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Topics in perturbation theory

From IBP identities to integrands

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Abstract
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In this thesis we present different topics in perturbation theory. We start by introducing the method of integration by parts identities, which reduces a generic Feynman integral to a linear combination of a finite basis of master integrals. In our analysis we make use of the Baikov representation as this form gives a nice framework for generating efficiently the identities needed to reduce integrals. In the second part of the thesis we briefly explain recent developments in the integration of Feynman integrals and present a method to bootstrap the value of p-integrals using constraints from certain limits of conformal integrals. We introduce also another method to obtain p-integrals at l-loops by cutting vacuum diagrams at l+1-loops. In the last part of the thesis we present recent developments in N=4 SYM to compute structure constants. We use perturbation theory to obtain new results that can be tested against this new conjecture. Moreover we use integrability based methods to constrain correlation function of protected operators.

Keywords: Perturbation theory, Feynman Integrals, Integrable field theories, Correlation functions.

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’ ὥστε [ἡ ἀπειρία] δυνάμει μὲν ἔστιν, ἐνεργείᾳ δ΄ οὔ ’
Physics, Aristotle

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I A. Georgoudis, K. J. Larsen and Y. Zhang, *Azurite: An algebraic geometry based package for finding bases of loop integrals*, Comput. Phys. Commun. **221** (2017) 203

II A. Georgoudis, V. Goncalves and R. Pereira, *Konishi OPE coefficient at the five loop order*, JHEP **1811** (2018) 184

III J. Böhm, A. Georgoudis, K. J. Larsen, M. Schulze and Y. Zhang, *Complete sets of logarithmic vector fields for integration-by-parts identities of Feynman integrals*, Phys. Rev. D **98** (2018) no.2, 025023

IV A. Georgoudis, V. Goncalves, E. Panzer and R. Pereira,*Five-loop massless propagator integrals*, arXiv:1802.00803 [hep-th].

V J. Böhm, A. Georgoudis, K. J. Larsen, H. Schönemann and Y. Zhang, *Complete integration-by-parts reductions of the non-planar hexagon-box via module intersections*, JHEP **1809** (2018) 024

VI D. Chicherin, A. Georgoudis, V. Gonçalves and R. Pereira, *All five-loop planar four-point functions of half-BPS operators in* $\mathcal{N} = 4$ *SYM*, JHEP **1811** (2018) 069

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Contents

1. Introduction

In order to describe nature we have to make predictions that are consistent with what is seen in experiments. In the theory of quantum interactions our best current approximation of the real world is the standard model of particle physics which describes how fundamental particles behave at energy scale below 1-10 TeV. The standard model has proven to possess an incredibly predictive power by surviving many high precision tests run at the LHC over the last decade, including the discovery of Higgs boson [1]. Beside these many excellent results a lot of open question and problems remain unanswered, most notably the quest for a dark matter candidate, unification of the fundamental forces and fine tuning problems related to the Higgs mass. A possible solutions is given by supersymmetry (SUSY), which is an extension to the standard space time symmetry which relates fermionic and bosonic particles.

A first check to the validity of extensions of the standard model would then be the discovery of supersymmetric particles. High precision calculations in perturbative quantum field theory (QFT), which are currently the next-to-next-to-leading-order (NNLO) quantum chromodynamics computation with several scales [2], are needed in order to possibly check these predictions. Independently of this phenomenological applications supersymmetric theories are interesting objects by themselves as the highly non trivial constraints coming from supersymmetry and the holographic correspondence [3, 4] pave a way to study QFTs in the strongly coupled, non-perturbative, regime. This could help us understand the dynamics of strongly correlated systems which can be usually only studied through numerical simulations. Especially in the recent years there has been a rich and intense development in both perturbative and non-perturbative results for QFTs. These developments have come from a better understanding of the mathematical structure of observables and by a clever use of symmetries and constraints.

In the case of perturbative computations the common method to keep track of all the contributions to a perturbative scattering relays on Feynman graphs and integrals, see for example tab. 1.1. The number of terms appearing in this perturbative calculation becomes very large very quickly, moreover this approach seems to break some symmetries of the theory which must re-emerge when extracting physical observables.

Starting from the work of Bern, Dixon, Dunbar and Kossower [5, 6], where unitarity based methods have been developed to efficiently construct integrands, in the last years a lot of new ideas have been developed

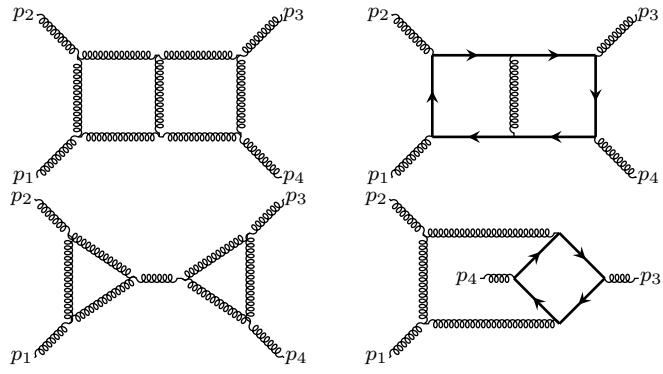


Table 1.1. *Some contributions to the $gg \to gg$ scattering at 2 loop.*

to simplify the standard approach to perturbative calculations[1]. For example using generalized unitarity [9] we are able to directly construct an ansatz for the amplitude and fixing it by solving the so called "cut equations". This avoids the problem of summing the individual contributions constructed from Feynman rules.

A lot of these developments have come from SUSY theories, more notably $\mathcal{N} = 4$ super Yang-Mills theory, which is a gauge theory with the maximal amount of supersymmetry, in the planar limit. In this limit we are sending the gauge group rank to infinity, suppressing the non planar contributions to the perturbative processes as they are subleading in rank. One example of such a contribution is the second element of the second column of tab. 1.1.

The rich structure of $\mathcal{N} = 4$ in the planar limit, which is characterized by its dual super conformal symmetry, allows one to obtain sharp constraints on the form of amplitudes [10]. It is then possible, for example, to construct an all loop integrand for any number of particles [11] and it has also been shown that a more compact local integrand form exists [12]. Moreover it is possible to represent amplitudes using more geometrical objects such as the Grassmanian [13] and the Amplituhedron[2] [15], which avoid the requirement of locality or unitary and exploit the full symmetries of the theory. Still in $\mathcal{N} = 4$ it was noted in [16] that Integrals appearing in amplitudes computations have an iterated structure. The symbol map, introduced in the same work, is a compact representation of such iterative structure and with physical constraints coming from special limits of scattering processes it can be used to bootstrap the amplitude directly [17]. Results at higher loops and multiplicity were obtained by imposing Steinmann relations [18,19], which are strin-

[1] For a detailed review see [7, 8]

[2] Of which there seems to exist a non planar extension [14].

gent constraints on the type of entries that can appear in the symbol. In the case of non planar contribution less is known as the structure is more involved nontheless some results were obtained [20, 21] for a special class of integrals by requiring cancellation of conformal anomalies. When the symmetries are not enough to fix the amplitude completely one has to directly compute Feynman integrals. An important part of this thesis is based on the computations of integral by parts (IBP) identities. These reduce the number of integrals appearing in an amplitude by relating a starting integral to a linear combination of a finite set of master integrals (MI). There exist algorithmic ways of constructing such identities [22, 23], for some easier examples it is actually possible to have analytic expressions for the coefficients appearing in front of the MI [24, 25]. For higher point and higher loop cases the computer power needed for the Gaussian elimination in the last step of the computation is a major bottleneck. To overcome this limitation one would like to reduce the number of integrals appearing in the intermediate step [26], and so the complexity of the generated system of equations, or to use a partial or complete numeric approach, based on the rational reconstruction method [27].

Lastly the actual computation of Feynman integrals has seen several development in the last years, most notably for integrals that can be expressed as multiple polylogarithms (Mpl) [28]. For this space of functions several methods exist to perform the computation in a somehow algorithmic way [29–31], as for some cases physical intuition and clever parametrization are required. In the case of multiple scales the space of functions is extended to elliptic multiple polylogarithms [32] which are not as well understood and for which methods for direct computation [33, 34] are currently being developed. A simpler class of Feynman integrals, those that evaluate to zeta and multiple zeta values, have shown to posses an interesting algebraic structure [35] closely related to number theory. Lastly it is important to mention that completely numerical implementation of unitarity method have been developed through the years [36] and have obtained several interesting results for NLO computations [37].

For non-perturbative results, in the last years several developments have been obtained for SUSY theories from localization [38, 39] and integrability [40]. Integrability requires that the scattering matrix factorizes in pairwise interactions. It can also be proven that for integrable scattering, in two dimension, there can not be particle production and the outgoing momenta are just a permutation of the incoming. In the planar limit $\mathcal{N} = 4$ is conjectured to be an integrabile theory and most notably this property has been used for obtaining the spectrum of the theory at finite coupling [41, 42]. In the past years a new fundamental object has been introduced, the hexagon form factor [43], which allows to compute

structure constants, in principle, at finite coupling. Structure constant, which are three point-functions coefficients, together with anomalous dimension fully characterize conformal field theories such as $\mathcal{N} = 4$. Form factors are fundamental objects in integrable theories and are defined as matrix element of local operator with the vacuum of the theory and an n-particle state. It has also been noticed that the hexagon form factor arises as a fundamental bulding block for four or higher-point functions [44] and also, surprisingly, in non-planar contributions [45].

The thesis is organized as follows. In chapter 2, which is based on papers I, III and V, we introduce the method of IBP identities. We focus on a particular representation of the Feynman integrals, known as the Baikov representation. In this framework we present the different tools used for obtaining IBP relations and introduce the notion of cuts and syzygy relations. We conclude by introducing the Laplace expansion of a symmetric matrix and a new efficient way for computing IBP identities. In chapter 3, based on the work done in paper IV, we focus our interest in computing Feynman integrals. We briefly introduce the different methods used in recent applications and then we tackle the problem of computing two-point integrals at five loops. We start by introducing a series of constraints that can be derived from IBP identities on the ϵ-expansion of the integrals in $D = 4 - 2\epsilon$ dimensions. We then explain how stronger constraints can be obtained by looking at particular limits of four-point conformal integrals. We conclude by mentioning a method for computing two-point integrals using constraints coming from vacuum diagrams. We conclude in chapter 4 by presenting some recent work, papers II and VI, aimed at obtaining perturbative results in $\mathcal{N} = 4$ SYM theory in $D = 4$ dimensions. We start by briefly explaining some property of this theory and how integrability in the planar limit allows for a bootstrap of different quantities. We then introduce the hexagon proposal for computing three-point functions. In the final part we explain how to compute structure constants from the four-point function of four protected operator. Moreover we explain a new method, using constraints coming form the hexagon proposal for three-point functions, for fixing the integrand of the four-point correlation function of protected operators of different weight, up to five loops.

2. Integration by parts identities

A fundamental tool for the study of amplitudes in perturbation theory are integration by parts (IBP) identities [22, 46]. The idea behind this method is to use algebraic relations, generated from the integration of a total derivative, to reduce integrals[1] appearing in the amplitude to a finite [47] set of master integrals (MI),

$$I = \sum_{i=1}^{N} c_i(\chi_j, D)\ I_i\,, \tag{2.1}$$

where the coefficients c_i depend in general on the external kinematical invariants χ_j, the dimension, D, and the I_i are dimensional regularized Feynman integral with L-loops, n propagators and E independent external momenta. These integrals have m independent scalar products generated by the combination of the loop and external momenta. As elements that appear both in the numerator and denominator simplify, we can define, for each set of n propagators, the $m-n$ remaining combinations as irreducible scalar products (ISP).

A generic integral takes the form

$$I(\nu_1,\ldots,\nu_m;D) \equiv \int \prod_{j=1}^{L} \frac{\mathrm{d}^D \ell_j}{\mathrm{i}\pi^{D/2}} \frac{N_{n,m}}{D_1^{\nu_1}\cdots D_k^{\nu_n}}\,, \quad \nu_i \geq 0\,, \tag{2.2}$$

where the numerator is defined as $N_{k,m} = D_{n+1}^{\nu_{n+1}}\cdots D_m^{\nu_m}$ and each propagator D_j is a quadratic combination of external and loop momenta. A total derivative can then be written explicitly as

$$0 = \int \frac{\mathrm{d}^D \ell_1}{\mathrm{i}\pi^{D/2}} \cdots \frac{\mathrm{d}^D \ell_L}{\mathrm{i}\pi^{D/2}} \sum_{j=1}^{L} \frac{\partial}{\partial \ell_j^\mu} \frac{v_j^\mu}{D_1^{\nu_1}\cdots D_m^{\nu_m}}\,, \tag{2.3}$$

where the vector v_j^μ is a polynomial in the scalar products of external and loop momenta.

IBP relations generated from (2.3) can be schematically represented as

$$\sum_k c_k I_k = 0\,, \tag{2.4}$$

[1] We assume that all our integrals are scalars, in the sense that they do not have any Lorentz index appearing in the numerator.

where the action of the total derivative on the starting integral, beside making explicit dependence on the χ_i and D appear, can lower or raise the indices ν_i of (2.2). To understand this better let us take as an example the 1-loop massive tadpole:

$$I(\nu_1) = \int \mathrm{d}^D l \frac{1}{(l^2 - m^2)^{\nu_1}}\,, \tag{2.5}$$

and consider the total derivative with $v_j^\mu = l^\mu$ and $\nu_1 = 1$,

$$\int \mathrm{d}^D l \frac{\partial}{\partial l_\mu}\left(\frac{l^\mu}{l^2 - m^2}\right) = \int \mathrm{d}^D l \left(\frac{D}{l^2 - m^2} - \frac{2l^2}{(l^2 - m^2)^2}\right) = \int \mathrm{d}^D l \left(\frac{D-2}{l^2 - m^2} + \frac{m^2}{(l^2 - m^2)^2}\right) = 0\,, \tag{2.6}$$

which can be compactly rewritten as

$$m^2 I(2) + (D-2) I(1) = 0\,. \tag{2.7}$$

In (2.7) we have integrals with higher propagators powers appearing and the coefficients of the reductions are dependent on the invariant m^2 and the dimension D. Such higher power propagator integrals are a spurious effect of IBP identities.

For more complicated cases the number of Integrals appearing and the form of the coefficients will be more involved. In general to generate reduction of the form (2.1) we need to first use (2.3) to generate IBP relations (2.4) and then row reduce the coefficient matrix extracted from the IBP relations.

	I_1	I_2	$\dots$	I_m
IBP_1	$c_{1,1}$	$c_{2,1}$	$\dots$	$c_{m,1}$
IBP_2	$c_{1,1}$	$c_{2,1}$	$\dots$	$c_{m,2}$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
IBP_N	$c_{1,N}$	$c_{2,N}$	$\dots$	$c_{m,N}$

Table 2.1. *General form of the coefficients matrix extracted from IBP identities. In general the matrix will be sparse.*

The solution of this problem in general is ill posed. To obtain reductions in a systematic way we have to introduce an ordering between different integrals[2] $I_1(a_1, \dots, a_m) \prec I_2(b_1, \dots, b_m)$. In general one wants to reduce integral to "simpler" ones[3], e.g. with less degree of propagators. An example of such ordering can be,using the definitions (2.5),

$$I(\nu_1) \prec I(\nu_2) \qquad \text{If} \quad \nu_1 < \nu_2 \quad \nu_1, \nu_2 > 0\,. \tag{2.8}$$

[2]This idea was first introduced in [23, 48]

[3]In Paper V was actually noted that a partial column swapping generated easier intermediate coefficients.

With an ordering it is possible to reduce the "most complicated" integral first and express the final relations in term of the "simplest" MIs. By applying this method recursively it is possible to construct IBP relations of the form (2.1).

There exists different implementations of this algorithm [49–55], in general the difficult part is to solve the system of equations as for higher loops and number of external momenta the number of unknown grows.In recent years the main focus has been into using rational coefficients over some finite field $\mathbb{Z}_p$[4], with p a prime number, to evaluate the IBPs reductions numerically and then reconstruct them.

2.1 Baikov representation

In this section we want to present another representation for Feynman integrals, the Baikov representation [57]. As it will be clear this choice is made for the purpose of studying IBP relations[5] and integrals on specific cuts[6] [59, 60]. Our starting point will be (2.2), we also have to remember that at each integral we can associate a corresponding graph, constructed following the Feynman rules. The propagators can be constructed from the graph by requiring momentum conservation at each vertex. This choice is not unique as the measure is invariant under shifts of the loop momenta.

To simplify the notation we introduce the vector:

$$V = (v_1, \dots, v_{E+L}) = (p_1, \dots, p_E, \ell_1, \dots, \ell_L)\,, \tag{2.9}$$

and express every D_i as a combination of scalar products $x_{i,j} = v_i \cdot v_j$. As the matrix V is symmetric we find that the total number of scalar products that can be constructed for L-loops and E external legs is $m = \frac{1}{2}L(2E+L+1)$. In order to change the integration variables to the $x_{i,j}$ we can perform the D dimensional solid-angle integration, as we are interested in the form of the integrals we can collect all of the prefactors[7] in a constant C. By performing the solid angle integration we have reduced the number of integrations from LD to the m independent components of V. In order to obtain the Baikov representation we can now perform a change of variables:

$$z_i = D_i\,, \tag{2.10}$$

[4] See for example [56]

[5] There have been some attempts to use the Baikov representation for calculating the integrals on cuts [58], the complications come from the region of integration which is defined for $F > 0$.

[6] Given by complexifing the loop momenta and evaluating a residue for a set of propagator $D_i = 0$.

[7] The form of this can be found, for example, in Paper III.

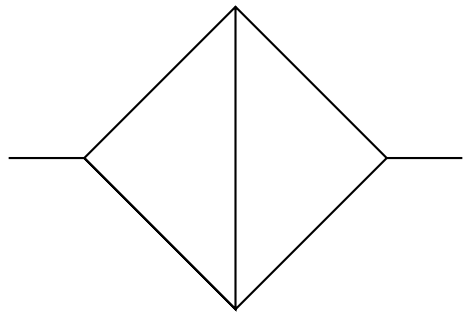


Figure 2.1. Feynman diagram for 2-loop two-point integral.

to obtain

$$I(\nu; D) = C \int \frac{\mathrm{d}z_1 \cdots \mathrm{d}z_m}{z_1^{\nu_1} \cdots z_k^{\nu_n}} F^{\frac{D-L-E-1}{2}} N_{n,m} \,, \tag{2.11}$$

where now $N_{k,m} = z_{n+1}^{\nu_{n+1}} \cdots z_m^{\nu_m}$.

The polynomial F is given as the determinant of the Gram matrix S generated from the vector V after we have imposed the change of variables (2.10):

$$S = \left(\begin{array}{ccc|ccc} x_{1,1} & \cdots & x_{1,E} & x_{1,E+1} & \cdots & x_{1,E+L} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{E,1} & \cdots & x_{E,E} & x_{E,E+1} & \cdots & x_{E,E+L} \\ \hline x_{E+1,1} & \cdots & x_{E+1,E} & x_{E+1,E+1} & \cdots & x_{E+1,E+L} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ x_{E+L,1} & \cdots & x_{E+L,E} & x_{E+L,E+1} & \cdots & x_{E+L,E+L} \end{array} \right) , \tag{2.12}$$

We can, as an example, construct the Baikov representation for a 2-loop two-point integral, represented in fig 2.1. The vector of external momenta and loop momenta is defined as $x = \{p, l_1, l_2\}$ and the Baikov variables are

$$\begin{aligned} z_1 &= l_1^2 = x_{2,2} \quad z_2 = (l_1 - p)^2 = x_{2,2} - 2x_{1,2} + x_{1,1} \\ z_3 &= l_2^2 = x_{3,3} \quad z_4 = (l_2 - p)^2 = x_{3,3} - 2x_{1,3} + x_{1,1} \\ z_5 &= (l_1 - l_2)^2 = x_{2,2} - 2x_{2,3} + x_{3,3} \,. \end{aligned} \tag{2.13}$$

By solving (2.13) we can construct the matrix S,

$$S = \left(\begin{array}{ccc} p^2 & \frac{1}{2}(p^2 + z_1 - z_2) & \frac{1}{2}(p^2 + z_3 - z_4) \\ \frac{1}{2}(p^2 + z_1 - z_2) & z_1 & \frac{1}{2}(z_1 + z_3 - z_5) \\ \frac{1}{2}(p^2 + z_3 - z_4) & \frac{1}{2}(z_1 + z_3 - z_5) & z_2 \end{array} \right) , \tag{2.14}$$

and its determinant

$$\begin{aligned} F =& \frac{1}{4}(-z_1^2 z_4 - z_2(z_2 z_3 + (z_3 - z_4)(z_3 - z_5)) + z_1(z_2(z_3 + z_4) + \\ & (z_3 - z_4)(z_4 - z_5)) - p^4 z_5 + p^2((-z_1 + z_3)(z_2 - z_4) + \\ & (z_1 + z_2 + z_3 + z_4)z_5 - z_5^2)) \,. \end{aligned} \tag{2.15}$$

We now want to clarify why the Baikov representation is optimal for the study of cuts. In this language the variables z_i represent propagators and numerators, differently from the usual representation they are all independent variables. To take a cut we have to set the associated propagator on shell, in the Baikov language this translates into taking the residue of the integral for a circle of radius ϵ around zero:

$$\int \frac{\mathrm{d}z_i}{z_i^{\nu_i}} \quad \xrightarrow{\text{cut}} \quad \oint_{\Gamma_\epsilon(0)} \frac{\mathrm{d}z_i}{z_i^{\nu_i}} \,. \tag{2.16}$$

In general we can compute a k-cut $\mathcal{C} = \{z_{c_1}, \ldots, z_{c_k}\}$ on (2.11) by taking the residues for the variables $\{z_{c_i}\}$ and setting them to zero in the polynomial F.

We can compute cuts also on integrals with $\nu_i > 1$ but they require to solve dimensional shift identities so for our treatment we will focus on $\nu_i = 1$.

2.2 Integration by parts

In order to construct IBP identities using the Baikov representation we will briefly shift to the language of forms. IBPs are constructed from a total derivative, so we have to generate an ansatz [61] for an $m-1$-form which under the action of an external derivative generates an integral of the form of (2.11):

$$\begin{aligned} 0 = \int \mathrm{d}\Big(\sum_{i=1}^{m} & \frac{(-1)^{i+1} a_i(z) F(z)^{\frac{D-L-E-1}{2}}}{z_1 \cdots z_n} \\ & \times \mathrm{d}z_{r_1} \wedge \cdots \wedge \widehat{\mathrm{d}z_{r_i}} \wedge \cdots \wedge \mathrm{d}z_{r_m} \Big) \,, \end{aligned} \tag{2.17}$$

where the a_i are general polynomials in z_i. By expanding the total derivative we obtain:

$$\begin{aligned} 0 = \int \Big[\sum_{i=1}^{m} \Big(& \overbrace{\frac{\partial a_i(z)}{\partial z_i}}^{(A)} + \overbrace{\frac{D-L-E-1}{2F(z)} a_i(z) \frac{\partial F}{\partial z_i}}^{(B)} \Big) - \sum_{i=1}^{n} \overbrace{\frac{a_i(z)}{z_i}}^{(C)} \Big] \\ & \times \frac{F(z)^{\frac{D-L-E-1}{2}}}{z_1 \cdots z_n} \mathrm{d}z_1 \cdots \mathrm{d}z_m \,. \end{aligned} \tag{2.18}$$

We can now analyze the terms obtained, (A) is a term in D dimension. This is the kind of relations we want to construct in order to generate IBP relations. The (B) part gives a term where the dimension is shifted to $D-2$, the shift is generated by the $1/F$ term. The last one, (C), gives an term in D dimension with propagators squared. In order to generate IBP which do not have terms contributing from lower dimensional identities we can require the polynomials $a_i(z)$ to satisfy:

$$bF + \sum_{i=1}^{m} a_i \frac{\partial F}{\partial z_i} = 0\,. \tag{2.19}$$

As we have seen applying cuts on the Baikov representation is more efficient on single propagators. Besides, as we have mentioned, integrals with double propagators do not appear from Feynman rules but are a spurious result of IBP identities[8]. So we can further constrain the coefficient $a_i(z)$ such that

$$z_i b_i = a_i\,, \tag{2.20}$$

which we can combine with (2.19) to obtain

$$bF + \sum_{i=1}^{n} z_i b_i \frac{\partial F}{\partial z_i} + \sum_{j=n+1}^{m} a_j \frac{\partial F}{\partial z_j} = 0\,, \tag{2.21}$$

where we have enforced (2.20) only on the variables associated with propagators.

2.2.1 Algebraic geometry tools

Before continuing in our description of IBP in Baikov representation is useful to define some basic elements of algebraic geometry[9].We define the ring of polynomials over a field[10] $\mathbb{F}$ as $\mathcal{P} = \mathbb{F}[x_1, \ldots, x_n]$.

Definition 1. An ideal is defined as a subset $\mathcal{I} \in \mathcal{P}$ that contains the null element and which is closed under the ring operation, $f_1, f_2 \in \mathcal{I}$ then $f_1 + f_2 \in \mathcal{I}$. Moreover for each element $h \in \mathcal{P}$ we have that $hf \in \mathcal{I}$ for each $f \in \mathcal{I}$.

[8]This requirement also reduces the number of identities involved, making solving the IBPs system easier.

[9]For a more detailed introduction and proof of the statement presented here see [62–64]

[10]Our main interest in this thesis is the closed field $\mathbb{C}$.

The ideal is generated by a finite basis of elements $\mathcal{I} = \langle f_1, \ldots, f_n \rangle$ and each element $g \in \mathcal{I}$ can be written as

$$g = \sum_i^n h_i f_i \,, \tag{2.22}$$

where $h_i \in \mathcal{P}$. In general the same ideal can be generated by different basis of polynomials f_i. From the point of view of algebraic geometry we want to study the locus of the zeros of $\mathcal{I}$.

Definition 2. Given a n dimensional $\mathbb{F}$ -Affine space $\mathbf{A}^n$, the alebraic set over $\mathbb{F}$ of $\mathcal{I} \in \mathcal{P}$ is defined as

$$\mathcal{Z}(\mathcal{I}) = \{p \in \mathbf{A}^n | f(p) = 0 \quad \forall f \in \mathcal{I}\} \,. \tag{2.23}$$

Before continuing is useful to introduce the notion of a Gröbner Basis (GB) of an ideal. As we have said the same ideal can be represented using different generators. In general if we want to test if a polynomial g is in $\mathcal{I}$ we have to perform a polynomial division and check that the reminder is zero. If the degree of g is lower than the degree of the generators f_i of $\mathcal{I}$ this is inconclusive. The GB is exactly a basis for which the membership problem is automatically solved, if the reminder is null then $g \in \mathcal{I}$. The GB is also useful to find the algebraic set of an ideal $\mathcal{Z}(\mathcal{I})$. In order to perform all this computations one has to define a monomial ordering.

Definition 3. Given all the monomials $\mathcal{M} \in \mathcal{P}$ an ordering $\prec$ in $\mathcal{M}$ has to satisfy the following conditions. It has to be a total ordering, such that any $m_1, m_2 \in \mathcal{M}$ can be sorted using $\prec$. The ordering has to be invariant under multiplication by an element m of $\mathcal{M}$, $m_1 \prec m_2$ also $mm_1 \prec mm_2$. The lowest element of the ordering is the constant monomial 1.

With a specific ordering we can define the notion of leading term

$$LT(f) = m_N \quad f_i = \sum_i^N c_i m_i \quad m_i \in \mathcal{M}, \quad c_i \in \mathbb{F} \,. \tag{2.24}$$

The GB of an ideal can be algorithmically constructed using the Buchenberg algorithm[11] which relies on the S-polynomial,

$$S(f,g) = \frac{LT(g)}{gcd(LT(f), LT(g))} f - \frac{LT(f)}{gcd(LT(f), LT(g))} g \,, \tag{2.25}$$

[11]Or more recent implementations such as the F4 and F5 algorithms by Faugere.

where $g, f \in \mathcal{P}$ and *gcd* is the great common divisor. In general computations of GB become more complicated as the number of polynomials and their degree grow.

Another interesting structure that comes up in our study of IBP identities is the notion of module.

Definition 4. A module $\mathbf{M}$ over a ring, $\mathcal{R}$ is defined as an Abelian group, with respect to the operation $+$, with an associative map $\mathcal{R} \times \mathbf{M} \to \mathbf{M}$.

For example we can take an ideal, $\mathcal{I} = \langle f_1, \ldots, f_n \rangle \in \mathcal{P}$, and define a module of the ideal as

$$\mathbf{M}(\mathcal{I}) = f_1 \mathbf{e}_1 + \cdots + f_n \mathbf{e}_n \,, \tag{2.26}$$

where $\mathbf{e}_i$ is a n-dimensional unit vector in the ith direction. For each module we can define a submodule, which we call a syzygy, which is generated by the elements satisfying

$$a_1 \cdot f_1 + \cdots + a_n \cdot f_n = 0 \,. \tag{2.27}$$

The syzygy of a module characterizes its independent generators. With this in mind we can look back at equations (2.19) and (2.21) and and recognize them as syzygys [26] of the modules generated by

$$\langle F, \frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_m} \rangle \quad \text{and} \quad \langle F, z_1 \frac{\partial F}{\partial z_1}, \ldots, z_k \frac{\partial F}{\partial z_k}, \ldots, \frac{\partial F}{\partial z_m} \rangle \,. \tag{2.28}$$

From a more geometrical point the solutions to (2.19) can be seen as the tangent vectors to the hypersurface $F = 0$ [65, 66], while solutions to (2.21) are tangent vectors to $z_1 \ldots z_n F = 0$. The problem of finding the generators of the IBP identities is then to find the generators of the syzygys of (2.28). In general this can be done by GB basis computations, by using for example the software SINGULAR[67], but as we have said the computations becomes more and more complicated as the generators of the modules grow in complexity. Already for 2-loops and five external legs we can not solve (2.21) completely.

2.2.2 On unitarity cuts

Up to now the treatment of IBP relations has been similar to the one described in the beginning of this chapter, with the extra requirement that no double propagators appear in order to reduce the number of integrals involved in the identities.

In [61] the authors describe a method for reconstructing IBPs identities by using a span of generating cuts. As we have seen the action of a cut on the Baikov representation is to calculates the residue

for a denominator of (2.11) at the origin. As the number of variables is reduced after a cut, the corresponding syzygy equations on the cut will be simpler. Let us study what happens when we perform a k-cut $\mathcal{C} = \{z_{c_1}, \ldots, z_{c_k}\}$. After taking the residues for the z_{c_i} variables associated with the cut, we are left with a subset of the starting variables $\mathcal{U} = \{z_{u_1}, \ldots, z_{u_{n-k}}, z_{n+1} \ldots, z_m\}$, and (2.21) becomes

$$bF + \sum_{i=1}^{n-k} z_{u_i} b_{u_i} \frac{\partial F}{\partial z_{u_i}} + \sum_{j=n+1}^{m} a_j \frac{\partial F}{\partial z_j} = 0\,. \tag{2.29}$$

IBPs generated by this system will not have contributions from MIs that have a propagator $z_i \in \mathcal{C}$ as they are nullified. The IBP relations constructed in this way will not be complete but by considering a set of cuts, and merging the final results together, we can reconstruct the IBPs reductions completely. For example in fig 2.2 are represented all the cuts needed to reconstruct the IBP identities for the non-planar hexagon box.

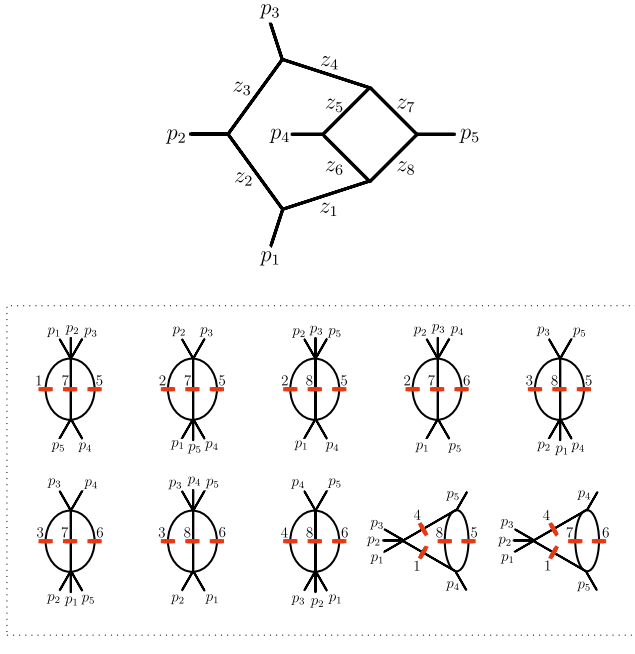


Figure 2.2. Spanning set of cuts for the non-planar hexagon box. This minimal set of cuts is constructed by finding master integrals that can not be collapsed, by deleting a propagator or a line in the associated graph, to another master integral.

The upshot of this method is that the computation of the solutions of (2.29) is simpler and faster than for the completely uncut case.

Following this reasoning the simpler cut to study is an n-cut, where we have nullified all the propagators, which we will call a maximal cut [68]. The equation for generating the IBPs will then be

$$0 = bF + \sum_{i=n}^{m} a_i \frac{\partial F}{\partial z_i}\,, \tag{2.30}$$

where the sum now runs only over the ISPs. The IBPs generated in this way will not have contributions from lower integrals as all the denominators are already nullified and they will only give the coefficients associated with the integral we started from. In Paper I we exploit those kind of relations to find a basis of MIs given a starting topology. The idea is to study every subintegral, where we have nullified one or more propagators, as a maximal cut in order to simplify the computation of the Syzygys solutions. We can then generate IBPs identities[12]for each sub integral and extract all the MIs contributing.

2.3 Laplace expansion

Up to now we have not used the property that the Baikov polynomial F is the determinant of a symmetric matrix[13]. We can start by considering the Laplace expansion of a matrix. For a generic matrix $M = (m_{i,j})_{i,j=1,\dots,n}$, the expansion on the ith row reads as:

$$\left[\sum_{k=1}^{n} m_{j,k} \frac{\partial(\det M)}{\partial m_{i,k}}\right] - \delta_{i,j} \det M = 0\,, \quad 1 \leq i,j \leq n\,. \tag{2.31}$$

For a symmetric matrix, $x_{i,j} = x_{j,i}$, as in (2.12), this becomes:

$$\left[\sum_{k=1}^{E+L} (1+\delta_{i,k}) x_{j,k} \frac{\partial F}{\partial x_{i,k}}\right] - 2\delta_{i,j} F = 0\,, \tag{2.32}$$

Though in principle we can generate $E + L$ equations from the Laplace expansion, we are interested in only the last L rows as the first E rows correspond to a derivative with respect to an external momenta.

The change of variables from $x_{i,j}$ to z_i can be obtained by acting with the chain rule. We can then write:

$$\sum_{\alpha=1}^{m} (a_{i,j})_\alpha \frac{\partial F}{\partial z_\alpha} + b_{i,j} F = 0\,, \tag{2.33}$$

[12]This method can be made faster by considering rational kinematics and by working in a specific rational dimension as the coefficients of the IBP reduction are not important for finding MIs.

[13]A similar approach was performed in [69]

where $a_{i,j}$ and $b_{i,j}$ are

$$(a_{i,j})_\alpha = \sum_{k=1}^{E+L} (1+\delta_{i,k}) \frac{\partial z_\alpha}{\partial x_{i,k}} x_{j,k} \quad \text{and} \quad b_{i,j} = -2\delta_{i,j}\,, \tag{2.34}$$

and where the indices run from $E+1 \leq i \leq E+L$, $1 \leq j \leq E+L$ and $1 \leq \alpha \leq m$. The $(a_{i,j})_\alpha$ generate solutions for (2.33), or in the previous language (2.19). This gives a way to compute the polynomial a_i without having to relay on any heavy computation[14].

Before extending this formalism to the cut and no double propagator case we want to remark on some properties of these solutions. The syzygy generators constructed from the Laplace expansion are at most linear in the z_i variables. This can be understood by looking at (2.34), the $x_{i,j}$ are by construction at most linear in the z_i. We can see an example of this in (2.14) where we have constructed the S matrix for a simple 2-loop example. Moreover the syzygy generators are homogeneous in the z_i and kinematics invariants.

As we have seen in section 2.2.1 the set of solutions $(a_{i,j})_\alpha$, from an algebraic geometry point of view can be seen as generators of a module, M_1, defined on the complex polynomials in the variables z_i, $\mathbb{C}[z_i]$. In the same way the solutions to the no double propagators requirement can themselves be seen as generators for a module, M_2, and can be written as

$$M_2 = \langle z_1 e_1, \ldots, z_k e_k, e_{k+1}, \ldots, e_m \rangle\,, \tag{2.35}$$

where the e_i is the m-dimensional unit vector in the ith direction. As we are interested in computing solutions that satisfy both equations, or equivalently (2.21), this correspond to finding the intersection $M_1 \cap M_2$ between the two modules. In Paper V we develop an efficient algorithm to obtain such intersection, by GB computation. As we have mentioned before the generators for the two modules are at most linear in the Baikov variables z_i which makes the computation more feasible than with the previous method, section 2.2.

Intersection on cut

As before we are interested in studying the IBP relations on a spanning cut and not on the complete set of propagators which we have seen is more computationally efficient. The application of a cut to the modules M_1 and M_2 is as before setting the associated variable $z_i = 0$. In general the cut solution $\widehat{M_1}$,where the $\widehat{\ }$ sets the cut propagators to zero, by itself is not a solution as F on a specific cut is not a determinant of a matrix.

[14]This kind of solutions are known as the basic canonical IBP identities.

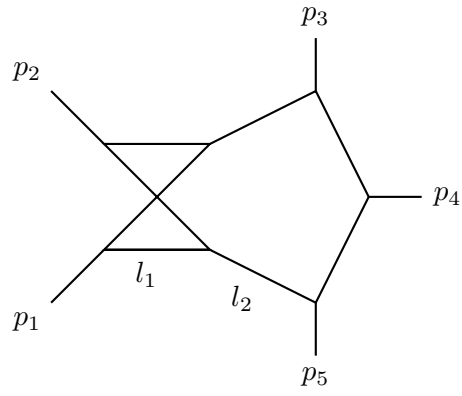


Figure 2.3. One of the non planar contributions to the five-point amplitude at two loops.

To obtain solutions on a cut we then have to compute:

$$\widehat{M}_1 \cap \widehat{M}_2 \,. \tag{2.36}$$

Using the method here described in Paper V we computed the reductions, up to numerators of degree 4, for one of the integrals contributing, see fig 2.3, to the five gluon amplitude at two loop.

3. Bootstrap of p-Integrals

The computation of Feynman diagrams is important in the evaluation of scattering amplitudes, correlation functions, structure constant in conformal field theories and anomalous dimensions. In the last decade several steps forward have been taken in trying to automatize this difficult task.

The method of differential equations [29, 30, 70–78] relies on the fact that Feynman integrals will, in general, be functions of the external kinematical invariants, $\{\chi_i\}$, and of the dimension D. We can then write for a given set of MIs $\mathcal{I}$ the following differential equation for any of the invariants χ_i

$$\frac{\partial \mathcal{I}(\{\chi_i\}, D)}{\partial \chi_i} = A_i(\{\chi_i\}, D)\mathcal{I}(\{\chi_i\}, D)\,. \tag{3.1}$$

where we have applied IBP relations to express the right-hand side of eq. (3.1) in terms of the MIs $\mathcal{I}$.

In general this system of differential equations is unsolvable. It was pointed out that for a canonical choice of the master integrals $\mathcal{G}$ [74], which depends on the dimension D, the system can be put in Fuchsian form and solved by iterative integrals

$$\frac{\partial \mathcal{G}(\{\chi_i\}, \epsilon)}{\partial \chi_i} = \epsilon A_i(\{\chi_i\})\mathcal{G}(\{\chi_i\}, \epsilon)\,, \tag{3.2}$$

where we have set $D = d - 2\epsilon$ and ϵ is the dimension regulator. A different approach is direct integration in parametric representation. This was already used for simpler cases but it was discovered in [31] that for integrals which are linearly reducible and finite, up to a Γ-function pre factor, they can be algorithmically computed as iterated integrals over single valued hyperlogarithms. One implementation of this algorithm is `HyperInt`[79]. For p-Integrals the method of differential equations can not be applied as the dependence on the scale is trivially obtained but the value of the integral comes from boundary information[1]. It could be in principle possible to integrate them using `HyperInt` but at 5-loops the required computational power and the problem of finding a set of finite MIs makes it rather complicated.

[1] It could be possible to apply the method of Dimensional recurrence and analyticity [80–82], but the high number of MI appearing makes it a very difficult task.

In Paper IV we derived a method for computing position space p-Integrals that relies on a conformal theory setup to generate a set of algebraic constraints to bootstrap their value, we have also implemented at 5-loops a method developed in [83] for computing the non-planar 5-loop momentum p-integrals.

3.1 Parametric representation

We now want to introduce and derive the parametric (Schwinger) representation. It will be useful for studying convergence and for direct integration of Feynman integrals. To derive it we start from a property of Γ functions known as the Schwinger trick:

$$\prod_i^n \frac{1}{A_i^{\nu_i}} = \prod_i^n \frac{1}{\Gamma(\nu_i)} \int_0^\infty \mathrm{d}x_i \; x_i^{\nu_i - 1} \mathrm{e}^{-x_i A_i} \,. \tag{3.3}$$

If now the A_i are the denominators of a generic Feynman integral I in D dimensions we can study the structure of the arguments of the exponent:

$$\sum_\alpha x_\alpha A_\alpha = \sum_{k,w} a^{kw} l_k \cdot l_w + 2 \sum_k b^k \cdot l_k + c \,, \tag{3.4}$$

where b_k is a vector of external momenta and c is a combination of the masses and square of external momenta. We can now perform the integration over the internal loop momenta by solving the D dimensional Gaussian integral to obtain:

$$I(\nu_1, \ldots, \nu_n, D) = \frac{1}{\prod_i^n \Gamma(\nu_i)} \left(\prod_i^n \int_0^\infty \mathrm{d}x_i \; x_i^{\nu_i - 1} \right) \mathcal{U}^{-\frac{D}{2}} \mathrm{e}^{-\frac{\mathcal{F}}{\mathcal{U}}} \,, \tag{3.5}$$

where we have defined

$$\mathcal{U} = det(a) \qquad \mathcal{F} = det(a) c - (a_{\mathrm{Adj}})^{ij} b_i b_j \,, \tag{3.6}$$

with $a_{\mathrm{Adj}} = det(a) a^{-1}$ is the adjoint matrix of a. The polynomials $\mathcal{U}$ and $\mathcal{F}$ are uniform and of degree L and $L+1$ in the Schwinger parameters. By using this property we can perform a further integration:

$$\lambda = \sum_i^n x_i \qquad \alpha_i = \frac{x_i}{\lambda} \,, \tag{3.7}$$

$$\prod_i^n \int_0^\infty \mathrm{d}x_i = \int_0^\infty \mathrm{d}\lambda \; \lambda^{n-1} \prod_i^n \int_0^\infty \mathrm{d}\alpha_i \; \delta(1 - \sum_i^n \alpha_i) \,. \tag{3.8}$$

Having defined the superficial degree of divergence as $\omega = \sum_i \nu_i - \frac{L}{2}(D)$ (3.5) becomes[2]:

$$I(\nu_1, \ldots, \nu_n, D) = \Gamma(\omega) \left(\prod_i^n \int_0^\infty \frac{\alpha_i^{\nu_i - 1} \mathrm{d}\alpha_i}{\Gamma(a_i)} \right) \frac{\mathcal{U}^{\omega - d/2}}{\mathcal{F}^\omega} \delta(1 - H(\alpha)) \,. \tag{3.9}$$

As we can see from (3.9) inserting a double propagator corresponds to multiplying the integral by its associated parameter α_i. A integral with a numerator, $\nu_j < 0$, will be identified as:

$$I(\nu_1, \ldots, \nu_j, \ldots, \nu_n, D) = \Gamma(\omega) \left(\prod_{i \neq j}^n \int_0^\infty \frac{\alpha_i^{\nu_i - 1} \mathrm{d}\alpha_i}{\Gamma(\nu_i)} \right) \left(\frac{\partial}{\partial \alpha_j} \right)^{\nu_j} \frac{\mathcal{U}^{\omega - d/2}}{\mathcal{F}^\omega} \delta(1 - H(\alpha)) \Big|_{\alpha_j = 0} \,, \tag{3.10}$$

where the variable α_j is the one associated to the numerator.

3.1.1 Direct integration

As one example, consider the 1-loop bubble graph, where the integration can be performed directly,

$$\begin{aligned} \int \mathrm{d}^D l \, \frac{1}{l^{\nu_1} (l - p)^{\nu_2}} &= \frac{\Gamma(\nu_1 + \nu_2 - \frac{D}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)} \int_0^\infty \mathrm{d}\alpha_1 \frac{\alpha_1^{\frac{D}{2} - \nu_2 - 1}}{(\alpha_1 + 1)^{D - \nu_1 - \nu_2}} \\ &= \frac{\Gamma(\nu_1 + \nu_2 - \frac{D}{2})}{\Gamma(\nu_1)\Gamma(\nu_2)} \frac{\Gamma(\frac{D}{2} - \nu_1)\Gamma(\frac{D}{2} - \nu_2)}{\Gamma(D - \nu_1 - \nu_2)} \,. \end{aligned} \tag{3.11}$$

More generally every integral that contains a sub-bubble in their graph, see for example fig 3.1, can have the loop momenta associated to it integrated:

$$\int \frac{\mathrm{d}^D l}{(l)^{\nu_1} (l - q)^{\nu_2}} f(p_i, l_i) = \frac{\Gamma(\frac{D}{2} - \nu_1)\Gamma(\frac{D}{2} - \nu_2)}{\Gamma(D - \nu_1 - \nu_2)} \int \frac{1}{(q^2)^{\nu_1 + \nu_2 - d/2}} f(p_i, l_i) \,, \tag{3.12}$$

where q represent the rest of the external and loop momenta that are flowing inside the sub-bubble graph and the function $f(p_i, l_i)$ is the integral stripped of the momentum l.

In general cases the integration is far more involved, as at higher loops the number of parameters α_i grows and for higher external points the space of functions which can be obtained becomes more complicated. As we have mentioned in the introduction, fro several masses

[2]By the Cheng-Wu theorem [84] we can choose a generic Hypersurface $H(\alpha)$ and not only $1 - \sum_i^n \alpha_i$

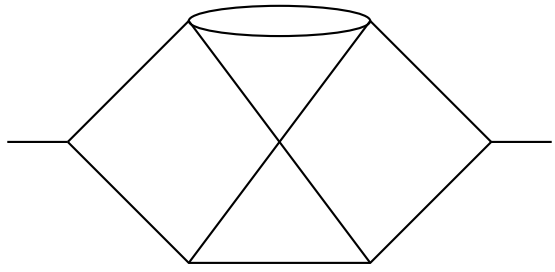


Figure 3.1. An example of a graph with a sub-bubble.

the class of functions appearing, in 4 dimensions, are elliptic multiple polylogarithms for which integration techniques are currently on a case by case example.

An algorithmic way of performing direct integrations was developed by Francis Brown [31] and it relies on a class of special functions known as hyperlogarithms[3]. This method works for a specific class of integrals that have the following properties, there has to exists an ordering of the parameters α_i such that after each integration the function

$$f_k(\alpha_{k+1};\ldots) = \int_0^\infty \mathrm{d}\alpha_k \; f_{k-1}(\alpha_k;\alpha_{k+1},\ldots)\,, \tag{3.13}$$

can be written as a hyperlogartihm in the next integration variable α_{k+1}. The parametric representation (3.9) has to be convergent, up to the overall $\Gamma(\omega)$ factor[4]. Lastly, the starting integrand has to be a rational function of the α_i, so we need to do an ϵ expansion prior to the integration. The integrals computed in this way will be represented as linear combination of hyperlogarithms. Following [31] we start to define these functions by taking a finite set of points, that can also be non constant, $\sigma_i \in \Sigma \subset \mathbb{C}$. We can then construct the symbol $\mathcal{A} = \{a_0, a_1, \ldots, a_n\}$, where in the entries a_i the subscript refers to the point σ_i and $a_0 = 0$. Hyperlogarithms are then defined recursively from the differential equation,

$$\partial_z L_{a_i\omega}(z) = \frac{1}{z-\sigma_i} L_\omega(z)\,, \tag{3.14}$$

where ω is a combination of the elements of $\mathcal{A}$. We also have to impose the boundary conditions

$$L_{a_0^n}(z) = \frac{1}{n!}\mathrm{Log}(z)^n \quad \text{and} \quad \lim_{z\to 0} L_\omega(z) = 0 \quad \text{for} \quad \omega \neq a_0^n\,. \tag{3.15}$$

The algorithm is then to find the primitive $P(\alpha_i)$, which derived gives the integrand at the previous step, then evaluate $P(\infty) - P(0)$ and

[3] A different approach is developed in [85, 86] and was implemented in [87].

[4] In [79] is described how to regularize divergent integrals in parametric representation.

express the result as hyperlogarithms. We want to present an example,

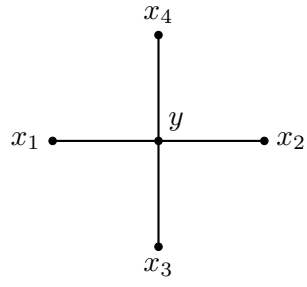


Figure 3.2. Four-point conformal integral at 1-loop. The external points $\{x_1, x_2, x_3, x_4\}$ are fixed by conformal symmetry to be $\{1, z, \infty, 0\}$ and the edges connecting two vertices are associated to a propagator $(x_i - x_j)^2$. The integration is performed over the internal point y.

fig. 3.2, of conformal four-point integral at 1-loop that can be easily integrated using the method described. The $\mathcal{U}$ and $\mathcal{F}$ polynomial are

$$\mathcal{U} = \alpha_1 + \alpha_2 + \alpha_3 \quad \mathcal{F} = \alpha_2\alpha_3(z-1)(\bar{z}-1) + \alpha_1\alpha_2 + \alpha_1\alpha_3 z\bar{z}\,. \tag{3.16}$$

Using the method described in [31] we find that the integral is linearly reducible and that an order that satisfies (3.13) is $\{\alpha_1, \alpha_2, \alpha_3\}$. The integrand, in $D = 4$ and following (3.9) having set $\alpha_3 = 1$ and with $\omega = 1$, reads

$$\int \mathrm{d}\alpha_1 \mathrm{d}\alpha_2 \frac{1}{(1+\alpha_1+\alpha_2)\left(\alpha_2\left(\alpha_1 + (z-1)(\bar{z}-1)\right) + \alpha_1 z\bar{z}\right)}\,, \tag{3.17}$$

which can be integrated directly in α_1 to obtain

$$\int \mathrm{d}\alpha_2 \frac{\mathrm{Log}(1+\alpha_2) - \mathrm{Log}(\alpha_2(z-1)(\bar{z}-1)) + \mathrm{Log}(\alpha_2 + z\bar{z})}{(\alpha_2+z)(\alpha_2+\bar{z})}\,. \tag{3.18}$$

In order to rewrite the integrand into simpler objects we need to perform partial fractions, in this case we have

$$\frac{1}{(\alpha_2+z)(\alpha_2+\bar{z})} = \frac{1}{z-\bar{z}}\left(\frac{1}{\alpha_2+\bar{z}} - \frac{1}{\alpha_2+z}\right)\,. \tag{3.19}$$

After further integration in α_2, and using properties of hyperlogarithms such as the shuffle relations, we obtain

$$\frac{1}{z-\bar{z}}\left(L_1(z)L_0(\bar{z}) - L_0(z)L_1(\bar{z}) + L_{1,0}(z) + L_{0,1}(\bar{z}) - L_{1,0}(\bar{z}) - L_{0,1}(z)\right)\,, \tag{3.20}$$

which can be rewritten, after a change of basis, as the known result for the 1-loop ladder in terms of classical polylogarithms

$$\frac{\text{Log}(z\bar{z})(\text{Log}(1-z)-\text{Log}(1-\bar{z}))+2\text{PolyLog}(2,z)-2\text{PolyLog}(2,\bar{z})}{z-\bar{z}}\,. \tag{3.21}$$

In the cases we are interested in studying, p-integrals, we obtain hyperlogarithms with arguments 1 which can be written as linear combinations of zeta and multiple zeta values (MZVs),

$$\zeta_{n_1,\ldots,n_d} := \sum_{1\leq k_1<\cdots<k_d} \frac{1}{k_1^{n_1}\cdots k_d^{n_d}}, \tag{3.22}$$

3.1.2 Check for convergence

The parametric representation (3.9) is also useful for checking the convergence of Feynman integrals in a specific dimension D. In [88] a simple algorithm for testing convergence has been introduced. Here we briefly present the main ideas as they will be useful for the Cut and Glue method, which we are going to describe later, and in general for the bootstrap of position space integrals.

Given a Feynman integral in parametric representation (3.9) we define the set of all propagators as $\mathcal{P}$. We take two disjoint subsets $\mathcal{I},\mathcal{V}\subset\mathcal{P}$ and define the following transformation on the α_k

$$\begin{aligned} &i\in\mathcal{I} \quad \alpha_i\to\lambda\alpha_i \\ &j\in\mathcal{V} \quad \alpha_i\to\frac{1}{\lambda}\alpha_j\,. \end{aligned} \tag{3.23}$$

For $\lambda\to 0$ this defines a degree of "vanishing", which we identify as $d_{i,j}$, for the integrand of (3.9) when the $\alpha_i\to 0$ and the $\alpha_j\to\infty$. For each of these subsets we can then associate a new superficial degree of divergence

$$\omega_{i,j} = |\mathcal{I}| - |\mathcal{V}| + d_{i,j}\,. \tag{3.24}$$

Then it can be proven that given the positiveness of all the coefficients of $\mathcal{F}$, if $\omega_{i,j}>0$ for all subsets $\mathcal{I},\mathcal{V}\subset\mathcal{P}$ with $\emptyset\neq\mathcal{I}\sqcup\mathcal{V}\subseteq\mathcal{P}$, then the integral is absolutely convergent.

3.2 p-Integrals

Up to now what we have presented methods that can be applied to generic Feynman integrals. For the rest of the chapter we will focus on a particular type, massless two-point configurations. We are interested in

diagrams appearing both in momentum and position space. In position space we can construct all possible contributing diagrams by considering propagators

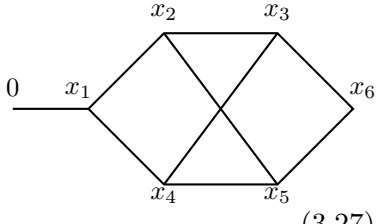
$$x_{ij} = x_i - x_j \,, \tag{3.25}$$

and construct integrals where the integration is performed over some internal points x_i and we have two external points defined such that $x_0 = 0$ and $x_6^2 = 1$. Generically a 5-loop integral can be written as

$$I(a) = \int \prod_{k=1}^{5} \mathrm{d}^D x_k \prod_{0 \le i < j \le 6} \frac{1}{x_{ij}^{2a_{ij}}}, \tag{3.26}$$

where a represents the possible exponents of the 20 propagators. Again we can associate a graph to an integral $I(a)$ where the propagators x_{ij} represent an edge connecting two vertices i and j, for example

$$\int \frac{\mathrm{d}^D x_1 \mathrm{d}^D x_2 \mathrm{d}^D x_3 \mathrm{d}^D x_4 \mathrm{d}^D x_5}{x_{01}^2\, x_{12}^2\, x_{23}^2\, x_{25}^2\, x_{36}^2\, x_{34}^2\, x_{14}^2\, x_{45}^2\, x_{56}^2} =$$



. (3.27)

In the case of momentum space integrals we can not construct a generic basis [5] that will contain all the possible topologies. In general one has to construct all possible trivalent graphs[6] and associate to each of them a set of propagators[7]. For example a non planar topology contributing

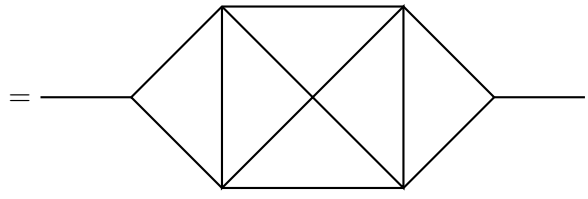
[5] At five Loops

[6] From a simple graph computation we know that at L-loops for E external momenta and trivalent vertices we will have maximum $P = 3L + E - 3$ propagators, which for our case gives $P = 14$.

[7] It is possible in principle by considering a Fourier transformation to obtain the value of the non-planar momentum or position integrals form one an other, but it requires to evaluate them with indices depending on ϵ.

at five loops is:

$$\int \frac{\mathrm{d}^D l_1 \mathrm{d}^D l_2 \mathrm{d}^D l_3 \mathrm{d}^D l_4 \mathrm{d}^D l_5}{l_1^2\ (l_1-p)^2\ l_2^2\ (l_2+p)^2\ l_4^2\ (l_3+l_4)^2\ l_5^2} \times \frac{1}{(l_1+l_3+l_4+l_5-p)^2\ (l_1+l_4+l_5)^2\ (l_2-l_3+p)^2} =$$



(3.28)

In the case of planar integrals the two are related by a dual transformation which acts trivially on the propagators, as such we have the choice of computing them in momentum or position space.

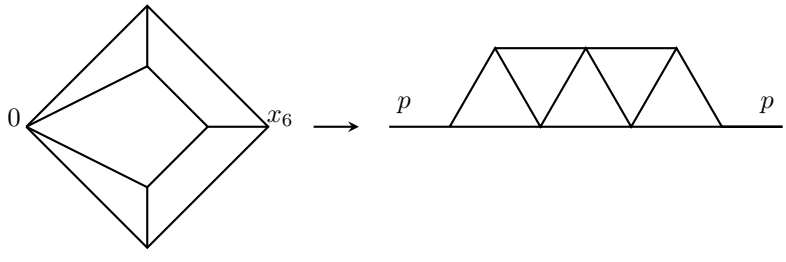


Figure 3.3. Duality transformation for a planar 5-loop two-point integral. To obtain the dual graph we draw an internal point for each loop. We draw two points outside each graph between the external points. Finally we connect points which are separated by a single edge.

3.3 Bootstrap of master integrals

In order to compute all the MIs contributing at five loops for two-point integrals we want to present different methods that do not require any explicit integration. The idea consists of generating linear equations that relate different ϵ orders of the MIs. By solving such systems we can then constrain values of their ϵ expansion.

3.3.1 Constraints from IBP identities

A first set of constraints can be extracted directly from IBP identities. In general Feynman Integrals will diverge, after dimensional regularization

with $D = d - 2\epsilon$, up to ϵ^{-2L}, in the case of p-integral the total degree is actually only L [89, 90].

For two-point integrals the c_i coefficients of (2.1) will depend only on the dimension D and in general can have poles in the dimensional regulating parameter ϵ [8]. In general the MIs can be written, ϵ expanded as

$$I_k = \sum_{i=0}^{\infty} \epsilon^{-L+i} I_{k,i} \,, \tag{3.29}$$

we can then rewrite a generic IBP identity (2.1) as:

$$I(a) = \sum_{j=0}^{\infty} \alpha_j(I_{k,i}, c_j)\, \epsilon^{-L+j-m} \,, \tag{3.30}$$

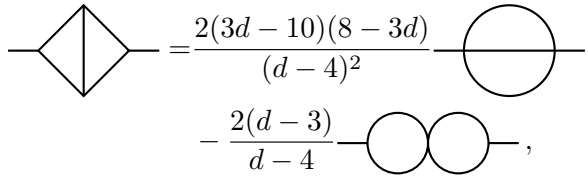
where the α_j are now linear combination of the numeric IBP coefficients c_j and the master integral coefficients $I_{k,i}$.

In (3.3.1) we have taken into account that the IBP coefficients can have divergences in ϵ, the starting integral on the other hand has to diverge, at maximum, as ϵ^{-L}. We can then write

$$\alpha_j = 0, \quad \text{for} \quad 0 \leq j < m \tag{3.31}$$

and by solving this system of constraints, we can find relations or vanishing orders of different MIs coefficients.

In general MIs at l-loops will contain integrals that are products of lower-loop MIs, that have sub-bubbles or that are easily computed. By plugging in this data into (3.31) we obtain a non-homogeneous system of equations and are able to obtain numerical results for the coefficients of the MIs. For example at two loop we can obtain all the MIs appearing up to order $O(\epsilon^2)$. We start by using IBP relations to express the topology



, (3.32)

in terms of two master integrals $\{I_1, I_2\}$. The second MI is a product of two bubble integrals (3.11) and as such can be directly evaluated to[9] $I_2 = \frac{1}{\epsilon^2}$. By using the method of sec. 3.1.2 we can moreover show that the starting integral is convergent. We can now insert the value for the

[8] For previous work see [91, 92].

[9] Given the usual normalization, known as the G-scheme, where the single bubble is set to be $\frac{1}{\epsilon}$.

second MI into (3.32), expand in $D = 4 - 2\epsilon$, and by requiring that the terms which have poles in ϵ disappear, we obtain for the first MI:

$$I_1 = -\frac{1}{4\epsilon} - \frac{5}{8} - \frac{27}{16}\epsilon\,, \tag{3.33}$$

3.3.2 Constraints from conformal integrals

Constraints coming from IBP relations are not enough to obtain all the required coefficients appearing in the ϵ expansions of the 5-loop MIs.

In order to obtain more relations we take advantage of the fact that two-point integrals appear in the asymptotic expansions [93,94] of higher point conformal integrals. We can for example consider four-point conformal integrals in position space, which are of the same form as (3.26) but with four external points that we label $x_1, \ldots, x_4$.

Conformal invariance allows us to set, for example, x_1 to the origin and send x_4 to infinity. We can moreover rewrite the integral as functions of two conformal invariant cross ratios, which we can choose as

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad \text{and} \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}. \tag{3.34}$$

In general these integrals are not known functions. For our purposes we just need to assume that they are finite. A way to generate finite integrals is, for example, by studying the $\mathcal{N} = 4$ correlation function of four $\frac{1}{2}$-BPS operators, see sec. 4.4.

We now take the limit for the points $x_2 \to x_1 = 0$, in this configuration we have two scales, $x_2^2 \ll x_3^2$ and we can use the technique of asymptotic expansions to expand the integral. For each integration variable x_i we can split the integration into two regions, one where x_i is of order x_2 or x_3. In these two regimes the denominators are

$$\begin{aligned} \frac{1}{(x_2 - x_i)^2} &= \sum_{n=0}^{\infty} \frac{(2x_2 \cdot x_i - x_2^2)^n}{(x_i^2)^{n+1}} \quad (\text{if } x_2^2 < x_i^2), \\ \frac{1}{(x_2 - x_i)^2} &= \sum_{n=0}^{\infty} \frac{(2x_2 \cdot x_i - x_i^2)^n}{(x_2^2)^{n+1}} \quad (\text{if } x_2^2 > x_i^2). \end{aligned} \tag{3.35}$$

In each of these regions we obtain a product of a k and $(5-k)$-loop p-Integral.

We can now extend the integration in each region to go beyond the radius of convergence of the Taylor expansion and extend the domain of integration to the whole space. By doing so we are considering dimensionally regularized integrals.

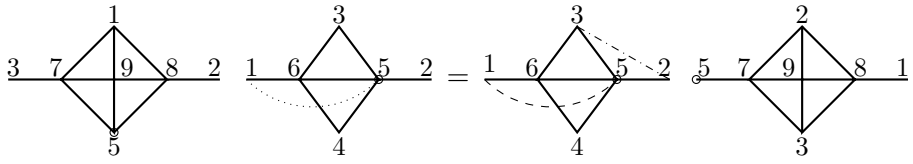


Figure 3.4. Magic identity for a 5-loop integral. In the left hand-side we have exchanged the points (1 2)(3 5) obtaining a different 3-loop subintegral. Dashed lines represent numerators, dotted and dosh-dotted represent square and cubic powers.

By performing an IBP reduction of the p-Integrals we can express the starting conformal integral as a combination of 5-loop MIs. The crucial step in this approach is that, the starting integral being conformal, its expansion has to be expressible in terms of the cross-ratios (3.34), which requires that all dependence on the spurious scales x_2^2 and x_3^2 must vanish. The expansion for the conformal integral to lowest order[10] in u and $1-v$ can then be expressed as

$$\sum_{n=0}^{5}\sum_{k=0}^{n}\sum_{l=0}^{n-k}\beta_{nkl}\,\log^k(x_3^2)\log^l(u)\epsilon^{-5+n}\,, \tag{3.36}$$

where the β_{nkl} are as before linear combination of the ϵ expansion of the MIs. Vanishing of the spurious scales and finiteness of the starting integrals give us

$$\begin{aligned}\beta_{nkl}&=0\,, \qquad \text{where } n<5 \text{ and } k\geq 0,\\ \beta_{5kl}&=0\,, \qquad \text{for } k>0.\end{aligned} \tag{3.37}$$

We can actually also cast constraints on the finite part of the integral. If we look at (3.34) we notice that by acting with a pairwise transformation we leave the u and v cross ratios invariant. Conformal symmetry constrains them to be invariant under the Klein group

$$\{id,(1\ 2)(3\ 4),(1\ 3)(2\ 4),(1\ 4)(2\ 3)\}\cong\mathbb{Z}_2\times\mathbb{Z}_2\,, \tag{3.38}$$

where $(i\ j)$ represent the permutation of the external point i with j. If we apply this transformation to a given four-point integral we will obtain an apparently different asymptotic expansion. By requiring that it has to match the one of the starting integral, we then obtain further constraints.

We can also apply this kind of transformation to each four-point sub integral of our starting integral (see fig 3.4 for an example). These kind of relations are known in the literature as magic identities [95].

[10]One could also expand to higher orders of u and v but the main bottleneck is the IBPs reductions of such complicated integrals.

3.3.3 Constrains from Cut and Glue symmetry

The methods described up to now work for integrals that are in position space. As stated in sec. 3.2, the planar sector is common between the two different representations. If we want to compute the non-planar contributions to the momentum space integral we can apply the method of the cut and glue symmetry (GaC) [83].

We start by studying mass regulated vacuum integrals, χ , at $L+1$-loops. We will concentrate on vacuum integrals without subdivergences and with superficial degree of divergence $\omega = 0$. We consider a single propagator of the starting vacuum integral, which by a change of variables can always be set to p^2, and where we add a mass regulating term m^2. We can in general write the vacuum integral as

$$\chi = \int \mathrm{d}^D p \frac{1}{(p^2+m^2)} \mathcal{I}(p)\,, \tag{3.39}$$

where with $\mathcal{I}$ we identify an L-loop p-integral obtained by cutting from the graph of χ the propagator with momentum p.

In $D = 4-2\epsilon$ the integral $\mathcal{I}$ depends on p as $(p^2)^{-1-L\epsilon}$. By using (3.12) we can then formally compute the divergent part of χ to be

$$\chi = \frac{C}{(L+1)\epsilon} + O(\epsilon^0)\,, \tag{3.40}$$

where C is the value of the p-integral. We then see that by cutting the integral with different propagators we can generate convergent L-loop p-integrals that have the same finite value C, see fig. 3.5.

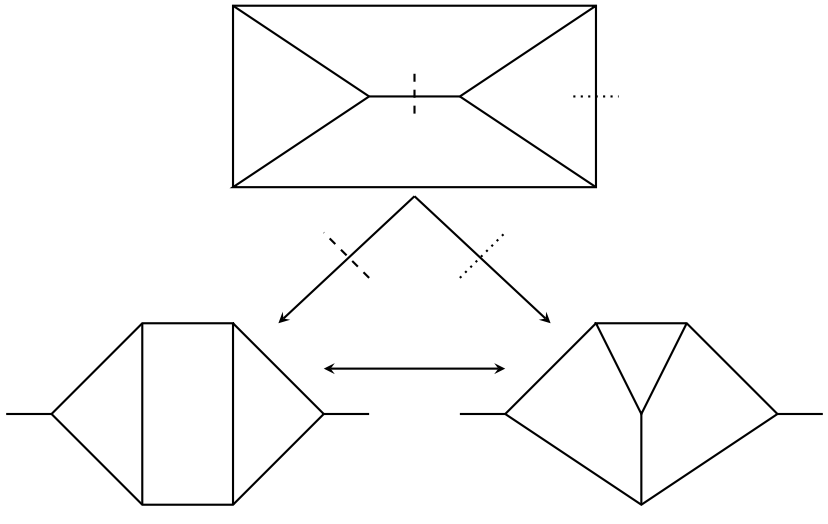


Figure 3.5. Different cuts of 4-loop integrals which give equivalent convergent 3-loop p-integrals.

This methods gives us relations between different integrals but does not compute their value C. We can, though, by performing an IBP reduction of the different convergent p-integrals and matching them order by order in ϵ obtain constraints on the ϵ expansions of the MIs. This is again a homogeneous system of equations, but as before we can plug in lower loop data or integrals with sub bubbles. Having already computed the planar sector of the 5-loop MIs we can use them as a starting point for this method, either to reduce the number of equations required, or as a cross check.

4. Correlation functions in $\mathcal{N} = 4$

$\mathcal{N} = 4$ SYM theory in the planar limit is the ideal playground to develop new tools and obtain interesting result in a 4 dimensional interacting conformal field theory (CFT). In the planar limit the theory is conjectured to be integrable, and it is related to a string theory on a $AdS_5 \times S^5$ background by a weak-strong duality [3]. Since the discovery of the integrable structure in 2002 [96] there has been a lot of developments in how to efficiently use integrability to compute structure constants and anomalous dimensions at finite coupling. In particular, a new framework based on hexagon form factors has been developed for the evaluation of both structure constants and higher-point functions. In Paper II we computed a structure constant at the 5-loop order. This order, which is important as a cross check of the integrability framework, where novel contributions to the hexagons arise at this order. In Paper VI, using constraints coming from hexagon form factors, we constructed the integrands for four-point functions of protected 1/2-BPS operators for any weight k up to five loops.

4.1 CFT

We will start by describing some features of CFTs, which will therefore be relevant for $\mathcal{N} = 4$ SYM.

Conformal invariance is characterized by transformations that leave the metric invariant up to a scalar factor [97]. They are generated by translation P_μ, dilatations D , Lorentz transformations, which comprise both boosts L_{0i} and rotations L_{ij}, $L_{\mu\nu}$ and special conformal transformations K_μ

$$\begin{aligned} P_\mu &= -i\partial_\mu & D &= -ix^\mu\partial_\mu \\ L_{\mu\nu} &= i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) & K_\mu &= -2ix_\mu(x\cdot\partial) + ix^2\partial_\mu \end{aligned} \tag{4.1}$$

In the case of conformal field theories we have states that are labelled by the dilatation eigenvalue, conformal dimension Δ, and the Lorentz group $SO(1, d-1)$ charges. General states can be described by the insertion of operators at the origin, which simplifies the form of the commutator with the generator (4.1), and then evolve them with

$$\mathcal{O}(x) = e^{iP\cdot x}\mathcal{O}(0)e^{-iP\cdot x}\,. \tag{4.2}$$

The generators P_μ and K_μ act as raising and lowering operators, their action on an operator $\mathcal{O}(x)$ creates a new operator with scaling dimension shifted accordingly. The operators which are annihilated by the K_μ, and so have the lowest scaling dimension, are called primaries. The operators constructed by acting on primaries with P_μ are called descendants. Requiring that the norm of two state is positive definite gives a bound on the dimension of scalars[1] operators $\Delta > \frac{d}{2} - 1$.

Beside organizing the operators of the theory in conformal multiplets, conformal symmetry also casts constraints on correlators of primary operators $\mathcal{O}(x)$. A conformally invariant two-point function can be shown to be of the form

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\rangle = \frac{\delta_{\Delta_1,\Delta_2}}{|x_1 - x_2|^{2\Delta_1}}\,. \tag{4.3}$$

Also three-point functions can be completely fixed up to a constant,

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\rangle = \frac{C_{123}}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{13}|^{\Delta_1+\Delta_3-\Delta_2}}\,, \tag{4.4}$$

where $x_{ij} = x_i - x_j$ and C_{123} is called structure constant. If we now try and constrain four-point functions of four operator with the same weight Δ we obtain

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \frac{F(u,v)}{|x_{12}|^{2\Delta}|x_{34}|^{2\Delta}}\,, \tag{4.5}$$

which depends on an unknown function of the cross ratios $\{u, v\}$, which are conformal invariant scalar quantities defined as

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \quad \text{and} \quad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\,. \tag{4.6}$$

For higher-point functions the constraints coming from conformal symmetry become less and less stringent but like in QFTs, in a CFT we can introduce the notion of operator product expansion (OPE), where we approximate two operators as a series of local operators. While in QFTs this is an asymptotic series, for CFTs it is actually convergent. We can write in general the product of two operators as

$$\mathcal{O}_1(x_1)\mathcal{O}_2(0) = \sum_j C_{12j}|x|^{\Delta_j-\Delta_1-\Delta_2}(\mathcal{O}_j(0) + \text{Descendants})\,, \tag{4.7}$$

where the sum is over all the possible primary operators that can be exchanged. This gives us a way to express n function as an infinite sum over $n-1$-point functions. By repeating this procedure we notice that the knowledge of scaling dimensions, Δ, and OPE coefficients, C_{123}, is in principle sufficient to obtain any higher-point function.

[1] A similar bound can be found for traceless symmetric representations $\Delta_l \geq l + d - 2$.

4.2 $\mathcal{N} = 4$ SYM theory and $N_c \to \infty$

We will now briefly describe the properties of $\mathcal{N} = 4$ SYM theory [98] and its planar limit . The fields of the theory organize in a vector multiplet, with one gauge field A_μ four chiral and anti-chiral spinors $\{\Psi_\alpha^Y, \bar{\Psi}_{\dot{\alpha}Y}\}$ and six scalars Φ_I, in the adjoint representation of a $SU(N)$ gauge group. The scalars also transform in a six dimensional representation of the $SO(6)$ R-symmetry group, while the fermions transform in the fundamental and anti-fundamental of $SU(4)$.

The bosonic symmetries of the theory are the conformal group $SU(2,2)$ and the $SU(4)$ R-symmetry, with supersymmetry they get enhanced to $PSU(2,2|4)$. The primaries operator will transform in an infinite dimensional, irreducible representation of this algebra. A representation is shortened when the lowest weight is annihilated by a set of supercharges, in those case we are left with what are called short representation. A set of this representations that we want to study are the ones annihilated by half of the supercharges, also known as $\frac{1}{2}$-BPS.

The $\frac{1}{2}$-BPS operators transform in a traceless and symmetric representation of the R-symmetry, we can enforce this by contracting a scalar single-trace operator with harmonic variables y_I obeying the null condition $y \cdot y = 0$. We can write the lowest component of the $\frac{1}{2}$-BPS multiplet as

$$\mathcal{O}_L(x, y) = y_{I_1} \dots y_{I_L} \mathrm{Tr}(\Phi^{I_1} \dots \Phi^{I_L}) \,, \tag{4.8}$$

where the choice of the scalar is be encoded in the polarization vector y_I and L is the length of the operator. In the case of $L = 2$ we obtain a supermultiplet which contains conserved currents and also the Lagrangian of the theory. Since the bottom component is protected from quantum correction this implies that also the Lagrangian is protected, the β functions is then zero and the theory is conformal also at the quantum level.

We now want to take the large N limit where we consider contributions in perturbation theory coming only for planar diagrams. We also take the limit for the coupling g_{YM} being small and define a new expansion parameter, the t'Hooft coupling,

$$\lambda = g_{\mathrm{YM}}^2 N \,. \tag{4.9}$$

As we mention there is a string theory dual to $\mathcal{N} = 4$ SYM, on the string side the planar limit is equal to taking the string coupling to zero which constraints the topology of the string world-sheet to have genus zero. In this large N limit the theory is conjectured to be integrable.

In [96] the authors mapped the problem of finding the anomalous dimension of single trace operators into finding the spectrum of a spin chain. A single trace operator can then be seen as a spin chain where each site transforms in a fundamental representation of $PSU(2,2|4)$,

$$\mathcal{V}_1 \otimes \dots \otimes \mathcal{V}_L \,. \tag{4.10}$$

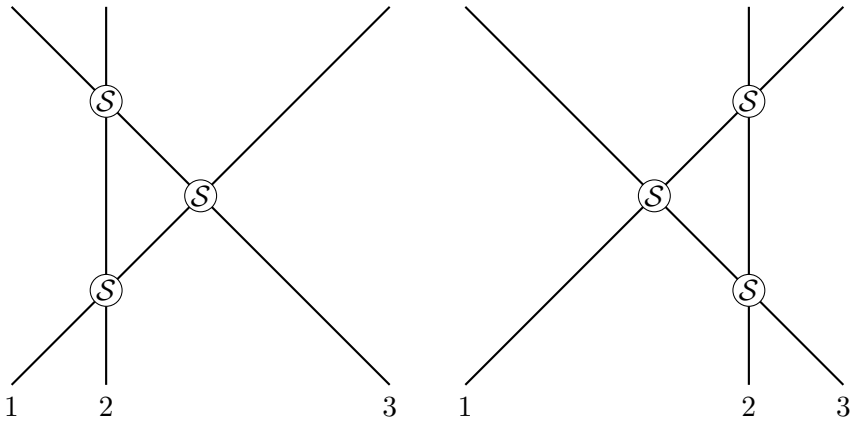


Figure 4.1. Two equivalent ways of representing a $3 \to 3$ scattering by pairwise interaction.

The cyclicity of the trace requires the spin chain to have a shift symmetry.

Before continuing it is convenient to recast the scalar fields into complex scalars

$$\begin{aligned} Z &= \Phi_1 + i\Phi_2 \qquad Y = \Phi_3 + i\Phi_4 \qquad X = \Phi_5 + i\Phi_6 \\ \bar{Z} &= \Phi_1 - i\Phi_2 \qquad \bar{Y} = \Phi_3 - i\Phi_4 \qquad \bar{X} = \Phi_5 - i\Phi_6 \,. \end{aligned} \tag{4.11}$$

We choose the vacuum to be the $\frac{1}{2}$-BPS single trace operator

$$\mathcal{O} = Tr[Z^L] \,, \tag{4.12}$$

we can then construct different single trace operators by inserting some element of the vector multiplet inside the trace of $\mathcal{O}$. These fluctuations are organized into representation of $\mathfrak{su}(2|2)^2$.

As they will be useful later we will introduce some machinery and tools that come from integrability. Integrability enforces that scattering of excitations in the spin-chain must factorize into pairwise scattering. If we take a three particle interaction we have two different ways of representing it with a $2 \to 2$ scattering, see fig. 4.1. The consistency between the two representations gives,

$$\mathcal{S}_{23}\mathcal{S}_{13}\mathcal{S}_{12} = \mathcal{S}_{12}\mathcal{S}_{13}\mathcal{S}_{23} \,, \tag{4.13}$$

where the indices $\{1, 2, 3\}$ represent the three particles that are scattering. This type of relation is known in the literature as the Yang-Baxter equation [99] (YBE).

In the case of $\mathcal{N} = 4$ SYM, the scattering $\mathcal{S}$-matrix can be fixed, up to a prefactor S_0 by requiring that it commutes with all the generators

of the centrally extended $\mathfrak{psu}(2|2)$ algebra[2] [100]. The prefactor S_0 can be fixed by imposing crossing symmetry and unitarity [101] to be

$$S_0(u_1,u_2)=\frac{x_1^+-x_2^-}{x_1^--x_2^+}\frac{1-\frac{1}{x_1^-x_2^+}}{1-\frac{1}{x_1^+x_2^-}}\frac{1}{\sigma^2}\,, \tag{4.14}$$

where σ is the BES dressing phase and x_i are the Zhukowsky variables. These variables are useful to parameterize the energy and momentum

$$e^{ip}=\frac{x^+}{x^-}\quad E_p=-i\frac{\sqrt{\lambda}}{4\pi}\left(x^+-\frac{1}{x^+}-x^-+\frac{1}{x^-}\right)\,, \tag{4.15}$$

where $x(u)^\pm=\frac{u+\sqrt{(u\pm\frac{i}{2})^2-4g^2}}{2g}$ and $g=\sqrt{\lambda}/2\pi$. A crossing transformation, which inverts the sign of the momentum and energy, acts on the x_i as

$$x^\pm\to\frac{1}{x^\pm}\,. \tag{4.16}$$

Up to now we have determined how general excitation on the spin chain scatter but for physical states we need a further constraint. As we mentioned physical states are mapped to gauge invariant operators and we have to require that the spin chain wave-function satisfies periodic boundary conditions. Let us start with a single excitation that goes around the spin chain with no interaction then the wave function would pick up a phase e^{ipL}, where L is the length of the chain. By imposing periodic boundary condition we require

$$e^{ipL}=1 \tag{4.17}$$

which gives a quantization condition on the momenta. In general states, where we have more than one excitation, we have to consider also interaction between the different momenta. This requirement give rise, when applied to each of the excitations, to what are known in the literature as the Bethe equations

$$e^{ip_iL}\prod_{j\neq i}\mathcal{S}(p_i,p_j)=1\,. \tag{4.18}$$

Solutions of the Bethe equations [102,103], Bethe roots, represent physical quantised momenta[3]. Once we have obtained the Bethe roots we can obtain the energy of the spin chain, which maps to the anomalous dimension. So whenever we want to represent an operator we have to solve the associated Bethe equations.

[2] As we have seen before the vacuum breaks the $\mathfrak{psu}(2,2|2)$ into $\mathfrak{su}(2|2)^2$ but for this algebra the energy is a constant. In order to obtain the $\mathcal{N}=4$ dispersion relation we have to add a central extension to the algebra.

[3] In general one has to solve a more involved system of equations, known as the nested Bethe equations [104] when the $\mathcal{S}$-matrix is not a simple phase.

4.3 Integrability of three-point functions

Planar $\mathcal{N} = 4$ SYM is believed to be integrable and in this framework a lot of tools have been developed to compute different quantities, such as correlation functions[4] [44, 106, 107], amplitudes[108] and anomalous dimensions [41, 42]. In this section we want to focus our attention on proposal to compute correlation functions using hexagons form factors [43]. The main idea of this method is to consider a string three-point amplitude, depicted in fig 4.2, and perform three cuts along the non open sides to obtain a pair of hexagons. The form of the hexagons can then be obtained, as in the case of the $\mathcal{N} = 4$ Beisert $\mathcal{S}$-matrix, by using residual symmetries and also form factor axioms. The full three-point function can then be obtained by "gluing" the two hexagons back together. We now proceed and bootstrap the form of the hexagons

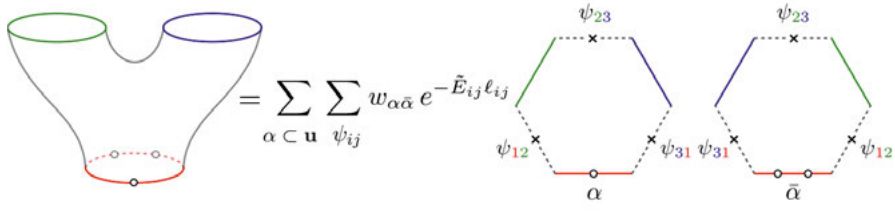


Figure 4.2. Cutting a three-point function gives us a pair of hexagons. Excitation living on the edges of the three-point function are then split between the two hexagons. In order to obtain the starting function we need to sum over the mirror excitations.

form factor and the structure constants. By cutting the three edges we are dividing the excitations present on the three operators between the left and right hexagons. We define the set of Bethe roots for the three different operators as

$$\{u\} = \alpha \cup \bar{\alpha} \qquad \{v\} = \beta \cup \bar{\beta}\,, \qquad \{w\} = \gamma \cup \bar{\gamma}\,, \tag{4.19}$$

where the bar and unbar quantities are inserted in the right and left hexagon respectively. The proposal for the form of the structure constants is

$$C_{123} = \sum_{\psi_{ij}} \sum_{\{u,v,w\}} \omega_{l_{13}}(\alpha,\bar{\alpha})\omega_{l_{12}}(\beta,\bar{\beta})\omega_{l_{23}}(\gamma,\bar{\gamma})\mathcal{H}_{\psi_{ij}}(\alpha,\beta,\gamma)\mathcal{H}_{\psi_{ij}}(\bar{\alpha},\bar{\beta},\bar{\gamma})\,, \tag{4.20}$$

where we have defined the bridge length l_{ij} as the number of wick contractions between the operators $\mathcal{O}_i$ and $\mathcal{O}_j$,

$$l_{ij} = \frac{(L_i + L_j - L_k)}{2}\,, \tag{4.21}$$

[4] And there are some results also for the non planar corrections [45, 105].

and the L_i are the length of the operators. The splitting factors

$$\omega_{l_{ji}}(\alpha,\bar{\alpha}) = \prod_{u_i\in\bar{\alpha}} e^{il_{ij}p(u_i)} \prod_{\substack{u_i\in\alpha\\ u_j\in\bar{\alpha}\\ u_i<u_j}} S(u_j,u_i)\,, \tag{4.22}$$

take into account that when an excitation in $\bar{\alpha}$ is moved from one hexagon to the other they have to interact with the other Bethe rapidities. The final element of the proposal are the Hexagon form factors $\mathcal{H}$ which are dependent on the set of rapidities appearing on physical edge and on the states ψ_{ij}. The latter came from the resolution of the identity which we have to insert when we glue the hexagon form factors together. We have to sum over all possible, mirror states ψ_{ij} that can be exchange between the two hexagons. In general resumming this type of mirror excitations is very complicated and this is one of the main obstacles to obtaining a finite coupling result from this approach[5].

We now want to focus our attention on the Hexagon form factor $\mathcal{H}$. We can think of this as a generic interaction vertex $\mathfrak{h}$ that is contracted with the three different spin chains states

$$\langle\mathfrak{h}|(1\rangle\otimes 2\rangle\otimes 3\rangle)\,. \tag{4.23}$$

In general this seems like a very complicated problem as we have excitations on the three different states.

The Zhukowsky variables functions $x(u)^{\pm}$ have two branch cuts. We can now consider what happens if we cross one of the two cuts, which we define symbolically as $u\to u^{\gamma}$, then the Zhukowsky variables will be analytically continued to

$$x^-\to x^- \qquad x^+\to 1/x^+\,. \tag{4.24}$$

This analytical continuation can be understood from the Hexagon form factor as sending an excitation to the next edge, fig. 4.3. Then, by applying different mirror transformations, we can consider an easier hexagon where all excitations are on a single edge.

Now we have obtained a simpler object that we can constrain, as in the case of the Beisert $\mathcal{S}$-matrix, by using the residual symmetry, in this case a single $\mathfrak{psu}(2|2)$. This means that the hexagon form factor has to be annihilated by all the generators, $\mathfrak{g}$, of the algebra when acting on a state χ,

$$\langle\mathfrak{h}|\mathfrak{g}|\chi\rangle = 0\,. \tag{4.25}$$

[5]In some recent work based on the results of F. Coronado [109, 110], the authors of [111] were able to provide a determinant formula for gluing back two edges of the hexagons. There is a previous result for strong coupling [112]

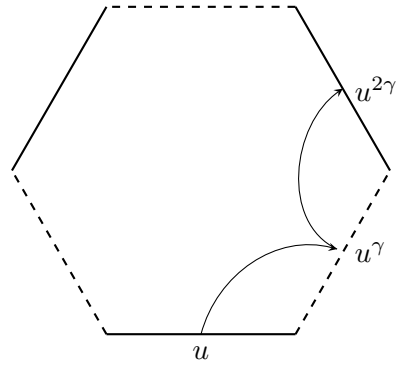


Figure 4.3. Example of two consecutive mirror transformation. The first transformation send an excitation to the next edge. The second transformation is equivalent to a crossing (energy and momentum change sign).

In this way we are able to determine the hexagon form factor for both the single and the two particle case completely. For multi-particle states the symmetry is not enough but we can make the generic ansatz,

$$\mathfrak{h}^{A_1\dot{A}_1\dots A_n\dot{A}_n} = (-1)^f \prod_{i<j} h_{ij} \langle \chi_N^{\dot{A}_n} \dots \chi_1^{\dot{A}_1} | \mathcal{S} | \chi_n^{A_n} \dots \chi_1^{A_1} \rangle \,, \tag{4.26}$$

based on the form of the two and single particle case. The only unknown factor is h_{ij} as $\mathcal{S}$ is the matrix part of the Beisert scattering matrix. In order to fix it completely we require crossing symmetry and that it satisfies the Watson equation [113] for form factors[6], obtaining,

$$h_{12} = \frac{x^- - x^-}{x^- - x^+} \frac{1 - \frac{1}{x^- x^+}}{1 - \frac{1}{x^+ x^+}} \frac{1}{\sigma} \,, \tag{4.27}$$

where σ is again the BES dressing factor and the x_i are the Zhukowsky variables (4.15).

We now want to focus our interest in the closed $SU(2)$ and $SL(2)$ sectors, where we have a single type of excitation, $\{Y, \bar{Y}\}$, or the covariant derivative, $\{D, \bar{D}\}$, respectively. By using conformal symmetry we can set the 3 operators to be located along a line at $\{0, 1, \infty\}$. We make this choice for the three BMN vacua,

$$\mathcal{O}_1 = Tr[Z^{L_1}](0) \quad \mathcal{O}_2 = Tr[\bar{Z}^{L_2}](\infty) \quad \mathcal{O}_3 = Tr[(Z + \bar{Z} + Y - \bar{Y})^{L_3}](1) \,. \tag{4.28}$$

The third operator is simply a rotated BMN vacuum and excitations inserted in $\mathcal{O}_1$ can only be wick contracted with it, see an example in

[6] Another test for the multi-particle ansatz is that it satisfies the decoupling condition for a bound state.

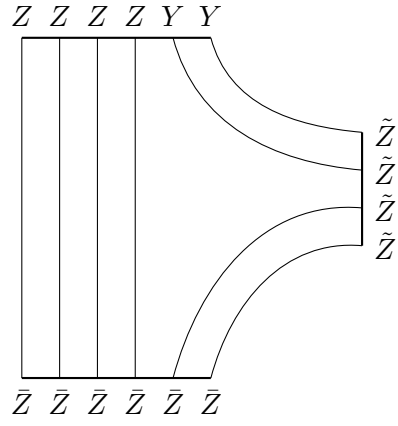


Figure 4.4. An example where we have excitations only on one operator $\mathcal{O}_1$, the top one, in the $SU(2)$ sector. We can see that the excitations can only wick contract with the rotated BPS vacuum $\mathcal{O}_3$.

fig 4.4. With this choice we are able to preserve the $psu(2|2)$ diagonal symmetry. It is extremely important to choose these vacua as we need this residual symmetry to fix the hexagon form factors.

We can now re-write (4.20) using the form of the hexagons obtained. In order to avoid normalizaton factors that come from combinatorics we can divide the structure constant by the BPS structure constant,

$$\frac{C^{\bullet\circ\circ}}{C^{\circ\circ\circ}} = \sqrt{\frac{\prod_k \mu(u_k)}{\mathrm{Det}\partial_{u_i}\phi_j \prod_{i<j} S(u_i,u_j)}}\mathcal{A}\,. \tag{4.29}$$

The measure $\mu(u_k)$, which come from the hexagon form factor, is the cost of producing a single excitation. The Gaudin determinant is the usual factor that appears in form factors computation and it is the normalization of the non-protected state in $\mathcal{O}_1$. The product of S-matrices in the denominator ensures the ansatz is independent of the order chosen for the rapidites in the external operator.. The $\mathcal{A}$ factor contains the different contribution coming from the mirror excitations,

$$\mathcal{A} = \mathcal{A}_{(0,0,0)} + \mathcal{A}_{(1,0,0)} + \mathcal{A}_{(0,0,1)} + \mathcal{A}_{(0,1,0)} + \dots\,, \tag{4.30}$$

where the indices $\{i,j,k\}$ count the number of mirror particles inserted at each edge. The first term $\mathcal{A}_{(0,0,0)}$ correspond to the insertion of the mirror vacuum, with zero excitations when gluing back the mirror edges.

For $Sl(2)$ excitations we can write

$$\mathcal{A}_{(0,0,0)} = \sum_{\alpha\cup\bar{\alpha}=\{u\}} (-1)^{|\bar{\alpha}|} \prod_{\substack{u_l\in\alpha\\ u_k\in\bar{\alpha}}} \omega(u_l,u_k) \prod_{i<j} \prod_{\substack{u_l\in\alpha\\ u_k\in\bar{\alpha}\\ u_l<u_k}} h_{ij}(u_l,u_k) \prod_{\substack{u_t\in\bar{\alpha}\\ u_r\in\alpha\\ u_t<u_r}} h_{ij}(u_t,u_r)\,, \tag{4.31}$$

where the splitting factor ω is the same as (4.22) and h_{ij} is the hexagon form factor in this sector. When the bridges l_{ij} are very large this is the only contribution as the other mirror excitations are suppressed because they come with a factor $e^{-l_{ij}E}$

4.3.1 Opposite and adjacent bridge Contribution

We want to present the form of the integrands for the single mirror excitation in the adjacent and opposite bridge [7].

In general the two different contributions will be proportional, for a single particle, to

$$\sum_{a\geq 1} \int \frac{du}{2\pi} \mu_a^{\gamma}(u) e^{-l_{ij}E_a(u)} Int(u,u_i)\,, \tag{4.32}$$

where we have a sum over the bound states a and integrate over the mirror rapidity u. As before l_{ij} represents the length of the bridge and here we have μ_a^{γ} which is the measure for a mirror bound state. Now we want to explicitly compute the integrand $Int(u,u_i)$ for the case of a particle in the adjacent and opposite mirror edge. Pictorially, following the notation in figure 4.2, we consider the physical edge to be the red one the adjacent excitations are $\{\psi_{12},\psi_{13}\}$ while the opposite is ψ_{23}. From (4.20) the integrand will be proportional to

$$Int(u,u_i) \propto \sum_{\chi=Y,\bar{Y},D,\bar{D}} h_{\chi} h_{\bar{\chi}}\,, \tag{4.33}$$

where χ represent a particle/anti-particle pair of virtual excitation living in one of the mirror edges. For the adjacent contribution this term is proportional to the transfer matrix[8] $T(u^{-\gamma}|u_i)$. The same argument can be done for the opposite bridge where we obtain again a quantity proportional to the transfer matrix $T(u^{\gamma}|u_i)$, but with a different analytic continuation for the mirror rapidity. In principle one could think

[7] In [114] a derivation for the multi particle mirror contribution is obtained.

[8] Where the γ accounts for the mirror transformation.

that the two effects are of the same order, but the different analytic continuations appearing in the argument of the transfer matrix, $-\gamma$ and γ, make them contribute at different orders. They are suppressed up to $l_{ij}+1$-loops for the opposite bridge, while for the adjacent bridge the corrections kick in at $l_{ij}+2$. It is important to mention that the opposite bridge contribution actually completely factorizes[9], for every number of mirror particles, as

$$\mathcal{A}_{(n_i,n_j,n_l)} = \mathcal{A}_{(n_i,0,n_l)}\mathcal{B}_{n_j}\,, \tag{4.34}$$

where $\{n_i, n_j\}$ are the number of adjacent mirror particles and n_j the opposite. Also $\mathcal{B}_{n_j}$ only depends on the length of the opposed bridge.

4.4 Four-Point correlator

We will now shift our attention to four-point function of protected operators. These are the first correlators of protected operator that are not fixed by symmetry,

$$\langle \mathcal{O}_{L_1}(x_1,y_1)\mathcal{O}_{L_2}(x_2,y_2)\mathcal{O}_{L_3}(x_3,y_3)\mathcal{O}_{L_4}(x_4,y_4)\rangle\,, \tag{4.35}$$

where we have used the notation of eq. (4.8). The correlator can be expanded in the t'Hooft coupling λ and, using the method of lagrangian insertions [115], it can be rewritten at every loop order ℓ as a $4+\ell$ tree level integrated correlator

$$\begin{aligned}&\langle \mathcal{O}_{L_1}(x_1,y_1)\mathcal{O}_{L_2}(x_2,y_2)\mathcal{O}_{L_3}(x_3,y_3)\mathcal{O}_{L_4}(x_4,y_4)\rangle_\ell \\ &\sim \int d^4x_5\ldots d^4x_{4+\ell}\,\langle \mathcal{O}_{L_1}\ldots\mathcal{O}_{L_4}\mathcal{L}(x_5)\ldots\mathcal{L}(x_{4+\ell})\rangle_{\text{Born}}\,,\end{aligned} \tag{4.36}$$

where the $\mathcal{L}$ insertions come from a derivative with respect to the coupling λ.

The tree level correlator is a polynomial in the harmonic variables y and a rational functions of x_i. The external points carry harmonic and conformal weight L_i. The internal points, where we have the Lagrangian insertions, carry only conformal weight 4. Super-conformal invariance of the correlator [116–119] imposes that it is proportional to:

$$\begin{aligned}R(1,2,3,4) = {}& d_{12}^2 d_{34}^2 x_{12}^2 x_{34}^2 + d_{13}^2 d_{24}^2 x_{13}^2 x_{24}^2 + d_{14}^2 d_{23}^2 x_{14}^2 x_{23}^2 \\ &+ d_{12}d_{23}d_{34}d_{14}(x_{13}^2x_{24}^2 - x_{12}^2x_{34}^2 - x_{14}^2x_{23}^2) \\ &+ d_{12}d_{13}d_{24}d_{34}(x_{14}^2x_{23}^2 - x_{12}^2x_{34}^2 - x_{13}^2x_{24}^2) \\ &+ d_{13}d_{14}d_{23}d_{24}(x_{12}^2x_{34}^2 - x_{14}^2x_{23}^2 - x_{13}^2x_{24}^2)\,.\end{aligned} \tag{4.37}$$

[9] Possible regularization of the Hexagons can spoil this properties.

As we require the correlator (4.35) to have harmonic weight L_i at the point x_i we need to absorb the missing $L_i - 2$ weight on the external points by a suitable combinations of propagators

$$d_{ij} = \frac{y_{ij}^2}{x_{ij}^2}\,, \qquad y_{ij}^2 \equiv y_i \cdot y_j\,, \qquad x_{ij}^2 \equiv (x_i - x_j)^2\,, \tag{4.38}$$

connecting the external points. The generic form of the $4+\ell$ tree level correlator $G^{(\ell)}_{L_1,L_2,L_3,L_4}$ is then

$$G^{(\ell)}_{L_1,L_2,L_3,L_4} \propto R(1,2,3,4) \sum_{\{b_{ij}\}} \left(\prod_{\substack{i<j \\ i,j\in\mathcal{E}}} (d_{ij})^{b_{ij}} \right) \frac{P^{(\ell)}_{\{b_{ij}\}}(x_1,\ldots,x_{4+\ell})}{\prod\limits_{\substack{p\in\mathcal{E} \\ q\in\mathcal{I}}} x_{pq}^2 \prod\limits_{\substack{p<q \\ p,q\in\mathcal{I}}} x_{pq}^2}\,, \tag{4.39}$$

where we have defined two sets of points, external $\mathcal{E}$ and internal $\mathcal{I}$. Moreover the sum in (4.39) is performed over

$$\{b_{ij}\}^{i<j}_{i,j\in\mathcal{E}} = \{b_{12}, b_{13}, b_{14}, b_{23}, b_{24}, b_{34}\} \tag{4.40}$$

such that $\sum_{j\neq i} b_{ij} = L_i - 2$ for all the external points $i \in \mathcal{E}$. The functions $P^{(\ell)}_{\{b_{ij}\}}$, which are polynomials in the x_i, have conformal weight $(1-\ell)$ at each point in $\mathcal{E}\cup\mathcal{I}$. Finally the polynomials $P^{(\ell)}_{\{b_{ij}\}}$ are invariant under permutation group that leaves invariant the factor

$$\prod_{\substack{i<j \\ i,j\in\mathcal{E}}} (d_{ij})^{b_{ij}}\,. \tag{4.41}$$

At each loop order we can enumerate the possible polynomials $P^{(\ell)}_{\{b_{ij}\}}$ that can appear, so in order to fix the integrand we just have to find the numerical constants multiplying them. In order to do it is useful to rewrite the integrated $P^{(\ell)}_{\{b_{ij}\}}$ in terms of conformal four-point integrals at ℓ-loops , $I^{(\ell)}_k(1,2,3,4)$,

$$\int d^4x_1 \ldots d^4x_{4+\ell} \frac{P^{(\ell)}_{\{b_{ij}\}}(x_1,\ldots,x_{4+\ell})}{\prod\limits_{\substack{p\in\mathcal{E} \\ q\in\mathcal{I}}} x_{pq}^2 \prod\limits_{\substack{p<q \\ p,q\in\mathcal{I}}} x_{pq}^2} = \sum_k c^{(k)}_{\{b_{ij}\}} I^{(\ell)}_k(1,2,3,4)\,, \tag{4.42}$$

where the unknowns are the numerical prefactors $c^{(k)}_{\{b_{ij}\}}$. As we have seen in sec. 3.3.2 four-point conformal integral are invariant under pairwise exchange of the external points, by imposing this and the magic identities as constraints we can reduce the number of integrals appearing in the correlation function.

4.4.1 Structure constants

As we have seen structure constants can be computed by analyzing three-point functions. Structure constant arise also in the OPE expansion of four-point functions, in the case of protected external operators they are usually easier to compute and analyze[10]. In order to extract structure constants from the four-point correlator we have to consider the OPE expansion of the fields contained in the correlator. If we take (4.7) and apply it to the fields at position $\{x_1, x_2\}$ and $\{x_3, x_4\}$ we obtain

$$\langle \mathcal{O}(x_1)\mathcal{O}(x_2)\mathcal{O}(x_3)\mathcal{O}(x_4)\rangle = \sum_k C_k \frac{F_{\Delta_k,S_k}}{x_{12}^4 x_{34}^4}\,, \tag{4.43}$$

where the sum runs over the possible primary operators that can be exchanged and the F_{Δ_k,S_k} are called conformal blocks and are fixed by conformal symmetry. In four dimensions they are known functions.

We are interested in extracting the structure constant of the Konishi operator with the shortest $\frac{1}{2}$-BPS operators[11]. The type of operators that can be exchanged in an OPE depends on the harmonic weight of the external operators. By choosing the polarization y_I accordingly we can single out the $20'$ contributions in the OPE, in this channel only the $SL(2)$ descendant of the Konishi appears[12]. The $SL(2)$ descendant of the Konishi, with twist $\tau = \Delta - S = 2$, can be studied in a small u limit of the four-point function as small twist operators are the leading contributions.

$$\mathcal{O}_{20'}(x, y_2)\mathcal{O}_{20'}(0, y_1) \sim \big(\text{protected}\big) + c_{20'20'\mathcal{K}} \frac{y_{12}^4}{(x^2)^{1-\gamma_{\mathcal{K}}/2}} \mathcal{K}(x) + \dots\,, \tag{4.44}$$

Up to now we have just rewritten the four-point function in a specific limit but in order to extract the value of the structure constant we need to evaluate the integrals in that limit. To do so we use the fact that for external operators of length 2, the integrand is known in the planar limit up to 10-loops[13] [120–123]. In this case there is only one $\{b_{ij}\} = \{0, 0, 0, 0, 0, 0\}$ and the polynomial P is invariant under the

[10] We can also extract more data from the OPE than from a single three-point function at the cost of obtaining sum rules rather than individual OPE data.

[11] This quantity is interesting from the Hexagon proposal as at 4-loops it is necessary to regularize the contribution coming from $\mathcal{A}_{1,0,1}$. This could in principle make the $\mathcal{A}_{1,1,1}$ contribution appear earlier than the expected six loops

[12] As this operator is in the supermultiplet of the Konishi the two structure constant are related by a multiplicative factor.

[13] This has been computed by studying the light-cone limit $x_{12}^2, x_{23}^2, x_{34}^2, x_{14}^2 \to 0$ where the correlator exponentiates and it is possible to relate different orders of the perturbative expansion.

permutation group $\mathcal{S}_{4+\ell}$ of the points x_i [119]. We are interested in the 5-loop contribution

$$\begin{aligned}
P^{(5)} = &-\frac{1}{2}x_{13}^2x_{16}^2x_{18}^2x_{19}^2x_{24}^4x_{26}^2x_{29}^2x_{37}^2x_{38}^2x_{39}^2x_{47}^2x_{48}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2 \\
&+\frac{1}{4}x_{13}^2x_{16}^2x_{18}^2x_{19}^2x_{24}^4x_{26}^2x_{29}^2x_{37}^4x_{39}^2x_{48}^4x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2 \\
&+\frac{1}{4}x_{13}^4x_{17}^2x_{19}^2x_{24}^2x_{26}^2x_{27}^2x_{29}^2x_{36}^2x_{39}^2x_{48}^6x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2 \\
&+\frac{1}{6}x_{13}^2x_{16}^2x_{19}^4x_{24}^4x_{28}^2x_{29}^2x_{37}^4x_{38}^2x_{46}^2x_{47}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{68}^2 \\
&-\frac{1}{8}x_{13}^4x_{16}^2x_{18}^2x_{24}^4x_{28}^2x_{29}^2x_{37}^2x_{39}^2x_{46}^2x_{47}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{69}^2x_{78}^2 \\
&+\frac{1}{28}x_{13}^2x_{17}^2x_{18}^2x_{19}^2x_{24}^8x_{36}^2x_{38}^2x_{39}^2x_{56}^2x_{57}^2x_{58}^2x_{59}^2x_{67}^2x_{69}^2x_{78}^2 \\
&+\frac{1}{12}x_{13}^2x_{16}^2x_{17}^2x_{19}^2x_{26}^2x_{27}^2x_{28}^2x_{29}^2x_{35}^2x_{38}^2x_{39}^2x_{45}^2x_{46}^2x_{47}^2x_{49}^2x_{57}^2x_{58}^2x_{68}^2 \\
&+S_9 \text{ perm}\,. \qquad (4.45)
\end{aligned}$$

The 5-loop integrals appearing in the correlation function are currently unknown but we can study a particular limit where 2 point coincide, as we have described in sec 3.3.2 , where the integrals simplify. From the point of view of the four-point function, when we take this limit we single out the contribution of a single operator in the OPE of (4.7). We can then match the two expressions

$$\sum_{n\geq 0}\lambda^n F^n(x_i) \underbrace{\to}_{x_1\to x_2,\, x_3\to x_4} \frac{1}{6x_{13}^4}(c_{\mathcal{K}}^2(a)u^{\frac{\gamma_{\mathcal{K}}}{2}}-1)(1+O(u)+O(1-v))\,, \qquad (4.46)$$

where in the right hand-side we have the leading contribution of (4.44), and for the left-hand side we introduced the notation

$$F^{(n)} = \frac{x_{12}^2x_{13}^2x_{14}^2x_{23}^2x_{24}^2x_{34}^2}{n!(-4\pi^2)^n}\int d^dx_5\dots d^dx_{4+n}\,\frac{P^{(n)}(x_i)}{\prod_{1\leq i<j\leq 4+n}x_{ij}^2}\,. \qquad (4.47)$$

We are then able to extract the contribution to the structure constant of the Konishi operator $\mathcal{K}$ at 5-loops

$$\begin{aligned}
(C_{\mathcal{K}}^2)_{(5)} = -64(7364&+1812\zeta_3-414\zeta_3^2+2688\zeta_5 \\
&+864\zeta_3\zeta_5+3717\zeta_7+5292\zeta_9)\,. \qquad (4.48)
\end{aligned}$$

4.4.2 Fixing the integrand

Up to now we have fixed the correlator of four protected operators up to a set of numerical factors $c_{\{b_{ij}\}}^{(k)}$ multiplying the conformal integrals

appearing at a loop order ℓ. A first simplification of the integrand is to consider it in the planar limit as only a subset of the conformal integrals will contribute. Secondly we can constrain the form of the integrand by studying the light cone OPE limit [124], $x_{12}^2 \to 0$. In this limit we can relate the light-cone singularities of correlators with different weight and thus reduce the number of coefficients $c^{(k)}_{\{b_{ij}\}}$. Using this constraints the authors of [124] were able to obtain integrand for different protected operators in the planar limit up to 3-loops. The authors also argued that there should exist a saturation bound κ for the coefficients b_{ij}. At higher loops the light-cone OPE is not enough to fix all the coefficients. In order to fix the remaining unknowns in Paper VI we used data coming from integrability. In order to obtain structure constants from the integrand, see sec. 4.4.1, we have to perform an asymptotic expansion. In general this require the knowledge of the ℓ-loop p-Integrals appearing so we can effectively constrain the four-point function integrand up to 5-loops.

After the asymptotic expansion the starting conformal integrals are expressed as

$$F^{\ell}_{\{b_{ij}\}} = \sum_k c^{(k)}_{\{b_{ij}\}} I^{(\ell)}_k(1,2,3,4) \sim \sum_{k=0}^{\ell} \alpha_k \mathrm{Log}(u)^k \,, \tag{4.49}$$

where α_k depend on the unknown coefficients $c^{(k)}_{\{b_{ij}\}}$. By looking at (4.46), as the anomalous dimension depends on the coupling λ, the higher Log powers depend only on lower loop OPE data. As we have seen in sec 4.3.1 in the integrability picture we have the notion of mirror excitations. We know that single mirror excitations are suppressed up to $l_ij + 1$-loops, for opposite bridge, and $l_{ij} + 2$-loops, for adjacent bridge, where l_ij is the length of the adjacent/opposite bridge. With this in mind let us then consider a generic correlator $\langle \mathcal{O}_{L_1}\mathcal{O}_{L_2}\mathcal{O}_{L_3}\mathcal{O}_{L_4}\rangle^{(l)}$, and take the OPE limit in some channel where we have picked the leading twist contributions, $\tau = L$. The same contribution in the integrability picture has an opposite bridge length of

$$l_{\text{opp}} = \frac{L_1 + L_2 - L}{2} \,. \tag{4.50}$$

If we now pick the correlator $\langle \mathcal{O}_{L_1+n}\mathcal{O}_{L_2+n}\mathcal{O}_{L_3}\mathcal{O}_{L_4}\rangle^{(l)}$ and proceed to extract the same OPE contributions we obtain that in this case the opposite bridge is of length $l_{\text{opp}} + n$. The two contributions must then have the same $\mathrm{Log}(u)^k$ coefficients[14] for $k \geq l - l_{\text{opp}}$ given that the opposite bridge start contributing only at $l_{\text{opp}} + 1$-loops.

[14] One has to be careful that only non extremal operators appear as extremal configurations can not be computed with the hexagon approach.

In (4.34) we have seen that the opposite bridge contribution factorizes. We can use this property to enforce further constraints on the coefficients of the four-point integrand. Let us start with a correlator $\langle \mathcal{O}_{L_1}\mathcal{O}_{L_2}\mathcal{O}_{L_3}\mathcal{O}_{L_4}\rangle^{(l)}$ such that $L_2 - L_1 > L_4 - L_3$. In this particular case we have that bridges appearing in the two structure constants C_{12O} and C_{34O} are distinct. Given the factorization property of the opposite contribution we can define a new correlator $\langle \mathcal{O}_{L'_1}\mathcal{O}_{L'_2}\mathcal{O}_{L'_3}\mathcal{O}_{L'_4}\rangle^{(l)}$ with lengths

$$\begin{aligned} L'_1 &= l_{01} + l_{34}\,, & L'_2 &= l_{02} + l_{34}\,, \\ L'_3 &= l_{03} + l_{12}\,, & L'_4 &= l_{04} + l_{12}\,, \end{aligned} \tag{4.51}$$

where l_{ij} is the same as (4.21) and the index 0 represent the operator appearing in the OPE. This type of equality gives constraints on all the $\mathrm{Log}(u)^k$ powers.

This type of constraints is not enough to fix the complete form of the integrand. To fix the remaining coefficients we have to plug in data from integrability computation and directly compare it with the asymptotic expansion. We fix the last coefficient by matching our results with the four-point function computed in [110] using integrability.

Triple wrapping

As we have mentioned it would be useful to understand how the renormalization procedure introduce introduced in [125] for computing the adjacent wrapping contribution, $\mathcal{A}_{(1,0,1)}$, affects the next wrapping effect $\mathcal{A}_{(1,1,1)}$. The structure constant where the triple wrapping should first appear is for the twist 2 operators with two shortest BPS operators at 6-loops.

Having obtained the integrand for the four-point function we can look at the OPE contributions coming from correlators of the form

$$\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_n\mathcal{O}_n\rangle\,, \tag{4.52}$$

for various n. For operators 3 and 4 of length n the opposed bridge in their OPE has length $n-1$, so the opposed wrapping contributes at n-loops. Using $n = 6$ we are able, by using consistency conditions on the sum rules extracted from the correlator, to extract the contribution of adjacent wrappings $\mathcal{A}_{(1,0,0)} + \mathcal{A}_{(0,0,1)} + \mathcal{A}_{(1,0,1)}$ up to five loops. By using sum rules which probe the different opposite bridge lengths we are able to notice a mismatch when all the bridges have length 1 obtaining a prediction for the new triple wrapping effect

$$\begin{aligned} \mathcal{A}_{\{1,1,1\}} = \lambda^5 \big(&11016\zeta_3 - 16200\zeta_5 - 5184\zeta_3^2 + 32130\zeta_7 \\ &-14256\zeta_3\zeta_5 - 9072\zeta_9\big)\,. \end{aligned} \tag{4.53}$$

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Svensk Sammanfattning

De senaste åren har bjudit på en rik och omfattande utveckling av resultat inom kvantfältteorier både med och utan störningsteori. I de fall då störningsteori används, till exempel i beräkningar för partikelacceleratorn LHC som kräver mycket hög precision, bygger den senaste utvecklingen på verktyg från många olika områden inom matematik och datavetenskap för att lösa de beräknings-relaterade problemen som uppstår vid komplicerade interaktioner mellan partiklar. Generellt sett är det flera svårigheter som måste övervinnas när man beräknar kvantiteter relaterade till en spridningsprocess. För det första måste man lyckas generera en integral för själva spridningsprocessen. För det andra måste man reducera antalet integraler man har genom att använda identiteter inom partialintegration, vilket är vad vissa delar av arbetet i denna avhandling har fokuserat på. Resultatet av att använda dessa metoder med partialintegration är att de ursprungliga integralerna kan uttryckas som en linjärkombination av en ändlig mängd "master" integraler.

I Artikel I utvecklar vi en effektiv algoritm för att hitta en sådan bas av "master" integraler genom att använda algebraisk geometri och symmetrier av Feynmandiagram. Förutom att veta denna bas är det också nödvändigt att beräkna de koefficienter som förekommer i identiteterna inom partialintegration. I Artikel III och V utvecklar vi en effektiv metod för att beräkna dessa koefficienter som är baserad på olika moderna metoder såsom den rationella rekonstruktionen av ett ändligt fält $\mathbb{Z}_p$. Tanken bakom detta är att vi kan beräkna koefficienterna som kommer från partialintegrationen numeriskt, och att vi sedan kan rekonstruera koefficienternas analytiska form genom att upprepa beräkningen med andra numeriska värden. Vi utvecklade även ett effektivt sätt att beräkna vad som kallas "module intersection", vilket behövs för att generera startpunkten för identiteterna inom partialintegration. Sista steget i att beräkna en amplitud är att räkna ut dessa "master" integraler. Detta uppnås vanligtvis med hjälp av metoder med differentialekvationer eller om möjligt genom direkt integration. I Artikel IV utvecklar vi en metod för att beräkna värdet av en särskild klass integraler som bara har två externa momenta eller punkter genom en metod som kallas "bootstrap", och vi använder detta för att beräkna integralerna vid fem loopar.

Den andra delen av denna avhandling fokuserar på en specifik teori: $\mathcal{N} = 4$ SYM i fyra dimensioner. Denna teori är känd för att vara

integrerbar i ett visst gränsvärde. Inom detta ramverk finns åtskilliga förslag på hur man kan beräkna kvantiteter vid ändlig kopplingskonstant. I Artikel II använder vi störningsteori för att beräkna så kallade "strukturkonstanter" relaterade till en specifik operator. Detta resultat kan användas som ett intressant test i beräkningen av strukturkonstanter genom att använda vad som kallas "Hexagons". Det förmodas även att en ny effekt börjar verka vid detta antal loopar. I Artikel VI härleder vi hur integranden bör se ut i fallet med fyra externa punkter och fyra särskilda ("protected") operatorer av olika längd med fyra och fem loopar. För att lyckas med detta använder vi en ansatz för integralen som utgångspunkt, för att därefter använda oss av information från strukturkonstanter och integrabilitet för att fastställa de resterande koefficienterna. Intressant nog kunde vi därefter isolera och beräkna hur detta bidrar till "Hexagons".

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