Implementing a One Qubit Holonomic Quantum Gate in a Bosonic Environment

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Abstract

Holonomic quantum gates are know to be resilient against classical errors in a set of parameters that define its dynamics, though this might not translate to errors due to a quantum environment. For that reason it is interesting to study how these gates behave under a quantum bath.

In particular, a recent paper has shown that under a dephasing noise it is possible to obtain assistance from the environment, meaning the presence of a quantum environment can improve the implementation of the gate [1].

In this work a one qubit holonomic quantum gate will be studied coupled to a bosonic environment. In particular it will be shown that a damping noise can also result in an environment assisted implementation of the gate.

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1 Introduction

The idea of using properties of quantum mechanics to process information was proposed by Feynman in his seminal paper [2]. About a decade later Shor's algorithm was developed [3], which showed quantum computers had an exponential gain for factorization, a very relevant problem in computation with daily applications especially in cryptography. This indicated that quantum computation could have significant applications. Quantum information has already produced interesting protocols such as quantum cryptography (the idea of using quantum mechanics to distribute an encryption key safely), but still, after so many years we have not reached *quantum supremacy*, when quantum computers perform calculation which would be impossible (practically speaking) on a classical computer.

In the same spirit as classical computation a discrete, or digital, language is currently used. The idea is to implement a set of N two level systems, and with a finite number of gates $U_{\gamma_i(k)}$ be able to arbitrarily approximate a unitary with $\left\|U - \prod_{k=1}^{N} U_{\gamma_i(k)}\right\| \leq \epsilon$. It turns out that three types of gates are enough to achieve this task, which is know as *universality* [4].

Implementing a quantum computer ended up being a very daunting task, especially due to decoherence and noise. Studying noise is a central theme in quantum information, especially because quantum systems are in general a lot harder to screen from the environment compared to classical systems. For that reason a series of protocols were created to tackle this problem, in particular error correction, which involves using more qubits than ideally needed to fix errors.

In this project a type of quantum computation that has some in-built noise protection against classical errors in a set of parameters that define its dynamics, known as holonomic quantum computation, will be studied and described with interactions to a bosonic environment. This is motivated by the expectation that the environment could assist in the implementation of the gate.

2 Holonomic quantum gates

2.1 Berry's phase and Wilczek-Zee phase

Suppose a system evolves with a time dependent Hamiltonian. The Schrödinger equation for an eigenstate will be given by

$$H(t)|n;t\rangle = E_n(t)|n;t\rangle. \tag{1}$$

For a general state the solution can be decomposed in eigenstates

$$|\alpha;t\rangle = \sum_n c_n(t) e^{i\theta_n(t)} |n;t\rangle \tag{2}$$

with $\hbar = 1$ from now on and

$$\theta_n(t) \equiv -\int_0^t E_n\left(t'\right) dt', \tag{3}$$

where the phase given by θ_n is called the *dynamical phase*, and will be useful later. Inputing this state in Schrödinger's equation results in

$$\sum_n e^{i\theta_n(t)} \left[\dot{c}_n(t)|n;t\rangle + c_n(t)\frac{\mathrm{d}}{\mathrm{d}t}|n;t\rangle\right] = 0. \tag{4}$$

Assuming that there is no degeneracy, looking at one particular amplitude in the eigenbasis gives

$$\dot{c}_m(t) = -c_m(t)\langle m;t|\left[\frac{\mathrm{d}}{\mathrm{d}t}|m;t\rangle\right] - \sum_n c_n(t) e^{i(\theta_n-\theta_m)} \frac{\langle m;t|\dot{H}|n;t\rangle}{E_n - E_m}. \tag{5}$$

If the Hamiltonian evolves very slowly in time compared with the differences between energy levels, then the second term can be neglected, which results in the solution

$$\begin{aligned} &c_n(t) = e^{i\gamma_n(t)} c_n(0), \\ &\gamma_n(t) \equiv i\int_0^t \langle n;t'|\left[\tfrac{\mathrm{d}}{\mathrm{d}t'}|n;t'\rangle\right] dt'. \end{aligned} \tag{6}$$

This extra phase is known as *Berry's phase* [5]. This phase took many years to be considered relevant, in particular because it was not clear if it could be measured. It was later discovered that it plays a major role in cyclic time evolutions.

If the Hamiltonian's time dependence can be described by a vector of parameters $\mathbf{R}(t)$ which belong to a smooth manifold, then the integrand in eq. 5, using the chain rule, can be rewritten as

$$i\langle n;\mathbf{R}|\left[\frac{\mathrm{d}}{\mathrm{d}t}|n;\mathbf{R}\rangle\right] = i\langle n;\mathbf{R}|\,\nabla_{\mathbf{R}}|n;\mathbf{R}\rangle\,\cdot\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t} \equiv \mathbf{A}_n(\mathbf{R})\cdot\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}t}. \tag{7}$$

This allows the integral to be evaluated along a curve in parameter space by changing variables in the integral in eq. 6:

$$\gamma_n(T) = i\int_0^T \mathbf{A}_n(\mathbf{R})\cdot\frac{\mathrm{d}\mathbf{R}}{dt}dt = \int_C \mathbf{A}_n(\mathbf{R})\cdot\mathrm{d}\mathbf{R}. \tag{8}$$

If the time evolution is cyclic, with a period T (meaning $\mathbf{R}(0) = \mathbf{R}(T)$), then using Stokes' theorem results in

$$\gamma_n(T) = \oint_C \mathbf{A}_n(\mathbf{R})\cdot\mathrm{d}\mathbf{R} = \int [\nabla_{\mathbf{R}}\times\mathbf{A}_n(\mathbf{R})]\cdot\mathrm{d}\mathbf{a}, \tag{9}$$

where $\mathrm{d}\mathbf{a}$ is a vector of measures for each parameter.

The fact that the phase only depends on the total flux $\nabla_{\mathbf{R}}\times\mathbf{A}_n(\mathbf{R})$ indicates that this phase is robust against random errors in $\mathbf{R}$ (for an illustration of this, see fig. 1 with an example where the parameter manifold is a unit sphere).

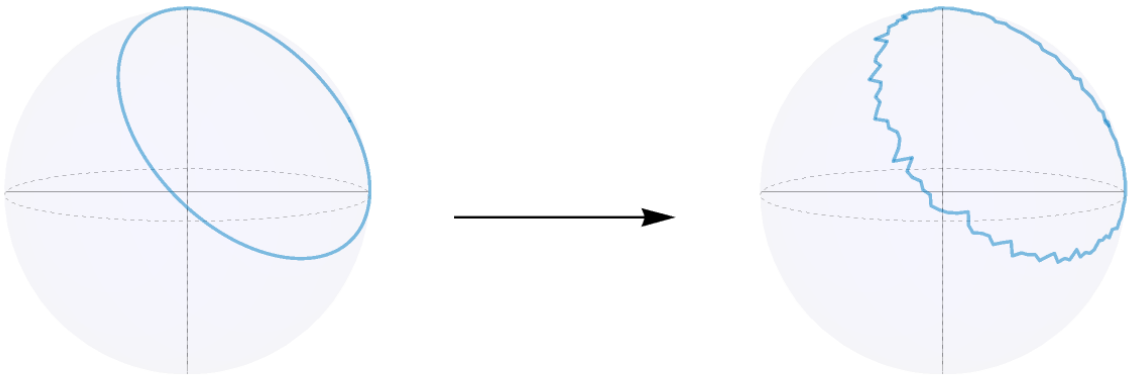

Figure 1: Illustrative example of how random noise in the parameter space will not change the phase significantly, since it only depends on the flux through the curve.

This vector field $\mathbf{A}$ is analogous to a vector potential when considering the transformation

$$\begin{aligned} |n;t\rangle &\longrightarrow e^{i\delta(\mathbf{R})}|n;t\rangle \\ \mathbf{A}_n(\mathbf{R}) &\longrightarrow \mathbf{A}_n(\mathbf{R}) - \nabla_{\mathbf{R}}\delta(\mathbf{R}) \end{aligned} \tag{10}$$

which in turn implies the curl of $\mathbf{A}$ is analogous to a magnetic field.

Assuming the initial state belongs to a degenerate subspace, using the previous results one can conclude that the system will stay in this as seen in eq. 6. Then the time evolution of the coefficient will have an extra sum

$$\dot{c}_m(t) = -\sum_n c_n(t)\langle m;t|\left[\frac{\mathrm{d}}{\mathrm{d}t}|n;t\rangle\right] \equiv iM_{mn}(t)c_n(t). \tag{11}$$

The solution for the vector of coefficients c_m will now be a matrix exponential

$$c_m(t) = T\exp\left\{i\int_0^t \mathrm{d}tM_{mn}(t)\right\}c_n(0). \tag{12}$$

where T is the time ordering operator. It is also possible, using the chain rule again, to define a matrix vectors potential analogous to eq. 7

$$\mathbf{A}_{mn}(\mathbf{R}) = \langle m;\mathbf{R}|\ \nabla_{\mathbf{R}}|n;\mathbf{R}\rangle\ . \tag{13}$$

This implies that a unitary is implemented in the subspace, given by

$$U(T) = P\exp\left\{i\int_C \mathbf{A}_{mn}(\mathbf{R}).\mathrm{d}\mathbf{R}\right\}, \tag{14}$$

where P is the path ordering. This unitary is also robust to noise if the curve described by $\mathbf{R}$ is closed for the same reason as in the Berry's phase case. The "phase" in the degenerate case is know as Wilzcek-Zee phase [6].

In analogy to the gauge transformation in eq. 10, instead of a time dependent phase one can apply a time dependent unitary to the degenerate subspace, which in turn implies the matrix $\mathbf{A}$ transforms as

$$\begin{aligned}\left|n';t\right\rangle =&\Omega(t)\left|n;t\right\rangle\\ \mathbf{A}'(t) =&\dot{\Omega}\Omega^{-1} + \Omega\mathbf{A}\Omega^{-1}\end{aligned} \tag{15}$$

which is how *gauge fields* tranform. In particular, the holonomies are analogous to *Wilson loops*, which are gauge invariant.

2.1.1 Adiabatic holonomic quantum gates

The protocol of generating a geometric phase in a cyclic evolution is know as a *holonomy*, in particular in the degenerate case a *non-Abelian holonomy*, after all the resulting unitary matrices might not commute. In this section a one-qubit quantum gate using only holonomies will be described.

In order to obtain a universal quantum computer a set of two one-qubit gates and one two-qubit gate is necessary. In this project only one-qubit gates will be described, though it is also possible to obtain a two-qubit gate using only non-Abelian holonomies as well.

Starting with a tripod system (see fig. 2) described by the Hamiltonian [7]

$$H = \left[\left|e\right\rangle\left(\Omega_0\left\langle 0\right|+\Omega_1\left\langle 1\right|+\Omega_a\langle a|\right) + \mathrm{h.c.}\right]\ . \tag{16}$$

This Hamiltonian has a doubly degenerate dark state (meaning they do not contain the exited state, $|e\rangle$) subspace where the non-Abelian holonomy will be implemented.

Assuming the degenerate subspace is spanned by the vectors $|D_1\rangle$ and $|D_2\rangle$, $\mathbf{A}$ as in eq. 13 will be given by

$$\mathbf{A}_{ij} = \langle D_i|\ \nabla_{\mathbf{R}}|D_j\rangle \tag{17}$$

and the resulting non-Abelian holonomy for a closed path

$$U(C) = P\exp\left\{i\oint_C \mathbf{A}(\mathbf{R})\cdot\mathrm{d}\mathbf{R}\right\}, \tag{18}$$

where P is the path ordering operator.

As an example, if we parametrize $\Omega_0 = \Omega\sin\theta\cos\phi$, $\Omega_1 = \Omega\sin\theta\sin\phi$ and $\Omega_a = \Omega\cos\theta$, the degenerate subspace will be spanned by $|D_1\rangle = \cos\theta\left(\cos\phi|0\rangle + \sin\phi\left|1\right\rangle\right) - \sin\theta\left|a\right\rangle$ and $|D_2\rangle = \cos\phi\left|1\right\rangle - \sin\phi\left|0\right\rangle$, and each component of the vector $\mathbf{A}$ will be given by

$$A_\theta = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \quad \text{and} \quad A_\phi = \sigma_y\cos\theta \tag{19}$$

resulting in the holonomy

$$U(C) = \exp\left\{i\sigma_y\oint_C \cos\theta\mathrm{d}\phi\right\}, \tag{20}$$

where the path ordering is no longer needed, and the integral is simply the solid angle generated by the path of θ and ϕ. Suppose the solid angle is $\pi/4$, then the implemented unitary will be

$$U(C) = \exp\left\{i\pi\sigma_y/4\right\} = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix} = \mathrm{H}.X \tag{21}$$

where H is the *Hadamard gate* and X is a bit flip.

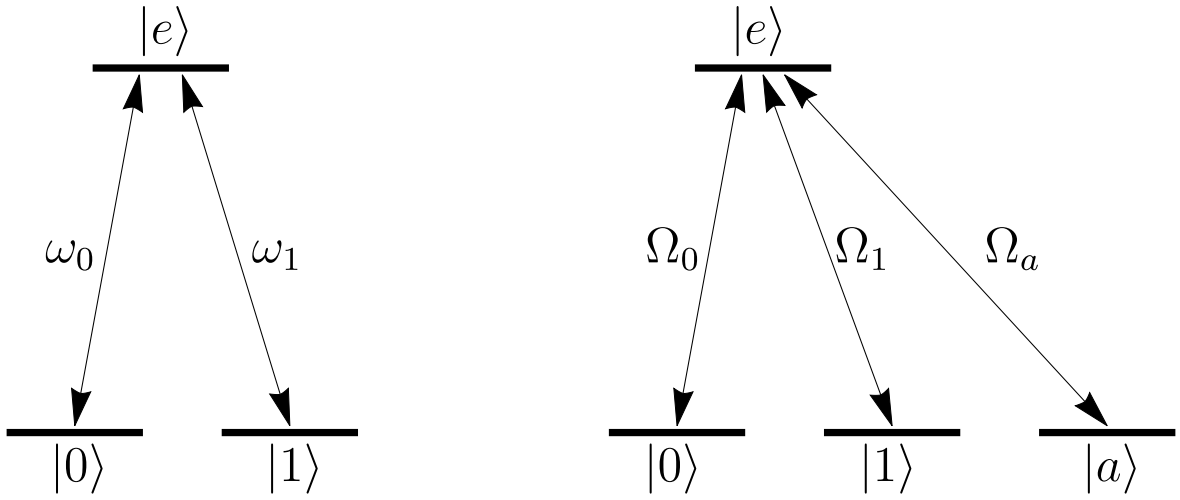

Figure 2: Energy level and couplings setup required for non-adiabatic and adiabatic (respectively a *lambda system* and a *tripod system*) implementations of a one-qubit holonomic quantum gate [7].

To make sure the gate is implemented in the computational basis the parameters at the beginning of the path can be chosen in such a way that the dark states are initially $|0\rangle$ and $|1\rangle$. One can show, with a different choice of parameters, that another gate that, joined with this one, generate a universal one-qubit gate.

This approach can also be generalized by using a $N+2$ level system, which in turn has a N dimensional degenerate subspace where a non-Abelian holonomy can be implemented [8]. This could, for instance, be used for a two-qubit gate.

2.2 Non-adiabatic holonomic quantum gates

Although holonomic quantum gates are robust against some errors, being adiabatic is a disadvantage if one is interested in implementing a quantum gate since adiabaticity restricts how fast it can be implemented and also exposes the system to the environment for, in some sense, a long time. For this reason the gate described above is not easy to implement experimentally, though it has been done [9]. More recently a non-adiabatic version, using only three energy levels was proposed, only requiring one dark state.

2.2.1 Non-adiabatic holonomies

The geometric and dynamical contribution to the dynamics can be separated by writing the time evolution operator as

$$U(t) = T\exp\left\{\int_0^t \mathrm{i}(A-K)\mathrm{d}t\right\} \tag{22}$$

where the matrices A and K are given by

$$A_{ab} = i\langle\psi_a|\frac{\mathrm{d}}{\mathrm{d}t}|\psi_b\rangle \quad \text{and} \quad K_{ab} = \langle\psi_a|H|\psi_b\rangle\,, \tag{23}$$

and $|\psi_a(t)\rangle$ is an orthonormal basis.

The interpretation of A as a geometrical comes from the fact that it does not depend on the Hamiltonian, but only in geometrical properties of the Hilbert space [10]. With this in mind a non-adiabatic holonomic quantum gate can be implemented by finding a process and a Hamiltonian with $K_{ab}=0$ for all t, then, with a cyclic evolution, a similar robustness to error as in the adiabatic case should be expected.

It is interesting to note that this matrix also has the same gauge structure as the adiabatic case, with a unitary Ω taking one basis to the other, the geometric and dynamical contributions transform as

$$A \to \mathrm{i}\Omega^\dagger\hat{\Omega} + \Omega^\dagger A\Omega \quad \text{and} \quad K \to \Omega^\dagger K\Omega. \tag{24}$$

2.2.2 Zero detuning implementation

One possible way to implement a non-adiabatic quantum gate is to start with the Hamiltonian [11]

$$H(t) = \Omega(t)\left(\omega_0|e\rangle\langle 0| + \omega_1|e\rangle\langle 1| + \text{h.c.}\right) \tag{25}$$

with $|\omega_0|^2 + |\omega_1|^2 = 1$, which has a dark state $|d\rangle = -\omega_1|0\rangle + \omega_0|1\rangle$ and a bright one $|b\rangle = \omega_0^*|0\rangle + \omega_1^*|1\rangle$. The Hamiltonian can then be rewritten as

$$H(t) = \Omega(t)(|e\rangle\langle b| + h.c.) \rightarrow \Omega(t)\tau_x \tag{26}$$

which is simply a Rabi oscillation of a qubit spanned by $\{|e\rangle, |b\rangle\}$, with τ_x being the Pauli operator in the e-b subspace. In this basis the time evolution operator is given by

$$U(t) = \exp\left\{-i\tau_x \int_0^t \mathrm{d}t\Omega(t)\right\}. \tag{27}$$

By choosing $\int_0^t \mathrm{d}t\Omega(t) = \pi$ the resulting gates simply does the operation $|b\rangle \rightarrow -|b\rangle$, $|e\rangle \rightarrow -|e\rangle$ and $|d\rangle \rightarrow |d\rangle$. In order to figure out what unitary will be implemented in the computational basis one can interpret this operation as a σ_z applied to the basis $|d\rangle$ and $|b\rangle$. Using the parametrization $\omega_0 = \sin(\theta/2)e^{i\phi}$,$\omega_1 = -\cos(\theta/2)$, which in turn defines a unit vector $\mathbf{n} = \{\sin\theta\sin\phi, \sin\theta\cos\phi, \cos\theta\}$ one can then define a change of basis matrix

$$\exp\left\{i\mathbf{n}.\vec{\sigma}\frac{\pi}{2}\right\}\begin{pmatrix}|1\rangle\\|0\rangle\end{pmatrix} = \begin{pmatrix}|b\rangle\\|d\rangle\end{pmatrix}, \tag{28}$$

which implies

$$\sigma_z \rightarrow \exp\left\{-i\mathbf{n}.\vec{\sigma}\frac{\pi}{2}\right\}.\sigma_z.\exp\left\{i\mathbf{n}.\vec{\sigma}\frac{\pi}{2}\right\} = \mathbf{n}.\vec{\sigma} \tag{29}$$

in the computational basis. This can be seen as a purely geometric evolution since the dynamical phase , given by $\langle\psi_k(t)|H(t)|\psi_l(t)\rangle = \langle k|H(t)|l\rangle$ (because $[U(t), H] = 0$, for all t) is zero throughout the evolution. The unitary holonomy in eq. 29 are traceless, but it is possible to implement a general unitary by using two gates, with $U(C) = U(C_\mathrm{m})\,U(C_\mathrm{n}) = \mathbf{n}\cdot\mathbf{m} - \mathrm{i}\sigma\cdot(\mathbf{n}\times\mathbf{m})$.

2.2.3 Single-shot realization

Performing two gates can be a problem since the system will be exposed to the environment for twice the time. For that reason a single-shot implementation would be interesting. It turns out to be possible to implement a geometric gate as long as one has a controllable detuning, and it has even been experimentally implemented in Xmon qutrits [12]. The single shot implementation is done with the Hamiltonian

$$\mathcal{H}(t) = \Omega(t)\sin\alpha|e\rangle\langle e| + \frac{1}{2}\Omega(t)\cos\alpha(|e\rangle\langle b| + |b\rangle\langle e|) \tag{30}$$

with

$$\int_0^\tau \frac{\Omega(t)}{2}dt = \pi, \tag{31}$$

which generate a time evolution operator $U(\tau) = |d\rangle\langle d| + e^{-i\pi(1+\sin\alpha)}|b\rangle\langle b\,\Big| + e^{-i\pi(1+\sin\alpha)}|e\Big\rangle\langle e|$, that can be projected in the qubit subspace resulting on the universal single qubit gate $U_L(\tau) = e^{-i\gamma n\cdot\sigma/2}$, with $\gamma = 1 + \sin\alpha$.

3 Physical model for non-adiabatic quantum gate

This section is dedicated to deriving a physically reasonable master equation where the quantum gate can be implemented, using Redfield's equation. To justify the use of this equation the Nakajima-Zwanzig projection operator method will be derived. Some authors choose to justify Redfield's equation with a series of approximations that are not obvious, but using Nakajima-Zwanzig, though cumbersome, makes the equation obtainable with few and reasonable approximations. It is also a powerful tool to derive master equations, in particular non-Markovian ones, with a variety of approximations.

3.1 Introduction to density matrix theory

Density matrices are a powerful tool to describe loss of information to the environment. By definition, they allow description of quantum states with a classical ignorance, by adding a classical probability to a quantum state. They are defined as positive matrices with unit trace, and can be written as

$$\rho = \sum_i p_i \left|\psi_i\right\rangle\!\left\langle\psi_i\right|, \tag{32}$$

where p_i are probabilities. It is easy to see that averages of operators can be calculated with $\langle A\rangle = \mathrm{tr}[A\rho]$, and that a unitary time evolution is given by $\dot{\rho} = -i[\rho, H]$, known as the *von-Neumann* equation.

Density matrices also appear naturally when considering subsystems. For example, take two entangled qubits in the state $(|00\rangle + |11\rangle)/\sqrt{2}$. If one only measures one of the qubits in any basis the two possible outcomes will have equal probabilities, meaning this cannot be a pure state. Instead it can be described by the density matrix

$$\rho = \mathbb{I}/2, \tag{33}$$

which is basis independent, and also correctly describes the 50-50 probabilities in all basis.

One key concept in density matrix theory is an entropic quantity called *purity*, defined as $P[\rho] = \mathrm{tr}\left[\rho^2\right]$, and is 1 if the state is pure and minimal if the state is maximally mixed. Incidentally this is a measure of how entangled the subsystem is to the other part (assuming the composite state is pure), as in the example above, a maximally entangled state resulted in a maximally mixed subsystem, with purity $1/2$.

Just like a subsystem cannot necessarily be described by a pure state, its evolution cannot necessarily be described by a unitary operator. Instead it is a general evolution that takes positive matrices with unit trace to positive matrices with unit trace, know as Completely-Positive-Trace-Preserving(CPTP) maps.

A positive map takes positive matrices to positive matrices, but this is not enough for a map to be physically acceptable. Complete positivity implies not only that a map M is positive, but $M \otimes \mathbb{I}$ is also positive, where $\mathbb{I}$ is the identity in any dimension. This is necessary since the map $M \otimes \mathbb{I}$ is simply applying M and doing nothing in an arbitrary system, which should still be positive.

3.2 General method for deriving a master equation

Suppose [13] a system interacts with another via H_I, with a parameter α. The von-Neumann equation in the interaction picture will be

$$\frac{\partial}{\partial t}\rho(t) = -i\alpha\left[H_I(t), \rho(t)\right] \equiv \alpha\mathcal{L}(t)\rho(t). \tag{34}$$

A central concept in open quantum systems is of a system interacting with a “large” bath, that is only slightly perturbed when they interact. With this in mind it is possible to define a set of projection super-operators

$$\begin{aligned} \rho \mapsto \mathcal{P}\rho = \mathrm{tr}_B\{\rho\} \otimes \rho_B \equiv \rho_S \otimes \rho_B, \\ \mathcal{Q}\rho = \rho - \mathcal{P}\rho, \end{aligned} \tag{35}$$

that are projections because of the properties

$$\begin{aligned} \mathcal{P} + \mathcal{Q} &= I, \\ \mathcal{P}^2 &= \mathcal{P}, \\ \mathcal{Q}^2 &= \mathcal{Q}, \\ \mathcal{P}\mathcal{Q} = \mathcal{Q}\mathcal{P} &= 0. \end{aligned} \tag{36}$$

These projectors capture the idea that the bath is almost in the thermal state, so in order to derive an equation for the reduced system assuming the bath is in the thermal state is a good starting point. Of course the full state is not really $\rho_S \otimes \rho_B$, after all if that were the case the system would not be entangled with the bath, implying it would always be pure, which would not properly thermalize.

Each projection will follow the equations of motion

$$\begin{aligned} \frac{\partial}{\partial t}\mathcal{P}\rho(t) = \mathcal{P}\frac{\partial}{\partial t}\rho(t) = \alpha\mathcal{P}\mathcal{L}(t)\rho(t) = \alpha\mathcal{P}\mathcal{L}(t)\mathcal{P}\rho(t) + \alpha\mathcal{P}\mathcal{L}(t)\mathcal{Q}\rho(t), \\ \frac{\partial}{\partial t}\mathcal{Q}\rho(t) = \mathcal{Q}\frac{\partial}{\partial t}\rho(t) = \alpha\mathcal{Q}\mathcal{L}(t)\rho(t) = \alpha\mathcal{Q}\mathcal{L}(t)\mathcal{P}\rho(t) + \alpha\mathcal{Q}\mathcal{L}(t)\mathcal{Q}\rho(t), \end{aligned} \tag{37}$$

where in the last step the identity $\mathbb{I} = \mathcal{P}+\mathcal{Q}$ was inserted. Since the objective is to describe the evolution of ρ_S, the projection from $\mathcal{Q}$ does not really need a detailed solution, instead one can formally solve it by introducing a Green's function

$$\mathcal{Q}\rho(t) = \mathcal{G}(t,t_0)\mathcal{Q}\rho(t_0) + \alpha\int_{t_0}^{t} ds\mathcal{G}(t,s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}\rho(s), \tag{38}$$

where

$$\mathcal{G}(t,s) \equiv T\exp\left[\alpha\int_{s}^{t} ds'\mathcal{Q}\mathcal{L}(s')\right] \tag{39}$$

and T is a time ordering operator. The Green's function satisfies the equation

$$\frac{\partial}{\partial t}\mathcal{G}(t,s) = \alpha\mathcal{Q}\mathcal{L}(t)\mathcal{G}(t,s). \tag{40}$$

Inserting this formal solution in the $\mathcal{P}$ part results in what is known as Nakajima-Zwanzing equation.

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{P}\rho(t) =&\alpha\mathcal{P}\mathcal{L}(t)\mathcal{G}(t,t_0)\mathcal{Q}\rho(t_0) + \alpha\mathcal{P}\mathcal{L}(t)\mathcal{P}\rho(t)\\ &+\alpha^2\int_{t_0}^{t} ds\mathcal{P}\mathcal{L}(t)\mathcal{G}(t,s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}\rho(s).\end{aligned} \tag{41}$$

Assuming the initial condition is factorized, meaning $\rho(0) = \rho_S(0)\otimes\rho_B(0)$, the first term in eq. 41 is zero, because $\mathcal{Q}\rho_S(0)\otimes\rho_B(0) = 0$. This equation is not particularly easy to solve, especially because it has a *memory kernel* $\mathcal{K}(t,s) \equiv \alpha^2\mathcal{P}\mathcal{L}(t)\mathcal{G}(t,s)\mathcal{Q}\mathcal{L}(s)\mathcal{P}$. Under weak coupling it can be simplified by expanding the Green's function and truncating the expression at order α^2, which can be done by substituting $\mathcal{G}(t,s) = 1 + O(\alpha)$, resulting in the equation

$$\frac{\partial}{\partial t}\mathcal{P}\rho(t) = \alpha\mathcal{P}\mathcal{L}(t)\mathcal{P}\rho(t) + \alpha^2\int_{t_0}^{t} ds\mathcal{P}\mathcal{L}(t)\mathcal{Q}\mathcal{L}(s)\mathcal{P}\rho(s). \tag{42}$$

Finally, using the explicit definition of the projection operator one arrives at the equation

$$\frac{\partial}{\partial t}\rho_S(t) = -i\alpha\,\mathrm{tr}_B[H_I(t),\rho_S(t)\otimes\rho_B] - \alpha^2\int_{t_0}^{t} ds\,\mathrm{tr}_B\left[H_I(t),\left[H_I(s),\rho_S(s)\otimes\rho_B\right]\right]. \tag{43}$$

One term was omitted in order to simplify the results. Though it is not easy to justify removing the term this equation is commonly used(see ref. [14]). This equation is also not easy to solve because it is an integro-differential equation. It is possible to show that under weak coupling approximation substituting $\rho_S(s)\to\rho_S(t)$ in the integral generates errors of the same order in the interaction, but to do so is not trivial. Instead this substitution is sometimes called the *Markov approximation*, even though it does not lead to a Markovian equation either, nor really is an approximation. It does result in a local in time equation, which is viable to solve, known as the Redfield equation

$$\frac{\partial}{\partial t}\rho_S(t) = -i\alpha\,\mathrm{tr}_B[H_I(t),\rho_S(t)\otimes\rho_B] - \alpha^2\int_{t_0}^{t} ds\,\mathrm{tr}_B\left[H_I(t),\left[H_I(s),\rho_S(t)\otimes\rho_B\right]\right]. \tag{44}$$

These weak coupling equations have the downside that if the approximations are not true the resulting map can be negative, meaning the resulting density matrix could have negative eigenvalues.

3.3 Lambda system in an optical cavity

The setup chosen to derive a physically reasonable equation to describe the non-adiabatic quantum gate will be of an atom in a cavity. Cavity environments are a way experimentalists control how the electromagnetic field behaves when coupling to a system. In particular it has a frequency ω_c and only allows dynamics with frequencies near resonance, allowing the atom to be approximated to have only a few levels. The interaction of an atom to the electromagnetic field is usually described by [15]

$$H = H_A + H_E - e\mathbf{r}\cdot\mathbf{E} \tag{45}$$

where H_A (H_E) is the atom's (field's) Hamiltonian. The field Hamiltonian is given by a collection of harmonic oscilators

$$H_E = \sum_{\mathbf{k}} \nu_{\mathbf{k}}\left(a^{\dagger}_{\mathbf{k}}a_{\mathbf{k}} + \frac{1}{2}\right) \tag{46}$$

and the field interacts with the atom through its displacement operator

$$\mathbf{E} = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}} \mathscr{E}_{\mathbf{k}} \left(a_{\mathbf{k}} + a_{\mathbf{k}}^{\dagger} \right), \tag{47}$$

where $\hat{\epsilon}_{\mathbf{k}} \mathscr{E}_{\mathbf{k}}$ are the interaction for different polarizations. The dipole operator can be decomposed in the atom's eigenbasis with the identity $\sum_i |i\rangle\langle i| = \mathbb{I}$

$$e\mathbf{r} = \sum_{i,j} e|i\rangle\langle i|\mathbf{r}|j\rangle\langle j| \equiv \sum_{i,j} \gamma_{ij}\sigma_{ij} \tag{48}$$

where $\gamma_{ij} = \langle i| \mathbf{r} |j\rangle$ and $\sigma_{ij} = |i\rangle\langle j|$, resulting in the Hamiltonian

$$H = \sum_{\mathbf{k}} v_k a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}} + \sum_i E_l \sigma_{ii} + \sum_{iJ} \sum_{\mathbf{k}} g_{\mathbf{k}}^{ij} \sigma_{ij} \left(a_{\mathbf{k}} + a_{\mathbf{k}}^{\dagger} \right) \tag{49}$$

with the coupling $g_{\mathbf{k}}^{ij} = -\gamma_{ij} \hat{\epsilon}_{\mathbf{k}} \mathscr{E}_{\mathbf{k}}$.

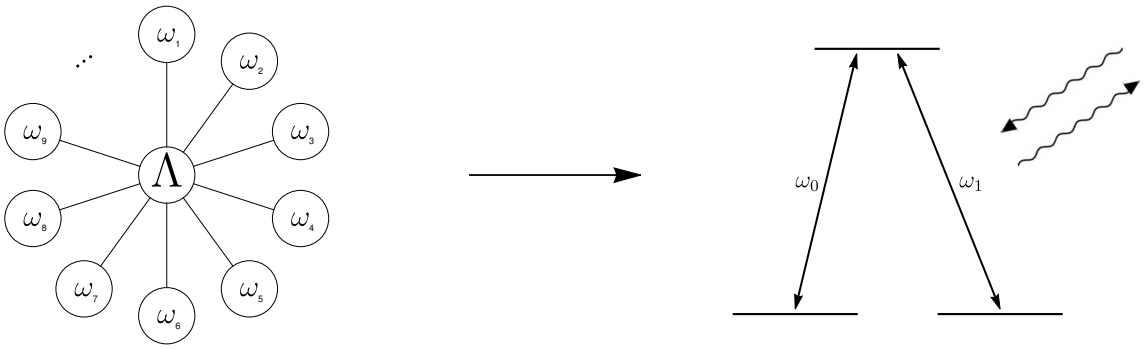

Figure 3: The system starts with a bare lambda hamiltonian coupled to a bosonic environment and after tracing out ends up with interactions between energy levels and dissipative photon emissions.

The quantum gate will be implemented in a Lambda system coupled to a bosonic bath through a dipolar interaction, assuming the atom only has three relevant levels. The Hamiltonians of the system and interaction will be given by

$$H = H_{\Lambda} + H_I + H_E \tag{50}$$

$$H_{\Lambda} = \begin{pmatrix} \delta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{51}$$

$$H_I = \sum_{k,i} (|e\rangle\langle i| + |i\rangle\langle e|) \otimes (\gamma_k a_{k,i}^{\dagger} + \gamma_k^* a_{k,i}) \tag{52}$$

where i is the index for different polarizations of the bath and also for the qubit subspace of the Hamiltonian.

If the problem was a two level atom there would be an exact solution to the dynamics, but instead the master equation will be derived under weak coupling, using eq. 44. Ignoring the first term for now leads to

$$\dot{\rho}(t) = -\operatorname{tr}_E \left\{ \int_0^t \mathrm{d}t' [H_I(t), [H_I(t'), \rho(t) \otimes \rho_{th}]] \right\}, \tag{53}$$

where the environment is assumed to be in a thermal state. In that case the ignored term would be exactly zero, though later the bath's state will have a coherent state added. Changing variables in eq 53 from t' to $t - t'$ results in

$$\dot{\rho}(t) = -\operatorname{tr}_E \left\{ \int_0^t \mathrm{d}t' [H_I(t), [H_I(t - t'), \rho(t) \otimes \rho_{th}]] \right\} \tag{54}$$

and expanding the commutators,

$$\dot{\rho}(t) = -\operatorname{tr}_E \left\{ \int_0^t \mathrm{d}t' \Big(H_I(t) H_I(t-t') \rho(t) \otimes \rho_{th} - H_I(t-t') \rho(t) \otimes \rho_{th} H_I(t) \Big) + \text{h.c.} \right\}. \tag{55}$$

This equation is in the interaction picture, so it is convenient to decompose the interaction in eigenoperators defined in the Schrödinger picture by the commutation relation

$$[H_0, S(\omega)] = \omega S(\omega) \Longrightarrow S_I(\omega, t) = \exp\{i\omega t\} S(\omega), \tag{56}$$

which implies $S(\omega) = S(-\omega)^\dagger$. In particular, for the proposed physical system the interaction is already decomposed in eigenoperators with $\omega = \delta$ $(-\delta)$ being the eigenvalue for the eigenoperator $|e\rangle\langle i|$ $(|i\rangle\langle e|)$.

For finite systems the eigenoperators are very easy to find. First note that the eigenoperators and their eigenvalues will be given by $\left|\lambda_i\right\rangle\!\left\langle\lambda_j\right|$, with $\omega_{ij} = \lambda_i - \lambda_k$. To obtain an operator decomposition just insert two identities as

$$A = \sum_{i,k} |\lambda_i\rangle\langle\lambda_i| A |\lambda_k\rangle\langle\lambda_k| = \sum_{i,k} (\langle\lambda_i| A |\lambda_k\rangle) |\lambda_i\rangle\langle\lambda_k| \tag{57}$$

The next step is to decompose the interaction in eigenoperators. Due to the propriety $S(\omega) = S(-\omega)^\dagger$ and the fact that the sum runs over all omega the decomposition is made such that one of the interactions is conjugate transposed and the other is not. This has two advantages: it makes the secular approximation a lot more convenient; it makes obtaining a Lindblad equation easier. In particular the polarization of light will be ignored, as the two channels simply do not couple, this derivation will be of only one of them. Defining $E(t) = \sum_k (\gamma_k^* a_k + \gamma_k a_k^\dagger)$

$$\begin{aligned} \dot{\rho}(t) = -\operatorname{tr}_E \Big\{ \sum_{\omega,\omega'} \int_0^t \mathrm{d}t' \Big(& S^\dagger(\omega'(t)) S(\omega(t-t')) \rho \otimes E^\dagger(t) E(t-t') \rho_{th} + \\ & - S(\omega(t-t')) \rho S^\dagger(\omega'(t)) \otimes E(t-t') \rho_{th} E^\dagger(t) \Big) + \text{h.c.} \Big\}, \end{aligned} \tag{58}$$

using the time dependence of the eigenoperators and joining the terms which belong to each Hilbert space from the system and bath,

$$\begin{aligned} \dot{\rho}(t) = \sum_{\omega,\omega'} \int_0^t \mathrm{d}t' \exp\{i(\omega' - \omega)t\} \exp\{i\omega(t')\} \Big(& S^\dagger(\omega') \rho S(\omega) \operatorname{tr}\Big\{ E(t-t') \rho_{th} E^\dagger(t) \Big\} + \\ & - S^\dagger(\omega') S(\omega) \rho \operatorname{tr}\Big\{ E^\dagger(t) E(t-t') \rho_{th} \Big\} \Big) + \text{h.c.}, \end{aligned} \tag{59}$$

results in

$$\dot{\rho}(t) = \sum_{\omega,\omega'} \Gamma(t) \exp\{i(\omega' - \omega)t\} \Big(S(\omega') \rho S^\dagger(\omega) - S^\dagger(\omega) S(\omega') \rho \Big) + \text{h.c.}, \tag{60}$$

where

$$\Gamma(t) = \int_0^t \mathrm{d}t' \exp\{i\omega t'\} \operatorname{tr}\Big\{ E^\dagger(t) E(t-t') \rho_{th} \Big\}. \tag{61}$$

At this point it is necessary to state what the bath looks like in order to calculate the trace in the last expression. If the bath is a at zero temperature the trace becomes

$$\operatorname{tr}\Big\{ E^\dagger(t) E(t-t') \rho_{th} \Big\} = \sum_{k,k'} \exp\{-i\omega_k t\} \exp\Big\{ i\omega_k'(t-t') \Big\} \gamma_k \gamma_{k'}^* \left\langle a_k a_{k'}^\dagger \right\rangle = \int_0^\infty \mathrm{d}\nu J(\nu) \exp\{-i\nu t'\}. \tag{62}$$

To get a simpler equation one can perform the secular approximation, which assumes that cross terms between ω's oscillate fast and can be *smoothed out*, to get the equation:

$$\dot{\rho}(t) = \sum_{\omega} \Gamma(t) \Big(S(\omega) \rho S^\dagger(\omega) - S^\dagger(\omega) S(\omega) \rho \Big) + \text{h.c.} \tag{63}$$

One can further simplify this equation by separating Γ in real γ and imaginary λ terms, resulting in the equation

$$\frac{d}{dt}\rho(t) = -i\left[H_{LS}, \rho(t)\right] + \mathcal{D}\left(\rho(t), S(\omega)\right), \tag{64}$$

where

$$H_{LS} = \sum_{\omega} \lambda(t) S^{\dagger}(\omega) S(\omega) \tag{65}$$

and

$$\mathcal{D}(\rho, S(\omega)) = \sum_{\omega} \gamma(t) \left(S(\omega) \rho S^{\dagger}(\omega) - \frac{1}{2} \left\{ S^{\dagger}(\omega) S(\omega), \rho \right\} \right). \tag{66}$$

Since the physical system is supposed to be a cavity, its spectral density J will be given by a sharp Lorentzian, which allows Γ to be evaluated analytically. It also allows one extra approximation, that only near-resonant terms contribute to the dynamics, which leaves only two terms, one for each channel. A Lorentzian has two parameters

$$J(\nu) = \frac{\alpha^2}{\pi} \frac{\Gamma/2}{(\nu - \omega_c)^2 + (\Gamma/2)^2} \tag{67}$$

a resonant frequency ω_c and its sharpness Γ (α is just the bath's coupling constant). If $\Gamma << \omega_c$ then the integral in eq. 62 can be taken from $-\infty$ to ∞, which is unphysical because it considers negative frequency modes, but mathematically sound since the spectral density is small in the region where the frequency is negative. This in turn results in

$$\Gamma(t) = \int_0^t \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}\nu J(\nu) \exp\{-i(\nu - \omega)t'\} = \int_0^t \mathrm{d}t' \int_{-\infty}^{\infty} \mathrm{d}\nu \frac{\alpha^2}{\pi} \frac{\Gamma/2}{(\nu - (\omega_c - \omega))^2 + (\Gamma/2)^2} \exp\{-i\nu t'\} \tag{68}$$

The integral in ν is just a Fourier transform of a Lorentzian, and is given by

$$\mathcal{F}_{\nu} \left[\frac{1}{\pi} \frac{\Gamma/2}{(\nu - \omega_0)^2 + (\Gamma/2)} \right] (s) = \exp\{-2\pi i s \omega_0 - \Gamma\pi|s|\} \tag{69}$$

and the time integral can also be evaluated analytically. Instead of including the results a brief description will be given. Both real and imaginary parts of $\Gamma(t)$ are a combination of an oscillating sine term(with phases) with frequency $\omega - \omega_c$ multiplied by a decreasing exponential. They both also have a constant term. This indicates that for long periods of time both terms become constants, which is know as the *Markovian limit*. In figure 4 we can observe the behavior of γ and λ with somewhat arbitrary parameters.

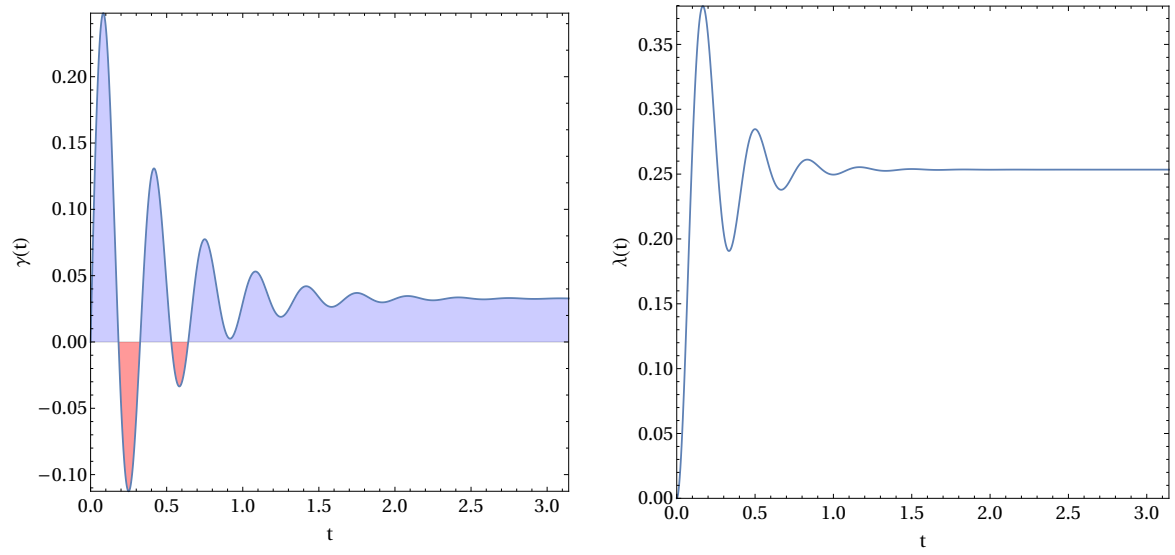

Figure 4: Real and imaginary part of $\Gamma(t)$, with arbitrary parameters, showing both their convergence and the negativity of the real part, which implies non-Markovianity.

λ is always positive, and there is not anything interesting in its behavior. γ is sometimes negative. This negativity has major implication in the physics involved. It can be shown that equations in the form of eq. 64 generate Markovian (divisible) maps if γ is always positive. When it is negative the system can recover coherence lost to the bath, and the evolution is non-Markovian.

Finally, by setting a resonant mode (with the same frequency as the channel) to a coherent state a *laser Hamiltonian* is obtained from the term that was ignored in eq. 53

$$H_L = (\alpha_L S(\omega) + \alpha_L^* S^\dagger(\omega)) \tag{70}$$

where the constant α_L depends on the displacement of the coherent state and constants related to the coupling to the mode.

4 Environment assisted implementation

One of the main problems related to the non-adiabatic holonomic quantum gate described in this project is that the qubit couples to an excited state. This means that when noise or parametric errors are added there is a chance that the non-qubit level is occupied in the end of the implementation. Since the zero temperature environment proposed above causes decay from the exited state to the qubit subspace it seems reasonable to believe that it could be used to fix errors.

In this section the traceless version of the holonomic quantum gate will be implemented using square pulses with a gaussian noise in the pulse time. This imposes a smaller than 1 fidelity even when uncoupled to the environment.

4.1 Initial states and pulse error

The gate fidelity will be averaged out on random initial states uniformly distributed on the Bloch sphere. The random unit vector that describes the state can be obtained by generating three normally distributed numbers defining a vector and then normalizing it [16]. A square pulse will be implemented with a runtime of π plus a gaussian noise with some arbitrary variance, which will result in a lower than 1 fidelity even when uncoupled to the environment.

4.2 Master equation

The X gate was arbitrarily chosen for illustration purposes.The master equation in the interaction picture to implement an X gate is

$$\dot{\rho} = -i[H_L, \rho] - i\lambda(t)[H_{LS}, \rho] + \gamma(t)(\mathcal{D}(\rho, |0\rangle\langle e|) + \mathcal{D}(\rho, |1\rangle\langle e|)) \tag{71}$$

with

$$H_L = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \tag{72}$$

The Lamb-shift term will be ignored since it can be effectively removed in a rotating frame and tuning the laser properly.

4.3 Markovian limit noise

In the markovian limit it is assumed that the system's dynamics is so much slower than the bath's that the time integral in eq. 68 can be approximated as having a limit in infinite time. This can be shown to generate a time evolution that forms a semi-group, which is the definition of a Markovian evolution in quantum mechanics [13]. Markovianity plays a central role in open quantum systems, in particular because there are theorems that guarantee CPTP in very general and well understood situations.

The resulting master equation will be

$$\dot{\rho} = -i[H_L, \rho] + \gamma(\infty)(\mathcal{D}(\rho, |0\rangle\langle e|) + \mathcal{D}(\rho, |1\rangle\langle e|)) \tag{73}$$

In fig. 5 the average fidelity $\mathcal{F}$ is plotted as a function to the coupling to the environment $\gamma(\infty) \equiv \gamma$. In this case the environment will always harm the fidelity.

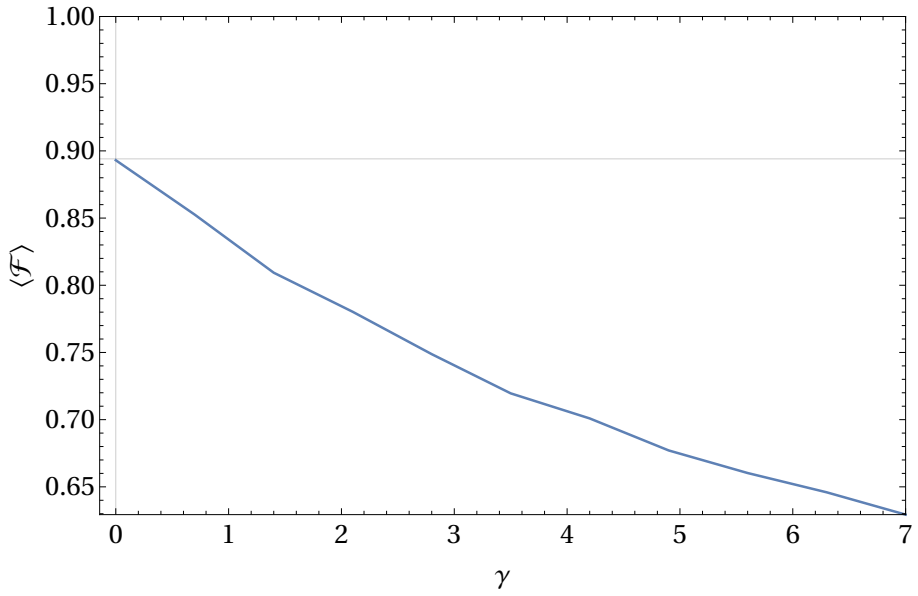

Figure 5: Average fidelities for an X gate coupled to a markovian environment

4.4 Including a relaxation time

A possible way of using the environment to increase the fidelity is to leave the system in the environment for it to be projected in the qubit subspace for a relaxation time ("$\mathcal{R}$") after the implementation of the gate. In a practical implementation this would have drawbacks since there usually are more sources of noises such as dephasing. For that reason it is interesting to look at different finite relaxation times, such as in fig 6.

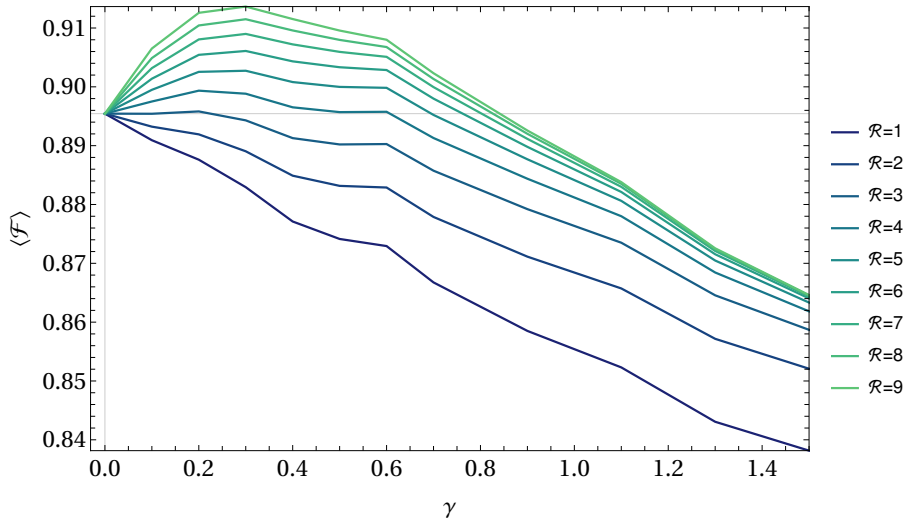

Figure 6: Average fidelities for an X gate coupled to a Markovian environment with diferent relaxation times $\mathcal{R}$.

This result is interesting because the gate becomes more accurate with the environment than without it, which is known as environmental assistance. There is a competition between the environment reducing the fidelity during the implementation but also helping it by projecting in the qubit subspace during the relaxation, which generates an optimal coupling strength. If the interaction is very weak then the implementation is optimal, but the necessary relaxation time to allow assistance becomes large. On the other hand if the interaction is too strong then no relaxation time is able to provide assistance since the fidelity during the implementation becomes too low.

5 Conclusion and future perspectives

An overview on holonomic quantum computation and open quantum systems was made. It was shown that a Markovian environment could assist in the implementation of a non-adiabatic holonomic quantum gate by addressing a relevant cause of errors due to the population of the exited state.

We believe that this protocol could be further improved if more properties such as non-Markovianity (negativity of the damping coefficient) are considered (see fig. 7), though there are technical problems involving non-Markovianity and weak coupling derivations, such as the lack of *strong non-Markovianity*, which is the case where there is a back-flow of information from the bath to the system.

Other possible approaches could involve simpler interactions to the environment, such as a $|e\rangle\langle b|$ coupling with the intent to obtain exact solutions, or more abstract models for the bath, such as collisional models.

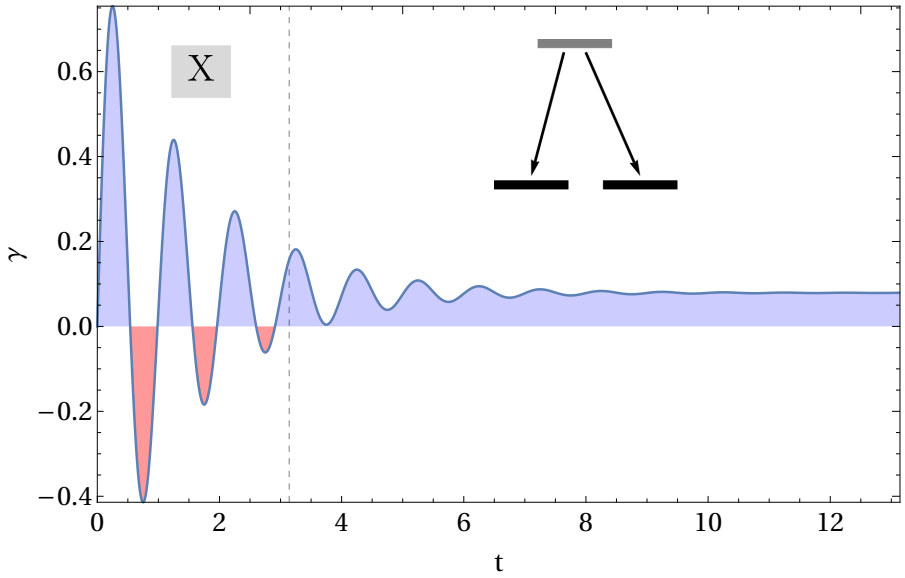

Figure 7: Illustration of the protocol, where the X gate in implemented when the evolution is non-Markovian and then left to decay into the qubit subspace in the Markovian regime.

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