Geometric Phases in Classical and Quantum Systems

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Abstract

We are accustomed to think the phase of single particle states does not matter. After all, the phase cancels out when calculating physical observables. However, the geometric phase can cause interference even in single particle states and can be measured. Berry’s phase is a geometric phase the system accumulates as its time-dependent Hamiltonian is subjected to closed adiabatic excursion in parameter space. In this report, we explore how Berry’s phase manifests itself in various fields of physics, both classical and quantum mechanical. The Hannay angle is a classical analogue to Berry’s phase and they are related by a derivative. The Aharonov-Bohm effect is a manifestation of Berry’s phase. Net rotation of deformable bodies in the language of gauge theory can be translated as a Berry phase. The well-known Born-Oppenheimer approximation is a molecular Aharonov-Bohm effect and is another manifestation of Berry’s Phase.

Sammanfattning

## Contents

1. **Introduction**  
2. **Berry’s phase**  
   2.1 Berry’s phase of a particle of spin \( s \) in an adiabatically rotating magnetic field  
3. **The Hannay angle**  
   3.1 Generalized harmonic oscillator  
   3.2 Relation between Berry’s phase and Hannay’s angle  
4. **Aharonov-Bohm effect**  
   4.1 Double slit experiment with a vector potential  
5. **Net rotation of deformable bodies with zero total angular momentum**  
   5.1 Rotating concentric spheres  
   5.2 Net rotation of falling cats  
6. **Born-Oppenheimer Approximation**  
7. **Conclusion**  
8. **Appendix**  
   8.1 Derivation of \( c(t) \)  
9. **References**
1 Introduction

An elementary effect in quantum mechanics had evaded detection for decades, and was discovered by M.V. Berry in 1983. The effect, now named Berry’s phase, is about how the evolution of a system is affected by adiabatic changes in its Hamiltonian. We are accustomed to think the phase of a single particle system is irrelevant, physical observables are proportional to $|\Psi|^2$, and the phase cancels out. Berry’s phase is a proof of the contrary, a particle can accumulate a geometric phase causing interference even in single particle states.

Consider a quantum system whose Hamiltonian $H(q,p;X(t))$ have some time-dependent parameters $X(t)$. If the change happens adiabatically (much slower than any other time scale in the system), a system starting out in the ground state will remain in the ground state throughout the evolution. In addition to the dynamical phase all quantum systems receive, it will pick up another phase which depends only on the path traveled in parameter space. While Berry’s phase does not explicitly depend on time, the change needs to happen very slowly for the adiabatic approximation to hold. If the change happen faster, one would need to consider more terms in the approximation and use adiabatic renormalization.

As a small example of Berry’s phase, consider an electron in the state $\psi$. Now, we split the beam and let one side go through an adiabatically changing electromagnetic field and acquire Berry’s phase before we recombine the beam. The wave function is now $\Psi = (\psi + e^{i\phi}\psi)/\sqrt{2}$ which gives us $|\Psi|^2 = |\psi|^2cos^2(\frac{\phi}{2})$. It can clearly be seen that for a given $\phi \neq 2\pi(n + \frac{1}{2})$, there will be destructive interference caused by the extra phase.

Functions that are path-dependent will have variables that fail to return to initial value when the function is driven around a closed loop. These variables are called anholonomies (some physicists calls these just holonomies, a notation M.V. Berry found to be barbarism, which we shall agree with) and Berry’s phase is an example of this, as pointed out by B. Simon.

The case of parallel transportation of a vector is an example of a phase anholonomy. Let $v$ be a vector with unit length and positioned at $r$ on a unit sphere, see figure. We make the demand that $v \cdot r = 0$ which makes $v$ tangent to the sphere. This vector will be transported along a closed loop $C$ on the surface of the sphere. At every point on the curve, we define $v' = r \wedge v$ to be a vector perpendicular to $r$ and $v$. These three vectors form a triad whose angular velocity is $\Omega$. The condition for parallel transport is $\Omega \cdot r = 0$,
i.e. \( \mathbf{v} \) does not twist. That is, \( \Omega \) lies in the \( \mathbf{v}, \mathbf{v}' \)-plane at all points along the curve.

Figure 1: Parallel transporting a vector \( \mathbf{v} \) along a closed loop on a sphere causes it to accumulate a phase \( \gamma \).

One can describe the orientation of \( \mathbf{v} \) and \( \mathbf{v}' \) using a complex vector \( \psi \) defined as \( \psi = (\mathbf{v} + i\mathbf{v}')/\sqrt{2} \). The condition of parallel transport now tells us that \( \text{Im}(\psi^* \cdot d\psi) = 0 \), where \( d\psi \) is a change in \( \psi \) for a given change \( dr \). To find the phase \( \gamma \), we need to define a local basis at a given \( r \). Let \( \mathbf{x} \) be a unit vector at \( r \) tangent to the surface, pointing to the east and let \( \mathbf{y} \) be a unit vector perpendicular to \( \mathbf{u} \), pointing northward. Just as \( \psi \) describes \( \mathbf{v} \) and \( \mathbf{v}' \), we use \( \mathbf{n} = (\mathbf{x} + i\mathbf{y})/\sqrt{2} \) to describe \( \mathbf{x} \) and \( \mathbf{y} \). \( \psi \) and \( \mathbf{n} \) are then related by \( \psi = \mathbf{n}e^{-i\gamma} \). Every time \( r \) moves a length \( dr \), \( \mathbf{v} \) accumulates an infinitesimal part of the phase \( d\gamma = \text{Im}(\mathbf{n}^*d\mathbf{n}) \). Integrating this phase over the entire loop yields \( \gamma \). The extra phase \( \mathbf{v} \) received is proportional to the solid angle subtended by area enclosed by the curve \( C \), as seen from the center of the sphere. \( \mathbf{v} \) has picked up a phase as it traveled along a loop, even though it never rotated.

Moving from classical parallel transport to quantum mechanical is straightforward. The vector \( \psi \) is now the quantum state \( \psi_n \) and will be transported around a loop in parameter space with the condition that \( \text{Im} \langle \psi_n | d\psi_n \rangle = 0 \). For the quantum mechanical case, Berry’s phase can be seen as the flux of a 2-form through the surface \( C \) that is enclosed by the loop \( \partial C \) in parameter space.

Berry’s phase has a classical analogue called Hannay’s angle. By using a parameter-dependent generating function to switch to the action-angle variables, one finds an extra term upon calculating the angle frequency. This
extra term is Hannay’s angle, which causes the system to progress the torus in phase space at a different rate compared to constant parameters. Hannay’s angle and Berry’s phase are closely related. In fact, Hannay’s angle is the negative derivative of Berry’s phase with respect to the quantum number n.

Berry pointed out the well-known Aharonov-Bohm effect is a manifestation of Berry’s phase. A particle traveling around a magnetic field inside a solenoid, while not under the presence of an electromagnetic field itself, will pick up a phase. The phase can easily be calculated using a gauge transformation and is proportional to the magnetic flux inside the solenoid. A simple experiment can show the effect of this; by setting up the double-slit experiment with a solenoid behind the two slits, one can see how the pattern in the back wall shifts because of the magnetic flux inside the solenoid.

Berry’s phase also manifests itself in more exotic areas of physics. Free-falling bodies can undergo deformations, starting and ending with the same shape, and receive a net rotation. The parameter space is now shape space and the phase difference is a net rotation of the body. Two concentric globes can become displaced when one globe is rotating with respect to the other. The theory can be applied to the falling cat phenomenon, where a cat dropped upside-down can flip mid-air and land on its feet. The cat is approximated by two cylinders and the shape make a closed excursion in shape space, which is parametrized by the angle the individual cylinders have rotated around their own axes. As the cylinders return to their initial angle, the cat’s body have turned around.

Berry’s phase is a ubiquitous phenomenon and have important applications in many fields of physics, including quantum field theory, and many of them are described in [4]. The purpose of this report is to see what happens to a physical system as its Hamiltonian is driven adiabatically around a loop in parameter space. We shall describe Berry’s phase using mathematical language and find out where some of its most interesting applications and manifestations lie, in both classical and quantum mechanics. The applications were picked to create a diverse range of topics while keeping it at a level fit for undergrad physics students.
2 Berry’s phase

In the case of time-independent Hamiltonian, a particle starting out in the n-th state,
\[ \hat{H} |n\rangle = E_n |n\rangle, \]  
will remain in the n-th state as time evolves, but accumulates a phase factor,
\[ \Psi = e^{-iE_n t} |n\rangle, \]  
where the exponential is the dynamical phase for n-th state \[^5\]. Note that we’re using natural units, implying \( \hbar = 1 \). The dynamical phase is just a rotation of the wave function and does nothing to physical observables of single particle states. What would equation 2.2 look like if the Hamiltonian were dependent on time? As the Hamiltonian changes, its eigenvalues and eigenstates will also change accordingly, and equation 2.1 has to be modified to account for these changes. For a given time \( t \), we choose a set of basis such that
\[ \hat{H}(t) |n(t)\rangle = E_n(t) |n(t)\rangle. \]  
The full wave function \( \Psi(t) \) will solve the time-dependent Schrödinger equation
\[ i \frac{\partial \Psi(t)}{\partial t} = \hat{H}(t)\Psi(t). \]  
In the case of time-dependent Hamiltonian, \( |n(t)\rangle \) still forms an orthogonal set
\[ \langle m(t)|n(t)\rangle = \delta_{mn}, \]  
and is complete. Since \( |n(t)\rangle \) is complete, we can form the full wave function, including the phase factor, as a sum of all states
\[ \Psi(t) = c_n(t)e^{-i\theta_n(t)} |n(t)\rangle, \]  
where summation over all \( n \) is implied. It is important to notice that the dynamical phase has changed since \( E_n \) is now a function of time \( E_n(t) \). Generalizing the dynamical phase in equation 2.2 gives us the dynamical phase for a time-dependent Hamiltonian
\[ E_n t \longrightarrow \int_0^t E_n(s)ds \equiv \theta_n(t). \]
$c_n(t)$ is needed since equation 2.6 does not solve the time-dependent Schrödinger equation without it. We omit the derivation of $c_n(t)$ here, but it can be found in the appendix. For the case when the Hamiltonian loops after some time $T$ (i.e. $R(0) = R(T)$, where $R$ are the parameters), the solution is

$$\Psi(T) = e^{i\gamma_n} e^{-i\theta_n(T)} |n(0)\rangle,$$

(2.8)

where

$$\gamma_n = i \oint_{\partial C} \langle n(R)|\nabla n(R)\rangle \, dR,$$

(2.9)

and $\partial C$ is the closed curve in parameter space. The integrand is called the Berry connection. This result often only holds if the adiabatic approximation ($\dot{H} \ll 1$) hold, but this requirement can be relaxed for many systems. The adiabatic approximation is proven in the appendix.

Equation 2.9 is called Berry’s phase [1]. Berry’s phase differs from the dynamical phase in the sense that it is independent of time. The dynamical phase is dependent on time while Berry’s phase cares only about the path taken (assuming the adiabatic theorem still hold). Since we are considering large $T$, the dynamical phase will dominate the evolution of the system. We make two remarks about the Berry phase; firstly, if only one parameter in the Hamiltonian is time dependent, Berry’s phase is zero. This can easily be seen in equation 2.9 as you’re integrating a scalar function of one variable around a closed loop. Only when two or more parameters change can the Hamiltonian loop and $\gamma_n$ be non-zero. Therefore, we assume more than one parameter is changing for the rest of the report. Secondly, because of the normalization, the integrand in equation 2.9 is purely imaginary. Proof:

$$\nabla \langle n|n\rangle = \langle \nabla n|n\rangle + \langle n|\nabla n\rangle = 0 \rightarrow \langle n|\nabla n\rangle = -\langle n|\nabla n\rangle^*,$$

(2.10)

where the star indicates complex conjugate.

The property of the integrand in equation 2.9 being purely imaginary is called parallel transport. Equation 2.9 is an integral of a continuous function over a closed curve and can be expressed, using Stokes’ theorem, as

$$\gamma_n = -\text{Im} \int_C \nabla \wedge \langle n|\nabla n\rangle \, dS$$

(2.11)

where $dS$ is an area element in parameter space. The wedge product can be rewritten as

$$\gamma_n = -\text{Im} \int_C \langle \nabla n| \wedge |\nabla n\rangle \, dS,$$

(2.12)
\[
= -\text{Im} \int_C \sum_{m \neq n} \langle \nabla n|m \rangle \wedge \langle m|\nabla n \rangle \, dS. \tag{2.13}
\]

Using equation 8.8 from the appendix, equation 2.13 becomes
\[
\gamma_n = -\int\int_C V_n(R) dS. \tag{2.14}
\]

where
\[
V_n(R) = \text{Im} \sum_{m \neq n} \frac{\langle n(R)|\nabla \hat{H}(R)|m(R) \rangle \wedge \langle m(R)|\nabla \hat{H}(R)|n(R) \rangle}{(E_n(R) - E_m(R))^2}. \tag{2.15}
\]

One can see that while \(\langle n|\nabla n \rangle\) depends on the phase of \(|n\rangle\), equation 2.15 does not. If we make the gauge transformation
\[
|n(R)\rangle \rightarrow e^{i\mu(R)}|n(R)\rangle, \tag{2.16}
\]
then
\[
\langle n(R)|\nabla n(R) \rangle \rightarrow \langle n(R)|\nabla n(R) \rangle + i\nabla \mu. \tag{2.17}
\]
\(\langle n|\nabla n \rangle\) is shifted by a gauge \(i\nabla \mu\). Equation 2.11 eliminates this gauge dependency since the curl of a gradient is zero and is therefore gauge invariant. One can think of \(\text{Im} \langle n|\nabla n \rangle\) as being a ‘vector potential’ and 2.15 as the ‘magnetic field’. Berry’s phase is then the flux of the ‘magnetic field’ through the area \(C\) that \(\partial C\) encloses.

### 2.1 Berry’s phase of a particle of spin \(s\) in an adiabatically rotating magnetic field

Consider a particle with spin \(1/2\) in a spin up state in a magnetic field \(B\). The particle is in an eigenstate of \(S_z\) and is always aligned with \(B\). We now let \(B\) change direction, but not strength, and is therefore our parameter for this system. As the direction goes in a loop, what is the resulting Berry phase? We can directly compute Berry’s phase from equation 2.12. The eigenstate of a spin up fermion in the direction of \(B\) is given by
\[
\chi_+ = \begin{bmatrix} \cos \theta/2 \\ e^{i\phi} \sin \theta/2 \end{bmatrix}. \tag{2.18}
\]
Computing the gradient in spherical coordinates, and the cross-product, gives
\[ \nabla \chi_+ = \frac{1}{2r} \left[ -\sin \frac{\theta}{2} \right] \hat{\theta} + \frac{1}{r \sin \theta} \left[ ie^{i\phi} \sin \frac{\theta}{2} \right] \hat{\phi}, \]  
(2.19)

\[ \langle \nabla \chi_+ \rangle \times |\nabla \chi_+\rangle = \frac{i \sin \frac{\theta}{2} \cos \frac{\theta}{2}}{r^2 \sin \theta} \hat{r} = i \frac{1}{2r^2} \hat{r}. \]  
(2.20)

Finally, computing Berry’s phase using 
\[ dS = r^2 \hat{r} d\Omega \]

\[ \gamma_n = -lm \int_C \frac{i}{2r^2} \hat{r} dS = -\frac{1}{2} \int_C d\Omega = -\frac{\Omega}{2}. \]  
(2.21)

Berry’s phase is therefore proportional to the solid angle subtended from the point \( \mathbf{B} = 0 \).

This result can be generalized for spin \( s \) particles to find \( \gamma_n = -n\Omega \). However, one cannot write down the state for a spin \( s \) particle as simple as a spinor in equation 2.18. Instead, one uses equation 2.15 to determine how the state of the particle is coupled to all the other states.

Consider a particle of spin \( s \) in a magnetic field \( \mathbf{B} \neq 0 \) that is described by a Hamiltonian \( \hat{H}(\mathbf{B}) = \kappa \mathbf{B} \cdot \hat{s} \), where \( \kappa \) is a constant and \( \hat{s} \) the spin operator \([1]\). The particle is in an eigenstate of \( s_z \) and is at all times aligned with \( \mathbf{B} \). The eigenvalues of this Hamiltonian are \( E_n = \kappa Bn \), where \( n \) are the \( 2s + 1 \) different eigenvalues of \( \hat{s} \). For this problem, the parameters \([R \text{ in earlier sections}]\) are the components of the magnetic field \( \mathbf{B} \). The direction of \( \mathbf{B} \) will change adiabatically around a loop \( C \) and the spin of the particle follows. To evaluate \( \mathbf{V}_n(\mathbf{B}) \), we exploit the fact that this quantity is gauge invariant and rotate the axes so \( z \)-axis line up with \( \mathbf{B} \). We apply the spin ladder operations on \( |n, s\rangle \) to get

\[ \hat{s}_+ |n, s\rangle = (\hat{s}_x + i\hat{s}_y) |n, s\rangle = \sqrt{s(s+1) - n(n+1)} |n + 1, s\rangle, \]

\[ \hat{s}_- |n, s\rangle = (\hat{s}_x - i\hat{s}_y) |n, s\rangle = \sqrt{s(s+1) - n(n-1)} |n - 1, s\rangle. \]  
(2.22)

We can see that the \( n \)-th state is only coupled to the states \( n+1 \) and \( n-1 \), which means there are only be two non-zero terms in equation 2.15. Multiplying the equations in 2.22 with \( \langle n \pm 1, s| \), and doing some algebra, we get

\[ \langle n \pm 1, s| \hat{s}_x |n, s\rangle = \frac{1}{2} \sqrt{s(s+1) - n(n \pm 1)} \]

\[ \langle n \pm 1, s| \hat{s}_y |n, s\rangle = \mp i \frac{1}{2} \sqrt{s(s+1) - n(n \pm 1)} \]  
(2.23)
Calculating $V_n$, we get (using $\nabla \hat{H} = \kappa \mathbf{s}$)

$$
\begin{align*}
V_{n,x} &= 0, \\
V_{n,y} &= 0,
\end{align*}
$$

(2.24)

because they both involve the term $\langle n | \hat{s}_z | n \pm 1 \rangle = 0$ or its conjugate. The only non-zero term of $V_n$ will be its z-component,

$$
V_{n,z} = \lim_{B \to 0} \frac{B}{B^2} (\langle n | \hat{s}_x | n + 1 \rangle \langle n + 1 | \hat{s}_y | n \rangle - \langle n | \hat{s}_y | n + 1 \rangle \langle n + 1 | \hat{s}_x | n \rangle \\
+ \langle n | \hat{s}_x | n - 1 \rangle \langle n - 1 | \hat{s}_y | n \rangle - \langle n | \hat{s}_y | n - 1 \rangle \langle n - 1 | \hat{s}_x | n \rangle)
$$

(2.25)

$$
= \frac{n}{B^2},
$$

where the $s$ is dropped from the bra and ket vectors. Restoring the axis to any other position, we get

$$
V_n(B) = \frac{nB}{B^3}.
$$

(2.26)

Using this in equation 2.14 one can see that $\gamma_n$ can be identified as the "magnetic flux" of a "magnetic monopole", positioned in the origin, of strength $-n$ through C. This result can be interpreted as the view of the area C from the point $B = 0$, i.e. $\gamma_n = -n\Omega(C)$, where $\Omega$ is the solid angle. For a given $n$, one can find a solid angle that gives a phase factor of -1. For example, if the solid angle is $2\pi$, fermions (half-integer $n$) receive a phase factor of -1. Since there is a $2s + 1$ fold degeneracy at $B = 0$, this explains the sign change of a wave function as it rotates around a degeneracy. The same thing can be said for bosons (integer $n$), but requiring a different solid angle. For $\Omega = \pi$, and $n = 1$, one would get a phase factor $-1$.

In this case, the Berry phase comes from some "magnetic monopole", which radiates radially. The analogy works here since $V_n(R)$ has a singularity at the degeneracy. The degeneracy is positioned at $B=0$ and is $2s+1$ degenerate, and the magnetic field splits the energy levels to lift this degeneracy. The degeneracy acts as some source of a 2-form that is generating a vector potential $i \langle n | dn \rangle$.

$V_n(R)$ being radiated as a magnetic monopole is not always the case, as its curl is generally not zero, implying the magnetic monopole-analogue is partially flawed. The sources of $V_n(R)$ can, for example, be "currents" flowing in parameter space [1].
3 The Hannay angle

The angle anholonomy discussed in previous sections have a classical (ℏ → 0) analogy, first proposed by J.H. Hannay, called Hannay’s angle [6]. This section loosely follows [7]. Consider an integrable Hamiltonian \( H(\theta, J; X) \) expressed in the action-angle variables [8]. In classical mechanics, the adiabatic theorem states that if the changes in the parameters \( X \) happen very slowly (\( \dot{H}(t) << 1 \)), the system initially moving on a torus \( J = \{ J_i \} \) will continue to do so, provided the Hamiltonian stays integrable [9]. The area enclosed by the curve in phase space (q,p) is constant and the changes in parameters will only cause the curve in phase space to stretch, rotate or deform. An important distinction between the quantum and the classical case is the fact that the Hamiltonian needs to be integrable in the classical case to ensure the action-angle variables exist. As the parameters go in a closed excursion in parameter space, i.e. \( X(0) = X(T) \), the system will have accumulated an extra angle \( \Delta \theta \), in addition to the angle it normally receives as time elapses. The total angle after one loops is

\[
\theta(T) = \theta(0) + \int_0^T \frac{\partial H(J;X)}{\partial J} \, dt + \Delta \theta.
\] (3.1)

The first term is the initial angle, the second term is the evolution of the system for a constant X and \( \Delta \theta \) is the extra angle originating from the path taken in parameter space, and is the subject of this section.

Consider a classical system whose Hamiltonian \( H(q, p; X) \) is integrable. Then there exists a canonical transformation to a new Hamiltonian \( \mathcal{H}(\theta, J; X) \) in action-angle variables \( (\theta, J) \) governed by a generating function \( S^\alpha(q, J; X) \). The superscript \( \alpha \) is to distinguish between different points on the curve in phase space that correspond to the same \( q \), see figure 2. The Hamiltonian \( \mathcal{H}(\theta, J; X) \) is related to the Hamiltonian \( H(q, p; X) \) by

\[
\mathcal{H}(\theta, J; X) = \bar{H}(J; X) + \frac{\partial S^\alpha}{\partial X} \frac{dX}{dt},
\] (3.2)

where \( \bar{H}(J; X) = H(q(\theta, J; X), p(\theta, J; X); X) \) is the angle-independent Hamiltonian corresponding to constant X. To find an expression for \( \frac{\partial S^\alpha}{\partial X} \), we define a single-valued function \( \mathcal{G} \) as

\[
\mathcal{G}(\theta, J; X) = S^\alpha(q(\theta, J; X), J; X) \quad 0 \leq \theta < 2\pi.
\] (3.3)
Figure 2: Curve in phase space. The superscript is to distinguish between different points on the curve that correspond to the same q.

Then,

\[
\frac{\partial S^\alpha}{\partial X} = \frac{\partial G}{\partial X} - \frac{\partial S^\alpha}{\partial q} \frac{\partial q}{\partial X}. \tag{3.4}
\]

Putting this into equation 3.2 and using the Hamilton equation \( p = \frac{\partial S^\alpha}{\partial q} \) gives

\[
\mathcal{H}(\theta, J; X) = \bar{H}(J; X) + \frac{dX}{dt} \left( \frac{\partial G}{\partial X}(\theta, J; X) - p(\theta, J; X) \frac{\partial q(\theta, J; X)}{\partial X} \right). \tag{3.5}
\]

This Hamiltonian is single-valued since \( p \) and \( q \) are periodic functions of \((\theta, J; X)\).

One can see the action variable \( J \) is constant for closed adiabatic excursion of parameters in parameter space by using Hamilton’s equation for the change in action

\[
\frac{dJ}{dt} = \frac{\partial \mathcal{H}(\theta, J; X)}{\partial \theta} = \frac{\partial \bar{H}(J; X)}{\partial \theta} + \frac{\partial}{\partial \theta} \frac{dX}{dt} \left( \frac{\partial G}{\partial X} - p \frac{\partial q}{\partial X} \right). \tag{3.6}
\]

The first term is zero, as the action is independent of the angle variable. One can apply the theorem of averaging [9] to find an average of the second term.
in equation 3.6 over one loop;

\[ \frac{1}{2\pi} \oint \frac{\partial}{\partial \theta} \frac{dX}{dt} \left( \frac{\partial g}{\partial X} - p \frac{\partial q}{\partial X} \right) d\theta = 0. \]  
(3.7)

Thus,

\[ \frac{dJ}{dt} = 0. \]  
(3.8)

Hamilton’s equation for the angle frequency is

\[ \frac{d\theta}{dt} = \frac{\partial H(\theta, J; X)}{\partial J} = \frac{\partial \bar{H}(J; X)}{\partial J} + \frac{\partial}{\partial J} \frac{dX}{dt} \left( \frac{\partial g}{\partial X} - p \frac{\partial q}{\partial X} \right). \]  
(3.9)

The first term in equation 3.9 gives the evolution of the system for constant $X$, while the second term gives $\Delta \theta$ from 3.1. Integrating the second term to find the angle difference over one loop

\[ \int_{0}^{T} \frac{\partial}{\partial J} \frac{dX}{dt} \left( \frac{\partial g}{\partial X} - p \frac{\partial q}{\partial X} \right) dt = \oint \frac{\partial}{\partial J} \left( \frac{\partial g}{\partial X} - p \frac{\partial q}{\partial X} \right) dX \]  
(3.10)

where $\partial C$ is a closed loop in parameter space. The first term is zero when integrated along a closed loop and only the second term will contribute to the angle. This term can be simplified with Stokes’ theorem

\[ -\oint_{\partial C} \frac{\partial}{\partial J} \left( p \frac{\partial q}{\partial X} \right) dX = -\frac{\partial}{\partial J} \int_{C} dp \wedge dq, \]  
(3.11)

where $C$ is any surface whose boundary is $\partial C$ and the d’s are the differentials with respect to the parameters. Equation 3.11 is often difficult to compute as the integrand is dependent on time both implicitly and explicitly through changes in $\theta$ and $X$. One can use theorem of averaging again, which yields

\[ \Delta \theta = -\frac{\partial}{\partial J} \int_{C} W(I; X) \cdot dC, \]  
(3.12)

where

\[ W(I; X) = \frac{1}{(2\pi)^N} \oint dp \wedge dq \, d\theta. \]  
(3.13)
3.1 Generalized harmonic oscillator

As an example of Hannay’s angle, consider a one dimensional Hamiltonian for a generalized harmonic oscillator given by

\[ H = \frac{Xq^2 + 2Yqp + Zp^2}{2}, \]  

(3.14)

where X, Y and Z are the parameters driven in a loop in parameter space \([7]\). We shall assume \(XZ > Y^2\) \(\forall t\) for the Hamiltonian to take the form of a rotated ellipse, see figure 3. The area is

\[ A = \frac{2\pi E}{\sqrt{XZ - Y^2}} = 2\pi J, \]  

(3.15)

where J is the action variable. Solving for E and putting it equal to equation 3.14 yields

\[ \bar{H} = J\omega, \]  

(3.16)

where \(\omega = \sqrt{XZ - Y^2}\). To find \(dp\) and \(dq\) as a function of the action-angle variables to use in equation 3.12 one needs to solve the system using Hamilton-Jacobi [8] equation for constant parameters \(X = (X, Y, Z)\);

\[ H(q, \frac{\partial S}{\partial q}; X) + \frac{\partial S}{\partial t} = 0. \]  

(3.17)
Making the ansatz \( S(q, \alpha; t) = W(q, \alpha) - \alpha t \) gives

\[
Xq^2 + 2Yq \frac{\partial W}{\partial q} + Z\left( \frac{\partial W}{\partial q} \right)^2 = 2\alpha.
\] (3.18)

Solving for \( \frac{\partial W}{\partial q} \) gives

\[
\frac{\partial W}{\partial q} = -\frac{Y}{Z} q \pm \sqrt{\frac{2\alpha}{Z} - \frac{q^2}{Z^2} \omega^2}.
\] (3.19)

We wish to find \( \frac{\partial S}{\partial \alpha} \), but at this point it is more convenient to find \( \frac{\partial^2 W}{\partial \alpha \partial q} \) and then integrating with respect to \( q \).

\[
\frac{\partial^2 W}{\partial \alpha \partial q} = \pm \frac{1}{\sqrt{2\alpha Z} \sqrt{1 - \frac{q^2}{2\alpha Z} \omega^2}}.
\] (3.20)

The plus or minus sign correspond to positive and negative extreme q-motion. Choosing the angle variable to be at the short side of the ellipse at \( \theta = 0 \) gives the minus sign. Integrating this with respect to \( q \) yields

\[
\frac{\partial W}{\partial \alpha} = \frac{1}{\omega} \cos^{-1} \left( q \sqrt{\frac{\omega^2}{2\alpha Z}} \right).
\] (3.21)

The conjugate momentum to \( \alpha \) is

\[
\beta = \frac{\partial S}{\partial \alpha} = \frac{\partial W}{\partial \alpha} - t = \frac{1}{\omega} \cos^{-1} \left( q \sqrt{\frac{\omega^2}{2\alpha Z}} \right) - t.
\] (3.22)

Solving for \( q \)

\[
q = \sqrt{\frac{2ZZ}{\omega}} \cos \theta,
\] (3.23)

and plugging it into equation 3.19 gives

\[
p = -\sqrt{\frac{2ZZ}{\omega}} \left( \frac{Y}{Z} \cos \theta + \frac{\omega}{Z} \sin \theta \right).
\] (3.24)

Putting this result into equation 3.12 gives

\[
\Delta \theta = \frac{\partial}{\partial J} \int_{C} \int \frac{1}{2\pi} \oint (d \sqrt{\frac{2ZZ}{\omega}} \wedge d \sqrt{\frac{2ZZ}{\omega}} Y) \cos(\theta)^2 d\theta =
\] (3.25)
\[ \iint_C \frac{d(Z)}{\omega} \wedge d\left(\frac{Y}{Z}\right), \quad (3.26) \]

which reduces to
\[ \iint_C \frac{-1}{4} (XZ - Y^2)^{-3/2} (XdY \wedge dZ + YdZ \wedge dX + ZdX \wedge dY) = \iint_C \frac{-1}{4} (XZ - Y^2)^{-3/2} X \, dA, \quad (3.27) \]

where \(dA\) is an area element on the surface \(C\). The symmetry of the parameters in the 2-form shows this vector field is purely radial. This confirms the reoccurring fact that the phase difference is proportional to the solid angle subtended in parameter space.

### 3.2 Relation between Berry’s phase and Hannay’s angle

Hannay’s angle is more than just a analogue to Berry’s phase, the two can be related mathematically. We follow the derivation of [7]. We shall construct a wave function following the method of Maslov and Fedoriuk [10], simplified by Berry [11]. We want to construct a wave function \(\psi(q)\) and associate it to the tori \(\{J_i\}\) by projection from phase space to q-space. The result is

\[ \psi(q; X) = \sum_\alpha a_\alpha(q, J; X) e^{i S_\alpha(q, J; X)}, \quad (3.28) \]

where \(S^\alpha\) is the generating function for different points \(\alpha\) in phase space corresponding to same \(q\) (see figure [2]), and \(\hbar\) is no longer set to 1. The amplitude is given by the determinant of the projection Jacobian

\[ a_\alpha^2 = \frac{1}{(2\pi)^N} \text{det} \left( \frac{d\theta_i}{dq_j} \right). \quad (3.29) \]

Putting this into the Berry connection (integrand of equation [2.11]) gives

\[ V_n(R) = \frac{1}{\hbar} \nabla_X \wedge \int dq \frac{1}{(2\pi)^N} \sum_\alpha \frac{d\theta^\alpha}{dq} \nabla_X S^\alpha(q, J; X). \quad (3.30) \]
Using equation 3.4, this can be rewritten as

\[ V_n(R) = \frac{1}{\hbar(2\pi)^N} \nabla X \wedge \oint d\theta (\nabla X g - p \nabla X q) \]

\[ = -\frac{1}{\hbar(2\pi)^N} \oint d\theta \nabla X p \wedge \nabla X q \]

\[ = -\frac{1}{\hbar} W(I; X), \tag{3.31} \]

where we switched the integration variable from q to θ and W is the angle 2-form from equation 3.13. For equation 3.28 to be an acceptable approximation for the quantum wave function Ψ, it is required to be single-valued. Therefore, if S is multi-valued, then

\[ \Delta S = \hbar(n + i \Delta \ln(a)) = J, \tag{3.32} \]

which is the corrected Bohr-Sommerfeld rule [12]. Using the equations for Hannay’s angle (3.12) and Berry’s phase (2.14), one gets the result

\[ \Delta \theta(J; X) = -\frac{\partial \gamma_n(C)}{\partial n}, \tag{3.33} \]

The two quantities are simply related by the negative derivative with respect to the quantum number n. The minus sign is there because the phase in quantum mechanics and the angle in classical mechanics traverse in opposite directions.

4 Aharonov-Bohm effect

In classical mechanics, particles moving in an electromagnetic field are described using the electric field E, the magnetic field B, and the scalar- and vector potentials. The potentials are mathematical constructs, rather than physical ones, and does not directly affect the system. In quantum mechanics, the potentials play a vital role. Aharonov and Bohm showed that a particle’s phase will change as it travels under the influence of a potential [13]. Berry pointed out many years later that this is an example of Berry’s phase. Consider a particle with charge q at a position r from an infinitely long solenoid, moving in a circle with radius b around the solenoid. The electromagnetic
field is fully contained inside the solenoid and the scalar potential is zero outside the solenoid, while the vector potential is not. The Hamiltonian for the system is

\[ H = \frac{1}{2m} \left[ \frac{\hbar}{i} \nabla - qA(r) \right]^2, \]  

(4.1)

where \( A(r) = \frac{q}{2\pi\Phi} \hat{\theta} \) is the vector potential, and \( \Phi \) the flux of the magnetic field inside the solenoid \( [5] \). Putting this into the time dependent Schrödinger equation gives

\[ \frac{1}{2m} \left[ \frac{\hbar}{i} \nabla - qA(r) \right]^2 \Psi = i\hbar \frac{\partial \Psi}{\partial t}. \]  

(4.2)

This can be simplified by the \( r \)-dependent gauge transformation

\[ \Psi = e^{ig} \Psi', \quad g = \frac{q}{\hbar} \int_{r_0}^{r} A(s) ds, \]  

(4.3)

where \( r_0 \) is some reference point. Then,

\[ \nabla \Psi = i \frac{q}{\hbar} A(r) e^{ig} \Psi' + e^{ig} \nabla \Psi'. \]  

(4.4)

We can plug equation (4.3) back into (4.4) and rearrange it to arrive at

\[ \left[ \frac{\hbar}{i} \nabla - qA(r) \right] \Psi = \frac{\hbar}{i} e^{ig} \nabla \Psi'. \]  

(4.5)

We can apply the left-hand side operator again,

\[ \left[ \frac{\hbar}{i} \nabla - qA(r) \right]^2 \Psi = -\frac{\hbar^2}{2m} e^{ig} \nabla^2 \Psi', \]  

(4.6)

and plug it back into equation (4.2) and cancel out the common term \( e^{ig} \) to arrive at

\[ -\frac{\hbar^2}{2m} \nabla^2 \Psi' = i\hbar \frac{\partial \Psi'}{\partial t}. \]  

(4.7)

Equation (4.7) gives the solution for a system without a vector potential in terms of \( \Psi' \). Since \( \Psi = e^{ig} \Psi' \), one can solve the system in terms of \( \Psi \) by just attaching the phase factor \( e^{ig} \) to \( \Psi' \). Now, if the particle go in a closed loop around the solenoid, what would \( g \) be? Using equation (4.3)

\[ g = \frac{q}{\hbar} \int_{r_0}^{r} A(s) ds = \frac{q}{\hbar} \oint A(s) ds = \frac{q}{\hbar} \iint_C \nabla \times A(s) \, dC = \frac{q}{\hbar} \iint_C B \, dC = \frac{q\Phi}{\hbar}, \]  

(4.8)
where we used Stokes’ theorem, and where $\mathbf{B}$ is the magnetic field. The particle will have acquired an additional phase, equal to $q\Phi/\hbar$, in addition to the dynamical phase. The time taken to go loop around the solenoid is insignificant, it does not have to happen adiabatically.

The question we would like to ask ourselves is this; can this phase be interpreted as a Berry phase? As Berry himself pointed out, this is indeed a case of Berry’s phase and can be easily calculated for a particle going around a solenoid [1]. There is, however, a small change we need to make to the system in order to have a proper setup for finding the Berry phase. Let the electron at $\mathbf{r}$ be trapped in a potential $V(\mathbf{r}-\mathbf{R})$ whose position is $\mathbf{R}$. The potential will be adiabatically transported in a loop around the solenoid and the electron follows. The electron it kept in the same state throughout the transportation. Using the same procedure as above. Schrödinger’s time-dependent equation now gives

$$\frac{1}{2m}\left[\frac{\hbar}{i}\nabla - q\mathbf{A}(\mathbf{r})\right]^2\Psi + V(\mathbf{r}-\mathbf{R})\Psi = i\hbar\frac{\partial\Psi}{\partial t}. \quad (4.9)$$

This is simplified by the gauge transformation

$$\Psi = e^{ig}\Psi', \quad g = \frac{q}{\hbar} \int_{\mathbf{R}}^{\mathbf{r}} \mathbf{A}(s)ds. \quad (4.10)$$

We wish to find $\nabla_{\mathbf{R}}\Psi$ to use in equation (4.9)

$$\nabla_{\mathbf{R}}\Psi = -\frac{iq}{\hbar}\mathbf{A}(\mathbf{R})e^{ig}\Psi' + e^{ig}\nabla_{\mathbf{R}}\Psi', \quad (4.11)$$

and so

$$\langle\Psi|\nabla_{\mathbf{R}}\Psi\rangle = -\frac{iq}{\hbar}\mathbf{A}(\mathbf{R}) - \int \Psi''\nabla\Psi'd^3\mathbf{r}, \quad (4.12)$$

where the second term is zero since it’s the expectation value of momentum of the electron. Calculating Berry’s phase gives

$$\gamma = i\oint \frac{-iq}{\hbar}\mathbf{A}(\mathbf{R})d\mathbf{R} = \frac{q}{\hbar} \int \int_{C} \nabla_{\mathbf{R}} \times \mathbf{A} dC = \frac{q\Phi}{\hbar}, \quad (4.13)$$

as expected.

This was an example of where $\mathbf{V}_n(\mathbf{R})$ is not some simple magnetic monopole, but instead something else. The system contains no degeneracies and the energy is certainly not dependent on the parameter.
4.1 Double slit experiment with a vector potential

Consider the double slit experiment, where an electron goes through two slits in a wall and create an interference pattern in the back wall. If we place a solenoid with flux $\Phi$ behind the first wall, between the two slits, the electron will pick up a phase as it travels around under the presence of the vector potential.

![Figure 4: Double slit experiment with a solenoid. The phase of the electron will be affected by the vector potential from the solenoid.](image)

Let $\Psi$ be the total wave function at the back wall. Then,

$$\Psi = \psi_1 e^{i\frac{q}{\hbar} \int_{C_1} A(r) dr} + \psi_2 e^{i\frac{q}{\hbar} \int_{C_2} A(r) dr} \equiv \psi_1 e^{ig_1} + \psi_2 e^{ig_2}, \quad (4.14)$$

where $C_{1,2}$ are the two different paths, see figure 4. We can rewrite this as

$$\Psi = (\psi_1 e^{i(g_1 - g_2)} + \psi_2) e^{ig_2}, \quad (4.15)$$

and use the fact that

$$g_1 - g_2 = \frac{q}{\hbar} \int_{C_1} A(r) dr - \frac{q}{\hbar} \int_{C_2} A(r) dr =$$

$$\frac{q}{\hbar} \int_{C_1} A(r) dr + \frac{q}{\hbar} \int_{-C_2} A(r) dr = \frac{q}{\hbar} \oint_{C} A(r) dr \quad (4.16)$$

$$= \frac{q\Phi}{\hbar},$$

to find

$$|\Psi|^2 = |\psi_1 e^{i\frac{q\Phi}{\hbar}} + \psi_2|^2, \quad (4.17)$$
which is independent of the path, since it will always enclose the magnetic flux. The vector potential shifts the interference pattern on the back wall in a way which depends on the flux in the solenoid.

5 Net rotation of deformable bodies with zero total angular momentum

We now turn to the subject of rotation of deformable bodies, where a body can receive a net rotation by executing a series of deformations starting and ending at the same shape, without having any initial angular momentum. The theory was developed by Wilczek and Shapere [14] [15]. A diver can jump from a springboard and while total angular momentum remains zero, he will have rotated his body [16]. A cat dropped upside down from a reasonable height will rotate and land on its feet unharmed [17]. The rotation is a geometrical phenomenon of the series of deformations, and is invariant under time scalings [14].

At heart, conservation of angular momentum is behind the effect. As some parts of the body rotate, other parts need to rotate in the opposite direction for the angular momentum to add up to zero. One way to solve such a system is using conservation of angular momentum. However, a significant problem often arises when trying to define coordinate axes as the shape changes, making calculation of angular momentum difficult.

We define the manifold $M_0$ to be shape space, the space where each point corresponds to a shape of the object. Let the manifold $M$ be oriented shape space, the orientation space for each shape. Shape space can be thought of as an infinitely dimensional space of all possible shapes of an object, while the orientated shape space is how the object is placed in the real world. As an object makes a closed excursion in shape space, what is its corresponding movement in oriented shape space?

Let $S_0(t)$ be an element in shape space and $S(t)$ to be a physically oriented shape. The two are related by a rotational matrix

$$S(t) = R(t)S_0(t). \quad (5.1)$$

Define $A$ to be an infinitesimal rotation resulting from an infinitesimal deformation of $S_0(t)$,

$$\dot{R}(t) = R \left( R^{-1} \dot{R} \right) \equiv R(t)A(t). \quad (5.2)$$
This differential equation has a solution

\[ R(t) = \bar{P} \exp \int_0^t A(t') dt', \quad (5.3) \]

where the \( \bar{P} \) indicates that the expansion is time-ordered;

\[ R(t) = 1 + \int_{0<s<t} A(s) ds + \int_{0<s<u<t} A(s)A(u) ds du + ... \quad (5.4) \]

As one takes the derivative of equation 5.4, one will receive \( A(t) \) to the right, which is in agreement with equation 5.2.

One can write \( A(t) \) in a purely geometric form, as a gauge potential. Let \( A \) now be a vector field defined on the tangent space \( TM_{0,S_0} \) at a point \( S_0 \) on the manifold \( M_0 \). \( A \) can be evaluated in a direction \( \dot{S}_0 \) in which the shape is changing. Then,

\[ A(t) \equiv A_{\dot{S}_0(t)}[S_0(t)]. \quad (5.5) \]

If \( \omega_i \in TM_{0,S_0} \), then \( A \) will have a component \( A_i \) for each \( \omega_i \). Using these definitions, equation 5.3 can be rewritten as

\[ R(t) = \bar{P} \exp \int_{\dot{S}_0(t)} A_{\dot{S}_0(t)}[S_0(t)] dS_0. \quad (5.6) \]

The freedom of choosing coordinate axes for each point in shape space becomes a freedom of gauge of \( S_0 \), since the shape is independent of choice of axes. We now show that \( A \) is a non-abelian gauge potential. If one rotates an arbitrary shape by \( \Omega[S_0] \);

\[ S_0 \rightarrow \Omega[S_0]S_0, \quad (5.7) \]

then \( R \) transforms as

\[ R(t) \rightarrow R(t)\Omega^{-1}[S_0]. \quad (5.8) \]

Using equation 5.2, \( A \) transforms as

\[ A = R^{-1}\dot{R} \rightarrow \Omega R^{-1}\dot{R}\Omega^{-1} + \Omega R^{-1}R\dot{\Omega} = \Omega A\Omega^{-1} + \Omega\dot{\Omega}^{-1}. \quad (5.9) \]

Thus, \( A \) transforms as a non-abelian gauge potential.

We are only interested in closed excursions in shape space, that is, when an objects initial and final shape is the same. For such system, one can ignore high order terms in the time-ordered approximation (equation 5.4).
To evaluate $R$, a more explicit expression for $A$ is needed. The path $S_0(t)$ can be expressed as

$$S_0(t) = S_0 + s(t).$$ \hspace{1cm} (5.10)

$s(t)$ can be expanded as a sum of all $\omega_i$'s;

$$s(t) = \sum_i \alpha_i(t) \omega_i.$$ \hspace{1cm} (5.11)

Thus, the position in shape space is

$$S_0(t) = S_0 + \sum_i \alpha_i(t) \omega_i,$$ \hspace{1cm} (5.12)

with velocity

$$\dot{S}_0(t) = \sum_i \dot{\alpha}_i(t) \omega_i.$$ \hspace{1cm} (5.13)

Using these two last equations, one can expand $A$ as follows;

$$A \dot{S}[S_0 + s(t)] = A \dot{S}[S_0] + \sum_i \frac{\partial A \dot{S}_0}{\partial \omega_i} \dot{\alpha}_i$$

$$= \sum_j (A_j \dot{\alpha}_j + \sum_i \frac{\partial A_j}{\partial \omega_i} \alpha_i \dot{\alpha}_j).$$ \hspace{1cm} (5.14)

The first term does not give any contribution in equation 5.4, since it is a gradient. $R$ is gauge covariant, and so every term in the time-ordered expansion is also gauge covariant. $R$ can be written as

$$R = \bar{P} \exp(\oint A dt) = 1 + \frac{1}{2} \oint \sum_{ij} F_{ij} \alpha_i \dot{\alpha}_j dt,$$ \hspace{1cm} (5.15)

where

$$F_{ij} = \partial_j A_i - \partial_i A_j + [A_i, A_j]$$ \hspace{1cm} (5.16)

is the curvature, or field strength tensor. $F$ contains all information about rotations due to deformations at $S_0$. 

21
5.1 Rotating concentric spheres

Consider two concentric spheres having radius \( r \) and \( R \), with \( R > r \). The spheres are connected by some mechanism allowing each sphere to rotate without any external torque. If the inner sphere rotate, the outer sphere needs to rotate in the opposite direction for angular momentum to be conserved. Since the spheres are of different sizes, and assuming different moment of inertia, if one undergoes a series of rotations starting and ending with the same orientation, the other sphere has received a net rotation.

Let shape space \( S_0 \) be the relative orientation of the two spheres, meaning the configuration is \( \text{SO}(3) \). In oriented shape space, one needs to also consider the orientation of the outer sphere, so configuration for the system is \( \text{SO}(3) \times \text{SO}(3) \). The net rotation of the system will simply be the total rotation of the outer sphere when the inner sphere has returned to its original relative orientation.

Let the inner sphere undergo a series of rotations; a rotation of \( \epsilon \) around \( x \)-axis, a rotation of \( \eta \) around \( y \)-axis, then a rotation of \(-\epsilon\) and \(-\eta\) around \( x \)-axis and \( y \)-axis respectively. The outer sphere needs another rotation of \(-\epsilon \nu\) around \( z \)-axis for the curve in shape space to be closed. How much has the outer sphere’s orientation changed as a consequence of the inner sphere’s closed excursion in shape space?

By conservation of angular momentum, the outer sphere’s response to rotation of inner sphere is given by

\[
A = -\alpha \Omega, \tag{5.17}
\]

where \( \alpha \) is some constant, and \( \Omega \) is the rotation of the inner sphere with some generators \( J_i \). The negative sign is because the inner sphere is rotating in the other direction. Using the Baker–Campbell–Hausdorff formula, ignoring high order terms,

\[
e^X e^Y = e^{X+Y + \frac{1}{2}[X,Y]}, \tag{5.18}
\]

one finds the orientation of the inner sphere

\[
e^{-i\epsilon J_x} e^{-i\eta J_y} e^{-i\epsilon J_x} e^{i\eta J_y} e^{i\epsilon J_x} = 1, \tag{5.19}
\]

as expected. Terms higher than \( \eta, \epsilon \) squared are ignored. The outer sphere’s orientation is then

\[
e^{i\alpha \eta J_x} e^{i\alpha \eta J_y} e^{i\alpha \epsilon J_x} e^{-i\alpha \eta J_y} e^{-i\alpha \epsilon J_x} = e^{i(\alpha - \alpha^2)\eta J_z}. \tag{5.20}
\]
The net rotation of the system is thus \((\alpha - \alpha^2)\eta\) in the z-direction. One can read of \(F\) off from the exponent as
\[
F_{xy} = (\alpha - \alpha^2)J_z, \tag{5.21}
\]
and by symmetry,
\[
F_{ij} = (\alpha - \alpha^2)\epsilon_{ijk}J_k. \tag{5.22}
\]
\(\alpha\) can easily be calculated using conservation of momentum
\[
L = I\dot{\theta} + I'\dot{\theta}' = 0, \tag{5.23}
\]
where \(I\) and \(J\) are the moment of inertia, \(\dot{\theta}\) and \(\dot{\theta}'\) is the angle of the inner sphere and the outer sphere, respectively, both being in the plane of rotation. Subtracting \(I\dot{\theta}'\) from both sides, and arrange it, we get
\[
\dot{\theta}' = -\frac{I}{I + I'}(\dot{\theta} - \dot{\theta}'), \tag{5.24}
\]
where \(\dot{\theta} - \dot{\theta}'\) is the relative orientation of the two spheres, which is equal to \(\Omega\). Thus, comparison with equation 5.17 gives
\[
\alpha = \frac{I}{I + I'}. \tag{5.25}
\]
The fact that \(\alpha\) is dependent on the moment of inertia of the two spheres should not be a surprise to anyone. \(\alpha\) takes values from 0 to 1. If the outer sphere’s moment of inertia is much greater than the inner \(I' \gg I\) then \(\alpha = 0\), and so the outer sphere hardly reacts to the movement of the inner sphere. On the other hand, if \(I \gg I'\) then \(\alpha = 1\) and the outer sphere mimics the rotation of the inner sphere. In both extremes, \(F = 0\) and there will be no extra angle.

5.2 Net rotation of falling cats

As a cat is dropped upside down from a reasonable height, it will execute a series of deformations with the same initial and final form and land on its feet, unharmed [17]. The purpose of this section is to give an explanation to why and how the cat rotates.

The deformation of the cat happens in three phases. Starting from an upside down position, phase one begins by the cat contracting its stomach
muscles and bending forward. Phase two is when its front body and back body rotates simultaneously around their own axes, without twisting. Lastly, phase three is when the cat stretches out again. One can argue phase three happens when its feet come in contact with the ground. That, however, is a moot discussion; the net rotation happens in phase two.

We approach this problem a bit differently compared to the case of concentric spheres rotating with respect to each other. The field strength tensor will be complicated to calculate because the cylinders’ center-of-mass does not coincide, and the asymmetry of the system make shape space harder to define. We choose instead an alternative solution, simply using conservation of angular momentum, loosely following that of Kane and Scher [17] and Frolich [16]. A solution using gauge theory is done by Richard Montgomery [18]. The cat’s shape will make a closed excursion in shape space. This loop will be parametrized by the angle \( \alpha \), going from 0 to \( 2\pi \).

The cat and its movements will be simplified by the following approximations.

1) The body of the cat will be approximated by two cylinders, one for each of its body halves, connected at one single point. Feet, tail and head will be ignored. One might argue the tail and legs of the cat is responsible for the rotation. While this will certainly contribute to the effect, it is not the main cause of net rotation. This is easily proven by observing that even bobcats can rotate mid-air and the fact that the mass of the tail is much smaller than that of the body and the tail have to turn much more than is possible for the body to flip. Therefore, the two-cylinder model is an acceptable approximation describing the system.

2) The two cylinders with both make an angle \( \theta \) with the z-axis. \( \theta \) does not change during phase two.

3) Both cylinders rotate around their own axes with the same angular velocity \( \dot{\alpha} \). If one rotated faster than the other, the cat’s body would twist, which would harm the cat.

Let the angle between the top cylinder and the z-axis be \( \theta \). The angle between bottom cylinder and the negative z-axis is also \( \theta \). This configuration stays constant through out phase two. The top and bottom cylinder will rotate with an angular velocity \( \dot{\alpha} \) in the \( \hat{i}_T \) and \( \hat{i}_B \) direction, respectively. For the angular momentum to add up to zero, the two-cylinder system needs a rotation common to both cylinders, which is the net rotation along the z-axis. Let \( \dot{\beta} \) be the angular velocity the two cylinders have in the z-direction.
Figure 5: Coordinate system for a falling cat, using the two-cylinder approximation. Cat is falling in the -y-direction. The body of the cat is turning around z-axis with an angle $\beta$.

The angular momentum is then
\[ L_T = (\dot{\alpha} - \dot{\beta} \cos \theta) I \dot{i}_T + \dot{\beta} \sin \theta J \dot{j}_T, \]
\[ L_B = (-\dot{\alpha} + \dot{\beta} \cos \theta) I \dot{i}_B - \dot{\beta} \sin \theta J \dot{j}_B, \]  
\hspace{1cm} (5.26)

where $I$ and $J$ are the moments of inertia around the principal axes. Adding the two angular momenta together gives
\[ L_T + L_B = (\dot{\alpha} - \dot{\beta} \cos \theta) I (\dot{i}_T - \dot{i}_B) + \dot{\beta} \sin \theta J (\dot{j}_T - \dot{j}_B) = 0. \]  
\hspace{1cm} (5.27)

Finding $\dot{i}_T - \dot{i}_B$ and $\dot{j}_T - \dot{j}_B$ and solving for $\dot{\beta}$ yields
\[ \dot{\beta} = \frac{I \dot{\alpha} \cos \theta}{I \cos^2(\theta) + J \sin^2(\theta)}. \]  
\hspace{1cm} (5.28)

When integrated and setting integration constant to 0, this yields
\[ \beta = \frac{I \alpha \cos \theta}{I \cos^2(\theta) + J \sin^2(\theta)}. \]  
\hspace{1cm} (5.29)
\( \beta \) is linear in \( \alpha \) and so the angle the whole cat has turned is proportional to the angle each halves have rotated in their own coordinate system. If \( I = J \), this simplifies to

\[ \beta = \alpha \cos \theta. \quad (5.30) \]

If \( \theta \) is, for example \( \pi/3 \), and \( \alpha \) goes from 0 to \( 2\pi \), then the cat will have turned around and can land safely.

6 Born-Oppenheimer Approximation

Consider a system operating under two time scales; a fast one and a slow one. In this section, we let the slow variable be the position of a nucleus and the fast variable to be the position of the electrons. While the nucleus can affect the energy of the system, it do not induce state transitions in electrons. The eigenstates of the electrons will, however, change adiabatically as the slow variables evolve. Solving a system for a large atom with many nucleons and electrons will be impossible without any approximations. Therefore, it is necessary to use the Born-Oppenheimer approximation. First done by Born and Oppenheimer [19], one assume the wave function of the nucleons and the electrons can be separated into two parts. One then use the adiabatic approximation and solve the wave function for the electrons using a fix nucleus and find an effective Hamiltonian for the wave function of the nucleus.

The Hamiltonian for such a system is

\[
H = -\frac{1}{2m_N} \nabla_R^2 - \frac{1}{2m_e} \nabla_r^2 + V_N(R) + V_e(r, R) \quad (6.1)
\]

where \( m_N, m_e, r, R \) are the mass of the nucleus, mass of the electron, position of the electron and position of the nucleus, respectively. The full wave function is separated into two;

\[
\psi_n = \phi_n(r, R) \Phi_n(R) \quad (6.2)
\]

where \( \phi_n(r, R) \) is the solution to the Schrödinger equation when the nucleus is fixed

\[
[-\frac{1}{2m_e} \nabla_r^2 + V_N(R) + V_e(r, R)]\phi_n(r, R) = \epsilon_n(R)\phi_n(r, R) \quad (6.3)
\]
Now, to obtain an effective Hamiltonian for the nucleus, we sandwich equation 6.1 between two electron states:

\[ H_{lm} \equiv \langle \phi_l | H | \phi_m \rangle = -\frac{1}{2m_N} \langle \phi_l | \nabla^2_R | \phi_m \rangle \]

\[ -\frac{1}{2m_e} \langle \phi_l | \nabla^2_r | \phi_m \rangle + \langle \phi_l | V_N(R) | \phi_m \rangle + \langle \phi_l | V_e(r,R) | \phi_m \rangle \]  

(6.4)

The three last terms are equal to the energy \( \epsilon_n \) of the electrons. Now, equation 6.4 is still an operator, which means the gradients are operating on the ket vector to the right, multiplied with the function it’s operating on. Therefore, we can use chain rule with a test function \( f \);

\[ \langle \phi_l | \nabla^2_R( | \phi_m \rangle f) = \langle \phi_l | \nabla^2_R | \phi_m \rangle f + 2 \langle \phi_l | \nabla_R | \phi_m \rangle \nabla_R f + \delta_{lm} \nabla^2_R f. \]  

(6.5)

The Hamiltonian is then

\[ H_{lm} = -\frac{1}{2m_N} [\langle \phi_l | \nabla^2_R | \phi_m \rangle + 2 \langle \phi_l | \nabla_R | \phi_m \rangle \nabla_R + \delta_{lm} \nabla^2_R] + \delta_{lm} \epsilon_m. \]  

(6.6)

At this point, one uses the adiabatic approximation and ignores the off-diagonal terms, leaving

\[ H_m = -\frac{1}{2m_N} [\langle \phi_m | \nabla^2_R | \phi_m \rangle + 2 \langle \phi_m | \nabla_R | \phi_m \rangle \nabla_R + \nabla^2_R] + \epsilon_m. \]  

(6.7)

If one defines \( A_m(R) \) as

\[ A_m(R) = i \langle m | \nabla_R | m \rangle, \]  

(6.8)

equation 6.7 can be rewritten as

\[ H_m = -\frac{1}{2m_N} \left( \nabla_R - iA_m(R) \right)^2 + \epsilon_m. \]  

(6.9)

The effect the electrons have on the nucleus is therefore approximated by a vector potential. The gauge potential \( A_m(R) \) is, of course, the Berry connection. This effect can be seen as a molecular Aharonov-Bohm effect.
7 Conclusion

The essence of the adiabatic theorem tells us this; if the time-scale of the changes in the Hamiltonian is much smaller than any other time-scale, the system stays in the same state. As a Hamiltonian is driven adiabatically in a closed loop in parameter space, the eigenstate will have been parallel transported and has accumulated a geometric phase, called Berry’s phase, as it returns. The condition of adiabaticity is only to ensure parallel transport. If parallel transport is ensured any other way, the condition can be relaxed.

We have seen how one can be wrong in thinking the phase of a single particle system is irrelevant. Indeed, the geometrical phase can have a crucial role in how a particle behave. In the presence of a degeneracy, $V_n(R)$ contains some singularity, causing the geometric phase to be proportional to some solid angle as seen from the degeneracy. In such a case, the vector potential stem from some "magnetic monopole" positioned at the degeneracy, radiating out a 2-form. However, the "magnetic monopole" analogy is flawed, the curl of $V_n(R)$ is generally not zero. Despite this, one can always see $V_n(R)$ as some "magnetic field" whose vector potential is Berry’s connection.

8 Appendix

8.1 Derivation of $c(t)$

We follow the derivation of [5]. Plugging equation 2.6 into equation 2.4 we get

$$i(\dot{c}_n(t)e^{-i\theta_n(t)}|n(t)) - iE_n(t)c_n(t)e^{-i\theta_n(t)}|n(t))$$

$$+c_n(t)e^{-i\theta_n(t)}|\dot{n}(t)) = \hat{H}c_n(t)e^{-i\theta_n(t)}|n(t)) .$$

(8.1)

Where we used the fact that $\dot{\theta}_n(t) = -iE_n(t)$. Since equation 2.3 still holds, the second term cancels out the right-hand side. Moving the third term to the right-hand side and canceling out $i$ gives us

$$\dot{c}_n(t)e^{-i\theta_n(t)}|n(t)) = -c_n(t)e^{-i\theta_n(t)}|\dot{n}(t)) .$$

(8.2)

Now we can take the inner product of $|n(t))$ with $|m(t))$ from the left

$$\dot{c}_n(t)e^{-i\theta_n(t)}\langle m(t)|n(t)) = -c_n(t)e^{-i\theta_n(t)}\langle m(t)|\dot{n}(t)) ,$$

(8.3)
and since n-th and m-th state are orthogonal, we’re left with
\[ \dot{c}_m(t) = -c_n(t) e^{i(\theta_m(t) - \theta_n(t))} \langle m(t) | \dot{n}(t) \rangle. \] (8.4)

We can split this equation into two parts; one for \( n = m \), and one for \( n \neq m \),
\[ \dot{c}_m(t) = -c_m(t) \langle m(t) | \dot{m}(t) \rangle - \sum_{n \neq m} c_n(t) e^{i(\theta_m(t) - \theta_n(t))} \langle m(t) | \dot{n}(t) \rangle. \] (8.5)

To obtain an expression for \( \langle m(t) | \dot{n}(t) \rangle \), we take the derivative of equation 2.3 with respect to time,
\[ \dot{\hat{H}}(t) | n(t) \rangle + \hat{H} | \dot{n}(t) \rangle = \hat{E}_n(t) | n(t) \rangle + \dot{E}_n(t) | \dot{n}(t) \rangle, \] (8.6)
and multiplying by \( \langle m(t) | \) from the left,
\[ \langle m(t) | \dot{\hat{H}}(t) | n(t) \rangle + \langle m(t) | \hat{H} | \dot{n}(t) \rangle = \hat{E}_n(t) \langle m(t) | n(t) \rangle + \dot{E}_n(t) \langle m(t) | \dot{n}(t) \rangle. \] (8.7)

Simplifying the expression by using orthogonality of states (and the fact that \( n \neq m \)), together with hermiticity of \( \hat{H} \),
\[ \langle m(t) | \dot{n}(t) \rangle = \frac{\langle m(t) | \dot{\hat{H}}(t) | n(t) \rangle}{\hat{E}_n(t) - \hat{E}_m(t)}. \] (8.8)

Plugging this back into equation 8.5,
\[ \dot{c}_m(t) = -c_m(t) \langle m(t) | \dot{m}(t) \rangle - \sum_{n \neq m} \frac{\langle m(t) | \dot{\hat{H}}(t) | n(t) \rangle}{\hat{E}_n(t) - \hat{E}_m(t)} c_n(t) e^{i(\theta_m(t) - \theta_n(t))}. \] (8.9)

We can see that the time derivative of the m-th new term in our ansatz depends on the m-th state (and its derivative) as well as a sum over all possible states. This is where the adiabatic approximation comes in. If
\[ | \dot{\hat{H}}(t) | \ll 1, \] (8.10)
then the second term can be dropped and equation 8.9 becomes
\[ \dot{c}_n(t) = -c_n(t) \langle n(t) | \dot{n}(t) \rangle, \] (8.11)
where we have used index n again (no sum this time). This equation has solution

\[ c_n(t) = c_n(0) \exp \left( - \int_0^t \langle n(s)|\dot{n}(s) \rangle \, ds \right), \]  

(8.12)

where we set the constant \( c_n(0) \) in front of the exponential to 1 because of the normalization and the fact that the system is kept in the same state over this process.
References


