partial x_i}|\nabla u|,
\end{aligned}$$

but

$$\frac{\partial}{\partial x_i}|\nabla u| = \frac{\partial}{\partial x_i}\sqrt{\sum_{j=1}^n\left(\frac{\partial u}{\partial x_j}\right)^2} = \frac{1}{2|\nabla u|}\sum_{j=1}^n 2\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}$$

which gives us

$$\frac{\partial}{\partial x_i}(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}) = |\nabla u|^{p-2}\frac{\partial^2 u}{\partial x_i^2} + (p-2)|\nabla u|^{p-4}\sum_{j=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}.$$

Hence

$$\Delta_p u = |\nabla u|^{p-2}\Delta u + (p-2)|\nabla u|^{p-4}\sum_{i,j=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j},$$

which gives us (2.3) by factoring out $|\nabla u|^{p-4}$. The ∞-Laplacian is defined as

$$\Delta_\infty u = \frac{1}{|\nabla u|^2}\sum_{i,j=1}^n \frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_i\partial x_j}$$

which comes from considering

$$\Delta_p u = |\nabla u|^{p-4} \left\{ |\nabla u|^2 \Delta u + (p-2) \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i \partial x_j} \right\} = 0, \tag{2.3}$$

dividing out $|\nabla u|^{p-2}$, dividing by $(p-2)$, and passing the limit $p \to \infty$ in

$$\Delta_\infty u = \lim_{p\to\infty} \frac{\Delta u}{p-2} + \frac{1}{|\nabla u|^2} \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i \partial x_j} = \frac{1}{|\nabla u|^2} \sum_{i,j=1}^{n} \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial u}{\partial x_i \partial x_j} = 0,$$

which gives us $\Delta_\infty u = 0$.

2.2 As a minimizer

Here, we follow [10]. Minimization of the p-Dirichlet energy (2.1) among all $u \in S$ shows that the first variation must vanish. That is

$$\int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \eta \rangle dx = 0$$

for all $\eta \in C_0^\infty(\Omega)$, which is equivalent to

$$\int_\Omega \eta \operatorname{div}(|\nabla u|^{p-2} \nabla u) dx = 0, \tag{2.4}$$

given the right boundary data. The requirement that (2.4) must hold for all test functions $\eta \in C_0^\infty(\Omega)$ it follows that

$$\operatorname{div}(|\nabla u|^{p-2} \nabla u) = 0$$

in Ω as in the previous section. This means that the p-Laplace is the Euler Lagrange equation for the p-Dirichlet energy.

It turns out that the class of strong solutions is too narrow for the treatment of the aforementioned problem. The concept of weak solutions is used instead, whose definition requires the definition of Sobolev spaces, Banach spaces, and the L^p-norm.

Definition 2.1. *The space L^p is defined as*

$$L^p(\Omega) = \{ f : f \text{ is measurable and } \|f\|_p = \left(\int_\Omega |f|^p d\mu \right)^{1/p} < \infty \},$$

where $1 < p < \infty$ and $\|f\|_p$ is called the ***standard norm*** *on $L^p(\Omega)$.*

Definition 2.2. *A* ***Banach space*** *is a normed vector space X over $\mathbb{R}$ or $\mathbb{C}$ which is complete under the metric associated with the norm.*

This means that for every Cauchy sequence $\{x_n\} \in X$ there exists an element $x \in X$ such that

$$\lim_{n\to\infty} x_n = x,$$

with respect to the norm of the vector space.

Definition 2.3. *The **Sobolev space** $W^{1,p}(\Omega)$ consists of functions u such that u and its weak derivatives*

$$\nabla u = \left(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \dots, \frac{\partial u}{\partial x_n} \right)$$

belong to the space $L^p(\Omega)$. Equipped with the norm

$$\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$$

it is a Banach space.

Every prerequisite for the definition of a weak solution is now provided.

Definition 2.4. *Let Ω be a domain in $\mathbb{R}^n$. A function $u \in W^{1,p}(\Omega)$ is called **weak solution** of the p-Laplacian in Ω if*

$$\int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \eta \rangle dx = 0 \tag{2.5}$$

for all $\eta \in C_0^\infty(\Omega)$. If $\Delta_p u$ is continuous too, then u is called a p-harmonic function.

The following fundamental result is now formulated and proved.

Theorem 2.1. *The following conditions are equivalent for $u \in W^{1,p}(\Omega)$.*

1. *u is minimal*

$$\int |\nabla u|^p dx \leq \int |\nabla \hat{u}|^p dx \quad \text{for all} \quad \hat{u} - u \in W^{1,p}(\Omega).$$

2. *The first variation vanishes*

$$\int \langle |\nabla u|^{p-2} \nabla u, \nabla \eta \rangle dx = 0 \quad \text{for all} \quad \eta \in W^{1,p}(\Omega).$$

If $\Delta_p u$ is continuous too, then the above conditions are equivalent to $\Delta_p u = 0$ in Ω.

Proof. ($1 \Rightarrow 2$) Assume that $u(x)$ is a local minimum. Let

$$\hat{u}(x) = u(x) + \epsilon \eta(x),$$

where $\epsilon > 0$ and $\eta \in C_0^\infty(\Omega)$. As

$$J(\epsilon) = \int |\nabla (u + \epsilon \eta)|^p dx$$

attains its minimum for $\epsilon = 0$, since $u(x)$ is a minimum, it follows that $J'(0) = 0$. This is 2.

($2 \Rightarrow 1$) The inequality

$$|b|^p \geq |a|^p + p \langle |a|^{p-2} a, b - a \rangle$$

holds for vectors due to convexity given that $p \geq 1$. This follows from

$$\begin{aligned}
\langle |a|^{p-2} a, b - a \rangle + |a|^p &= \langle |a|^{p-2} a, b \rangle + \langle |a|^{p-2}, -a \rangle + |a|^p \\
&= \langle |a|^{p-2} a, b \rangle - |a|^{p-2} |a|^2 + |a|^p \\
&= \langle |a|^{p-2} a, b \rangle - |a|^p + |a|^p \\
&= \langle |a|^{p-2} a, b \rangle \leq \frac{p-1}{p} |a|^p + \frac{1}{p} |b|^p \Leftrightarrow \\
&\Leftrightarrow p \langle |a|^{p-2} a, b - a \rangle + p|a|^p \leq (p-1)|a|^p + |b|^p \\
&= p|a|^p - |a|^p + |b|^p \Leftrightarrow |b|^p \geq |a|^p + p \langle |a|^{p-2} a, b - a \rangle,
\end{aligned}$$

where it was used that

$$\begin{aligned}\langle |a|^{p-2}a, b\rangle &\leq |a|^{p-2}|a||b| \\ &= |a|^{p-1}|b| \\ &\leq \frac{(|a|^q)^{p-1}}{q} + \frac{|b|^p}{p} \\ &= \frac{(|a|^{\frac{p}{p-1}})^{p-1}}{\frac{p}{p-1}} + \frac{|b|^p}{p} \\ &= \frac{p-1}{p}|a|^p + \frac{|b|^p}{p}\end{aligned}$$

by Cauchy-Schwarz inequality, and Young's inequality for products

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

with Hölder exponents $p, q \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = 1.$$

Consequently

$$\int_\Omega |\nabla \hat{u}|^p dx \geq \int_\Omega |\nabla u|^p dx + p \int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla(\hat{u} - u)\rangle dx.$$

Assuming that the first variation vanishes, choose $\eta = \hat{u} - u$. Then

$$\int_\Omega |\nabla \hat{u}|^p dx \geq \int_\Omega |\nabla u|^p dx.$$

This is the desired result. □

Note that if (2.5) holds for all $\eta \in C_0^\infty(\Omega)$, then it also holds for all $\eta \in W_0^{1,p}(\Omega)$ given that $u \in W^{1,p}(\Omega)$. This means that the minimizers are the same as the weak solutions.

2.3 Uniqueness of its solutions

In this section we show uniqueness of the solutions to the p-Laplacian. In order to do this, we need the following definition. We still follow [10] here.

Definition 2.5. *A function $v \in W_{loc}^{1,p}(\Omega)$ is called* ***weak supersolution*** *in Ω if*

$$\int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla \eta\rangle dx \geq 0$$

for all nonnegative $\eta \in C_0^\infty(\Omega)$. The inequality is simply reverse for the definition of ***weak subsolutions****.*

We now state and prove the main result of this section.

Theorem 2.2. *Assume that u and v are p-harmonic functions in the bounded domain Ω. If at each $\xi \in \partial\Omega$*

$$\limsup_{x \to \xi} u(x) \leq \liminf_{x \to \xi} v(x),$$

excluding the cases when $\infty \leq \infty$ and $-\infty \leq -\infty$, then $u \leq v$ in Ω.

Proof. The set

$$D_\epsilon = \{x : u(x) > v(x) + \epsilon\} \quad \text{for} \quad \epsilon > 0$$

is open, and it is either empty or $\Omega_\epsilon \subset\subset \Omega$. Since the functions are p-harmonic it follows that

$$\int_\Omega \langle |\nabla u|^{p-2} \nabla u, \nabla \eta \rangle dx = 0 \tag{2.6}$$

and

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v, \nabla \eta \rangle dx = 0. \tag{2.7}$$

Consequently, subtraction of the equation of (2.6) from (2.7) yields

$$\int_\Omega \langle |\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u, \nabla \eta \rangle dx = 0$$

for all $\eta \in W_0^{1,p}(\Omega)$. The choice

$$\eta(x) = \min\{v(x) - u(x) + \epsilon, 0\}$$

yields

$$\int_{D_\epsilon} \langle |\nabla v|^{p-2} \nabla v - |\nabla u|^{p-2} \nabla u, \nabla v - \nabla u \rangle dx = 0,$$

which only is possible whenever $\nabla u = \nabla v$ a.e. in Ω_ϵ since the integrand is positive whenever $\nabla u \neq \nabla v$. Thus, $u(x) = v(x) + C$ in Ω_ϵ.

Moreover, $C = \epsilon$ because $u(x) = v(x) + \epsilon$ on $\partial\Omega_\epsilon$. This is the case since when x goes from being a member to not being a member of Ω_ϵ, then $u(x)$ goes from $u(x) > v(x) + \epsilon$ to $u(x) \leq v(x) + \epsilon$, which happens when $u(x) = v(x) + \epsilon$. This means that $\Omega_\epsilon = \emptyset$ since $\Omega_\epsilon := \{x | u(x) > v(x) + \epsilon\}$.

The fact that Ω_ϵ would contain all x such that $u(x) > v(x) + \epsilon$ and is empty together with $\Omega_\epsilon \subset\subset \Omega$ means that $u(x) \leq v(x) + \epsilon$ for all $x \in \Omega$. This means that $u \leq v$ for all $x \in \Omega$, as $\epsilon > 0$ can be chosen arbitrarily small. □

The uniqueness then comes from simply changing places between u and v in the assumtion of the theorem. Because if both $u \leq v$ and $v \leq u$, then $u = v$ in Ω.

2.4 Existence of its solutions

Both methods studied here come from [10]. We will only examine the proofs of the first method, and state the other as it shows the relation to viscosity solutions.

2.4.1 Method 1: Weak solutions and Sobolev spaces

Theorem 2.3. *Assume that Ω is a bounded domain in $\mathbb{R}^n$, and that $g \in W^{1,p}(\Omega)$. There exists a unique $u \in W^{1,p}(\Omega)$ with the boundary values $u - g \in W_0^{1,p}(\Omega)$ such that*

$$\int_\Omega |\nabla u|^p dx \leq \int_\Omega |\nabla v|^p dx$$

for all similar v. Thus, u is a weak solution. If $g \in C(\bar{\Omega})$ and the boundary $\partial\Omega$ is regular enough as well, then $u \in C(\bar{\Omega})$ and $u\Big|_{\partial\Omega} = g\Big|_{\partial\Omega}$.

Proof. If the minimizer were not unique, then there would exist two minimizers u_1 and u_2. A third function

$$v = \frac{u_1 + u_2}{2}$$

could then be created. This is the average value of the two claimed minimizers. This allows us to use the triangle inequality, which would yield

$$\left|\frac{\nabla u_1 + \nabla u_2}{2}\right|^p \leq \frac{|\nabla u_1|^p + |\nabla u_2|^p}{2}$$

If $\nabla u_1 \neq \nabla u_2$ in a set of postive measure, then the this inequality would be strict, which would mean that

$$\begin{aligned}\int_\Omega |\nabla u_2|^p dx \leq \int_\Omega \left|\frac{\nabla u_1 + \nabla u_2}{2}\right|^p dx \leq \int_\Omega \frac{|\nabla u_1|^p + |\nabla u_2|^p}{2} \\ < \frac{1}{2}\left(\int_\Omega |\nabla u_1|^p + \int_\Omega |\nabla u_2|^p\right) = \int_\Omega |\nabla u_2|^p dx,\end{aligned}$$

which is a contradiction. Thus $\nabla u_1 = \nabla u_2$ a.e. in Ω. Consequently, $u_1 = u_2 + C$, and $C = 0$ as $u_2 - u_1 \in W_0^{1,p}(\Omega)$. Thus, $u_1 = u_2$. This proves the uniqueness of the minimizer.

The existence of a minimizer is shown as follows. Let

$$I_0 = \inf \int_\Omega |\nabla v|^p dx \leq \int_\Omega |\nabla g|^p dx < \infty.$$

Then, $0 \leq I_0 < \infty$. Choose admissable functions v_j such that

$$\int_\Omega |\nabla v_j|^p dx < I_0 + \frac{1}{j} \quad \text{for all} \quad j = 1, 2, 3, \ldots. \tag{2.8}$$

This sequence will now be bounded by $\|v_j\|_{W^{1,p}(\Omega)}$. The Poincaré inequality

$$\|w\|_{L^p(\Omega)} \leq C_\Omega \|\nabla w\|_{L^p(\Omega)}$$

holds for all $w \in W_0^{1,p}(\Omega)$. Let $w = v_j - g$. Consequently

$$\begin{aligned}\|v_j - g\|_{L^p(\Omega)} &\leq C_\Omega\{\|\nabla v_j\|_{L^p(\Omega)} + \|\nabla g\|_{L^p(\Omega)}\} \\ &\leq C_\Omega\{(I_0 + 1)^{\frac{1}{p}} + \|\nabla g\|_{L^p(\Omega)}\}.\end{aligned}$$

Here, it was used that

$$\|\nabla g\|_{L^p(\Omega)} = \left(\int_\Omega |\nabla g|^p dx\right)^{1/p} < \infty,$$

by the previous assumption. Thus

$$\|v_j\|_{L^p(\Omega)} = \|v_j - g\|_{L^p(\Omega)} - \|g\|_{L^p(\Omega)} < M \quad \text{for} \quad j = 1, 2, 3, \ldots \tag{2.9}$$

where the Poincaré inequality for $\|g\|_{L^p(\Omega)}$, and the constant M is dependent on the index of j. Together, (2.8) and (2.9) consistute the desired bound.

Weak convergence now tells us that there exists a function $u \in W^{1,p}(\Omega)$ and a subsequence such that

$$v_{j_v} \rightharpoonup u \quad \text{and} \quad \nabla v_j \rightharpoonup \nabla u \quad \text{weakly in} \quad L^p(\Omega)$$

We have that $u - g \in W_0^{1,p}(\Omega)$ due to $W_0^{1,p}(\Omega)$ being closed under weak convergence. This means that u is an admissable function. It is also the sought after minimizer since

$$\int_\Omega |\nabla v_{j_\nu}|^p dx \leq \lim_{j_\nu \to \infty} \int_\Omega |\nabla v_{j_\nu}|^p dx = I_0,$$

due to lower semicontinuity of v_j. $\square$

2.4.2 Method 2: Viscosity solutions and Perron's method

We start this section with the following definition.

Definition 2.6. *A function $v : \Omega \to (-\infty, \infty]$ is called p-superharmonic in Ω if*

1. *v is lower semi-continuous in Ω,*
2. *$v \neq \infty$ in Ω, and*
3. *for each domain $D \subset\subset \Omega$ the comparison principle holds.*

Note that a function $u : \Omega \to [-\infty, \infty)$ is called p-subharmonic if $v = -u$ is p-superharmonic. Consider the Dirichlet boundary value problem

$$\begin{cases} \Delta h = 0 & \text{in } \Omega \\ h = g & \text{on } \partial\Omega. \end{cases}$$

The treatment considered here will be for the p-Laplacian, of which the p-superharmonic and p-subharmonic functions are fundamental. The statements in this section will be given without proofs, but are found in [10].

Let Ω be bounded domain in $\mathbb{R}^n$, and $g : \partial\Omega \to [-\infty, \infty]$ denote the boundary values that we desire. The boundary value problem for the p-Laplacian is solved by considering the aforementioned p-subharmonic and p-superharmonic functions called the Perron subsolution $\underline{h}$ and the Perron supersolution $\bar{h}$ respectively. These functions fulfill the following properties and more.

1. $\underline{h} \leq \bar{h}$ in Ω
2. $\underline{h}$ and $\bar{h}$ are p-harmonic functions
3. $\underline{h} = \bar{h}$ if g is continuous
4. If there exists a p-harmonic function h in Ω such that

$$\lim_{x \to \xi} h(x) = g(\xi)$$

at each $\xi \in \partial\Omega$, then $h = \underline{h} = \bar{h}$.

We will however restrict our attention to these properties in order to prove uniqueness and existence for the viscosity solutions. We begin by constructing two classes of functions. Namely, the class of upper functions $\mathcal{U}_g$, and the class of lower functions $\mathcal{L}_g$. The upper class $\mathcal{U}_g$ consists of all the functions $v : \Omega \to (-\infty, \infty]$ such that

1. v is p-superharmonic in Ω,
2. v is bounded from below, and
3. $\liminf_{x \to \xi} v(x) \geq g(\xi)$ when $\xi \in \partial\Omega$

The lower class $\mathcal{L}_g$ consists of all the functions $u : \Omega \to [-\infty, \infty)$ such that

1. u is p-subharmonic in Ω,
2. u is bounded from above, and
3. $\limsup_{x \to \xi} u(x) \leq g(\xi)$, when $\xi \in \partial\Omega$.

Differentiability is not assumed for the p-subharmonic or p-superharmonic functions as if $v_1, v_2, \ldots, v_k \in \mathcal{U}_g$ then the pointwise minimum $\min\{v_1, v_2, \ldots, v_k\}$ also is a member of $\mathcal{U}_g$. Likewise, if $u_1, u_2, \ldots, u_k \in \mathcal{L}_g$ then $\max\{u_1, u_2, \ldots, u_k\}$ is a member of $\mathcal{L}_g$.

At every point in Ω the **upper solution** is defined as

$$\bar{h}_g(x) = \inf_{v \in \mathcal{U}_g} v(x),$$

and the **lower solution** is defined as

$$\underline{h}_g(x) = \sup_{u \in \mathcal{L}_g} u(x),$$

at every point in Ω. The subscript g is often omitted in the literature. Henceforth, we will follow this convention and write $\bar{h}$ instead of $\bar{h}_g$. We now state the theorem and a lemma that is needed to prove it, but without proof. We will also state Wiener's resolutivity theorem as well without proof.

Theorem 2.4. *The function $\bar{h}$ satisfies one of the following conditions*

1. *$\bar{h}$ is p-harmonic in Ω,*
2. *$\bar{h} \equiv \infty$ in Ω, or*
3. *$\bar{h} \equiv -\infty$ in Ω.*

Similar results hold for $\underline{h}$.

The main result of this section is now stated. It is called Wiener's resolutivity theorem.

Theorem 2.5. *If $g : \partial\Omega \to \mathbb{R}$ is continuous, then $\bar{h}_g = \underline{h}_g$ in Ω.*

2.5 In the complex plane

2.5.1 The nonlinear Cauchy-Riemann equations

Here, we still follow [10]. The nonlinear Cauchy-Riemann equations for a p-harmonic function u are as follows. If u is a p-harmonic function in a simply connected domain Ω in the complex plane, then there exists a unique up to a constant q-harmonic function v, which is called the p-harmonic conjugate, such that

$$v_x = -|\nabla u|^{p-2} u_y \quad \text{and} \quad v_y = |\nabla u|^{p-2} u_x$$

or, equivalently

$$u_x = |\nabla v|^{q-2} v_y \quad \text{and} \quad u_y = |\nabla v|^{q-2} v_x.$$

This equivalence comes from the following computations. First

$$\begin{aligned}
|\nabla v|^2 = v_x^2 + v_y^2 &= (|\nabla u|^{p-2}|u_y|)^2 + (|\nabla u|^{p-2}|u_x|)^2 \\
&= |\nabla u|^{2(p-2)}|u_y|^2 + |\nabla u|^{2(p-2)}|u_x|^2 \\
&= |\nabla u|^{2(p-2)}(|u_y|^2 + |u_x|^2) \\
&= |\nabla u|^{2(p-2)}|\nabla u|^2 = |\nabla u|^{2(p-2+1)} = |\nabla u|^{2(p-1)}.
\end{aligned}$$

Second, square root both sides

$$\sqrt{|\nabla v|^2} = |\nabla v| = \sqrt{|\nabla u|^{2(p-1)}} = |\nabla u|^{p-1}.$$

Third, using the assumption that p and q are **conjugate Hölder exponents**, which means that p and q are such that

$$\frac{1}{p}+\frac{1}{q}=1 \Leftrightarrow \frac{1}{q}=1-\frac{1}{p} \Leftrightarrow q=1\Big/\left(1-\frac{1}{p}\right)=1\Big/\frac{p-1}{p}=\frac{p}{p-1}$$

it follows that

$$|\nabla v|^q=|\nabla v|^{\frac{p}{p-1}}=|\nabla u|^{p-1\frac{p}{p-1}}=|\nabla u|^p.$$

This is the desired result. Moreover

$$\langle\nabla u,\nabla v\rangle=u_xv_x+u_yv_y=-u_xu_y|\nabla u|^{p-2}+u_xu_y|\nabla u|^{p-2}=0,$$

which means that the stream lines of the p-harmonic function are orthogonal to that of the q-harmonic complex conjugate.

2.5.2 Quasiconformal maps

The Jacobian becomes

$$J(f)=\begin{bmatrix}\frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y}\\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y}\end{bmatrix}=\begin{bmatrix}u_x & u_y\\ v_x & v_y\end{bmatrix}=\begin{bmatrix}|\nabla v|^{q-2}v_y & -|\nabla v|^{q-2}v_x\\ v_x & v_y\end{bmatrix}=\begin{bmatrix}u_x & u_y\\ -|\nabla u|^{p-2}u_y & |\nabla u|^{p-2}u_x\end{bmatrix}.$$

This is seen geometrically as a circle being mapped to an ellipse instead of another circle. Thise comes from the fact that the imaginary and real part can differ by a factor of $|\nabla u|^{p-2}$ or, equivalently $|\nabla v|^{q-2}$, as shown before. This allows for semiaxes of different length, which gives us an ellipse.

If f has a derivative at z we may use the complex differential operators

$$\begin{aligned}f_z&=\frac{1}{2}(f_x-if_y)=\frac{1}{2}(u_x+v_y)+\frac{i}{2}(v_x-v_y)\\ f_{\bar z}&=\frac{1}{2}(f_x+if_y)=\frac{1}{2}(u_x-v_y)+\frac{i}{2}(v_x+u_y).\end{aligned}$$

The Jacobian of f is

$$J(z,f)=|f_z|^2-|f_{\bar z}|^2,$$

which comes from

$$\begin{aligned}|f_z|^2-|f_{\bar z}|^2&=\left(\frac{1}{2}(u_x+v_y)\right)^2+\left(\frac{1}{2}(v_x-u_y)\right)^2-\left(\frac{1}{2}(u_x-v_y)\right)^2-\left(\frac{1}{2}(v_x+u_y)\right)^2\\ &=\frac{1}{4}(u_x^2+2u_xv_y+v_y^2)+\frac{1}{4}(v_x^2-2u_yv_x+u_y^2)-\frac{1}{4}(u_x^2-2u_xv_y+v_y^2)-\frac{1}{4}(v_x^2+2u_yv_x+u_y^2)\\ &=u_xv_y-u_yv_x=\begin{vmatrix}u_x & u_y\\ v_x & v_y\end{vmatrix}=\det(J(f))\end{aligned}$$

with

$$\begin{aligned}f_z&=\frac{1}{2}(f_x+if_y)=\frac{1}{2}(u_x+iv_x+i(u_y+iv_y)=\frac{1}{2}(u_x-v_y)+\frac{i}{2}(v_x+u_y)\\ f_{\bar z}&=\frac{1}{2}(f_x+if_y)=\frac{1}{2}(u_x+iv_x-i(u_y+iv_y)=\frac{1}{2}(u_x+v_y)+\frac{i}{2}(v_x-u_y).\end{aligned}$$

The following discussion is due to [1]. If f preserves orientation, then $|f_{\bar z}|<|f_z|$. If it does not, then $|f_{\bar z}|>|f_z|$. The following formulations are obtained by computing the inverse image of the unit circle. This is done by computing the real curve of the equation $\|f\|=1$. Let

$$a:=f_z \quad\text{and}\quad b:=f_{\bar z}.$$

Write

$$z := re^{i\theta}, \quad a := |a|e^{i\alpha}, \quad \text{and} \quad b := |b|e^{i\beta}.$$

The equation $\|f\| = 1$ becomes in polar coordinates

$$\left|(|a| + |b| \cos\left(\theta + \frac{\alpha - \beta}{2}\right) + i(|a| - |b|) \sin\left(\theta + \frac{\alpha - \beta}{2}\right)\right| = \frac{1}{r}.$$

This is the equation of an ellipse with major axis at polar angle $\frac{\beta-\alpha}{2}$ of semi-length $\frac{1}{|a|+|b|}$, and with minor axis at polar angle $\frac{\beta-\alpha}{2} + \frac{\pi}{2}$ of semi-length $\frac{1}{||a|-|b||}$. In this context, $\|f\|$ is the inverse of the semi-length of the of the minor axis, and $\det(f)$ is the ratio of the area of the unit circle to its preimage up to sign.

The ratio between the axes is

$$\frac{|a| + |b|}{|a| - |b|} \leq \frac{1 + k}{1 - k} = K,$$

where $0 \leq k < 1$ and $K \geq 1$. This is what was describing earlier. An orientation preserving diffeomorphism whose derivative maps infinitesimal circles to infinitesimal ellipses whose eccentricity is at most K.

That $K = 1$ means that it would be conformal is seen as

$$\frac{|a| + |b|}{|a| - |b|} = 1 \Leftrightarrow |a| + |b| = |a| - |b| \Leftrightarrow |b| = -|b|,$$

which only is possible when $|b| = 0$. Thus

$$\frac{1}{|a| - |b|} = \frac{1}{|a|} = \frac{1}{|a| + |b|}.$$

Hence, a circle. Here, it was used that

$$\frac{|a| + |b|}{|a| - |b|} \leq 1 \Rightarrow \frac{|a| + |b|}{|a| - |b|} = 1,$$

since the axes would otherwise be erroneously labeled, and would yield $K \geq 1$ when relabeled. Since they were not erroneously labeled it means that only

$$\frac{|a| + |b|}{|a| - |b|} = 1$$

is possible. Moreover

$$|b| = |f_{\bar{z}}| = 0,$$

which means that f is holomorphic. This comes from

$$\begin{aligned} f_{\bar{z}} &= \frac{1}{2}(f_x + if_y) = \frac{1}{2}(u_x + iu_x + i(u_y + iv_y)) \\ &= \frac{1}{2}(u_x + iv_x + iu_y - v_y) = 0 \Leftrightarrow u_x + iv_x = v_y - iu_y, \end{aligned}$$

which only is true when $\text{Re}(u_x + iv_x) = \text{Re}(v_y - iu_y)$ and $\text{Im}(u_x + iv_x) = \text{Re}(v_y - iu_y)$. This is the case when $u_x = u_y$ and $u_y = -v_x$. The derivatives f_z and $f_{\bar{z}}$ exist in the sense that for $w = u + iv = f(z) = f(x + iy)$ the derivative is

$$dw = du + idv = u_x dx + u_y dy + i(v_x dx + v_y dy)$$

which is written as $dw = f_z dz + f_{\bar{z}} d\bar{z}$ with the aforementioned formulas for f_z and $f_{\bar{z}}$.

If $U, v \subset \mathbb{C}$ are open and $f : U \to V$ is a continuous map whose derivatives are locally in L^2, then

$$J(f) = |f_z|^2 - |f_{\bar{z}}|^2 \quad \text{and} \quad |f'(z)|^2 = (|f_z| + |f_{\bar{z}}|)^2$$

are locally in L^2. This motivates the following definition of quasiconformal maps from [9].

Definition 2.7. *Let U, V be open subsets of $\mathbb{C}$, and take $K \geq 1$. A map $f : U \to V$ is called **quasiconformal** if*

1. *it is a homeomorphism,*
2. *its distributive derivatives are locally in L^2, and*
3. *its distributive derivatives satisfy*

$$J(f) \geq \frac{|f'(z)|^2}{K} \quad \textit{locally in } L^1.$$

Last criteria means that f is orientation preserving as the Jacobian is positive. This might be reformulated as the following definition.

Definition 2.8. *Let U and V be open subsets of $\mathcal{C}$. Let $K \geq 1$, and set*

$$k := \frac{K-1}{K+1}.$$

*Thus, $0 \leq k < 1$. A map $f : U \to V$ is called K**-quasiconformal** if it is a homeomorphism whose distributional partial derivatives are in L^2_{loc} and satisfy*

$$|f_{\bar{z}}| \leq k|f_z|$$

*in L^2_{loc}. A map is called **quasiconformal** if it is K-quasiconformal for some K.*

Here, the smallet $K \geq 1$ such that a map is f is K-quasiconformal is called the **quasiconformal constant** of the f, and is denoted as $K(f)$. It is also referred to as the **quasiconformal dilatation** or the **quasiconformal norm**. It measures how close the map is to being conformal. The closer it is to one, the more conformal it is. It is an upper bound of the eccentricity.

The computation

$$\begin{aligned}\det(J(f)) &= |\nabla v|^{q-2}v_x^2 + |\nabla v|^{q-2}v_y^2 = |\nabla v|^{q-2}(v_x^2 + v_y^2) \\ &= |\nabla u|^{p-2}u_x^2 + |\nabla u|^{p-2}u_y^2 = |\nabla u|^{p-2}(u_x^2 + u_y^2)\end{aligned}$$

shows that its Jacobian only is a rescaling of the Jacobian in the conformal case. As claimed earlier, this means that a quasiconformal map is orientation preserving whenever the derivative is non-zero.

The nonlinear Cauchy-Riemann equations for a function u imply that u is p-harmonic. This follows from

$$\Delta_p u = \frac{\partial}{\partial x}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial y}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial y}\right) = \frac{\partial v}{\partial x \partial y} - \frac{\partial v}{\partial y \partial x} = 0.$$

The behavior of quasiconformal maps for the p-Laplacian will now be investigated. Namely, what the choice of $2 < p < \infty$ means for the map. We call a non-injective quasiconformal mapping a **quasiregular mapping**.

The connection between the complex gradient $f = \frac{1}{2}(u_x - iu_y)$ and the p-Laplacian sets important criteria for the quasiregular mapping which will be shown now.

Theorem 2.6. *The complex gradient $f = \frac{1}{2}(u_x - iu_y)$ of a p-harmonic function u is a quasiregular mapping which satisfies the system*

$$\begin{aligned} f_{\bar{z}} &= \left(\frac{1}{p} - \frac{1}{2}\right)\left(\frac{\bar{f}}{f} f_z + \frac{f}{\bar{f}}\overline{f_z}\right), \\ |f_{\bar{z}}| &\leq \left(1 - \frac{2}{p}\right)|f_z| \quad \textit{for almost every } z \in \Omega. \end{aligned}$$

Proof. Let $f = u_x - iu_y$ and $|F_a| = |f|^a f$ for $a > -1$. We know that $F_a \in W^{1,2}_{loc}(\Omega)$ for $a = \frac{p-2}{2}$ (See Theorem 16.3.1 in [9]). This is in fact valid for any $a > -1$ as we will show in the end.

We start with the observation that $2u_x = |F_a|^{-\frac{a}{a+1}}(F_a + \bar{F}_a)$ and $2u_y = i|F_a|^{-\frac{a}{a+1}}(F_a - \bar{F}_a)$ since

$$|F_a|^{-\frac{a}{a+1}}(F_a + \bar{F}_a) = f + \bar{f} = u_x - iu_y + u_x + iu_y = 2u_x$$

and

$$i|F_a|^{-\frac{a}{a+1}}(F_a - \bar{F}_a) = i(f - \bar{f}) = i(u_x - iu_y - u_x - iu_y) = 2u_y$$

where

$$|F_a|^{-\frac{a}{a+1}}F_a = (|f|^a|f|)^{-\frac{a}{a+1}}F_a = |f|^{(a+1)(-\frac{a}{a+1})}F_a = |f|^{-a}F_a = |f|^{-a}|f|^a f = f.$$

Since $(u_x)_y = (u_y)_x$ it follows that

$$\frac{\partial}{\partial y}[|F_a|^{-\frac{a}{a+1}}(F_a + \bar{F}_a)] = i\frac{\partial}{\partial x}[|F_a|^{-\frac{a}{a+1}}(F_a - \bar{F}_a)].$$

This is equivalent to

$$\operatorname{Im}\frac{\partial}{\partial \bar{z}}(|F_a|^{-\frac{a}{a+1}}F_a) = 0$$

as it means that the real and imaginary components are orthogonal. It is also seen from

$$\operatorname{Im}\frac{\partial}{\partial \bar{z}}(|F_a|^{-\frac{a}{a+1}}F_a) = \operatorname{Im}\frac{\partial}{\partial \bar{z}}f = \operatorname{Im}\frac{1}{2}\left(\frac{\partial}{\partial x}f + i\frac{\partial}{\partial y}f\right) = \operatorname{Im}\frac{1}{2}(u_{xx} - iu_{yx} + iu_{xy} + u_{yy}) = 0.$$

This implies that

$$\operatorname{Im}\frac{\partial}{\partial z}F_a - \operatorname{Im}\overline{\frac{\partial}{\partial \bar{z}}F_a} = -\frac{a}{a+2}\left[\frac{\bar{F}_a}{F_a}\frac{\partial}{\partial z}F_a - \frac{F_a}{\bar{F}_a}\overline{\frac{\partial}{\partial z}F_a}\right] \tag{2.10}$$

since

$$\begin{aligned}
0 = \operatorname{Im}\left[\frac{\partial}{\partial \bar{z}}(|F_a|^{-\frac{a}{a+1}}F_a)\right] = |F_a|^{-\frac{a}{a+1}}\operatorname{Im}\left[\frac{\partial}{\partial \bar{z}}F_a\right] + \operatorname{Im}\left[F_a\frac{\partial}{\partial \bar{z}}|F_a|^{-\frac{a}{a+1}}\right] \Leftrightarrow \\
\operatorname{Im}\left[\frac{\partial}{\partial \bar{z}}F_a\right] = -|F_a|^{\frac{a}{a+1}}\operatorname{Im}\left[F_a\frac{\partial}{\partial \bar{z}}|F_a|^{-\frac{a}{a+1}}\right]
\end{aligned}$$

where it was used that $|F_a|^{-\frac{a}{a+1}}$ is a scalar, which gives us

$$\begin{aligned}
\operatorname{Im}\frac{\partial}{\partial \bar{z}}F_a &= -|F_a|^{\frac{a}{a+1}}\operatorname{Im}\left[F_a\frac{\partial}{\partial \bar{z}}|F_a|^{-\frac{a}{a+1}}\right] \\
&= -|F_a|^{\frac{a}{a+1}}\left(-\frac{a}{a+1}\right)|F_a|^{-\frac{a}{a+1}}|F_a|^{-1}\operatorname{Im}\left[F_a\frac{\partial}{\partial \bar{z}}|F_a|\right] \\
&= -|F_a|^{\frac{a}{a+1}}\left(-\frac{a}{a+1}\right)|F_a|^{-\frac{a}{a+1}}|F_a|^{-1}\operatorname{Im}\left[F_a\frac{\partial}{\partial \bar{z}}(|F_a|^2)^{1/2}\right] \\
&= \frac{a}{a+1}\frac{1}{|F_a|}\operatorname{Im}\left[F_a\frac{\partial}{\partial \bar{z}}(|F_a|^2)^{1/2}\right]
\end{aligned}$$

Consider $g = (|F_a|^2)^{1/2} = |F_a|$. We have

$$\frac{\partial}{\partial \bar{z}}g^2 = 2g\frac{\partial g}{\partial \bar{z}} \Leftrightarrow \frac{\partial g}{\partial \bar{z}} = \frac{1}{2g}\frac{\partial g^2}{\partial \bar{z}} = \frac{1}{2|F_a|}\frac{\partial |F_a|^2}{\partial \bar{z}}.$$

This gives us

$$\begin{aligned}
\operatorname{Im}\frac{\partial}{\partial z}F_a =& \frac{a}{a+1}\frac{1}{|F_a|}\operatorname{Im}\left[\frac{F_a}{2|F_a|}\frac{\partial}{\partial \bar{z}}(|F_a|^2)\right] \\
=& \frac{a}{a+1}\frac{1}{|F_a|}\operatorname{Im}\left[\frac{F_a}{2|F_a|}\frac{\partial}{\partial \bar{z}}(F_a\cdot\bar{F}_a)\right] \\
=& \frac{a}{2(a+1)}\frac{1}{|F_a|^2}\operatorname{Im}\left[F_a\bar{F}_a\frac{\partial}{\partial \bar{z}}F_a + F_a^2\frac{\partial}{\partial \bar{z}}\bar{F}_a\right] \\
=& \frac{a}{2(a+1)}\frac{1}{|F_a|^2}\left(|F_a|^2\operatorname{Im}\left[\frac{\partial}{\partial \bar{z}}F_a\right] + \operatorname{Im}\left[F_a^2\frac{\partial}{\partial \bar{z}}\bar{F}_a\right]\right) \\
=& \frac{a}{2(a+1)}\frac{1}{F_a\bar{F}_a}\left(|F_a|^2\operatorname{Im}\left[\frac{\partial}{\partial \bar{z}}F_a\right] + \operatorname{Im}\left[F_a^2\frac{\partial}{\partial \bar{z}}\bar{F}_a\right]\right) \\
=& \frac{a}{2(a+1)}\operatorname{Im}\left[\frac{\partial}{\partial \bar{z}}F_a\right] + \frac{a}{2(a+1)}\operatorname{Im}\left[\frac{F_a}{\bar{F}_a}\frac{\partial}{\partial \bar{z}}\bar{F}_a\right].
\end{aligned}$$

We manipulate this to obtain the following

$$\begin{aligned}
\operatorname{Im}\frac{\partial}{\partial \bar{z}}F_a &= \frac{a}{2(a+1)}\operatorname{Im}\left[\frac{\partial}{\partial \bar{z}}F_a\right] + \frac{a}{2(a+1)}\operatorname{Im}\left[\frac{F_a}{\bar{F}_a}\frac{\partial}{\partial \bar{z}}\bar{F}_a\right] \Leftrightarrow \\
\left(1-\frac{a}{2(a+1)}\right)\operatorname{Im}\left[\frac{\partial}{\partial \bar{z}}F_a\right] &= \frac{a}{2(a+1)}\operatorname{Im}\left[\frac{F_a}{\bar{F}_a}\overline{\frac{\partial}{\partial z}F_a}\right] \Leftrightarrow \\
\frac{a+2}{2(a+1)}\operatorname{Im}\left[\frac{\partial}{\partial \bar{z}}F_a\right] &= \frac{a}{2(a+1)}\operatorname{Im}\left[\frac{F_a}{\bar{F}_a}\overline{\frac{\partial}{\partial z}F_a}\right] \Leftrightarrow \\
\operatorname{Im}\left[\frac{\partial}{\partial \bar{z}}F_a\right] &= \frac{a}{a+2}\operatorname{Im}\left[\frac{F_a}{\bar{F}_a}\overline{\frac{\partial}{\partial z}F_a}\right].
\end{aligned}$$

Now we get (2.10) as follows

$$\begin{aligned}
\operatorname{Im}\frac{\partial}{\partial \bar{z}}F_a - \operatorname{Im}\overline{\frac{\partial}{\partial \bar{z}}F_a} &= \frac{a}{a+2}\left(\frac{F_a}{\bar{F}_a}\overline{\frac{\partial}{\partial z}F_a} - \overline{\frac{F_a}{\bar{F}_a}\overline{\frac{\partial}{\partial z}F_a}}\right) \\
&= \frac{a}{a+2}\left(\frac{F_a}{\bar{F}_a}\overline{\frac{\partial}{\partial z}F_a} - \frac{\bar{F}_a}{F_a}\frac{\partial}{\partial z}F_a\right) \\
&= -\frac{a}{a+2}\left(\frac{\bar{F}_a}{F_a}\frac{\partial}{\partial z}F_a - \frac{F_a}{\bar{F}_a}\overline{\frac{\partial}{\partial z}F_a}\right).
\end{aligned}$$

We acquire the vector field

$$2|\nabla u|^{p-2}\nabla u = |F_a|^{\frac{p-2-a}{a+1}}((F_a+\bar{F}_a)+i(F_a-\bar{F}_a))$$

The fact that f is p-harmonic means we have

$$\frac{\partial}{\partial x}[|F_a|^{\frac{p-2-a}{a+1}}(F_a+\bar{F}_a)] + i\frac{\partial}{\partial y}[|F_a|^{\frac{p-2-a}{a+1}}(F_a-\bar{F}_a)] = 0.$$

This means that

$$\operatorname{Re}\frac{\partial}{\partial \bar{z}}(|F_a|^{\frac{p-2-a}{a+1}}F_a) = 0,$$

which implies that

$$\operatorname{Re}\frac{\partial}{\partial z}F_a - \operatorname{Re}\overline{\frac{\partial}{\partial \bar{z}}F_a} = -\frac{2-p-a}{a+p}\left[\frac{\bar{F}_a}{F_a}\frac{\partial}{\partial z}F_a - \frac{F_a}{\bar{F}_a}\overline{\frac{\partial}{\partial z}F_a}\right] \tag{2.11}$$

since

$$0 = \operatorname{Re} \frac{\partial}{\partial \bar{z}} (|F_a|^{\frac{p-2-a}{a+1}} F_a) = |F_a|^{\frac{p-2-a}{a+1}} \operatorname{Re}\left[F_a \frac{\partial}{\partial \bar{z}} F_a\right] + \operatorname{Re}\left[F_a \frac{\partial}{\partial \bar{z}} |F_a|^{\frac{p-2-a}{a+1}}\right]$$

again since $|F_a|^{-\frac{p-2-a}{a+1}}$ is a scalar, and

$$\begin{aligned}
0 = |F_a|^{\frac{p-2-a}{a+1}} \operatorname{Re}\left[F_a \frac{\partial}{\partial \bar{z}} F_a\right] + \operatorname{Re}\left[F_a \frac{\partial}{\partial \bar{z}} |F_a|^{\frac{p-2-a}{a+1}}\right] &\Leftrightarrow \\
\operatorname{Re} \frac{\partial}{\partial \bar{z}} F_a = -|F_a|^{-\frac{p-2-a}{a+1}} \operatorname{Re}\left[F_a \frac{\partial}{\partial \bar{z}} |F_a|^{\frac{p-2-a}{a+1}}\right] & \\
= -|F_a|^{-\frac{p-2-a}{a+1}} \left(\frac{p-2-a}{a+1}\right) |F_a|^{\frac{p-2-a}{a+1}} |F_a|^{-1} \operatorname{Re}\left[F_a \frac{\partial}{\partial \bar{z}} |F_a|\right] & \\
= -\frac{p-2-a}{a+1} \frac{1}{|F_a|} \operatorname{Re}\left[F_a \frac{\partial}{\partial \bar{z}} (F_a \cdot \bar{F}_a)^{1/2}\right] & \\
= -\frac{p-2-a}{a+1} \frac{1}{|F_a|} \operatorname{Re}\left[\frac{F_a}{2|F_a|} \frac{\partial}{\partial \bar{z}} (F_a \cdot \bar{F}_a)\right] & \\
= -\frac{p-2-a}{2(a+1)} \frac{1}{|F_a|^2} \operatorname{Re}\left[F_a \bar{F}_a \frac{\partial}{\partial \bar{z}} F_a + F_a^2 \frac{\partial}{\partial \bar{z}} F_a\right] & \\
= -\frac{p-2-a}{2(a+1)} \frac{1}{|F_a|^2} \left(F_a \bar{F}_a \operatorname{Re}\left[\frac{\partial}{\partial \bar{z}} F_a\right] + \operatorname{Re}\left[F_a^2 \frac{\partial}{\partial \bar{z}} F_a\right]\right) & \\
= -\frac{p-2-a}{2(a+1)} \operatorname{Re}\left[\frac{\partial}{\partial \bar{z}} F_a\right] - \frac{p-2-a}{2(a+1)} \operatorname{Re}\left[\frac{F_a}{\bar{F}_a} \frac{\partial}{\partial \bar{z}} \bar{F}_a\right]. &
\end{aligned}$$

Again

$$\begin{aligned}
\left(1 + \frac{p-2-a}{2(a+1)}\right) \operatorname{Re}\left[\frac{\partial}{\partial \bar{z}} F_a\right] = -\frac{p-2-a}{2(a+1)} \operatorname{Re}\left[\frac{F_a}{\bar{F}_a} \overline{\frac{\partial}{\partial z} F_a}\right] &\Leftrightarrow \\
\Leftrightarrow \frac{p+a}{2(a+1)} \operatorname{Re}\left[\frac{\partial}{\partial \bar{z}} F_a\right] = -\frac{p-2-a}{2(a+1)} \operatorname{Re}\left[\frac{F_a}{\bar{F}_a} \overline{\frac{\partial}{\partial z} F_a}\right] &\Leftrightarrow \\
\Leftrightarrow \operatorname{Re}\left[\frac{\partial}{\partial \bar{z}} F_a\right] = -\frac{p-2-a}{a+p} \operatorname{Re}\left[\frac{F_a}{\bar{F}_a} \overline{\frac{\partial}{\partial z} F_a}\right]. &
\end{aligned}$$

Thus

$$\begin{aligned}
\operatorname{Re} \frac{\partial}{\partial \bar{z}} F_a + \operatorname{Re} \overline{\frac{\partial}{\partial \bar{z}} F_a} &= -\frac{p-2-a}{a+p} \left(\frac{F_a}{\bar{F}_a} \overline{\frac{\partial}{\partial z} F_a} + \overline{\frac{F_a}{\bar{F}_a} \overline{\frac{\partial}{\partial z} F_a}}\right) \\
&= -\frac{p-2-a}{a+p} \left(\frac{F_a}{\bar{F}_a} \overline{\frac{\partial}{\partial z} F_a} + \frac{\bar{F}_a}{F_a} \frac{\partial}{\partial z} F_a\right) \\
&= -\frac{p-2-a}{a+p} \left(\frac{\bar{F}_a}{F_a} \frac{\partial}{\partial z} F_a + \frac{F_a}{\bar{F}_a} \overline{\frac{\partial}{\partial z} F_a}\right)
\end{aligned}$$

Adding (2.10) and (2.11) together we get

$$\frac{\partial}{\partial \bar{z}} F_a = q_1 \frac{\partial}{\partial \bar{z}} F_a + q_2 \overline{\frac{\partial}{\partial \bar{z}} F_a} \tag{2.12}$$

where

$$q_1 = -\frac{1}{2}\left(\frac{p-2-a}{a+p} + \frac{a}{a+2}\right)\frac{\bar{F}_a}{F_a} \quad \text{and} \quad q_2 = -\frac{1}{2}\left(\frac{p-2-a}{a+p} - \frac{a}{a+2}\right)\frac{F_a}{\bar{F}_a} \tag{2.13}$$

since

$$
\begin{aligned}
2\frac{\partial}{\partial \bar z}F_a &= \left(\frac{\partial}{\partial \bar z}F_a - \overline{\frac{\partial}{\partial \bar z}F_a}\right) + \left(\frac{\partial}{\partial \bar z}F_a + \overline{\frac{\partial}{\partial \bar z}F_a}\right)\\
&= -\frac{a}{a+2}\left(\frac{\bar F_a}{F_a}\frac{\partial}{\partial z}F_a - \frac{F_a}{\bar F_a}\overline{\frac{\partial}{\partial z}F_a}\right) - \frac{p-2-a}{a+p}\left(\frac{\bar F_a}{F_a}\frac{\partial}{\partial z}F_a + \frac{F_a}{\bar F_a}\overline{\frac{\partial}{\partial z}F_a}\right)\\
&= -\frac{a}{a+2}\frac{\bar F_a}{F_a}\frac{\partial}{\partial z}F_a + \frac{a}{a+2}\frac{F_a}{\bar F_a}\overline{\frac{\partial}{\partial z}F_a} - \frac{p-2-a}{a+p}\frac{\bar F_a}{F_a}\frac{\partial}{\partial z}F_a - \frac{p-2-a}{a+p}\frac{F_a}{\bar F_a}\overline{\frac{\partial}{\partial z}F_a} \Leftrightarrow\\
\frac{\partial}{\partial \bar z}F_a &= -\frac{1}{2}\left(\frac{a}{a+2} + \frac{p-2-a}{a+p}\right)\frac{\bar F_a}{F_a}\frac{\partial}{\partial z}F_a - \frac{1}{2}\left(\frac{p-2-a}{a+p} + \frac{a}{a+2}\right)\frac{F_a}{\bar F_a}\overline{\frac{\partial}{\partial z}F_a}.
\end{aligned}
$$

The system is elliptic as shown below for $p \geq 2$ and $a > -1$ since

$$
|q_1| + |q_2| = \max\left[\left|\frac{p-2-a}{a+p}\right|, \left|\frac{a}{a+2}\right|\right] < 1.
$$

That the inequality holds as the numerators are smaller than the denominators for said a and p as is seen by inspection. This bound and the fact that $F_a \in W^{1,2}_{loc}(\Omega)$ for $a = \frac{p-2}{2}$ implies that F_a is a quasiregular mapping. To improve this result to $a > -1$ we use the quasiconformal radial stretching

$$
\zeta \to \zeta|\zeta|^{\alpha-1} \quad \text{and} \quad \alpha > 0 \Leftrightarrow \alpha - 1 > -1,
$$

where

$$
\alpha = \frac{2(a-1)}{p}.
$$

Since $F_a = |f|^a f = f$ for $a = 0$ we obtain

$$
\begin{aligned}
\frac{\partial f(z)}{\partial \bar z} &= -\frac{1}{2}\frac{p-2}{p}\frac{\bar f}{f}\frac{\partial f(z)}{\partial z} - \frac{1}{2}\frac{p-2}{p}\frac{f}{\bar f}\overline{\frac{\partial f(z)}{\partial z}}\\
&= -\frac{1}{2}\left(1-\frac{2}{p}\right)\frac{\bar f}{f}\frac{\partial f(z)}{\partial z} - \frac{1}{2}\left(1-\frac{2}{p}\right)\frac{f}{\bar f}\overline{\frac{\partial f(z)}{\partial z}}\\
&= \left(\frac{1}{p}-\frac{1}{2}\right)\frac{\bar f}{f}\frac{\partial f(z)}{\partial z} + \left(\frac{1}{p}-\frac{1}{2}\right)\frac{f}{\bar f}\overline{\frac{\partial f(z)}{\partial z}}\\
&= \left(\frac{1}{p}-\frac{1}{2}\right)\left(\frac{\bar f}{f}\frac{\partial f(z)}{\partial z} + \frac{f}{\bar f}\overline{\frac{\partial f(z)}{\partial z}}\right)
\end{aligned}
$$

from (2.13), and

$$
\begin{aligned}
\left|\frac{\partial f(z)}{\partial \bar z}\right| &\leq \left|\frac{1}{2}\left(1-\frac{2}{p}\right)\frac{\bar f(z)}{f(z)}\frac{\partial f(z)}{\partial z} + \frac{1}{2}\left(1-\frac{2}{p}\right)\frac{f(z)}{\bar f(z)}\overline{\frac{\partial f(z)}{\partial z}}\right|\\
&= \frac{1}{2}\left(1-\frac{2}{p}\right)\left(\left|\frac{\bar f(z)}{f(z)}\frac{\partial f(z)}{\partial z}\right| + \left|\frac{f(z)}{\bar f(z)}\overline{\frac{\partial f(z)}{\partial z}}\right|\right)\\
&= \frac{1}{2}\left(1-\frac{2}{p}\right)\cdot 2\left|\frac{\partial f(z)}{\partial z}\right| = \left(1-\frac{2}{p}\right)\left|\frac{\partial f(z)}{\partial z}\right|
\end{aligned}
$$

for almost every $z \in \Omega$, where we used the triangle inequality, that $|f| = |\bar f|$ for all $f \in \mathbb{C}$, and that $F_a = |f|^a f = f$ for $a = 0$. □

2.6 In the asymptotic expansion

The following discussion comes from [6]. Here, we will first show the asymptotic expansion for the p-Laplacian, and then for the ∞-Laplacian, as they will be needed for the Tug-of-War games.

Theorem 2.7. *Let $2 \leq p \leq \infty$ and u be a continuous function in a domain $\Omega \subset \mathbb{R}^n$. There are constants α and β such that the asymptotic expansion*

$$u(x) = \frac{\alpha}{2}\left(\max_{\bar{B}_\epsilon(x)} u + \min_{\bar{B}_\epsilon(x)} u\right) + \beta \fint_{B_\epsilon(x)} u(y)dy + o(\epsilon^2) \quad as \quad \epsilon \to 0,$$

holds for all $x \in \Omega$ in the viscosity sense if and only if

$$\Delta_p u(x) = 0$$

in the viscosity sense.

Proof. Expand the p-Laplacian as follows

$$\Delta_p u = (p-2)|\nabla u|^{p-4}\langle D^2 u \nabla u, \nabla u\rangle + |\nabla u|^{p-2}\Delta u \tag{2.14}$$

Assume that u is a smooth function such that $\Delta u \neq 0$. The formal expansion (2.14) shows that u is a solution to $\Delta_p u = 0$ if and only if

$$(p-2)\Delta_\infty u + \Delta u = 0 \tag{2.15}$$

as shown earlier. Moreover

$$u(x) - \fint_{B_\epsilon(x)} u dy = -\epsilon^2 \Delta u(x) \frac{1}{N} \fint_{B(0,1)} |z|^2 dz + o(\epsilon^2), \tag{2.16}$$

and

$$u(x) - \frac{1}{2}\left\{\max_{\bar{B}_\epsilon(x)} u + \min_{\bar{B}_\epsilon(x) u}\right\} \approx u(x) - \frac{1}{2}\left\{u\left(x + \epsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right) + u\left(x - \epsilon\frac{\nabla u(x)}{|\nabla u(x)|}\right)\right\} = \epsilon^2 \Delta_\infty u(x) + o(\epsilon^2) \tag{2.17}$$

is obtained by Taylor expansion and using a Lagrange multiplier argument for the growth of the function. Denote the volume of the unit ball in $\mathbb{R}^N$ by Ω_N, and the area of the unit sphere in $\mathbb{R}^N$ by σ_{N-1}. Since

$$\frac{\sigma_{N-1}}{\Omega_N} = N$$

it follows that

$$\frac{1}{N}\fint_{B(0,1)} |z|^2 dz = \frac{1}{N+2}.$$

Multiply (2.16) and (2.17) by suitable constants α and β such that it adds up to (2.15). We start the next part of the proof by introducing the the viscosity characterizations of p-harmonic functions for $2 \leq p < \infty$ and for then $p = \infty$.

Definition 2.9. *Consider the equation*

$$-\operatorname{div}(|\Delta u|^{p-2}\nabla u) = 0$$

for $2 \leq p < \infty$.

1. *A* ***lower semi-continuous function*** *u is a viscosity supersolution if*

$$-(p-2)\Delta_\infty \phi(x) - \Delta\phi(x) \geq 0$$

for every $\phi \in C^2$ such that $u - \phi$ has a strict minimum at $x \in \Omega$ and $\Delta\phi(x) \neq 0$.

2. *A* ***upper semi-continuous function*** *u is a viscosity subsolution if*

$$-(p-2)\Delta_\infty \phi(x) - \Delta\phi(x) \leq 0$$

for every $\phi \in C^2$ such that $u - \phi$ has a strict maximum at $x \in \Omega$ and $\Delta\phi(x) \neq 0$.

3. *u is a viscosity solution if it is both a supersolution and a subsolution.*

For the case $p = \infty$ the test functions must restricted to the set

$$S(x) = \left\{ \phi \in C^2 : \nabla\phi(x) \neq 0 \vee \left(\nabla\phi(x) = 0 \wedge \exists \lim_{y \to x} \frac{\phi(y) - \phi(x)}{|y-x|^2} = \frac{\Delta_\infty \phi(x)}{2} \right) \right\}.$$

Definition 2.10. *Consider the equation $-\Delta_\infty u = 0$.*

1. *A* ***lower semi-continuous function*** *u is a viscosity supersolution if*

$$-\Delta_\infty \phi(x) \geq 0$$

for every $\phi \in S(x)$ such that $u - \phi$ has a strict minimum at $x \in \Omega$.

2. *A* ***upper semi-continuous function*** *u is a viscosity subsolution if*

$$-\Delta_\infty \phi(x) \leq 0$$

for every $\phi \in S(x)$ such that $u - \phi$ has a strict maximum at $x \in \Omega$.

3. *u is a viscosity solution if it is both a viscosity supersolution and a viscosity subsolution.*

We start by considering asymptotic expansions for smooth functions that involve the ∞-Laplacian and the regular Laplacian. Choose a point $x \in \Omega$ and $\phi \in B_\epsilon(x)$. Let x_1^ϵ and x_2^ϵ be the points where $\phi(x)$ attains its maximum and minimum in $\bar{B}_\epsilon(x)$ respectively. This is written as

$$\phi(x_1^\epsilon) = \max_{y \in \bar{B}_\epsilon(x)} \phi(y) \quad \text{and} \quad \phi(x_2^\epsilon) = \min_{y \in \bar{B}_\epsilon(x)} \phi(y)$$

The Taylor expansion of ϕ at x is

$$\phi(y) = \phi(x) + \nabla u(x) \cdot (y-x) + \frac{1}{2}\langle D^2\phi(x)(y-x), (y-x)\rangle + o(|y-x|^2) \quad \text{as} \quad |y-x| \to 0.$$

Evaluation at x_1^ϵ yields

$$\phi(x_1^\epsilon) = \phi(x_1^\epsilon) + \nabla u(x) \cdot (x_1^\epsilon - x) + \frac{1}{2}\langle D^2\phi(x)(x_1^\epsilon - x), (x_1^\epsilon - x)\rangle + o(\epsilon^2) \quad \text{as} \quad \epsilon \to 0. \tag{2.18}$$

Evaluation at the antipodal point of x_1^ϵ with respect to x, which is given by $\tilde{x}_1^\epsilon = 2x - x_1^\epsilon$, yields

$$\phi(\tilde{x}_1^\epsilon) = \phi(x_1^\epsilon) - \nabla u(x) \cdot (x_1^\epsilon - x) + \frac{1}{2}\langle D^2\phi(x)(x_1^\epsilon - x), (x_1^\epsilon - x)\rangle + o(\epsilon^2) \quad \text{as} \quad \epsilon \to 0. \tag{2.19}$$

Adding (2.18) and (2.19) yields

$$\phi(\tilde{x}_1^\epsilon) + \phi(x_1^\epsilon) - 2\phi(x) = \langle D^2\phi(x)(x_1^\epsilon - x), (x_1^\epsilon - x)\rangle + o(\epsilon^2).$$

Since x_1^ϵ is the point where the minimum of ϕ is attained it follows that

$$\phi(\tilde{x}_1^\epsilon) + \phi(x_1^\epsilon) - 2\phi(x) \le \max_{y \in \bar{B}_\epsilon(x)} \phi(y) + \min_{y \in \bar{B}_\epsilon(x)} \phi(y) - 2\phi(x).$$

Thus

$$\frac{1}{2}\left\{ \max_{y \in \bar{B}_\epsilon(x)} \phi(y) + \min_{y \in \bar{B}_\epsilon(x)} \phi(y) \right\} - \phi(x) \ge \frac{1}{2}\langle D^2\phi(x)(x_1^\epsilon - x), (x_1^\epsilon - x)\rangle + o(\epsilon^2). \tag{2.20}$$

The same process for x_2^ϵ yields

$$\frac{1}{2}\left\{ \max_{y \in \bar{B}_\epsilon(x)} \phi(y) + \min_{y \in \bar{B}_\epsilon(x)} \phi(y) \right\} - \phi(x) \le \frac{1}{2}\langle D^2\phi(x)(x_1^\epsilon - x), (x_1^\epsilon - x)\rangle + o(\epsilon^2). \tag{2.21}$$

The analoguous derivation for the expansion of the regular Laplacian is now performed. Taking the average of both sides of the Taylor expansion of ϕ at x yields

$$⨍_{B_\epsilon(x)} \phi(y)dy = \phi(x) + \sum_{i,j=1}^{N} \frac{\partial^2 \phi}{\partial x_i^2}(x) ⨍_{B_\epsilon(0)} \frac{1}{2} z_i z_j dz + o(\epsilon^2), \tag{2.22}$$

where the integral is zero when $i \neq j$. Moreover

$$⨍_{B_\epsilon(0)} z_i^2 dz = \frac{1}{N} ⨍_{B_\epsilon(0)} |z|^2 dz = \frac{1}{N\Omega_N \epsilon^N} \int_0^\epsilon \int_{\partial B_\rho} \rho^2 dS d\rho = \frac{\sigma_{N-1}\epsilon^2}{N(N+2)\Omega_N} = \frac{\epsilon^2}{N+2}.$$

Thus (2.22) becomes

$$⨍_{B_\epsilon(x)} \phi(y)dy - \phi(x) = \frac{\epsilon^2}{2(N+2)} \Delta\phi(x) + o(\epsilon^2). \tag{2.23}$$

Assume that $p \ge 2$. Thus $\alpha \ge 0$. Multiply (2.20) with α and (2.23) with β, and add them together. Then

$$\frac{\alpha}{2}\left\{ \max_{y \in \bar{B}_\epsilon(x)} \phi(y) + \min_{y \in \bar{B}_\epsilon(x)} \phi(y) \right\} + \beta ⨍_{B_\epsilon(x)} \phi(y)dy - \phi(x) \ge \tag{2.24}$$

$$\ge \frac{\beta\epsilon^2}{2(N+2)} \left((p-2)\langle D^2\phi(x) \left(\frac{x_1^\epsilon - x}{\epsilon} \right), \left(\frac{x_1^\epsilon - x}{\epsilon} \right) \rangle + \Delta\phi(x) \right) + o(\epsilon^2) \tag{2.25}$$

Note that $x_1^\epsilon \in \partial B_\epsilon(x)$ for small enough $\epsilon > 0$ whenever $\nabla\phi(x) \neq 0$. Assume the contrary. Then there exists a subsequence $x_1^{\epsilon_j} \in B_{\epsilon_j}(x)$ of minimium points of ϕ. Thus, $\nabla\phi(x_1^{\epsilon_j}) = 0$, and by continuity, with $x_1^{\epsilon_j}$ as $\epsilon_j \to 0$, it follows that $\nabla\phi(x) = 0$. Using Lagrange multipliers now shows that

$$\lim_{\epsilon \to 0} \frac{x_1^\epsilon - x}{\epsilon} = -\frac{\nabla\phi}{|\nabla\phi|}(x). \tag{2.26}$$

The prerequisites are needed to show that if the asymptotic mean value formula holds, then u is a viscosity solution. Assume to this end that u is a function which satisfies the asymptotic mean value formula in the viscosity sense.

Definition 2.11. *A continuous function u satisfies*

$$u(x) = \frac{\alpha}{2}\left\{ \max_{y \in \bar{B}_\epsilon(x)} \phi(y) + \min_{y \in \bar{B}_\epsilon(x)} \phi(y) \right\} + \beta ⨍_{B_\epsilon(x)} \phi(y)dy + o(\epsilon^2) \quad to \quad \epsilon \to 0$$

in the viscosity sense if

1. we have

$$0 \geq -\phi(x) + \frac{\alpha}{2}\left\{\max_{y\in \bar{B}_\epsilon(x)} \phi(y) + \min_{y\in \bar{B}_\epsilon(x)} \phi(y)\right\} + \beta \fint_{B_\epsilon(x)} \phi(y)dy + o(\epsilon^2)$$

for every $\phi \in C^2$ such that $u - \phi$ has a strict minimum at $x \in \bar{\Omega}$.

2. we have

$$0 \leq -\phi(x) + \frac{\alpha}{2}\left\{\max_{y\in \bar{B}_\epsilon(x)} \phi(y) + \min_{y\in \bar{B}_\epsilon(x)} \phi(y)\right\} + \beta \fint_{B_\epsilon(x)} \phi(y)dy + o(\epsilon^2)$$

for every $\phi \in C^2$ such that $u - \phi$ has a strict maximum at $x \in \bar{\Omega}$.

Hence, consider a smooth function ϕ such that $u - \phi$ is a strict minimum at x and $\phi \in S(x)$ if $p = \infty$. Now

$$0 \geq -\phi(x) + \frac{\alpha}{2}\left\{\max_{y\in \bar{B}_\epsilon(x)} \phi(y) + \min_{y\in \bar{B}_\epsilon(x)} \phi(y)\right\} + \beta \fint_{B_\epsilon(x)} \phi(y)dy + o(\epsilon^2).$$

Thus

$$0 \geq \frac{\beta\epsilon^2}{2(N+2)}\left((p-2)\langle D^2\phi(x)\left(\frac{x_1^\epsilon - x}{\epsilon}\right), \left(\frac{x_1^\epsilon - x}{\epsilon}\right)\rangle + \Delta\phi(x)\right) + o(\epsilon^2),$$

by (2.24). If $\nabla\phi(x) \neq 0$, then take $\epsilon \to 0$. It follows by (2.26) that

$$0 \geq \frac{\beta}{2(N+2)}((p-2)\Delta_\infty\phi(x) + \Delta\phi(x)).$$

The proof that $\Delta_\infty\phi(x) \leq 0$ is needed. This is done from

$$0 \geq \frac{1}{2}\left\{\max_{\bar{B}_\epsilon(x)} \phi + \min_{\bar{B}_\epsilon(x)} \phi\right\} - \phi(x)$$

The assumption that $\phi \in S(x)$ implies that

$$(\Delta_\infty\phi(x) - \delta)\epsilon^2 \leq \phi(x) - \phi(y) \leq (\Delta_\infty\phi(x) + \delta)\epsilon^2$$

where $\delta > 0$. Thus

$$0 \geq \frac{1}{2}\left\{\max_{\bar{B}_\epsilon(x)}(\phi(y) - \phi(x)) + \min_{\bar{B}_\epsilon(x)} \phi(x) - \phi(y)\right\} \geq (\Delta_\infty\phi(x) - \delta)\epsilon^2.$$

Division by ϵ yields that $\Delta_\infty\phi(x) - \delta \leq 0$, and since δ is arbitrary it follows that $\Delta_\infty\phi(x) \leq 0$. This proves that u is a viscosity supersolution.

The proof that u is a viscosity subsolution is done by considering a the maximum point of the test function to deduce the reverse inequality of (2.24), which is done by using (2.21) and (2.23) and choosing a test function which touches the solution from above.

The reverse implication is proved as follows. Assume that u is a viscosity supersolution. Let ϕ be a smooth test function such that $u - \phi$ has a strict local maximum at $x \in \Omega$. If $p = \infty$, then it is also assumed that $\phi \in S(x)$. If $\nabla\phi(x) \neq 0$, then

$$-(p-2)\Delta_\infty\phi(x) - \Delta\phi(x) \leq 0. \tag{2.27}$$

To prove the result the following equation needs to be proven

$$\limsup_{\epsilon \searrow 0} \frac{1}{\epsilon^2}\left(\frac{\alpha}{2}\left\{\max_{y\in \bar{B}_\epsilon(x)} \phi + \min_{y\in \bar{B}_\epsilon(x)} \phi\right\} + \beta \fint_{B_\epsilon(x)} \phi(y)dy - \phi(x)\right) \geq 0.$$

This follows from (2.24) by dividing with ϵ^2 and use (2.26) and (2.27) to show that the limit is bounded from below by zero.

Assume that $\phi \in S(x)$ and $\Delta_\infty \phi(x) \geq 0$. If $\nabla\phi(x) = 0$ and $p = \infty$, then

$$\limsup_{\epsilon \searrow 0} \frac{1}{\epsilon^2} \left(\frac{\alpha}{2} \left\{ \max_{y \in \bar{B}_\epsilon(x)} \phi + \min_{y \in \bar{B}_\epsilon(x)} \phi \right\} - \phi(x) \right) \geq 0.$$

The argument for the supersolution is analogous. □

We now proceed to the asymptotic expansion for the ∞-Laplacian.

Theorem 2.8. *Assume that $\phi \in C^2(\Omega)$ and that $\nabla\phi(x_0) \neq 0$ at $x_0 \in \Omega$. Then the asymptotic expansion*

$$\phi(x_0) = \frac{1}{2} \left(\max_{\bar{B}_\epsilon(x_0)} \phi + \min_{\bar{B}_\epsilon(x_0)} \phi \right) - \frac{\Delta_\infty \phi(x_0)}{|\nabla\phi(x_0)|^2} \epsilon^2 + o(\epsilon^2)$$

holds.

Proof. Since $\nabla\phi(x_0) \neq 0$ it is possible to choose $\epsilon > 0$ so small that $|\nabla\phi(x_\epsilon)| > 0$ when $|x_0 - x_\epsilon| \leq \epsilon$. The extremal values of ϕ in $\bar{B}_\epsilon(x_0)$ are attained at points for which

$$x = x_0 \pm \epsilon \frac{\nabla\phi(x)}{|\nabla\phi(x)|} \quad \text{where} \quad |x - x_0| = \epsilon. \tag{2.28}$$

This is seen using a Lagrangian multiplier argument. The idea is that we are moving in the direction which the function increases or decreases the most. This is an approximates of the maximum and minimum points. This means that they are approximately antipodal points, which will be important in this discussion.

Moreover

$$\frac{\nabla\phi(x_\epsilon)}{|\nabla\phi(x_\epsilon)|} = \frac{\nabla\phi(x)}{|\nabla\phi(x)|} + o(\epsilon) \quad \text{as} \quad \epsilon \to 0. \tag{2.29}$$

Let $x_1 \in \partial B_\epsilon(x)$ and $\tilde{x}_1 = 2x - x_1$ denote the exact antipodal points. Adding the Taylor expansions

$$\phi(x_1) = \phi(x) + \nabla\phi(x) \cdot (x_1 - x) + \frac{1}{2}\langle D^2\phi(x)(x_1 - x), (x_1 - x)\rangle + o(|x_1 - x|^2) \quad \text{as} \quad |x_1 - x| \to 0$$

and

$$\phi(\tilde{x}_1) = \phi(x) - \nabla\phi(x) \cdot (\tilde{x}_1 - x) + \frac{1}{2}\langle D^2\phi(x)(\tilde{x}_1 - x), (\tilde{x}_1 - x)\rangle + o(|\tilde{x}_1 - x|^2) \quad \text{as} \quad |\tilde{x}_1 - x| \to 0.$$

together yields

$$\phi(x_1) + \phi(\tilde{x}_1) = 2\phi(x) + \langle D^2\phi(x)(x_1 - x), (x_1 - x)\rangle + o(|x_1 - x|^2) \quad \text{as} \quad |x_1 - x| \to 0$$

Select x_1 as the maximal point

$$\phi(x_1) = \max_{|x - x_1| < \epsilon} \phi(x).$$

Inserting (2.28) with the approximation (2.29) yields

$$\begin{aligned}
\max_{B_\epsilon(x)} \phi + \min_{B_\epsilon(x)} \phi &\leq \phi(x_1) + \phi(\tilde{x}_1) \\
&= 2\phi(x) + \langle D^2\phi(x)(x_1 - x), (x_1 - x)\rangle + o(|x_1 - x|^2) \\
&= 2\phi(x) + \left\langle D^2\phi(x) \frac{\nabla\phi(x)}{|\nabla\phi(x)|}, \frac{\nabla\phi(x)}{|\nabla\phi(x)|} \right\rangle \epsilon^2 + o(\epsilon^2) \\
&= 2\phi(x) + \frac{\Delta_\infty \phi(x)}{|\nabla\phi(x)|^2} \epsilon^2 + o(\epsilon^2).
\end{aligned}$$

Select x_2 as the maximal point

$$\phi(x_2) = \min_{|x-x_2|} \phi(x).$$

The reverse inequality is now derived analoguously

$$\max_{B_\epsilon(x)} \phi + \min_{B_\epsilon(x)} \phi \geq \phi(\tilde{x}_2) + \phi(x_2).$$

This proves the asymptotic formula. □

2.7 As a Tug of War game with or without drift

At each instance k a game plays out as described below with probability α, and with probability a point in $B_\epsilon(x_k)$ is chosen instead, where $\alpha, \beta \in [0,1]$ are such that $\alpha + \beta = 1$.

We will now describe the just mentioned game. Here, we follow [13]. A Tug of War game is a zero sum game between two players. That the game is a zero sum game means that the gain $F(x_T)$ of the winner is the loss $F(x_T)$, or, gain $-F(x_T)$, of the looser, with $F(x_T)+(-F(x_T)) = 0$ as the zero sum result. The gains of the winner is the losses of the loser. Thus, the players wants to play the game according to a strategy which maximizes its own gain, and, consequently, minimizes the opponent's as it is zero sum game.

Here, $F : \Gamma_1 \to \mathbb{R}$ is a Lipschitz continuous function which is called the **payoff function** of the game, where $\Omega \subset \mathbb{R}^n$, $\Gamma_1 \subset \partial\Omega$, and $\Gamma_2 = \partial\Omega \backslash \Gamma_1$. The map from Γ_1 is considered simply due to definiteness. The choice does not matter as it is a zero sum game, we just need to know what amount is paid, and to whom.

A function f is called Lipschitz continuous if there exists a real valued constant $K \in \mathbb{R}$ such that $|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$. This value of the game is obtained when the token reaches the boundary $\partial\Omega$ at x_T the terminal time $T \in \mathbb{N}$ and the end games.

The game consists of a sequence of points $\{x_i\}_{i=0}^{T\in\mathbb{N}}$ for $x_0 \in \bar{\Omega}\backslash\Gamma_1$ and $x_T \in \Gamma_\epsilon$, where

$$\Gamma_\epsilon = \{x \in \mathbb{R}^N \backslash D : \text{dist}(x, B) \leq \epsilon\}.$$

When the game starts at the starting time $t = 0$ the token is placed at the starting position x_0. Then, a fair toin is tossed, and the winner of the coin toss gets to move the token to a new position $x_1 \in \overline{B_\epsilon(x_0)} \cap \bar{\Omega}$. Every turn until the game ends proceeds similarly. A coin is tossed and the winner gets to move the token from x_{k-1} to $x_k \in \overline{B_\epsilon(x_{k_1})} \cap \bar{\Omega}$. When the game reaches its terminal state x_T the game ends and the transaction is made.

Every state except for x_0 is a random variable depending on the result of the coin tosses and the strategies adopted by the players. The initial state x_0 of the token is known to both players. Then both each player $i \in \{1, 2\}$ chooses an **action** $a_0^i \in \overline{B_\epsilon(0)}$ that is announced to the other player. Together, these form an **action profile** $a_0 = \{a_0^1, a_0^2\} \in \overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)}$. Then, the next state x_1 of the game, which is obtained by performing the chosen action on the current state, is chosen using a probability distribution $p(\cdot|x_0, a_0)$ in $\bar{\Omega}$, which is given by the fair coin toss in this case.

At state k each player i chooses an action a_k^i based on the **history**

$$h_k = (x_0, a_0, x_1, \ldots, x_{k-1}, a_{k-1}, x_k).$$

If the game ends at $j < k$, then $x_m = x_j$ and $a_m = 0$ for all $j \leq m \leq k$. The current state x_k and the action profile $a_k = \{a_k^1, a_k^2\}$ determine probability distribution $p(\cdot|x_k, a_k)$ of the new state x_{k+1}.

Denote

$$H_k = (\bar{\Omega}\backslash\Gamma_1) \times (\overline{B_\epsilon(0)} \times \overline{B_\epsilon(0)} \times \bar{\Omega})^k$$

as the set of histories up to state k, and

$$H = \bigcup_{k \geq 1} H_k$$

as the set of all histories. H has a measurable structures as it is a product space. The complete history space H_∞ is the set of plays defined as infinite sequences $(x_0, a_0, \dots, a_{k-1}, x_k, \dots)$ endowed with the product topology. Thus, the final payoff function for induces a **Borel-measurable** function on H_∞.

A **pure strategy** $S_i = \{S_i^k\}_k$ for a player i is a sequence of mappings from histories to actions. This means that H maps to $\overline{B_\epsilon(0)}$ such that S_i^k is a Borel-measurable mapping from H_k to $\overline{B_\epsilon(0)}$ that maps histories ending with x_k to elements of $\overline{B_\epsilon(0)}$. This roughly means that a strategy gives the next movement, given that the player wins the coin toss, as a function of the current state and the history of the game.

The initial state x_0 and a profile of strategies $\{S_I, S_{II}\}$ define a unique probability $\mathbb{P}^{x_0}_{S_I, S_{II}}$ on the space of plays H_∞. This is seen by **Kolmogorov's extension theorem** which is proved in [2]. The corresponding expected value is given by $\mathbb{E}^{x_0}_{S_I, S_{II}}$. The expected payoff of the game for player I is defined as

$$V_{x_0,I}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_T)] & \text{if the game terminates almost surely,} \\ -\infty & \text{otherwise.} \end{cases}$$

The expected payoff for player II is defined analogously.

$$V_{x_0,II}(S_I, S_{II}) = \begin{cases} \mathbb{E}^{x_0}_{S_I, S_{II}}[F(x_T)] & \text{if the game terminates almost surely,} \\ +\infty & \text{otherwise.} \end{cases}$$

With this it is possible to define the ϵ-value of the game for player I as

$$u_I^\epsilon(x_0) = \sup_{S_I} \inf_{S_{II}} V_{x_0,I}(S_I, S_{II}),$$

and the ϵ-value of the game for player II as

$$u_{II}^\epsilon(x_0) = \sup_{S_{II}} \inf_{S_I} V_{x_0,II}(S_I, S_{II}).$$

These are the least possible outcomes that each player can expect from the ϵ-game starts at x_0, where games which do not end are severely punished. If $u_I^\epsilon = u_{II}^\epsilon := u_\epsilon$, then the game is said to have a value.

When the game position reaches a strip Γ_ϵ around the boundary of width ϵ, again, namely

$$\Gamma_\epsilon = \{x \in \mathbb{R}^N \backslash D : \operatorname{dist}(x, B) \le \epsilon\},$$

it stops and player I gains the payoff, which player I has tried to maximize the payoff, and player II tried to minimize. This gives a hint that

$$u_\epsilon(x) = \frac{1}{2}\alpha \left(\sup_{\bar{B}_\epsilon(x)} u_\epsilon + \inf_{\bar{B}_\epsilon(x)} u_\epsilon \right) + \beta \fint_{B_\epsilon(x)} u_\epsilon dy,$$

describes the expected payoff of the game. This will be proved thoroughly, but the intuition behind is that it is the sum of the possible cases. The cases are that either player I or player II moves, or a random point is chosen, with the corresponding probabilities.

The reasoning here is that the value function for the players in the Tug of War game with homogeneous noise satisfies the asymptotic mean value property for p-harmonic functions. This is referred to as the dynamic programming principle for Tug of War games with homogeneous noise. This reasoning is valid for the ∞-Laplacian as well since $p = \infty$ simply means that $\beta = 0$. The only difference is that the game in that case proceeds without noise.

We start by stating the Dynamic Programming Principle, whose proof is found in [7].

Lemma 2.1. *The value function for player I satisfies*

$$\begin{cases} u_I^\epsilon(x_0) = \frac{\alpha}{2}\left(\sup_{\bar{B}_\epsilon(x)} u_\epsilon + \inf_{\bar{B}_\epsilon(x)} u_\epsilon\right) + \beta \fint_{B_\epsilon(x)} u_\epsilon u_\epsilon dy & \textit{if} \quad x_0 \in \Omega, \\ u_I^\epsilon(x_0) = F(x_0) & \textit{if} \quad x_0 \in \Gamma_\epsilon \end{cases}$$

The value function for player II satisfies the analoguous equation

We now draw the connection to the previous discussion of p-harmonious functions. See [10] for details.

Definition 2.12. *u_ϵ is a p-harmonious function in Ω whose boundary values are a bounded Borel function $F : \Gamma_\epsilon \to \mathbb{R}$ if*

$$u_\epsilon(x) = \frac{1}{2}\alpha\left(\sup_{\bar{B}_\epsilon(x)} u_\epsilon + \inf_{\bar{B}_\epsilon(x)} u_\epsilon\right) + \beta \fint_{B_\epsilon(x)} u_\epsilon dy,$$

where

$$\alpha = \frac{p-2}{p+N} \quad \textit{and} \quad \beta = \frac{2+N}{p+N}, \tag{2.30}$$

and

$$u_\epsilon(x) = F(x) \quad \textit{for all} \quad x \in \Gamma_\epsilon.$$

This is motivated as follows. If $\alpha = 0$ and $\beta = 1$, then

$$u_\epsilon(x) = \fint_{B_\epsilon(x)} u_\epsilon dy,$$

and, if $\alpha = 1$ and $\beta = 0$, then

$$u_\epsilon(x) = \frac{1}{2}\left(\sup_{\bar{B}_\epsilon(x)} u_\epsilon + \inf_{\bar{B}_\epsilon(x)} u_\epsilon\right).$$

Moreover, when $p = 2$, then

$$\alpha = \frac{p-2}{p+N} = 0 \quad \text{and} \quad \beta = \frac{2+N}{p+N} = 1,$$

and, when $p = \infty$, then

$$\alpha = \lim_{p\to\infty} \frac{p-2}{p+N} = \lim_{p\to\infty} \frac{1-\frac{2}{p}}{1+\frac{N}{p}} = 1,$$

and

$$\beta = \lim_{p\to\infty} \frac{2+N}{p+N} = \lim_{p\to\infty} \frac{\frac{2}{p}+\frac{N}{p}}{1+\frac{N}{p}} = 0. \tag{2.31}$$

That α and β as defined in (2.30) fulfill $\alpha + \beta = 1$ since

$$\alpha + \beta = \frac{p-2}{p+N} + \frac{2+N}{p+N} = \frac{p+N}{p+N} = 1.$$

Moreover

$$\frac{\alpha}{\beta} = \frac{p-2}{N+2},$$

is required as

$$\frac{p-2}{N+2} = \frac{\alpha}{\beta} \Leftrightarrow p-2 = \frac{\alpha}{\beta\frac{1}{N+2}}. \tag{2.32}$$

The relationship in (2.32) is needed since we need to have

$$\frac{1}{2}\alpha\left(\sup_{\bar{B}_\epsilon(x)} u + \inf_{\bar{B}_\epsilon(x)} u\right) + \beta\fint_{B_\epsilon(x)} u dy = u(x) + \alpha\Delta_\infty u(x) + \beta\frac{1}{N+2}\Delta u(x) + o(\epsilon^2) \quad \text{as} \quad \epsilon \to 0,$$

where we want to rewrite the second order operator as

$$\alpha\Delta_\infty u(x) + \beta\frac{1}{N+2}\Delta u(x) = \beta\frac{1}{N+2}\left(\Delta u(x) + \frac{\alpha}{\beta\frac{1}{N+2}}\Delta_\infty u(x)\right),$$

since

$$\Delta u(x) + \frac{\alpha}{\beta\frac{1}{N+2}}\Delta_\infty u(x) = |\nabla u|^{2-p}\{(p-2)\Delta_\infty u(x) + \Delta u(x)\},$$

if

$$p - 2 = \frac{\alpha}{\beta\frac{1}{N+2}}.$$

This is fulfilled by (2.30) as

$$\frac{\alpha}{\beta} = \frac{p-2}{p+N}\Big/\frac{2+N}{p+N} = \frac{p-2}{p+N}\cdot\frac{p+N}{2+N} = \frac{p-2}{N+2}.$$

2.8 Uniqueness of the solutions

This discussion and further details are found in [13]. We start by defining some key concepts used throughout this discussion. We start by defining the notion of martingales.

Definition 2.13. *Let $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ be a probability space. A* ***martingale*** *is a sequence of random variables $\{M_k(\omega)\}_{k=1}^\infty$ where $\omega \in \mathcal{O}$ with respect to the sub-σ fields $\mathcal{F}_k \subset \mathcal{F}$ for $k = 1, 2, \dots$ if it fulfills the following*

1. *Every random variable M_k is measurable with respect to the corresponding σ field $\mathcal{F}_k$, and $\mathbb{E}(|M_k|) < \infty$.*
2. *The inclusion of the σ fields is increasing. That is, $\mathcal{F}_k \subset \mathcal{F}_{k+1}$ for $k = 1, 2, \dots$.*
3. *Every random variable M_k fulfills the relation*

$$\mathbb{E}[M_k|\mathcal{F}_{k-1}] = M_{k-1}$$

almost surely with respect to $\mathbb{P}$.

The sequence is a submartingale or supermartingale if instead

$$\mathbb{E}[M_k|\mathcal{F}_{k-1}] \geq M_{k-1} \quad \text{or} \quad \mathbb{E}[M_k|\mathcal{F}_{k-1}] \leq M_{k-1}$$

almost surely with respect to $\mathbb{P}$, respectively.

The optimal stopping theorem will also come in handy. The details are found in [15].

Theorem 2.9. *Let $\{M_k\}_{k=1}^\infty$ be a martingale, and τ be a bounded stopping time. Then*

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0].$$

If the sequence instead is a submartingale or supermartingale, then

$$\mathbb{E}[M_\tau] \geq \mathbb{E}[M_0] \quad \text{and} \quad \mathbb{E}[M_\tau] \leq \mathbb{E}[M_0].$$

We will also use Fatou's lemma. The details here are found in [8].

Lemma 2.2. *If f_k is nonnegative and Lebesgue measurable for every k, then*

$$\int \liminf_k f_k d\lambda \leq \liminf_k \int f_k d\lambda.$$

The following theorem is a comparison principle, which means that u_I^ϵ and u_{II}^ϵ are the smallest and largest p-harmonious functions, respectively. We use martingale methods similar to those in [16].

The idea of the first proof is that the choice of a strategy according to the minimal values of the test function the second player can make the process a supermartingale. The optional stopping theorem then implies that the expectation of the process is bounded by the test function. This yields an upper bound for the value of the first player.

Theorem 2.10. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and open set. If v_ϵ is a p-harmonious function with boundary values F_v in Γ_ϵ such that $F_v \geq F_{u_I^\epsilon}$, then $v \geq u_I^\epsilon$.*

Proof. Let the first player follow any strategy S_I, and the second player follow a strategy S_{II}^0 such that, at $x_{k-1} \in \Omega$, the player chooses to step to a point that almost minimizes v. That is, to a point $x_k \in \bar{B}_\epsilon(x_{k-1})$ such that

$$v(x_k) \leq \inf_{\bar{B}_\epsilon(x_{k-1})} v + \eta 2^{-k}$$

for some fixed value $\eta > 0$. Moreover, x_0 is the starting point. It follows that

$$\begin{aligned}
&\mathbb{E}^{x_0}_{S_I, S^0_{II}}(v(x_k) + \eta 2^{-k} | x_0, \dots, x_{k-1}) \\
\leq& \frac{\alpha}{2}\left(\inf_{\bar{B}_\epsilon(x_{k-1})} v + \eta 2^{-k} + \sup_{\bar{B}_\epsilon(x_{k-1})} v \right) + \beta \fint_{B_\epsilon(x_{k-1})} v dy + \eta 2^{-k} \\
\leq& v(x_{k-1}) + \eta 2^{-(k-1)},
\end{aligned}$$

where it was used that supremum estimates the strategy of the first player and that v is p-harmonious. Hence

$$M_k = v(x_k) + \eta 2^{-k},$$

is a supermartingale. Moreover, since $F_v \geq F_{u_I^\epsilon}$ at Γ_ϵ, it follows that

$$\begin{aligned}
u_I^\epsilon(x_0) &= \sup_{S_I} \inf_{S_{II}} \mathbb{E}^{x_0}_{S_I, S_{II}}[F_{u_I^\epsilon}(x_T)] \\
&\leq \sup_{S_I} \mathbb{E}^{x_0}_{S_I, S_{II}}[F_v(x_T) + \eta 2^{-T}] \\
&\leq \sup_{S_I} \liminf_{k \to \infty} \mathbb{E}^{x_0}_{S_I, S_{II}}[v(x_{T \wedge k}) + \eta 2^{-(T \wedge k)} \\
&\leq \sup_{S_I} \mathbb{E}^{x_0}_{S_I, S_{II}}[M_0] = v(x_0) + \eta.
\end{aligned}$$

Again, here the optimal stopping theorem for M_k and Fatou's lemma was used. The result follows by choosing $\eta = 0$, which is possible since η was arbitrary. ☐

Similarly, we may prove that u_{II}^ϵ is the largest p-harmonious function. If player II follows a strategy and player I always chooses to step to the point where v is almost maximized, then this means that $v(x_k) - \eta 2^{-k}$ is a submartingale. Again, Fatou's lemma and the optimal stopping theorem proves the claim.

The following theorem will now show that the game has a value. The comparison principle that we just proved together with this result will prove uniqueness of p-harmonious functions with given boundary data.

Theorem 2.11. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and open set, and F be the boundary boundary values on Γ_ϵ. Then $u_I^\epsilon = u_{II}^\epsilon$.*

Proof. Clearly $u_I^\epsilon \leq u_{II}^\epsilon$. Hence, it is only required to show that $u_{II}^\epsilon \leq u_I^\epsilon$. Use the same approach as before. Let the first player follow any strategy S_I, and the second player follow a strategy S_{II}^0 such that, at $x_{k-1} \in \Omega$, the player chooses to step to a point that almost minimizes u_I^ϵ. That is, to a point $x_k \in \bar{B}_\epsilon(x_{k-1})$ such that

$$u_I^\epsilon(x_k) \leq \inf_{\bar{B}_\epsilon(x_{k-1})} v + \eta 2^{-k}$$

for some fixed value $\eta > 0$. Moreover, x_0 is the starting point. It follows that

$$\begin{aligned}
&\mathbb{E}_{S_I,S_{II}^0}^{x_0}(u_I^\epsilon(x_k) + \eta 2^{-k} | x_0, \ldots, x_{k-1}) \\
&\leq \frac{\alpha}{2}\left(\sup_{\bar{B}_\epsilon(x_{k-1})} u_I^\epsilon + \inf_{\bar{B}_\epsilon(x_{k-1})} u_I^\epsilon + \eta 2^{-k}\right) + \beta \fint_{B_\epsilon(x_{k-1})} u_I^\epsilon dy + \eta 2^{-k} \\
&\leq u_I^\epsilon(x_{k-1}) + \eta 2^{-(k-1)},
\end{aligned}$$

where it was used that supremum estimates the strategy of the first player and that v is p-harmonious. Hence

$$M_k = u_I^\epsilon(x_k) + \eta 2^{-k},$$

is a supermartingale. Moreover

$$\begin{aligned}
u_{II}^\epsilon(x_0) &= \sup_{S_I I} \inf_{S_I} \mathbb{E}_{S_I,S_{II}}^{x_0}[F(x_T)] \\
&\leq \sup_{S_I} \mathbb{E}_{S_I,S_{II}}^{x_0}[F(x_T) + \eta 2^{-T}] \\
&\leq \sup_{S_I} \liminf_{k\to\infty} \mathbb{E}_{S_I,S_{II}}^{x_0}[u_I^\epsilon(x_{T\wedge k}) + \eta 2^{-(T\wedge k)} \\
&\leq \sup_{S_I} \mathbb{E}_{S_I,S_{II}}^{x_0}[u_I^\epsilon(x_0) + \eta] = u_I^\epsilon(x_0) + \eta.
\end{aligned}$$

As analoguous to the previous proof. It was also used that the game ends almost surely. Again, choosing $\eta = 0$ completes the proof. □

Now we obtain the main result of this section.

Theorem 2.12. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded and open set. Then there exists a unique p-harmoniuos function in Ω with given boundary conditions F.*

Proof. Existence follows from the fact that the game functions are p-harmonious as they are described by the Dynamic Programming Principle. Uniqueness is implied by Theorem 2.10 and Theorem 2.11. □

Corollary 2.1. *The value of the game with payoff function F coincides with the p-harmonious function with boundary values F.*

We refer to [13] for details about convergence as $\epsilon \to 0$ and that the game ends almost surely. We end with an example which shows that the value functions are discontinuous in general.

Consider the domain $\Omega = (0, 1)$, and

$$u_I^\epsilon(x) = \begin{cases} 1 & \text{if } x \geq 1 \\ 0 & \text{if } x \leq 0. \end{cases}$$

The strategies here are clear. Player I should move ϵ to the right, and player II ϵ to the left. The probability of reaching $x \geq 1$ is uniformly bounded from below in $(0,1)$ by $C = (2/\alpha)^{-(1/1+\epsilon)}$. This is the probability the player I wins every coin toss until the game ends when $x \geq 1$. Hence, $u_I^\epsilon > C > 0$ in $(0,1)$. This gives us a discontinuity at $x = 0$, and, consequently, a discontinuity at $x = \epsilon$. This is seen by noting that u_I^ϵ non-decreasing due

$$u_I^\epsilon(\epsilon-) = \lim_{x \nearrow \epsilon} \frac{1}{2}\alpha \sup_{|x-y|\leq\epsilon} u_I^\epsilon(y) + \frac{\beta}{2\epsilon}\int_0^{2\epsilon} u_I^\epsilon dy = \frac{1}{2}\alpha u_I^\epsilon(2\epsilon-) + \frac{\beta}{2\epsilon}\int_0^{2\epsilon} u_I^\epsilon dy$$

since

$$\sup_{|x-y|\leq\epsilon} u_I^\epsilon(y) = u_I^\epsilon(x+\epsilon) \quad \text{and} \quad \inf_{|x-y|\leq\epsilon} u_I^\epsilon = 0$$

for $x \in (0,\epsilon)$, but

$$u_I^\epsilon(\epsilon+) \geq \frac{1}{2}\alpha C + \lim_{x \searrow \epsilon} \frac{1}{2}\alpha \sup_{|x-y|\leq\epsilon} u_I^\epsilon(y) + \frac{\beta}{2\epsilon}\int_0^{2\epsilon} u_I^\epsilon dy \geq \frac{1}{2}\alpha C + u_I^\epsilon(\epsilon-)$$

since

$$\sup_{|x-y|\leq\epsilon} u_I^\epsilon(y) = u_I^\epsilon(x+\epsilon) \geq u_I^\epsilon(2\epsilon-) \quad \text{and} \quad \inf_{|x-y|\leq\epsilon} u_I^\epsilon = C$$

for $x > \epsilon$.

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