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Irreducible representations of finite monoids

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Introduction

This paper is a literature study of the 2016 book Representation Theory of Finite Monoids by Benjamin Steinberg [3]. As this book contains too much interesting material for a simple master thesis, we have narrowed our attention to chapters 1, 4 and 5. This thesis is divided into three main parts: Theory, Application and Appendix.

Within the Theory chapter, we (as the name might suggest) develop the necessary theory to assist with finding irreducible representations of finite monoids.

Finite monoids and their structure gives elementary definitions as regards to finite monoids, and expands on the basic theory of their structure. This part corresponds to chapter 1 in [3].

The theory of an idempotent develops just enough theory regarding idempotents to enable us to state a key result, from which the principal result later follows almost immediately. This part corresponds to chapter 4 in [3].

Irreducible representations of a finite monoid holds the principal result frequently referred to: This theorem states (among a few other things) that finding irreducible representations of finite monoids is equivalent to finding irreducible representations of finite groups. As the reader is most likely aware, finding irreducible representations of finite groups is a problem for which we have tools readily available. This part corresponds to chapters 5.1 - 5.2 in [3].

The Application chapter defines the symmetric inverse monoid, and then proceeds to calculate in detail the irreducible representations of a particular such monoid, I_3. Full tables for how I_3 acts on its irreducible representations are provided at the end of this chapter.

The appendix contains Basic definitions drawn from [1] and [2], as well as Finite dimensional algebras (drawn from Appendix A in [3]): The definitions and handful of results found here are no doubt familiar to most readers, but have been included in the interest of being thorough.

Note that not all results are proven below: We elect to prove our three main theorems 1, 2 and 3, the latter's immediate corollary, Corollary 11, and the results necessary to in turn prove these. The sceptical reader requiring certainty that the numerous other results stated do in fact hold is cordially invited to peruse [3], wherein can be found the desired proofs.

Theory

Finite monoids and their structure

This section shall provide us with the definitions and theory for finite monoids, and their structure, which we shall need to arrive at our fundamental result. In addition, we introduce some notation that will be used frequently.

Introductory notions

The majority of the content below should be familiar to readers who have studied algebra.

Definition 1. A semigroup consists of a set, possibly empty, and an associative binary operation (the latter typically referred to as multiplication).
A subsemigroup of a semigroup is a subset, possibly empty, which is closed under multiplication.

Definition 2. A monoid is a semigroup with an identity element, with the identity element often written as 1.
A submonoid of a monoid is a subsemigroup containing the identity.

Example 1. $\mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ are all monoids when equipped with either addition or multiplication.
$\mathbb{N}$ equipped with multiplication is a monoid; however, $\mathbb{N} \setminus \{1\}$ equipped with multiplication is a semigroup, but not a monoid.

The identity is unique, and as a consequence of the definition, monoids are nonempty. From here on out, unless otherwise specified, S denotes a semigroup, G denotes a subsemigroup of S, and M denotes a monoid. We do not require that the identity of a semigroup G is the same as the identity of the semigroup S if S is a monoid.

Definition 3. An element $m \in M$ is called a unit if there exists $m^{-1} \in M$ such that $mm^{-1} = 1 = m^{-1}m$. The units form a group called the group of units of M, which is the unique maximal subgroup of M that is also a submonoid.

Definition 4. An element $z \in S$ is called a zero element (or simply a zero) if $sz = z = zs \; \forall s \in S$.

Example 2. Any group is clearly its own group of units (groups by definition have inverses).
$\mathbb{Z}_4 = \{0, 1, 2, 3\}$ equipped with multiplication modulo 4 is a monoid with group of units $G = \{1, 3\}$, which is a submonoid of Z_4.
$\mathbb{Z}$ equipped with multiplication has group of units $G = \{-1, 1\}$, which is a submonoid of $\mathbb{Z}$.

Note that zero elements are necessarily unique: Let $z_1, z_2 \in S$ be such that $sz_1 = z_1 = z_1s$, $sz_2 = z_2 = z_2s \; \forall s \in S$. Then $z_1 = z_2z_1 = z_2$.

Definition 5. For S a semigroup and $A, B \subseteq S$, define

$$AB = \{ab : a \in A, b \in B\}$$

Definition 6. For S a semigroup, denote by S^{op} the opposite semigroup, which has the same underlying set as S but whose binary operation $*$ is given by $s * t = ts$.
The same notation will be used for opposite rings, posets and categories.

Definition 7. Let M, N be monoids. A homomorphism of monoids $\varphi : M \to N$ is a mapping such that, for all $m_1, m_2 \in M$,

(i) $\varphi(1) = 1$

(ii) $\varphi(m_1 m_2) = \varphi(m_1)\varphi(m_2)$

A bijective homomorphism is called an isomorphism, where the inverse of an isomorphism is again an isomorphism.
Two monoids are said to be isomorphic if there exists an isomorphism between them. Semigroup homomorphisms and isomorphisms are defined in the same fashion.

Example 3. As every monoid contains the identity, we always have the trivial homomorphism

$$\varphi : M \to N$$
$$\varphi(m) = 1_N$$

For G a submonoid of M, we have the inclusion homomorphism

$$\varphi : G \to M$$
$$\varphi(g) = g$$

And for M a monoid, we have the identity homomorphism

$$\varphi : M \to M$$
$$\varphi(m) = m$$

Definition 8. A congruence on a monoid M is an equivalence relation $\equiv$ such that $m_1 \equiv m_2$ implies $um_1v \equiv um_2v \ \forall u, v \in M$. The equivalence or congruence class of m is denoted as $[m]_\equiv$.

Definition 9. Let $A \subseteq M$, where M is a monoid. The submonoid $\langle A \rangle$ generated by A is the smallest submonoid of M containing A. The subsemigroup generated by a subset of a semigroup is defined and denoted in the same way.

Example 4. Let M be the monoid $\mathbb{Z}_9 = \{0, 1, 2, \ldots, 8\}$ equipped with multiplication modulo 9. Then $\langle 3 \rangle = \{0, 1, 3\}$ equipped with multiplication modulo 9 is the submonoid generated by 3.

Definition 10. For S, T semigroups, their direct product $S \times T$ is a semigroup with coordinate-wise binary operation $(s, t)(s', t') = (ss', tt')$.

Example 5. Note that the direct product of two monoids is a monoid using the coordinate-wise binary operation above: Let M, N be monoids. As M, N are monoids, they each contain an identity element, making the identity element of $M \times N = (1_M, 1_N)$. As they are semigroups, their individual operations are associative.
As an example, consider $M = Z_2$, $N = Z_3$ equipped with multiplication modulo 2 and 3 respectively. Then $Z_2 \times Z_3$ is a (finite) monoid equipped with coordinate-wise binary operation, and has identity element $(1, 1)$.

Definition 11. An element e of a semigroup S is an idempotent if $e = e^2$. The set of idempotents of a subset $X \subseteq S$ is denoted $E(X)$.
Note that, if $e \in E(S)$, then eSe is a monoid with respect to the binary operation of S with identity e. Then the group of units of eSe is denoted by G_e and is called the maximal subgroup of S at e: This is the unique subgroup with identity e that is maximal with respect to containment.
Define also $I_e = eSe \setminus G_e$.

Definition 12. There is a natural partial order on the set $E(S)$ of idempotents of a semigroup S defined by $e \leq f$ if $ef = e = fe$, or equivalently, $e \leq f \Leftrightarrow eSe \subseteq fSf$.

Example 6. Take the finite monoid $\mathrm{M} = \mathbb{Z}_{12}$ equipped with multiplication modulo 12. Then $E(M) = \{0, 1, 4, 9\}$. Pick $e = 4$, and we have $eMe = 4 \cdot Z_{12} \cdot 4 = \{0, 4, 8\}$, which is again a monoid under multiplication modulo 12, and clearly has identity 4. This monoid has group of units $G_4 = \{4, 8\}$. Thus, $I_4 = 4 \cdot \mathbb{Z}_{12} \cdot 4 \setminus G_4 = \{0\}$.

We note that, for $E(M)$, $4 \leq 1$, as $4 \cdot \mathbb{Z}_{12} \cdot 4 = \{0, 4, 8\} \subseteq 1 \cdot \mathbb{Z}_{12} \cdot 1 = \mathbb{Z}_{12} = \{0, 1, 2, \ldots, 11\}$.
By the same token, we have $9 \leq 1$, as $9 \cdot \mathbb{Z}_1 2 \cdot 9 = \{0, 3, 6, 9\} \subseteq 1 \cdot \mathbb{Z}_1 2 \cdot 1$.
Note, however, that 4 and 9 are not in relation to each other.

If M is a monoid, then G_1 is the group of units of M: This should be obvious from the definition.

Finally, note that the image of an idempotent under a homomorphism is an idempotent.

Cyclic semigroups

Here we define the index and period of an element in a finite semigroup, and thus equipped, proceed to specify what a cyclic subsemigroup generated by such an element is, as well as state a few results. This is mainly used for developing the character theory of finite monoids (which lies beyond our scope); however, the four results herein will all be required in order to prove our main theorem later on.

Definition 13. Let S be a finite semigroup, and $s \in S$. Then there exists a smallest positive integer c, called the index of s, such that $s^c = s^{c+d}$ for some $d > 0$. The smallest possible such d is called the period of s.

Note that $s^c = s^{c+qd}$ for all $q \geq 0$:

$$s^{c+qd} = s^c \underbrace{s^d \cdots s^d}_{q \text{ times}} = s^{c+d} \underbrace{s^d \cdots s^d}_{q-1 \text{ times}} \stackrel{\text{def}}{=} s^c \underbrace{s^d \cdots s^d}_{q-1 \text{ times}} = \cdots = s^c$$

Proposition 1. Let S be a finite semigroup, and $s \in S$ have index c and period d. Then $s^i = s^j \Leftrightarrow i = j$ or $i, j \geq c$ and $i \equiv j \mod d$.

Definition 14. Let S be a finite semigroup, and $s \in S$. We call $\langle s \rangle = \{s^n : n \geq 1\}$ the cyclic subsemigroup generated by s.

Note that, by Proposition 1 above, $\langle s \rangle = \{s, s^2, \ldots, s^{c+d-1}\}$ (c, d index and period of s), and that the elements of $\langle s \rangle$ are distinct.

Example 7. Take as finite monoid the permutations of 3 elements under composition of permutations

$$S_3 = \left\{ \left(\begin{smallmatrix} 1&2&3\\1&2&3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3\\1&3&2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3\\2&1&3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3\\2&3&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3\\3&2&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3\\3&1&2 \end{smallmatrix}\right) \right\}$$

Then $\left(\begin{smallmatrix} 1&2&3\\3&1&2 \end{smallmatrix}\right) \in S$ has index 1 and period 3, as

$$\left(\begin{smallmatrix} 1&2&3\\3&1&2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1&2&3\\3&1&2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1&2&3\\3&1&2 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1&2&3\\3&1&2 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1&2&3\\2&3&1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1&2&3\\1&2&3 \end{smallmatrix}\right)$$

We then get that

$$\langle \left(\begin{smallmatrix} 1&2&3\\3&1&2 \end{smallmatrix}\right) \rangle = \left\{ \left(\begin{smallmatrix} 1&2&3\\2&3&1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3\\3&1&2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1&2&3\\1&2&3 \end{smallmatrix}\right) \right\}$$

Corollary 1. Let S be a finite semigroup, and $s \in S$ have index c and period d. Then the subsemigroup $C = \{s^n : n \geq c\}$ of $\langle s \rangle$ is a cyclic group of order d. The identity of C, denoted s^ω, is the unique idempotent in $\langle s \rangle$ given by s^m, where $m \geq c$ and $m \equiv 0 \mod d$. If $s^{\omega+1} = s^\omega s$, then $C = \langle s^{\omega+1} \rangle$.

Corollary 1 has a very interesting consequence, which we shall have use for later: Any nonempty finite semigroup has at least one idempotent.

Corollary 2. A nonempty finite semigroup contains an idempotent.

This corollary can be reformulated into a stronger result, again for later use, which is sometimes called the pumping lemma.

Lemma 1. Let S be a semigroup of order $n > 0$. Then $S^n = SE(S)S$. Equivalently, $s \in S^n \Leftrightarrow s = uev$, where $e \in S$ is an idempotent and $u, v \in S$.

Green's relations

Green's relations are three equivalence relations based on the ideal structure of monoids. First, let us recall the definition of an ideal, as well as introduce some notation.

Definition 15. Let M be a finite monoid.
A left ideal of M is a nonempty subset I such that $MI \subseteq I$.
A right ideal of M is a nonempty subset I such that $IM \subseteq I$.
A two-sided ideal of M is a nonempty subset I such that $MIM \subseteq I$.

By Corollary 2, as any left, right or two-sided ideal of M is a subsemigroup, each ideal contains an idempotent. Furthermore, M has a unique minimal ideal: If $I_1, I_2, \ldots, I_n$ are ideals of M, then $I_1 I_2 \cdots I_n$ is an ideal of M by definition, and is contained in all other ideals. Also, the union of two ideals is an ideal.

Definition 16. Let M be a finite monoid, and $m \in M$. Then Mm, mM and MmM are the principal left, principal right and principal two-sided ideals generated by m. We introduce the notation

$$I(m) = \{s \in M : m \notin MsM\}$$

If $I(m) \neq \emptyset$, it is an ideal: For each $s \in I(m)$, by definition we have that $m \notin MsM$, and so $m \notin MI(m)M$.

We also have that $I(m) = \emptyset$ if and only if m belongs to the minimal ideal of M: If $I(m) = \emptyset$, there can by definition exist no ideal that does not contain m. If on the other hand m belongs to the minimal ideal of M, we have by the discussion above that this ideal is unique and contained in every other ideal, and so there can exist no $s \in M$ such that $m \notin MsM$.

We are now ready to state Green's relations.

Definition 17 (Green's relations)**.** Let M be a finite monoid, and $m_1, m_2 \in M$. We define:

(i) $m_1 \mathscr{J} m_2$ if and only if $Mm_1M = Mm_2M$

(ii) $m_1 \mathscr{L} m_2$ if and only if $Mm_1 = Mm_2$

(iii) $m_1 \mathscr{R} m_2$ if and only if $m_1M = m_2M$

The $\mathscr{J}$-class, $\mathscr{L}$-class and $\mathscr{R}$-class of an element m are denoted by J_m, L_m and R_m respectively.

Definition 18. A monoid M is called $\mathscr{R}$-trivial if $mM = nM$ implies $m = n$ (the $\mathscr{R}$-relation is equality) with $\mathscr{L}$-triviality being defined in the same fashion. A monoid is $\mathscr{J}$-trivial if $MmM = MnM$ implies that $m = n$. Note that $\mathscr{J}$-trivial monoids are both $\mathscr{R}$-trivial and $\mathscr{L}$-trivial.

Example 8. Let us take as finite monoid M the full transformation monoid under composition of transformations

$$T_2 = \left\{ \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 2 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right) \right\}$$

We label the transformations thusly: $m_1 = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$, $m_2 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 2 \end{smallmatrix}\right)$, $m_3 = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ and $m_4 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$. Note that $m_1 \mathscr{L} m_2$, $m_1 \mathscr{J} m_2$, but we do **not** have $m_1 \mathscr{R} m_2$:

$$Mm_1 = Mm_1M = \left\{ \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 2 & 2 \end{smallmatrix}\right) \right\} = Mm_2M = Mm_2$$
$$m_1M = \left\{ \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right) \right\} \neq \left\{ \left(\begin{smallmatrix} 1 & 2 \\ 2 & 2 \end{smallmatrix}\right) \right\} = m_2M$$

As for m_3 and m_4, we do have $m_3 \mathscr{J} m_4$, $m_3 \mathscr{L} m_4$ and $m_3 \mathscr{R} m_4$:

$$m_3M = Mm_3 = Mm_3M = \left\{ \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 2 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right) \right\} = Mm_4M = Mm_4 = m_4M$$

Finally, we note that, as we have at least one of each Green relation between different elements, $M = T_2$ is neither $\mathscr{J}$-, $\mathscr{L}$- nor $\mathscr{R}$-trivial.

As alluded to earlier, these relations shall be quite helpful in proving our main theorem later on (indeed, every single definition and result is stated with that goal in mind). Green's relations are preserved by application of idempotents.

Lemma 2. Let M be a finite monoid, $e \in E(M)$ and $m_1, m_2 \in eMe$. If $\mathscr{K}$ is one of Green's relations $\mathscr{J}, \mathscr{L}$ or $\mathscr{R}$, then $m_1 \mathscr{K} m_2$ in M if and only if $m_1 \mathscr{K} m_2$ in eMe.

We shall need a definition of M-equivariant mappings, and toward that end, define an M-set.

Definition 19. Let M be a finite monoid. A left M-set consists of a set X along with a mapping $M \times X \to X$, written $(m, x) \mapsto mx$ (called a left action), such that

(i) $1x = x$

(ii) $m_1(m_2x) = (m_1m_2)x$

for all $x \in X$, $m_1, m_2 \in M$.
Right M-sets are defined in like fashion, or can be identified with left M^{op}-sets.
An M-set is said to be faithful if, for all $x \in X$, $mx = m'x$ implies $m = m'$ for $m, m' \in M$.

Definition 20. A mapping $\varphi : X \to Y$ of M-sets is said to be M-equivariant if $\varphi(mx) = m\varphi(x)$ for all $m \in M, x \in X$. If φ is bijective, φ is called an isomorphism of M-sets, where the inverse of such an isomorphism is also an isomorphism.
X is isomorphic to Y, denoted $X \cong Y$, if there is an isomorphism between them. $\mathrm{Hom}_M(X, Y)$ denotes the set of all M-equivariant mappings from X to Y, while $\mathrm{End}_M(X)$, $\mathrm{Aut}_M(X)$ denote the endomorphism monoid and automorphism group of an M-set.

Example 9. Let M be the finite monoid T_2 defined previously. Then $X = \left\{ \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 2 & 2 \end{smallmatrix}\right) \right\}$ is a right ideal of M, and also a right M-set under composition of transformations (indeed, each right ideal of a finite monoid is a right M-set and vice versa for left ideals).

Example 10. Let M be any finite monoid, and let $M = X = Y$. M acts on itself in the obvious way:

$$\begin{aligned} M \times M &\to M \\ (m, x) &\mapsto mx \end{aligned}$$

Now, for any $q \in M$, define

$$\begin{aligned} \varphi_q : M &\to M \\ x &\mapsto xq \end{aligned}$$

Then φ_q is M-equivariant, as

$$\varphi_q(mx) = mxq = m\varphi_q(x)$$

The following theorem exhibits the stability property of finite monoids: It allows (to some extent) for cancellativity. An alternative formulation is that $xm \in J_m \Rightarrow xm \in L_m$, and $mx \in J_m \Rightarrow mx \in R_m$.

Theorem 1. Let M be a finite monoid and $m, x \in M$. Then

$$MmM = MxmM \Leftrightarrow Mm = Mxm$$
$$MmM = MmxM \Leftrightarrow mM = mxM$$

In other words, for $m \in M$, we have $J_m \cap Mm = L_m$ and $J_m \cap mM = R_m$.

Proof. We prove the first equivalence, as the proof of the second is all but identical. Clearly, $Mm = Mxm$ implies $MmM = MxmM$. Now, assume $m = uxmv$. We have $Mxm \subseteq Mm \subseteq Mxmv$. However, we also have $|Mxmv| \leq |Mxm|$, as we have the surjective map $z \mapsto zv$ from Mxm to $Mxmv$. So, $Mxm = Mm = Mxmv$. □

This theorem allows for reformulating what $\mathscr{J}$-equivalence means (which may seem obvious, but cancellativity is far from certain in this case: We did need the above theorem):

Corollary 3. Let M be a finite monoid and $m_1, m_2 \in M$. Then TFAE:

(i) $Mm_1M = Mm_2M$

(ii) There exists $r \in M$ such that $Mm_1 = Mr$ and $rM = m_2M$

(iii) There exists $s \in M$ such that $m_1M = sM$ and $Ms = Mm_2$

We also have that the set of non-units of a finite monoid (if nonempty) is an ideal.

Corollary 4. Let M be a finite monoid with group of units G. Then $G = J_1$ and $M \setminus G$, if nonempty, is an ideal of M.

Finally, note that the group of units of eMe is exactly the intersection of eMe and J_e, where we recall that J_e is the $\mathscr{J}$-class of e.

Corollary 5. Let M be a finite monoid, and $e \in E(M)$. Then $J_e \cap eMe = G_e$. In particular, $I_e = eI(e)e$ and is an ideal of eMe (if nonempty).

Proof. By Lemma 2, $J_e \cap eMe$ is the $\mathscr{J}$-class of e in eMe. But this is the group of units G_e of eMe by Corollary 4 (recall that e is the identity of eMe), and so $J_e \cap eMe = G_e$. Note that $eI(e)e$ is an ideal of eMe (if nonempty) by the discussion at the start of this chapter, and $eI(e)e \subseteq I_e$. Conversely, if $m \in I_e = eMe \setminus G_e$, then $m \notin J_e$ by the first part of this corollary, and since $m \in MeM$, we can conclude that $e \notin MmM$. Thus, $m \in I(e)$ and therefore $m = eme \in eI(e)e$. □

von Neumann regularity

In our final foray into the basic tenets of finite monoids and their structure, we define regularity of a monoid element and the monoid itself. This definition then allows us to state three important results.

So, let us begin by defining regularity.

Definition 21. Let M be a monoid. An element $m \in M$ is said to be (von Neumann) regular if $m = mam$ for some $a \in M$, i.e. $m \in mMm$.
M is (von Neumann) regular if all of its elements are regular.

Example 11. All monoids that are also groups are regular, as any element x in a group is regular ($x = xx^{-1}x$).
The T_2 monoid defined previously is also regular: Clearly $m_1 = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$, $m_2 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 2 \end{smallmatrix}\right)$ and $m_3 = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right)$ are regular (they are idempotents). As for $m_4 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)$, we have that

$$m_4 m_4 m_4 = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1 & 2 \\ 1 & 2 \end{smallmatrix}\right) = m_4$$

$\mathbb{N}$ equipped with multiplication is not regular: The only regular element is 1.

The following proposition facilitates checking for regularity, connecting it with idempotents.

Proposition 2. Let M be a finite monoid. For $m \in M$, TFAE:

(i) m is regular.

(ii) $Mm = Me$ for some idempotent $e \in E(M)$.

(iii) $mM = eM$ for some idempotent $e \in E(M)$.

(iv) $MmM = MeM$ for some idempotent $e \in E(M)$.

Definition 22. A $\mathscr{J}$-class containing an idempotent is called a regular $\mathscr{J}$-class.

Example 12. In Example 8, we note that both $\mathscr{J}$-classes are regular, as m_1, m_2 and m_3 are idempotents.

Proposition 2 above states that a $\mathscr{J}$-class is regular if and only if all its elements are regular, allowing for easy verification of whether or not this holds, given a $\mathscr{J}$-class.

The following proposition provides a connection between what can be called idempotent ideals and idempotent elements.

Proposition 3. Let M be a finite monoid and I an ideal of M. Then $I^2 = I$ if and only if I is generated (as an ideal) by idempotents, that is $I = ME(I)M = IE(I)I$.

As a consequence of Proposition 2, we can reformulate Proposition 3, taking advantage of idempotents.

Corollary 6. Let M be a finite monoid. If $m \in M$, then $(MmM)^2 = MmM$ if and only if $MmM = MeM$ for some $e \in E(M)$, that is if and only if m is regular.

Proof. Assume that $(MmM)^2 = MmM$. By Proposition 3, $m \in MeM$ for some $e \in E(MmM)$, and so $MeM = MmM$ (MmM is an ideal of M). So, m is regular by Proposition 2. Conversely, $(MeM)^2 = MeMMeM = MeeM = MeM$. □

The theory of an idempotent

In this section, our previous work is used as a foundation for expanding on the theory of idempotents. We shall define five functors with the help of idempotents and reiterate several important results as pertains to them, then proceed to embellish the theory tying idempotents to simple modules.

For this section, let A be a finite dimensional algebra over a field $\Bbbk$, and let $e \in A$ be an idempotent. A few things to note: As usual, for $X, Y \subseteq A$,

$$XY = \left\{ \sum_{i=1}^{n} x_i y_i : x_i \in X, y_i \in Y \right\}$$

Furthermore, AeA is an ideal of A, and eAe is an algebra with identity e.

The five functors $\mathbf{Ind}_e, \mathbf{Coind}_e, \mathbf{Res}_e, T_e$ and N_e

We now define five functors, towards our end goal: Finding a bijection between irreducible representations for finite monoids, and irreducible representations for finite groups.

Definition 23. We define the following functors:

$$\begin{gathered}
\mathrm{Ind}_e : eAe\text{-mod} \to A\text{-mod} \\
\mathrm{Ind}_e(V) = Ae \otimes_{eAe} V \\
\mathrm{Coind}_e : eAe\text{-mod} \to A\text{-mod} \\
\mathrm{Coind}_e(V) = \mathrm{Hom}_{eAe}(eA, V) \\
\mathrm{Res}_e : A\text{-mod} \to eAe\text{-mod} \\
\mathrm{Res}_e(V) = eV \cong \mathrm{Hom}_A(Ae, V) \cong eA \otimes_A V \\
T_e : A\text{-mod} \to A\text{-mod} \\
T_e(V) = AeV \\
N_e : A\text{-mod} \to A\text{-mod} \\
N_e(V) = \{v \in V : eAv = 0\}
\end{gathered}$$

and name them, respectively, the induction functor, the coinduction functor, the restriction functor. The submodule $T_e(V)$ is called the trace of the projective module Ae in V (leaving the final functor without a nomenclature).

Again, a few things to note: $eV \cong \mathrm{Hom}_A(Ae, V)$ follows by Proposition 17. $eV \cong eA \otimes_A V$ is given by the mappings

$$\begin{gathered}
eA \otimes_A V \to eV \\
a \otimes_A v \mapsto av \\
eV \to eA \otimes_A V \\
v \mapsto e \otimes_A v
\end{gathered}$$

We observe that Ind_e is left adjoint to Res_e, and Coind_e is right adjoint to Res_e, as evidenced by the following proposition.

Proposition 4. Let V be an A-module and W an eAe-module. Then there are natural isomorphisms $\mathrm{Hom}_A(\mathrm{Ind}_e(W), V) \cong \mathrm{Hom}_{eAe}(W, \mathrm{Res}_e(V))$ and $\mathrm{Hom}_A(V, \mathrm{Coind}_e(W)) \cong \mathrm{Hom}_{eAe}(\mathrm{Res}_e(V), W)$.

Proof. Note that, for $X \in A$-mod-B (i.e. X left A-module and right B-module), we have from homological algebra that $\mathrm{Hom}_A(X \otimes_B M, N) \cong \mathrm{Hom}_B(M, \mathrm{Hom}_A(X, N))$. Note further that this prerequisite holds in the two statements above, and we have the result. □

Res_e is exact; given a few prerequisites, so are Ind_e and Coind_e.

Proposition 5. The functor Res_e is exact. If Ae respectively eA is a flat right respectively projective left eAe-module, then Ind_e respectively Coind_e is exact.

Intriguingly, Res_e is (in some sense) almost distributive with respect to tensor product and hom functors.

Proposition 6. Let A, B be finite dimensional algebras and e an idempotent of A. Suppose that U is a $B-A$-bimodule and V is an $A-B$-bimodule. Then Ue is a $B-eAe$-bimodule and eV is an $eAe-B$-bimodule by restricting the actions. Furthermore, for every B-module W there are natural isomorphisms of eAe-modules

$$e(V \otimes_B W) \cong eV \otimes_B W$$
$$e\ \mathrm{Hom}_B(U, W) \cong \mathrm{Hom}_B(Ue, W)$$

Another interesting property emerges as a result of Proposition 6: Ind_e and Coind_e are right quasi-inverse to Res_e.

Proposition 7. There are, for any eAe-module V, natural isomorphisms

$$\mathrm{Res}_e \circ \mathrm{Ind}_e(V) \cong V \cong \mathrm{Res}_e \circ \mathrm{Coind}_e(V)$$

Ind_e and Coind_e preserve isomorphisms of hom-sets.

Proposition 8. The functors Ind_e and Coind_e are fully faithful (i.e. induce isomorphisms of hom-sets).

As a corollary of Proposition 5, we are able to determine how Res_e acts on T_e and N_e.

Corollary 7. Let V be an A-module. Then $\mathrm{Res}_e(N_e(V)) = 0, \mathrm{Res}_e(T_e(V)) = \mathrm{Res}_e(V), \mathrm{Res}_e(V/N_e(V)) \cong \mathrm{Res}_e(V)$ and $\mathrm{Res}_e(V/T_e(V)) \cong 0$.

Continuing to link our five functors, we deduce how T_e and N_e act on Ind_e and Coind_e respectively.

Proposition 9. Let V be an eAe-module. Then $T_e(\mathrm{Ind}_e(V)) = \mathrm{Ind}_e(V)$ and $N_e(\mathrm{Coind}_e(V)) = 0$.

Proof. For the first equality, note that

$$T_e(\mathrm{Ind}_e(V)) = Ae(Ae \otimes_{eAe} V) = Ae(eAe) \otimes_{eAe} V = Ae \otimes_{eAe} V = \mathrm{Ind}_e(V)$$

as submodules of $Ae \otimes_{eAe} V$. For the second equality, let $\varphi : eA \to V$ be an eAe-module homomorphism such that $eA\varphi = 0$ (i.e. $\varphi \in N_e(\mathrm{Coind}_e(V))$. For $a \in A$, we have $\varphi(ea) = \varphi(eea) = (ea\varphi)(e) = 0$ as $ea\varphi = 0$. So, $\varphi = 0$ and therefore $N_e(\mathrm{Coind}_e(V)) = 0$. □

We now connect T_e and N_e to a pair of homomorphisms corresponding to the identity mapping $1_{eV} \in \mathrm{Hom}_{eAe}(\mathrm{Res}_e(V), \mathrm{Res}_e(V))$ where V is an A-module.

Proposition 10. Let V be an A-module, and $\alpha : \mathrm{Ind}_e(eV) \to V$ as well as $\beta : V \to \mathrm{Coind}_e(eV)$ be given by

$$\alpha(a \otimes v) = av \text{ for } a \in Ae, v \in eV$$
$$\beta(v)(a) = av \text{ for } a \in eA, v \in V$$

Then $T_e(V) = \alpha(\mathrm{Ind}_e(eV))$ and $N_e(V) = \ker \beta$. Furthermore, $\ker \alpha \subseteq N_e(\mathrm{Ind}_e(eV))$ and $T_e(\mathrm{Coind}_e(eV)) \subseteq \beta(V)$.

We conclude with a corollary: Note that for W an eAe-module, $e\ \mathrm{Ind}_e(W) \cong W \cong e\ \mathrm{Coind}_e(W)$ by Proposition 7, and thus the identity map 1_W corresponds to a homomorphism $\varphi : \mathrm{Ind}_e(W) \to \mathrm{Coind}_e(W)$. We now make this corresponding homomorphism explicit, and deduce its kernel/image.

Corollary 8. Let W be an eAe-module and let

$$\varphi : \mathrm{Ind}_e(W) \to \mathrm{Coind}_e(W)$$

be given by

$$\varphi(a \otimes w)(b) = baw$$

for $a \in Ae$, $b \in eA$ and $w \in W$. Then $\varphi(\mathrm{Ind}_e(W)) = T_e(\mathrm{Coind}_e(W))$ and $\ker \varphi = N_e(\mathrm{Ind}_e(W))$. Thus there is an isomorphism

$$\mathrm{Ind}_e(W)/N_e(\mathrm{Ind}_e(W)) \cong T_e(\mathrm{Coind}_e(W))$$

of A-modules.

Idempotents and simple modules

Recall that $\mathrm{Res}_e(V) = eV$ for V an A-module. We then have that Res_e takes simple modules to simple modules, or to the zero module.

Lemma 3. Let S be a simple A-module. Then either $eS = 0$ or eS is a simple eAe-module.

Proof. If $eS = 0$, we are done, so assume $eS \neq 0$. Then, for $v = es \in eS$, $eAev = eAees = eAes = eAv = eS$ since $Av = S$ by simplicity of S. Thus, eS is simple. □

As a corollary, we have that Res_e in a sense preserves composition factors for finite dimensional modules.

Corollary 9. Let V be a finite dimensional A-module with composition factors $S_1, \ldots, S_m$ (with multiplicities). Then the composition factors of eV as an eAe-module are the nonzero entries of the list $eS_1, \ldots, eS_m$ (with multiplicities).

Proof. Let

$$0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_m = V$$

be a composition series with $V_i/V_{i-1} \cong S_i$ for $i = 1, \ldots, m$. Then since Res_e is exact (Proposition 5), we have that $eV_i/eV_{i-1} \cong e(V_i/V_{i-1}) = eS_i$. By Lemma 3, each nonzero eS_i is simple. It then follows that after removing the repetitions from the series

$$eV_0 \subseteq eV_1 \subseteq \cdots \subseteq eV_m = eV$$

we get a composition series for eV whose composition factors are the nonzero entries of the list $eS_1 \ldots, eS_m$. □

The radicals of A and eAe can be related as follows.

Proposition 11. $\mathrm{rad}(eAe) = e\,\mathrm{rad}(A)e$. In particular, if A is semisimple, then eAe is also semisimple.

We can now provide some insight as to the radical and socle of induced and coinduced modules.

Proposition 12. Let V be a finite dimensional A-module.

(i) If $V = T_e(V)$, then $N_e(V) \subseteq \mathrm{rad}(V)$. Equality holds if and only if eV is semisimple.

(ii) If $N_e(V) = 0$, then $\mathrm{soc}(V) \subseteq T_e(V)$. Equality holds if and only if eV is semisimple.

Proof. Let $R = \mathrm{rad}(A)$.

(i) Let M be a maximal submodule of V. If $N_e(V) \nsubseteq M$, then $N_e(V) + M = V$, and so $eV = e(N_e(V) + M) = eM$. From this we get that $M \supseteq AeM = AeV = V$, which is a contradiction: Thus, $N_e(V) \subseteq M$ and therefore, as M is arbitrary, we have $N_e(V) \subseteq \mathrm{rad}(V)$. Now, as $\mathrm{rad}(V) = RV$ by Theorem 4 part *(iv)*, and $eRe = \mathrm{rad}(eAe)$ by Proposition 11, it follows that $e\,\mathrm{rad}(V) = eRV = eRAeV = eReeV = \mathrm{rad}(eAe)eV = \mathrm{rad}(eV)$, and so eV is semisimple if and only if $e\,\mathrm{rad}(V) = 0$. Then $\mathrm{rad}(V) \subseteq N_e(V)$ if and only if eV is semisimple.

(ii) Let $S \subseteq V$ be a simple module. As $N_e(V) = 0$, $0 \neq eS \subseteq eV$ and therefore $S = AeS \subseteq AeV = T_e(V)$. It follows that $\text{soc}(V) \subseteq T_e(V)$. Note that $e \text{ rad}(AeV) = eRAeV = eReeV = \text{rad}(eAe)eV = \text{rad}(eV)$, again by Theorem 4, and so eV is semisimple if and only if $e \text{ rad}(T_e(V)) = 0$, i.e. $\text{rad}(T_e(V)) \subseteq N_e(V) = 0$. But we have $\text{rad}(T_e(V)) = 0$ if and only if $T_e(V) \subseteq \text{soc}(V)$, and so eV is semisimple if and only if $T_e(V) = \text{soc}(V)$.

□

The following proposition is almost a corollary of Proposition 12, expanding further on the radical and socle of induced/coinduced modules.

Proposition 13. Let V be an A-module.

(i) If $V = T_e(V)$ and eV is simple, then $V/\text{rad}(V) = V/N_e(V)$ is simple.

(ii) If $N_e(V) = 0$ and eV is simple, then $\text{soc}(V) = T_e(V)$ is simple.

Proposition 13 can now be applied to induced/coinduced modules, yielding the existence of isomorphisms.

Corollary 10. Let V be a semisimple eAe-module. Then

$$\text{Ind}_e(V)/\text{rad}(\text{Ind}_e(V)) \cong \text{soc}(\text{Coind}_e(V))$$

Moreover, if V is simple, then $\text{Ind}_e(V)/\text{rad}(\text{Ind}_e(V))$ and $\text{soc}(\text{Coind}_e(V))$ are isomorphic simple modules, and there are isomorphisms

$$\text{Res}_e(\text{Ind}_e(V)/\text{rad}(\text{Ind}_e(V))) \cong V \cong \text{Res}_e(\text{soc}(\text{Coind}_e(V)))$$

of eAe-modules.

Proof. By Propositions 7, 9 and 12 we have $\text{rad}(\text{Ind}_e(V)) = N_e(V)$ and $\text{soc}(\text{Coind}_e(V)) = T_e(\text{Coind}_e(V))$. The first statement now follows from Corollary 8. The second statement follows from Propositions 9, 13, Corollary 7 and the isomorphisms $e \text{ Ind}_e(V) \cong V \cong e \text{ Coind}_e(V)$ from Proposition 7. □

The foundations thus laid, we can now state an immediate precursor to our main theorem, connecting isomorphism classes of a certain class of simple modules with isomorphism classes of simple A-modules, utilizing idempotents.

While not immediately obvious, through careful application of the five functors we've defined previously, we shall see that this ultimately allows for the discovery of irreducible representations of finite monoids.

Theorem 2. There is a bijection between isomorphism classes of simple eAe-modules and isomorphism classes of simple A-modules not annihilated by e, induced by

$$\begin{aligned} V \mapsto &\mathrm{Ind}_e(V)/N_e(\mathrm{Ind}_e(V)) = \mathrm{Ind}_e(V)/\mathrm{rad}(\mathrm{Ind}_e(V)) \\ &\cong \mathrm{soc}(\mathrm{Coind}_e(V)) = T_e(\mathrm{Coind}_e(V)) \\ S \mapsto &\mathrm{Res}_e(S) = eS \end{aligned}$$

for V a simple eAe-module and S a simple A-module, where $eS \neq 0$. Consequently, there is a bijection between the set of isomorphism classes of simple A-modules and the disjoint union of the sets of isomorphism classes of simple eAe-modules and of simple A/AeA-modules.

Proof. A few things to note: eV is simple and therefore semisimple, as is V. Furthermore, $V = T_e(V) = AeV$, and $N_e(\mathrm{Coind}_e(V)) = \{v \in \mathrm{Hom}_{eAe}(eA, V) : eAv = 0\} = 0$.

We have $\mathrm{Ind}_e(V)/N_e(\mathrm{Ind}_e(V)) = \mathrm{Ind}_e(V)/\mathrm{rad}(\mathrm{Ind}_e(V))$ by Proposition 12 part (i).
Next, we have $\mathrm{Ind}_e(V)/\mathrm{rad}(\mathrm{Ind}_e(V)) \cong \mathrm{soc}(\mathrm{Coind}_e(V))$ by Corollary 10.
By Proposition 9, we have $N_e(\mathrm{Coind}_e(V)) = 0$, and therefore by Proposition 12 part (ii), $\mathrm{soc}(\mathrm{Coind}_e(V)) = T_e(\mathrm{Coind}_e(V))$. Both mappings are thus well-defined.

Note that $\mathrm{Res}_e(\mathrm{soc}(\mathrm{Coind}_e(V))) = e\,\mathrm{soc}(\mathrm{Coind}_e(V)) \cong V$ by Corollary 10. Thus, we need to show that $S \cong \mathrm{soc}(\mathrm{Coind}_e(eS))$. By Proposition 4, we have isomorphisms $\mathrm{Hom}_A(S, \mathrm{soc}(\mathrm{Coind}_e(eS))) \cong \mathrm{Hom}_A(S, \mathrm{Coind}_e(eS)) \cong \mathrm{Hom}_{eAe}(eS, eS) \neq 0$.

Now, as eS is simple by Lemma 3, $\mathrm{soc}(\mathrm{Coind}_e(eS)))$ is simple by Corollary 10, and therefore $S \cong \mathrm{soc}(\mathrm{Coind}_e(eS))$ by Lemma 4.

The final statement follows from what we just proved, and the observation that simple A-modules annihilated by e are precisely the simple A/AeA-modules. □

Irreducible representations of a finite monoid

Our preparatory work is almost concluded. In the first subsection below, we introduce a slew of definitions relating to the algebra of a monoid, as well as a single proposition ensuring us that we can uniquely extend a monoid homomorphism to a (monoid) algebra homomorphism.

In the second subsection, we move on to state, and prove in detail, our principal result.

Monoid algebras

We define monoid representations, what their equivalence means, and in particular a matrix representation.

Definition 24. Let M be a monoid and $\Bbbk$ a field. A representation of M on a $\Bbbk$-vector space V is a homomorphism $\rho : M \to \mathrm{End}_{\Bbbk}(V)$, where $\dim V$ is the degree of ρ.
Two representations $\rho : M \to \mathrm{End}_{\Bbbk}(V)$ and $\psi : M \to \mathrm{End}_{\Bbbk}(W)$ are equivalent if there is a vector space isomorphism $T : V \to W$ such that $T^{-1}\psi(m)T = \rho(m)$ for all $m \in M$.
A representation ρ is faithful if it is injective.

Definition 25. Let M be a monoid and $\Bbbk$ a field. A matrix representation of M over $\Bbbk$ is a homomorphism $\rho : M \to M_n(\Bbbk)$ for some $n \geq 0$.
Two matrix representations $\rho, \psi : M \to M_n(\Bbbk)$ are equivalent if there is an invertible matrix $T \in M_n(\Bbbk)$ such that $T^{-1}\psi(m)T = \rho(m)$ for all $m \in M$.

Note that equivalence classes of degree n representations of M are in bijection with equivalence classes of matrix representations $\rho : M \to M_n(\Bbbk)$; thus, we shall not distinguish in terminology between representations on finite dimensional vector spaces and matrix representations.

We move on to explicit construction of a monoid algebra.

Definition 26. The monoid algebra $\Bbbk M$ of a monoid M over a field $\Bbbk$ is the $\Bbbk$-algebra constructed as follows: As a vector space, $\Bbbk M$ has basis M. So the elements of $\Bbbk M$ are formal sums $\sum_{m \in M} c_m m$ with $m \in M, c_m \in \Bbbk$ and with only finitely many $c_m \neq 0$. In practice, M will be finite and so this last constraint is unnecessary. The product is given by

$$\left(\sum_{m \in M} c_m m\right) \cdot \left(\sum_{m \in M} d_m m\right) = \sum_{m,n \in M} c_m d_n mn$$

When G is a group, $\Bbbk G$ is called the group algebra. Note that $\Bbbk M$ is finite dimensional if and only if M is finite.

As alluded to above, we may uniquely extend a monoid homomorphism to a $\Bbbk$-algebra homomorphism (i.e. a monoid algebra homomorphism).

Proposition 14. Let $\Bbbk$ be a field, A a $\Bbbk$-algebra and M a monoid. Then every monoid homomorphism $\varphi : M \to A$ extends uniquely to a $\Bbbk$-algebra homomorphism $\phi : \Bbbk M \to A$.

Let $\Bbbk$ be a field, A a $\Bbbk$-algebra, and V an A-module. Then V is the same thing as a $\Bbbk$-vector space V together with a $\Bbbk$-algebra homomorphism $\rho : A \to \mathrm{End}_{\Bbbk}(V)$. It follows from Proposition 14 that a representation of a monoid M on a $\Bbbk$-vector space V is the same thing as a $\Bbbk M$-module structure on V, and that two representations are equivalent if and only if the corresponding $\Bbbk M$-modules are isomorphic. Explicitly, if $\rho : M \to \mathrm{End}_{\Bbbk}(V)$ is a representation, then the $\Bbbk M$-module structure on V, for $v \in V$, is given by

$$\left(\sum_{m\in M} c_m m\right) \cdot v = \sum_{m\in M} c_m \rho(m) v$$

Example 13. For any field $\Bbbk$ and any monoid M we have a $\Bbbk M$-module via

$$\varphi : \Bbbk M \times \Bbbk M \to \Bbbk M$$
$$\varphi((m_1, m_2)) = 1_{\Bbbk M}(m_2) = m_2$$

for $m_1, m_2 \in \Bbbk M$.

Let S_n be the permutations of n elements, and take $\mathbb{C}^n$ with standard basis $\underline{e} = \{e_1, e_2, \ldots, e_n\}$. Then we get the natural S_n-module via permuting the indices of the basis elements thusly:

$$\varphi : S_n \times \mathbb{C}^n \to \mathbb{C}^n$$
$$\varphi((\pi, e_i)) = e_{\pi(i)}$$

for $\pi \in S_n, e_i \in \underline{e}$.

Let T_2 be as defined earlier, and $X = \left\{\left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 \\ 2 & 2 \end{smallmatrix}\right)\right\}$. Then X is a T_2-module via

$$\varphi : T_2 \times X \to X$$
$$\varphi((t, x)) = tx$$

for $t \in T_2, x \in X$ (any ideal can be made into a module in like fashion).

Definition 27. Let $\Bbbk$ be a field and M a monoid. We say that the $\Bbbk M$-module V affords the corresponding representation $\rho : M \to \mathrm{End}_{\Bbbk}(V)$ or, if we choose a basis for V, the corresponding matrix representation $\rho : M \to M_n(\Bbbk)$.

We conclude this section with some notation and (most likely familiar) definitions.

Definition 28. Let $\Bbbk$ be a field, and M a monoid. The set of isomorphism classes of simple $\Bbbk M$-modules will be denoted $\mathrm{Irr}_{\Bbbk}(M)$. The isomorphism class of a module V will typically be written $[V]$.

Definition 29. Let M be a monoid and $\Bbbk$ a field. We call the representation $\rho : M \to \Bbbk$ given by $\rho(m) = 1$ for all $m \in M$ the trivial representation. We call $\Bbbk$, equipped with the corresponding module structure, the trivial module.

Definition 30. Let $\Bbbk$ be a field, M a monoid, and V, W $\Bbbk M$-modules. The tensor product $V \otimes_{\Bbbk} W$, often simply denoted $V \otimes W$, becomes a $\Bbbk M$-module by defining $m(v \otimes w) = mv \otimes mw$ for $m \in M, v \in V, w \in W$.

Definition 31. Let $\Bbbk$ be a field. If $\varphi : M \to N$ is a homomorphism of monoids, then any $\Bbbk N$-module V can be viewed as a $\Bbbk M$-module via the action $mv = \varphi(m)v$ for $m \in M, v \in V$. We then say that the $\Bbbk M$-module V is the inflation of the $\Bbbk N$-module V to M. More generally, if $\psi : A \to B$ is a homomorphism of finite dimensional algebras and V is a B-module, we call the A-module structure on V given by $av = \psi(a)v$ for $a \in A, v \in V$ the inflation of V along ψ.

Example 14. Take as M, N any monoids where M is a submonoid of N, and let $\varphi : M \to N$ be the inclusion homomorphism, i.e.

$$\begin{aligned} \varphi : M &\to N \\ m &\mapsto m \end{aligned}$$

Then any N-module V becomes an M-module via

$$\begin{aligned} \varphi_\star : M \times V &\to V \\ (m, v) &\mapsto \varphi(m)v \end{aligned}$$

i.e. the inflation of V along φ.

Clifford-Munn-Ponizovskiĭ theory

We stand at the apex (pun intended, as the reader shall soon observe) of our efforts. In this section, we at last state, and prove in detail, our principal result.

First a basic definition.

Definition 32. Let M be a finite monoid, $\Bbbk$ a field, V a $\Bbbk M$-module, $X \subseteq M, Y \subseteq V$. Then

$$XY = \left\{ \sum_{i=1}^{n} k_i x_i y_i : k_i \in \Bbbk, x_i \in X, y_i \in Y \right\}$$

Let $\Bbbk X$ denote the $\Bbbk$-linear span of X in $\Bbbk M$.

We now define an apex (recall that $I_e = eMe \setminus G_e$).

Definition 33. Let M be a finite monoid, $\Bbbk$ a field, and S a simple $\Bbbk M$-module. An idempotent $e \in E(M)$ is an apex for S if $eS \neq 0$ and $I_e S = 0$.

Example 15. Let our finite monoid M be T_3, the full transformation monoid for three elements under composition of transformations. Also let $\Bbbk$ be any field, and take as our simple $\Bbbk M$-module S the trivial module over T_3, which acts as the identity. Let $e = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right)$: e is clearly idempotent, but also an apex. Note that $eS = \{\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right)\} \neq 0$, and

$$\begin{aligned} I_e S &= \big(eMe \setminus G_e\big)S \\ &= \Big(\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right) T_3 \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right) \setminus G_{\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right)}\Big)S \\ &= \big(\{\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right)\} \setminus \{\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right)\}\big)S \\ &= 0 \cdot S = 0 \end{aligned}$$

Now take as simple $\Bbbk M$-module S over $M = T_3$ the trivial module for $m \in S_3$, and the zero module for $m \notin S_3$:

$$\varphi : \Bbbk M \times M \to M$$
$$k \cdot m \mapsto \begin{cases} m & \text{if } m \in S_3 \\ 0 & \text{if } m \notin S_3 \end{cases}$$

In this case, $e = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \end{smallmatrix}\right)$ is not an apex, since $eS = 0$.

Next, we establish a few properties of apexes.

Proposition 15. Let M be a finite monoid, $\Bbbk$ a field and S a simple $\Bbbk M$-module with apex $e \in E(M)$.

(i) $I(e) = \{m \in M : mS = 0\}$

(ii) If $f \in E(M)$, then f is an apex for S if and only if $MeM = MfM$.

Proof. (i) Since S is simple and $eS \neq 0$, we have $MeS = S$ and therefore $I(e)S = I(e)MeS = I(e)eS$. Now suppose that $I(e)S \neq 0$ (in particular, $I(e) \neq \emptyset$), and note that, as $I(e)$ is an ideal of M, $I(e)S$ is a submodule of S. Since S is simple, we have $I(e)eS = I(e)S = S$, and so $eS = eI(e)eS = I_eS = 0$, a contradiction. We can therefore conclude that $I(e)S = 0$. Now if $e \in MmM$, then $0 \neq eS \subseteq MmMS = MmS$, and so $mS \neq 0$.

(ii) If $f \in E(M)$ with $fS \neq 0$, then $f \notin I(e)$ by (i) above, and so $MeM \subseteq MfM$. It follows by symmetry that if f is another apex, then $MeM = MfM$. Conversely, if $f \in E(M)$ with $MfM = MeM$, then $fS \neq 0$ and $I_fS = 0$, again by (i) above, since $f \notin I(e)$ and $I_f \subseteq I(e)$. Thus, f is an apex for S.

□

We shall need some notation, and a definition of left and right module structures on the L_e and R_e classes of an idempotent e.

Definition 34. Let M be a monoid and I an ideal of M. We identify the coset $m + \Bbbk I$ with m for $m \notin I$ and thus write $(\Bbbk M/\Bbbk I)m$ instead of $(\Bbbk M/\Bbbk I)(m + \Bbbk I))$.

Definition 35. Let M be a monoid and $\Bbbk$ a field. Fix an idempotent $e \in E(M)$, and let $A_e = \Bbbk M/\Bbbk I(e)$. Note that, as $\Bbbk$-vector spaces, $A_e e \cong \Bbbk L_e$, and $eA_e \cong \Bbbk R_e$. The corresponding left $\Bbbk M$-module structure on $\Bbbk L_e$ for $m \in M, l \in L_e$ is defined by

$$m \odot l = \begin{cases} ml & \text{if } ml \in L_e \\ 0 & \text{else} \end{cases}$$

From now on, we omit the symbol $\odot$. The right $\Bbbk M$-module structure on $\Bbbk R_e$ is defined in like fashion.

Now for an observation: As mentioned when we introduced it, Theorem 2 bears a startling resemblance to our principal result, the latter soon to be stated. We would thus of course like to use Theorem 2 (indeed, we want to make use of the whole theory of idempotents we expended effort on introducing).

Towards this end, let M be a finite monoid and fix an idempotent $e \in E(M)$. We observe that $eA_ee = e\,[\Bbbk M/\Bbbk\{m \in M : e \notin MmM\}]\,e \cong \Bbbk[eMe]/\Bbbk[e\{m \in M : e \notin MmM\}e] = \Bbbk[eMe]/\Bbbk I_e \overset{\text{Cor 5}}{=} \Bbbk[eMe]/\Bbbk[eMe \setminus G_e] \cong \Bbbk G_e$ by applying Corollary 5.

By Proposition 15 above, a simple $\Bbbk M$-module has apex e if and only if it is annihilated by $I(e)$ but not by e. That is to say, the simple $\Bbbk M$-modules with apex e are exactly the inflations of simple A_e-modules S, where $eS \neq 0$. We may thus apply the idempotent theory previously established.

By the observation above, we have that if V is a $\Bbbk G_e$-module, then

$$\mathrm{Hom}_{eA_ee}(eA_e, V) \cong \mathrm{Hom}_{\Bbbk G_e}(\Bbbk R_e, V) \cong \mathrm{Hom}_{G_e}(R_e, V)$$

where $\mathrm{Hom}_{G_e}(R_e, V)$ denotes the vector space of G_e-equivariant mappings $\varphi : R_e \to V$.

We may now redefine the five familiar functors previously introduced, in terms of the group of units of an idempotent.

Definition 36. Let M be a finite monoid, $\Bbbk$ a field and $e \in E(M)$.
The action of $m \in M$ on $\varphi \in \mathrm{Hom}_{G_e}(R_e, V)$ is given by

$$(m\varphi)(r) = \begin{cases} \varphi(rm) & \text{if } rm \in R_e \\ 0 & \text{else} \end{cases}$$

Define:

$$\begin{gathered} \mathrm{Ind}_{G_e} : \Bbbk G_e\text{-mod} \to \Bbbk M\text{-mod} \\ \mathrm{Ind}_{G_e}(V) = A_e e \otimes_{eA_e e} V = \Bbbk L_e \otimes_{\Bbbk G_e} V \\ \mathrm{Coind}_{G_e} : \Bbbk G_e\text{-mod} \to \Bbbk M\text{-mod} \\ \mathrm{Coind}_{G_e}(V) = \mathrm{Hom}_{eA_e e}(eA_e, V) = \mathrm{Hom}_{G_e}(R_e, V) \\ \mathrm{Res}_{G_e} : \Bbbk M\text{-mod} \to \Bbbk G_e\text{-mod} \\ \mathrm{Res}_{G_e}(V) = eV \\ T_e : \Bbbk M\text{-mod} \to \Bbbk M\text{-mod} \\ T_e(V) = \Bbbk MeV = MeV \\ N_e : \Bbbk M\text{-mod} \to \Bbbk M\text{-mod} \\ N_e(V) = \{v \in V : e\Bbbk Mv = 0\} = \{v \in V : eMv = 0\} \end{gathered}$$

We now state and prove our principal result in all its glory.

Theorem 3. Let M be a finite monoid and $\Bbbk$ a field.

(i) There is a bijection between isomorphism classes of simple $\Bbbk M$-modules with apex $e \in E(M)$ and isomorphism classes of simple $\Bbbk G_e$-modules induced by

$$\begin{aligned} S &\mapsto \mathrm{Res}_{G_e}(S) = eS \\ V &\mapsto V^\sharp = \mathrm{Ind}_{G_e}(V)/N_e(\mathrm{Ind}_{G_e}(V)) = \mathrm{Ind}_{G_e}(V)/\mathrm{rad}(\mathrm{Ind}_{G_e}(V)) \\ &\cong \mathrm{soc}(\mathrm{Coind}_{G_e}(V)) = T_e(\mathrm{Coind}_{G_e}(V)) \end{aligned}$$

where S is a simple $\Bbbk M$-module with apex e and V is a simple $\Bbbk G_e$-module.

(ii) Every simple $\Bbbk M$-module has an apex (unique up to $\mathscr{J}$-equivalence).

(iii) If V is a simple $\Bbbk G_e$-module, then every composition factor of $\mathrm{Ind}_{G_e}(V)$ and $\mathrm{Coind}_{G_e}(V)$ has apex f with $MeM \subseteq MfM$. Moreover, $V^\sharp$ is the unique composition factor of either of these two modules with apex e and it appears in both with multiplicity one.

Proof. We shall prove each statement in turn.

(i) We previously established that simple $\Bbbk M$-modules with apex e are the same thing as inflations of simple A_e-modules S with $eS \neq 0$, and that $eA_e e \cong \Bbbk G_e$. Thus, apply Theorem 2 to A_e, and the statement follows immediately.

(ii) Let S be a simple $\Bbbk M$-module, and let I be the minimal ideal of M such that $IS \neq 0$. Since IS is a submodule of S, we have $IS = S$ and so $I^2 S = IS = S \neq 0$. Thus, $I^2 = I$ by minimality of I ($I^2 \subseteq I$).
Now choose $m \in I$ such that $mS \neq 0$, which is possible as $IS \neq 0$. Then $MmMS \neq 0$ and therefore $I = MmM$, again by minimality of I.
By Corollary 6 there exists an idempotent $e \in E(M)$ such that $I = MeM$: This e is in fact an apex of S. First, observe that $S = IS = MeMS = MeS$, where $MS = S$ as S is simple and MS is a submodule of S. We can then conclude that $eS \neq 0$. Next, since $MI_eM = MeI(e)eM \subsetneq MeM = I$ by Corollary 5, we have $MI_eMS = 0$ again by minimality of I, and so $I_eS = 0$. Thus e is an apex by definition, and from applying Proposition 15 part (ii) we have that it is unique up to $\mathscr{J}$-equivalence, again by definition.

(iii) Let $f \in E(M), f \in I(e)$. Since $f \operatorname{Ind}_{G_e}(V) = 0 = f \operatorname{Coind}_{G_e}(V)$, Corollary 9 yields that f annihilates each composition factor of $\operatorname{Ind}_{G_e}(V)$ and $\operatorname{Coind}_{G_e}(V)$, and therefore cannot be an apex for any of these composition factors by definition. So, we must have, for any apex g for the composition factors, that $g \notin I(e)$: I.e. $MeM \subseteq MgM$.
Furthermore, as $e \operatorname{Ind}_{G_e}(V) = V = e \operatorname{Coind}_{G_e}(V)$, again by Corollary 9 it follows that $V^\sharp$ is the only composition factor of either module with apex e, and appears with multiplicity one in both.

□

Thus, we have shown that the isomorphism classes of the irreducible representations of a finite monoid are in bijection with the isomorphism classes of the irreducible representations of its maximal subgroups. Therefore, we have reduced the problem of finding irreducible representations of a finite monoid to finding irreducible representations of finite groups.

As an interesting corollary, we get a parametrization of the irreducible representations of a finite monoid.

Corollary 11. Let $e_1, \ldots, e_s$ be a complete set of idempotent representatives of the regular $\mathscr{J}$-classes of M. Then there is a bijection between $\operatorname{Irr}_\Bbbk(M)$ and the disjoint union $\cup_{i=1}^{s} \operatorname{Irr}_\Bbbk(G_{e_i})$.

Proof. Follows from Theorem 3 parts (i) and (ii), and the observation that each simple $\Bbbk M$-module V has a unique apex amongst $e_1, \ldots, e_s$. □

Application

The symmetric inverse monoid

Theorem 3 allows for computing the irreducible representations for full classes of finite monoids. For an example of such, we refer to [3] chapter 5.3, wherein the irreducible representations for the finite monoid T_n are deduced.

For a more explicit example, we shall use Theorem 3 and Corollary 11 to compute the irreducible representations for a particular finite monoid. A few definitions are thus required.

Definition 37. A monoid M is called an inverse monoid if, for all $m \in M$, there exists a unique element $m^* \in M$, called the inverse of m, such that $mm^*m = m$ and $m^*mm^* = m^*$.

Definition 38. Let X be a set. A partial mapping (or partial transformation) $f : X \to X$ is a mapping from a subset $\mathrm{dom}(f)$ (the domain of f) of X to X. We denote the set of all partial transformations of X as PT_X; note that this set is a monoid with composition of mappings as its operation. If $X = \{1, \ldots, n\}$, we write PT_n. The empty partial mapping is the zero element of PT_X. If $Y \subseteq X$, we write 1_Y for the partial mapping with domain Y that fixes Y pointwise.

Definition 39. Let X be a set. The symmetric inverse monoid I_X on X is the inverse monoid of all partial injective mappings $f : X \to X$ with respect to composition of partial mappings. We write I_n for the symmetric inverse monoid on $\{1 \ldots, n\}$.
The rank (often denoted rk) of a partial injective mapping f is the cardinality of its image (or, equivalently, that of its domain).

The symmetric inverse monoid thus defined, we can proceed with finding the irreducible representations for one particular such monoid.

Calculating the irreducible representations of I_3

We calculate the irreducible representations of the symmetric inverse monoid

$$\begin{aligned} I_3 = \{ & \left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 1&\emptyset&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&\emptyset&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&\emptyset&\emptyset\end{smallmatrix}\right), \\ & \left(\begin{smallmatrix}1&2&3\\ \emptyset&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&2&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&2\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&3\end{smallmatrix}\right), \\ & \left(\begin{smallmatrix}1&2&3\\ 1&2&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 1&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&2&\emptyset\end{smallmatrix}\right), \\ & \left(\begin{smallmatrix}1&2&3\\ 1&\emptyset&2\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 1&\emptyset&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&\emptyset&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&\emptyset&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&\emptyset&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&\emptyset&2\end{smallmatrix}\right), \\ & \left(\begin{smallmatrix}1&2&3\\ \emptyset&1&2\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&1&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&2&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&2&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&3&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&3&2\end{smallmatrix}\right), \\ & \left(\begin{smallmatrix}1&2&3\\ 1&2&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 1&3&2\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&1&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&3&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&2&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&1&2\end{smallmatrix}\right) \} \end{aligned}$$

over the field $\mathbb{C}$, where for instance $\left(\begin{smallmatrix}1&2&3\\ \emptyset&1&2\end{smallmatrix}\right)$ is the partial mapping that maps the second element to the first, the third element to the second, and does not map the first element to anything.

Now, Corollary 11 suggests that our first step should be to find the $\mathscr{J}$-classes of I_3. Note that any partial injective mapping which maps exactly n elements is in $\mathscr{J}$-relation with any other partial injective mapping which maps exactly n elements (denote this $\mathscr{J}$-class by J_n). Thus, for I_3, we will have exactly four $\mathscr{J}$-classes:

$$\begin{aligned} J_0 &= \{ \left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&\emptyset\end{smallmatrix}\right) \} \\ J_1 &= \{ \left(\begin{smallmatrix}1&2&3\\ 1&\emptyset&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&\emptyset&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&\emptyset&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&1&\emptyset\end{smallmatrix}\right), \\ & \quad \left(\begin{smallmatrix}1&2&3\\ \emptyset&2&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&2\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&3\end{smallmatrix}\right) \} \\ J_2 &= \{ \left(\begin{smallmatrix}1&2&3\\ 1&2&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 1&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&2&\emptyset\end{smallmatrix}\right), \\ & \quad \left(\begin{smallmatrix}1&2&3\\ 1&\emptyset&2\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 1&\emptyset&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&\emptyset&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&\emptyset&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&\emptyset&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&\emptyset&2\end{smallmatrix}\right), \\ & \quad \left(\begin{smallmatrix}1&2&3\\ \emptyset&1&2\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&1&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&2&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&2&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&3&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ \emptyset&3&2\end{smallmatrix}\right) \} \\ J_3 &= \{ \left(\begin{smallmatrix}1&2&3\\ 1&2&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 1&3&2\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&1&3\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 2&3&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&2&1\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\ 3&1&2\end{smallmatrix}\right) \} \end{aligned}$$

Note that J_0, J_1, J_2, J_3 are all regular, since they all contain at least one idempotent. Let us select, from J_0, J_1, J_2, J_3 respectively, the idempotent representatives $\left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&\emptyset\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&2&3\\ 1&\emptyset&\emptyset\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}1&2&3\\ 1&2&\emptyset\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}1&2&3\\ 1&2&3\end{smallmatrix}\right)$, labeling them e_0, e_1, e_2, e_3 in keeping with our notation.

Applying Corollary 11, we now need to find the irreducible representations (i.e. simple modules) of the group of units for these four idempotent representatives, as they are in bijection with the irreducible representations of the finite monoid I_3. Once we have these irreducible representations of each group of units, we can explicitly calculate the irreducible representations of I_3 by applying Theorem 3. Let us proceed forthwith, and recall that S_n is the group of permutations of n elements, as well as that for G_{e_i}, e_i must be the identity.

e_0:

$$\left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&\emptyset\end{smallmatrix}\right) \cdot I_3 \cdot \left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&\emptyset\end{smallmatrix}\right) = \{ \left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&\emptyset\end{smallmatrix}\right) \}$$
$$\Rightarrow G_{e_0} = \{ \left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&\emptyset\end{smallmatrix}\right) \}$$

Since we always have the trivial module, and G_{e_0} is 1-dimensional, that is all we have; i.e. $\mathbb{C}_{\text{triv}}^{G_{e_0}}$.

e_1:

$$\left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right)\cdot I_3\cdot\left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right)\cdot\left\{\left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\2&\emptyset&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\3&\emptyset&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&\emptyset\end{smallmatrix}\right)\right\}=\left\{\left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&\emptyset\end{smallmatrix}\right)\right\}$$
$$\Rightarrow G_{e_1}=\left\{\left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right)\right\}\cong\left\{\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\right\}=S_1$$

We note that S_1 has only the trivial module (which is simple), denoted by $\mathbb{C}^{S_1}_{\text{triv}}$.

e_2:

$$\begin{aligned}&\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right)\cdot I_3\cdot\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right)\\
&=\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right)\cdot\left\{\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\2&\emptyset&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\3&\emptyset&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\\emptyset&1&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\\emptyset&2&\emptyset\end{smallmatrix}\right),\right.\\
&\qquad\left.\left(\begin{smallmatrix}1&2&3\\\emptyset&3&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\1&3&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\2&3&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\3&1&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\3&2&\emptyset\end{smallmatrix}\right)\right\}\\
&=\left\{\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\2&\emptyset&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\\emptyset&1&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\\emptyset&2&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right)\right\}\\
&\Rightarrow G_{e_2}=\left\{\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right)\right\}\cong\left\{\left(\begin{smallmatrix}1&2\\1&2\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2\\2&1\end{smallmatrix}\right)\right\}=S_2\end{aligned}$$

Recall that S_2 has the simple modules $\mathbb{C}^{S_2}_{\text{triv}}$ and $\mathbb{C}^{S_2}_{\text{sign}}$, where the sign module acts by changing the sign.

e_3:

$$\left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right)\cdot I_e\cdot\left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right)=\left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right)\cdot I_3=I_3$$
$$\Rightarrow G_{e_3}=\left\{\left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right),\left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)\right\}=S_3$$

Finally, we recall that the simple modules for S_3 are $\mathbb{C}^{S_3}_{\text{triv}}$, $\mathbb{C}^{S_3}_{\text{sign}}$, and $\mathcal{S}^{(2,1)}$, the last being the Specht module for the partition $(2,1)$ (the reader will hopefully forgive us for not expanding on the definitions for Specht modules, partitions and the like here).

So, by Corollary 11, we have that the irreducible representations of the symmetric inverse monoid I_3 over the field $\mathbb{C}$ are isomorphic to

$$\left(\mathbb{C}^{G_{e_0}}_{\text{triv}}\right)^\sharp,\left(\mathbb{C}^{S_1}_{\text{triv}}\right)^\sharp,\left(\mathbb{C}^{S_2}_{\text{triv}}\right)^\sharp,\left(\mathbb{C}^{S_2}_{\text{sign}}\right)^\sharp,\left(\mathbb{C}^{S_3}_{\text{triv}}\right)^\sharp,\left(\mathbb{C}^{S_3}_{\text{sign}}\right)^\sharp,\left(\mathcal{S}^{(2,1)}\right)^\sharp$$

It remains to explicitly calculate these modules, using Theorem 3. Thus, for each $\mathbb{C}G_{e_i}$-module V above, we need to perform five steps:

1. Determine L_{e_i}.
2. Calculate $\text{Ind}_{G_{e_i}}(V)=\mathbb{C}L_{e_i}\otimes_{\mathbb{C}G_{e_i}}V$.
3. Calculate $N_{e_i}(\text{Ind}_{G_{e_i}}(V))=\{v\in\text{Ind}_{G_{e_i}}(V):e_iI_3v=0\}$.
4. Calculate the quotient $\text{Ind}_{G_{e_i}}(V)/N_{e_i}(\text{Ind}_{G_{e_i}}(V))$.
5. Determine the dimension of the quotient $\text{Ind}_{G_{e_i}}(V)/N_{e_i}(\text{Ind}_{G_{e_i}}(V))$ by finding its basis, as well as how I_3 acts on the quotient's basis elements (recall that we defined the left $\mathbb{C}I_3$-module structure on $\mathbb{C}L_{e_i}$ above, see Definition 35).

We note, however, that in all these cases, $N_{e_i}(\text{Ind}_{G_{e_i}}(V))=\{0\}$, meaning that $\text{Ind}_{G_{e_i}}(V)/N_{e_i}(\text{Ind}_{G_{e_i}}(V))=\text{Ind}_{G_{e_i}}(V)/\{0\}=\text{Ind}_{G_{e_i}}(V)$, thus leaving us with three required steps for each V.

On to the calculations. Full tables depicting how I_3 acts on basis elements for these modules are provided at the end of this chapter.

$(\mathbb{C}_{\mathrm{triv}}^{G_{e_0}})^\sharp$:

1.

$$e_0 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{smallmatrix}\right) \Rightarrow I_3 \cdot e_0 = \left\{\left(\begin{smallmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{smallmatrix}\right)\right\} \Rightarrow L_{e_0} = \left\{\left(\begin{smallmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{smallmatrix}\right)\right\}$$

2.

$$\mathrm{Ind}_{G_{e_0}}(\mathbb{C}_{\mathrm{triv}}^{G_{e_0}}) = \mathbb{C}L_{e_0} \otimes_{\mathbb{C}G_{e_0}} \mathbb{C}_{\mathrm{triv}}^{G_{e_0}} \cong \mathbb{C}L_{e_0}$$

3. $\mathbb{C}L_{e_0}$ is 1-dimensional with basis $\left\{\left(\begin{smallmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{smallmatrix}\right)\right\}$. All elements in I_3 act as the identity, since for any $\sigma \in I_3$, $\sigma \cdot \left(\begin{smallmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{smallmatrix}\right)$.

$(\mathbb{C}_{\mathrm{triv}}^{S_1})^\sharp$:

1.

$$e_1 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & \emptyset & \emptyset \end{smallmatrix}\right) \Rightarrow I_3 \cdot e_1 = \left\{\left(\begin{smallmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & \emptyset & \emptyset \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & \emptyset & \emptyset \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & \emptyset & \emptyset \end{smallmatrix}\right)\right\} \Rightarrow L_{e_1} = \left\{\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & \emptyset & \emptyset \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & \emptyset & \emptyset \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & \emptyset & \emptyset \end{smallmatrix}\right)\right\}$$

2.

$$\mathrm{Ind}_{G_{e_1}}(\mathbb{C}_{\mathrm{triv}}^{S_1}) = \mathbb{C}L_{e_1} \otimes_{\mathbb{C}G_{e_1}} \mathbb{C}_{\mathrm{triv}}^{S_1} \cong \mathbb{C}L_{e_1}$$

3. $\mathbb{C}L_{e_1}$ is 3-dimensional with basis $\left\{\left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & \emptyset & \emptyset \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & \emptyset & \emptyset \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & \emptyset & \emptyset \end{smallmatrix}\right)\right\}$. For $\left(\begin{smallmatrix} 1 & 2 & 3 \\ \alpha & \beta & \gamma \end{smallmatrix}\right) \in I_3$, $\left(\begin{smallmatrix} 1 & 2 & 3 \\ i & \emptyset & \emptyset \end{smallmatrix}\right) \in B$, we have

$$\left(\begin{smallmatrix} 1 & 2 & 3 \\ \alpha & \beta & \gamma \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 & 2 & 3 \\ i & \emptyset & \emptyset \end{smallmatrix}\right) = \begin{cases} 0 & \text{if } i = 1, \alpha = \emptyset \\ \left(\begin{smallmatrix} 1 & 2 & 3 \\ \alpha & \emptyset & \emptyset \end{smallmatrix}\right) & \text{if } i = 1, \alpha \neq \emptyset \\ 0 & \text{if } i = 2, \beta = \emptyset \\ \left(\begin{smallmatrix} 1 & 2 & 3 \\ \beta & \emptyset & \emptyset \end{smallmatrix}\right) & \text{if } i = 2, \beta \neq \emptyset \\ 0 & \text{if } i = 3, \gamma = \emptyset \\ \left(\begin{smallmatrix} 1 & 2 & 3 \\ \gamma & \emptyset & \emptyset \end{smallmatrix}\right) & \text{if } i = 3, \gamma \neq \emptyset \end{cases}$$

$(\mathbb{C}^{S_2}_{\text{triv}})^\sharp$:

1.

$$\begin{aligned} e_2 &= \left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right) \\ \Rightarrow I_3 \cdot e_2 &= \Big\{\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\2&\emptyset&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\3&\emptyset&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\\emptyset&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\\emptyset&2&\emptyset\end{smallmatrix}\right), \\ &\quad \left(\begin{smallmatrix}1&2&3\\\emptyset&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\1&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\2&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\3&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\3&2&\emptyset\end{smallmatrix}\right)\Big\} \\ \Rightarrow L_{e_2} &= \Big\{\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\2&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\3&2&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\1&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\3&1&\emptyset\end{smallmatrix}\right)\Big\} \end{aligned}$$

2.

$$\text{Ind}_{G_{e_2}}(\mathbb{C}^{S_2}_{\text{triv}}) = \mathbb{C}L_{e_2} \otimes_{\mathbb{C}G_{e_2}} \mathbb{C}^{S_2}_{\text{triv}}$$

3. $\mathbb{C}L_{e_2}$ has the basis L_{e_2}, whilst $\mathbb{C}^{S_2}_{\text{triv}}$ has basis $\left\{\left(\begin{smallmatrix}1&2\\1&2\end{smallmatrix}\right) + \left(\begin{smallmatrix}1&2\\2&1\end{smallmatrix}\right)\right\}$, so $\mathbb{C}L_{e_2} \otimes_{\mathbb{C}G_{e_2}} \mathbb{C}^{S_2}_{\text{triv}}$ has the basis $B = \left\{\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right) + \left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\2&3&\emptyset\end{smallmatrix}\right) + \left(\begin{smallmatrix}1&2&3\\3&2&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\1&3&\emptyset\end{smallmatrix}\right) + \left(\begin{smallmatrix}1&2&3\\3&1&\emptyset\end{smallmatrix}\right)\right\}$ and is therefore 3-dimensional. $\sigma \in I_3 \setminus \{J_0 \cup J_1\}$ act by permuting the basis vectors where defined (and otherwise act as 0), whilst $\sigma \in J_0 \cup J_1$ act as 0.

$(\mathbb{C}^{S_2}_{\text{sign}})^\sharp$:

1. As before, we have

$$L_{e_2} = \Big\{\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\2&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\3&2&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\1&3&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\3&1&\emptyset\end{smallmatrix}\right)\Big\}$$

2.

$$\text{Ind}_{G_{e_2}}(\mathbb{C}^{S_2}_{\text{sign}}) = \mathbb{C}L_{e_2} \otimes_{\mathbb{C}G_{e_2}} \mathbb{C}^{S_2}_{\text{sign}}$$

3. $\mathbb{C}^{S_2}_{\text{sign}}$ has the basis $\left\{\left(\begin{smallmatrix}1&2\\1&2\end{smallmatrix}\right) - \left(\begin{smallmatrix}1&2\\2&1\end{smallmatrix}\right)\right\}$, so $\mathbb{C}L_{e_2} \otimes_{\mathbb{C}G_{e_2}} \mathbb{C}^{S_2}_{\text{sign}}$ has the basis $\left\{\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right) - \left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\2&3&\emptyset\end{smallmatrix}\right) - \left(\begin{smallmatrix}1&2&3\\3&2&\emptyset\end{smallmatrix}\right), \left(\begin{smallmatrix}1&2&3\\1&3&\emptyset\end{smallmatrix}\right) - \left(\begin{smallmatrix}1&2&3\\3&1&\emptyset\end{smallmatrix}\right)\right\}$ and is therefore 3-dimensional. Again, $\sigma \in I_3 \setminus \{J_0 \cup J_1\}$ act by permuting the basis vectors where defined (and otherwise act as 0), whilst $\sigma \in J_0 \cup J_1$ act as 0.

$(\mathbb{C}^{S_3}_{\text{triv}})^\sharp$:

1.

$$e_3 = \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right) \Rightarrow I_3 \cdot e_3 = I_3$$
$$\Rightarrow L_{e_3} = \left\{ \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right) \right\}$$

2. Note that $\mathbb{C}L_{e_3} = \mathbb{C}G_{e_3}$, as $L_{e_3} = S_3 = G_{e_3}$. Therefore, we have

$$\operatorname{Ind}_{G_{e_3}}(\mathbb{C}^{S_3}_{\text{triv}}) = \mathbb{C}L_{e_3} \otimes_{\mathbb{C}G_{e_3}} \mathbb{C}^{S_3}_{\text{triv}} \cong \mathbb{C}^{S_3}_{\text{triv}}$$

3. $\mathbb{C}^{S_3}_{\text{triv}}$ has the basis $\left\{ \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right) \right\}$ and is therefore 1-dimensional. $\sigma \in I_3 \setminus S_3$ act as 0, while $\sigma \in S_3$ act as the identity.

$(\mathbb{C}^{S_3}_{\text{sign}})^\sharp$:

1. As above, we have

$$L_{e_3} = \left\{ \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right) \right\}$$

2. Again, we have

$$\operatorname{Ind}_{G_{e_3}}(\mathbb{C}^{S_3}_{\text{sign}}) = \mathbb{C}L_{e_3} \otimes_{\mathbb{C}G_{e_3}} \mathbb{C}^{S_3}_{\text{sign}} \cong \mathbb{C}^{S_3}_{\text{sign}}$$

3. $\mathbb{C}^{S_3}_{\text{sign}}$ has the basis $\left\{ \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right) - \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right) + \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right) \right\}$ and is therefore 1-dimensional. $\sigma \in I_3 \setminus S_3$ act as 0, and $\sigma \in S_3$ act by changing the sign.

$(\mathcal{S}^{(2,1)})^\sharp$:

1. Yet again, we have

$$L_{e_3} = \left\{ \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{smallmatrix}\right) \right\}$$

2. For the same reason as before,

$$\operatorname{Ind}_{G_{e_3}}(\mathcal{S}^{(2,1)}) = \mathbb{C}L_{e_3} \otimes_{\mathbb{C}G_{e_3}} \mathcal{S}^{(2,1)} \cong \mathcal{S}^{(2,1)}$$

3. $\mathcal{S}^{(2,1)}$ has the basis $\left\{ \begin{smallmatrix} \overline{1\ 2} \\ \underline{3} \end{smallmatrix} - \begin{smallmatrix} \overline{3\ 2} \\ \underline{1} \end{smallmatrix}, \begin{smallmatrix} \overline{1\ 3} \\ \underline{2} \end{smallmatrix} - \begin{smallmatrix} \overline{2\ 3} \\ \underline{1} \end{smallmatrix} \right\}$ and is therefore 2-dimensional.
$\sigma \in I_3 \setminus S_3$ act as 0, whilst $\sigma \in S_3$ act as in the Specht module of the partition $(2,1)$ (see the table for explicit actions).

Table 1: I_3 acting on $(\mathbb{C}^{G_{e_0}}_{\text{triv}})^\sharp$

$I_3 \times (\mathbb{C}^{G_{e_0}}_{\text{triv}})^\sharp$	$v = \begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & \emptyset & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & \emptyset & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & \emptyset & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 1 & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 2 & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 3 & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 1 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 2 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 3 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & \emptyset \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & \emptyset & 2 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & \emptyset & 3 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & \emptyset & 1 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & \emptyset & 3 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & \emptyset & 1 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & \emptyset & 2 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 1 & 2 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 1 & 3 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 2 & 1 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 2 & 3 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 3 & 1 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 3 & 2 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	v
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	v

Table 2: I_3 acting on $(\mathbb{C}^{S_1}_{\text{triv}})^\sharp$

$I_3 \times (\mathbb{C}^{S_1}_{\text{triv}})^\sharp$	$v_1 = \begin{pmatrix} 1 & 2 & 3 \\ 1 & \emptyset & \emptyset \end{pmatrix}$	$v_2 = \begin{pmatrix} 1 & 2 & 3 \\ 2 & \emptyset & \emptyset \end{pmatrix}$	$v_3 = \begin{pmatrix} 1 & 2 & 3 \\ 3 & \emptyset & \emptyset \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & \emptyset \end{pmatrix}$	0	0	0
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & \emptyset & \emptyset \end{pmatrix}$	v_1	0	0
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & \emptyset & \emptyset \end{pmatrix}$	v_2	0	0
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & \emptyset & \emptyset \end{pmatrix}$	v_3	0	0
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 1 & \emptyset \end{pmatrix}$	0	v_1	0
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 2 & \emptyset \end{pmatrix}$	0	v_2	0
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 3 & \emptyset \end{pmatrix}$	0	v_3	0
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 1 \end{pmatrix}$	0	0	v_1
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 2 \end{pmatrix}$	0	0	v_2
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & \emptyset & 3 \end{pmatrix}$	0	0	v_3
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & \emptyset \end{pmatrix}$	v_1	v_2	0
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & \emptyset \end{pmatrix}$	v_1	v_3	0
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & \emptyset \end{pmatrix}$	v_2	v_1	0
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & \emptyset \end{pmatrix}$	v_2	v_3	0
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & \emptyset \end{pmatrix}$	v_3	v_1	0
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & \emptyset \end{pmatrix}$	v_3	v_2	0
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & \emptyset & 2 \end{pmatrix}$	v_1	0	v_2
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & \emptyset & 3 \end{pmatrix}$	v_1	0	v_3
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & \emptyset & 1 \end{pmatrix}$	v_2	0	v_1
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & \emptyset & 3 \end{pmatrix}$	v_2	0	v_3
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & \emptyset & 1 \end{pmatrix}$	v_3	0	v_1
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & \emptyset & 2 \end{pmatrix}$	v_3	0	v_2
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 1 & 2 \end{pmatrix}$	0	v_1	v_2
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 1 & 3 \end{pmatrix}$	0	v_1	v_3
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 2 & 1 \end{pmatrix}$	0	v_2	v_1
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 2 & 3 \end{pmatrix}$	0	v_2	v_3
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 3 & 1 \end{pmatrix}$	0	v_3	v_1
$\begin{pmatrix} 1 & 2 & 3 \\ \emptyset & 3 & 2 \end{pmatrix}$	0	v_3	v_2
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$	v_1	v_2	v_3
$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$	v_1	v_3	v_2
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$	v_2	v_1	v_3
$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$	v_2	v_3	v_1
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$	v_3	v_2	v_1
$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$	v_3	v_1	v_2

Table 3: I_3 acting on $(\mathbb{C}^{S_2}_{\text{triv}})^\sharp$

$I_3 \times (\mathbb{C}^{S_2}_{\text{triv}})^\sharp$	$v_1 = \binom{1\,2\,3}{1\,2\,\emptyset} + \binom{1\,2\,3}{2\,1\,\emptyset}$	$v_2 = \binom{1\,2\,3}{2\,3\,\emptyset} + \binom{1\,2\,3}{3\,2\,\emptyset}$	$v_3 = \binom{1\,2\,3}{1\,3\,\emptyset} + \binom{1\,2\,3}{3\,1\,\emptyset}$
$\binom{1\,2\,3}{\emptyset\,\emptyset\,\emptyset}$	0	0	0
$\binom{1\,2\,3}{1\,\emptyset\,\emptyset}$	0	0	0
$\binom{1\,2\,3}{2\,\emptyset\,\emptyset}$	0	0	0
$\binom{1\,2\,3}{3\,\emptyset\,\emptyset}$	0	0	0
$\binom{1\,2\,3}{\emptyset\,1\,\emptyset}$	0	0	0
$\binom{1\,2\,3}{\emptyset\,2\,\emptyset}$	0	0	0
$\binom{1\,2\,3}{\emptyset\,3\,\emptyset}$	0	0	0
$\binom{1\,2\,3}{\emptyset\,\emptyset\,1}$	0	0	0
$\binom{1\,2\,3}{\emptyset\,\emptyset\,2}$	0	0	0
$\binom{1\,2\,3}{\emptyset\,\emptyset\,3}$	0	0	0
$\binom{1\,2\,3}{1\,2\,\emptyset}$	v_1	0	0
$\binom{1\,2\,3}{1\,3\,\emptyset}$	v_3	0	0
$\binom{1\,2\,3}{2\,1\,\emptyset}$	v_1	0	0
$\binom{1\,2\,3}{2\,3\,\emptyset}$	v_2	0	0
$\binom{1\,2\,3}{3\,1\,\emptyset}$	v_3	0	0
$\binom{1\,2\,3}{3\,2\,\emptyset}$	v_2	0	0
$\binom{1\,2\,3}{1\,\emptyset\,2}$	0	0	v_1
$\binom{1\,2\,3}{1\,\emptyset\,3}$	0	0	v_3
$\binom{1\,2\,3}{2\,\emptyset\,1}$	0	0	v_1
$\binom{1\,2\,3}{2\,\emptyset\,3}$	0	0	v_2
$\binom{1\,2\,3}{3\,\emptyset\,1}$	0	0	v_3
$\binom{1\,2\,3}{3\,\emptyset\,2}$	0	0	v_2
$\binom{1\,2\,3}{\emptyset\,1\,2}$	0	v_1	0
$\binom{1\,2\,3}{\emptyset\,1\,3}$	0	v_3	0
$\binom{1\,2\,3}{\emptyset\,2\,1}$	0	v_1	0
$\binom{1\,2\,3}{\emptyset\,2\,3}$	0	v_2	0
$\binom{1\,2\,3}{\emptyset\,3\,1}$	0	v_3	0
$\binom{1\,2\,3}{\emptyset\,3\,2}$	0	v_2	0
$\binom{1\,2\,3}{1\,2\,3}$	v_1	v_2	v_3
$\binom{1\,2\,3}{1\,3\,2}$	v_3	v_2	v_1
$\binom{1\,2\,3}{2\,1\,3}$	v_1	v_3	v_2
$\binom{1\,2\,3}{2\,3\,1}$	v_2	v_3	v_1
$\binom{1\,2\,3}{3\,2\,1}$	v_2	v_1	v_3
$\binom{1\,2\,3}{3\,1\,2}$	v_3	v_1	v_2

Table 4: I_3 acting on $(\mathbb{C}^{S_2}_{\text{sign}})^\sharp$

$I_3 \times (\mathbb{C}^{S_2}_{\text{sign}})^\sharp$	$v_1 = \left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right) - \left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right)$	$v_2 = \left(\begin{smallmatrix}1&2&3\\2&3&\emptyset\end{smallmatrix}\right) - \left(\begin{smallmatrix}1&2&3\\3&2&\emptyset\end{smallmatrix}\right)$	$v_3 = \left(\begin{smallmatrix}1&2&3\\1&3&\emptyset\end{smallmatrix}\right) - \left(\begin{smallmatrix}1&2&3\\3&1&\emptyset\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&\emptyset\end{smallmatrix}\right)$	0	0	0
$\left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right)$	0	0	0
$\left(\begin{smallmatrix}1&2&3\\2&\emptyset&\emptyset\end{smallmatrix}\right)$	0	0	0
$\left(\begin{smallmatrix}1&2&3\\3&\emptyset&\emptyset\end{smallmatrix}\right)$	0	0	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&1&\emptyset\end{smallmatrix}\right)$	0	0	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&2&\emptyset\end{smallmatrix}\right)$	0	0	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&3&\emptyset\end{smallmatrix}\right)$	0	0	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&1\end{smallmatrix}\right)$	0	0	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&2\end{smallmatrix}\right)$	0	0	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&3\end{smallmatrix}\right)$	0	0	0
$\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right)$	v_1	0	0
$\left(\begin{smallmatrix}1&2&3\\1&3&\emptyset\end{smallmatrix}\right)$	v_3	0	0
$\left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right)$	$-v_1$	0	0
$\left(\begin{smallmatrix}1&2&3\\2&3&\emptyset\end{smallmatrix}\right)$	v_2	0	0
$\left(\begin{smallmatrix}1&2&3\\3&1&\emptyset\end{smallmatrix}\right)$	$-v_3$	0	0
$\left(\begin{smallmatrix}1&2&3\\3&2&\emptyset\end{smallmatrix}\right)$	$-v_2$	0	0
$\left(\begin{smallmatrix}1&2&3\\1&\emptyset&2\end{smallmatrix}\right)$	0	0	v_1
$\left(\begin{smallmatrix}1&2&3\\1&\emptyset&3\end{smallmatrix}\right)$	0	0	v_3
$\left(\begin{smallmatrix}1&2&3\\2&\emptyset&1\end{smallmatrix}\right)$	0	0	$-v_1$
$\left(\begin{smallmatrix}1&2&3\\2&\emptyset&3\end{smallmatrix}\right)$	0	0	v_2
$\left(\begin{smallmatrix}1&2&3\\3&\emptyset&1\end{smallmatrix}\right)$	0	0	$-v_3$
$\left(\begin{smallmatrix}1&2&3\\3&\emptyset&2\end{smallmatrix}\right)$	0	0	$-v_2$
$\left(\begin{smallmatrix}1&2&3\\\emptyset&1&2\end{smallmatrix}\right)$	0	v_1	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&1&3\end{smallmatrix}\right)$	0	v_3	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&2&1\end{smallmatrix}\right)$	0	$-v_1$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&2&3\end{smallmatrix}\right)$	0	v_2	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&3&1\end{smallmatrix}\right)$	0	$-v_3$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&3&2\end{smallmatrix}\right)$	0	$-v_2$	0
$\left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right)$	v_1	v_2	v_3
$\left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right)$	v_3	$-v_2$	v_1
$\left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right)$	$-v_1$	v_3	v_2
$\left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)$	v_2	$-v_3$	$-v_1$
$\left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right)$	$-v_2$	$-v_1$	$-v_3$
$\left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)$	$-v_3$	v_1	$-v_2$

Table 5: I_3 acting on $(\mathbb{C}^{S_3}_{\text{triv}})^\sharp$

$I_3 \times (\mathbb{C}^{S_3}_{\text{triv}})^\sharp$	$v = \left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right) + \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right) + \left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right) + \left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right) + \left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right) + \left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\2&\emptyset&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\3&\emptyset&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&1&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&2&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&3&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&1\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&2\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&3\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&3&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\2&3&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\3&1&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\3&2&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&\emptyset&2\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&\emptyset&3\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\2&\emptyset&1\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\2&\emptyset&3\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\3&\emptyset&1\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\3&\emptyset&2\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&1&2\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&1&3\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&2&1\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&2&3\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&3&1\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&3&2\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right)$	v
$\left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right)$	v
$\left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right)$	v
$\left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)$	v
$\left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right)$	v
$\left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)$	v

Table 6: I_3 acting on $(\mathbb{C}^{S_3}_{\text{sign}})^\sharp$

$I_3 \times (\mathbb{C}^{S_3}_{\text{sign}})^\sharp$	$v = \left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right) - \left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right) - \left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right) + \left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right) - \left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right) + \left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)$
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&\emptyset&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\2&\emptyset&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\3&\emptyset&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&1&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&2&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&3&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&1\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&2\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&\emptyset&3\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&2&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&3&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\2&1&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\2&3&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\3&1&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\3&2&\emptyset\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&\emptyset&2\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&\emptyset&3\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\2&\emptyset&1\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\2&\emptyset&3\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\3&\emptyset&1\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\3&\emptyset&2\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&1&2\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&1&3\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&2&1\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&2&3\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&3&1\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\\emptyset&3&2\end{smallmatrix}\right)$	0
$\left(\begin{smallmatrix}1&2&3\\1&2&3\end{smallmatrix}\right)$	v
$\left(\begin{smallmatrix}1&2&3\\1&3&2\end{smallmatrix}\right)$	$-v$
$\left(\begin{smallmatrix}1&2&3\\2&1&3\end{smallmatrix}\right)$	$-v$
$\left(\begin{smallmatrix}1&2&3\\2&3&1\end{smallmatrix}\right)$	v
$\left(\begin{smallmatrix}1&2&3\\3&2&1\end{smallmatrix}\right)$	$-v$
$\left(\begin{smallmatrix}1&2&3\\3&1&2\end{smallmatrix}\right)$	v

Table 7: I_3 acting on $(\mathcal{S}^{(2,1)})^\sharp$

$I_3 \times (\mathcal{S}^{(2,1)})^\sharp$	$v_1 = \overline{\underline{\begin{smallmatrix}1 & 2\\ 3\end{smallmatrix}}} - \overline{\underline{\begin{smallmatrix}3 & 2\\ 1\end{smallmatrix}}}$	$v_2 = \overline{\underline{\begin{smallmatrix}1 & 3\\ 2\end{smallmatrix}}} - \overline{\underline{\begin{smallmatrix}2 & 3\\ 1\end{smallmatrix}}}$
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 1&\emptyset&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 2&\emptyset&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 3&\emptyset&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&1&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&2&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&3&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&1\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&2\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&\emptyset&3\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 1&2&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 1&3&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 2&1&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 2&3&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 3&1&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 3&2&\emptyset\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 1&\emptyset&2\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 1&\emptyset&3\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 2&\emptyset&1\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 2&\emptyset&3\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 3&\emptyset&1\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 3&\emptyset&2\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&1&2\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&1&3\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&2&1\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&2&3\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&3&1\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ \emptyset&3&2\end{smallmatrix}\right)$	0	0
$\left(\begin{smallmatrix}1&2&3\\ 1&2&3\end{smallmatrix}\right)$	v_1	v_2
$\left(\begin{smallmatrix}1&2&3\\ 1&3&2\end{smallmatrix}\right)$	v_2	v_1
$\left(\begin{smallmatrix}1&2&3\\ 2&1&3\end{smallmatrix}\right)$	$v_1 - v_2$	$-v_2$
$\left(\begin{smallmatrix}1&2&3\\ 2&3&1\end{smallmatrix}\right)$	$-v_2$	$v_1 - v_2$
$\left(\begin{smallmatrix}1&2&3\\ 3&2&1\end{smallmatrix}\right)$	$-v_1$	$v_2 - v_1$
$\left(\begin{smallmatrix}1&2&3\\ 3&1&2\end{smallmatrix}\right)$	$v_2 - v_1$	$-v_1$

Appendix: Prerequisite theory

Basic definitions

The following is a collection of basic definitions, no doubt familiar to any reader who has studied algebra.

Definition 40. A group is an ordered pair of a set G and one binary operation on the set G such that

(i) The operation is associative

(ii) There is an identity element

(iii) Every element x of G has an inverse, i.e. $\forall x \in G\ \exists y \in G$: $xy = yx = 1$

Definition 41. A subgroup of a group G is a subset H of G such that

(i) $1 \in H$

(ii) $x \in H \Rightarrow x^{-1} \in H$

(iii) $x, y \in H \Rightarrow xy \in H$

Definition 42. A ring is an ordered triple $(R, +, \cdot)$ of a set R and two binary operations on R (addition and multiplication) such that

(i) $(R, +)$ is an abelian group

(ii) $(R, \cdot)$ is a semigroup

(iii) The multiplication is distributive

If in addition the multiplicative semigroup $(R, \cdot)$ has an identity element, we say that $(R, +\cdot)$ is a ring with identity or a unital ring.

Definition 43. A field is a commutative ring $F \neq 0$ such that $F \setminus \{0\}$ is a ring under multiplication.

Definition 44. Let $\Bbbk$ be a field. The set V equipped with addition $+ : V \times V \to V$ and scalar multiplication $\cdot : \Bbbk \times V \to V$ is called a vector space if, $\forall u, v, w \in V$, $\forall a, b \in \Bbbk$,

(i) $u + (v + w) = (u + v) + w$

(ii) $u + v = v + u$

(iii) $\exists 0 \in V : v + 0 = v$

(iv) $\exists -v \in V : v + (-v) = 0$

(v) $a(bv) = (ab)v$

(vi) $1v = v$, where 1 is the multiplicative identity in $\Bbbk$

(vii) $a(u + v) = au + av$

(viii) $(a + b)v = av + bv$

Definition 45. Let $\Bbbk$ be a field, and $(A, \cdot)$ a vector space over $\Bbbk$ with an additional binary multiplication over A. Then A is an algebra over $\Bbbk$ if, $\forall x, y, z \in A$, $\forall a, b \in \Bbbk$:

(i) $(x + y) \cdot z = x \cdot z + y \cdot z$

(ii) $x \cdot (y + z) = x \cdot y + x \cdot z$

(iii) $(ax) \cdot (by) = (ab)(x \cdot y)$

Definition 46. Let A be an algebra over the field $\Bbbk$, where $A \neq 0$. A is a division algebra if, for any $a, b \in A$ where $b \neq 0$, $\exists! x, y \in A : a = bx, a = yb$.

Definition 47. Let R be a unital ring with multiplicative identity 1. A left R-module M consists of an abelian (commutative) group $(M, +)$ and an operation $\cdot : R \times M \to M$ such that, $\forall r, s \in R$, $\forall x, y \in M$,

(i) $r \cdot (x + y) = r \cdot x + r \cdot y$

(ii) $(r + s) \cdot x = r \cdot x + s \cdot x$

(iii) $(rs) \cdot x = r \cdot (s \cdot x)$

(iv) $1 \cdot x = x$

where the ring multiplication is denoted by juxtaposition. A right R-module is defined similarly, with scalars from R acting on M from the right, and the above requirements rewritten in the same way. Note that if R is a commutative unital ring, left R-modules are the same as right R-modules (in which case we simply call them R-modules).

Definition 48. Let M be an R-module. $N \subseteq M$ is a submodule of M if N is closed under the module operation of M, and we denote this by $N \leq M$. N is a proper submodule if $N \neq M, N \neq 0$.

Definition 49. Let R and S be unital rings. An abelian group M is an R-S-bimodule if

(i) M is a left R-module and a right S-module

(ii) $\forall r \in R, s \in S, m \in M : (rm)s = r(ms)$

An R-R-bimodule is called an R-bimodule.

Definition 50. Relative to a subgroup H of a group G, the left coset of an element x of G is the subset xH of G; the right coset of an element x of G is the subset Hx of G. These sets are also called left and right cosets of H.

Definition 51. Let V, W be vector spaces over a field $\Bbbk$. The tensor product $V \otimes_{\Bbbk} W$ is defined as the quotient of the vector space with basis $\{v \otimes w : v \in V, w \in W\}$ modulo the subspace spanned by all

$$(v_1 + v_2) \otimes w - v_1 \otimes w - v_2 \otimes w$$
$$v \otimes (w_1 + w_2) - v \otimes w_1 - v \otimes w_2$$
$$(\lambda v) \otimes w - v \otimes (\lambda w)$$

where $v, v_1, v_2 \in V,\ w, w_1, w_2 \in W,\ \lambda \in \Bbbk$.

Definition 52. Let M be an R-module, where R is a commutative unital ring. The nth tensor power $T^n(M)$ (often denoted $M^{\otimes n}$) is defined as follows:

$$M^{\otimes 0} = R$$
$$M^{\otimes 1} = M$$
$$M^{\otimes n} = \underbrace{M \otimes \cdots \otimes M}_{n \text{ times}} \text{ for } n \geq 2$$

Definition 53. A category $\mathcal{C}$ has

(i) A class whose elements are the objects of $\mathcal{C}$

(ii) A class whose elements are the morphisms (arrows) of $\mathcal{C}$

(iii) Two class functions that assign to every morphism of $\mathcal{C}$ a domain and a codomain, which are objects of $\mathcal{C}$

(iv) A class function that assigns to certain pairs (α, β) of morphisms of $\mathcal{C}$ their composition or product $\alpha\beta$. This product is a morphism of $\mathcal{C}$ and has the following properties:

  (a) $\alpha\beta$ is defined if and only if the domain of α is the codomain of β, and then the domain of $\alpha\beta$ is the domain of β and the codomain of $\alpha\beta$ is the codomain of α

  (b) For every object A of $\mathcal{C}$ there exists an identity morphism 1_A whose domain and codomain are A, such that $\alpha 1_A = \alpha$ whenever A is the domain of α and $1_A\beta = \beta$ whenever A is the codomain of α

  (c) If $\alpha\beta$ and $\beta\gamma$ are defined, then $\alpha(\beta\gamma) = (\alpha\beta)\gamma$

Definition 54. A covariant functor F from a category $\mathcal{A}$ to a category $\mathcal{C}$ assigns to each object A of $\mathcal{A}$ an object $F(A)$ of $\mathcal{C}$, and assigns to each morphism α of $\mathcal{A}$ a morphism $F(\alpha)$ of $\mathcal{C}$ such that

(i) If $\alpha : A \to B$, then $F(\alpha) : F(A) \to F(B)$

(ii) $F(1_A) = 1_{F(A)}$ for every object A of $\mathcal{A}$

(iii) $F(\alpha\beta) = F(\alpha)F(\beta)$ whenever $\alpha\beta$ is defined

If nothing else is specified, assume any functors are covariant.

Definition 55. A contravariant functor F from a category $\mathcal{A}$ to a category $\mathcal{C}$ assigns to each object A of $\mathcal{A}$ an object $F(A)$ of $\mathcal{C}$, and assigns to each morphism α of $\mathcal{A}$ a morphism $F(\alpha)$ of $\mathcal{C}$ such that

(i) If $\alpha : A \to B$, then $F(\alpha) : F(B) \to F(A)$

(ii) $F(1_A) = 1_{F(A)}$ for every object A of $\mathcal{A}$

(iii) $F(\alpha\beta) = F(\beta)F(\alpha)$ whenever $\alpha\beta$ is defined

Definition 56. A finite or infinite sequence $\cdots M_i \xrightarrow{\varphi_i} M_{i+1} \xrightarrow{\varphi_{i+1}} M_{i+2} \cdots$ of module homomorphisms is exact when $\text{Im } \varphi_i = \text{Ker } \varphi_{i+1}$ for all i. A short exact sequence is an exact sequence $0 \to A \to B \to C \to 0$.

Definition 57. A covariant functor is exact when it transforms short exact sequences into short exact sequences.

Definition 58. Let F and G be functors from $\mathcal{A}$ to $\mathcal{B}$. A natural transformation $\tau : F \to G$ assigns to each object A of $\mathcal{A}$ a morphism $\tau_A : F(A) \to G(A)$, so that $G(\alpha)\tau_A = \tau_B F(\alpha)$ for every morphism $\alpha : A \to B$ of $\mathcal{A}$. If every τ_A is an isomorphism, then τ is a natural isomorphism.

Definition 59. We denote by A-mod the category of finite dimensional left A-modules, and A-Mod denotes the category of all left A-modules.

Definition 60. A module P is projective if and only if, for every surjective module homomorphism $f : N \to M$ and every module homomorphism $g : P \to M$, there exists a homomorphism $h : P \to N$ such that $f \circ h = g$.

Definition 61. A left R-module M is flat when the functor $- \otimes_R M$ is exact.

Finite dimensional algebras

Below can be found a collection of definitions, as well as a few results, necessary for the main content of this paper. As with the basic definitions stated previously, these should all be familiar. For this section, let $\Bbbk$ be a field and A a finite dimensional $\Bbbk$-algebra.

Semisimple modules and algebras

Definition 62. An A-module S is simple if $S \neq 0$ and the only submodules of S are 0 and S. Equivalently, a nonzero module S is simple if $Av = S$ for all nonzero vectors $v \in S$. Note that $S = Av$, for $v \neq 0$, implies that S is a quotient of A and so every simple A-module is finite dimensional.

Lemma 4 (Schur's Lemma)**.** If S, S' are simple A-modules, then every nonzero homomorphism $\varphi : S \to S'$ is an isomorphism. In particular, $\mathrm{End}_A(S)$ is a finite dimensional division algebra over $\Bbbk$. If $\Bbbk$ is algebraically closed, then $\mathrm{End}_A(S) = \Bbbk \cdot 1_s \cong \Bbbk$.

Definition 63. An A-module M is semisimple if $M = \bigoplus_{\alpha \in F} S_\alpha$ for some family of simple submodules $\{S_\alpha : \alpha \in F\}$.

Proposition 16. Let M be an A-module. Then TFAE:

(i) M is semisimple.

(ii) $M = \sum_{\alpha \in F} S_\alpha$ with $S\alpha \leq M$ simple for all $\alpha \in F$.

(iii) For each submodule $N \leq M$, there is a submodule $N' \leq M$ such that $M = N \oplus N'$.

Definition 64. It follows that every A-module V has a unique maximal semisimple submodule $\mathrm{soc}(V)$, called the socle of V. Note also that $\mathrm{soc}(V)$ is the sum of all the simple submodules of V, and is semisimple by Proposition 16.

Definition 65. A maximal submodule of a module V is a proper submodule which is maximal with respect to the inclusion ordering on the set of proper submodules of V. If V is an A-module, we define $\mathrm{rad}(V)$ to be the intersection of all maximal submodules of V.

Definition 66. We can view A as a finite dimensional A-module called the regular module. Maximal submodules of A are just maximal left ideals and so $\mathrm{rad}(A)$ is the intersection of all maximal left ideals of A.

Definition 67. An ideal I of an algebra A is nilpotent if $I^n = 0$ for some $n \geq 1$.

Theorem 4. Let A be a finite dimensional $\Bbbk$-algebra.

(i) $\mathrm{rad}(A)$ is the intersection of all maximal right ideals of A.

(ii) $\mathrm{rad}(A)$ is a two-sided ideal.

(iii) $\mathrm{rad}(A)$ is the intersection of the annihilators of the simple A-modules.

(iv) If V is a finite dimensional A-module, then $\mathrm{rad}(V) = \mathrm{rad}(A) \cdot V$.

(v) An A-module M is semisimple if and only if $\mathrm{rad}(A) \cdot M = 0$.

(vi) $\mathrm{rad}(A)$ is nilpotent.

(vii) $\mathrm{rad}(A)$ is the largest nilpotent ideal of A.

Definition 68. A finite dimensional $\mathbb{k}$-algebra A is semisimple if the regular module A is a semisimple module.

Definition 69. An element $a \in A$ is nilpotent if $a^n = 0$ for some $n \geq 1$.

Definition 70. An A-module V is faithful if its annihilator is 0, i.e. $aV = 0$ implies $a = 0$.

Definition 71. Let V be a finite dimensional A-module. A composition series for V is an unrefinable chain of submodules

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_n = V$$

We call n the length of the composition series and the simple modules V_i/V_{i-1} the composition factors.

Definition 72. Let V be a finite dimensional A-module. We define the length of V to be the length of a composition series for V. For S a simple A-module, we define its multiplicity as a composition factor in V as the number $[V : S]$ of composition factors in some composition series for V that are isomorphic to S.

Indecomposable modules

Definition 73. A nonzero A-module M is indecomposable if $M = M' \oplus M''$ implies $M' = 0$ or $M'' = 0$. Note that every simple module is indecomposable, but the converse only holds when A is semisimple.

Definition 74. A finite dimensional algebra is said to be of finite representation type if it has only finitely many isomorphism classes of finite dimensional indecomposable modules.

Definition 75. Let V be a finite dimensional A-module and P a finite dimensional projective A-module. An epimorphism $\varphi : P \to V$ is called a projective cover if $W + \ker\,\varphi = P$ implies $W = P$ for W a submodule of P. Note that this is equivalent to $\ker\,\varphi \subseteq \mathrm{rad}(P)$.

An introduction to idempotents

Definition 76. Let $E(A)$ be the set of idempotents of the finite dimensional algebra A. Two idempotents $e, f \in E(A)$ are orthogonal if $ef = 0 = fe$.

Definition 77. A collection $\{e_1, \ldots, e_n\}$ of pairwise orthogonal idempotents of A is called a complete set of orthogonal idempotents if $1 = e_1 + \cdots + e_n$.

Proposition 17. Let $e \in E(A)$ and let M be an A-module.

(i) $\mathrm{Hom}_A(Ae, M) \cong eM$ via $\varphi \to \varphi(e)$

(ii) $\mathrm{End}_A(Ae) \cong (eAe)^{op}$

In particular, $\mathrm{End}_A(A) \cong A^{op}$.

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