Dewey and mathematical practice: revisiting the distinction between procedural and conceptual knowledge

Tove Österman & Kajsa Bråting

To cite this article: Tove Österman & Kajsa Bråting (2019): Dewey and mathematical practice: revisiting the distinction between procedural and conceptual knowledge, Journal of Curriculum Studies, DOI: 10.1080/00220272.2019.1594388

To link to this article: https://doi.org/10.1080/00220272.2019.1594388

© 2019 The Author(s). Published by Informa UK Limited, trading as Taylor & Francis Group.

Published online: 22 Mar 2019.
Dewey and mathematical practice: revisiting the distinction between procedural and conceptual knowledge

Tove Österman and Kajsa Bråting

Uppsala University, Sweden

ABSTRACT

We identify a recent trend in school mathematics as well as in some of the research literature in mathematics education: an emphasis on the practical uses of mathematics and an increased emphasis on verbalizations as opposed to numerical and computational skills. With tools provided by John Dewey, an early advocate of contextual and practical knowledge, we analyse the common research framework for discussing mathematical knowledge in terms of the procedural and the conceptual. We argue that procedural and conceptual knowledge should not be seen as opposites, and that the tendency to treat them as such might be avoided by emphasising the notion of operational skill. We argue that this is important in order for the students to gain both the contextual knowledge and the computational skill entailed in mathematical knowledge.

KEYWORDS

Mathematics education; Dewey; mathematical knowledge; operational skill

1. Introduction

During the 1990s, an international reform movement in mathematics education emerged in several countries (Boesen et al., 2014; Niss, Bruder, Planas, Turner, & Villa-Ochoa, 2016). Traditionally, curriculum documents have been based on what mathematical content that should be studied, that is, notions, concepts, theories, methods and results (Boesen et al., 2014; Österholm, 2018). The new trend instead aims to clarify the relationship between mathematical content and practice by introducing generic competencies such as conceptual understanding, problem solving, reasoning and communication skills, which can be noted in the mathematics curricula in several countries, for instance the Nordic countries (e.g. Kilpatrick, Swafford, & Findell, 2001; National Council of Teachers of Mathematics [NCTM], 2000; Niss, 2003; Niss & Höjgaard-Jensen, 2002). One expression of this trend is the increasing emphasis on verbal knowledge as well as applied mathematics, sometimes referred to as ‘real-world mathematics’, often motivated by a need to make sure that the students learn something that they can use outside the test situation, i.e. that the students understand how the mathematics they learn in school connects to life outside the classroom (Gainsburg, 2008; Grønmo, 2011; Pongsakdi, Laine, Veermans, Hannula-Sormunen, & Lehtinen, 2016). This is often explained in terms of making sure that the students learn mathematics creatively, based on conceptual knowledge, rather than just memorizing empty formulas (e.g. Boesen et al., 2014; Jonsson, Norqvist, Liljkvist, & Lithner, 2014).

The attempt to involve the students and contextualize the information is not new. In fact, it has been central for many school reformers, among them the American philosopher and progressive educationalist John Dewey. Even if Dewey came to be seen as the main progressive ideologue in many countries, he was not always understood and applied the way he would have preferred. Later in his career he came to criticize parts of the progressive movement quite severely (Dewey, 1988/
1930), and he in turn was criticized for being too academically oriented as well as for lacking a sense of direction (Karier, 1986). The latter was an effect of Dewey not wanting to blueprint a new social order; he thought that education should prepare the students to create their own future, to become active participants in a democratic society open and in the making (Dewey, 1993/1919). This is why the method captured in the slogan Dewey has become famous for, ‘learning by doing’, was so central to his educational philosophy. After the Second World War and the rise of fascism, as strengthening democracy became widely recognized as an important aim for education in many Western countries (Raywid, 1992), Dewey had an impact on the school systems of e.g. Britain, Austria, Germany, and Sweden (Biesta & Miedema, 1996; Brehony, 1997; Hartman, Lundgren, & Hartman, 2004). In Sweden, Dewey has been very influential at least since 1946 (Hartman et al., 2004), to such an extent that some writers speak of a ‘state progressivism’ (Broady, 2011). Lately, the allegedly ‘soft’ educational ideals of progressivism have received increasing criticism in Sweden. Many popular debaters blame the perceived problems, such as lack of discipline and low achievement, on a progressive influence with its alleged relativization of knowledge and placing more value on student participation and activation than on imparting robust knowledge (Enkvist, 2016; Kornhall, 2014; Larsson, 2011). A similar criticism has been mounted by e.g. Michael Young (2009) against the modern British educational system.

Despite the criticism against democratic classrooms and student participation, the concern to provide contextual knowledge that can be used in life outside the classroom remains, and has even been accentuated in the past decades. For instance, in many countries there has been increased focus on the results of the PISA-test (Programme for International Student Assessment), where, in mathematics, real world and narrative word problems take precedence over more traditional classroom mathematics that focus on numerical skills (Organization for Economic Cooperation and Development [OECD], 2009). A related tendency, visible in the research as well as in current mathematical curriculum documents, is an increased emphasis on verbal skills such as reasoning, discussing and communicating mathematics, coupled with less emphasis on the actual ‘doing’ of mathematics through symbolic manipulations, equation solving, and the use of algorithms (Bråting & Österman, 2017). The theoretical underpinnings for this development have been provided by, among other things, a certain interpretation of the distinction between conceptual and procedural knowledge. This distinction has become a platform for empirical studies as well as theoretical frameworks, where the distinction has often gained the status of a dichotomy (Kieran, 2013).

Our aim in this paper is to illuminate the development from numerical and computational skills toward increased emphasis on verbalizations in mathematics education, through contrasting it to the educational philosophy of John Dewey. More specifically, we will use Dewey’s distinction between the psychological and logical aspects of a subject (Dewey, 1959b/1902) in our analysis of the theoretical framework in which mathematical knowledge is described as a product of conceptual and procedural knowledge. It can be helpful to spell out Dewey’s philosophy of education in contrast to the more recent theoretical developments within educational research, in order to gain a better view of the latter. We will examine the philosophical assumptions underlying many scholars’ view of what constitutes mathematical understanding, and argue that the resulting opposition that we can discern between conceptual and procedural knowledge has led to an increased emphasis on verbal as opposed to numerical and computational skills. Since we argue that this opposition is problematic for the aim of learning mathematics, we propose to emphasize operational skill, the ‘doing’ of mathematics. The operational aspect could be said to be embedded in the existing framework, however, we argue that it has been increasingly neglected over time and therefore stands in need of more attention. We believe that this would sharpen the conceptual tools needed to discuss mathematical knowledge, and allow researchers to discuss the importance of numerical and computational skills.

We will concretize our discussion with examples from the Nordic countries, which are interesting to study both from educational research and policy perspectives. As neighbouring countries, the educational systems in the Nordic countries share common values and ideologies for
geographic, cultural and historical reasons, and during the last 50 years the Nordic welfare state has been established as a unique model with a strong emphasis on equity of access to high-quality education (Yang Hansen, Gustafsson, & Rosén, 2011). The implementation of generic competencies in the curriculum documents was developed relatively early in the Nordic countries. Already in the late 1990s, the Danish KOM-project (Kompetencer og matematiklæring) was initiated, with the aim to characterize mathematical knowledge in terms of generic competencies (Niss, 2003; Niss & Højgaard-Jensen, 2002). Furthermore, in the Nordic countries as well as in the English-speaking countries, applied mathematics (or ‘real world mathematics’) has been a driving force in the development of school mathematics, in contrast to countries in Eastern Europe and East Asia (Grønmo, 2011). The strong emphasis on applied mathematics is revealed in the results of international evaluations such as TIMSS (Trends in International Mathematics and Science Study) and PISA, where the performance of students from the Nordic countries has been relatively weak on tasks in pure mathematics, especially in algebra, and better on tasks that have a daily life context (Bråting, Madej, & Hemmi, 2019; Grønmo, 2011). However, it is interesting that the overall results on these evaluations have differed between the Nordic countries over the years.

2. Education according to Dewey

The practical use of mathematics is one of the main concerns of school mathematics today (OECD 2009), just as it was for Dewey, who objected to a view of abstract knowledge as an end in itself and stressed the connection between education and real life, ‘the organic relation of theory and practice’ (Dewey, 1959a/1899, p. 84). Only if the knowledge the students gain in school is important to them, if it addresses their concerns and interests, will it truly be educative, he claimed. This could be misunderstood as advocating a school where the students set the agenda and consequently spend their time playing or socializing rather than learning, and where the teachers lack the authority to discipline them. However, this is not what Dewey had in mind. He specifically argued that ‘student interest’ was not to be identified with whatever the students happened to want at a specific time and place:

It will do harm if child-study leave in the popular mind the impression that a child of a given age has a positive equipment of purposes and interests to be cultivated just as they stand. Interests in reality are but attitudes toward possible experiences; they are not achievements; their worth is in the leverage they afford, not in the accomplishment they represent. To take the phenomena presented at a given age as in any way self-explanatory or self-contained is inevitably to result in indulgence and spoiling. Any power, whether of child or adult, is indulged when it is taken on its given and present level of consciousness. Its genuine meaning is in the propulsion it affords toward a higher level (Dewey, 1959b/1902, pp. 99–100).

Dewey sees children as naturally active, and it is the teacher’s job to see to it that this energy is channelled toward realizing their full potential. In order to do so, the teacher needs to have thorough knowledge of the subject matter, and know each individual student well enough to be able to direct him or her in the right way. This requires psychological and social understanding as well as expertise in the subject matter, and is more demanding than applying ready-made schemata that ignore the individuality of the students, their different backgrounds, abilities and interests (Dewey, 1997/1938). However, the difficulty reflects the importance of the task. For Dewey, the educational ideal is after all a societal ideal—to foster able young adults who can think critically and actively participate in, and form, the society they are a part of (Karier, 1986). This means that the freedom essential for educational purposes is not freedom from external control, but rather the freedom resulting in power of self control (Dewey, 1997/1938). Education should foster a spirit of criticism and cultivate a habit of inquiry (Dewey, 2008/1922), which cannot be conveyed as truths, but require an active undertaking on the part of the students. The Deweyan slogan ‘learning by doing’ is therefore not merely a heuristic device, but necessary for fostering the critical thinking that he considered essential for members of a modern democracy.
In order to address the students, to speak to them rather than at them, the teacher needs to have an understanding of their lives, of their thoughts and interests. She needs to listen to them and attend to them. And in order to help them develop, to reach higher and to continually learn something new, she needs to have thorough knowledge of the subject matter as well as the ability to direct the students’ attention to ever more abstract and difficult matters. Dewey illustrates this as a movement from the psychological aspects of the subject to the logical. To emphasise the psychological aspect is to stress student interest and experiences and to relate the subject matter to what the student is familiar with; to emphasise the logical aspect is instead to focus on the subject as it appears to the expert, as an abstract body of knowledge driven by its internal laws and rules (Dewey, 1959b/1902). The teacher needs to be familiar with both of these aspects in order to teach the students; the teacher of mathematics must master mathematics as an abstract body of knowledge, but also know how to make it intelligible and interesting to the students. Dewey’s point is that these aspects cannot be separated, they are both part of a well functioning education. The child’s interest (the psychological aspect) and the subject matter (the logical aspect) are two limits that define a single process: ‘Just as two points define a straight line, so the present standpoint of the child and the facts and truths of studies define instruction’ (Dewey, 1959b/1902, p. 97). This means that the teacher ideally starts from what interests the students, but that which gives direction to the instruction, the goal, is the ‘organized bodies of truth’—mathematics as abstract knowledge. Education, then, is what takes place in between these two defining points, the movement from student experience to abstract subject matter.

In mathematics, this movement could be exemplified by how children first learn to count concrete objects, and how in school small children are first presented with everyday examples involving familiar objects. During the years to come the instruction will become more and more demanding and move toward what Dewey calls the logical aspect of mathematics: toward mathematics as a goal in itself without a connection to practical interests. However, it is important to keep in mind that these two aspects are interdependent. They are two sides of the same coin, impossible to categorically separate. The logical aspect, the part of mathematics that is driven forward by the internal development of the field rather than by the need of practical applications, would be unthinkable unless mathematics also played a practical role in human life. And the practical applications of mathematics (in for example technology) are dependent on the progress made in the theoretical field of mathematics research. In school these two aspects need to go hand in hand as well, in order for the students to see the education as meaningful as well as be able to develop mathematically.

We propose that there are similarities between the didactical perspective on mathematics and Dewey’s psychological aspect, where student interest, context and practical mathematics are central, whereas the logical perspective entails the context-independent, timeless aspects of mathematics such as symbolic manipulation (i.e. the ‘pure mathematics’ mentioned in footnote 2 at page 5 above). As we have seen, Dewey thought that these aspects go hand in hand in the educational process—it is imperative that the mathematics taught in the classroom connects to the students’ experiences and to other subjects. This is similar to the aspirations of modern educational theorists, who investigate how mathematics could be made meaningful, how to connect the mathematics taught in the classroom with practical tasks in the real world (Gainsburg, 2008; Pongsakdi et al., 2016). Just like Dewey, they want to avoid an educational system where the students only learn empty formulas and rules by rote, without real understanding. However, the theoretical framework of Dewey and modern educational theorists differ in important respects. Whereas Dewey saw the practical aspects of mathematics as inseparable from the timeless aspects of mathematics, such as the operational aspects, and thought it important that both be included in the learning process, we will argue that many modern educational theorists tend to separate these aspects (e.g. Hiebert & Lefevre, 1986).
3. Mathematical knowledge in educational theory

How mathematical knowledge should be characterized is a recurring issue within the research field of mathematics education. Over the years this issue has, among other things, given rise to a distinction between procedural and conceptual knowledge which has served as a platform for empirical studies as well as the formation of theoretical frameworks (cf. Kieran, 2013). This research has also influenced the content of mathematics curricula in several countries, not least in the Nordic countries. Some of this research stems from the beginning of the 1970s. Here, ‘instrumental understanding’ denotes what happens when students learn a number of fixed and specific plans to solve a specific kind of task (Skemp, 1976). Skemp is critical of this type of understanding, since the students do not get an overall understanding of the relationship between the individual stages and the final goal of the exercise. Instead, he emphasizes ‘relational understanding’, where the students gradually form a conceptual structure that can produce an unlimited number of plans for getting from any starting point within a student’s schema to any finishing point (Skemp, 1976). According to Skemp, the understanding will then become the goal in itself, and the plans are no longer fixed and immediately tied to a particular class of problems, as in the case of instrumental understanding.

Hiebert and Lefevre (1986) make a distinction between ‘conceptual knowledge’ and ‘procedural knowledge’. The latter is described by Hiebert and Lefevre as knowledge consisting of procedures used for solving mathematical problems, where the procedures are ‘chains of prescriptions for manipulating symbols’ (Hiebert & Lefevre, 1986, pp. 7–8). This description is similar to Skemp’s (1976) explanation of instrumental understanding cited in the above paragraph. Skemp uses the phrase ‘learning a number of fixed and specific plans’ which is similar to the term ‘procedure’ and ‘chains of prescriptions’ used by Hiebert and Lefevre (1986). The term conceptual knowledge, on the other hand, is described as knowledge rich in relationships that ‘can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions, so that all pieces of information are linked to some network’ (Hiebert & Lefevre, 1986, pp. 3–4). This is similar to Skemp’s description of relational understanding. Skemp speaks of a ‘connected web’ and a ‘network’ of knowledge, whereas Hiebert and Lefevre (1986) use the expression ‘conceptual structure’ in order to characterize this kind of knowledge. Both describe a situation in which the student not only uses specific mathematical concepts, but also has an understanding of the whole system of concepts.

Building on the ideas above, Lithner (2008) has developed a theoretical framework for mathematical reasoning. Lithner refers to, among others, Skemp’s (1976) and Hiebert and Lefevre (1986) frameworks, but points out that none of these are actually ‘aiming at characterizing the reasoning itself’ (Lithner, 2008) and therefore Lithner intends to fill this gap. On the basis of his own empirical studies, Lithner argues that rote learning, understood as a process of learning something by repeating it until it is memorized, is a main factor behind learning and achievement difficulties (Lithner, 2004, 2008). To specify and communicate the findings of his empirical studies Lithner distinguishes between two different types of mathematical reasoning: ‘imitative’ and ‘creative’ reasoning, where the former is connected to rote learning. By imitative reasoning, Lithner means recollection of a complete answer to a specific task, for instance when a student memorizes every step of a proof (but is unable to explain the meaning of each step). It can also be described as a strategy where one recalls a specific solution algorithm, which is similar to Hiebert and Lefevre (1986) description of procedural knowledge in terms of learning ‘chains of prescriptions’ and Skemp’s explanation of instrumental understanding as ‘learning a number of fixed and specific plans’.

In contrast, Lithner (2008) wants to emphasize the importance of creative reasoning. This takes place when a student creates a reasoning sequence novel to him or her (or recreates a forgotten one), where there are arguments supporting the strategy choice, arguments that are anchored in
intrinsic mathematical properties of the components involved. This is related to Hiebert and Lefevre (1986) explanation of conceptual knowledge as ‘a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information’ as well as Skemp’s (1976) description of relational understanding as an ‘overall understanding of the relationship between the individual stages and the final goal of an exercise’. The main difference between imitative and creative reasoning is that in the latter case the solution is created, whereas in the former the solution follows a known path or is immediate through recollection (Lithner, 2008).

Even though Skemp (1976) focuses on understanding, Hiebert and Lefevre (1986) on knowledge, and Lithner (2008) on reasoning, we believe that their common aim is to account for the difference between merely learning something by heart without any real understanding and without knowing how to apply what one has learned, on the one hand, and actually understanding a mathematical operation and being able to apply that knowledge in new situations as well as understanding the connections between the different concepts involved, on the other. Since both researchers and teachers agree that teaching students empty formulas without conveying any understanding of their meaning is not a desirable educational goal, theorists (e.g. Hiebert & Lefevre, 1986; Jonsson et al., 2014) try to develop methods to improve the students’ conceptual knowledge and minimize procedural knowledge.

There are also studies that go against this trend and instead focus on the positive outcomes of instrumental understanding, procedural knowledge or imitative reasoning. For instance, Star (2005) argues for a renewed focus on procedural knowledge in the research of mathematics education, based on the following three points: (1) The development of students’ procedural knowledge has not received enough attention; (2) one possible explanation for this is that current characterizations of procedural knowledge reflect limiting assumptions about how procedures are known; and (3) reconceptualising procedural knowledge to remedy these assumptions would have important implications for both research and practice. Star (2005) also points out that procedural knowledge was widely studied within cognitive psychology in the 1980s (e.g. Brown & VanLehn, 1980) and thereafter the relationship between procedural and conceptual knowledge has been a research topic in developmental psychology (Rittle-Johnson & Alibali, 1999; Rittle-Johnson, Siegler, & Alibali, 2001; Schneider, Rittle-Johnson, & Star, 2011).

Furthermore, Carolyn Kieran has written a critical paper where she claims that the distinction between conceptual understanding and procedural skills is a ‘false dichotomy in mathematics education’ (Kieran, 2013, p. 153). She argues that conceptual understanding and procedural skills cannot be seen as separate entities since the elaboration and constitution of procedures are conceptual in nature and contribute to the understanding of the mathematical objects treated. Moreover, procedural skill contains a significant conceptual component in itself since procedures, even those that function as automatized, need to be regularly updated, revised and extended (Kieran, 2013). According to Kieran (2013), the dichotomy between procedural skills and conceptual understanding has been particularly damaging in algebra, since algebra has come to be viewed as a domain dominated by procedures of symbolic manipulations where the conceptual is generally absent. On the basis of her own project on algebra learning Kieran (2013) illustrates algebraic examples of the co-emergence of procedural (or technical) and conceptual components as well as the interplay between them. However, we will not go into details on algebraic expressions, instead we will focus on the trend that Kieran is critical of, i.e. the view of the procedural as inferior to the conceptual. We will try to trace the reasons for the dichotomy and spell out what has been lost in the process, what we call the operational aspect of mathematics.

4. The goal of teaching mathematics

Dewey and educational theorists today (e.g. Hiebert & Lefevre, 1986; Pongsakdi et al., 2016) agree that students should not learn a separate classroom-mathematics that they only ever use in school,
but a mathematics connected to their daily activities, something that can help them manage and make sense of other areas of their lives. School mathematics has been criticized for being boring and too focused on calculations without context and rote learning of algorithms, which leads to students only gaining surface knowledge, i.e. techniques that they can only use in the classroom:

Much time in mathematics classes is spent on learning and rehearsing algorithms, which are supposed to provide students with a quick and reliable way to cope with many of the tasks ahead (Boesen et al., 2014; Hiebert, 2003). There are, however, doubts as to whether these algorithms actually give rise to any deeper understanding of the principles of mathematics, or whether the extensive use of algorithms is counter-productive (Hiebert, 2003) (Jonsson et al., 2014, p. 21).

The extensive use of algorithms, as described above, only leads to superficial, procedural knowledge. This is generally thought to be the case when the teacher walks the students through a solution with the help of an algorithm, and the students are subsequently asked to solve a series of problems using the same algorithm. This teaching method is strongly criticized by many researchers: ‘There are extensive concerns pertaining to the idea that students do not develop sufficient mathematical competence. This problem is at least partially related to the teaching of procedure-based learning’ (Jonsson et al., 2014, p. 20).

Imitative reasoning is what Dewey tried to avoid by stressing the importance of ‘psychologizing’ the subject, making it interesting to the students through relating it to their everyday lives outside the classroom, and then gradually moving in a direction of more abstract knowledge. Abstract thinking is indeed a goal for the modern educational researchers we have reviewed above as well, but they do not share Dewey’s understanding of ‘abstract thinking’. For Dewey, it was an ability to solve problems mathematically, to engage in mathematics as does a researcher, who is interested in the proofs, refutations and calculations as ends in themselves, rather than as a means to achieve practical ends. Practical applications are of course important in people’s everyday lives, as well as in the professional lives of economists, engineers and physicists, to name a few, but it is not the subject matter of higher mathematics. According to Dewey, not every student should become a professional mathematician, but it is important that the goal is always set high (in every subject), so that the students are pushed to ever more difficult tasks, in order to develop their thinking and reach further than they might have expected (Dewey, 1966/1916).

Today, however, there is a growing tendency to understand abstract thinking, the deep mathematical understanding, as that which is expressed in arguments, modelling and contextualization (e.g. Pongsakdi et al., 2016). This tendency is also visible in the curricula of many countries today (NCTM, 2000), where mathematical knowledge is first and foremost tied to verbal abilities: to be able, in words, to explain why an operation is preferable to another, why a solution is correct etc. For instance, in the current Swedish national curriculum for mathematics (Läroplan för grundskolan, förskoleklassen och fritidshemmet 2011) the knowledge demands are based on generic competencies, such as the ability to describe mathematical concepts and to communicate mathematics in various contexts, rather than the ability to calculate and reach the right results. The commentary material to the Swedish national curriculum emphasizes that students should learn to use ‘metacognitive reflections in order to think out loud, look for alternative solutions as well as discuss and evaluate solutions, methods, strategies and results’ (National Agency for Education, 2011, p. 1). This stands in contrast to the Swedish curriculum of 1980 (Läroplan för grundskolan 1980) where the importance of teaching the students numerical and computational skills was stressed by citing specific areas of mathematics that the students needed to master in order to manage important practical affairs and to become able citizens. Moreover, the Swedish 1980 curriculum prescribed that the students should learn algorithms, while in the 1994 and 2011 curriculums the term algorithm was removed. Simultaneously, there are signs that the students’ operational skills have suffered. In a recent study, Norwegian students were shown to have weaker numeracy skills compared to earlier generations (Billington & Gabrielsen, 2017). In Sweden, the algebraic tasks in educational material for upper secondary school have become less complex.
compared to the 1960s, 1970s and 1980s (Jakobsson-Åhl, 2008). Furthermore, Filipsson and Thunberg (2008) argue that a discrepancy in what is considered important mathematical knowledge has emerged between the Swedish secondary school and the universities. At the university level, routine skills in arithmetic and algebraic computations are considered absolutely necessary when learning mathematics, whereas in the secondary school mathematics curricula, focus lies on mathematics problem solving skills and how students understand concepts, rather than on practising computational skill (Brandell, Hemmi, & Thunberg, 2008; Filipsson & Thunberg, 2008). This becomes problematic in cases where the computational complexity of exercises at the university level is way above the level of complexity that the students are used to from secondary school (Filipsson & Thunberg, 2008). Consequently, many students encounter serious problems when they begin to study mathematics at university level, and initial problems may lead to prolonged or unsuccessful studies or even cause students to quit their studies in mathematics (Brandell et al., 2008). It seems that these problems are not unique for Sweden, there are similar problems in other countries, for instance in England (Kouvela, Hernandez-Martinez, & Craft, 2018) and Canada (Kajander & Lovric, 2005). Here we argue that an increased focus on operational skills in secondary school, not least in the mathematics curriculum documents, is necessary in order to facilitate the transition from secondary school to university level mathematics.

5. What is mathematical understanding?

To better understand the development from numerical and computational skills toward verbalizations, we have to look more closely at how algorithms are perceived. A common view is that the use of algorithms does not further mathematical understanding. The advantage of using algorithms is that they make it easier for students to reach the right conclusion, but the ability to use an algorithm is not an indication of any deeper understanding of the principles of mathematics (Hiebert, 2003). Brousseau (1997, p. 130) is often quoted in this context: ‘The reason why an algorithm is regarded as efficient in solving a task (but not for learning) is that it is designed to avoid meaning’. The application of algorithms is often considered mechanical and it is sometimes argued that only carelessness can lead to a mistake (Lithner, 2008). In fact, imitative reasoning, the application of rules and algorithms, is at times considered the opposite of creative reasoning:

The aspect of creativity that is emphasized in this framework is not ‘genius’ or ‘exceptional novelty,’ but the creation of mathematical task solutions that can be modest but that are original to the individual who creates them. Thus, creative is the opposite of imitative (Bergqvist & Lithner, 2012, p. 256).

If imitative reasoning—the application of rules and algorithms—is taken as the opposite of (desirable) creative reasoning, educators might become weary of numerical skill more generally. If the important thing is the thought process behind the calculations, rather than the calculations themselves, verbal explanations are needed to show whether the students really understand the tasks that they have completed. The calculations could, after all, be a result of merely mechanically applying rules or algorithms. On this approach, the efforts of a student who uses an algorithm to solve a problem and can check whether she, on the basis of the algorithm, has the right answer, is equated with the efforts of a student who copies the task from another student and has no idea what algorithm was used, nor how to make sure that the answer is correct.

What is the reason for the negative conception of using algorithms, and the idea that the mathematical knowledge should be expressed verbally? In the following, we will discuss ‘understanding’ rather than ‘knowledge’, since we want to focus on the process taken to underlie mathematical knowledge, rather than the exhibited competence. It seems to us that what is at stake here is the philosophical assumptions made about understanding. A common dualistic view is adopted, where understanding is taken to be a mental process, a process of which the calculation is the result. The result is seen to be contingent on the actual understanding, which is hidden —just like for the mind-body dualist, the bodily expression of a feeling is contingent: we can hide
our feelings or force a smile even when we are sad. We cannot trust the facial expressions to reveal a person’s true feelings, as we cannot trust the calculations on a paper to reveal the mental process that lies behind them. Thus, we cannot know how the student ended up with the numbers that she has written down. This view of the performed calculation as empty in itself, coupled with an idea of two distinct kinds of reasoning, one desirable (creative) and the other not (imitative), results in a ‘suspicion’ toward the mathematical operations the student has written down as potential expressions of merely memorizing empty rules or procedures. Therefore, we need something else, something in addition to a traditional mathematical task to make sure that the calculations originate in real understanding—and what we end up with is a verbalized account.

That understanding is primarily thought of as a mental phenomenon is shown by the importance attached to brain activity, in studies investigating how different teaching methods affect the brain. For instance, in a Swedish research study (Karlsson Wirebring, Lithner, Jonsson, & Liljekvist, 2015), a group of students got to solve mathematical problems using only given algorithms and rules, through imitative reasoning, another group solved the same problems using what was characterized as creative reasoning. After working with the tasks for a week, their brain activity was monitored and the results of the tasks were compared between the groups (Karlsson Wirebring et al., 2015).

One problem with sharply separating understanding as mental activity and the completion of a mathematical task as the result of this understanding, is that the gap between the test situation and real life problems widens. This becomes apparent in a Finnish study, concerned precisely with making mathematics applicable to real world situations. One way of reaching this aim, it is argued, is to create realistic, narrative word problems for the students to solve, like the following:

Paula is preparing some food and drinks for her birthday party. She buys two packets of chips (1 packet costs 2.50 euros), a big packet of mixed candies (1 packet costs 3.60 euros), and 4 bottles of lemonade (1 bottle costs 1.25 euros). Three friends come to the party. How much does the snacks and drinks cost for each participant? (correct answer: 3.4) (Pongsakdi et al., 2016, p. 32)

A realistic feature of the problem is that the students have to figure out that they should divide the total cost with all four participants, not just the three guests. In this task, which was used to evaluate the students’ reasoning ability, full points were awarded for students who modelled the task correctly, even if they made computation errors and thus arrived at the wrong answer (Pongsakdi et al., 2016). Since the students’ test results improved after rehearsing with realistic, narrative word problems, the researchers concluded that word problems have a positive effect on students’ problem solving skills. But it is notable that ‘problem solving skill’ here only refers to the ability to model the task, not the ability to actually complete it. Imagine a real life situation, where Paula was in charge of calculating the per person admission fee to an event. To the people depending on her it would not suffice that she thought correctly, unless she also got her sums right. In fact, in most real life situations, whether we are concerned with paying taxes or figuring out what medical dose to take, we need to arrive at the correct solution to avoid suffering negative consequences. In the most common sense of ‘problem solving’, the students who arrived at the wrong mathematical result failed to solve the problem. That the students’ mastery of pure mathematics (in this case exemplified by calculations with decimal numbers) is the basis for all types of applied mathematics, is illustrated in the mathematical cycle of applied mathematics defined in the US Standards from 1989 (National Council of Teachers of Mathematics [NCTM], 1989), as well as in the PISA framework. Nevertheless, in English-speaking and Nordic countries, this has, to some extent, been misunderstood to mean that applied mathematics can be seen as an alternative to pure mathematics (Gardiner, 2004; Grønmo, 2011).

Again, it can be useful to remind ourselves of Dewey’s advice to not separate the psychological and the logical aspects of mathematics: in order to solve mathematical problems successfully you need the reasoning ability, context-sensitivity and situational understanding as well as an ability to perform the calculations needed. In Pongsakdi et al. (2016) example above it turned out that the
students’ reasoning ability did not suffice to solve the problem at hand, they also needed to be able to perform the calculations (calculating with decimal numbers). For Dewey, this insistence is connected to his rejection of dualism and the conception of intelligence as a purely mental phenomenon. The idea that we could know whether someone is a deep thinker through examining their brain is therefore ludicrous to him: thinking is internally connected to its expression, to how we manage in our practical activities. To suggest that ‘problem solving skill’ could be evaluated independently of whether the skill actually solves the problem it was set out to solve would not make sense to him.

6. Learning by doing: operational skill

We argue that the distinction between procedural and conceptual knowledge has led to a dichotomy where operational skill gradually over the years has fallen between chairs. In light of our hypothesis that, following Dewey, the different aspects of mathematics need to go hand in hand, we find it particularly important to emphasize operational skill, which combines procedural and conceptual elements. The operational aspect of mathematics is a methodological aspect that the students learn by engaging in mathematical activity, for instance performing calculations, solving equations, manipulating the symbols in a mathematical expression, proving a mathematical statement, etcetera. It is not a matter of mechanically applying formulae, and it need not be identified with a distinct mental process, but is part of a practice that the mathematician needs to master in order to develop his or her mathematical intuition and see relevant connections where others would not. Even if mastering the operational aspect requires practice, it entails a context-free, timeless dimension (and is therefore abstract in Dewey’s sense). The key to operational skill lies in mastering the mathematical symbolism and the rules that have been worked out through centuries of mathematical progression. An example of this would be to understand the right way to use an algorithm.

One way of understanding the operational aspect of mathematics is to describe it in terms of the symbolic nature of mathematics, in the sense that mathematics has been perceived ever since the late 16th century. On this view, mathematics is not something we first learn and then do, but rather something we need to learn by doing. A symbolism is not merely a system of notation in the typographical or linguistic sense, but is connected to the mathematical activity itself. It is the operational aspect of a symbol, its function in a calculus, its role in the manipulation and transformation of expressions, which can be said to constitute it as a symbol (Stenlund, 2015). Dewey’s logical aspect of mathematics can thus be identified as the symbolic conception of mathematics: to master the logical aspect is to master a symbolic calculus.

We argue that in the received conceptual framework for explaining mathematical knowledge, through procedural and conceptual knowledge, the different aspects of mathematical knowledge are often treated as clearly demarcated and separate sets of skills. This leads to a neglect of the operational aspect of mathematics, which could be said to fall into both categories. Particularly when the procedural and the conceptual are perceived of as opposites, there is very little room left for conceptualizing or evaluating the reflective use of algorithms and rules. Procedural knowledge need not only be a matter of rote learning; in learning a method for solving a problem the students learn to master a calculus and solve a problem mathematically. However, the ability to use received notations, symbols and rules requires a lot of practice, rather than the ‘novelty’ and ‘originality’ associated with conceptual knowledge. In this sense it combines aspects of both the conceptual and the procedural and resolves the apparent opposition between them.

In short, we propose to highlight the actual doing of mathematics, that we have termed operational skill. We find ourselves in agreement with Kieran, who argues that the dichotomy between procedural skill and conceptual understanding is misplaced since ‘procedures are conceptual in nature’ (Kieran, 2013, p. 161). The risk of teaching the students mindless rules seems to be one of the reasons for why the distinction procedural/conceptual has turned into something of
a dichotomy, and we realize the importance of avoiding this kind of rote learning. But it is important to point out that merely memorizing procedures or imitating solutions without understanding might not properly be called (procedural) knowledge at all, but simply failed education. However, practicing rules and procedures is often a necessary step in learning to use them operationally and with understanding, and should therefore not be equated with mere imitation and regarded as unimportant for learning mathematics, since gaining mathematical knowledge is dependent on the practical mastery of symbols and procedures. Mathematical knowledge gained by means of analogies, visualizations and verbal explanations are important means for creating a setting in which the student finds herself at home in the material, but verbal explanations, often identified as evidence of conceptual knowledge, cannot replace the symbolic calculations that the student needs to master in order to solve a mathematical problem. We therefore want to emphasise operational skill, in order to capture the importance of numerical and computational skills for mathematical knowledge, not as mindless rule-following, but as a thoughtful use and application of skills often necessary for understanding the mathematical solution.

7. Conclusion

We have argued that the existing framework for explaining mathematical knowledge in terms of procedural and conceptual knowledge is often interpreted as a dichotomy between the two aspects essential for education: practical modelling and verbal explanations (Dewey’s psychological aspect), on the one hand, and the operational, symbolic aspects that are free of context (Dewey’s logical aspect), on the other. Partly as a result of how procedural and conceptual knowledge are defined, numerical and computational skills have become less important in mathematics education, and meta-mathematical, communicative skills are seen as more important, as embodying what is thought of as ‘real’ mathematical knowledge.

As mentioned, there are signs that students’ numerical and computational skills have suffered over the years: (Billington & Gabrielsen, 2017; Filipsson & Thunberg, 2008; Jakobsson-Åhl, 2008). There are many reasons for these developments. The emphasis on practical uses of mathematics and ‘real-world mathematics’ at the expense of numerical and computational skills could be one of them. Sutherland (2002) shows that countries that emphasize real life (contextual) mathematical understanding tend to be less concerned about numerical and computational skills. Therefore, we suggest a new approach to the research framework, with more emphasis placed on operational skill. As the latter includes mathematics as a practical activity that the students need to engage in, at the same time as it is concerned with the symbolic, context-free nature of mathematics, it is indeed a call to revive the Deweyan slogan of ‘learning by doing’ in mathematics education.

Notes

1. From its start in the 1890’s, progressivism was not a unitary movement but a multi-faceted protest against pedagogical narrowness and one-sidedness (Cremin, 1959).
2. ‘Pure mathematics’ should here be understood as the context-independent, timeless aspects of mathematics, or as a mathematical activity that does not explicitly take its application into consideration. A concrete example would be symbolic manipulations in algebra.
3. Cf. Michal Young’s concept of ‘powerful knowledge’ as that which can help the students make sense of the world and take them beyond their own experiences (Young, 2014).
4. Dewey’s psychological and logical aspects of a subject is in some respects similar to Basil Bernstein’s horizontal and vertical discourse (Bernstein 1999).
5. ‘Abstract’ for Dewey refers to a subject as self-contained, driven by its own rules rather than by practical applications. The outcome, the abstract to which education is to proceed, is an interest in intellectual matters for their own sake, a delight in thinking for the sake of thinking.’ (Dewey, 1986/1933, p. 298).
7. Mind-body dualism is intuitive up to a point, hence the attraction it casts on laymen as well as philosophers. However, when consistently applied it can lead to problems such as scepticism about other minds. Mind-body dualism was rejected, among others, by Dewey (1986/1933) and Wittgenstein (1953).

8. This could be one reason for the Nordic countries' weak TIMSS-results in algebra which was mentioned in the introduction section above (Bråting et al., 2019; Grønmo, 2011). Algebra is a topic where symbolic manipulations and context-independent problems (i.e. typical examples of pure mathematics) frequently appear.

9. ‘Thinking is not a separate mental process; it is an affair of the way in which the vast multitude of objects that are observed and suggested are employed, [...] Consequently, any subject, topic, question, is intellectual not per se but because of the part it is made to play in directing thought in the life of any particular person’ (Dewey, 1986/1933, pp. 156–157).

Disclosure statement

No potential conflict of interest was reported by the authors.

Notes on contributors

Tove Österman is a researcher in theoretical philosophy at Uppsala University, Sweden, e-mail: tove.osterman@filosofi.uu.se. Her interests are educational philosophy and philosophy of mathematics. She is author of ‘What Does Calculating Have to Do With Mathematics? Wittgenstein, Dewey, and Mathematics Education in Sweden’, Peters Michael A., Stickney, Jeff (eds.) A Companion to Wittgenstein on Education (New York: Springer, 2017).

Kajsa Bråting is Associate professor (docent) in mathematics education and Director of postgraduate studies at Uppsala University, Sweden, e-mail: kajsa.brating@edu.uu.se. Her interests are mathematics education, epistemology and history of mathematics. She is author of ‘On the relations between historical epistemology and students’ conceptual developments in mathematics.’ Educational Studies in Mathematics, 89(2), 251–265.

ORCID

Tove Österman http://orcid.org/0000-0001-9329-1537

References


