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Category O for Takiff $\mathfrak{sl}_2$

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Abstract. We study basic properties of a natural analogue of the BGG category O for the non-semi-simple Takiff Lie algebra of $\mathfrak{sl}_2$. The main emphasis is on trying to understand the structure of Verma modules and the Gabriel quiver for the blocks of this category.

1 Introduction

The Takiff algebra is defined to be a Lie algebra over a truncated polynomial ring. Takiff algebras were used by Takiff in 1971 to describe, in some cases, the ring of polynomial over an algebra invariant under the adjoint action.

In this thesis we take the construction of the original Takiff algebra of $\mathfrak{sl}_2$ and study the natural analogue of the category O, or sometimes called category BGG. For the original definition of the category O we will need a semi-simple Lie algebra, but in our case it turns out that the Takiff algebra of $\mathfrak{sl}_2$ is not semi-simple, and therefore we have to alter our definition of category O to ensure that the Verma modules are in fact in this category. This is due to the fact that the study of Verma modules is a center piece when studying the category O. In our case we no longer require that the objects are weight modules, we instead take them to be generalized weight modules.

The main focus of this paper lies in the study of Verma modules. One of the reasons why the Verma modules are so important is because when we later trying to decompose our category in terms of the simples objects, we will need some theory for the simple objects to compute everything explicitely. This is where the Verma modules comes in. It turns out that every simple object is a quotient of some Verma module. Not only do we compute the block decomposition for our category, but we also compute the Gabriel quivers for all the different blocks.

The outline for this paper is as follows: In section 2 we define our Takiff algebra and find its triangular decomposition. In section 3 we define the universal enveloping algebra together with some of its properties. Section 4 studies the Casimir elements in our universal enveloping algebra for Takiff $\mathfrak{sl}_2$. In section 5 we define the thick category $\tilde{O}$ and proving some basic properties of this category. Section 6 introduces Verma modules and deducing a formula for all submodules of the Verma modules. In section 7 we will derive the Hasse diagrams for our Verma modules. In section 8 we study the block decomposition for our category. Lastly, section 9 we will introduce all the Gabriel quivers for all of the different blocks in our category.

2 The Takiff algebra of $\mathfrak{sl}_2$

Definition 2.1. *The generelized Takiff algebra, or sometimes called the truncated current Lie algebra,* $\mathfrak{g}$ *of the Lie algebra* $\mathfrak{sl}_2$ *is defined as*

$$\mathfrak{g} = \mathfrak{sl}_2 \otimes_{\mathbb{C}} \mathbb{C}[x] \Big/ \mathbb{C}\langle x^{n+1}\rangle, \tag{1}$$

with the Lie bracket defined as $[a \otimes x^i, b \otimes x^j] = [a, b] \otimes x^{i+j}$.

We get the original Takiff algebra by setting $n = 1$, which will be the main focus in this paper. Since $\mathfrak{sl}_2$ is of complex dimension 3 we get that $\mathfrak{g}$ is of complex dimension 6. The Lie algebra $\mathfrak{g}$ will have the basis $\{f, h, e, \bar{f}, \bar{h}, \bar{e}\}$, where f, h and e is the standard basis for $\mathfrak{sl}_2$ and $\bar{f} = f \otimes x$, $\bar{h} = h \otimes x$ and $\bar{e} = e \otimes x$. Note that the subalgebra $span\{\bar{f}, \bar{h}, \bar{e}\}$ is a non-zero abelian ideal, implying that $\mathfrak{g}$ is not semi-simple.

Let $\mathfrak{g}_0 = span\{f, h, e\}$ and $\mathfrak{g}_1 = span\{\bar{f}, \bar{h}, \bar{e}\}$. Then $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$. With this our $\mathfrak{g}$ will in fact be a super Lie algebra. Following [2], for $i \in \mathbb{Z}_2$ we can define Δ_i as the set $\alpha \in \mathfrak{h}_0^*$ such that $\alpha \neq 0$ and $\mathfrak{g}_i^{(\alpha)} \neq 0$. Here, using the notation in [2], $\mathfrak{g}_i^{(\alpha)}$ is the maximal subspace of $\mathfrak{g}$ such that $h - \alpha(h)$ acts locally nilpotent. Setting $\Delta = \Delta_0 \cup \Delta_1$. The elements in Δ will be the roots of $\mathfrak{g}$. Now the root lattice Q is the subgroup of $\mathfrak{h}^*$ generated by Δ.

Note that $Q = 2\mathbb{Z}h^*$. Thus taking the linear map $l : Q \to \mathbb{Z}$ where $2nh^* \mapsto 2n$ we get a triangular decomposition of $\mathfrak{g}$ as $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{n}_0 \oplus \mathfrak{n}_-$, where $\mathfrak{n}_+ = \bigoplus_{l(\alpha)>0} \mathfrak{g}^{(\alpha)}$, $\mathfrak{n}_0 = \bigoplus_{l(\alpha)=0} \mathfrak{g}^{(\alpha)}$ and $\mathfrak{n}_- = \bigoplus_{l(\alpha)<0} \mathfrak{g}^{(\alpha)}$. Note that $\mathfrak{n}_0 = \mathfrak{h}$. We will denote $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$.

3 Universal enveloping algebra

Definition 3.1. *Let* $\mathfrak{g}$ *be a Lie algebra over* $\mathbb{C}$. *The universal enveloping algebra of* $\mathfrak{g}$ *is defined as the quotient*

$$U(\mathfrak{g}) = T(\mathfrak{g})/I(\mathfrak{g}), \tag{2}$$

where $T(\mathfrak{g})$ *is the tensor algebra of* $\mathfrak{g}$ *and* $I(\mathfrak{g})$ *the two-sided ideal of* $T(\mathfrak{g})$ *generated by all elements of the form* $xy - yx - [x, y]$, *where* x *and* y *are elements of* $\mathfrak{g}$.

Definition 3.2. *A filtered ring is a ring S with a family $\{F_n | n = 0, 1, 2, \dots\}$ of subgroups of S^+ such that*

1. *for each i, j, $F_i F_j \subset F_{i+j}$*

2. *for $i < j$, $F_i \subset F_j$*

3. *$\bigcup F_n = S$*

The family $\{F_n\}$ is called a filtration of S.

Note that $U(\mathfrak{g})$ is an extension of the subring $\mathbb{C}$, with generators $\{f, h, e, \bar{f}, \bar{h}, \bar{e}\}$. With this we can define a filtration for $U(\mathfrak{g})$ in the following way. An element in $U(\mathfrak{g})$ of the form

$$cx_1x_2 \dots x_n \tag{3}$$

where $c \in \mathbb{C}$ and $x_i \in \{f, h, e, \bar{f}, \bar{h}, \bar{e}\}$ for all $i = 1, 2, \dots, n$ is called a word of length n. Let us define F_i as the additive subgroup of $U(\mathfrak{g})$ consisting of words of length i or less, $F_0 = \mathbb{C}$. This family $\{F_i\}$ is a filtration of $U(\mathfrak{g})$.

Definition 3.3. *A graded ring is a ring T together with a family $\{T_n | n = 0, 1, 2, \dots\}$ of subgroups of T^+ such that*

1. *$T_i T_j \subset T_{i+j}$, and*

2. *$T = \oplus_n T_n$, as abelian groups.*

The family $\{T_n\}$ is called a grading of T. A non-zero element of T_n is said to be homogeneous of degree n.

Let S be a filtered ring. Then one can construct a graded ring T in the following way. Define $T_n = F_n / F_{n-1}$ and $T = \oplus T_n$. It is enough to consider the homogeneous elements when defining the multiplication. Let $a \in F_n \backslash F_{n-1}$, then a have degree n and $\bar{a} = a + F_{n-1}$ is the leading term of a. Now let b be of degree m. We can define $\bar{a}\bar{b} = ab + F_{n+m-1}$. This is well-defined and makes T into a graded ring called $\mathrm{gr}(S)$, which is denoted by the associated graded ring of S.

The filtration discussed above leads to the associated graded ring $\mathrm{gr}(U(\mathfrak{g}))$.

Theorem 3.4. *If S is a filtered ring and $gr(S)$ is left or right Noetherian then S is left or right Noetherian respectively.*

Proof. Let $I \subset S$ be a right ideal of S. Then we can define the right ideal $\mathrm{gr}(I)$ of $\mathrm{gr}(S)$ by setting $(\mathrm{gr}(I))_n = (I + F_{n-1}) \cap F_n/F_{n-1}$, and

$$\mathrm{gr}(I) = \bigoplus_n (\mathrm{gr}(I))_n. \tag{4}$$

Note that $(I + F_{n-1}) \cap F_n/F_{n-1} \simeq (I \cap F_n)/(I \cap F_{n-1})$. If $\bar{a} \in (\mathrm{gr}(I))_n$ and $\bar{r} \in (\mathrm{gr}(I))_m$ then

$$\bar{a}\bar{r} = ar + F_{n+m-1} \in (I + F_{n-1}) \cap F_n/F_{n-1}. \tag{5}$$

This implies that $\mathrm{gr}(I)$ is in fact a right ideal of $\mathrm{gr}(S)$. Similarly, every left ideal $I \subset S$ one can define a left ideal $\mathrm{gr}(I) \subset \mathrm{gr}(U(\mathfrak{g}))$.

Hence if $\mathrm{gr}(S)$ does not have any strictly ascending chain of left or right ideals, neither has S. □

Proposition 3.5. *The associated graded ring $gr(U(\mathfrak{g}))$ is a commutative $\mathbb{C}$-algebra, generated over $\mathbb{C}$ by the set $\{f, h, e, \bar{f}, \bar{h}, \bar{e}\}$.*

Proof. The relation $x_i x_j - x_j x_i = [x_i, x_j]$ in $U(\mathfrak{g})$ shows that

$$\bar{x}_i \bar{x}_j - \bar{x}_j \bar{x}_i = [x_i, x_j] + F_1 \tag{6}$$

in F_2/F_1. The fact that $\mathfrak{g}$ is a Lie algebra implies that $[x_i, x_j] \in F_1$, forcing $\bar{x}_i \bar{x}_j = \bar{x}_j \bar{x}_i$. In other words, $\mathrm{gr}(U(\mathfrak{g}))$ is commutative. The rest follows from the definition of the filtration of $U(\mathfrak{g})$. □

Corollary 3.6. *Let $\mathfrak{g}$ be a Lie algebra. If $\mathfrak{g}$ is finite dimensional then $U(\mathfrak{g})$ is a Noetherian ring.*

Proof. It is known that every finitely generated commutative algebra over $\mathbb{C}$ is Noetherian. Thus by Proposition 3.5 we have that $\mathrm{gr}(U(\mathfrak{g}))$ is Noetherian. By Theorem 3.4 $U(\mathfrak{g})$ is Noetherian. □

The following theorem will be useful when we are working with universal algebras.

Theorem 3.7. *(Poincaré–Birkhoff–Witt theorem) The set of monomials*

$$\{f^{k_1} \bar{f}^{k_2} h^{k_3} \bar{h}^{k_4} e^{k_5} \bar{e}^{k_6} | k_1, k_2, \ldots, k_6 \in \mathbb{Z}_{>0}\} \tag{7}$$

forms a basis for $U(\mathfrak{g})$.

4 Casimir elements of $U(\mathfrak{g})$

Following [8]. Let E_{ij} for $i, j = 1, 2$ be the standard basis for $\mathfrak{gl}_2$ and let $E_{ij}^{(k)} = E_{ij} \otimes x^k$, where $k = 0, 1$, be the basis for $\mathfrak{g}_2 = \mathfrak{gl}_2 \otimes_{\mathbb{C}} \mathbb{C}[x] \big/ \mathbb{C}\langle x^{n+1}\rangle$. Then the elements

$$\begin{aligned}
\theta_1 &= E_{11}^{(0)} + E_{22}^{(0)} - 2 \\
\theta_2 &= E_{11}^{(1)} + E_{22}^{(1)} \\
\theta_3 &= E_{11}^{(0)} E_{22}^{(1)} + E_{11}^{(1)} (E_{22}^{(0)} - 2) - E_{21}^{(0)} E_{12}^{(1)} - E_{21}^{(1)} E_{12}^{(0)} \\
\theta_4 &= E_{11}^{(1)} E_{22}^{(1)} - E_{21}^{(1)} E_{12}^{(1)}
\end{aligned} \tag{8}$$

generate $Z(\mathfrak{g}_2)$.

Proposition 4.1. *The elements*

$$\begin{aligned}
c_1 &= \frac{1}{4}\bar{h}^2 + \frac{1}{2}\bar{h}(\frac{1}{2}h - 2) - f\bar{e} - \bar{f}e \\
c_2 &= \frac{1}{4}\bar{h}^2 - \bar{f}\bar{e}
\end{aligned} \tag{9}$$

generate $Z(\mathfrak{g})$.

Proof. To get the elements in $Z(\mathfrak{g})$ from $Z(\mathfrak{g}_2)$ we can set $E_{11} + E_{22} = 0$ since we want the trace to be zero. From this we have that $E_{11}^{(k)} = \frac{1}{2}h \otimes x^k$, $E_{22}^{(k)} = \frac{1}{2}h \otimes x_k$, $E_{12}^{(k)} = e \otimes x^k$ and $E_{21}^{(k)} = f \otimes x^k$. $\square$

5 Defining the Thick Category $\tilde{O}$

Definition 5.1. *The category $\tilde{O}$ for $\mathfrak{g}$ is defined as the full subcategory of U-mod consisting of modules M on which the action of $U(\mathfrak{b})$ is locally finite in the sense that* $\dim U(\mathfrak{b})v < \infty$ *for all $v \in M$.*

The first thing we need to prove are some basic properties of $\tilde{O}$.

Proposition 5.2. *Let $M \in \tilde{O}$. If $N \subset M$ is a submodule, then $N \in \tilde{O}$.*

Proof. Let $N \subset M$ be a submodule. By definition of M is finitely generated. Let $\{m_1, m_2, \ldots, m_n\}$ generate M. Then for all $i = 1, 2, \ldots, n$ there exists a subalgebra $W_i \subset U(\mathfrak{g})$ such that $W_i(m_i) = U(m_i) \cap N$. Now by corollary 3.6 our subring W_i is finitely generated. From this we can conclude that $\bigcup_{i=1}^n \operatorname{Gen}(W_i)m_i$ generate N and that $|\bigcup_{i=1}^n \operatorname{Gen}(W_i)m_i| < \infty$. $\square$

Proposition 5.3. *Let $M \in \tilde{O}$. If $N \subset M$ then $M/N \in \tilde{O}$.*

Proof. Let $\{m_1, m_2, \ldots, m_n\}$ be the generating set for M. Then $M/N \in U$*-mod* follows from the fact that $\{m_1 + N, m_2 + N, \ldots, m_n + N\}$ generates M/N.

To show that $U(\mathfrak{b})$ acts locally finite on M/N we need to show that $\dim U(\mathfrak{b})(v+N) < \infty$ for all $v + N \in M/N$. Since N is closed under the action of $U(\mathfrak{g})$ we get that $U(\mathfrak{b})(v+N) = \{w + N | w \in U(\mathfrak{b})v\}$. In other words, $U(\mathfrak{b})(v+N) = U(\mathfrak{b})v/N$. Which has lower dimension than $U(\mathfrak{b})$. Thus $\dim U(\mathfrak{b})(v+N) < \dim U(\mathfrak{b})v < \infty$. $\square$

Proposition 5.4. *Let $0 \xrightarrow{f} X \to Y \xrightarrow{g} Z \to 0$ be a short exact sequence. If $X, Z \in \tilde{O}$ then $Y \in \tilde{O}$.*

Proof. Let $\{x_1, x_2, \dots, x_n\}$ and $\{z_1, z_2, \dots, z_m\}$ be the generating set for X and Z respectively. Exactness implies that g is surjective. Thus there exists $y_i \in Y$ such that $g(y_i) = z_i$ for all $i = 1, 2, \dots, m$. Now I claim that $\{f(x_1), f(x_2), \dots, f(x_n), y_1, y_2, \dots, y_m\}$ is a generating set for Y. Let $v \in Y$. Now

$$g(v) = \sum_{i=1}^{m} r_i z_i \tag{10}$$

where $r_i \in U(\mathfrak{g})$ for all $i = 1, 2, \dots, m$. Using the fact that g is a homomorphism we can rewrite the right hand to

$$\sum_{i=1}^{m} r_i z_i = \sum_{i=1}^{m} r_i g(y_i) = g\left(\sum_{i=1}^{m} r_i y_i\right). \tag{11}$$

Clearly we have that $g\left(v - \sum_{i=1}^{m} r_i y_i\right) = 0$, and thus $v - \sum_{i=1}^{m} r_i y_i \in \ker g$. Exactness at Y yields that $v - \sum_{i=1}^{m} r_i y_i \in \operatorname{Im} f$. Hence there exists $s_k \in U(\mathfrak{g})$ for all $k = 1, 2, \dots, n$ such that $f\left(\sum_{k=1}^{n} s_k x_k\right) = v - \sum_{i=1}^{m} r_i y_i$. Using the fact that f is an homomorphism and solving for v gives us

$$v = \sum_{k=1}^{n} s_k f(x_k) + \sum_{i=1}^{m} r_i y_i. \tag{12}$$

Hence Y is generated by $\{f(x_1), f(x_2), \dots, f(x_n), y_1, y_2, \dots, y_m\}$.

Left to show that for any $v \in Y$ we have that $\dim U(\mathfrak{b})v < \infty$. By above we can write $v = \sum_{k=1}^{n} s_k f(x_k) + \sum_{i=1}^{m} r_i y_i$. Then

$$\dim U(\mathfrak{b})v \leq \dim U(\mathfrak{b}) \sum_{k=1}^{n} s_k f(x_k) + \dim U(\mathfrak{b}) \sum_{i=1}^{m} r_i y_i. \tag{13}$$

Here f and g will induce an isomorphisms between

$$U(\mathfrak{b}) \sum_{k=1}^{n} s_k x_k \simeq U(\mathfrak{b}) \sum_{k=1}^{n} s_k f(x_k) \tag{14}$$

and the inequality

$$\dim U(\mathfrak{b}) \sum_{i=1}^{m} r_i y_i \leq \dim U(\mathfrak{b}) \sum_{i=1}^{m} r_i z_i \tag{15}$$

respectively. Since $X, Z \in \tilde{O}$ we have that the left hand side in the above expressions are finite, which implies that $\dim U(\mathfrak{b})v < \infty$. $\square$

Definition 5.5. *A non-empty full subcategory T of an abelian category A (of modules over a ring R) is a Serre subcategory if for any exact sequence*

$$0 \to M \to M' \to M'' \to 0 \tag{16}$$

then M' is in T if and only if M and M'' are in T.

Corollary 5.6. *The category $\tilde{O}$ is a Serre subcategory of $U(\mathfrak{g})$-mod.*

Proof. It follows directly from Proposition 5.2, Proposition 5.3 and Proposition 5.4. $\square$

Definition 5.7. *Let $\lambda \in \mathbb{C}$ and $M \in \tilde{O}$. The generalized weight space M_λ is defined as*

$$M_\lambda = \{v \in M | (h - \lambda(h))^k v = 0, \forall h \in \mathfrak{h}, k \gg 0\} \tag{17}$$

A $\mathfrak{g}$-module M is called a generalized weight module if $M = \bigoplus_\lambda M_\lambda$. We will denote the support of M by $\mathrm{Supp}(M)$, which is the set of all $\lambda \in \mathbb{C}$ such that $M_\lambda \neq 0$. The following two propositions will give us a clue on how the generalized weight modules behave in our objects of $\tilde{O}$.

Remark 5.8. *By the PBW-theorem each generator v for an object $M \in \tilde{O}$ can be written as $v = \sum_{i=1}^n v_i$ where each $v_i \in M_{\lambda_i}$ for some $\lambda_i \in \mathbb{C}$. Now since all the generalized weight spaces are invariant under the action of $\mathfrak{h}$ we can use powers of $h - \lambda_i(h)$ for all $i = 1, \ldots, n$ to prove that all $v_i \in M$ for all $i = 1, \ldots, n$.*

Proposition 5.9. *Let $M \in \tilde{O}$. There exists $\lambda_1, \lambda_2, \ldots, \lambda_k$ such that*

$$\mathrm{Supp}(M) \subset \bigcup_{i=1}^n (\lambda_i + 2\mathbb{Z}). \tag{18}$$

Proof. Let $\{m_1, m_2, \ldots, m_n\}$ be the generating set for M. By the remark 5.8 we can assume that all $m_i \in M_{\lambda_i}$ for some $\lambda_i \in \mathbb{C}$. The roots in $\mathfrak{g}$ implies that

$$\begin{aligned} h(em_i) &= (\lambda_i + 2)em_i \\ h(\bar{e}m_i) &= (\lambda_i + 2)\bar{e}m_i \\ h(fm_i) &= (\lambda_i - 2)fm_i \\ h(\bar{f}m_i) &= (\lambda_i - 2)\bar{f}m_i. \end{aligned} \tag{19}$$

This immediately yields that $\mathrm{Supp}(M) \subset \bigcup_{i=1}^n (\lambda_i + 2\mathbb{Z})$. $\square$

Proposition 5.10. *Let $M \in \tilde{O}$. For all $\lambda \in \mathbb{C}$ we have that $\dim M_\lambda < \infty$.*

Proof. Let $m = \{m_1, m_2, \dots, m_n\}$ be the generating set for M. By the definition of $M \in \tilde{O}$ we have that $\dim U(\mathfrak{b})(m_i) < \infty$, which implies that $\dim U(\mathfrak{b})(m) < \infty$. With this we can write $M = U(\mathfrak{n}_-)U(\mathfrak{b})(m)$. Without loss of generality we can assume that m is a basis for $U(\mathfrak{b})(m)$. Thus $M = U(\mathfrak{n}_-)(m)$. Let $U(\mathfrak{n}_-)_{\mu-\lambda_i} = \{f^a \bar{f}^b | a + b = \mu - \lambda\}$. Clearly $\dim U(\mathfrak{n}_-)_{\mu-\lambda_i} < \infty$ and together with the fact that

$$M_\mu = \sum_{i=1}^{n} U(\mathfrak{n}_-)_{\mu-\lambda_i} m_i, \tag{20}$$

shows that $\dim M_\mu < \infty$. □

Proposition 5.11. *For all $M, N \in \tilde{O}$, the dimension* $\dim Hom(M, N) < \infty$.

Proof. Let $\{m_1, m_2, \dots, m_n\}$ be the generating set for M. A homomorphism $f : M \to N$ must satisfy that if $v \in M$ and $u \in U(\mathfrak{g})$ then $f(uv) = uf(v)$, since it is a module homomorphism. This implies that f is completely defined by what it does on the generating set. Using Remark 5.8 let $\lambda_1 = h(m_1)$. By Proposition 5.10 we have that $\dim N_{\lambda_i} < \infty$ for all $i = 1, 2, \dots, n$. Thus $\dim \mathrm{Hom}(U(\mathfrak{g})(m_i), N_{\lambda_i}) < \infty$. This is true for all $i = 1, 2, \dots, n$, hence $\dim \mathrm{Hom}(M, N) < \infty$. □

Proposition 5.12. *If $M = U\langle v_1, v_2, \dots, v_n\rangle$, where $v_1, v_2, \dots, v_n \in M$, and if for every $i = 1, 2, \dots, n$ the dimension* $\dim U(\mathfrak{b})v_i < \infty$ *then $M \in \tilde{O}$.*

Proof. By the remark 5.8 we can assume that all $v_i \in M_{\lambda_i}$ for some $\lambda_i \in \mathbb{C}$. Since e and $\bar{e}$ raises the weight by 2 and that all M_λ are h-invariant and $\bar{h}$-invariant we have that $\dim U(\mathfrak{g})v_i < \infty$ implies that there exists $N_i \in \mathbb{Z}$ such that if $\lambda \geq \lambda_i + N_i$ then $U(\mathfrak{g})v_i \cap M_\lambda = 0$. Let $N = \inf\{\lambda_i + N_i | i = 1, 2, \dots, n\}$. Then if $\lambda \geq N$ then $M \cap M_\lambda = 0$ since M is finitely generated. Using Proposition 5.10 we can conclude that $\dim U(\mathfrak{g})v < \infty$ if $v \in M_\lambda$ for some $\lambda \in \mathbb{C}$.

Now let $v \in M$ be an arbitrary element. We can write $v = \sum_{i=1}^{m} w_i$ for some $w_i \in M_{\lambda_{w_i}}$. Now

$$\dim U(\mathfrak{g})v = \dim U(\mathfrak{g}) \sum_{i=1}^{m} w_i \leq \dim \sum_{i=1}^{m} U(\mathfrak{g})w_i < \infty \tag{21}$$

since $\dim U(\mathfrak{g})w_i < \infty$ for all $i = 1, 2, \dots, m$ □

6 Verma Modules

For $\lambda \in \mathfrak{h}^*$, define $\mathbb{C}_\lambda$ to be the one-dimensional $\mathfrak{b}$-module with generator v_λ such that

$$\mathfrak{n}_+ v_\lambda = 0, \quad Hv_\lambda = \lambda(H)v_\lambda, \quad \forall H \in \mathfrak{h}. \tag{22}$$

Definition 6.1. *The Verma module is defined as follows:*

$$\Delta(\lambda) = \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathbb{C}_\lambda \cong U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda. \tag{23}$$

From the definition and the PBW-theorem it is clear that $\{f^i\bar{f}^j \otimes v_\lambda | i, j \in \mathbb{Z}_{\geq 0}\}$ forms a basis for $\Delta(\lambda)$. Let us begin by writing down how the basis for $\mathfrak{g}$ acts on $\Delta(\lambda)$. For an arbitrary basis vector for $\Delta(\lambda)$ we get that

$$\begin{aligned} hf^i\bar{f}^j \otimes v_\lambda &= (\lambda(h) - 2i - 2j)f^i\bar{f}^j \otimes v_\lambda \\ \bar{h}f^i\bar{f}^j \otimes v_\lambda &= \lambda(\bar{h})f^i\bar{f}^j \otimes v_\lambda - 2if^{i-1}\bar{f}^{j+1} \otimes v_\lambda \\ ef^i\bar{f}^j \otimes v_\lambda &= i(\lambda(h) - i - 2j + 1)f^{i-1}\bar{f}^j \otimes v_\lambda + j\lambda(\bar{h})f^i\bar{f}^{j-1} \otimes v_\lambda \\ \bar{e}f^i\bar{f}^j \otimes v_\lambda &= i\lambda(\bar{h})f^{i-1}\bar{f}^j \otimes v_\lambda - i(i-1)f^{i-2}\bar{f}^{j+1} \otimes v_\lambda. \end{aligned} \tag{24}$$

With some abusive notation let v_λ be $1 \otimes v_\lambda$.

Lemma 6.2. *For any $\lambda \in \mathfrak{h}^*$, $\Delta(\lambda) \in \tilde{O}$.*

Proof. It is finitely generated since v_λ generates $\Delta(\lambda)$.

Left to show that the action of $U(\mathfrak{b})$ is locally finite. Let $v \in \Delta(\lambda)$. We can write

$$v = \sum_{p=1}^{m} \sum_{\substack{i,i\geq 0 \\ i+j=k}} a_{ij}f^i\bar{f}^j \otimes v_\lambda, \tag{25}$$

where $a_{ij} \in \mathbb{C}$ and $m \in \mathbb{Z}_{\geq 0}$. Let W be a vector space with the basis

$$\{f^i\bar{f}^j \otimes v_\lambda | 0 \leq i + j \leq m\}. \tag{26}$$

Clearly W is finite dimensional and also $v \in W$. From (24) we get that W is $U(\mathfrak{b})$-invariant. Thus $\dim U(\mathfrak{b})(v) \leq \dim W < \infty$. $\square$

The goal of this section is to examine the submodules of $\Delta(\lambda)$ for all $\lambda \in \mathfrak{h}^*$.

Lemma 6.3. *Let $T \in Mat_{n\times n}(\mathbb{C})$ and let $W \in \mathbb{C}^n$ be a non-zero subspace of $\mathbb{C}^n$ such that $T(W) \subset W$. Then W contains a weight vector of T.*

Proof. Let $V \in Mat_{n\times r}(\mathbb{C})$ such that the columns of V form a basis for W. As a T-invariant subspace we have that $TV = VU$, where $\in Mat_{r\times r}(\mathbb{C})$. A consequence of the fundamental theorem of algebra we know that every linear operator must have a non-zero weight vector. Let x be a non-zero weight vector for U with weight λ. Now $Ux = \lambda x$ and acting with V gives us $TVx = VUx = \lambda Vx$. Hence Vx is a weight vector of T and we also have that $Vx \neq 0$ because V is linearly independent, also $Vx \in W$ by the construction of V. $\square$

Note that $\Delta(\lambda)_{\lambda-2kh^*} = span\{f^i\bar{f}^j \otimes v_\lambda | i + j = k\}$. With this we can write our Verma modules as $\Delta(\lambda) = \bigoplus_{k\in\mathbb{N}} \Delta(\lambda)_{\lambda-2kh^*}$.

Lemma 6.4. *If $\lambda \in \mathbb{C}$ and $N \subset \Delta(\lambda)$ is a non-zero submodule, then there exists a non-zero $v \in N$ such that $\bar{h}v = \lambda(\bar{h})v$.*

Proof. Let N be a non-zero sub-module of $\Delta(\lambda)$. Then there exists $\mu \in \mathfrak{h}^*$ such that $N_\mu \neq 0$. From Proposition 5.10 we have that $\dim N_\mu < \infty$. From the relations (24) we have that N_λ is $\bar{h}$-invariant. Using Lemma 6.3 we get that there exists a weight vector v of $\bar{h}$. Last thing we need to check is that the weight for v must be $\lambda(\bar{h})$. This follows from the fact that the action of $\bar{h}$ onto $\Delta(\lambda)_{\lambda-2kh^*}$ in matrix form is

$$[\bar{h}] = \begin{bmatrix} \lambda(\bar{h}) & & & & \\ -2k & \lambda(\bar{h}) & & & \\ & -2(k-1) & \lambda(\bar{h}) & & \\ & & \ddots & \ddots & \\ & & & -2 & \lambda(\bar{h}) \end{bmatrix}, \tag{27}$$

in the basis $\{f^k \otimes v_\lambda, f^{k-1}\bar{f} \otimes v_\lambda, \dots, \bar{f}^k \otimes v_\lambda\}$. This is true for all $k \in \mathbb{N}$, thus forcing the weight to be $\lambda(\bar{h})$. □

Furthermore, a weight vector for $\bar{h}$ in $\Delta(\lambda)_{\lambda-2kh^*}$ is a linear combination of elements of the form $\bar{f}^k \otimes v_\lambda$. With this we can prove our first main result.

Theorem 6.5. *$\Delta(\lambda)$ is simple if and only if $\lambda(\bar{h}) \neq 0$.*

Proof. Let $N \subset \Delta(\lambda)$ be a non-zero submodule. By Lemma 6.4 there exists a weight vector $v \in N$ of $\bar{h}$. This weight vector will be on the form

$$v = p(\bar{f}) \otimes v_\lambda, \tag{28}$$

where $p(t) \in \mathbb{C}[t]$ with degree $l \in \mathbb{Z}$ because how $\bar{h}$ acts on $\Delta(\lambda)_{\lambda-2kh^*}$. Acting with e^l on v yields

$$e^l v = l\frac{(l+1)}{2}\lambda(\bar{h})^l p_l v_\lambda, \tag{29}$$

where p_l is the leading coefficient of $p(t)$. By assumption $\lambda(\bar{h}) \neq 0$ implies that $v_\lambda \in N$, but v_λ generates $\Delta(\lambda)$, thus $N = \Delta(\lambda)$. Hence $\Delta(\lambda)$ is simple. □

Theorem 6.6. *If $\lambda(\bar{h}) = 0$ and $\lambda(h) \notin \mathbb{Z}_{\geq 0}$ then all submodules of $\Delta(\lambda)$ are of the form $U(\mathfrak{n}_-)(\bar{f}^j \otimes v_\lambda)$ for some $j \in \mathbb{Z}_{\geq 0}$.*

Proof. Let $N \subset \Delta(\lambda)$ be a non-zero submodule. By Lemma 6.4 we have a weight vector $v \in N$ for $\bar{h}$. As before we can write it as $v = p(\bar{f}) \otimes v_\lambda$ where $p(t) \in \mathbb{C}[t]$ with degree $l \in \mathbb{Z}$. We also have that there exists an $a \in \mathbb{N}$ such that $p(t) = t^a\bar{p}(t)$, where $\bar{p}(t)$ has a constant term. Let us now act with h on v.

$$hv = \sum_{k=a}^{l} p_k(\lambda(h) - 2k)\bar{f}^k \otimes v_\lambda \tag{30}$$

Where the p_k's are the coefficients in $p(t)$. Since $\lambda(h) \notin \mathbb{Z}$ we have that $\lambda(h) - 2k \neq 0$ for all $k \in \mathbb{Z}$. By taking

$$v - \frac{1}{\lambda(h) - 2l} hv \tag{31}$$

we get a polynomial with

$$\deg(v - \frac{1}{\lambda(h) - 2l} hv) < l. \tag{32}$$

Inductively we can reduce our weight vector to a weight vector of the form $\bar{f}^a \otimes v_\lambda$. Hence $\bar{f}^a \otimes v_\lambda \in N$. Let $j = \inf\{a | \bar{f}^a \otimes v_\lambda \in N\}$. We want to show that $\bar{f}^j \otimes v_\lambda$ generates N. Assume that it does not generate N. Then we can assume that there exists

$$w \notin U(\mathfrak{g})(\bar{f}^j \otimes v_\lambda). \tag{33}$$

Since all the terms with higher exponents over $\bar{f}$ than j lies in $U(\mathfrak{g})(\bar{f}^j \otimes v_\lambda)$ we can rewrite w as

$$w = \sum_{k=0}^{a-1} q_k(f) \bar{f}^k \otimes v_\lambda \tag{34}$$

where $q_k \in \mathbb{C}[t]$ for all $k = 0, \dots, a-1$. Now let $b = \sup\{\deg q_k | k = 0, \dots, a-1\}$. By assumption we have that $\lambda(\bar{h}) = 0$ which implies that acting with e only reduces the exponent on f. Making $e^b w$ a weight vector for $\bar{h}$ since all the terms has a zero exponent on f. By above we can reduce this weight vector to a weight vector of the form $\bar{f}^g \otimes v_\lambda$ where $g < j$. This contradicts the minimality of j. Now (24) shows that $N = U(\mathfrak{n}_-)(\bar{f}^j \otimes v_\lambda)$. $\square$

Theorem 6.7. *If $\lambda(\bar{h}) = 0$ and $\lambda(h) \in \mathbb{Z}_{\geq 0}$, then for all submodules $N \subset \Delta(\lambda)$ we have one of the following cases*

1. *$N = U(\mathfrak{n}_-)(\bar{f}^j \otimes v_\lambda)$ for some $j \in \mathbb{Z}_{\geq 0}$.*
2. *$N = U(\mathfrak{g})(f^{\lambda(h)-2k+1} \bar{f}^k \otimes v_\lambda)$ for some $k \in \mathbb{Z}_{\geq 0}$, and $\lambda(h) - 2k + 1 > 0$.*
3. *$N = U(\mathfrak{g})\{\bar{f}^j \otimes v_\lambda, f^{\lambda(h)-2k+1} \bar{f}^k \otimes v_\lambda\}$ for some $j, k \in \mathbb{Z}_{\geq 0}$ where $\lambda(h) - 2j + 1 > 0$ and $\lambda(h) - 2k + 1 > 0$.*

Proof. Let $N \subset \Delta(\lambda)$ be a non-zero submodule. By Lemma 6.4 there exists a weight vector for $\bar{h}$ in N. If we do the same procedure as in the proof of Theorem 6.6 we have that there exists $j \in \mathbb{Z}_{\geq 0}$ such that $\bar{f}^j \otimes v_\lambda$, and $\bar{f}^i \otimes v_\lambda \notin N$ for all $i < j$, where $i \in \mathbb{Z}_{\geq 0}$. If $N = U(\mathfrak{n}_-)(\bar{f}^j \otimes v_\lambda)$ we are in the first case.

Note that $e f^n \bar{f}^m \otimes v_\lambda = 0$ when $n = \lambda(h) - 2m + 1$. Then for any $v \in N$ there exists $\alpha \in \mathbb{N}$ such that we can reduce $f^{\lambda(h)+1} v$ to the form

$$e^\alpha f^{\lambda(h)+1}v = \sum_{\substack{p=1 \\ \lambda(h)-2p+1>0}} a_p f^{\lambda(h)-2p+1}\bar{f}^p \otimes v_\lambda + \sum_{\lambda(h)-2p+1\leq 0} b_p \bar{f}^p \otimes v_\lambda, \tag{35}$$

where $a_p, b_p \in \mathbb{C}$. Also note that

$$e\bar{h}f^{\lambda(h)-2p+1}\bar{f}^p \otimes v_\lambda = \\ 2(\lambda(h)-2p+1)(\lambda(h)-2p)(4p+1)f^{\lambda(h)-2(p+1)+1}\bar{f}^{p+1} \otimes v_\lambda. \tag{36}$$

By (36) we can reduce $e^\alpha f^{\lambda(h)+1}v$ to a weight vector of $\bar{h}$. With this and the arguments from Theorem 6.6 we can assume that $e^\alpha f^{\lambda(h)+1}v$ is

$$e^\alpha f^{\lambda(h)+1}v = \sum_{\substack{p\geq 0 \\ \lambda(h)-2p+1>0}} a_p f^{\lambda(h)-2p+1}\bar{f}^p \otimes v_\lambda \tag{37}$$

All the terms in that sum lives in different weight spaces, thus as in proof of Theorem 6.6, using h we can show that $f^{\lambda(h)-2p+1}\bar{f}^p \otimes v_\lambda \in N$ whenever $a_p \neq 0$. Let $k = \inf\{p | f^{\lambda(h)-2p+1}\bar{f}^p \otimes v_\lambda \in N\}$. If N is generated by $f^{\lambda(h)-2k+1}\bar{f}^k \otimes v_\lambda$ we are in the second case.

If neither of the first cases, then assume that there exists $v \in N$ such that no term in v lies in $U(\mathfrak{g})(f^{\lambda(h)-2k+1}\bar{f}^k \otimes v_\lambda, \bar{f}^j \otimes v_\lambda)$. Assume that $f^{\lambda(h)+1}v - w \neq 0$ where w is the part of $f^{\lambda(h)+1}v$ that lies in $U(\mathfrak{g})(f^{\lambda(h)-2k+1}\bar{f}^k \otimes v_\lambda)$. Doing the same procedure as above we arrive at a contradiction with the minimality of k. Thus $v = \sum_{p=k}^{j} a_p f^{c_p}\bar{f}^p \otimes v_\lambda$ for some $c_p \in \mathbb{C}$, where $\lambda(h) - 2p + 1 < 0$. But now by the same reasoning as in the proof of Theorem 6.6, we arrive at a contradiction with the minimality of j. □

Definition 6.8. *Let $\lambda \in \mathfrak{h}^*$. Define $L(\lambda) = \Delta(\lambda)/\max(\Delta(\lambda))$, where*

$$\max(\Delta(\lambda)) = \sum_{\substack{N\subset\Delta(\lambda) \\ N_\lambda=0}} N. \tag{38}$$

Lemma 6.9. *The submodule* $\max(\Delta(\lambda))$ *is a unique maximal submodule of* $\Delta(\lambda)$.

Proof. The construction gives us that $\max(\Delta(\lambda))_\lambda = 0$, since it is a sum of submodules N that satisfies $N_\lambda = 0$. This shows that $\max(\Delta(\lambda)) \neq \Delta(\lambda)$. Let $M \subset \Delta(\lambda)$ such that $\max(\Delta(\lambda)) \subset M$. If $M_\lambda \neq 0$ then $v_\lambda \in M$. Since v_λ generates $\Delta(\lambda)$ we have that $M = \Delta(\lambda)$.

If $M_\lambda = 0$, then $M \subset \max(\Delta(\lambda))$ by definition and thus $M = \max(\Delta(\lambda))$. This proves maximality of $\max(\Delta(\lambda))$.

Let us now prove uniqueness. Assume that $M \subset \Delta(\lambda)$ is maximal. Then $M_\lambda = 0$, thus $M \subset \max(\Delta(\lambda))$ by the definition of $\max(\Delta(\lambda))$. Since M is maximal and the fact that $\max(\Delta(\lambda)) \neq \Delta(\lambda)$ implies that $M = \max(\Delta(\lambda))$. □

Corollary 6.10. *For all $\lambda \in \mathfrak{h}^*$ the simple quotient $L(\lambda)$ is unique.*

Proof. The submodule $\max(\Delta(\lambda))$ is unique and therefore $L(\lambda)$ is unique. $\square$

Proposition 6.11. *Let $M \in \tilde{O}$. If M is simple then $M \simeq L(\lambda)$ for some λ.*

Proof. From Proposition 5.9 we have that there exists a highest weight λ. The weight space is finite dimensional from Proposition 5.10. By the fact that it is the highest weight vector implies that $eM_\lambda = \bar{e}M_\lambda = 0$. In other words, every element in M_λ is a weight vector for e and $\bar{e}$ with the weights being 0. We also have that M_λ is an $\mathfrak{h}$-invariant subspace and using the fact that $[h, \bar{h}] = 0$, we know that there exists a common weight vector $v \in M_\lambda$ for h and $\bar{h}$. Note that $U(\mathfrak{b})(v) = U(\mathfrak{h})(v) = \mathbb{C}v$. Furthermore $U(\mathfrak{b})(v) \simeq \mathbb{C}_\lambda$. Now using the fact that $(\mathrm{Ind}_{U(\mathfrak{b})}^{U(\mathfrak{g})}, \mathrm{Res}_{U(\mathfrak{b})}^{U(\mathfrak{g})})$ is a pair of adjoint functors, where $\mathrm{Res}_{U(\mathfrak{b})}^{U(\mathfrak{g})}$ is defined as the restriction of $U(\mathfrak{g})$ onto $U(\mathfrak{b})$, yields

$$0 \neq \mathrm{Hom}_{U(\mathfrak{b})}(\mathbb{C}_\lambda, \mathrm{Res}_{U(\mathfrak{b})}^{U(\mathfrak{g})} M) = \mathrm{Hom}_{U(\mathfrak{g})}(\mathrm{Ind}_{U(\mathfrak{b})}^{U(\mathfrak{g})} \mathbb{C}_\lambda, M), \tag{39}$$

but $\mathrm{Ind}_{U(\mathfrak{b})}^{U(\mathfrak{g})} \mathbb{C}_\lambda = \Delta(\lambda)$. Every image of a homomorphism is a submodule. Thus every non-zero $f : \Delta(\lambda) \to M$ is surjective since M is simple. Taking the quotient with the kernel yields that $L(\lambda) \simeq M$. $\square$

7 Hasse diagrams for Verma modules

Proposition 7.1. *The Hasse diagram for $\Delta(\lambda)$, where $\lambda(\bar{h}) \neq 0$, is just a point.*

Proof. Follows immediately from the fact that $\Delta(\lambda)$ is simple. $\square$

Proposition 7.2. *The Hasse diagram for $\Delta(\lambda)$, where $\lambda(\bar{h}) = 0$ and $\lambda(h) \notin \mathbb{Z}_{\geq 0}$, is*

$$\Delta(\lambda) \;\text{——}\; U(\mathfrak{g})(\bar{f} \otimes v_\lambda) \;\text{——}\; U(\mathfrak{g})(\bar{f}^2 \otimes v_\lambda) \;\text{——}\; U(\mathfrak{g})(\bar{f}^3 \otimes v_\lambda) \;\text{——}\; \cdots \tag{40}$$

Proof. Since all submodules are of the form $U(\mathfrak{g})(\bar{f}^j \otimes v_\lambda)$ and that

$$\cdots \subset U(\mathfrak{g})(\bar{f}^3 \otimes v_\lambda) \subset U(\mathfrak{g})(\bar{f}^2 \otimes v_\lambda) \subset U(\mathfrak{g})(\bar{f} \otimes v_\lambda) \subset U(\mathfrak{g})(v_\lambda) \tag{41}$$

we have our desired Hasse diagram. $\square$

Lemma 7.3. *Let $\xi \in \mathfrak{h}^*/Q$ such that $\lambda(\bar{h}) = 0$ for all $\lambda \in \xi$. Then there is an inclusion*

$$\Delta(\lambda - 2h^*) \to \Delta(\lambda) \tag{42}$$

for all $\lambda \in \xi$.

Proof. Since $\lambda(\bar{h}) = 0$ we have that $e\bar{f} \otimes v_\lambda$ is zero and thus $U(\mathfrak{n}_+)\bar{f} \otimes v_\lambda = 0$. This implies that there is an inclusion from $\Delta(\lambda - 2h^*)$ to $\Delta(\lambda)$ defined as $v_{\lambda - 2h^*} \mapsto \bar{f}v_\lambda$, which extends to a homomorphism. $\square$

Let $m_\lambda \in \mathbb{Z}_{\geq 0}$ be the biggest number such that $\lambda(h) - 2m_\lambda + 1 > 0$.

Proposition 7.4. *The Hasse diagram for $\Delta(\lambda)$, where $\lambda(\bar{h}) = 0$ and $\lambda(h) \in \mathbb{Z}_{\geq 0}$, is*

$$\begin{tikzcd}
\Delta(\lambda) \arrow[d, dash] \\
U(\mathfrak{g})\{f^{\lambda(h)+1}, \bar{f}\} \arrow[r, dash] \arrow[d, dash] & \Delta(\lambda - 2h^*) \arrow[d, dash] \\
\vdots \arrow[r, dash] \arrow[d, dash] & \vdots \arrow[r, dash] \arrow[d, dash] & \ddots \arrow[d, dash] \\
U(\mathfrak{g})\{f^{\lambda(h)+1}, \bar{f}^{m_\lambda}\} \arrow[r, dash] \arrow[d, dash] & U(\mathfrak{g})(f^{\lambda(h)-2+1}\bar{f}^{m_\lambda}) \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] \arrow[d, dash] & \Delta(\lambda - 2m_\lambda) \arrow[d, dash] \\
U(\mathfrak{g})(f^{\lambda(h)+1}) \arrow[r, dash] & U(\mathfrak{g})(f^{\lambda(h)-2+1}) \arrow[r, dash] & \cdots \arrow[r, dash] & U(\mathfrak{g})(f^{\lambda(h)-2m_\lambda+1}) \arrow[r, dash] & \cdots
\end{tikzcd} \tag{43}$$

where the height of this diagram is $\lambda(h) + 1$.

Proof. We will prove this inductively. We will start with the case where $\lambda(h) = 0$. From Lemma 7.3 we have that $\Delta(\lambda - 2h^*) \subset \Delta(\lambda)$, and thus from Proposition 7.2 we can write out its diagram. Using Theorem 6.7 we have two submodules left, namely $U(\mathfrak{g})(f \otimes v_\lambda)$ and $\Delta(\lambda)$. Here it is easy to see that $\Delta(\lambda - 2h^*) \subset U(\mathfrak{g})(f \otimes v_\lambda) \subset \Delta(\lambda)$. So we arrive at the following Hasse diagram

$$\begin{tikzcd}
\Delta(\lambda) \arrow[d, dash] \\
U(\mathfrak{g})(f \otimes v_\lambda) \arrow[r, dash] & \Delta(\lambda - 2) \arrow[r, dash] & \cdots
\end{tikzcd} \tag{44}$$

which is the desired Hasse diagram.

Using induction on $\lambda(h)$ we only need to look at the submodules that are not a submodule of $\Delta(\lambda - 2h^*)$. Using Theorem 6.7 we know that a submodule $N \subset \Delta(\lambda)$ where $N \not\subset \Delta(\lambda - 2h^*)$ then N is one of the following three cases

1. $N = \Delta(\lambda)$,

2. $N = U(\mathfrak{g})\{f^{\lambda(h)+1} \otimes v_\lambda, \bar{f}^j \otimes v_\lambda\}$ for some $j \in \mathbb{N}$ where $\lambda(h) - 2j + 1 > 0$,

$$
\begin{tikzcd}
\Delta(\lambda) \arrow[d, dash] & & & & \\
U(\mathfrak{g})\{f^{\lambda(h)+1}, \bar{f}\} \arrow[r, dash] \arrow[d, dash] & \Delta(\lambda - 2h^*) \arrow[d, dash] & & & \\
\vdots \arrow[r, dash] \arrow[d, dash] & \vdots \arrow[r, dash] \arrow[d, dash] & \ddots \arrow[d, dash] & & \\
U(\mathfrak{g})\{f^{\lambda(h)+1}, \bar{f}^{m_\lambda}\} \arrow[r, dash] \arrow[d, dash] & U(\mathfrak{g})(f^{\lambda(h)-2+1}\bar{f}^{m_\lambda}) \arrow[r, dash] \arrow[d, dash] & \cdots \arrow[r, dash] \arrow[d, dash] & \Delta(\lambda - 2m_\lambda) \arrow[d, dash] & \\
U(\mathfrak{g})(f^{\lambda(h)+1}) \arrow[r, dash] & U(\mathfrak{g})(f^{\lambda(h)-2+1}) \arrow[r, dash] & \cdots \arrow[r, dash] & U(\mathfrak{g})(f^{\lambda(h)-2m_\lambda+1}) \arrow[r, dash] & \cdots
\end{tikzcd}
\tag{47}
$$

3. $N = U(\mathfrak{g})(f^{\lambda(h)+1} \otimes v_\lambda)$.

Note that

$$
\begin{aligned}
U(\mathfrak{g})(f^{\lambda(h)+1} \otimes v_\lambda) \subset U(\mathfrak{g})\{f^{\lambda(h)+1} \otimes v_\lambda, \bar{f} \otimes v_\lambda\} \subset U(\mathfrak{g})\{f^{\lambda(h)+1} \otimes v_\lambda, \bar{f}^2 \otimes v_\lambda\} \\
\subset \cdots \subset U(\mathfrak{g})(f^{\lambda(h)+1} \otimes v_\lambda).
\end{aligned}
\tag{45}
$$

and that

$$
\begin{aligned}
&U(\mathfrak{g})\{f^{\lambda(h)-2+1}\bar{f} \otimes v_\lambda, \bar{f}^j \otimes v_\lambda\} \subset U(\mathfrak{g})\{f^{\lambda(h)+1} \otimes v_\lambda, \bar{f}^j \otimes v_\lambda\} \\
&U(\mathfrak{g})(f^{\lambda(h)-2+1}\bar{f} \otimes v_\lambda) \subset U(\mathfrak{g})(f^{\lambda(h)+1} \otimes v_\lambda).
\end{aligned}
\tag{46}
$$

Hence the induction on $\lambda(h)$ implies that the Hasse diagram for $\Delta(\lambda)$ is (47) which is the desired form.

$\square$

8 Block decomposition

In this section we will study the block decomposition of the category $\tilde{O}$.

First we want to introduce a duality in $\tilde{O}$. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be defined as

$$
\begin{aligned}
\sigma(e) &= -f \quad \sigma(\bar{e}) = -\bar{f} \\
\sigma(h) &= h \quad \sigma(\bar{h}) = \bar{h} \\
\sigma(f) &= e \quad \sigma(\bar{f}) = \bar{e}
\end{aligned}
\tag{48}
$$

.
This is clearly an involution. Let $M \in \tilde{O}$, using that $M = \bigoplus_{\lambda \in \mathfrak{h}} M_\lambda$ we define the duality functor $\odot$ as

$$M^{\odot} = \bigoplus_{\lambda \in \mathfrak{h}} M_\lambda^* \tag{49}$$

such that if $g \in M^{\odot}$, $x \in M$ and $a \in \mathfrak{g}$ then $a \cdot g(x) = g(\sigma(a) \cdot x)$.

If $M \in \tilde{O}$ and let m be a generating set for M. Then $m^{\odot}$ is a generating set for $M^{\odot}$, furthermore the definition of σ implies that $\dim U(\mathfrak{b})v < \infty$ where $v \in M^{\odot}$. Indeed for all M_λ there exists a $N \in \mathbb{Z}_{\geq 0}$ such that $M_{\lambda+2nh^*} = 0$ for all $n \geq N$. Thus for all M_λ^* there exists a $N \in \mathbb{Z}_{\geq 0}$ such that $M^*_{\lambda+2nh^*} = 0$ for all $n \geq N$. Since every element is a linear combination of elements in different M_λ^* we have $\dim U(\mathfrak{b})v < \infty$ where $v \in M^{\odot}$. If $f : M \to N$, then we define $\odot(f) : N^{\odot} \to M^{\odot}$ by $\alpha \mapsto \alpha \circ f$. This makes the duality functor $\odot$ a contravariant and involutive functor on $\tilde{O}$.

Proposition 8.1. *The duality functor $\odot : \tilde{O} \to \tilde{O}$ is exact.*

Proof. It is enough to prove that short exact sequences maps to short exact sequences. Let

$$0 \to M \xrightarrow{f} X \xrightarrow{g} N \to 0 \tag{50}$$

be a short exact sequence. We want to show that the sequence

$$o \to N^{\odot} \xrightarrow{\odot(g)} X^{\odot} \xrightarrow{\odot(f)} M^{\odot} \to 0 \tag{51}$$

is exact.

Exactness at $N^{\odot}$: Since $\odot(g)(\alpha) = \alpha \circ g$, then exactness at $N^{\odot}$ comes from the fact that g is an epimorphism.

Exactness at $X^{\odot}$: The composition $\odot(f) \odot (g) = g \circ f \circ \_ = 0$ by exactness of (50), therefore $\mathrm{Im}(\odot(g)) \subset \ker(\odot(f))$. Now let $\alpha \in \ker(\odot(f))$. Define $\beta \in N^{\odot}$ such that $\beta(n) = \alpha(x)$, where $g(x) = n$. Then $\odot(g)(\beta) = \beta \circ g = \alpha$. Hence it is exact at $X^{\odot}$.

Exactness at $M^{\odot}$: Since $\odot(f)(\alpha) = \alpha \circ f$, then exactness at $M^{\odot}$ comes from the fact that f is an monomorphism.

Thus 51 is exact. $\square$

Proposition 8.2. *Let $M \in \tilde{O}$, then* $\mathrm{Supp}(M) = \mathrm{Supp}(M^{\odot})$.

Proof. Let $\lambda, \mu \in \mathrm{Supp}(M)$ and $\alpha \in M_\lambda^*$. Then

$$(h - \mu(h))\alpha = 0 \quad \iff \quad \lambda = \mu \tag{52}$$

.
Thus $M^{\odot}_{\lambda} \neq 0$ and $\lambda \in \mathrm{Supp}(M^{\odot})$. The fact that this functor is involutive implies that $\mathrm{Supp}(M) = \mathrm{Supp}(M^{\odot})$. $\square$

By Corollary 6.10 and Proposition 6.11 the simple objects in our category are fully determined by its highest weight and its support. Thus by Proposition 8.2 the duality functor will preserve the isomorphism classes of the simple objects.

We will decompose $\tilde{O}$ using its simple objects. There is a one-to-one correspondence between simple objects $L(\lambda) \in \tilde{O}$ and pairs $(\lambda(h), \lambda(\bar{h}))$ up to isomorphism of the $L(\lambda)$. Now consider the smallest equivalence relation $\sim$ on simple object in $\tilde{O}$ such that $\sim_e \subset \sim$, where $\sim_e$ is defined as

$$L \sim_e L' \quad \iff \quad \mathrm{Ext}^1_{\tilde{O}}(L, L') \neq 0. \tag{53}$$

With this relation we can decompose the category such that two simple objects are in the same direct summand if they are equivalent under $\sim$.

First of all, morphisms in $\tilde{O}$ must preserve the action of Casimir elements. Thus taking two object M and M' with highest weights λ and λ' respectively. Then using c_2 in Proposition 4.1 we have the following decomposition of $\tilde{O}$

$$\tilde{O} = \bigoplus_{\xi \in h^*/Q} \tilde{O}[\xi], \tag{54}$$

as Serre categories, where $\tilde{O}[\xi]$ be the full subcategory of $\tilde{O}$ consisting of all M such that $\mathrm{Supp}(M) \subset \xi$. In order for us to understand this decomposition we must understand $\tilde{O}[\xi]$ for all $\xi \in \mathfrak{h}/Q$.

Proposition 8.3. *Let $\xi \in \mathfrak{h}^*/Q$ such that $\lambda(\bar{h}) = 0$ for all $\lambda \in \xi$. The category $\tilde{O}[\xi]$ is indecomposable in the sense of Serre categories.*

Proof. The inclusion map from Lemma 7.3 together with the fact that $\Delta(\lambda)$ is indecomposable implies that $L(\lambda)$ and $L(\lambda - 2h^*)$ lies in the same indecomposable direct summand of $\tilde{O}[\xi]$ for any $\lambda \in \xi$. $\square$

Lemma 8.4. *Let $M \in \tilde{O}$. Then M is projective if and only if $\mathrm{Ext}^1_{\tilde{O}}(M, N) = 0$ for all $N \in \tilde{O}$.*

Proof. Given an exact short sequence $0 \to X \to L \to N \to 0$. The functor $\mathrm{Ext}^1_{\tilde{O}}$ induces by definition a long exact sequence

$$0 \to \mathrm{Hom}_{U(\mathfrak{g})}(M, K) \to \mathrm{Hom}_{U(\mathfrak{g})}(M, L) \to \mathrm{Hom}_{U(\mathfrak{g})}(M, N) \to \mathrm{Ext}^1_{\tilde{O}}(M, K) \to \dots \tag{55}$$

Now M is projective if and only if $\mathrm{Hom}_{U(\mathfrak{g})}(M, \_)$ is exact. Since $\mathrm{Ext}^i_{\tilde{O}}(M, N) = 0$ for all $i = 1, 2, \dots$ and all $N \in \tilde{O}$, by the definition of $\mathrm{Ext}^1_{\tilde{O}}$, we have that $\mathrm{Hom}_{U(\mathfrak{g})}(M, \_)$ is exact if and only $\mathrm{Ext}^1_{\tilde{O}}(M, N) = 0$ for all $N \in \tilde{O}$. $\square$

Proposition 8.5. *If $\lambda(\bar{h}) = 0$ then $\Delta(\lambda)$ is not projective.*

Proof. Let $\nabla(\lambda + h^*)$ be the dual module of $\Delta(\lambda + h^*)$. Firstly, $\Delta(\lambda + h^*)$ has $L(\lambda + h^*)$ as simple top, which implies that $\nabla(\lambda + h^*)$ has $L(\lambda + h^*)$ as simple socle. Now consider the following short exact sequence

$$0 \to L(\lambda + h^*) \xrightarrow{f} \nabla(\lambda + h^*) \xrightarrow{g} \operatorname{coker}(f) \to 0. \tag{56}$$

Here we have that $L(\lambda + h^*)$ is not the simple socle for $\operatorname{coker}(f)$, thus

$$\operatorname{Hom}_{U(\mathfrak{g})}(\Delta(\lambda), \operatorname{coker}(f)) \neq 0. \tag{57}$$

Since λ is the highest weight of $\Delta(\lambda)$ we get that $\operatorname{Hom}_{U(\mathfrak{g})}(\Delta(\lambda), L(\lambda + h^*)) = 0$. Also $L(\lambda + h^*)$ is the socle of $\nabla(\lambda + h^*)$ which yields that $\operatorname{Hom}_{U(\mathfrak{g})}(\Delta(\lambda), \nabla(\lambda + h^*)) = 0$. From this we can conclude that $\operatorname{Hom}_{U(\mathfrak{g})}(\Delta(\lambda), \_)$ is not exact, in other words, $\Delta(\lambda)$ is not projective. $\square$

Lemma 8.6. *For two simple objects $\Delta(\lambda), \Delta(\mu) \in \tilde{O}$ where $\lambda \neq \mu$, $\lambda(\bar{h}) \neq 0$ and $\mu(\bar{h}) \neq 0$ we have*

$$\operatorname{Ext}^1_{\tilde{O}}(\Delta(\lambda), \Delta(\mu)) = 0. \tag{58}$$

Proof. Since we have the adjunction between Res and Ext it is enough to show that $\operatorname{Ext}^1_{\tilde{O}}(\mathbb{C}_\lambda, \operatorname{Res}^{U(\mathfrak{g})}_{U(\mathfrak{b})}\Delta(\mu)) = 0$. With the use of the isomorphism $\mathbb{C}_\lambda \to \mathbb{C}_0$ where $v_\lambda \mapsto v_0 + \lambda(h)$ we only need to show that $\operatorname{Ext}^1_{\tilde{O}}(\mathbb{C}_0, \operatorname{Res}^{U(\mathfrak{g})}_{U(\mathfrak{b})}\Delta(\mu)) = 0$. Now if $\operatorname{Res}^{U(\mathfrak{g})}_{U(\mathfrak{b})}\Delta(\mu)$ is injective, then it would automatically imply the desired result. Taking an element from $\operatorname{Ext}^1_{\tilde{O}}(\Delta(\lambda), \Delta(\mu))$, that is a short exact sequence of the form

$$0 \to \Delta(\mu) \to X \to \Delta(\lambda) \to 0. \tag{59}$$

Now if $\operatorname{Res}^{U(\mathfrak{g})}_{U(\mathfrak{b})}\Delta(\mu)$ is injective then every short exact sequence of that form will split, that is, it will exists an element in $x \in X_\mu$ such that $\mathfrak{n}_+ x = 0$. Now if $\mathfrak{n}_+ x = 0$ then $\mathfrak{n}_+\mathfrak{h}x = 0$. Thus it will be enough to show that $\operatorname{Res}^{U(\mathfrak{g})}_{U(\mathfrak{b})}\Delta(\mu)$ is injective as an $\mathfrak{n}_+$-module.

Consider the module $\mathbb{C}[e, \bar{e}]$ which is a $\mathfrak{n}_+$-module. This is a free module in the category of $\mathbb{C}[e, \bar{e}]$-fgmod, which is the caregory of finitely graded $\mathbb{C}[e, \bar{e}]$-modules, and where $\mathbb{C}[e, \bar{e}]_i$ is the set of all homogeneous polynomials of degree i. Since it is a free module it is also a projective module. The duality $\odot$ can be extended to a duality in $\mathbb{C}[e, \bar{e}]$-fgmod in the standard way, therefore we will denote it by the same symbol. Now the dual module $\mathbb{C}[e, \bar{e}]^\odot$, which is by definition

$$\mathbb{C}[e, \bar{e}]^\odot = \bigoplus_{i=-\infty}^{\infty} (\mathbb{C}[e, \bar{e}]_{-i})^*, \tag{60}$$

an injective module. We would like to show that $\mathbb{C}[e, \bar{e}]^\odot$ is isomorphic to $\operatorname{Res}^{U(\mathfrak{g})}_{U(\mathfrak{b})}\Delta(\mu)$ as $\mathfrak{n}_+$-modules. Since $\mathbb{C}$ is the socle for both $\operatorname{Res}^{U(\mathfrak{g})}_{U(\mathfrak{b})}\Delta(\mu)$ and $\mathbb{C}[e, \bar{e}]^\odot$, we have the following diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, hook] \arrow[d, hook] & \mathrm{Res}^{U(\mathfrak{g})}_{U(\mathfrak{b})}\Delta(\mu) \arrow[dl, Rightarrow, "\exists\phi"] \\
\mathbb{C}[e,\bar{e}]^{\odot} &
\end{tikzcd} \tag{61}$$

The morphism ϕ exists because $\mathbb{C}[e,\bar{e}]^{\odot}$ is injective. It is easy to see that the dimension of $(\mathbb{C}[e,\bar{e}]_{-i})^*$ and $\delta(\mu)_{\mu(h)-2i}$ are the same, which implies that ϕ is an isomorphism. Hence $\mathrm{Res}^{U(\mathfrak{g})}_{U(\mathfrak{b})}\Delta(\mu)$ is injective.

$\square$

Corollary 8.7. *We can decompose $\tilde{O}_{\bar{h}\neq 0}$ in the following way*

$$\tilde{O}[\xi] = \bigoplus_{\lambda\in\xi} \tilde{O}[\xi]_\lambda \tag{62}$$

where $\tilde{O}[\xi]_\lambda$ is the Serre subcategory generated by $\Delta(\lambda)$.

Proof. By Theorem 6.5 we have that $L(\lambda) = \Delta(\lambda)$. Then the decomposition follows from Lemma 8.6. $\square$

9 Gabriel Quivers

In this section we are going to compute the Gabriel quivers for the blocks in $\tilde{O}$.

Computing Gabriel quivers for our blocks, it is convenient to introduce the full subcategory $\tilde{O}[\xi,\lambda]$ of $\tilde{O}[\xi]$ consisting of all modules M of the form $M_{\lambda+2ih^*} = 0$ for all $i \in \mathbb{Z}_{\geq 0}$ for some $\lambda \in \xi$. The reason we introduce these subcateories is that from the definition we have that $\tilde{O}[\xi,\lambda] \hookrightarrow \tilde{O}[\xi,\lambda+2h^*]$, and thus we can compute the quiver for $\tilde{O}[\xi]$ inductively using $\tilde{O}[\xi,\lambda]$. Hence the problem is reduce to computing the quiver for $\tilde{O}[\xi,\lambda]$.

Lemma 9.1.

1. *If $\lambda(\bar{h}) \neq 0$, then* $\mathrm{Ext}^1_{\tilde{O}}(L(\lambda),L(\lambda)) \cong \mathbb{C}^2$.

2. *If $\lambda(\bar{h}) = 0$ and $\lambda(h) \notin \mathbb{Z}_{\geq 0}$, then* $\mathrm{Ext}^1_{\tilde{O}}(L(\lambda),L(\lambda)) \cong \mathbb{C}^2$.

3. *If $\lambda(\bar{h}) = 0$ and $\lambda(h) \in \mathbb{Z}_{\geq 0}$, then* $\mathrm{Ext}^1_{\tilde{O}}(L(\lambda),L(\lambda)) = 0$.

Proof. Consider the non-split short exact sequence

$$0 \to \Delta(\lambda) \to M \to \Delta(\lambda) \to 0 \tag{63}$$

Let $m_1 \in \Delta(\lambda)_\lambda$ be the elements that generates $\Delta(\lambda)$ and let $m_2 \in M_\lambda$ be the element that maps to m_1 under the quotient map. With this $\dim M_\lambda = 2$ and has the basis $\{m_1, m_2\}$. Since the sequence does not split and the fact that M_λ is a generalized weight space implies that

$$[h]_{M_\lambda} = \begin{bmatrix} \lambda(h) & \alpha \\ 0 & \lambda(h) \end{bmatrix} \quad [\bar{h}]_{M_\lambda} = \begin{bmatrix} \lambda(\bar{h}) & \beta \\ 0 & \lambda(\bar{h}) \end{bmatrix}, \tag{64}$$

where $\alpha, \beta \in \mathbb{C}$. This is due to the fact that $(H - \lambda(H))m_i = 0$ for all $i = 1, 2$ and all $H \in \mathfrak{h}$ together with the fact that $\dim U(\mathfrak{b})m_i < \infty$ for all $i = 1, 2$. Since α and β are free parameters, we have that $\dim \mathrm{Ext}^1_{\bar{O}}(\Delta(\lambda), \Delta(\lambda)) \leq 2$. Let us now examine for which values on α and β gives isomorphic extensions.

Consider the following diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Delta(\lambda) & \xrightarrow{f_1} & M & \xrightarrow{g_1} & \Delta(\lambda) & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle\gamma} & & \| & & \\ 0 & \longrightarrow & \Delta(\lambda) & \xrightarrow{f_2} & N & \xrightarrow{g_2} & \Delta(\lambda) & \longrightarrow & 0 \end{array} \tag{65}$$

where

$$\begin{aligned} [h]_{M_\lambda} &= \begin{bmatrix} \lambda(h) & \alpha_1 \\ 0 & \lambda(h) \end{bmatrix} \quad [\bar{h}]_{M_\lambda} = \begin{bmatrix} \lambda(\bar{h}) & \beta_1 \\ 0 & \lambda(\bar{h}) \end{bmatrix} \\ [h]_{N_\lambda} &= \begin{bmatrix} \lambda(h) & \alpha_2 \\ 0 & \lambda(h) \end{bmatrix} \quad [\bar{h}]_{N_\lambda} = \begin{bmatrix} \lambda(\bar{h}) & \beta_2 \\ 0 & \lambda(\bar{h}) \end{bmatrix}. \end{aligned} \tag{66}$$

By the commutativity of the first square we get that

$$\gamma(m_1) = \gamma(f_1(v_\lambda)) = f_2(v_\lambda) = n_1. \tag{67}$$

Similarly for the other square we have that $\gamma(m_2) = n_2$. The fact that γ is a homomorphism implies that $\alpha_1 = \alpha_2$ and $\beta_1 = \beta_2$. Indeed

$$\begin{aligned} h\gamma(m_2) &= \alpha_2 n_1 + \lambda(h) n_2 \\ \bar{h}\gamma(m_2) &= \beta_2 n_1 + \lambda(\bar{h}) n_2 \end{aligned} \tag{68}$$

but on the other hand we have

$$\begin{aligned} h\gamma(m_2) &= \gamma(hm_2) = \gamma(\alpha_1 m_1 + \lambda(h) m_2) = \alpha_1 n_1 + \lambda(h) n_2, \\ \bar{h}\gamma(m_2) &= \gamma(\bar{h}m_2) = \gamma(\beta_1 m_1 + \lambda(\bar{h}) m_2) = \beta_1 n_1 + \lambda(\bar{h}) n_2. \end{aligned} \tag{69}$$

Now 1 follows from Theorem 6.5.

Now if $\lambda(\bar{h}) = 0$ and $\lambda \notin \mathbb{Z}_{\geq 0}$, then recall that we can view $L(\lambda)$ as a $\mathfrak{sl}_2$ module via $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\overline{\mathfrak{sl}}_2$. Consider the short exact sequence

$$0 \to L(\lambda) \to M \to L(\lambda) \to 0, \tag{70}$$

where M is constructed in a similar way as before. Note that since $L(\lambda)$ is a quotient of $\Delta(\lambda)$ we have that M is a quotient of $\Delta(V)$. In other words, there exists a short exact sequence of the form

$$0 \to K \to \Delta(V) \to M \to 0. \tag{71}$$

If we let $\alpha \neq 0, \beta = 0$, then we can pick

$$K = U(\mathfrak{n}_-) \otimes \mathbb{C}\langle \bar{f} \otimes m_1, \bar{f} \otimes m_2 \rangle, \tag{72}$$

which will give us a non-trivial extension. To find the another non-trivial extension we will set $\alpha = 0, \beta \neq 0$. It is easy to see that K has to be generated by elements in $\Delta(V)_{\lambda - 2h^*}$. We find the highest weight vectors in $\Delta(V)_{\lambda - 2h^*}$ by acting with n_+ on an arbitrary element.

$$\begin{aligned} &e(af \otimes m_1 + b\bar{f} \otimes m_1 + cf \otimes m_2 + d\bar{f} \otimes m_2) = a\lambda(h)m_1 + \alpha b m_2 + c\lambda(h)m_2 = 0 \\ &\bar{e}(af \otimes m_1 + b\bar{f} \otimes m_1 + cf \otimes m_2 + d\bar{f} \otimes m_2) = \alpha a m_2 = 0 \end{aligned} \tag{73}$$

where $a, b, c, d \in \mathbb{C}$. In this system of equation we will have two free parameters c and d, which implies that there exists a two dimensional subspace of $\Delta(V)_{\lambda - 2h^*}$ which is generated highest weight vectors. Taking K to be generated by these weight vectors gives us another non-trivial extension of $L(\lambda)$ with itself. These two extensions are not isomorphic by the same argument as in the first case. This shows 2.

In the third case where $\lambda(\bar{h}) = 0$ and $\lambda(h) \in \mathbb{Z}_{\geq 0}$ we also have that $\dim L(\lambda) < \infty$. Since $L(\lambda)$ can be viewed as an $\mathfrak{sl}_2$ module we can use the Weyl theorem that says that every finite dimensional $\mathfrak{sl}_2$ module is semi-simple. This shows 3. □

Theorem 9.2. *The subcategory $\tilde{O}[\xi]_\lambda$, where $\lambda(\bar{h}) \neq 0$, has the following Gabriel quiver*

$$\begin{tikzcd} \bullet \arrow[loop left, "a"] \arrow[loop right, "b"] \end{tikzcd} \tag{74}$$

Proof. The result follows from the fact that the subcategory has only one simple object $L(\lambda)$ and the first case in Lemma 9.1. □

Let us consider the following quivers

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "a"] & 6 \arrow[l, bend left, "b"] \arrow[r, bend left, "a"] \arrow[loop, out=120, in=60, distance=3em, "c"] \arrow[loop, out=110, in=70, distance=1.5em, "d"'] & 4 \arrow[l, bend left, "b"] \arrow[r, bend left, "a"] \arrow[loop, out=120, in=60, distance=3em, "c"] \arrow[loop, out=110, in=70, distance=1.5em, "d"'] & 2 \arrow[l, bend left, "b"] \arrow[r, bend left, "a"] \arrow[loop, out=120, in=60, distance=3em, "c"] \arrow[loop, out=110, in=70, distance=1.5em, "d"'] & 0 \arrow[l, bend left, "b"]
\end{tikzcd} \tag{75}$$

and

$$\begin{tikzcd}
\cdots \arrow[r, bend left, "a"] & -4 \arrow[l, bend left, "b"] \arrow[r, bend left, "a"] \arrow[loop, out=120, in=60, distance=3em, "c"] \arrow[loop, out=110, in=70, distance=1.5em, "d"'] & -2 \arrow[l, bend left, "b"] \arrow[r, bend left, "a"] \arrow[loop, out=120, in=60, distance=3em, "c"] \arrow[loop, out=110, in=70, distance=1.5em, "d"'] & 0 \arrow[l, bend left, "b"] \arrow[r, bend left, "a"] \arrow[loop, out=120, in=60, distance=3em, "c"] \arrow[loop, out=110, in=70, distance=1.5em, "d"'] & 2 \arrow[l, bend left, "b"] \arrow[r, bend left, "a"] \arrow[loop, out=120, in=60, distance=3em, "c"] \arrow[loop, out=110, in=70, distance=1.5em, "d"'] & 4 \arrow[l, bend left, "b"] \arrow[r, bend left, "a"] \arrow[loop, out=120, in=60, distance=3em, "c"] \arrow[loop, out=110, in=70, distance=1.5em, "d"'] & \cdots \arrow[l, bend left, "b"]
\end{tikzcd} \tag{76}$$

which we will denote ${}_{\infty}Q$ and ${}_{\infty}Q_{\infty}$ respectively.

Lemma 9.3. *The quiver ${}_{\infty}Q$ is the Gabriel quiver for $\tilde{O}[\xi, \lambda]$, where $\lambda(\bar{h}) = 0$ and $\lambda(h) \notin \mathbb{Z}$ for all $\lambda \in \xi$.*

Proof. The simples in $\tilde{O}[\xi, \lambda]$ are $L(\lambda - 2ih^*)$ for all $i \in \mathbb{Z}_{\geq 0}$. Let us first assign $L(\lambda - 2ih^*)$ to the vertex i in ${}_{\infty}Q$. We have to show that if $j < i$, then

$$\operatorname{Ext}^1_{\tilde{O}}(L(\lambda - 2ih^*), L(\lambda - 2jh^*)) = \begin{cases} \mathbb{C}, & \text{if } i = j + 1 \\ 0, & \text{otherwise} \end{cases}. \tag{77}$$

To prove this consider the short exact sequence

$$0 \to L(\lambda - 2jh^*) \to X \to L(\lambda - 2ih^*) \to 0. \tag{78}$$

Here X must have highest weight $\lambda - 2i$ and thus must be a quotient of $\Delta(\lambda - 2ih^*)$. By Theorem 6.6 and Lemma 7.3 we have that the radical of $\Delta(\lambda - 2ih^*)$ is unique and is equal to $\mathrm{Rad}(\Delta(\lambda - 2\mathrm{ih}^*)) = \Delta(\lambda - 2(i+1)h^*)$. This together with the fact that $\Delta(\lambda - 2ih^*)$ is indecomposable for all $i \in \mathbb{Z}_{\geq 0} \geq 0$ implies (77).

Now using the fact that the duality $\odot$ is exact and preserves the category $\tilde{O}[\xi, \lambda]$ and isomorphism classes of simple modules we get that

$$\operatorname{Ext}^1_{\tilde{O}}(L(\lambda - 2ih^*), L(\lambda - 2jh^*)) \cong \operatorname{Ext}^1_{\tilde{O}}(L(\lambda - 2jh^*), L(\lambda - 2ih^*)). \tag{79}$$

It remains to show that $\operatorname{Ext}^1_{\tilde{O}}(L(\lambda - 2ih^*), L(\lambda - 2ih^*)) \cong \mathbb{C}$ for all $i \in \mathbb{Z}_{\geq 0}$, but this follows from the second case in Lemma 9.1. $\square$

Theorem 9.4. *The Gabriel quiver for $\tilde{O}[\xi]$, where ξ is non-integral and $\lambda(\bar{h}) = 0$ for all $\lambda \in \xi$, is ${}_{\infty}Q_{\infty}$.*

Here ξ being integral means that $\lambda(h) \in \mathbb{Z}$ for all $\lambda \in \xi$.

Proof. This follows directly from Lemma 9.3 and by taking the inductive limit. $\square$

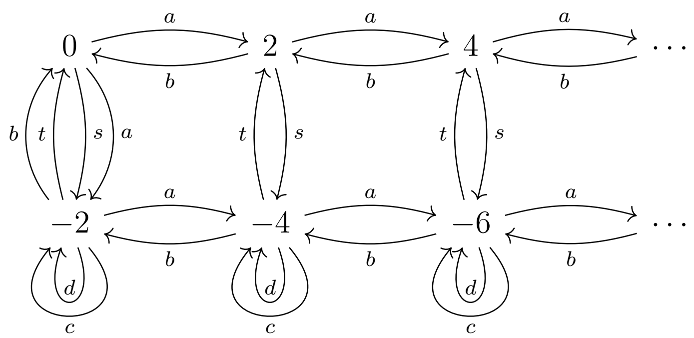
It remains to compute the Gabriel quiver for when ξ is integral and $\lambda(\bar{h}) = 0$ for all $\lambda \in \xi$. When ξ is integral recall that the action of e in $\Delta(\lambda)$ looks like

$$ef^i \bar{f}^j \otimes v_\lambda = i(\lambda(h) - i - 2j + 1)f^{i-1}\bar{f}^j \otimes v_\lambda. \tag{80}$$

This implies that we have an $\mathfrak{sl}_2$ extension of $L(\lambda)$ with $L(r \cdot \lambda)$, in the sense that $\mathfrak{g} \twoheadrightarrow \mathfrak{g}/\overline{\mathfrak{sl}}_2 \cong \mathfrak{sl}_2$. Here $L(r \cdot \lambda) = L(-\lambda - 2)$. It can also be seen by the following short exact sequence

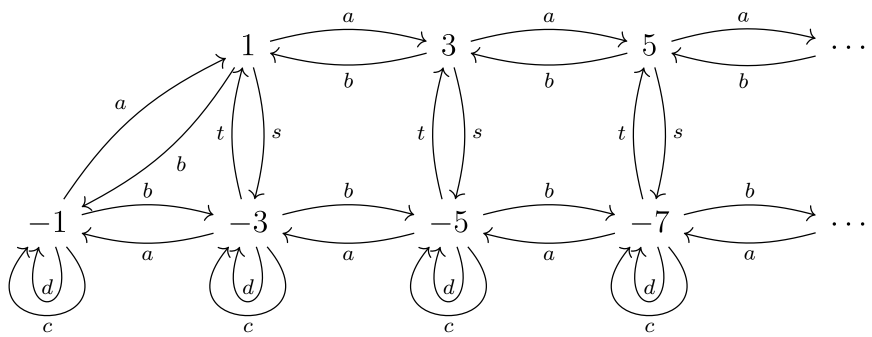
$$0 \to L(r \cdot \lambda) \to \Delta(\lambda)/\Delta(\lambda - 2) \to L(\lambda) \to 0. \tag{81}$$

Now consider the following two quivers



(82)

and



(83)

which we will denote by Γ^{even} and Γ^{odd} respectively. Let us also define Γ^{even}_n (Γ^{odd}_n) as the full subquiver of Γ^{even} (Γ^{odd}) which consist of all vertices up to n.

Lemma 9.5. *The Gabriel quiver for $\tilde{O}[\xi, \lambda]$, where ξ is integral and $\lambda(\bar{h}) = 0$ for all $\lambda \in \xi$, is:*

1. Γ^{even}_n *when $\lambda(h) \in 2\mathbb{Z}$ for all $\lambda \in \xi$,*

2. Γ^{odd}_n *when $\lambda(h) \in 2\mathbb{Z} + 1$ for all $\lambda \in \xi$.*

Proof. Let us start with the first case. First we assign $L(\lambda - 2ih^*)$ to the vertex $n-2i$. We want to show that if $i<j$ and $\lambda - 2ih^* \neq 0$ then

$$\mathrm{Ext}^1_{\tilde{O}}(L(\lambda-2ih^*), L(\lambda-2jh^*)) = \begin{cases} \mathbb{C}, & \text{if } i = j+1 \\ \mathbb{C}, & \text{if } r\cdot(\lambda-2ih^*) = \lambda-2jh^* \\ 0, & \text{otherwise} \end{cases} \tag{84}$$

Consider the short exact sequence

$$0 \to L(\lambda-2jh^*) \to X \to L(\lambda-2ih^*) \to 0. \tag{85}$$

Since $i<j$, X must have $\lambda - 2ih^*$ as its highest weight, therefore X must be a quotient of $\Delta(\lambda-2ih^*)$. By Theorem 6.7, the radical of $\Delta(\lambda-2ih^*)$ must be either

$$\Delta(\lambda-2(i+1)h^*) \quad \text{or} \quad U(\mathfrak{g})\{\bar{f}\otimes v_\lambda, f^{\lambda(h)-2i-1}\otimes v_\lambda\}. \tag{86}$$

If $\mathrm{Rad}(\Delta(\lambda-2ih^*)) = \Delta(\lambda-2(i+1)h^*)$, then $\mathrm{Rad}(\Delta(\lambda-2(i+1)h^*)) = \Delta(\lambda-2(i+2)h^*)$. Now using the fact that the radical is unique for Verma modules we have that

$$\mathrm{Ext}^1_{\tilde{O}}(L(\lambda-2ih^*), L(\lambda-2(i+1)h^*)) \cong \mathbb{C}. \tag{87}$$

If $\mathrm{Rad}(\Delta(\lambda-2ih^*)) = U(\mathfrak{g})\{\bar{f}\otimes v_\lambda, f^{\lambda(h)-2i-1}\otimes v_\lambda\}$, then by Proposition 7.4 the module $U(\mathfrak{g})\{\bar{f}\otimes v_\lambda, f^{\lambda(h)-2i-1}\otimes v_\lambda\}$ has two maximal submodules namely

$$\Delta(\lambda-2(i+1)h^*) \quad \text{and} \quad U(\mathfrak{g})\{\bar{f}^2\otimes v_\lambda, f^{\lambda(h)-2i-1}\otimes v_\lambda\}. \tag{88}$$

The first one yields

$$\mathrm{Ext}^1_{\tilde{O}}(L(\lambda-2ih^*), L(\lambda-2jh^*)) \cong \mathbb{C}, \quad r\cdot(\lambda-2ih^*) = \lambda-2jh^* \tag{89}$$

and the second one yields

$$\mathrm{Ext}^1_{\tilde{O}}(L(\lambda-2ih^*), L(\lambda-2(i+1)h^*)) \cong \mathbb{C}. \tag{90}$$

This shows 84. When $\lambda - 2ih^* = 0$, then $r\cdot(\lambda-2ih^*) = \lambda - 2(i+1)h^*$ and with the argument above we get that

$$\mathrm{Ext}^1_{\tilde{O}}(L(\lambda-2ih^*), L(\lambda-2(i+1)h^*)) \cong \mathbb{C}^2. \tag{91}$$

Same as the previous theorem. By using that the duality $\odot$ preserves the isomorphism classes we get the reversed cases for free. This together with Lemma 8.6 proves the first case. The second case follows by the same arguments. □

Theorem 9.6. *The Gabriel quiver for $\tilde{O}[\xi,\lambda]$, where ξ is integral and $\lambda(\bar{h}) = 0$ for all $\lambda\in\xi$, is:*

1. Γ^{even} *when $\lambda(\bar{h}) = 0$ and $\lambda(h) \in 2\mathbb{Z}$ for all $\lambda \in \xi$,*

2. Γ^{odd} *when* $\lambda(\bar{h}) = 0$ *and* $\lambda(h) \in 2\mathbb{Z}+1$ *for all* $\lambda \in \xi$.

Proof. This follows by Lemma 9.5 and taking the inductive limit. $\square$

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