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Verma Modules, The Weyl Character Formula and Embedding Theorems

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Contents

1 Preliminaries

Unless otherwise specified, all vector spaces are over $\mathbb{C}$.

1.1 Lie Algebras

Definition 1.1. A vector space $\mathfrak{g}$ equipped with a bilinear operation $[\cdot,\cdot]$ (sometimes called a Lie bracket) such that:

(i) $[x, x] = 0$ for all $x \in \mathfrak{g}$;

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

is a **Lie algebra**.

If A is an associative algebra over a field, a Lie bracket may be defined by the commutator $[x, y] = xy - yx$ for all $x, y \in A$, where $(x, y) \mapsto xy$ is the multiplicative operation on A. Then $(A, [\cdot,\cdot])$ is called the **underlying Lie Algebra of** A, and A is referred to as an **enveloping algebra**. Every Lie algebra may be embedded into the underlying Lie algebra for some associative algebra.

Definition 1.2. A vector subspace $\mathfrak{g}'$ of $\mathfrak{g}$ is a **Lie subalgebra** if $[x, y] \in \mathfrak{g}'$ for all $x, y \in \mathfrak{g}'$.

Definition 1.3. An **ideal** is a subalgebra $\mathfrak{g}'$ of $\mathfrak{g}$ such that $[x, y] \in \mathfrak{g}'$ for all $x \in \mathfrak{g}$ and $y \in \mathfrak{g}'$. If $\mathfrak{g}'$ is an ideal, the bracket operation defines a bracket operation on $\mathfrak{g}/\mathfrak{g}'$, so $\mathfrak{g}/\mathfrak{g}'$ is a Lie algebra, called the **quotient Lie algebra**. If $\mathfrak{g}$ contains no proper nonzero ideals it is termed a **simple Lie algebra** and called a **semi-simple Lie algebra** if $\mathfrak{g}$ decomposes into simple components.

Let V be a vector space and $\mathrm{End}(V)$ be the algebra of endomorphisms of V. The underlying Lie algebra of $\mathrm{End}(V)$ is called $\mathfrak{gl}(V)$. Let V be finite dimensional with dimension n. Given a fixed basis of V, $\mathfrak{gl}(V)$ may be identified with the algebra of $n \times n$ matrices and denoted $\mathfrak{gl}_n$. The canonical basis of $\mathfrak{gl}_n$ in given by e_{ij} (the $n \times n$ matrix with 1 in the (ij) position and 0 elsewhere). The subalgebras of $\mathfrak{gl}(V)$ are referred to as the **linear Lie algebras**.

Example. Let V be a vector space of dimension n. The **special linear algebra** is the subalgebra of $\mathfrak{gl}_n$ consisting of endomorphisms with trace 0, denoted by $\mathfrak{sl}_n$. As trace is independent of basis, this subalgebra is well defined. Given that $\mathrm{Tr}(xy) = \mathrm{Tr}(yx)$ and $\mathrm{Tr}(x+y) = \mathrm{Tr}(x)+\mathrm{Tr}(y)$, clearly $\mathfrak{sl}_n$ is a Lie subalgebra of $\mathfrak{gl}_n$. A basis of $\mathfrak{sl}_n$ may be formed by matrices of the form e_{ij} for $i \neq j$ and $e_{ii} - e_{i+1,i+1}$ for $1 \leq i < n$, thus having dimension $n^2 - 1$.

Example. Let V be a vector space of odd dimension $n = 2k + 1$ with a bilinear form on V defined by the matrix

$$s = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & I_k \\ 0 & I_k & 0 \end{bmatrix}$$

where I_k is the $k \times k$ identity matrix. The special orthogonal algebra is the subalgebra of $\mathfrak{gl}_n$ consisting of endomorphisms x such that $sx = -x^t s$, and is denoted $\mathfrak{so}_n$. If x is partitioned as s is, then an equivalent condition is that x must be of the form

$$x = \begin{bmatrix} 0 & b_1 & b_2 \\ -b_2^t & m & n \\ -b_1^t & p & -m^t \end{bmatrix}$$

for arbitrary $1 \times k$ matrices b_1, b_2 and $k \times k$ matriced m, n, p with the added constraints $n^t = -n$ and $p^t = -p$. A basis may be constructed by restricting to each of the associated partitions. The matrices $e_{1,k+i+1} - e_{i+1,1}$ and $e_{1,i+1} - e_{k+i+1,1}$ for $1 \leq i \leq k$ form b_2 and b_1. Corresponding to m, n and p respectively, the matrices $e_{i+1,j+1} - e_{k+i+1,k+j+1}$ for $1 \leq i \neq j \leq k$, $e_{i+1,k+j+1} - e_{j+1,k+i+1}$ for $1 \leq i < j \leq k$ and $e_{k+i+1,j+1} - e_{k+j+1,i+1}$ for $1 \leq i < j \leq k$ form a partial basis. With the addition of diagonal matrices of the form $e_{ii} - e_{k+i,k+i}$ for $2 \leq i \leq k+1$ yields a basis of $\mathfrak{so}_n$ with dimension $2k^2 + k$.

In the case of an even dimensional vector space, the orthogonal algebra is constructed similarly, where the bilinear form on V is of the form

$$s = \begin{bmatrix} 0 & I_k \\ I_k & 0 \end{bmatrix}.$$

Let $\mathfrak{g}$ be a Lie algebra, and define an operation $(x, y) \mapsto -[x, y]$. Then $\mathfrak{g}$ equipped with this new bracket operation becomes another Lie algebra called the **opposite** to $\mathfrak{g}$.

Definition 1.4. Let $\mathfrak{g}, \mathfrak{g}'$ be Lie algebras. A linear map $\phi : \mathfrak{g} \to \mathfrak{g}'$ is a **homomorphism** of Lie algebras if $\phi([x, y]) = [\phi(x), \phi(y)]$ for all $x, y \in \mathfrak{g}$.

Definition 1.5. Let X, Y be subsets of $\mathfrak{g}$. The set of elements of X that commute, i.e. the set of elements x such that $[x, y] = 0$, with all elements y of Y is called the **centralizer** of Y in X. The centralizer of $\mathfrak{g}$ in $\mathfrak{g}$ is an ideal and referred to as the **center** of $\mathfrak{g}$.

Definition 1.6. The **normalizer** $\mathfrak{n}$ of a subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ is the subalgebra consisting of all $x \in \mathfrak{g}$ such that $[x, \mathfrak{h}] \in \mathfrak{h}$. The subalgebra $\mathfrak{h}$ is an ideal of $\mathfrak{n}$.

Definition 1.7. Let $\mathfrak{g}$ be a Lie algebra. The **descending central series** of $\mathfrak{g}$ is the chain of subalgebras:

$$\mathfrak{g} = \mathfrak{g}^1 \supseteq \mathfrak{g}^2 = [\mathfrak{g}, \mathfrak{g}^1] \supseteq \mathfrak{g}^3 = [\mathfrak{g}, \mathfrak{g}^2] \supseteq \cdots \supseteq \mathfrak{g}^n = [\mathfrak{g}, \mathfrak{g}^{n-1}] \supseteq \ldots,$$

and the **derived series** is

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} = [\mathfrak{g}^{(0)}, \mathfrak{g}^{(0)}] \supseteq \mathfrak{g}^{(2)} = [\mathfrak{g}^{(1)}, \mathfrak{g}^{(1)}] \supseteq \cdots \supseteq \mathfrak{g}^{(n)} = [\mathfrak{g}^{(n-1)}, \mathfrak{g}^{(n-1)}] \supseteq \ldots.$$

For any semisimple Lie algebra $\mathfrak{g}$, the **derived subalgebra** $[\mathfrak{g}, \mathfrak{g}]$ is nonzero, and hence $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ given that $\mathfrak{g}$ is a sum of simple Lie algebras.

Definition 1.8. A Lie algebra $\mathfrak{g}$ is **nilpotent** if it satisfies one of the following equivalent conditions:

(i) there exists $k \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{g}^k = 0$;

(ii) there exists $k \in \mathbb{Z}_{\geq 0}$ such that $[x_1, [x_2, [\ldots, [x_{k-1}, x_k], \ldots,]]] = 0$ for all $x_1, \ldots, x_k \in \mathfrak{g}$;

(iii) there exists a decreasing series of ideals of $\mathfrak{g}$:

$$\mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots \supseteq \mathfrak{g}_n$$

such that $\mathfrak{g}_1 = \mathfrak{g}$, $\mathfrak{g}_n = 0$ and $[\mathfrak{g}, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+1}$ for $i < n$.

Definition 1.9. A Lie algebra $\mathfrak{g}$ is **solvable** if it satisfies one of the following equivalent conditions:

(i) there exists $k \in \mathbb{Z}_{\geq 0}$ such that $\mathfrak{g}^{(k)} = 0$;

(ii) there exists a decreasing series of ideals of $\mathfrak{g}$:

$$\mathfrak{g}_1 \supseteq \mathfrak{g}_2 \supseteq \cdots \supseteq \mathfrak{g}_n$$

such that $\mathfrak{g}_1 = \mathfrak{g}$, $\mathfrak{g}_n = 0$ and $[\mathfrak{g}_i, \mathfrak{g}_i] \subseteq \mathfrak{g}_{i+1}$ for $i < n$.

By definition of the descending central series and derived series, $\mathfrak{g}^{(k)} \subset \mathfrak{g}^{n+1}$. Hence if the descending central series terminates, so must the derived series, i.e. every nilpotent subalgebra is solvable.

Example. The subalgebra of $\mathfrak{gl}_n$ consisting of strictly upper triangular matrices is nilpotent, and thus solvable. Conversely, the subalgebra consisting of upper triangular matrices is solvable only.

1.2 Modules

Definition 1.10. Let V be a vector space. A homomorphism $\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$ is termed a **representation** of $\mathfrak{g}$ in V. The **dimension** of the representation ϱ is the dimension of the space V, and V is referred to as a **$\mathfrak{g}$-module**.

The labels representation and module will be used interchangably. For simplicity, $\varrho(x)v$ is denoted xv unless specification is necessary.

Example. Let $\mathfrak{g}$ be any Lie algebra. Then $\mathfrak{g}$ functions as a $\mathfrak{g}$-module via the action $\varrho(x)y = [x, y]$. This is termed the **adjoint representation**. The endomorphism $\varrho(x)$ is denoted ad x.

Example. Given $\mathfrak{g} = \mathfrak{sl}_n$, the **natural representation** is the representation of $\mathfrak{sl}_n$ in $\mathbb{C}^n$ such that each $x \in \mathfrak{sl}_n$ acts by normal matrix multiplication on vectors. This representation holds for any other Lie subalgebra of $\mathfrak{gl}_n$.

Definition 1.11. Given representations ϱ, ϱ' of $\mathfrak{g}$ in V, V'; a linear map $\phi : V \to V'$ is a **$\mathfrak{g}$-homomorphism** if $\varrho'(x)\phi(v) = \phi(\varrho(x)v)$ for all $x \in \mathfrak{g}$.

Definition 1.12. Two representations ϱ, ϱ' of $\mathfrak{g}$ in V, V' are **equivalent** or **isomorphic** if there exists a **$\mathfrak{g}$-isomorphism** from V to V'. The equivalence class $[V]$ is generally referred to as a representation instead a class of representations.

Definition 1.13. Let ϱ_i be representations of $\mathfrak{g}$ in a family of vector spaces $(V_i)_{i \in I}$ and $V = \bigoplus_{i \in I} V_i$. Then $\varrho : \mathfrak{g} \to \mathfrak{gl}(V)$, defined by $\varrho(x) = \bigoplus_{i \in I} \varrho_i(x)$, is a representation of $\mathfrak{g}$ in V called the **direct sum of representations**. If all the representations ϱ_i are equivalent to a representation π of $\mathfrak{g}$ in W, then ϱ is a **multiple** of π and V is a multiple of W.

Definition 1.14. Given a representation ϱ of $\mathfrak{g}$ in V and a vector subspace $W \subset V$, W is **stable** under ϱ if $\varrho(W) \subset W$. Given a stable subspace W, $\varrho\big|_W(x) = \varrho(x)\big|_W$ is a **subrepresentation** of ϱ and W is a **submodule**.

Definition 1.15. Given a submodule $W \subset V$, with representation ϱ on V, let π be the endomorphism of V/W induced by ϱ. Then π is a representation termed the **quotient representation** and V/W is a **quotient module**.

Definition 1.16. A representation ϱ of $\mathfrak{g}$ in V is **simple** or **irreducible** if V contains no proper nonzero submodules.

Lemma 1.17 (Schur's Lemma, [Ser05])**.** *If ϱ is an irreducible representation of $\mathfrak{g}$ in V, and $\psi \in End(V)$ commutes with $\varrho(x)$ for all $x \in \mathfrak{g}$, then ψ acts via scalar multiplication.*

Definition 1.18. A representation ϱ is **semi-simple** or **completely reducible** if any of the following equivalent conditions are satisfied:

(i) ϱ is a direct sum of simple representations;

(ii) V is a direct sum of simple submodules;

(iii) for each submodule W of V, there exists a complement W' of W that is a submodule of V.

Every finite dimensional representation of a semisimple Lie algebra is completely reducible. Given that the proof of this statement requires a few more definitions, it will be given later.
Under the adjoint representation, clearly the ideals of $\mathfrak{g}$ yield the submodules. Thus any simple (resp. semi-simple) Lie algebra $\mathfrak{g}$ is a simple (resp. semi-simple) module under the adjoint representation.

1.3 The Cartan Subalgebra and the Root System

Definition 1.19. An endomorphism u of a finite dimensional vector space V is **diagonalizable** if there exists a basis B of V such that the matrix of u is diagonal with respect to B

Definition 1.20. An endomorphism u of a finite dimensional vector space V is **triangularizable** if any of the following three equivalent conditions hold:

(i) there exists a basis B of V such that the matrix of u is upper triangular with respect to B;

(ii) there exists a basis B of V such that the matrix of u is lower triangular with respect to B;

Definition 1.21. A nilpotent subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, such that $\mathfrak{h}$ equals its normalizer in $\mathfrak{g}$, is called a **Cartan subalgebra**.

Theorem 1.22. *Let $\mathfrak{h}$ be a nilpotent subalgebra of $\mathfrak{g}$. Consider the adjoint representation of $\mathfrak{h}$ in $\mathfrak{g}$. Assume ad x is triangularizable for all $x \in \mathfrak{h}$, and define for $\lambda \in \mathfrak{h}^*$:*

$$\mathfrak{g}^\lambda = \{x \in \mathfrak{g} \ : \ [h, x] = \lambda(h)x\}.$$

Then:

(i) $\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}^\lambda$;

(ii) $[\mathfrak{g}^\lambda, \mathfrak{g}^\mu] \subseteq \mathfrak{g}^{\lambda+\mu}$;

(iii) $\mathfrak{g}^0$ *is a subalgebra of* $\mathfrak{g}$ *such that* $\mathfrak{h} \subseteq \mathfrak{g}^0$;

(iv) $\mathfrak{h}$ *is a Cartan subalgebra if and only if* $\mathfrak{h} = \mathfrak{g}^0$;

(v) $\mathfrak{g}^0$ *is a Cartan subalgebra if it is nilpotent.*

Definition 1.23. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. If ad x is triangularizable for all $x \in \mathfrak{h}$, then $\mathfrak{h}$ is a **splitting Cartan subalgebra**. If $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$, set of $\lambda \in \mathfrak{h}^*$ such that $\mathfrak{g}^\lambda \neq 0$ is called the **root system** of $\mathfrak{g}$ with respect to $\mathfrak{h}$. The set of **simple roots** $\Delta \in \Phi$ is a base for Φ and a basis for $\mathfrak{h}^*$

Definition 1.24. Let $\mathfrak{g}$ be a semi-simple Lie algera and $\mathfrak{h}$ a splitting Cartan subalgebra of $\mathfrak{g}$. The pair $(\mathfrak{g}, \mathfrak{h})$ is a *split semi-simple Lie algebra* and the set of roots of $\mathfrak{g}$ for this Cartan subalgebra is denoted $\Phi(\mathfrak{g}, \mathfrak{h})$.

Theorem 1.25. *Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra with correspodning set of roots $\Phi = \Phi(\mathfrak{g}, \mathfrak{h})$. Then:*

(i) $\mathfrak{g} = \mathfrak{h} \oplus \left(\bigoplus_{\alpha \in \Phi} \mathfrak{g}^\alpha\right)$ *where dim* $\mathfrak{g}^\alpha = 1$ *for each* $\alpha \in \Phi$;

(ii) If $\alpha \in \Phi$ *then* $-\alpha \in \Phi$, *and* $\mathfrak{h}_\alpha = [\mathfrak{g}^\alpha, \mathfrak{g}^{-\alpha}]$ *is a one dimensional vector subspace of* $\mathfrak{h}$;

(iii) Φ *spans the vector space* $\mathfrak{h}^*$;

(iv) For any $\alpha \in \Phi$, $\mathfrak{s}_\alpha = \mathfrak{h}_\alpha \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$ *is a subalgebra of* $\mathfrak{g}$ *isomorphic to* $\mathfrak{sl}_2$.

Given a basis of $\mathfrak{h}$ such that the adjoint representation is triangularizable, let $(\cdot, \cdot)$ be the standard inner product on the vector space $\mathfrak{h}^*$ with respect to the dual basis. Define the operator $\langle \cdot, \cdot \rangle : \mathfrak{h}^* \times \mathfrak{h}^* \to k$ by $\langle \beta, \alpha \rangle = 2\frac{(\beta,\alpha)}{(\alpha,\alpha)}$.

Example. Let $\mathfrak{g} = \mathfrak{sl}_n$ and e_{ij} the $n \times n$ matrix with 1 in the (i, j) position and 0 elsewhere. Let $\mathfrak{h}$ be the subalgebra of diagonal matrices spanned by $e_{ii} - e_{i+1,i+1}$. Then $\mathfrak{h}$ is a splitting Cartan subalgebra of $\mathfrak{g}$. With respect to $\mathfrak{h}$, the set of roots of $\mathfrak{g}$ is of the form $\epsilon_i - \epsilon_j$, where $\epsilon_i \in \mathfrak{h}^*$ maps e_{ii} to 1 and $e_{jj} = 0$ for $j \neq i$. For $\alpha = \epsilon_i - \epsilon_j$, the root space $\mathfrak{g}^\alpha$ is the one dimensional space spanned by e_{ij}.

Example. Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra, and α a root of $\mathfrak{g}$. For nonzero $x_\alpha \in \mathfrak{g}^\alpha$, choose $x_{-\alpha} \in \mathfrak{g}^{-\alpha}$ such that $\alpha(h_\alpha) = 2$ for $h_\alpha = [x_\alpha, x_{-\alpha}]$. Then for any $\lambda \in \mathfrak{h}^*$, the operator $\langle \lambda, \alpha \rangle$ is equivalent to $\lambda(h_\alpha)$. Let V be the subalgebra of $\mathfrak{g}$ spanned by $x_\alpha, x_{-\alpha}$ and h_α. Let ϕ be a homomorphism from $\mathfrak{sl}_2$ to V defined by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \mapsto h_\alpha, \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mapsto x_\alpha, \quad \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \mapsto x_{-\alpha}.$$

Then ϕ is an isomorphism, so for any $\alpha \in \Phi$, the subalgebra $\mathfrak{s}^\alpha = \mathfrak{g}^\alpha \oplus \mathfrak{h}^\alpha \oplus \mathfrak{g}^{-\alpha}$ may be identified with $\mathfrak{sl}_2$, where $\mathfrak{h}^\alpha$ is the one dimensional subspace spanned by h_α. For any representation ϱ of $\mathfrak{g}$, the restriction $\varrho\big|_{\mathfrak{s}^\alpha}$ may be identified with an $\mathfrak{sl}_2$ representation.

Proposition 1.26. *Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra with root system Φ. Let Δ be a base for Φ. Given $\alpha, \beta \in \Phi$:*

(i) $\langle \beta, \alpha \rangle \in \mathbb{Z}$;

(ii) the only roots that are scalar multiples of α are α and $-\alpha$;

(iii) $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$ *where all* $n_\alpha \in \mathbb{Z}_{\geq 0}$ *or all* $n_\alpha \in \mathbb{Z}_{\leq 0}$.

Definition 1.27. For a split semi-simple Lie algebra $(\mathfrak{g}, \mathfrak{h})$ and root system Φ, fix a base Δ. Then:

(i) a root $\beta = \sum_{\alpha \in \Delta} n_\alpha \alpha$ is **positive (resp. negative)** if $n_\alpha \in \mathbb{Z}_{\geq 0}$ (resp. $n_\alpha \in \mathbb{Z}_{\leq 0}$);

(ii) $\mathfrak{n}_+ = \bigoplus_{\alpha \in \Phi_+} \mathfrak{g}^\alpha$ is termed the **positive root space**, where Φ_+ is the set of positive roots, and similarly $\mathfrak{n}_- = \bigoplus_{\alpha \in \Phi_-} \mathfrak{g}^\alpha$ is termed the **negative root space**;

(iii) the decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{n}_+ \oplus \mathfrak{n}_-$ is called the **triangular decomposition**;

(iv) $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ is termed the **Borel subalgebra**

with respect to $\mathfrak{h}$ and Δ.

If $\mathfrak{h}^{*\prime} \subset \mathfrak{h}^*$ is a subspace of codimension one such that $\mathfrak{h}^{*\prime}$ does not contain any $\alpha \in \Phi$, then $\mathfrak{h}^{*\prime}$ splits $\mathfrak{h}^*$ into two connected components containing positive and negative roots. In particular, as any subspace of codimension one is the kernel of a linear functional λ on $\mathfrak{h}^*$, the set of positive roots with respect to such a functional is

$$\Phi_+ = \{\alpha \mid \lambda(\alpha) > 0\}.$$

Example. Let $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{h}$ the Cartan subalgebra of $\mathfrak{g}$ consisting of diagonal matrices. Define a total order $\prec$ on $\{1, \ldots, n\}$ and let $s(i)$ be the successor of i with respect to the order $\prec$. Then the set of positive roots consists of the roots $\epsilon_i - \epsilon_j$ for $i \prec j$ and $\Delta = \{\epsilon_i - \epsilon_{s(i)}\}$. For such an order, the Borel subalgebra is of the form

$$\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\substack{\alpha = \epsilon_i - \epsilon_j \\ i \prec j \\ i,j \in \{1,\ldots,n\}}} \mathfrak{g}^\alpha.$$

The Borel subalgebras of $\mathfrak{sl}_n$ with respect to $\mathfrak{h}$ are in one-to-one correspondence with the total orders on $\{1, \ldots, n\}$.

Definition 1.28. Let σ_α be the reflection of $\mathfrak{h}^*$ across the hyperplane orthogonal to α, defined by $\lambda \mapsto \lambda - \langle \lambda, \alpha \rangle \alpha$. The **Weyl group** W of a split semi-simple Lie algebra $(\mathfrak{g}, \mathfrak{h})$ is the group generated by σ_α for all $\alpha \in \Phi$.

Proposition 1.29. *Given a basis Δ for a root system Φ, the Weyl group W is generated by the simple reflections σ_α for $\alpha \in \Delta$. Additionally W acts transitively on Φ.*

Let $l(w)$ be the size of the smallest set of simple reflections such that their product forms w for $w \in W$ and set $\text{sgn}(w) = -1^{l(w)}$.

1.4 The Universal Enveloping Algebra and the Poincaré Birkhoff Witt Theorem

All statements and proofs follow those provided in [Dix96].

Let $\mathfrak{g}$ be a Lie algebra and $T(\mathfrak{g})$ the tensor algebra, i.e.

$$T(\mathfrak{g}) = \bigoplus_{k=0}^{\infty} T^k(\mathfrak{g})$$

where $T^k(\mathfrak{g}) = \mathfrak{g}^{\otimes k}$. In particular $T^0(\mathfrak{g}) = k$ and $T^1(\mathfrak{g}) = \mathfrak{g}$. With the product being tensor multiplication, $T(\mathfrak{g})$ is an associative algebra. Let $J \subset T$ be the two-sided ideal generated by all terms of the form

$$x \otimes y - y \otimes x - [x, y]$$

for all $x, y \in \mathfrak{g}$. Given the generators of J, we have $J \subset T_+(\mathfrak{g})$ where $T_+(\mathfrak{g}) = \bigoplus_{k=1}^{\infty} T^k(\mathfrak{g})$.

Definition 1.30. The associative algebra $U(\mathfrak{g}) = T(\mathfrak{g})/J$ is called the **universal enveloping algebra** such that the mapping σ composed of the canonical mappings $\mathfrak{g} \to T(\mathfrak{g})$ and $T(\mathfrak{g}) \to U(\mathfrak{g})$ yields

$$\sigma(x)\sigma(y) - \sigma(y)\sigma(x) = \sigma([x, y])$$

for all $x, y \in \mathfrak{g}$.

The center of the universal enveloping algebra is denoted $Z(\mathfrak{g})$ and in the case of a commutative enveloping algebra, $U(\mathfrak{g}) = S(\mathfrak{g})$.
Let $U^0(\mathfrak{g})$ be the canonical image of $T^0(\mathfrak{g})$ in $U(\mathfrak{g})$ and $U_+(\mathfrak{g})$ the image of $T_+(\mathfrak{g})$. Given that $T(\mathfrak{g}) = T^0(\mathfrak{g}) \oplus T_+(\mathfrak{g})$ and $J \subset T_+(\mathfrak{g})$, we have

$$U(\mathfrak{g}) = U^0(\mathfrak{g}) \oplus U_+(\mathfrak{g})$$

where $U^0(\mathfrak{g}) = k$. Hence $U(\mathfrak{g})$ is generated by the image of 1 and $\mathfrak{g}$ in $U(\mathfrak{g})$.
Denote the canonical image of $\bigoplus_{k=0}^{p} T^p(\mathfrak{g})$ in $U(\mathfrak{g})$ by $U_p(\mathfrak{g})$. Fix a basis $x_1, \ldots, x_n$ for $\mathfrak{g}$ and denote the canonical image x_i in $U(\mathfrak{g})$ by y_i. For any finite sequence I let $y_I = y_{i_1} \ldots y_{i_p}$ for $I = (i_1, \ldots, i_p)$. Define the ordering $i \leq I$ if $i \leq i_1, \ldots, i_p$.

Lemma 1.31. *Let $a_1, \ldots, a_p \in \mathfrak{g}$, the mapping σ be the composition of the canonical mappings $\mathfrak{g} \to T(\mathfrak{g})$ and $T(\mathfrak{g}) \to U(\mathfrak{g})$ and π a permutation of $\{1, \ldots, p\}$. Then:*

$$\sigma(a_1) \ldots \sigma(a_p) - \sigma(a_{\pi(1)}) \ldots \sigma(a_{\pi(p)}) \in U_{p-1}(\mathfrak{g}).$$

Additionally the y_I generate the vector space $U_p(\mathfrak{g})$ for all increasing seqences I of length $\leq p$.

Proof. The permutation is generated by transpositions of the form $(j, j+1)$, so it is sufficient to prove this for transpositions of this form. Given that

$$\sigma(a_1) \ldots \sigma(a_p) - \sigma(a_1) \ldots \sigma(a_{j+1})\sigma(a_j) \ldots \sigma(a_p) = \sigma(a_1) \ldots \sigma([a_j, a_{j+1}]) \ldots \sigma(a_p),$$

the identity holds true. Thus for every increasing sequence I of length $\leq p$ and any permutation π, $y_{\pi(I)}$ is a linear combination of y_J for increasing sequences J of length $\leq p$. The vector space $U_p(\mathfrak{g})$ is therefore generated by the elements y_I for all increasing sequences I of length $\leq p$. □

Lemma 1.32. *Let P be the algebra of polynomials $k[z_1, \ldots, z_n]$ and P_i the set of elements with degree $\leq i$. Set $z_I = z_{i_1} \ldots z_{i_p}$ for $I = (i_1, \ldots, i_p)$. There exists a unique linear mapping $f_p : \mathfrak{g} \otimes P_p \to P$, for each $p \in \mathbb{Z}_{\geq 0}$, which satisfies the following conditions:*

(i) $f_p(x_i \otimes z_I) = z_i z_I$ *for* $i \leq I, z_I \in P_p$;

(ii) $f_p(x_i \otimes z_I) - z_i z_I \in P_q$ *for* $z_I \in P_q, q \leq p$;

(iii) $f_p(x_i \otimes f_p(x_j \otimes z_J)) = f_p(x_j \otimes f_p(x_i \otimes z_J)) + f_p([x_i, x_j] \otimes z_j)$ *for* $z_J \in P_{p-1}$.

Additionally

$$f_p\big|_{\mathfrak{g}\otimes P_{p-1}} = f_{p-1}.$$

Proof. Given $p = 0$, for all $i \in \{1, \ldots, n\}$, the map f_0 must be defined as $f_0(x_i \otimes 1) = z_i$ to satisfy condition ((i)). In this case, the conditions ((ii)) and ((iii)) follow immediately. We shall proceed by induction. Assume there exists some p, such that f_k is unique and satisfies conditions ((i)), ((ii)) and ((iii)) for all $k < p$. If there exists such an f_p, then $f_p\big|_{\mathfrak{g}\otimes P_{p-1}} = f_{p-1}$. Thus it is sufficient to show that f_{p-1} has only one linear extension f_p on $\mathfrak{g} \otimes P_p$ such that the conditions of the lemma are satisfied.
As P is a commutative algebra, we may take I to be an increasing sequence of p elements and define $f_p(x_i \otimes z_I)$. If $i \leq I$, then the choice of f_p is restricted to $f_p(x_i \otimes z_I) = z_i z_I$ by ((i)). If $i \not\leq I$, then I can be written as (j, J) where J is an increasing sequence of $p - 1$ elements and $j < 1, j \leq J$. Then $z_j z_J = f_{p-1}(x_j \otimes z_J)$, and following from conditions ((i)) and ((iii)):

$$\begin{aligned} f_p(x_i \otimes z_I) &= f_p(x_i \otimes f_{p-1}(x_j \otimes z_J)) \\ &= f_p(x_j \otimes f_{p-1}(x_i \otimes z_J)) + f_{p-1}([x_i, x_j] \otimes z_J). \end{aligned}$$

As $z_J \in P_{p-1}$ and ((ii)) holds true for f_{p-1}, we have

$$f_{p-1}(x_i \otimes z_J) - z_i z_J = w \in P_{p-1}$$

and then

$$\begin{aligned} f_p(x_i \otimes z_I) &= f_p(x_j \otimes f_{p-1}(x_i \otimes z_J)) + f_{p-1}([x_i, x_j] \otimes z_J) \\ &= f_p(x_j \otimes (z_i z_J + w)) + f_{p-1}([x_i, x_j] \otimes z_J) \\ &= z_j z_i z_J + f_{p-1}(x_j \otimes w) + f_{p-1}([x_i, x_j] \otimes z_J). \end{aligned}$$

Hence define $f_p(x_i \otimes z_I) = z_i z_I$ for $i \leq I$ and as above for $I = (j, J)$ with $j < i$ and $j \leq J$. This is a unique linear extension of f_{p-1} to $\mathfrak{g} \otimes P_p$ such that the conditions ((i)) and ((ii)) hold by definition. Hence we must show that f_p defined as such satisfies ((iii)), i.e.

$$f_p(x_i \otimes f_p(x_j \otimes z_J)) = f_p(x_j \otimes f_p(x_i \otimes z_J)) + f_p([x_i, x_j] \otimes z_j)$$

for arbitrary i, j and $z_J \in P_{p-1}$.
If $j < i$ and $j \leq J$, then the condition holds from the definition of f_p. As the bracket operation is anti-symmetric, the condition holds for $i < j$ and $i \leq J$ as well. If $i = j$ then the condition is trivially satisfied. Assume that $i, j \not\leq J$. Let $J = (k, K)$ such that $k < i$ and $k < j$. By the induction hypothesis, as $z_K \in P_{p-2}$,

$$\begin{aligned} f_p(x_j \otimes z_J) &= f_p(x_j \otimes f_p(x_k \otimes z_K)) \\ &= f_{p-1}(x_j \otimes f_{p-1}(x_k \otimes z_K)) \\ &= f_{p-1}(x_k \otimes f_{p-1}(x_j \otimes z_K)) + f_{p-1}([x_j, x_k] \otimes z_K) \\ &= f_p(x_k \otimes f_p(x_j \otimes z_K)) + f_p([x_j, x_k] \otimes z_K). \end{aligned}$$

As $z_K \in P_{p-2}$, condition ((ii)) yields $f_p(x_j \otimes z_K) = z_j z_K + w$ for some $w \in P_{p-2}$. Then

$$f_p(x_i \otimes f_p(x_j \otimes z_J)) = f_p(x_i \otimes f_p(x_k \otimes (z_j z_K + w))) + f_p(x_i \otimes f_p([x_j, x_k] \otimes z_K)).$$

For the first term on the right hand side of the equality, condition ((iii)) holds by the definition of f_p as $k < j$, $k \leq K$, and by the induction hypothesis as $w \in P_{p-2}$. The condition holds for the second term as $z_K \in P_{p-2}$. This yields

$$\begin{aligned} f_p(x_i \otimes f_p(x_j \otimes z_J)) &= f_p(x_i \otimes f_p(x_k \otimes f_p(x_j \otimes z_K)) + f_p(x_i \otimes f_p([x_j, x_k] \otimes z_K)) \\ &= f_p(x_k \otimes f_p(x_i \otimes f_p(x_j \otimes z_K)) + f_p([x_i, x_k] \otimes f_p(x_j \otimes z_K)) \\ &\quad + f_p([x_j, x_k] \otimes f_p(x_i \otimes z_K)) + f_p([x_i, [x_j, x_k]] \otimes z_K). \end{aligned}$$

Interchanging i, j and the using the properties of the bracket operation

$$\begin{aligned}
&f_p(x_i \otimes f_p(x_j \otimes z_J)) - f_p(x_j \otimes f_p(x_i \otimes z_J)) \\
&= f_p(x_k \otimes [f_p(x_i \otimes f_p(x_j \otimes z_K)) - f_p(x_j \otimes f_p(x_i \otimes z_K))]) + f_p([x_i, [x_j, x_k]] \otimes z_K) \\
&\quad - f_p([x_j, [x_i, x_k]] \otimes z_K) \\
&= f_p(x_k \otimes f_p([x_i, x_j] \otimes z_K)) + f_p([x_i, [x_j, x_k]] \otimes z_K) + f_p([x_j, [x_k, x_i]] \otimes z_K) \\
&= f_p([x_i, x_j] \otimes f_p(x_k \otimes z_K)) + f_p([x_k, [x_i, x_j]] \otimes z_K) + f_p([x_i, [x_j, x_k]] \otimes z_K) \\
&\quad + f_p([x_j, [x_k, x_i]] \otimes z_K) \\
&= f_p([x_i, x_j] \otimes f_p(x_k \otimes z_K)) \\
&= f_p([x_i, x_j] \otimes z_J).
\end{aligned}$$

Hence

$$f_p(x_i \otimes f_p(x_j \otimes z_J)) = f_p(x_j \otimes f_p(x_i \otimes z_J)) + f_p([x_i, x_j] \otimes z_j)$$

for all i, j and $z_J \in P_{p-1}$, so condition ((iii)) holds for f_p. ☐

Lemma 1.33. *The y_I form a basis for the vector space $U(\mathfrak{g})$ for every increasing sequence I.*

Proof. Combining the maps from the previous lemma for all p yields a bilinear man $f : \mathfrak{g} \times P \to P$ such that $f(x_i, y_I) = y_i y_I$ for $i \leq I$ and

$$f(x_i, f(x_j, z_J)) = f(x_j, f(x_i, z_J)) + f([x_i, x_j], z_J).$$

This defines a representation ϱ of $\mathfrak{g}$ on P such that $\varphi(x_i)z_I = z_i z_I$ for $i \leq I$. By universality of of $U(\mathfrak{g})$ there is a homomorphism $\phi : U(\mathfrak{g}) \to \mathrm{End}(P)$ such that $\phi(y_i)z_I = z_i z_I$ for $i \leq I$. By induction, if $I = (i_1, \ldots, i_p)$ is a increasing sequence then

$$\phi(y_I) = \phi(y_{i_1} \ldots y_{i_p}) \cdot 1 = z_{i_1} \ldots z_{i_p} = z_I.$$

As ϕ is a homomorphism, given that the z_I for each increasing sequence I are linearly independent, so are the y_I.

The y_I, for every increasing sequence I generate $U(\mathfrak{g})$ and are linearly independent. Hence they form a basis for the vector space $U(\mathfrak{g})$. ☐

Theorem 1.34 (Poincare-Birkhoff-Witt)**.** *Let $(x_1, \ldots, x_n)$ be a basis for $\mathfrak{g}$. Then a basis for $U(\mathfrak{g})$ can be formed by the set of elements $x_1^{v_1} x_2^{v_2} \ldots x_n^{v_n}$ where $v_1, \ldots, v_n \in \mathbb{Z}_{\geq 0}$.*

Corollary 1.35. *Assume that $\mathfrak{g}$ has a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, such that $(y_1, \ldots, y_n), (h_1, \ldots, h_n)$ and $(x_1, \ldots, x_n)$ are bases for $\mathfrak{n}_-, \mathfrak{h}$ and $\mathfrak{n}_+$ respectively. Then a basis for $U(\mathfrak{g})$ is formed by the set of elements $y_1^{a_1} \ldots y_n^{a_n} h_1^{b_1} \ldots h_n^{b_n} x_1^{c_1} \ldots x_n^{c_n}$ where $a_i, b_i, c_i \in \mathbb{Z}_{\geq 0}$, and*

$$U(\mathfrak{g} = U(\mathfrak{n}_-)U(\mathfrak{h})U(\mathfrak{n}_+).$$

1.5 Complete Reducibility Theorem

Theorem 1.36 ([Ser05])**.** *If $\mathfrak{g}$ is semisimple, and ϱ is a finite dimensional representation of $\mathfrak{g}$, then ϱ is completely reducible.*

Proof. The proof is completed in steps, following from proof given in [Ser05].

(i) Let $\mathfrak{g}$ be a semisimple Lie algebra and ϱ a finite dimensional representation of $\mathfrak{g}$ in V. We may assume that ϱ is a faithful representation, i.e. injective. If not we may quotient $\mathfrak{g}$ with $\ker(\varrho)$, which does not affect the reducibility of the representation. Define the bilinear form $B_\varrho(x, y) = \mathrm{Tr}_V \varrho(x)\varrho(y)$. As trace is invariant of basis, this form is well defined. Given that ϱ is injective, it is quite clear that B_ϱ is nondegenerate (i.e. if $B_\varrho(x, y) = 0$ for all y, then $x = 0$).

(ii) Let (x_i) be a set basis of $\mathfrak{g}$, and B a nondegenerate bilinear form. As B is nondegenerate, there exists a dual basis (y_i) of $\mathfrak{g}$ with respect to B, in other words $B(x_i, y_j) = \delta_{ij}$ where $\delta_{ij} = 1$ if $i = j$, and 0 otherwise. Define the element

$$b = \sum_{i,j} x_i y_j \in U(\mathfrak{g}).$$

This element is contained in the center of $U(\mathfrak{g})$ and is called the Casimir element corresponding to B.

(iii) Denote the Casimir element corresponding to B_ϱ by c_ϱ. As c_ϱ lies in the center of $U(\mathfrak{g})$, it commutes with $\varrho(\mathfrak{g})$, hence acts as an endomorphism of V as a $\mathfrak{g}$-module. Additionally, given the definition of B_ϱ, we have

$$\mathrm{Tr}(\varrho(c_\varrho)) = \sum_{i,j} \mathrm{Tr}(\varrho(x_i)\varrho(y_j)) = \sum_i \mathrm{Tr}(B_\varrho(x_i, y_i)) = \dim\, \mathfrak{g}.$$

(iv) Assume ϱ is an irreducible representation. Then c_ϱ acts via scalar multiplication on V. Given that $\mathrm{Tr}(c_\varrho) = \dim\, \mathfrak{g}$, $\varrho(c_\varrho)$ is zero if and only if $\mathfrak{g} = 0$. Hence c_ϱ acts via nonzero scalar multiplication on V.

(v) Define the exact sequence of $\mathfrak{g}$-modules

$$0 \longrightarrow W \longrightarrow V \longrightarrow \mathbb{C} \longrightarrow 0$$

where $\mathfrak{g}$ acts trivially on $\mathbb{C}$. Due to the semisimplicity of $\mathfrak{g}$, as $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ and $\mathbb{C}$ is one dimensional, $\mathfrak{g}$ must act trivially. We want to show that the sequence splits, i.e. there is a one dimensional submodule of V complementary to W that maps to $\mathbb{C}$.

(vi) We may reduce the case of the above short exact sequence to W being a simple submodule. As W is a submodule of codimension one in V, for any $X \subset W$, we have W/X is a submodule of codimension one in V/X. Hence there is some $\tilde{W}/X \subset V/X$ which is a complement to W/X in V/X. Then $W \oplus \tilde{W} = V$. Thus we have the following short exact sequence

$$0 \to X \to \tilde{W} \to \mathbb{C} \to 0$$

which yields by construction $\tilde{X} \subset \tilde{W}$ such that $X \oplus \tilde{X} = \tilde{W}$. Then

$$\tilde{X} \oplus W = \tilde{X} \oplus W \oplus X = W \oplus \tilde{W} = V,$$

so $\tilde{X}$ is a complementary subspace in V to W.

(vii) We may assume $\mathfrak{g}$ acts faithfully on W. If not, let $\mathfrak{a} \subset \mathfrak{g}$ be the subalgebra that acts trivially on W. As $\mathfrak{g}$ acts trivially on $\mathbb{C}$, and the homomorphisms of the short exact sequence are $\mathfrak{g}$ homomorphisms, $\mathfrak{a}(V) \subset W$. Since $\mathfrak{a}(W) = 0$, we have $[\mathfrak{a}, \mathfrak{a}]$ acting trivially on V. Since $\mathfrak{g}$ is semisimple, so is $\mathfrak{a}$, hence $[\mathfrak{a}, \mathfrak{a}] = \mathfrak{a}$. As $\mathfrak{a} \subset \mathrm{Ker}(\mathfrak{g} \to \mathrm{End}(V))$ we have $\mathfrak{g}/\mathfrak{a}$ acts on V and acts faithfully on W. Given that $\mathfrak{g}/\mathfrak{a}$ retains its semisimplicity, we may assume that $\mathfrak{g}$ acts on W faithfully.

(viii) Assume ϱ is a representation of $\mathfrak{g}$ in W such that ϱ acts faithfully and W is simple. Then the Casimir element c_ϱ corresponding to B_ϱ acts as a $\mathfrak{g}$ endomorphism on V where, $c_\varrho(V) \subset W$ as c_ϱ acts trivially on $\mathbb{C}$. Assuming that $\mathfrak{g}$ is nonzero we have $c_\varrho(W) = W$. Then $\mathrm{Ker}(c_\varrho) \subset V$ is a complement of W in V that is stable under $\mathfrak{g}$.

(ix) Let

$$0 \to W \to V \to \tilde{W} \to 0$$

be a short exact sequence of arbitrary $\mathfrak{g}$-modules. Viewing $\mathrm{Hom}_{\mathfrak{g}}(V, W)$ as a $\mathfrak{g}$-module, define the following submodules:

$$\mathcal{V} = \{\phi \in \mathrm{Hom}(V, W) \;:\; \phi\big|_W \in \mathbb{C}\}$$

$$\mathcal{W} = \{\phi \in \mathrm{Hom}(V, W) \;:\; \phi\big|_W = 0\}.$$

Thus there is a short exact sequence of $\mathfrak{g}$-modules

$$0 \to \mathcal{W} \to \mathcal{V} \to \mathbb{C} \to 0.$$

From above there exists some element $\phi \in \mathcal{V}$ invariant under $\mathfrak{g}$ that is mapped to 1. Hence $\ker(\phi) \subset V$ is invariant under $\mathfrak{g}$, and is therefore a complementary submodule to W. This yields the decomposition $V = W \oplus \ker(\phi)$, hence the sequence $0 \to W \to V \to \tilde{W} \to$ splits. □

1.6 Weight and weight modules

Definition 1.37. Let ϱ be a representation of $\mathfrak{g}$ on V with splitting Cartan subalgebra $\mathfrak{h}$. For $\lambda \in \mathfrak{h}^*$, a **weight space** is a subspace V_λ of V given by:

$$V_\lambda = \{v \in V \;:\; hv = \lambda(h)v \;\forall h \in \mathfrak{h}\}.$$

For $\lambda \in \mathfrak{h}^*$ such that V_λ is non-zero, λ is termed a **weight** of V and any non-zero $v \in V_\lambda$ is a weight vector of weight λ.

Definition 1.38. Given a weight λ of a $\mathfrak{g}$-module with splitting Cartan subalgebra $\mathfrak{h}$ and root system Φ, it is an **integral weight** if $\langle \lambda, \alpha \rangle \in \mathbb{Z}$ for all $\alpha \in \Phi$. A weight is **dominant integral** with respect to some base Δ if $\langle \lambda, \alpha \rangle \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Delta$, or equivalently all $\alpha \in \Phi_+$.

Definition 1.39. The set of **fundamental weights** of a split semi-simple Lie algebra $(\mathfrak{g}, \mathfrak{h})$ with root system Φ and base $\Delta = \{\alpha_1, \ldots, \alpha_n\}$ is the set $\omega_1, \ldots, \omega_n$ such that $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$ where

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Clearly the integral weights are combinations of fundamental weights with integer coefficients. The set of integral weights is called the **weight lattice**.

Definition 1.40. Let V be a module of $\mathfrak{g}$. If V admits a decomposition into weight spaces, i.e. $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, then V is a **weight module**

Lemma 1.41. *Let V be an arbitrary module of a split semisimple Lie algebra $(\mathfrak{g}, \mathfrak{h})$. Then $\bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$ is a submodule of V. If V is finite dimensional, then $V = \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda$, i.e. V is a weight module.*

Definition 1.42. Let V be a weight module of $\mathfrak{g}$ and v^+ a weight vector with weight λ. If $\mathfrak{n}_+ v^+ = 0$ then v^+ is called a **maximal vector** and and λ is a **highest weight**.

Definition 1.43. Let V be a module of a split semi-simple Lie algebra $(\mathfrak{g}, \mathfrak{h})$ with a chosen base $\Delta \subset \Phi$. Let $\mathfrak{b}_+ = \mathfrak{h} \oplus \mathfrak{n}_+$ be the Borel subalgebra corresponding to Δ. If V is generated by a maximal vector v^+ of weight λ, then it is termed a **$\mathfrak{b}$-highest weight module** with highest weight λ.

Lemma 1.44. *Let V be a $\mathfrak{b}$-highest weight module for a Lie algebra $\mathfrak{g}$. Then:*

(i) V is a weight module;

(ii) the weights of V are of the form $\lambda - \sum_{\alpha \in \Phi_+} n_\alpha \alpha$ where $n_\alpha \in \mathbb{Z}_{\geq 0}$;

(iii) $\dim V_\lambda = 1$.

Lemma 1.45. *Let V, W be irreducible $\mathfrak{b}$-highest weight modules of $\mathfrak{g}$ with the same highest weight λ. Then V and W are isomorphic as $\mathfrak{g}$-modules.*

Proposition 1.46. *Let V be a finite dimensional, irreducibe weight module of $\mathfrak{g}$. Then every weight of V is integral.*

1.7 Induced Representations

Definition 1.47. Let W be an $\mathfrak{h}$-module and $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Given that $U(\mathfrak{g})$ is a right $U(\mathfrak{h})$-module, we may form a left $U(\mathfrak{g})$-module $U(\mathfrak{g}) \otimes_{U(\mathfrak{h})} W$ called the **$\mathfrak{g}$-module induced by** W, denoted $\text{ind}(W, \mathfrak{g})$. If the representation of $\mathfrak{h}$ corresponding to W is ϱ, then we may refer to the **representation of $\mathfrak{g}$ induced by** ϱ, denoted $\text{ind}(\varrho, \mathfrak{g})$.

Definition 1.48. Given a finite dimensional vector space E with a vector subspace F stable under some endomorphism u. Define $\text{tr}_{E/F}u = \text{tr}u - \text{tr}(u|_F)$. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Define

$$\theta_{\mathfrak{g},\mathfrak{h}}(x) = \frac{1}{2}\text{tr}_{\mathfrak{g}/\mathfrak{h}}\text{ad}_{\mathfrak{g}}x$$

for all $x \in \mathfrak{h}$.

Definition 1.49. Let ϱ be a representation of $\mathfrak{h}$ in W and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. Define

$$\varrho^{\sim}(x) = \varrho(x) + \theta_{\mathfrak{g},\mathfrak{h}}(x) \cdot 1$$

for all $x \in \mathfrak{h}$. Then $\varrho^{\sim}$ is a representation of $\mathfrak{h}$ in W and $\text{ind}(\varrho^{\sim}, \mathfrak{g}) = \text{ind}^{\sim}(\varrho, \mathfrak{g})$ the **twisted representation of $\mathfrak{g}$ induced by** ϱ and the corresponding module, $\text{ind}^{\sim}(W, \mathfrak{g})$, is termed the **twisted $\mathfrak{g}$-module induced by the $\mathfrak{h}$-module** W.

2 The Weyl Character Formula

Given a split semi-simple Lie algebra $(\mathfrak{g}, \mathfrak{h})$ and $\lambda \in \mathfrak{h}^*$, one may construct a representation of $\mathfrak{g}$ such that λ is the highest weight with respect to a given Borel subalgebra. Additionally a simple highest weight module may be constructed for any $\lambda \in \mathfrak{h}^*$. In the case of dominant integral weights, one has a finite dimensional simple highest weight module. For such a module, the dimensions of the module and each of its weight spaces may be combinatorically determined via computation by the Weyl character formula.

2.1 Verma Modules

Let $(\mathfrak{g}, \mathfrak{h})$ be a split semi-simple Lie algebra with root system Φ and $\lambda \in \mathfrak{h}^*$. For a chosen base Δ and corresponding Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}_+$, define a one dimensional representation τ_λ of $\mathfrak{b}$ by $\tau_\lambda(h + n) = \lambda(h)$ for $h \in \mathfrak{h}$ and $n \in \mathfrak{n}_+$. Let ϱ be the twisted representation of $\mathfrak{g}$ induced by τ_λ:

$$\varrho = \text{ind}^{\sim}(\tau_\lambda, \mathfrak{g}).$$

From the definition of the twisted induced module:

$$\theta_{\mathfrak{g},\mathfrak{b}}(x) = \frac{1}{2}\text{tr}_{\mathfrak{g}/\mathfrak{b}}\text{ad}_{\mathfrak{g}}x,$$

so $\theta_{\mathfrak{g},\mathfrak{b}}(x) = \frac{1}{2}\text{tr ad}_{\mathfrak{g}}(x)\big|_{\mathfrak{n}_-}$ for $x \in \mathfrak{b}$. As $\text{ad}_{\mathfrak{g}}n$ is nilpotent for $n \in \mathfrak{n}_+$ and $\text{ad}_{\mathfrak{g}}h\big|_{\mathfrak{n}_-} = -2\rho(h)$ for $h \in \mathfrak{h}$, where ρ is the half sum of positive roots, $\theta_{\mathfrak{g},\mathfrak{b}}(h + n) = -\rho(h)$. Hence

$$\tau_\lambda^{\sim}(x) = \tau_\lambda(x) + \theta_{\mathfrak{g},\mathfrak{b}}(x) = \tau_{\lambda-\rho}.$$

The representation ϱ can be seen as the representation of $\mathfrak{g}$ induced by $\tau_{\lambda-\rho}$.

Definition 2.1. The $\mathfrak{g}$-module $M(\lambda)$, corresponding to the representation ϱ, is a $\mathfrak{b}$-highest weight module with highest weight $\lambda - \rho$, called the **Verma module**, i.e:

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}$$

where $\mathbb{C}$ is the module associated with the representation $\tau_{\lambda-\rho}$.

Proposition 2.2. *Let $M(\lambda)$ be a Verma module, $\alpha \in \Phi$ and $x_\alpha \in \mathfrak{g}_\alpha$. Let $\alpha_1, \ldots, \alpha_n \in \Phi_+$ be pairwise distinct roots and $\mathfrak{B}(\mu)$ the number of families $(n_\alpha)_{\alpha \in \Phi_+} \subset \mathbb{Z}_{\geq 0}$ such that $\mu = \sum_{\alpha \in \Phi_+} n_\alpha \alpha$. Then:*

(i) $M(\lambda) = \bigoplus_{\mu \in \mathfrak{h}^*} M(\lambda)_\mu$.

(ii) All weights μ of $M(\lambda)$ are given by

$$\mu = \lambda - \rho - \sum_{\alpha \in \Phi_+} n_\alpha \alpha$$

where $n_\alpha \in \mathbb{Z}_{\geq 0}$, and

$$dim M(\lambda)_\mu = \mathfrak{B}(\lambda - \rho - \mu).$$

(iii) For each weight μ:

$$M(\lambda)_\mu = \sum_{\substack{p_1, \ldots, p_n \in \mathbb{Z}_{\geq 0} \\ \lambda - \delta - p_1\alpha_1 - \cdots - p_n\alpha_n = \mu}} x_{-\alpha_1}^{p_1} \ldots x_{-\alpha_n}^{p_n} \otimes \mathbb{C}.$$

(iv) Additionally:

$$\begin{aligned} M(\lambda)_{\lambda-\rho} &= 1 \otimes \mathbb{C} \\ M(\lambda) &= U(\mathfrak{n}_-) M(\lambda)_{\lambda-\rho} \\ U(\mathfrak{n}_+) M(\lambda)_{\lambda-\rho} &= 0. \end{aligned}$$

Proof. To prove (iv), as $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}$, it is generated by $1 \otimes 1$. Clearly $M(\lambda)_{\lambda-\rho} = 1 \otimes \mathbb{C}$, and $\mathfrak{n}_+(1 \otimes 1) = 0$ by definition of $\tau_{\lambda-\rho}$, so $U(\mathfrak{n}_+) M(\lambda)_{\lambda-\rho} = 0$. By the Poincaré-Birkhoff-Witt Theorem, $U(\mathfrak{g}) = U(\mathfrak{n}_-) U(\mathfrak{h}) U(\mathfrak{n}_+)$, hence

$$M(\lambda) = U(\mathfrak{n}_-) M(\lambda)_{\lambda-\rho}.$$

For (iii), if $h \in \mathfrak{h}$ then

$$\begin{aligned} h(x_{-\alpha_1}^{p_1} \ldots x_{-\alpha_n}^{p_n} \otimes 1) &= [h, x_{-\alpha_1}^{p_1} \ldots x_{-\alpha_n}^{p_n}] \otimes 1 + x_{-\alpha_1}^{p_1} \ldots x_{-\alpha_n}^{p_n} h \otimes 1 \\ &= (-p_1\alpha_1 - \cdots - p_n\alpha_n)(h)(x_{-\alpha_1}^{p_1} \ldots x_{-\alpha_n}^{p_n} \otimes 1) \\ &\quad + (\lambda - \rho)(h)(x_{-\alpha_1}^{p_1} \ldots x_{-\alpha_n}^{p_n} \otimes 1) \\ &= (\lambda - \rho - p_1\alpha_1 - \cdots - p_n\alpha_n)(h)(x_{-\alpha_1}^{p_1} \ldots x_{-\alpha_n}^{p_n} \otimes 1). \end{aligned}$$

Both (i) and (ii) follow directly from (iii). □

Definition 2.3. Let V be a $\mathfrak{g}$-module. The **central character** of V is a Lie algebra homomorphism

$$\chi : Z(\mathfrak{g}) \to \mathbb{C}$$

such that for all $x \in Z(\mathfrak{g})$ and $v \in V$ the action of x is defined by $xv = \chi(x)v$.

Proposition 2.4. *Let V be a $\mathfrak{g}$-module generated by a weight vector v of weight λ, such that $\mathfrak{n}_+ v = 0$. Then:*

(i) There exist a unique homomorphism $\phi : M(\lambda + \rho) \to V$ such that $\phi(1 \otimes 1) = v$ and ϕ is surjective.

(ii) $V = U(\mathfrak{n}_-)v$.

(iii) Every endomorphism of V as a $\mathfrak{g}$-module is scalar.

(iv) V has a central character.

(v) The homomorphism ϕ is bijective if and only if $V \neq 0$ and u acts injectively on V for each nonzero $u \in U(\mathfrak{n}_-)$.

Proof. Both assertions (i) and (ii) follow from the construction of $M(\lambda+\rho)$ as an induced representation and as v generates V as a $\mathfrak{g}$-module. Assume φ is an endomorphism of V as a $\mathfrak{g}$-module. Then

$$h\varphi(v) = \varphi(hv) = \lambda(h)\varphi(v).$$

As V_λ is one dimensional, and $\varphi(v) \in V_\lambda$, there exists some $k \in \mathbb{C}$ such that $\varphi(v) = kv$. Since v generates V, for any $u \in U(\mathfrak{g})$

$$\varphi(uv) = u(\varphi(v)) = kuv.$$

Any element in $Z(\mathfrak{g})$ acts on V as an endomorphism, hence assertion (iv) follows. Finally, if ϕ is an isomorphism then clearly each $u \in U(\mathfrak{n}_-)$ acts injectively on V. If not, there must exist some nonzero $u \in U(\mathfrak{n}_-)$ such that $\phi(u \otimes 1) = 0$. Thus

$$uv = u\phi(1 \otimes 1) = \phi(u \otimes 1) = 0$$

so unless $V = 0$, u does not act injectively. □

Let the central character of $M(\lambda) = \chi_\lambda$.

Proposition 2.5. *Let $\lambda, \mu \in \mathfrak{h}^*$. Then the following statements are equivalent:*

(i) $\chi_\lambda = \chi_\mu$

(ii) $\lambda \in W\mu$.

Proposition 2.6. *Let $M(\lambda)$ be a Verma module of $\mathfrak{g}$. Every proper submodule of $M(\lambda)$ is contained in $M(\lambda)_+ = \sum_{\mu\neq\lambda-\rho} M(\lambda)_\mu$. Additionally there exists a maximal proper submodule K and the $\mathfrak{g}$-module $M(\lambda)/K = L(\lambda)$ is irreducible.*

Proof. Let W be a submodule of $M(\lambda)$ distinct from $M(\lambda)$. Given that $M(\lambda) = \bigoplus_{\mu\in\mathfrak{h}^*} M(\lambda)_\mu$, we have

$$W = \bigoplus_{\mu\in\mathfrak{h}^*} W \cap M(\lambda)_\mu \quad \text{and} \quad W \cap M(\lambda)_{\lambda-\rho} = 0$$

as dim $M(\lambda)_{\lambda-\rho} = 1$. Therefore W must be contained in $M(\lambda)_+$, so the sum of all proper submodules of $M(\lambda)$ is contained in $M(\lambda)_+$. The sum K of all proper submodules of $M(\lambda)$ is thus maximal and $L(\lambda) = M(\lambda)/K$ is simple. □

Proposition 2.7. *If V is a simple $\mathfrak{g}$-module such that there exists a nonzero $v \in V_{\lambda-\rho}$ annihilated by $\mathfrak{n}_+$, then V is isomorphic to $L(\lambda)$.*

Proof. Let $\phi : M(\lambda) \to V$ be the unique surjective homomorphism such that $\phi(1 \otimes 1) = v$. V is simple and $V \simeq M(\lambda)/\ker\phi$, thus $\ker\phi = K$ so

$$V = M(\lambda)/K = L(\lambda).$$

□

Proposition 2.8. *Let V be a finite dimensional simple $\mathfrak{g}$-module. Then*

(i) There exists unique $\lambda \in \mathfrak{h}^*$ such that V is isomorphic to $L(\lambda+\rho)$.

(ii) λ is a dominant integral weight and dim $V_\lambda = 1$.

(iii) If μ is a weight of V, then $\mu = \lambda - \sum_{\alpha\in\Phi_+} n_\alpha\alpha$ for $n_\alpha \in \mathbb{Z}_{\geq 0}$ and $(\mu,\mu) \leq (\lambda,\lambda)$.

Proof. (i) As V is finite dimensional, there must exist some weight λ such that $\lambda+\alpha$ is not a weight of V for all positive roots α. Additionally, as V is simple, it is generated by $v \in V_\lambda$. Thus V is isomorphic to $L(\lambda+\rho)$.

(ii) Let $\mathfrak{s}_\alpha = \mathfrak{h}_\alpha \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha}$ be the subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_\alpha$ where $\alpha \in \Phi_+$. If ϱ is the representation of $\mathfrak{g}$ in V, then $\varrho\big|_{\mathfrak{s}_\alpha}$ is isomorphic to a representation of $\mathfrak{sl}_2$. As V is finite dimensional and $x_\alpha V_\lambda = 0$, we have $\lambda(h_\alpha) = \langle\lambda,\alpha\rangle \in \mathbb{Z}_{\geq 0}$ and V_λ is one dimensional.

(iii) Let μ be a weight of V. There exists some $w \in W$ such that $w\mu$ is a dominant integral weight. Since every element in the Weyl orbit of μ is a weight of V, and the inner product on $\mathfrak{h}^*$ is stable under w, we may assume μ to be dominant integral. As λ is the highest weight, $\lambda - \mu$ is a sum of positive roots with positive integer coefficients. Hence $\langle \lambda + \mu, \lambda - \mu \rangle \geq 0$ so:

$$
\begin{aligned}
0 &\leq (\lambda + \mu, \lambda - \mu) \\
&= (\lambda, \lambda) + (\mu, \lambda) - (\lambda, \mu) - (\mu, \mu) \\
&= (\lambda, \lambda) - (\mu, \mu),
\end{aligned}
$$

and the assertion follows.

□

Lemma 2.9. *Let A be an associative algebra with an underlying Lie algebra defined by the bracket $[x, y] = xy - yx$. Let $x, y, h \in A$ be elements such that $[h, y] = -2y$ and $[x, y] = h$. For $m \in \mathbb{Z}_{\geq 0}$:*

$$[x, y^m] = m(h + m - 1)y^{m-1} = my^{m-1}(h - m + 1).$$

Proof. For $m = 0, 1$ the equality is obvious. Assuming the identity holds for m, then

$$
\begin{aligned}
[x, y^{m+1}] &= xy^{m+1} - y^{m+1}x \\
&= xyy^m - yxy^m + yxy^m - yy^mx \\
&= [x, y]y^m + y[x, y^m] \\
&= hy^m + ym(h + m - 1)y^{m-1} \\
&= hy^m + m(h + m - 1)y^m + [y, m(h + m - 1)y^{m-1}] \\
&= hy^m + m(h + m - 1)y^m + 2my^m \\
&= (m + 1)(h + m)y^m
\end{aligned}
$$

□

Proposition 2.10. *Let $\lambda \in \mathfrak{h}^*$ and $\alpha \in \Delta$. Assume $m = \langle \lambda, \alpha \rangle$ is a positive integer. Given the canonical generator $v = 1 \otimes 1$ of $M(\lambda)$, for nonzero $x_{-\alpha} \in \mathfrak{g}^{-\alpha}$, let V be the submodule of $M(\lambda)$ generated by $v' = x_\alpha^m v$. Then V is isomorphic to $M(\sigma_\alpha \lambda)$.*

Proof. As $v' \neq 0$ and $\sigma_\alpha \lambda = \lambda - m\alpha$, we have

$$v' \in M(\lambda)_{\lambda - \rho - m\alpha} = M(\lambda)_{\sigma_\alpha \lambda - \rho}.$$

If $\beta \in \Delta$ with $\beta \neq \alpha$, then $[\mathfrak{g}^\beta, \mathfrak{g}^\alpha] = 0$ and $\mathfrak{g}^\beta v = 0$ as $\mathfrak{g}^\beta \subset \mathfrak{n}_+$. Hence if $x_\beta \in \mathfrak{g}^\beta$ then

$$x_\beta v' = x_\beta x_{-\alpha}^m v = [x_\beta, x_{-\alpha}^m]v + x_{-\alpha}^m x_\beta v = 0.$$

Let $x_\alpha \in \mathfrak{g}^\alpha$ be such that $[x_\alpha, x_{-\alpha}] = h_\alpha$, where $\langle \lambda, \alpha \rangle = \lambda(h_\alpha)$. Following from the statement of (2.9)

$$
\begin{aligned}
x_\alpha v' &= x_\alpha x_{-\alpha}^m v \\
&= [x_\alpha, x_{-\alpha}^m]v + x_{-\alpha}^m x_\alpha v \\
&= mx_{-\alpha}^{m-1}(h_\alpha - m + 1)v \\
&= mx_{-\alpha}^{m-1}(\lambda(h_\alpha) - \rho(\alpha) - m + 1)v \\
&= 0.
\end{aligned}
$$

Hence $\mathfrak{n}_+ v' = 0$. Additionally as nonzero $u \in U(\mathfrak{n}_-)$ acts injectively on $M(\lambda)$, it does too on any submodule. Therefore V is isomorphic to $M(\sigma_\alpha \lambda)$. □

Lemma 2.11. *Let V be a be a $\mathfrak{g}$-module generated by an element v of V_λ such that $\mathfrak{n}_+ v = 0$. For $\alpha \in \Phi$, let x_α in $\mathfrak{g}^\alpha$. For all $\alpha \in \Delta$, assume $x_{-\alpha}^m v = 0$ for sufficiently large m. Then V is a finite dimensional, simple module.*

Lemma 2.12. *Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight, K the maximal proper submodule of $M(\lambda+\rho)$ and $v = 1\otimes 1$. For all $\alpha \in \Delta$, let $x_{-\alpha}$ be a nonzero element of $\mathfrak{g}^{-\alpha}$ and $m_\alpha = \langle \lambda, \alpha\rangle + 1$. Then*

$$K = \sum_{\alpha\in\Delta} U(\mathfrak{g}) x_{-\alpha}^{m_\alpha} v = \sum_{\alpha\in\Delta} U(\mathfrak{n}_-) x_{-\alpha}^{m_\alpha} v$$

and K has finite codimension.

Proof. As λ is a dominant integral weight, for each $\alpha \in \Delta$, $m_\alpha = \langle \lambda+\rho, \alpha\rangle$ is a positive integer. Thus for each α, the submodule generated by $x_{-\alpha}^{m_\alpha} v$ is

$$V_\alpha = U(\mathfrak{n}_+) x_{-\alpha}^{m_\alpha} v \simeq M(\sigma_\alpha(\lambda+\rho))$$

and is distinct from $M(\lambda+\rho)$. Thus $\sum_{\alpha\in\Delta} V_\alpha$ is distinct from $M(\lambda+\rho)$, and thereby contained in K. For sufficiently large m and each $\alpha \in \Delta$, we have $x_{-\alpha}^m v = 0$ in $M(\lambda+\rho)/\sum_{\alpha\in\Delta} V_\alpha$. Hence the quotient is simple and finite dimensional, so $K = \sum_{\alpha\in\Delta} U(\mathfrak{n}_-) x_{-\alpha}^{m_\alpha} v$ and has finite codimension. □

Theorem 2.13. *There is a one-to-one correspondence between the set of dominant integral weights and the set of classes of finite dimensional, simple $\mathfrak{g}$-modules.*

Proposition 2.14. *Let V be a finite dimensinal $\mathfrak{g}$-module, and $w_0 \in W$ such that $w_0(\Delta) = -\Delta$. Then*

(i) For any $\lambda \in \mathfrak{h}^*$ the orthogonal subspace of V_λ in V^* is $\sum_{\mu\neq-\lambda} V_\mu^*$, so $V_{-\lambda}^*$ can be identified with the dual of V_λ.

(ii) If V is simple and λ is its highest weight, then V^* is a simple module and has highest weight $-w_0\lambda$.

2.2 The Weyl Character Formula

Let $M(\lambda)$ be the Verma module of a split semi-simple Lie algebra $(\mathfrak{g}, \mathfrak{h})$ with root system Φ and base Δ. Define Q_+ to be the set of linear combinations of Φ_+ with nonnegative integer coefficients. Additionally define a partial order on weights by $\lambda \geq \mu$ if $\lambda - \mu \in Q_+$
Denote the additive group of mappings from $\mathfrak{h}^*$ to $\mathbb{Z}$ by $\mathbb{Z}^{\mathfrak{h}^*}$, the subgroup $\mathbb{Z}[\mathfrak{h}^*] \subset \mathbb{Z}^{\mathfrak{h}^*}$ of maps with finite support, and the subgroup $\mathbb{Z}\langle\mathfrak{h}^*\rangle \subset \mathbb{Z}^{\mathfrak{h}^*}$ of maps with support contained in a finite union of sets $v - Q_+$ for $v \in \mathfrak{h}^*$.

Definition 2.15. Let V be a weight module of $\mathfrak{g}$ such that each weight space is finite dimensional. The **character** of V is a map in $\mathbb{Z}^{\mathfrak{h}^*}$ defined by:

$$\text{Ch } V(\mu) = \dim V_\mu.$$

For any $\lambda \in \mathfrak{h}^*$, let $e^\lambda \in \mathbb{Z}^{\mathfrak{h}^*}$ be the map with support $\{\lambda\}$, which takes the value 1 at λ. Then all elements of $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ are of the form $\sum_{\lambda\in\mathfrak{h}^*} c_\lambda e^\lambda$ with $c_\lambda \in \mathbb{Z}$ for all $\lambda \in \mathfrak{h}^*$. Then $\mathbb{Z}\langle\mathfrak{h}^*\rangle$ may be viewed as a ring with multiplication defined by

$$\Big(\sum_{\lambda\in\mathfrak{h}^*} c_\lambda e^\lambda\Big)\Big(\sum_{\mu\in\mathfrak{h}^*} c'_\mu e^\mu\Big) = \sum_{v\in\mathfrak{h}^*}\Big(\sum_{\substack{\lambda+\mu=v\\ \lambda,\mu\in\mathfrak{h}^*}} c_\lambda c'_\mu e^v\Big),$$

specifically $e^\lambda e^\mu = e^{\lambda+\mu}$. The Weyl group W operates on $\mathbb{Z}^{\mathfrak{h}^*}$ by $w(e^\lambda) = e^{w\lambda}$.

Lemma 2.16. *Let V and W be $\mathfrak{g}$-modules. Then:*

(i) if V has a character and W is a submodule of V, both W and V/W have characters, and

$$Ch\ V = Ch\ W + Ch\ V/W;$$

(ii) if Ch V, Ch W $\in \mathbb{Z}\langle \mathfrak{h}^* \rangle$*, then the* $\mathfrak{g}$*-module* $V \otimes W$ *has a character, and*

$$Ch(V \otimes W) = (Ch\ V)(Ch\ W).$$

Lemma 2.17. *Let*

$$d = \sum_{w \in W} sgn(w) e^{w\rho} \in \mathbb{Z}[\mathfrak{h}^*]$$

and

$$K = \sum_{\gamma \in Q_+} \mathfrak{B}(\gamma) e^{-\gamma} \in \mathbb{Z}\langle \mathfrak{h}^* \rangle.$$

As an element of $\mathbb{Z}\langle \mathfrak{h}^* \rangle$*, d is invertible, and* $Ke^{-\rho}d = 1$.

Proposition 2.18. *Let* $\lambda \in \mathfrak{h}^*$*. The* $\mathfrak{g}$*-module* $M(\lambda)$ *has a character, and*

$$Ch\ M(\lambda) = d^{-1} e^{\lambda} \in \mathbb{Z}\langle \mathfrak{h}^* \rangle$$

for $d = \sum_{w \in W} sgn(w) e^{w\rho}$.

Proof. For any $\gamma \in \mathfrak{h}^*$, $\dim M(\lambda)_\gamma = \mathfrak{B}(\lambda - \rho - \gamma)$, so

$$\text{Ch } M(\lambda) = \sum_{\gamma \in \mathfrak{h}^*} \mathfrak{B}(\gamma) e^{\lambda - \rho - \gamma}.$$

As $Ke^{-\rho}d = 1$, and Ch $M(\lambda) = e^{\lambda - \rho}K$, we have

$$\text{Ch } M(\lambda) = d^{-1} e^{\lambda}.$$

□

Lemma 2.19. *Let M be an arbitrary* $\mathfrak{g}$*-module with central character* χ_{λ_0}*, for some* $\lambda_0 \in \mathfrak{h}^*$*, and character in* $\mathbb{Z}\langle \mathfrak{h}^* \rangle$*. Let* D_M *be the set of* $\lambda \in W\lambda_0$ *such that* $Supp(Ch\ M) \cap \{\lambda - \delta + Q_+\} \neq \emptyset$*. Then we have*

$$Ch\ M = \sum_{\lambda \in D_M} n_\lambda Ch\ M(\lambda),$$

where $n_\lambda \in \mathbb{Z}$ *for each* $\lambda \in D_M$*.*

Proof. If $M = 0$, then Supp(Ch M) $= D_M = \emptyset$. Hence assume $M \neq 0$. Let $\mu - \rho \in$ Supp Ch M be a maximal element with respect to the the Borel subalgebra defined by Δ. Assume $\dim M_{\mu-\rho} = m$. Let V be the submodule of M generated by any $v \in M_{\mu-\rho}$. Then there exists a unique surjective homomorphism ϕ such that $\phi(1 \otimes 1) = v$. Thus there must exist some $\mathfrak{g}$-homomorphism $\varphi : (M(\mu))^m \to M$ such that $(M(\mu)_{\mu-\rho})^m$ is mapped bijectively into $M_{\mu-\rho}$. As such, the central character of $M(\mu)$ must be χ_{λ_0}, so $\mu \in W\lambda_0$ and $\mu \in D_M$.
Let $\ker\varphi = L$ and $\text{coker}\varphi = N$. Then there exists an exact sequence of homomorphisms

$$0 \to L \to (M(\mu))^m \to M \to N \to 0.$$

Given that Ch $V =$ Ch $W +$ Ch V/W for any $\mathfrak{g}$-module V and submodule W, and Ch $(M(\mu)^m = m$ Ch $M(\mu)$, we have

$$\text{Ch } M = m \text{ Ch } M(\mu) - \text{Ch } L + \text{Ch } N.$$

As N is the co-kernel of φ, Supp Ch $N \subset$ Supp Ch M, so $D_N \subset D_M$. Since $\mu - \rho$ is a maximal weight with respect to Δ, $\{\mu - \rho + Q_+\} \cap$ Supp Ch $M = \mu - \rho$. Hence $D_N \subsetneq D_M$ given that $\mu - \rho \notin$ Supp Ch N.
If $\lambda \in D_L$, then $\{\lambda - \rho + Q_+\}$ must intersect with Supp Ch $M(\mu)$, as L is a submodule of $(M(\mu))^m$. Thus $\mu - \rho \in \{\lambda - \rho + Q_+\}$, and it follows that $D_L \subset D_M$. As L is the kernel of φ, which maps $(M(\mu)_{\mu-\rho})^m$ bijectively, $L \cap (M(\mu)_{\mu-\rho})^m = 0$. Therefore $\mu \notin D_L$, so $D_L \subsetneq D_M$.
Given that Ch $M \in \mathbb{Z}\langle \mathfrak{h}^* \rangle$, the set D_M must be finite. Hence D_L and D_N are subsets of D_M with strictly smaller cardinalities. As both L, N have central character χ_{λ_0}, induction on the cardinality of the sets D_L and D_N shows Ch M to be a $\mathbb{Z}$-linear combination of characters of $M(\lambda)$ for $\lambda \in W\lambda_0$. □

Theorem 2.20 (The Weyl Character Formula)**.** *Let V be a finite dimensional, irreducible $\mathfrak{g}$-module with highest weight λ. Then*

$$\Big(\sum_{w\in W} sgn(w)e^{w\rho}\Big)\, Ch\ V = \sum_{w\in W} sgn(w)e^{w(\lambda+\rho)}.$$

Proof. As V is a finite dimensional highest weight module, it is isomorphic to $L(\lambda+\rho)$ and has a finitely supported character. Additionally, it has central character $\chi_{\lambda+\rho}$. Since Ch $M(\mu) = d^{-1}e^{\mu}$ we have

$$\text{Ch } V = \sum_{w\in W} n_w d^{-1} e^{w(\lambda+\rho)}, \quad n_w \in \mathbb{Z}.$$

Therefore $d\text{Ch } V = \sum_{w\in W} n_w e^{w(\lambda+\rho)}$. For any $w \in W$, $\text{dim}V_\mu = \text{dim}V_{w\mu}$, so Ch V is stable under the action of W. On the other hand, $wd = \text{sgn}(w)d$, so dCh V is an alternating function. Hence

$$d\text{Ch } V = a\sum_{w\in W} \text{sgn}(w)e^{w(\lambda+\rho)}$$

for some $a \in \mathbb{Z}$. The weight space V_λ is one dimensional, so the coefficient of $e^{\lambda+\rho}$ must be 1 in dCh V, and so $a = 1$. This yields the character formula

$$\Big(\sum_{w\in W} \text{sgn}(w)e^{w\rho}\Big)\text{Ch } V = \sum_{w\in W} \text{sgn}(w)e^{w(\lambda+\rho)}.$$

□

Corollary 2.21. *Let $\mu \in \mathfrak{h}^*$ and V a simple finite dimensional module with highest weight λ. Then the multiplicity of μ as weight of V is*

$$\sum_{w\in W} sgn(w)\mathfrak{B}(w(\lambda+\rho) - (\mu+\rho)).$$

Proof. By the Weyl Character formula:

$$\begin{aligned}
\text{Ch}V &= d^{-1}\sum_{w\in W}\text{sgn}(w)e^{w(\lambda+\rho)}\\
&= Ke^{-\rho}\sum_{w\in W}\text{sgn}(w)e^{w(\lambda+\rho)}\\
&= \sum_{\gamma\in Q_+}\mathfrak{B}(\gamma)e^{-\gamma-\rho}\sum_{w\in W}\text{sgn}(w)e^{w(\lambda+\rho)}.
\end{aligned}$$

Thus $\text{Ch}V = \sum_{w\in W}\text{sgn}(w)\sum_{\gamma\in Q_+}\mathfrak{B}(\gamma)e^{w(\lambda+\rho)-\gamma-\rho}$. Applying this to μ yields

$$\text{Ch}(V)(\mu) = \sum_{w\in W}\text{sgn}(w)\mathfrak{B}(w(\lambda+\rho) - (\mu+\rho)).$$

□

2.2.1 The Weyl Dimension Formula

Theorem 2.22. *Let V be a finite dimensional irreducible module of $(\mathfrak{g}, \mathfrak{h})$ with highest weight λ. Then*

$$dimV = \frac{\Pi_{\alpha\in\Phi_+}\langle\lambda+\rho,\alpha\rangle}{\Pi_{\alpha\in\Phi_+}\langle\rho,\alpha\rangle}.$$

Proof. Let $\alpha \in \Phi_+$ and define $f_\alpha : \mathbb{Z}\langle\mathfrak{h}^*\rangle \to \mathbb{Z}\langle\mathfrak{h}^*\rangle$ by linearly extending $f(e^\mu) = \langle\mu,\alpha\rangle e^\mu$. Given that

$$f_\alpha(e^{\mu+\mu'}) = \langle\mu+\mu',\alpha\rangle e^{\mu+\mu'} = \langle\mu,\alpha\rangle e^{\mu+\mu'} + \langle\mu',\alpha\rangle e^{\mu+\mu'}$$

the map f_α is a derivation. As f_α and f_β commute for $\alpha \neq \beta$, let $f = \Pi_{\alpha\in\Phi_+} f_\alpha$. This is a differential operator but no longer a derivation.
As previously defined, $Ke^{-\rho}d = 1$, thus we have

$$d = \sum_{w\in W} \operatorname{sgn}(w)e^{w\rho} = e^{-\rho}\Pi_{\alpha\in_+}(e^\alpha - 1).$$

Let $v : \mathbb{Z}[\mathfrak{h}^*] \to \mathbb{Z}$ be the homomorphism mapping g to the sum of its values. Thus $v(\text{Ch } V) = \dim V$.
Using the product form of d, apply f and v consecutively, in that order, to $d\text{Ch}V$. As f_α is a derivation,

$$f_\alpha(d\text{Ch } V) = f_\alpha(d)\text{Ch } V + df_\alpha(\text{Ch } V).$$

Additionally $v(e^\alpha - 1) = 0$ for all $\alpha \in \Phi_+$, so for any proper subset $\Phi' \subset \Phi_+$ we have

$$v(\Pi_{\alpha\in\Phi'} f_\alpha(e^\alpha - 1)) = 0.$$

This yields $v(f(d\text{Ch } V)) = v(f(d))v(\text{Ch } V)$. To compute $f(v(d))$, we use use the linearity of f and v, and $d = \sum_{w\in W} \operatorname{sgn}(w)e^{w\rho}$. For e^ρ, clearly $v(f(e^{rho})) = \Pi_{\alpha\in\Phi_+}\langle\rho, \alpha\rangle$. The Weyl group acts transitively on the root system and preserves the usual inner product $(\cdot, \cdot)$ on $\mathfrak{h}^*$, so $\langle w\rho, \alpha\rangle = \langle\rho, w^{-1}\alpha\rangle$. Additionally the number of positive roots mapped to negative roots by $w \in W$ is $l(w) = l(w^{-1})$, $\operatorname{sgn}(w) = (-1)^{l(w)}$, and $\langle\rho, -\alpha\rangle = -\langle\rho, \alpha\rangle$ for any $\alpha \in \Phi_+$. Thus

$$v(f(e^{w\rho})) = (-1)^{l(w)}\Pi_{\alpha\in\Phi_+}\langle\rho, \alpha\rangle.$$

Hence

$$\begin{aligned} v(f(d)) &= \sum_{w\in W} \operatorname{sgn}(w)(-1)^{l(w)}\Pi_{\alpha\in\Phi_+}\langle\rho, \alpha\rangle \\ &= \sum_{w\in W} (-1)^{l(w)}(-1)^{l(w)}\Pi_{\alpha\in\Phi_+}\langle\rho, \alpha\rangle \\ &= |W|\Pi_{\alpha\in\Phi_+}\langle\rho, \alpha\rangle \end{aligned}$$

The right hand side of the Weyl dimension formula is the sum $\sum_{w\in W} \operatorname{sgn}(w)e^{w(\lambda+\rho)}$. From above, $v(f(e^{w(\lambda+\rho)})) = (-1)^{l(w)}\Pi_{\alpha\in\Phi_+}\langle\lambda + \rho, \alpha\rangle$, which yields

$$v(f(\sum_{w\in W} \operatorname{sgn}(w)e^{w(\lambda+\rho)})) = |W|\Pi_{\alpha\in\Phi_+}\langle\lambda + \rho, \alpha\rangle.$$

Hence applying f and consequently v to both sides of the Weyl character formula results in the dimension formula

$$\dim V = \frac{\Pi_{\alpha\in\Phi_+}\langle\lambda + \rho, \alpha\rangle}{\Pi_{\alpha\in\Phi_+}\langle\rho, \alpha\rangle}.$$

□

2.3 Examples

2.3.1 $\mathfrak{g} = \mathfrak{sl}_2$

Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sl}_2$ consisting of diagonal matrices, with corresponding roots $\epsilon_1 - \epsilon_2$ and $\epsilon_2 - \epsilon_1$. Given the normal linear ordering on the $\mathbb{Z}_{>0}$, the half sum of positive roots becomes $\rho = \frac{\epsilon_1-\epsilon_2}{2}$ which is also the fundamental weight. Thus the dominant integral weights are in one-to-one correspondence with nonnegative integers.
Let $\lambda = m\rho$ and V be a finite dimensional, irreducible representation of $\mathfrak{sl}_n$ with highest weight λ. The Weyl character formula yields

$$\begin{aligned} (e^\rho - e^{-\rho})\text{Ch } V &= \sum_{\mu\in\mathfrak{h}^*} c_\mu(e^{\mu+\rho} - e^{\mu-\rho}) \\ &= e^{(m+1)\rho} - e^{-(m+1)\rho}. \end{aligned}$$

Given that $(e^{(m+1)\rho} - e^{-(m+1)\rho})(\mu)$ is nonzero if and only if $\mu = (m+1)\rho, -(m+1)\rho$ we have the the following equalities:

$$c_{m\rho} - c_{(m+2)\rho} = 1$$
$$c_{-(m+2)\rho} - c_{-m\rho} = -1$$
$$c_{n\rho} - c_{(n+2)\rho} = 0, \quad n \neq m, -(m+2).$$

Thus for all integers i, j, $c_{(m+2i+1)\rho} = c_{(m+2j+1)\rho}$. By the initial condition, V is finite dimensional. Therefore there are only finitely many weight spaces, so $c_{(m+2i+1)\rho} = 0$ for all integers i. Additionally $c_{(m+2i)\rho} = c_{(m+2j)\rho}$ and $c_{-(m+2i)\rho} = c_{-(m+2j)\rho}$ for $i, j \geq 1$. Hence $c_{(m+2i)\rho} = c_{-(m+2i)\rho} = 0$ for all $i \geq 1$ from the finite dimensionality of V. Finally the formula results in

$$c_{-m\rho} = c_{-(m-2)\rho} = \cdots = c_{(m-2)\rho} = c_{m\rho}.$$

As $\lambda = m\rho$ is the highest weight of V, its weight space must be one dimensional. Hence the the character of V is given by

$$\text{Ch } V = e^{m\rho} + e^{(m-2)\rho} + \ldots e^{-(m-2)\rho} + e^{-m\rho}.$$

2.3.2 $\mathfrak{g} = \mathfrak{sl}_3$

As before, let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{sl}_3$ consisting of diagonal matrices. With the usual ordering on $\mathbb{Z}_{>0}$, the positive roots are $\alpha = \epsilon_1 - \epsilon_2$, $\beta = \epsilon_2 - \epsilon_3$ and $\gamma = \epsilon_1 - \epsilon_3$ with base $\Delta = \{\alpha, \beta\}$. Thus the half sum of the positive roots is $\rho = \gamma$ and the fundamental weights are $\omega_1 = \epsilon_1$, $\omega_2 = -\epsilon_3$. The Weyl group of $\mathfrak{sl}_3$ is

$$W = \{1, \sigma_\alpha, \sigma_\beta, \sigma_\alpha\sigma_\beta, \sigma_\beta\sigma_\alpha, \sigma_\alpha\sigma_\beta\sigma_\alpha\}.$$

While $\mathfrak{h}^*$ is two dimensional, the weights ϵ_1, ϵ_2, and ϵ_3 are constrained by $\epsilon_1 + \epsilon_2 + \epsilon_3 = 0$, so ω_1, ω_2 function as fundamental weights. Any dominant integral weight is of the form $\lambda = m_1\omega_1 + m_2\omega_2 = (m_1, m_2)$. Then

$$\langle \omega_1 + \omega_2, \alpha \rangle = 1, \quad \langle \omega_1 + \omega_2, \beta \rangle = 1 \quad \langle \omega_1 + \omega_2, \gamma \rangle = 2$$

and

$$\langle (m_1+1)\omega_1 + (m_2+1)\omega_2, \alpha \rangle = m_1 + 1, \quad \langle (m_1+1)\omega_1 + (m_2+1)\omega_2, \beta \rangle = m_2 + 1,$$
$$\langle (m_1+1)\omega_1 + (m_2+1)\omega_2, \gamma \rangle = m_1 + m_2 + 2.$$

Therefore the Weyl dimension formula in the case of $\mathfrak{sl}_3$ becomes

$$\text{dim} V = \frac{(m_1+1)}{1} \frac{(m_2+1)}{1} \frac{(m_1+m_2+2)}{2}.$$

2.3.3 The Natural Representation

Let V be the natural representation of $\mathfrak{sl}_3$. The highest weight is $\lambda = (1, 0) = \epsilon_1$. From the Weyl dimension formula

$$\begin{aligned}\text{dim } V &= \frac{(1+1)}{1} \frac{(0+1)}{1} \frac{(1+0+2)}{2} \\ &= 3.\end{aligned}$$

The dimension of a weight space is invariant under the Weyl orbit of a weight, and dim $V_\lambda = 1$ as λ is the highest weight. The Weyl orbit of λ is as follows

$$\sigma_\alpha(\epsilon_1) = \epsilon_2, \quad \sigma_\beta(\epsilon_1) = \epsilon_1, \quad \sigma_\beta\sigma_\alpha(\epsilon_1) = \epsilon_3, \quad \sigma_\alpha\sigma_\beta(\epsilon_1) = \epsilon_2, \quad \sigma_\alpha\sigma_\beta\sigma_\alpha(\epsilon_1) = \epsilon_3.$$

As dim$V = 3$, the Weyl orbit of ϵ_1 is the set of weights of V, and

$$\text{Ch}(V) = e^{\epsilon_1} + e^{\epsilon_2} + e^{\epsilon_3}.$$

Alternatively, the character of V may be computed with the Weyl character formula, given by

$$(e^{\rho}-e^{\alpha}-e^{\beta}+e^{-\alpha}+e^{-\beta}-e^{\rho})\mathrm{Ch}(V)=e^{2\epsilon_1-\epsilon_3}-e^{2\epsilon_2-\epsilon_3}-e^{2\epsilon_1-\epsilon_2}+e^{2\epsilon_3-\epsilon_2}+e^{2\epsilon_2-\epsilon_1}-e^{2\epsilon_3-\epsilon_1}.$$

Applying $\epsilon_2+\rho$ to both sides yields

$$c_{\epsilon_2}-c_{\epsilon_2+\rho-\alpha}-c_{\epsilon_2+\rho-\beta}+c_{\epsilon_2+\rho+\alpha}+c_{\epsilon_2+\rho+\beta}-c_{\epsilon_2+2\rho}=0.$$

Given that $\lambda=\epsilon_1$ is the highest weight of V, clearly $\epsilon_2+\rho-\alpha=2\epsilon_2-\epsilon_3, \epsilon_2+\rho+\alpha=2\epsilon_1-\epsilon_3, \epsilon_2+\rho+\beta=2\epsilon_2-2\epsilon_2+\epsilon_1, \epsilon_2+2\rho=\epsilon_2+2\epsilon_1-2\epsilon_3$ are not weights of V. Thus all that remains is

$$c_{\epsilon_2}-c_{\epsilon_1}=0$$

and so $c_{\epsilon_2}=1$. Similarly, applying $\epsilon_3+\rho$ to both sides yields $c_{\epsilon_3}=1$. As $\dim V=3$, we obtain the same result

$$\mathrm{Ch}(V)=e^{\epsilon_1}+e^{\epsilon_2}+e^{\epsilon_3}.$$

Of the set of weights of V, there are $\epsilon_1,\ \epsilon_2=\epsilon_1-\alpha,\ \epsilon_3=\epsilon_1-\alpha-\beta$. Thus the lowest weight of V is ϵ_3, so $-\epsilon_3$ is the highest weight of V^*, i.e. $V^*\simeq L(0,1)$.

2.3.4 The Adjoint Representation

The finite dimensional irreducible representation of $\mathfrak{sl}_3$ with highest weight $\lambda=(1,1)$ is the adjoint representation. Then

$$\begin{aligned}\dim(V)&=\frac{1+1}{1}\frac{1+1}{1}\frac{1+1+2}{2}\\&=8.\end{aligned}$$

The highest weight of V is $\gamma=\rho=\epsilon_1-\epsilon_3$. As the Weyl group acts transitively on the set of roots, $\dim V_\mu=1$ for all $\mu\in\Phi$. The Weyl orbit of ρ is as follows

$$\sigma_\alpha(\rho)=\beta,\quad \sigma_\beta(\rho)=\alpha,\quad \sigma_\alpha\sigma_\beta(\rho)=-\alpha,\quad \sigma_\beta\sigma_\alpha=-\beta,\quad \sigma_\alpha\sigma_\beta\sigma_\alpha(\rho)=-\rho.$$

Given that $\lambda+\rho=2\rho:$

$$\sum_{w\in W}\mathrm{sgn}(w)e^{w(\lambda+\rho)}=e^{2\rho}-e^{2\beta}-e^{2\alpha}+e^{-2\alpha}+e^{-2\beta}-e^{-2\rho}$$

and

$$d\mathrm{Ch}(V)=\sum_{\mu\in\mathfrak{h}^*}c_\mu(e^{(\mu+\rho)}-e^{(\mu+\beta)}-e^{(\mu+\alpha)}+e^{(\mu-\alpha)}+e^{(\mu-\beta)}-e^{(\mu-\rho)}).$$

Applying both sides of the Weyl character formula to ρ results with

$$d\mathrm{Ch}(V)(\rho)=c_0-c_{\rho-\beta}-c_{\rho-\alpha}+c_{\rho+\alpha}+c_{\rho+\beta}-c_{2\rho}=0.$$

The highest weight of V is ρ, so $\rho+\alpha,\rho+\beta$ and 2ρ are not weights of V. As $\rho-\alpha$ and $\rho-\beta$ are both roots of $\mathfrak{sl}_3$, $c_{\rho-\alpha}=c_{\rho-\beta}=1$. Hence $c_0=2$ and

$$\mathrm{Ch}(V)=2e^0+e^{\rho}+e^{\alpha}+e^{\beta}+e^{-\alpha}+e^{-\beta}+e^{-\rho}.$$

3 Homomorphisms Between Verma Modules

All theorems are taken from [Dix96] with the proofs supplemented by [Hum08].

3.1 Submodules and Subquotients of Verma Modules

Unless otherwise specified, $\mathfrak{g}$ will denote a finite dimensional, semisimple Lie algebra.

Definition 3.1. Let ϱ be a representation of $\mathfrak{g}$ in V. A series $(V_0, \dots, V_n)$ of submodules of V such that

$$V = V_0 \supset V_1 \supset \cdots \supset V_n = 0$$

is termed a **composition series**. If V_i/V_{i+1} is simple for each $i \in \{0, \dots, n-1\}$, then the series is termed a **Jordan-Hölder series**.

Definition 3.2. A ring is called a **Noetherian ring** if each every chain of ascending ideals terminates eventually. A module is termed a **Noetherian module** if it satisfies the following equivalent conditions:

(i) every chain of ascending submodules terminates eventually;

(ii) every submodule is finitely generated.

While it will not be proven here, we shall take from [MR01] that the universal enveloping algebra of a finite dimensional Lie algebra is a Noetherian ring.

Proposition 3.3. *Let $\lambda \in \mathfrak{h}^*$ and $M(\lambda)$ the associated Verma module of the split semi-simple Lie algebra $(\mathfrak{g}, \mathfrak{h})$. Then:*

(i) $M(\lambda)$ has a Jordan-Hölder series;

(ii) Every simple subquotient of $M(\lambda)$ is isomorphic to $L(\mu)$ for some $\mu \in W\lambda \cap (\lambda - Q_+)$.

Proof. (ii) Let N and N' be submodules of $M(\lambda)$ such that $N' \subset N$ and N/N' is simple. Given that the decomposition of $M(\lambda)$ into weight spaces is inherited by quotients, we have

$$N/N' = \bigoplus_{\mu \in \mathfrak{h}^*} (N/N')_\mu.$$

The set of weights of N/N' must be contained in the set of weights of $M(\lambda)$. Hence the weights of N/N' are contained in $\lambda - \rho - Q_+$. Thus there must exist some $\mu \in \lambda - Q_+$ such that $\mu - \rho$ is a weight of N/N' but $\mu - \rho + \alpha$ is not for all $\alpha \in \Delta$. Then for some nonzero $v \in (N/N')_{\mu-\rho}$, the module N/N' is generated by v and $\mathfrak{n}_+ v = 0$. Thus $N/N' \simeq L(\mu)$. Conversely, as N/N' is a subquotient of $M(\lambda)$, the two share a central character. Hence, as $\chi_\lambda = \chi\mu$, μ must lie in the Weyl orbit of λ. Therefore the simple subquotient N/N' is isomorphic to $L(\mu)$, where $\mu \in W\lambda \cap (\lambda - Q_+)$.

(i) Given that $M(\lambda)$ is generated by a single element and $U(\mathfrak{g})$ is a Noetherian ring, $M(\lambda)$ is a Noetherian $U(\mathfrak{g})$-module. Thus every nonzero submodule N of $M(\lambda)$ contains a maximal submodule N' such that N/N' is simple. If $M(\lambda)$ has no Jordan-Hölder series, then there would exist an infinite decreasing sequence of submodules $(N_0, N_1, \dots)$ such that N_i/N_{i+1} is simple. As a direct consequence of (ii), infinitely many of these subquotients would be isomorphic to $L(\mu)$ for some $\mu \in W\lambda \cap (\lambda - Q_+)$. This would imply that μ would be a weight of infinite multiplicity in $M(\lambda)$. Given that every weight space of $M(\lambda)$ must be finite dimensional, there must exist a Jordan-Hölder series of $M(\lambda)$. □

Proposition 3.4. *Let $\lambda, \mu \in \mathfrak{h}^*$. If $M(\mu)$ is isomorphic to a submodule of $M(\lambda)$ then $\mu \leq \lambda$ and $\mu \in W(\lambda)$.*

Proof. If $M(\mu)$ is isomorphic to a submodule of $M(\lambda)$ then $\mu - \rho \in \lambda - \rho - Q_+$, so $\lambda - \mu \in Q_+$ and $\mu \leq \lambda$. Additionally, as a submodule, $M(\mu)$ and $M(\lambda)$ share the same central character, so $\mu \in W\lambda$. □

Lemma 3.5. *Let R be a left Noetherian ring (i.e. each chain of ascending left ideals terminates eventually). Assume $x \in R$ is not a right zero divisor. The ideal Rx must intersect nontrivially with each left ideal I. In particular, if R has no right zero divisors, then any two nonzero left ideals must intersect nontrivially.*

Proof. Assume $I \subset R$ is a nontrivial left ideal, $x \in R$ is not a right divisor and $I \cap Rx = 0$. Hence $x \notin I$, so there is a chain of ideals

$$0 \subset I \subset I + Ix \subset I + Ix + Ix^2 \subset \dots$$

where each inclusion is proper, in contradiction to the Noetherian property of R. If the the sequence terminates then there must be some n such that there exists nonzero $a_i \in I$ where

$$a_0 + a_1x + a_2x^2 + \dots + a_nx^n = 0.$$

This implies that $a_0 = -(a_1 + a_2x + \dots a_nx^{n-1})x$ and therefore

$$a_0 = -(a_1 + a_2x + \dots a_nx^{n-1})x \in I \cap Rx = 0.$$

Since x is not a right zero divisor, this forces $a_1 + a_2x + \dots a_nx^{n-1} = 0$. Inductively, $a_i = 0$ for all $i \in \{0, \dots, n\}$, yielding a contradiction. Hence the intersection of I and Rx must be nontrivial. ☐

Proposition 3.6. *Let $\lambda \in \mathfrak{h}^*$ and $M(\lambda)$ the associated Verma module. Then*

(i) there exists a smallest nonzero submodule V in $M(\lambda)$;

(ii) the module V is isomorphic to $M(\mu)$ for some $\mu \in \mathfrak{h}^*$.

Proof. For any $\lambda \in \mathfrak{h}^*$, $M(\lambda)$ and $U(\mathfrak{n}_-)$ are isomorphic as $\mathfrak{n}_-$-modules when the latter is equipped with the left regular representation. From above, any two nonzero left ideals of $U(\mathfrak{n}_-)$ must intersect nontrivially. Given that any $\mathfrak{g}$-submodule of $M(\lambda)$ is also a $\mathfrak{n}_-$-submodule, any two such nonzero submodules must also intersect nontrivially. As $M(\lambda)$ has a Jordan-Hölder series, it must contain a minimal submodule V. This is the unique smallest submodule.

As V is the smallest nonzero submodule in the Jordan-Hölder series, $V/0$ is simple and therefore isomorphic to $L(\mu)$ for some μ. As a submodule of $M(\lambda)$ however, $U(\mathfrak{n}_-)$ acts injectively on V. Hence $V \simeq M(\mu) = L(\mu)$. ☐

3.2 Homomorphisms Between Verma Modules

Theorem 3.7. *Let $\lambda, \mu \in \mathfrak{h}^*$. Then*

(i) every nonzero homomorphism $\varphi : M(\mu) \to M(\lambda)$ is injective;

(ii) the space $Hom_{\mathfrak{g}}(M(\mu), M(\lambda))$ has dimension at most one.

Proof. (i) Let v_μ and v_λ be the canonical generators of $M(\mu)$ and $M(\lambda)$ respectively and assume $\varphi : M(\mu) \to M(\lambda)$ is a nonzero homomorphism of $\mathfrak{g}$-modules. Given that $M(\mu)$ and $M(\lambda)$ are isomorphic to $U(\mathfrak{n}_-)$ as left $U(\mathfrak{n}_-)$ modules, there must exist some $u \in U(\mathfrak{n}_-)$ such that $\varphi(v_\mu) = uv_\lambda$. Assume that φ is not injective. Then there exists nonzero $u' \in U(\mathfrak{n}_-)$ such that $u'v_\mu \neq 0$ but $\varphi(u'v_\mu) = 0$. Then

$$\varphi(u'v_\mu) = u'\varphi(v_\mu) = (u'u)v_\lambda = 0.$$

As u' is nonzero by assumption, and $U(\mathfrak{n}_-)$ contains no right zero divisors, $u = 0$. This yields

$$\varphi(M(\mu)) = \varphi(U(\mathfrak{n}_-)v_\mu) = U(\mathfrak{n}_-)\varphi(v_\mu) = U(\mathfrak{n}_-)uv_\lambda = 0,$$

so φ is trivial. Hence every nonzero $\mathfrak{g}$-homomorphism must be injective.

(ii) Let $\varphi_1, \varphi_2 : M(\mu) \to M(\lambda)$ be $\mathfrak{g}$-homomorphisms such that $\varphi_1(M(\mu)) = \varphi_2(M(\mu))$. Then there must exist some automorphism ϕ of $M(\mu)$ such that $\varphi_2 = \varphi_1 \circ \phi$. However all automorphisms of $M(\mu)$ act via scalar multiplication, hence φ_1 and φ_2 are linearly dependent. If $M(\mu)$ is simple and both φ_1 and φ_2 are nonzero, then both homomorphisms map $M(\mu)$ to the unique simple submodule of $M(\lambda)$, so $\varphi_1(M(\mu)) = \varphi_2(M(\mu))$. Finally we have the general case. As $M(\mu)$

contains a simple submodule isomorphic to $L(\vartheta) = M(\vartheta)$, there exists an injective homomorphism $\psi : M(\vartheta) \to M(\mu)$. As $M(\vartheta)$ is simple, we have already shown that $\mathrm{Hom}_{\mathfrak{g}}(M(\vartheta), M(\lambda))$ has dimension at most one. Define the linear mapping

$$f : \mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda)) \to \mathrm{Hom}_{\mathfrak{g}}(M(\vartheta), M(\lambda))$$

by $\varphi \mapsto \varphi \circ \psi$. Assume $f(\varphi) = 0$. Then $\varphi = 0$ as $\psi(M(\vartheta)) \neq 0$ and we have already shown that all nonzero homomorphisms are injective. Hence f is injective and

$$\dim \mathrm{Hom}_{\mathfrak{g}}(M(\mu), M(\lambda)) \leq 1.$$

□

3.3 Existence of Embeddings (Integral Case)

Proposition 3.8. *Let $\lambda \in \mathfrak{h}^*$ be a dominant integral weight and $w \in W$. If $w = \sigma_n \dots \sigma_1$ is a reduced decompostion of w, where σ_i is the reflection associated with $\alpha_i \in \Delta$, let*

$$\lambda_0 = \lambda,\ \lambda_1 = \sigma_1\lambda_0,\ \dots,\ \lambda_i = \sigma_i\lambda_{i-1},\ \dots,\ \lambda_n = w\lambda.$$

Then:

(i) $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n$ *and* $\langle \lambda_i, \alpha_{i+1} \rangle \in \mathbb{Z}_{>0}$ *for all* $i \in \{0, \dots, n-1\}$;

(ii) $M(\lambda_0) \supset M(\lambda_1) \supset \cdots \supset M(\lambda_n) = M(w\lambda)$.

Lemma 3.9. *Let $\mathfrak{a}$ be a nilpotent Lie algebra with universal enveloping algebra $U(\mathfrak{a})$. Let $x \in \mathfrak{a}$, $u \in U(\mathfrak{a})$ and p a nonnegative integer. Then there exists $l \in \mathbb{Z}_{\geq 0}$ such that*

$$x^l u \in U(\mathfrak{a})x^p.$$

Proof. Denote left (resp. right) multiplication by x on $U(\mathfrak{a})$ by l_x (resp. r_x). The adjoint action of $\mathfrak{a}$ on itself extends to $U(\mathfrak{a})$ by $\mathrm{ad}x = l_x - r_x$. Through straightforward computation it is obvious that l_x, r_x and $\mathrm{ad}x$ commute. As $\mathfrak{a}$ is nilpotent, there exists some nonegative integer q such that $(\mathrm{ad}x)^q = 0$. Let $u \in U(\mathfrak{a})$ and fix p. Then

$$\begin{aligned} x^l u &= l_x^l u \\ &= (r_x + \mathrm{ad}x)^l u \\ &= \sum_{i=0}^{l} \binom{l}{i} r_x^{l-i} (\mathrm{ad}x)^i u \\ &= \sum_{i=0}^{q} \binom{l}{i} r_x^{l-i} (\mathrm{ad}x)^i u \\ &= \sum_{i=0}^{q} \binom{l}{i} (\mathrm{ad}x)^i u x^{l-i} \\ &\in U(\mathfrak{a})x^{l-q}. \end{aligned}$$

Choose l such that $l \geq p + q$. Then the lemma holds. □

Lemma 3.10. *Let $\lambda, \mu \in \mathfrak{h}^*$ and $\alpha \in \Delta$. Assume*

$$M(\sigma_\alpha \mu) \subset M(\mu) \subset M(\lambda)$$

and $p = \langle \lambda, \alpha \rangle$ is an integer.

(i) If $p \leq 0$, then $M(\lambda) \subset M(\sigma_\alpha \lambda)$.

(ii) If $p > 0$, then $M(\sigma_\alpha \mu) \subset M(\sigma_\alpha \lambda) \subset M(\lambda)$.

Proof. (i) If $p \leq 0$, then
$$\langle \sigma_\alpha \lambda, \alpha \rangle = \langle \lambda, \sigma_\alpha \alpha \rangle = -\langle \lambda, \alpha \rangle \geq 0.$$
Thus $M(\lambda) \subset M(\sigma_\alpha \lambda)$ by (2.10).

(ii) Let v_μ and v_λ be the canonical generators of $M(\mu)$ and $M(\lambda)$ respectively. Choose $x_\alpha \in \mathfrak{g}^\alpha$ and $y_\alpha \in \mathfrak{g}^\alpha$ such that $[x_\alpha, y_\alpha] = h_\alpha$ where $\lambda(h_\alpha) = \langle \lambda, \alpha \rangle$. Given that $M(\sigma_\alpha \mu) \subset M(\mu)$, by (2.10), $M(\sigma_\alpha)$ is generated by $y_\alpha^m v_\mu$ where $m = \langle \mu, \alpha \rangle \in \mathbb{Z}_{\geq 0}$. Similarly $M(\sigma_\alpha \lambda)$ is a submodule of $M(\lambda)$ generated by $y_\alpha^p v_\lambda$.

Since $M(\mu) \subset M(\lambda)$, there exists some $u \in U(\mathfrak{n}_-)$ such that $v_\mu = u v_\lambda$. As $\mathfrak{n}_-$ is a nilpotent Lie algebra, there exists some $l \in \mathbb{Z}_{\geq 0}$ such that $y_\alpha u \in U(\mathfrak{n}_-) y_\alpha^p$. Hence
$$y_\alpha^l v_\mu = y_\alpha^l u v_\lambda \in U(\mathfrak{n}_-) y_\alpha^p v_\lambda \subset M(\sigma_\alpha \lambda).$$
If $l \leq m$ then we are done as then $y_\alpha^m u \in U(\mathfrak{n}_-) y_\alpha^p$. Therefore assume that $l \geq m$. From (2.9), as v_μ is a vector of weight $\mu - \rho$, we have
$$\begin{aligned}[x_\alpha, y_\alpha^l] v_\mu &= l y_\alpha^{l-1}(h_\alpha - l + 1) v_\mu \\ &= l(m-l) y_\alpha^{l-1} v_\mu.\end{aligned}$$
Given that v_μ is annihilated by $\mathfrak{n}_+$, we have
$$l(m-l) y_\alpha^{l-1} v_\mu = x_\alpha y_\alpha^l v_\mu - y_\alpha^l x_\alpha v_\mu = x_\alpha y_\alpha^l v_\mu \in M(\sigma_\alpha \lambda),$$
thus inductively $y_\alpha^m v_\mu \in M(\sigma_\alpha \lambda)$. The module $M(\sigma_\alpha \mu)$ is generated by $y_\alpha^m v_\mu$, hence
$$M(\sigma_\alpha \mu) \subset M(\sigma_\alpha \lambda) \subset M(\lambda).$$
□

Theorem 3.11 ([Hum08]). *Let $\lambda \in \mathfrak{h}^*$ be an integral weight and $\alpha \in \Phi_+$. Assume that $\mu = \sigma_\alpha \lambda \leq \lambda$. Then there exists an embedding*
$$M(\mu) \subset M(\lambda).$$

Proof. (i) Given that λ is an integral weight, μ must be integral as well. There exists some $w \in W$ such that $\mu' = w^{-1}\mu$ is a dominant integral weight. If we have a reduced decomposition $w = \sigma_n \dots \sigma_1$, by (3.8) there exists a sequence of embeddings relative to this decomposition such that
$$M(\mu') = M(\mu_0) \supset M(\mu_1) \supset \dots \supset M(\mu_n) = M(\mu),$$
where $\mu_k = \sigma_k \dots \sigma_1 \mu_0$ for $k \in \{1, \dots, n\}$. Additionally the weights are ordered by
$$\mu_0 \geq \mu_1 \geq \dots \geq \mu_n.$$

(ii) Define a parallel sequence of weights associated to λ, with $\lambda' = w^{-1}\lambda$. Then $\lambda' = \lambda_0$, and for $k \in \{1, \dots, n\}$ define $\lambda_k = \sigma_k \dots \sigma_1 \lambda_0$, so $\lambda_n = \lambda$.

(iii) We may assume without loss of generality that $\mu < \lambda$, as the theorem holds trivially with equality. This implies $\mu_k \neq \lambda_k$ for all $k \in \{0, \dots, n\}$ given that the Weyl group acts transitively on roots. Since $\mu = \sigma_n \dots \sigma_1 \mu_0$ and $\lambda = \sigma_n \dots \sigma_1 \lambda_0$, then $\sigma_n \dots \sigma_1 \mu_0 = \sigma_\alpha \sigma_n \dots \sigma_1 \lambda_0$. This results in
$$\mu_k = \sigma_k \dots \sigma_1 \mu_0 = \sigma_{k+1} \dots \sigma_n \sigma_\alpha \sigma_n \dots \sigma_1 \lambda_0 = (\sigma_n \dots \sigma_{k+1})^{-1} \sigma_\alpha (\sigma_n \dots \sigma_{k+1}) \lambda_k.$$
As a conjugate of a reflection, $(\sigma_n \dots \sigma_{k+1})^{-1} \sigma_\alpha (\sigma_n \dots \sigma_{k+1})$ is itself a reflection, hence $\mu_k = \sigma_{\beta_k} \lambda_k$ for some $\beta_k \in \Phi_+$. This implies that $\mu_k - \lambda_k = -\langle \lambda_k, \beta_k \rangle \beta_k$ is an integral multiple of β_k, as λ_k is an integral weight.

(iv) From (3.8), we have $\mu_k \geq \mu_{k+1}$ for all $k \in \{0, \dots, n-1\}$. Additionally μ' is the unique dominant integral weight in the Weyl orbit of μ and λ. By assumption $\mu < \lambda$ and $\mu' \geq \lambda'$. Hence there must exists a least $k \in \{0, \dots, n\}$ such that $\mu_k > \lambda_k$ but $\mu_{k+1} < \lambda_{k+1}$.

(v) Given such a k, by definition

$$\mu_{k+1} - \lambda_{k+1} = \sigma_{k+1} \dots \sigma_1 \mu_0 - \sigma_{k+1} \dots \sigma_1 \lambda_0 = \sigma_{k+1}(\mu_k - \lambda_k).$$

In this case $\mu_{k+1} - \lambda_{k+1}$ is a negative multiple of β_{k+1} and $\mu_k - \lambda_k$ a positive multiple of β_k. As σ_{k+1} is a simple reflection, the only positive root mapped to a negative root is α_{k+1}. This forces $\beta_k = \beta_{k+1} = \alpha_{k+1}$. By (2.10), this yields the inclusion

$$M(\mu_{k+1}) \subset M(\lambda_{k+1}).$$

(vi) Following (3.8) and the previous step, we have the chain of embeddings

$$M(\mu_{k+2}) = M(\sigma_{k+2}\mu_{k+1}) \subset M(\mu_{k+1}) \subset M(\lambda_{k+1}).$$

This fulfills the initial conditions of (3.10), and either alternative results in the same inclusion $M(\mu_{k+2}) = M(\sigma_{k+2}\mu_{k+1}) \subset M(\sigma_{k+2}\lambda_{k+1}) = M(\lambda_{k+2})$.

(vii) Inductively this argument yields

$$M(\mu) = M(\sigma_n \mu_{n-1}) \subset M(\sigma_n \lambda_{n-1}) = M(\lambda).$$

□

3.4 Existence of Embeddings (General Case)

The theorem of existence of embeddings of Verma modules also holds for any arbitrary weight $\lambda \in \mathfrak{h}^*$ and may be reformulated as follows.

Theorem 3.12 ([Hum08]). *Let $\lambda \in \mathfrak{h}^*$ be and $\alpha \in \Phi_+$. Assume that $\mu = \sigma_\alpha \lambda \leq \lambda$. Then there exists an embedding*

$$M(\mu) \subset M(\lambda).$$

While the previous theorem guarantees the existence of embeddings based on certain criterion, the following states the equivalence between embeddings of Verma modules and the ordering on weights within the Weyl orbit of the highest weight. While I will not be providing the proof, complete proofs can be found in both [Hum08] and [Dix96].

Theorem 3.13 ([Dix96]). *Let $\lambda, \mu \in \mathfrak{h}^*$. The following conditions are equivalent:*

(i) $M(\mu) \subset M(\lambda)$

(ii) There exists $\alpha_1, \dots, \alpha_n \in \Phi_+$ such that

$$\lambda \geq \sigma_{\alpha_1}\lambda \geq \sigma_{\alpha_2}\sigma_{\alpha_1}\lambda \geq \dots \geq \sigma_{\alpha_n} \dots \sigma_{\alpha_1}\lambda = \mu$$

3.5 Antidominant Weights and Simple Verma Modules

Definition 3.14. Let $\lambda \in \mathfrak{h}^*$. If, for all $\alpha \in \Phi_+$:

$$\langle \lambda, \alpha \rangle \notin \mathbb{Z}_{>0}$$

then λ is termed an **antidominant weight**.

Theorem 3.15 ([Dix96]). *Let $\lambda \in \mathfrak{h}^*$. Then the following statements are equivalent:*

(i) $M(\lambda) = L(\lambda)$;

(ii) λ *is an antidominant weight.*

Proof. Not (ii) $\Rightarrow$ not (i): Assume that λ is not antidominant. Then there exists some $\alpha \in \Phi_+$ such that $\langle \lambda, \alpha \rangle \in \mathbb{Z}_{>0}$. Hence

$$\lambda - \sigma_\alpha \lambda = \lambda - (\lambda - \langle \lambda, \alpha \rangle \alpha) = \langle \lambda, \alpha \rangle \alpha$$

so $\sigma_\alpha \lambda < \lambda$. By (3.12), we have a proper inclusion $M(\sigma_\alpha \lambda) \subset M(\lambda)$. Thus $M(\lambda)$ is not simple, so $M(\lambda) \neq L(\lambda)$.

Not (i) $\Rightarrow$ not (ii): If $M(\lambda) \neq L(\lambda)$ then by (3.6) there exists a smallest proper nonzero submodule of $M(\lambda)$ isomorphic to $M(\mu)$. Following from (3.4), $\mu < \lambda$ and $\mu \in W\lambda$, so there exists at least one $\alpha \in \Phi_+$ such that $\sigma_\alpha \lambda < \lambda$ which forces $\langle \lambda, \alpha \rangle \in \mathbb{Z}_{>0}$. Hence λ is not an antidominant weight. ☐

3.6 The example of $M(\rho)$

In the case of the $\mathfrak{sl}_3$, let $M(\rho)$ be the Verma module with highest weight 0. Given that ρ is a dominant integral weight:

$$M(w\rho) \subset M(\rho)$$

for all $w \in W$. Given the set of positive roots $\Phi_+ = \{\alpha, \beta, \rho\}$ and that the Weyl orbit of ρ corresponds to

$$\sigma_\alpha(\rho) = \beta, \quad \sigma_\beta(\rho) = \alpha, \quad \sigma_\alpha \sigma_\beta(\rho) = -\alpha, \quad \sigma_\beta \sigma_\alpha(\rho) = -\beta, \quad \sigma_\alpha \sigma_\beta \sigma_\alpha(\rho) = -\rho,$$

there exists the following chains of embeddings of Verma modules within $M(\rho)$.

3.6.1 The chains corresponding to α and β

Beginning with the element $\sigma_\beta = \alpha$, we have

$$\rho - \sigma_\beta(\rho) = \rho - \alpha = \beta \in Q_+$$

$$\alpha - \sigma_\alpha \sigma_\beta(\rho) = \alpha - -\alpha = 2\alpha \in Q_+$$

$$-\alpha - \sigma_\beta \sigma_\alpha \sigma_\beta(\rho) = -\alpha - -\rho = \beta \in Q_+$$

which yields the following order of weights $\rho \geq \alpha \geq -\alpha \geq -\rho$. Hence

$$M(\rho) \supset M(\alpha) \supset M(-\alpha) \supset M(-\rho).$$

Similarly, given $\sigma_\alpha(\rho) = \beta$

$$\rho - \sigma_\alpha(\rho) = \rho - \beta = \alpha \in Q_+$$

$$\beta - \sigma_\beta \sigma_\alpha(\rho) = \beta - -\beta = 2\beta \in Q_+$$

$$-\beta - \sigma_\alpha \sigma_\beta \sigma_\alpha(\rho) = -\beta - -\rho = \alpha \in Q_+$$

there exists a chain of embeddings $M(\rho) \supset M(\beta) \supset M(-\beta) \supset M(-\rho)$.

3.6.2 The chains containing both α and β

Given that $\alpha \not\leq \beta$ and $\beta \not\leq \alpha$, there is no embedding between $M(\alpha)$ and $M(\beta)$. Beginning with the embedding

$$M(\rho) \supset M(\alpha)$$

we have $\sigma_\beta \sigma_\alpha \sigma_\beta(\alpha) = -\beta$ and $\alpha \geq -\beta$, hence

$$M(\rho) \supset M(\alpha) \supset M(-\beta) \supset M(\rho).$$

Similarly, $\sigma_\alpha \sigma_\beta \sigma_\alpha(\beta) = -\alpha$, which yields the chain of embeddings

$$M(\rho) \supset M(\beta) \supset M(-\alpha) \supset M(-\rho).$$

Given that $-\alpha$ and $-\beta$ are not connected in the order prescribed on W, the only chains of ordered roots are the following:

$$\begin{aligned} \rho \geq \alpha \geq -\alpha \geq -\rho \\ \rho \geq \alpha \geq -\beta \geq -\rho \\ \rho \geq \beta \geq -\beta \geq -\rho \\ \rho \geq \beta \geq -\alpha \geq -\rho. \end{aligned}$$

Hence there are no further embeddings of Verma modules in $M(\rho)$.

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