On the Construction and Traversability of Lorentzian Wormholes

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In this literature review we discuss and describe the theoretically predicted phenomena known as wormholes, where two different regions of space-time are joined by a “throat” or a “bridge”. If information or even an observer could be sent through the wormhole we refer to it as traversable. We argue that traversable wormholes demands negative energy densities and display a number of different constructions found within the field. Among these the original construction by Einstein-Rosen and Morris-Thorne’s discussion on traversability. We also give an overview of the current state of the field by presenting more recently published papers: “Casimir Energy of a Long Wormhole Throat” by Luke Buther and “Traversable Wormholes via a Double Trace Deformation” by Ping Gao, Daniel Louis Jafferis and Aron C.Wall.
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I. INTRODUCTION

Wormholes are a fascinating phenomena predicted by the theoretical framework of general relativity. A wormhole is a region where the structure of space-time acts as a bridge between two spatially separated locations; no experimental evidence of such objects exists to date but the concepts have fascinated nobel prize winning physicists and science-fiction writers alike for several decades. This report gives a basic quantitative definition and description of wormholes by looking at what is considered the first wormhole construction, the Einstein-Rosen bridge\[1\], and the revolutionary paper of Morris and Thorne\[2\]. The current status of the field is also reviewed through the survey of two more recently published papers on the subject: Luke Butchers (2014) paper\[3\] which continues and expands the ideas of Morris and Thorne and P. Gao, D. L. Jafferis, and A. C. Wall (2017)\[4\] paper in which a wormhole is constructed in AdS-space.

The field of wormhole physics lies in the intersection of general relativity and quantum field theory (QFT). This reports main focus will be on the general relativity aspects with many QFT results either motivated heuristically or stated directly with reference to other literature. This, in combination with the section titled “Theoretical background” where the basics of general relativity are recounted, should make this report ideal for a reader possessing some familiarity with special relativity and the basic of quantum mechanics.

II. THEORETICAL BACKGROUND

A. Introduction to General Relativity

The material from this section is based on Sean Carroll’s “Lectures notes on general relativity”\[5\]. General relativity (GR) as a theory was first published in 1915 by Albert Einstein to expand upon and, as the name suggests, generalize his special relativity (SR) to include the notion of gravity. At the heart of the theory lies the insight that gravity can be modeled as an effect of the curvature and geometry of space-time and not as a Newtonian action at a distance. The geometry of space-time is in turn determined by its contained amount, and distribution of, matter and energy; two concepts already intimately linked by SR. GR being a theory concerned with such concepts as non-Euclidean geometry, curvature, coordinates of higher dimensions (space-time has three spacial- and one temporal dimension) it needs more versatile and general mathematical tools than those used in Newtonian physics. These tools are provided by the mathematical discipline of differential geometry. Below we give a brief overview of the mathematical and physical basics that will be needed to understand the content of this report.

B. Space-Time

The idea of not treating space and time as two entirely distinct concepts but as one more intertwined entity, space-time, is something already present in special relativity. It is a natural and necessary consequence of the transition from the Galilean to the Lorentzian transformations to also abandon the concept of a universal time and in the process create a more complex interplay between position in time and space. It is also very natural to give a geometrical interpretation to objects behavior in this space-time; to create a notion of distance between two “events”, the name given to the individual points of space-time. In Special relativity this notion of distance is given by the Minkowski metric

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$

(1)

One can notice that this metric has no dependence on position. No matter where one may happen to be located in time and space, the geometry is uniform and unchanging. This homogeneity of space-time allows one to sweep some of the more geometrical considerations under the rug in SR, but this is not the case in general relativity. In GR the metric in general has a dependence on both time and position which lends a much richer structure to space-time. Space-time itself we may think of as a four-dimensional (hyper)surface with a set of specific properties. Most important of which is that we want to be able to assign coordinates to this surface in a reasonable manner. These coordinates must not be valid for the entire surface but we demand that the entire surface can be covered by patches of these coordinates and that the patches may overlap. We also demand that for every point in space-time it should be possible for us to pick
a set coordinates that makes the metric take on the appearance of the Minkowski metric at that point; phrased differently we want space-time to be locally flat. This demand is one formulation of what is called Einstein’s equivalence principle.

All these somewhat vague demands we pose on our model of space-time are made precise by the mathematical notion of a pseudo-Riemannian manifold, but for the purpose of this report these heuristic descriptions will suffice.

### C. The Metric

We turn our attention to the metric as a mathematical object. As we previously mentioned the metric is in general a function of the coordinates of space-time of the form:

\[
ds^2 = g(\mathcal{X})_{\mu\nu}dx^\mu dx^\nu \tag{2}
\]

Here we have used the Einstein summation convention where repeated pairs of lowered and raised indices are summed over and \(\mathcal{X}\) is a vector of coordinate values. A set of coordinates can be thought of as a set of basis vectors protruding at every point in space-time (see figure 1). Making a good choice of coordinates could have a drastic effect on the complexity of calculations so it is important to know how to transform the metric between different coordinate systems. Just as in SR the metric is a tensor, more precisely a tensor field but a tensor at every coordinate point, and its components follows the familiar transformation law under change of coordinate.

The numerical value of this quantity, since it is a scalar, is something agreed upon by all observers no matter their reference system and the physical interpretation of its value is as a measure of the curvature of space-time. As one expect the Ricci scalar is identically zero in flat Minkowski space-time. The main use of all these quantities will be in the construction of the essential Einstein-tensor which will be central to our next topic of discussion.

### D. The Einstein Equations

Up until now we have assumed the existence of a metric but not said how one obtains this metric given knowledge of the the matter/energy content of a particular space-time. This transition from content to geometry is dictated by the Einstein equations (here, and in the rest of the report, we use geometrized units where the speed of light \(c\) and the gravitational constant \(G\) are made unitless and their value is set to one).

\[
G^{\mu\nu} = 8\pi T^{\mu\nu} + 2\Lambda \delta^{\mu\nu} \tag{8}
\]

Where \(G^{\mu\nu}\) is the above mentioned Einstein tensor. It is defined using quantities we defined in the above subsection:

\[
G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} \tag{9}
\]

This being the case the Einstein tensor is determined entirely by the metric. The symbol \(\Lambda\) is the cosmological constant.
constant, to put it simply it can be thought of as a parameter that determines the curvature of empty space-time. We will return to discuss this in more detail in the section on Anti-de Sitter space but for now it suffices to know that putting this parameter to zero is to demand that empty space should be flat. The other unfamiliar object on the right hand side of equation is the energy-stress tensor \( T \). We give a more detailed physical interpretation of its individual components, in its diagonal form, in section “Analysing the energy-stress tensor”; for now we will be content with saying that \( T \) holds information about pressure, energy density, momentum flux and shear stresses throughout space-time. Our last comment about the Einstein equation is that one should not be fooled by the neatness of Equation (8); this simple looking expression is in general a set of highly coupled second order partial differential equations and it requires a great deal of ingenuity to obtain analytic solutions for it, something we will expand upon in the section on the Morris and Thorne construction.

After this, admittedly somewhat minimalist, introduction to general relativity we start our survey of wormholes by looking at a specific example of a space-time, the Schwarzschild space-time. Historically this was one of the first instances of a structure that resembles a wormhole being described and it will serve as our motivating example.

III. INTRODUCTION TO UNTRAVERSABLE WORMHOLES

A. The Schwarzschild metric and the Einstein-Rosen bridge

Setting the cosmological constant \( \Lambda \) to zero in the Einstein equations one obtains

\[
G^{\mu \beta} = 8\pi T^{\mu \beta} \tag{10}
\]

If we set \( T^{\mu \beta} = 0 \) for all \( \mu, \beta \) we obtain the sourceless Einstein equations

\[
G^{\mu \beta} = 0 \tag{11}
\]

We will make a few assumptions in order to be able to find solution in terms of a metric. The first assumptions puts restrictions on the symmetry of the space time in consideration; we demand that he metric should be spherically symmetric and static. A metric is said to be spherically symmetric if all events in the space-time it describes are located on spatial hyper-surfaces of constant time whose line element can be made into the form

\[
dl^2 = r^2(d\theta^2 + \sin^2 \theta d\phi^2) \tag{12}
\]

Where \( r \in [0, \infty), \theta \in [0, \pi] \) and \( \phi \in [0, 2\pi) \) We introduce the following shorthand

\[
d\Omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2
\]

We call a metric static if the components are independent of the time coordinate and invariant under time reversal \((t, r, \theta, \phi) \mapsto (-t, r, \theta, \phi)\). Under these assumptions the solution to Equation (we will not give the precise derivation to this metric but it is a straightforward but tedious task to make sure that it solves the Einstein equations) is the metric

\[
ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{13}
\]

This is known as the Schwarzschild metric. The factor \( M \) enters the metric as a integration constant but needs to be interpreted as the Newtonian mass near the origin in order to yield the correct limiting behavior for large \( r \). One can also note that

\[
\lim_{r \rightarrow \infty} ds^2 \approx -dt^2 + dr^2 + r^2 d\Omega^2
\]

Which is simply the Minkowski metric expressed in spherical coordinates. This intuitive property is known as asymptotic flatness.

Just from inspection it is clear that this metric exhibits behaviors of special interest at two values of \( r \): one at \( r = 0 \) and one at \( r = 2M \). At both these values the metric displays a singularity; these singularities, while qualitative similar, have very different physical interpretations. In general relativity, as in all physical theories involving coordinates, two types of singularities are possible to encounter. The first type is coordinate singularities, these are simply instances were your rule for assigning labels to the events of space time break down or, in the language of differential topology, our coordinate map of choice dose not globally cover the manifold. The takeaway from this is that these singularities can be removed by a coordinate change. The other type of singularities are the true physical singularities, points were actual observables diverges. Our heuristic way of determining if a singularity is of the first or second type will be to

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1 This requirement is not strictly necessary since it can be shown that this follows from the requirement of spherical symmetry, See Carroll. Here it is done for clarity.

2 It is not a uncommon assessment among theoreticians that the presence of this type of singularities is a manifestation of our lack of understanding of quantized gravity.
look at the curvature at the point of interest to see if it diverges. To get a frame independent measure of the curvature we look at contractions of the Ricci-tensor. Take for example the contraction
\[
R^\mu\rho\sigma\nu R_{\mu\rho\sigma\nu} = \frac{12 M^2}{r^6}
\]
This tells us that curvature is in some sense diverging at \( r = 0 \) and that this singularity is a physical one. We will convince ourselves that the singularity at \( r = 2M \) is on the other hand of the coordinate type, but none the less very interesting and highly non-trivial, by looking at a coordinate change that removes it.

The particular coordinates that we shall consider were discovered by Albert Einstein and Nathan Rosen[1]. We let
\[
u^2 = r - 2M \Leftrightarrow u = \pm \sqrt{r - 2M} \quad (14)
\]
In these coordinates the metric becomes, with \( u \in (-\infty, \infty) \)
\[
ds^2 = -\frac{u^2}{u^2 + 2M} dt^2 + 4(u^2 + 2M) du^2 + (u^2 + 2M)^2 d\Omega^2 \quad (15)
\]
We can now observe that both previously mentioned singularities have disappeared from the metric but this is not in contradiction with the previous discussion. This coordinate change is not bijective and what we have done is covered the area \( r > 2M \) twice and discarded \( r < 2M \) along with it the inner singularity.

Since there is no angle dependence in the integrand we can write
\[
A(u) = 4\pi (u^2 + 2M)^2 \quad (17)
\]
We now observe that this area takes a minimal value of \( 16\pi M^2 \) when \( u = 0 \). This can be interpreted as a “throat” joining the two asymptotically flat regions \( u \in (0, \infty) \) and \( u \in (-\infty, 0) \). This is the motivating example of a wormhole and this property: a throat joining two asymptotically flat regions, where spherical surfaces take on minimum radius (or equivalently area) will be our working definition of what constitutes a wormhole. That being said there exists some disagreement in the literature concerning if one should classify the Einstein-Rose bridge has a proper wormhole. Matt Visser refers to the bridge as “only a coordinate artifact”[3] while Morris and Thorne writes about “Schwarzschild wormholes”[2].

One's stance on this distinction is, for the purpose of this report, of little consequence since the the Einstein-Rosen bridge is excluded from the possible list of traversable wormholes either way.

To see why this is the case we take another look at the schwarzschild metric. We previously stated that this metric is time-independent, by this we meant that Equation (13) has no dependence on the coordinate “t”. But in a more general setting we define the time coordinate, call it: \( x \), as the coordinate which corresponding component of the metric, \( g_{xx} \), has the opposite sign as compared to the rest of the diagonal components. For example, in the case of the Schwarzschild metric when \( (1 - \frac{2M}{r}) > 0 \) the coefficient of \( dt^2 \) is negative while all other components are positive. In this region it is therefor justified to refer to \( t \) as the time coordinate, but we also notice that inside the event horizon, when \( (1 - \frac{2M}{r}) < 0 \), “\( t \)” takes on the roll of the time coordinate. Since the metric is dependant on the coordinate \( r \) the region inside the event horizon actually evolves with time! Changing to a set of coordinates without a coordinate singularity at the event horizon: the Kruskal-Szekeres (T,X) coordinates, which we elaborate on in section VII, one can see the effect of this time evolution[14]. As can be seen in Figure [3], where the Kruskal time evolves forward from the left figure to the right, the bridge of the wormhole forms and then “pinches off”.

As we see in the above Fig. [2], where we have plotted \( u \) against \( r \) and \( \theta \), space time around the event horizon now has a cylindrical funnel-like structure. This statement can be made more precise by considering the areas of spheres in this metric, surfaces of constant \( u \) and \( t \).
\[
A(u) = \int_S (u^2 + 2M)^2 d\Omega \quad (16)
\]

As R. W. Fuller and J. A. Wheeler showed[7] the “pinching off” takes place so rapidly that no particle or photon can cross the bridge before it closes. This
The face of interest, the set of points where $\rho = 0$, and Schwarzschild radius when $a = 0$. There is another surface known as the Kerr metric. This vacuum solution is also a sign that the coordinate $u$ breaks down at the throat and that it therefore is unsuitable to describe passage through the wormhole. Before we move on to discuss traversable wormholes, we look at another type of black hole based wormhole.

B. The Kerr wormhole

The next major type of wormhole emerges from a metric known as the Kerr metric. This vacuum solution is achieved by relaxing the assumption of spherical symmetry to cylindrical symmetry while also demanding that the metric should be time independent. The Kerr metric has the form:

$$ds^2 = -dt^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2 + \frac{2Mr}{\rho^2} (\sin^2 \theta d\phi - dt)^2$$

Where $\Delta(r) = r^2 - 2Mr + a^2$ and $\rho(r, \theta) = r^2 + a^2 \cos^2 \theta$. This metric describes the space-time surrounding a rotating wormhole/black hole and the parameter $a$ can be seen as a measure of this rotation. The case $a = 0$ corresponds to the Schwarzschild solution, if one interprets $M$ once again as the mass. The geometry of the Kerr space-time is much more complex than the Schwarzschild and for some reasons we will list in the next section, this one is also highly non-traversable[6]. With this in mind, we will only give a brief description of this geometry. The behavior of the space-time is highly dependant on whether $a < M$, $a > M$ or $a = M$.

In the $a < M$ case there exists two horizons. These correspond to the zeros of the function $\Delta(r)$. These are

$$r_{1,2} = M \pm \sqrt{M^2 - a^2}$$

Observe that we recover the singularities of the Schwarzschild radius when $a = 0$. There is another surface of interest, the set of points where $\rho(r, \theta) = 0$. These are the points where $r = 0$ and $\theta = \frac{\pi}{2}$. This innermost singularity is not a point but a ring. It is the interior of this ring that comprises the throat of the wormhole and connects two asymptotically flat regions of space. While the prospect of, as inwards falling observer, dodge the singularity by disappearing into the ring and exit safely on the other side sounds exiting enough further analysis makes this a very unlikely occurrence. One would expect extreme levels of red-shift near this ring singularity and this in combination by the presence of a singularity and the additional complexity of the Kerr's black hole/ -wormhole as compered Schwarzschilds makes it unstable candidate; while this is the case it will be useful later in the report, when we create traversable wormholes by modifying untraversable ones, to know that their exist other untraversable wormholes than the Schwarzschild one.

IV. THE MORRIS AND THORNE CONSTRUCTION OF A TRAVERSABLE WORMHOLE

A. Criteria for traversability

The wormholes we have presented so far could hardly be categorized as traversable since traveling into them would constitute a sure-fire death sentence. The tidal forces poses a considerable problem, threatening to rip any reasonable traveler to pieces and even if this could be prevented the presence of a horizon (as in the case of the Einstein-Rosen bridge) is a definite deal-breaker. These problems of traversability inspired Morris and Thorne to formulate a list of criteria that a wormhole must satisfy in order to be deemed traversable[6]. The main criteria were:

1. No horizon, since this would prevent passage through the wormhole.
2. Survivable tidal forces.
3. The duration of the passage through the wormhole must be perceived as finite for both the traveler and any observer at either end of the wormhole.
4. Physically reasonable energy-stress tensor.

Morris and Thorne succeeded in finding solutions to the Einstein equations which at least could be said to have several of these properties and we will present parts of their construction here, focusing on the fourth criterion of the list. The main idea of the procedure is to decide on a desirable metric and then use the Einstein equation to “work backwards” to determine the necessary energy-stress tensor; backwards in the sense that one in Newtonian gravity usually calculates a gravitational field from a given mass-distribution, not the other way-around. Nevertheless the technique is highly useful.
B. Calculating the Einstein tensor

Morris and Thorne assume a static and spherically symmetric space-time in order to simplify the calculations. Their ansatz for the metric was

\[ ds^2 = -e^{2\Phi(r)} dt^2 + \frac{dr^2}{1 - \frac{b(r)}{r}} + r^2 d\Omega^2 \]  

(21)

Where \( r \in [0, \infty) \), \( \theta \in [0, \pi) \) and \( \phi \in [0, 2\pi) \). \( \Phi(r) \) and \( b(r) \) are some functions dependent on the radial coordinate later to be determined. The goal now is, as previously stated, to use Equation (10) to determine \( T \). We do this by determining the Einstein tensor \( G \). The Einstein tensor is defined to be

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \]  

(22)

Where \( g_{\mu\nu} \) is the metric and \( R_{\mu\nu} \) is referred to as the Ricci tensor and is defined to be

\[ R_{\mu\nu} = R^\lambda_{\nu\lambda\mu} \]  

(23)

The Ricci tensor can be further contracted using the matrix inverse of the metric to yield the so called Ricci scalar \( R \)

\[ R = g^{\mu\nu} R_{\mu\nu} \]  

(24)

The (3,1)-tensor that is used in Equation (23) is called the Riemann curvature tensor and has the following definition

\[ R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\alpha_{\lambda\gamma} \Gamma^\lambda_{\beta\delta} - \Gamma^\alpha_{\lambda\delta} \Gamma^\lambda_{\beta\gamma} \]  

(25)

We recall that each of the Christoffel symbols are defined as

\[ \Gamma^\alpha_{\beta\gamma} = \frac{1}{2} g^{\alpha\lambda} (g_{\lambda\beta,\gamma} + g_{\lambda\gamma,\beta} - g_{\beta\gamma,\lambda}) \]  

(26)

It is not difficult to realise that determining all the components of the Einstein tensor is quit the chore and this herculean task of algebra is best left to a computer; with this in mind we will only compute one of the components of the Riemann tensor to illustrate the procedure. Our component of choice will be \( R^\theta_{\phi\phi\theta} \). According to Equation (23) it has the form

\[ R^\theta_{\phi\phi\theta} = \Gamma^\theta_{\phi\phi,\theta} - \Gamma^\theta_{\phi\theta,\phi} + \Gamma^\theta_{\lambda\phi} \Gamma^\lambda_{\phi\theta} - \Gamma^\theta_{\lambda\phi} \Gamma^\lambda_{\phi\theta} \]  

(27)

We work through the expression one term at a time. Notice that due to the fact that the metric is diagonal the index \( \lambda \) in Equation (26) can be replaced by \( \alpha \). The first term therefore becomes

\[ \Gamma^\theta_{\phi\phi,\theta} = \frac{1}{2} \partial_\theta (g^{\theta\theta} (g_{\theta\phi,\phi} + g_{\theta\phi,\phi} - g_{\phi\phi,\phi})) \]

Reading of the the metric components

\[ \Gamma^\theta_{\phi\phi,\theta} = \frac{r^2}{2r^2} \partial_\theta (\partial_\theta (\sin^2 \theta)) \]

\[ \Gamma^\theta_{\phi\theta,\phi} = -\frac{1}{2} \partial_\theta (\sin \theta \cos \theta) \]

\[ \Gamma^\theta_{\phi\phi,\theta} = \sin^2 \theta - \cos^2 \theta \]  

(28)

We consider the second term

\[ \Gamma^\theta_{\phi\theta,\phi} = \frac{1}{2} \partial_\phi (g^{\theta\theta} (g_{\theta\phi,\phi} + g_{\theta\phi,\phi} - g_{\phi\phi,\phi})) \]

From inspection of the metric we can conclude that this term is zero; all three metric components in the innermost parenthesis are either off-diagonal or constant with respect to the variable of their corresponding derivative, more concretely \( \partial_\phi (g_{\theta\theta}(r)) = 0 \). So

\[ \Gamma^\theta_{\phi\theta,\phi} = 0 \]  

(29)

We move on to the third term. We can make a few observations that simplifies the calculations. When term three is expanded each its terms have a factor of the form \( \Gamma_{\lambda\phi}^\lambda \) where \( \lambda \) is a dummy index. Using the same logic as in term 2 and the fact that \( g_{\theta\theta} \) is only a function of \( r \) these factors are only one-zero when \( \lambda = r \). Therefore:

\[ \Gamma^\theta_{\lambda\phi} = \Gamma^\theta_{r\phi} \Gamma^\lambda_{\phi\phi} \]

\[ \Gamma^\theta_{\lambda\phi} \Gamma^\lambda_{\phi\phi} = -\frac{1}{r} \left( 1 - \frac{b(r)}{r} \right) \sin^2 \theta \]  

(30)

Finally we look at term four. An additional simplifying observation is that when one is working with a diagonal metric the Christoffel symbols are only non-zero when at least two of their three indexes are the same; otherwise they are a sum of off-diagonal metric elements and therefore zero. This in combination with the discussion for the previous term lets us conclude

\[ \Gamma^\theta_{\lambda\phi} \Gamma^\lambda_{\phi\phi} = \Gamma^\theta_{\phi\theta} \Gamma^\phi_{\phi\phi} \]

\[ \Gamma^\theta_{\lambda\phi} \Gamma^\lambda_{\phi\phi} = -\sin^2 \theta + \cos^2 \theta \]  

(31)
If we add up all the terms we obtain the final expression for the Riemann tensor component, in agreement with Morris and Thorne

\[ R^\theta_{\phi\theta\phi} = \frac{b(r)\sin^2 \theta}{r} \]  

(32)

The rest of the components can be calculated in a similar manner or, for the somewhat less masochistic reader, read off from Morris and Thorne’s paper. In order to simplify transition from the Riemann tensor to the Einstein tensor Morris and Thorne at this stage switches to an orthonormal set of basis vectors characterised by the transformation

\[ e_t = e^\phi e^i \]
\[ e_r = (1 - \frac{b(r)}{r})^{-1/2} e^r \]
\[ e_\theta = re_\theta \]
\[ e_\phi = rsin \theta e_\phi \]

The metric takes on an especially pleasant form in this basis, namely that of the standard metric from special relativity. Which can be confirmed by the formula

\[ g_{\alpha\beta} = e_\alpha \cdot e_\beta \]  

(33)

We transform our calculated component of the Riemann tensor to demonstrate. Since this transformation is only a rescaling of the basis vector the components will simply be rescaled as well. We perform the calculation

\[ R^\theta_{\phi\theta\phi} e_\theta \otimes e^\phi \otimes e^\theta \otimes e^\phi = \]
\[ R^\theta_{\phi\theta\phi}(re_\theta) \otimes (\frac{e^\phi}{\sin \theta}) \otimes (\frac{e^\theta}{r}) \otimes (\frac{e^\phi}{\sin \theta}) = \]
\[ \frac{R^\theta_{\phi\theta\phi}}{r^2 \sin^2 \theta} e_\alpha \otimes e_\alpha \otimes e_\alpha \otimes e_\alpha \]

From this we conclude that

\[ R^\theta_{\phi\theta\phi} = \frac{b(r)}{r^3} \]  

(34)

Similar calculations can be done for all components and one finally ends up with the following result for the non-zero components of the Einstein tensor

\[ G_{tt} = \frac{b'(r)}{r^2} \]  

(35)

\[ G_{\varphi\varphi} = -\frac{b(r)}{r^3} + 2\left(1 - \frac{b(r)}{r}\right)\frac{\Phi(r)}{r} \]  

(36)

\[ G_{\theta\theta} = \left(1 - \frac{b}{r}\right)[\Phi'' + \Phi'(\Phi' + \frac{1}{r})] - \frac{1}{2r^2}((b'r - b)(\Phi' + \frac{1}{r})) \]  

(37)

\[ G_{\theta\varphi} = G_{\varphi\theta} \]  

(38)

Now using the the Einstein equation we can from these components deduce the corresponding components of the energy-stress tensor

\[ T_{tt} = \frac{b'(r)}{8\pi r^2} \]  

(39)

\[ T_{r\varphi} = -\frac{b(r)}{8\pi r^3} + \frac{1}{4\pi} \left(1 - \frac{b(r)}{r}\right) \frac{\Phi(r)}{r} \]  

(40)

\[ T_{\theta\theta} = \frac{1}{8\pi} \left(1 - \frac{b}{r}\right)[\Phi'' + \Phi'(\Phi' + \frac{1}{r})] - \frac{1}{16\pi r^2}((b'r - b)(\Phi' + \frac{1}{r})) \]  

(41)

\[ T_{\varphi\varphi} = T_{\theta\varphi} \]  

(42)

Since we are in a reference frame where the energy-stress tensor is diagonal it is fairly straightforward to interpret the individual components physically but before we do this it is helpful to reflect on what information we have just gained. We started with a desired metric and have now determined what distribution of matter and energy must be present to give rise to such a metric. What remains to see is if this distribution has physically realizable properties or, put in another way, if criterion 4 for traversability is fulfilled.

C. Analysing the energy-stress tensor

In this section we derive some properties of the energy-stress tensor that will be useful after the more precise discussing on reasonability of matter in general relativity and on energy conditions. We start by relabeling and interpreting the components of T.

\[ T_{tt} = \rho \]  

(43)

\[ T_{r\varphi} = -\tau \]  

(44)

\[ T_{\varphi\varphi} = p \]  

(45)
We now want to understand what the different components of stress-tensor physically. This can be done by comparing the now diagonal stress-energy tensor to the stress-energy tensor of a perfect fluid. This is a standard procedure in general relativity in order to gain physical intuition; one assumes that the matter one is considering has a particular equation of state to interpret the individual components as measurable quantities. The stress-energy tensor for a perfect fluid is:

\[ T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \]  

Here \( U \) is the four velocity, \( \rho \) is the energy density and \( p \) is the pressure. Considering a perfect fluid in its rest frame we can therefore interpret our original \( \rho \) as the energy density, \( -\tau \) as the radial pressure (\( \tau \) is radial tension) and finally \( p \) is the transverse pressure. In order to make any statements of the behavior of these quantities we must first acquire some information about the function \( b(r) \), referred to as the shape function by Morris and Thorne for reasons we will see soon. One fact that remains true for any wormhole, by definition, is that there exist a “throat”. More precisely that means that the radius of spheres \( r \), centred at the origin, as a function of proper radial distance \( \ell \) has a minimum, which we call \( r_0 \). This fact in combination with definition of the radial proper distance from the throat \( r_0 \)

\[ \ell(r) = \int_{r_0}^{r} \frac{dr_s}{\sqrt{1 - \frac{b(r_s)}{r_s}}} \]  

lets us make the following chain of arguments

\[ \frac{d\ell}{dr} = \sqrt{1 - \frac{b(r)}{r}} \Rightarrow \frac{dr}{d\ell} = \sqrt{1 - \frac{b(r)}{r}} \]  

\[ \frac{d^2r}{d\ell^2} = \frac{d^2r}{d\ell} \frac{dr}{d\ell} \frac{dr}{d\ell} \frac{d^2r}{d\ell} = \frac{d}{d\ell} \left( \frac{dr}{d\ell} \right)^2 \]  

\[ \frac{d^2r}{d\ell^2} = \frac{1}{2r} \left( \frac{b(r)}{r} - b'(r) \right) \]  

Since \( r(\ell) \) has a minimum at \( r_0 \) that means \( \frac{d^2r}{d\ell^2} \geq 0 \) at \( r_0 \) and, due to the smoothness of the coordinate functions, that there exists an interval \( (r_0, r_0 + \epsilon) \) for some \( \epsilon > 0 \) where

\[ \frac{1}{2r} \left( \frac{b(r)}{r} - b'(r) \right) > 0 \]  

Equation (53) is the first of two results we need about the shape function. To obtain the second one we further consider the geometry of space-time close to the throat, or to be more specific the embedding of that geometry as a two dimensional surface living in a three dimensional euclidean space. Since we are dealing with a spherically symmetric and static space-time we can fix the \( t \) and \( \theta \) to some specific value without loss of generality. Doing this Equation (21) reduces to

\[ ds^2 = \frac{1}{1 - \frac{b(r)}{r}} dr^2 + r^2 d\phi^2 \]  

Since we have discarded the azimuthal coordinate, and time coordinate, we are left with a cylindrically symmetric surface only dependant on \( r \) which we denote \( Z(r) \). The metric of the euclidean space that we are embedding in, can in general be expressed as follows, if we introduce cylindrical coordinates

\[ ds^2 = dz^2 + dr^2 + r^2 d\phi^2 \]  

On our embedded surface we can rewrite this using the relation

\[ dz = \left[ \frac{dZ(r)}{dr} \right] dr \]  

The metric restricted to our surface can therefore be written

\[ ds^2 = (1 + \left[ \frac{dZ(r)}{dr} \right]^2) dr^2 + r^2 d\phi^2 \]  

If we compare this to Equation (54) we can conclude that

\[ 1 + \left[ \frac{dZ(r)}{dr} \right]^2 = \frac{1}{1 - \frac{b(r)}{r}} \]  

Solving for \( \frac{dZ(r)}{dr} \) we arrive at

\[ dZ(r) = \pm \frac{1}{\sqrt{\frac{r}{b(r)} - 1}} \]  

We now see that the shape of the embedded surface is entirely determined by the function \( b(r) \) which motivates the name of shape function. The final step in order to obtain the second result we realise that the presence of a throat, a minimum value for \( r \) which we call \( r_0 \), means that

\[ \frac{b(r)}{r} > b'(r) \]  

\[^3\text{Where integrand in the definition is simply } \sqrt{g_{rr}}\]
\[ \frac{1}{\sqrt{b(r_0)}} - 1 = 0 \Rightarrow b(r_0) = r_0 \]  

(61)

Which is the other result we needed. We now need to give an interpretation to the form function in terms of physical quantities, to do this we use Equations (43) and (44). We evaluate these expressions at \( r = r_0 \)

\[ \rho(r_0) = \frac{b'(r_0)}{8\pi r_0^2} \]  

(62)

\[ \tau(r_0) = \frac{b(r_0)}{8\pi r_0^3} \]  

(63)

We also evaluate Equation (53) at \( r_0 \)

\[ 1 > b'(r_0) \]  

(64)

Now we combine Equations (62), (63) and (64) and obtain a set of inequalities valid in a region near the throat

\[ 1 > \rho(r_0)8\pi r_0^2 \]  

(65)

\[ 1 > \frac{\rho(r_0)}{\tau_0} \Rightarrow \tau_0 > \rho(r_0) \]  

(66)

While this final inequality may look inconspicuous it will be of great importance in the following section when we discuss energy conditions and their violation; for now we can simply state that matter with this behavior is not anticipated in classical theories and is therefore referred to as “exotic matter” by Morris and Thorne. But before we move on to energy conditions we comment on the other criteria on the list required for traversability.

V. ENERGY CONDITIONS AND THEIR VIOLATION

A. The Energy conditions

There exist in general relativity a sets of conditions that all physically realisable stress-energy tensors are expected to fulfill; these conditions should be seen as different ways to formalise the concept of “reasonable” matter[6]. We list a few of these conditions here.

Given a stress-energy tensor of the form below, in some orthonormal frame

\[ T^{\lambda\mu} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p_1 & 0 & 0 \\ 0 & 0 & p_2 & 0 \\ 0 & 0 & 0 & p_3 \end{bmatrix} \]  

(67)

We have the following definitions, given both in coordinate-free form and in the the above given orthonormal frame.

Null energy condition
For any null vector \( k^\mu \)

\[ T_{\lambda\mu} k^\lambda k^\mu \geq 0 \]  

(68)

or equivalently

\[ \rho + p_i \geq 0 \quad \forall \ i \in [1, 2, 3] \]  

(69)

Weak energy condition
For any timelike vector \( V^\mu \)

\[ T_{\lambda\mu} V^\lambda V^\mu \geq 0 \]  

(70)

or equivalently

\[ \rho \geq 0 \quad \text{and} \quad \rho + p_i \geq 0 \quad \forall \ i \in [1, 2, 3] \]  

(71)
These two conditions are not independent of one another and one can by using continuity arguments show that the weak energy condition implies the null. The weak condition is also simpler to give a physical interpretation: it demands that any observer traveling less than the speed of light should not observe any negative energy densities (energy densities lower than that of the vacuum). One can weaken both these conditions by demanding that they should hold on average over any null/time-like curve, but still being allowed to be violated at individual points of space-time. More specifically, for the weak condition:

**Average weak energy condition**
The condition holds on a given timelike curve $\Gamma$ if

$$\int_{\Gamma} T_{\mu \nu} V^\mu V^\nu d\tau \geq 0$$

(72)

Where $\tau$ is the proper time that parametrizes the curve. We also have a similar analogue for the null condition.

**Average null energy condition (ANEC)**
For a given null curve $\Gamma$

$$\int_{\Gamma} T_{\mu \nu} V^\mu V^\nu d\tau \geq 0$$

(73)

With these definitions in mind we now see the importance of Equation (66). Remembering that $p_1 = -\tau$ in this case one sees that this inequality implies that null energy condition, and consequently the weak as well, is violated in a region near the throat of the Morris and Thorne wormhole. This would imply that if the null condition was a strict law of nature the Morris and Thorn construction could not be realised. Fortunately for our purpose there exists instances were quantum systems do violate these energy conditions. We now look at two important instances of this.

**B. Introducing the Casimir effect**

The Casimir effect is a experimentally observed quantum mechanical phenomena where two uncharged conducting plates separated by a small distance experiences a slight compressing force driving the plates together [6]. The effect can be understood using the concept of a zero point, or vacuum, energy from quantum field theory. In this view free space is permeated by electromagnetic fields of all wavelengths which, together with other fields, constitutes the uniform energy of the vacuum. But between two conducting plates there are boundary conditions imposed on the vacuum fields; the waves has to vanish at at both plates, see figure (3). If the plates are a distance $d$ apart and then the component of the field that is parallel to the normal of the plates has to have wave number of the form $\frac{n\pi}{d}$ where $n$ is some integer. This means that not all vacuum wavelengths can exist between the plates and that the energy density there is lower than the energy density outside the plates. It is this gradient of energy that gives rise to the inwards pushing force. This phenomena is therefore a prime candidate for a mechanism that allows us to generate negative energy densities. In order to confirm that the Casismir effect dose in fact violate the the previously mentioned energy conditions we derive, in a heuristic manner, the energy-stress tensor for this system of two plates.

**C. Deriving the Casimir energy-stress tensor**

In order to do this derivation we need to demonstrate and use a property of the energy-stress tensor - its behavior under a conformal transformation. A conformal transformation is a change of coordinates that simply rescales all metric elements by a scalar function

$$g_{\mu' \lambda'} = e^{\Omega(x)} g_{\mu \lambda}$$

(74)

More specifically we are interested in infinitesimal conformal transformations, for these transformations we assume that parameter $\Omega(x)$ to be small and expand in powers of it

$$e^{\Omega(x)} g_{\mu \lambda} = [1 + \Omega(x) + \frac{\Omega(x)^2}{2!} + O(\Omega(x)^3)] g_{\mu \lambda}$$

(75)

Now we consider the classical equation of motion in terms of the action $S(\eta_{\mu \lambda})$ as a function of the metric. Note that we use the Minkowski metric since we are dealing with QFT on a flat space-time. Classically particles follows trajectories such that the variation of the metric vanishes. Using the varational version of the chain rule we can conclude

$$\delta S(\eta_{\mu \lambda}) = 0 \iff \frac{\delta S(\eta_{\mu \lambda})}{\delta \eta_{\mu \lambda}} \delta \eta_{\mu \lambda} = 0$$

(76)
We recall the definition of the stress-energy tensor where

\[ T^{\mu\lambda} \propto \frac{\delta S(\eta_{\mu\lambda})}{\delta \eta_{\mu\lambda}} \]  

(77)

This identity is true for any transformation. For the case of conformal transformation we can find an explicit expression for \( \delta \eta_{\mu\lambda} \) by inspection of Equation (75) and identifying the variation of the metric as the term linear in \( \Omega(x) \). Thus we conclude that for a conformal transformation

\[ \delta \eta_{\mu\lambda} \propto \eta_{\mu\lambda} \]  

(78)

Finally this gives us

\[ T^{\mu\lambda} \eta_{\mu\lambda} = 0 \]  

(79)

Now we return to the Casimir stress-energy, which we will denote \( T_C \) in particular. We know that the \( T_C \) are dependent on the metric, through the Einstein equation. From symmetry considerations we can also conclude the \( T_C \) can only have spatial dependence in the direction of the plates unit normal vector which we call \( \hat{z} \). From experiments we also know there is a dependence on the distance \( d \) between the plates. Using dimensional analysis we can conclude that \( T_C \) must be of the form

\[ T^{\mu\lambda} \propto \frac{\hbar}{d^4}(f_1(z/d)\eta^{\mu\lambda} + f_2(z/d) + \hat{z}^{\mu}\hat{z}^{\lambda}) \]  

(80)

Where \( f_1 \) and \( f_2 \) are two unit less arbitrary functions. But we also know that stress-energy is conserved and since we are working in Minikowski space using Cartesian coordinates that means that:

\[ \Delta_{\lambda} T^{\mu\lambda} = 0 \iff \partial_{\lambda} T^{\mu\lambda} = 0 \]  

(81)

Since we only have \( z \)-dependence we can conclude that this means that \( f_1 \) and \( f_2 \) are constants. Now we make use of Equation (79). This equation tells us that classically the trace of \( T_C \) should disappear. Doing the explicit calculation this gives us the condition

\[ f_2 = -4f_1 \]  

(82)

We can now rewrite Equation (80) in to

\[ T^{\mu\lambda} \propto \frac{\hbar}{d^4}(\eta^{\mu\lambda} - 4\hat{z}^{\mu}\hat{z}^{\lambda}) \]  

(83)

There should be noted that there are some subtleties to consider when we utilize the conformal symmetry to \( T_C \).

What we are doing is assuming the presence of a classical symmetry in in an inherently quantum mechanical system. If one wants a more rigorous argument further quantum-field theoretical justification would be needed but this is beyond the scope of this report. Likewise, in order to establish the the exact proportionality constant one would need a more direct derivation but this too is not germane to this report and the value is widely available in the literature [6, 8]. Reading of this value, we arrive at

\[ T^{\mu\lambda} = \frac{\pi^2}{720d^4}(\eta^{\mu\lambda} - 4\hat{z}^{\mu}\hat{z}^{\lambda}) \]  

(84)

We can rewrite this in matrix form

\[ T^{\lambda\mu} = \frac{\pi^2\hbar}{720d^4} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -3 \end{bmatrix} \]  

(85)

It is now readily clear that that \( T_C \) violates the null energy condition, and thus also the the weak condition and any other stronger condition. This is very much a reassuring result for anyone tasked with the building of a traversable wormhole; we now have instances where a observed process violate the energy conditions in a similar manner to the throat of the Morris-Thorne wormhole. We now turn our attention to a variation of the Casimir effect.

D. The Topological Casimir effect

Reviewing our derivation of the general form of \( T_C \) we can note that the central assumption was the presence of boundary conditions that could limit the wavelengths of the fields. The actual plates were only of secondary importance and, for our purpose, even counterproductive since their mass contributes with a positive energy density. It is therefore natural to do away with the plates and try to find another way to induce the boundary conditions.

This can be done by considering a space-time manifold that is periodic in some direction, say the \( z \)-direction. A simple example would be to consider a cylindrical universe with circumference \( c \), this topology can be achieved by identifying two sides of a rectangle, see figure 4. The restriction on the \( z \) components wave number are in this case \( \pi n \) for some integer \( n \). All the arguments that lead up to Equation (80) are also valid in this setup but since the the boundary conditions are now periodic the proportionality constant is different. The result is [6]

\[ T^{\mu\lambda} = \frac{\pi^2}{45c^4}(\eta^{\mu\lambda} - 4\hat{z}^{\mu}\hat{z}^{\lambda}) \]  

(86)
FIG. 5. The construction of a topological cylinder through the identification of the borders of a rectangle

This result is even more encouraging than the ordinary Casimir effect for someone with wormhole construction in mind; one could expect this kind of topological effect in any wormhole throat and thus maybe the very shape of the wormhole could in and of itself help supply the negative energy density it needs.

VI. THE BUTCHER WORMHOLE

With the potential of the topological Casimir effect in mind and the knowledge, about spherically symmetric and time independent metrics, we gained from the Morris-Thorne discussion it only seems natural to combine the too. This is exactly what Luke Butcher did in his 2014 paper [9]. Butcher proposed a metric of the following form

$$ds^2 = -dt^2 + dz^2 + r(z)^2 d\Omega$$  (87)

$$r(z) = \sqrt{L^2 + z^2 - L + r_B}$$  (88)

Were L and $r_B$ are constants and $z \in [-\infty, \infty]$. If we look at the area of spheres centered at the origin, as we did for the Schwarzschild metric

$$A(z) = \int_S (\sqrt{L^2 + z^2 - L + r_B})^2 d\Omega$$  (89)

$$A(z) = 4\pi(\sqrt{L^2 + z^2 - L + r_B})^2$$  (90)

$$A_{min} = 4\pi r_B^2$$  (91)

This tells us that there, as would be expected for a wormhole, exists a throat at $z = 0$ and that the constant $r_B$ can be interpreted as the radius of the wormhole. Plotting $r$ as function of $z$ for different values of $L$ we observe that $L$ as a parameter controls the length of the wormholes throat, since $r(z)$ “flattens out” near the throat at $z = 0$ for larger values of $L$.

FIG. 6. $r(z)$ plotted for different values of $L$

In order to connect the Butcher wormhole to the Morris-Thorne discussion we make the following coordinate transformation

$$r = \sqrt{L^2 + z^2 - L + r_B}$$  (92)

The Butcher metric, Equation (87), now becomes

$$ds^2 = -dt^2 + \frac{dr^2}{1 - \frac{L^2}{(r + L - r_B)^2}} + r^2 d\Omega^2$$  (93)

From this we can identify that the Butcher metric is a special case of the Morris-Thorne metric with the two arbitrary functions chosen as follows

$$b(r) = \frac{L^2 r}{(r + L - r_B)}$$  (94)

$$\Phi(r) = 0$$  (95)

Since $\Phi(r)$ is identically zero the same result of C.V Vishshwara we used to analyse the general Morris-Thorne construction assures us that the metric exhibits no horizons. Proceeding as Butcher did originally we switch to a orthogonal set of basis vectors with coordinates $(t, z, \theta, \phi)$. We can express the energy-stress tensor as
\[ T_{\mu,\lambda} = F_o(z) \text{diag}(1, -1, \frac{r(z)}{\sqrt{L^2 + z^2}}, \frac{r(z)}{\sqrt{L^2 + z^2}}) + F_{EX}(z) \text{diag}(-1, 0, 0, 0) \]

\[ F_o(z) = \frac{L^2}{(L^2 + z^2)r(z)^28\pi} \]

\[ F_{EX}(z) = \frac{L^2}{(L^2 + z^2)^{\frac{1}{2}}r(z)4\pi} \]

If we assume that \( L \geq r_B \) we can notice that the first term obeys the null energy condition while the second term violates it. The second term therefore represents what we previously refereed to as exotic matter and the first is ordinary matter. We note that \( F_{EX}(z) \) and \( F_o(z) \) has a maximium at \( z = 0 \) where

\[ F_{EX}(0) = \frac{1}{Lr_B4\pi} \]

and

\[ F_o(0) = \frac{1}{r_B^28\pi} \]

From this we can conclude that taking a larger value of \( L \), corresponding to making a longer wormhole, see figure 4, decreases the need for exotic matter. Having made this observation Butcher went on to analyse what kinds of the energy-stress tensor is produced by the topological Casimir effect in such a long-throated wormhole.

FIG. 7.

His final result is:

\[ T_{\mu,\lambda} = \frac{1}{2880\pi r_B^2} \{ \text{diag}(0, 0, 1, 1) + 2\ln \left( \frac{r_B}{r_o} \right) \text{diag}(-1, 1, -1, -1) \} \]

where \( r_0 \) is a constant. We can once again observer that the first term constitutes the “exotic part”. If we now equate \( F_{EX} \), from Equation 98, to the term factor infront of the exotic term in the above equation we find a condition for the two to be of comparable size:

\[ \frac{1}{2880\pi r_B^2} 2\ln \left( \frac{r_B}{r_o} \right) = \frac{L^2}{(L^2 + z^2)^{\frac{1}{2}}r(z)4\pi} \]

For very large values of \( L \) we have

\[ \frac{\ln \left( \frac{r_B}{r_o} \right)}{1440\pi r_B^4} = \frac{1}{r_B4\pi} \]

or

\[ \frac{\ln \left( \frac{r_B}{r_o} \right)}{360\pi} = r_B^3 \]

Which can be satisfied for \( r_B \) much larger than the Planck distance, which is unity in natural units. While Butcher then goes on to conclude that this wormhole in particular is unstable, though it can be made to “collapse extremely slow[ly]”, this construction is of great importance since it shows that wormholes in principle should be able to supply their own source of exotic matter, simplifying many constructions a great deal. Up until now we have looked at constructions of traversable wormholes that amounts to finding a promising metric and then investigating its properties. But all these methods have the shortcoming of not answering: what physical process produces such a metric? The rest of the report will investigate another recent construction which tries to answer this question by creating traversability by perturbing black holes. But before we can do this we need to explain a few concepts, namely Carter-Penrose diagrams and Anti-de Sitter space.

VII. CARTER-PENROSE DIAGRAMS

When one is studying more complex space-times, such as those around wormholes, with features such as horizons and singularities it would be useful to have some method that lets us represent the entirety of the space time graphically to aid intuition. This is the purpose of the Carter-Penrose (C-P) diagram. To create a 2-dimensional C-P diagram from scratch one suppresses two spacial coordinates and then finds a transformation with two properties. Firstly the transformation should compactify the space-time, in other words it should map the infinite space-time to a bounded region of the plane. Secondly the transformation should be conformal, of the form of Equation 74. This guaranties that angles between light-cones stays the same or equivalently that the causal structure of the original space-time remains. To demonstrate this procedure we construct a C-P diagram for the now familiar Schwarzschild metric.
A. Carter-Penrose diagram for the Schwarzschild metric

We mentioned in our introduction to the Schwarzschild metric that neither the original Schwarzschild coordinates or the Einstein-Rosen coordinates covered the entire space-time manifold. A set of coordinates that does not have this problem is the Kruskal coordinates. In these coordinates the Schwarzschild metric has the form

$$ds^2 = \frac{32M^3}{r} e^{-\frac{r}{2M}} (-dT^2 + dX^2) + r^2 \Omega^2$$

(105)

Where $X, T \in (-\infty, \infty)$ and $-\infty < T^2 - X^2 < 1$. $r$ is related to $X$ and $T$ by the transcendental equation

$$X^2 - T^2 = \left(\frac{r}{2M} - 1\right) e^{-\frac{r}{2M}}$$

(106)

Since the Schwarzschild metric is spherically symmetric we can discard the angular coordinate without loss of information. We now move to light-cone coordinates given by

$$U = T - X$$

(107)

$$U = T + X$$

(108)

The metric now becomes, with the angular part suppressed

$$ds^2 = -\frac{32M^3}{r} e^{\frac{r}{2M}} dU dV$$

(109)

And $r$ is related to the new coordinates by

$$UV = (1 - \frac{r}{2M}) e^{\frac{r}{2M}}$$

(110)

We now make another coordinate transformation that compactifies the coordinates and gives them finite range; to do this we use the inverse tangent function. The new coordinates are given by

$$U' = \arctan \left( \frac{U}{\sqrt{2M}} \right)$$

(111)

$$V' = \arctan \left( \frac{V}{\sqrt{2M}} \right)$$

(112)

These coordinates have the range $-\pi/2 < U' < \pi/2$ and $-\pi/2 < V'$. This gives the metric the form

$$ds^2 = -\frac{32M^3}{r} \frac{2M}{\cos^2(U') \cos^2(V')} e^{\frac{r}{2M}} dU' dV'$$

(113)

Now if one compares this metric to spherically symmetric Minkowski space-time in radial light cone coordinates (with angular part again suppressed):

$$ds^2 = -dU dV$$

(114)

We can now see that Equation (113) is related to the Minkowski metric by a conformal factor. Multiplying by a conformal factor does not alter the causal structure of space-time so we can learn much about the original space-time by looking at a plot of the finite region of the $U,V$-plane that it is conformally related to. This region can be seen in Figure 8 (all points in the diagram corresponds to a 2-sphere):

![Carter-Penrose diagram for the Schwarzschild space-time (from [5])](image)

1. $i^+$ corresponds to future time like infinity, a point infinitely far into the future
2. $i^-$ is the past timelike infinity, a point infinitely far into the past
3. $i^0$ is the spacelike infinity, a point infinitely far away in the radial direction
4. $I^+$ is the future null infinity, a surface where all null future directed geodesics end.
5. $I^-$ is the past null infinity were all future directed null geodesics start.

We can also note that the curvature singularity $r = 0$ now corresponds to, not a point, but a vertical line. The topmost triangular region of the diagram, separated from the exterior by the event horizon, is the interior of the black hole. By observing the light cones inside this region one easily see that the geometry makes escape from the singularity impossible since all possible paths are pointed towards the $r = 0$.

VIII. ANTI-DE SITTER SPACE

Another topic that will be very relevant for our future discussion are the concepts of de Sitter and anti-de Sitter space. Up till now we are familiar with Minkowski
space, the flat space-time that is the setting of special relativity. De Sitter and anti-de Sitter are generalizations of this concept - space-times were the the scalar curvature is constant and positive or negative respectively. If one considers 2-dimensional surfaces then the surface of a sphere is de Sitter with constant positive curvature and that of a hyperboloid is anti-de Sitter. These concepts can be rephrased in the language of general relativity as vacuum solutions to the Einstein equations with different signs of the the cosmological constant. If the cosmological constant is zero Minkowski metric is a vacuum solution, a positive constant yields de Sitter and a negative Anti-de Sitter.$^{[5]}$

Interestingly enough, while it is de Sitter space that can be used to approximate our physical universe to a high degree it is in anti-de Sitter space times that recent developments in wormhole physics has been made. This is due to the fact that anti-de Sitter space offers a theoretical setting were some theoretical frame works, such as the current efforts to quantize gravity, simplifies. More specifically our next construction of a traversable wormhole will take place in three dimension, two spatial dimensions + one temporal, anti-de Sitter space; we will denote this $AdS_3$.

IX. THE BTZ BLACK HOLE/WORMHOLE

A. Introducing the BTZ black hole

The discovery of a black hole vacuum solution in $AdS_3$ in 1992 by M. Banados, C. Teitelboim, and J. Zanelli (BTZ) surprised the physics community greatly$^{[9]}$. It had then been believed for some time that 2+1 gravity in 1992 by M. Banados, C. Teitelboim, and J. Zanelli (BTZ) surprised the physics community greatly$^{[9]}$. It had then been believed for some time that 2+1 gravity simply offered too few degree’s of freedom for such constructions as black holes and wormholes - but this belief was proven wrong when the addition of a negative cosmological constant allowed BTZ to construct a black hole very similar to Schwarzschild’s. The metric for the BTZ black hole given in “Schwarzschild-esq” coordinates

$$ds^2 = -r^2 dt^2 + \frac{\ell^2}{r^2} dv^2 + r^2 d\phi^2$$

(115)

Where $\ell$ is the radius of curvature of the space-time, $r \in [0, \infty)$ and $\phi$ has period of $2\pi$. This metric is a solution to the Einstein equation with a cosmological constant set to $\Lambda = -\frac{1}{\ell^2}$. We observe that there is a horizon at $r = r_h$, were the metric has a coordinate singularity, analogous to the Schwarzschild radius, and that the BTZ black hole has a mass of

$$M = \frac{r_h^2}{8\ell^2}$$

(116)

But the geometry is not identical to that of the Schwarzschild case; for example the two exterior regions that the black hole joins, in an untraversable manner, are asymptotically $AdS_3$ instead of asymptotically Minkowski. There is also no curvature singularity at $r = 0$, we can note that the above metric is entirely well behaved at this point; but closer analysis reveals that space-time beyond this point exhibits pathological features such as closed timelike loops$^{[7]}$ which suggests these regions should be discarded and making $r = 0$ a “causal singularity”.

B. BTZ shock waves

What makes BTZ black holes interesting, as a possible basis for a traversable wormholes, is their behavior under perturbation; to be more specific, what happens to the metric when energy is added to the black hole. To demonstrate this we will be working with the BTZ metric and we will be doing so in Kruskal light cone coordinates as they cover the entirety of space-time. In these coordinate the BTZ metric becomes

$$ds^2 = \frac{-4\ell^2 du dv + r_h^2(1 - uv)^2 d\phi^2}{(1 + uv)^2}$$

(117)

The Carter-Penrose diagram for this space-time has the following appearance$^{[8]}$, see figure $[9]$

![FIG. 9. Penrose diagram of the BTZ space-time](image)

We now consider what would happen if someone positioned on the edge of the leftmost region were to throw a massive objective, with a mass $m << M$, at the horizon of the black hole, see figure $[10]$. This was done by S. H. Shenker and D. Stanford $^{[10]}$ and in more detail by T. Nikolakopoulou in $[8]$. We approximate the trajectory of the object as light like, a $45^\circ$ straight line in the Carter-Penrose diagram, which is an acceptable approximation for ultrarelativistic object. Now, the addition of a small mass to the interior of the black hole might not seem like something that would have noticeable effect on the overall geometry of the space time. But a closer analysis, which we will omit in this report, tells us that the energy of the object as measured for an observer at $t = 0$ (someone residing on the vertical line passing trough the center of the Carter-Penrose diagram) will be

$^{5}$ World lines that are closed in space-time, allowing particles to experience cyclical time.
From this we get the condition
\[ E \propto \frac{E_0 \ell}{r_h} e^{-\frac{tw}{r_h}} \] \hspace{1cm} (118)

Where \( E_0 \) is the original energy of the particle and \( t_w \) is the moment the past were the particle was released - it is important to keep in mind here that as consequence of our choice of coordinates the coordinate time in the left region runs backwards so \( t_w > 0 \) is in the casual past of \( r = 0 \). Picking a sufficiently large \( t_w \) makes the introduction of the particle a highly energetic event, akin to a shock wave in space-time.

![Diagram showing the introduction of an object, realised at time \( t_w \), into the event horizon](image)

This is the origin of the term “BTZ shock waves”. When the particle is inside the horizon the mass-energy of the black hole increases from \( M \) to \( M+E \) and horizon radius grows as well. Using the Equation (116) we calculate that the new radius \( \tilde{r}_h \) has the following relation to the old
\[ \tilde{r}_h = \sqrt{\frac{M+E}{M}}r_h \] \hspace{1cm} (119)

Since we are working in light cone coordinates the light-like path of the object is characterised by \( u = \) constant, we set this constant to be \( u_w = e^{-\frac{tr}{\ell}} \). We will obtain our new space-time by “stitching together” two BTZ black holes with radius \( r_h \) and \( \tilde{r}_h \) along this path. To do this we introduce new light cone Kruskal coordinates, \((\tilde{u}, \tilde{v})\) to the left of \( u_w \). To join these we two space times we impose two conditions: firstly that the time coordinate \( \tilde{t} \) should be continuous over the “stitch” \( u = u_w \) and from this we conclude that the path of the object in the new coordinates is \( u_w = e^{-\frac{tr}{\ell}} \). Secondly, since the light like path of the object is covered by both coordinate patches and that the two coordinate systems share angular coordinate \( \phi \) we demand that \( g_{\phi\phi}(u,v) = g_{\phi\phi}(\tilde{u}, \tilde{v}) \). From this we get the condition
\[ \frac{1 - u_w v}{1 + u_w v} = \tilde{r}_h \frac{1 - \tilde{u}_w \tilde{v}}{1 + \tilde{u}_w \tilde{v}} \] \hspace{1cm} (120)

For simplicity we use the fact that \( E << M \) and the first condition to set \( u_w = \tilde{u}_w \). We then turn to Equation (120) and define \( x = v u_w \) and \( y = \tilde{v} u_w \). We substitute and use Equation (119)
\[ \sqrt{1 + \frac{E}{M} \frac{1 - y}{1 - x}} = \frac{1 + y}{1 + x} \] \hspace{1cm} (121)

We now use the Maclaurin expansion of the square root discarding all powers of \( \frac{E}{M} \) higher than one.
\[ (1 + \frac{E}{2M}) \frac{1 - y + (x-x)}{1 - x} = \frac{1 + y + (x-x)}{1 + x} \] \hspace{1cm} (123)
\[ (1 + \frac{E}{2M})(1 + \frac{x - y}{1 - x}) = 1 + \frac{y - x}{1 + x} \] \hspace{1cm} (124)
\[ (1 + \frac{E}{2M})(1 + \frac{x - y}{1 - x}) - (1 + \frac{x - y}{1 - x}) = 1 + \frac{y - x}{1 + x} \frac{(1 + \frac{x - y}{1 - x})}{1 - x} \] \hspace{1cm} (125)

Further algebraic manipulation yields
\[ \frac{y - x}{1 + x} = \frac{E}{4M}(1 - y) \] \hspace{1cm} (126)

Undoing the substitution gives
\[ \tilde{v} u_w - v u_w = \frac{E}{4M}(1 - \tilde{v} u_w) \] \hspace{1cm} (127)
\[ \tilde{u} - v = \frac{E}{4M}(1 - \tilde{v} u_w)(u_w^{-1} + v) \] \hspace{1cm} (128)
\[ \tilde{v} - v = \frac{E}{4M}(1 - \tilde{v} u_w)(u_w^{-1} + v) \] \hspace{1cm} (129)

Solving for \( \tilde{v} \) we get
\[ \tilde{v} = v + \frac{E}{4M}(u_w^{-1} + v) \frac{1}{1 + \frac{E}{4M}(1 + u_w v)} \] \hspace{1cm} (130)

Again we Maclaurin expand the right hand side in powers of \( \frac{E}{M} \) and get
\[ \tilde{v} = v + \frac{E}{4M} u_w^{-1} + \frac{E}{4M} v^2 u_w + \mathcal{O}(\frac{E^2}{M^2}) \] \hspace{1cm} (131)
If we now use the relation $u_w = e^{\frac{-rhtw}{2\ell}}$ and let $t_w \to \infty$ the third term goes to zero. Discarding higher powers of $\frac{E}{M}$ we get the final result

\[ \tilde{v} = v + a, \quad a = \frac{E}{4M} e^{-\frac{rhtw}{2\ell}} \]  

(132)

So the relation between the two sets of Kruskal coordinates in the limit of small $\frac{E}{M}$ and large $t_w$, corresponding to a small perturbation sent from far in the past is a translation. This gives rise to Carter-Penrose diagram with the following appearance, see figure 11:

![FIG. 11.](image)

We can express the metric for both sides of the shock wave using the Heaviside function $H(x)$, which is equal to zero if $x < 0$ or equal to one if $x > 0$.

\[ ds^2 = -4\ell^2 du dv + r_h^2 (1 - u(v + \alpha H(u))^2 d\phi^2 \]

(133)

Now we consider what happens if we send a light signal from the other region of the black hole that intersects the patch of the perturbing object. The effect can be seen by modifying the above Carter-Penrose diagram by making the two horizon’s meet, figure 12:

![FIG. 12.](image)

The light ray from the right now gets shifted inwards, deeper inside the horizon. Now this effect is obviously not conducive to the construction of a traversable wormhole but not much imagination is needed to deduce that it is a shift outwards that would be of interest. Looking at Equation [132] we realise that what is needed, since it is a negative $\alpha$ we are after, is, quite remarkable, once again negative energy! If $E < 0$ then we get the following effect, see figure 13:

![FIG. 13.](image)

The light signal, or object, is now “ejected” in to the other side of the diagram. The shock wave has rendered the BTZ black hole a traversable wormhole.

**X. NON-LOCAL INTERACTIONS IN SCALAR FIELDS AS A SOURCE OF NEGATIVE ENERGY**

Seeing the potential of the BTZ shock wave we are of course interested in finding a source of negative energy that could give us the above described effect. We realise that the Casimir effect, as we have described it, is poorly suited for this purpose since we want a more localised source; we therefore prefer if the object entering the horizon were particles or, equivalently, fields. To gain insight into how this could be done we consider the simplest possible field in the simplest possible space-time, namely a massless scalar field $\phi$, in 1+1 Minkowski space, as was done in [8]. The action of such a field when propagating in free space is

\[ S = -\int \frac{1}{2} \partial_\mu \phi \partial^\mu \phi \]  

(134)

We now perturb this action by a term given by

\[ \delta S = g \phi_L \phi_R \]  

(135)

This term represents an interaction between the field and itself at two points of space time, $x_L$ and $x_R$, at time which we set to $t = 0$. It should be noted that this interaction is non-local. This means that the interaction allows space-time regions that should be causally disconnected, according to GR, to influence one another. For this section we disregard this violation and explore the consequences. The constant $g$ is a parameter that determines the strength of this interaction, we set its value to be very small. We are interested in the expectation value
of the energy-stress tensor to see if it displays the negative energy we are after, more precisely the expectation value for perturbed ground state of the field

$$|\Psi\rangle = e^{i\phi L \phi_R} |0\rangle \quad (136)$$

Doing the calculation in the first order of $g$ see \[3\], we find that the expectation values of the energy stress tensor in light cone coordinates $(u,v)$ are:

$$\langle \Psi | T_{uu}(u) |\Psi\rangle = -\frac{g}{4\pi} \left( \frac{\delta(u - u_R)}{u - u_L} + \frac{\delta(u - u_L)}{u - u_R} \right) \quad (137)$$

$$\langle \Psi | T_{vv}(v) |\Psi\rangle = -\frac{g}{4\pi} \left( \frac{\delta(v - v_R)}{v - v_L} + \frac{\delta(v - v_L)}{v - v_R} \right) \quad (138)$$

Here $\delta$ is the Dirac delta function. We can retrieve the total energy density by the relation:

$$T_{tt}(t,x) = T_{uu}(u) + T_{vv}(v) \quad (139)$$

Plotting this density in a space-time diagram we get the below figure:

FIG. 14. Blue corresponds to a region of negative energy density and green to positive energy density

We see that the interaction term has produced distinct regions of both positive and negative energy in the space-time which might be considered surprising given how mundane the individual components of this construction where, aside for the non-local behavior. Keeping with calculation made by T.Nikolakopoulou one can also perform the calculations to second order in $g$ in order to investigate if this has any unexpected effect. To be able to do this we can no longer investigate point-sources since this gives rise to divergences in the calculations. We therefore must “smear” the sources over a non-zero area in space-time. We replace $\phi_R$ and $\phi_L$ with

$$O_R = \int_{v_R + A}^{v_R - A} \int_{u_R - A}^{u_R + A} \phi(u,v) dudv \quad (140)$$

$$O_L = \int_{v_L - A}^{v_L + A} \int_{u_L - A}^{u_L + A} \phi(u,v) dudv \quad (141)$$

We have spread the source over an area of $4A$ in the space-time diagram. By doing this and performing similar calculations, as was done to obtain Equation (137) and (138) but now in second order in $g$, and plotting it we get, Figure 15

FIG. 15. Blue corresponds to a region of negative energy density and green to positive energy density

As one might expect the different regions are now also smeared but beyond this there is still regions of negative energy. We would now want now try to obtain a similar effect in the BTZ space-time but before we do this we need to introduce a few notions from quantum mechanics.

XI. PURE AND MIXED STATES

The resource for this section is Michael A. Nielsen and Isaac L. Chuangs “Quantum Computation and Quantum Information” \[1\]. In quantum mechanics one usually represents the state of a given system as a state vector $|\psi\rangle$, as was done above. All physical information about the system, such as expectation values for observables, can be extracted from this state vector. This state of affair, when all available information can be summarised by a single vector, is called a pure state. One can also consider a slightly more general situation. If the state of the system $|\psi\rangle$ is not known but we instead only have the knowledge that the system either is in state $|\psi_1\rangle$ with probability $p_1$ or state $|\psi_2\rangle$ with probability $p_2$ etc. Our information is no longer neatly summarised by a single vector, this situation is called a mixed state. When one is dealing with mixed states it is often helpful to introduce a new mathematical object called the density matrix, defined by

$$\rho = \sum_{i} p_i |\psi_i\rangle \langle \psi_i| \quad (142)$$
The density matrix is as the name suggests a matrix, or equivalently a linear operator, and one can reformulate the entirety of quantum mechanics in a “density matrix language”. For example, if one acts on the state with a measurement operator \( M \) associated with measurement result \( m \) then the probability of \( m \) being the outcome is given by

\[
p(m) = tr(M^\dagger M \rho)
\] (143)

Here \( tr() \) denotes the trace. It is important to stress that the density matrix formulation is totally equivalent to state vector formulation and one can describe pure states as density matrices as well. For a pure state \( |\psi\rangle \) the density matrix takes the form

\[
\rho = |\psi\rangle \langle \psi|
\] (144)

We also need to understand the technique of purification of a mixed state. To do this we recall that the state of a composite quantum system constructed by combining two subsystems \( A \) and \( B \) is described by the tensor product of two state vectors

\[
|\psi_{AB}\rangle = |\psi_{A}\rangle \otimes |\psi_{B}\rangle
\] (145)

Roughly speaking, by using the tensor product one can create a larger state space (Hilbert space) from two smaller. One could ask the question: given a mixed state could one somehow encode all of its information as a single state vector (a pure state) in some larger state space? The answer to this question is yes and the procedure to do so is what is referred to as purification. Before we describe the details one should note that the pure state that is produced is not unique and that some arbitrary choices has been made our example, but that this is not a concern. Given a mixed state density matrix in an orthogonal basis \( |\psi_i\rangle \) describing a system with state space \( A_1 \)

\[
\rho_{A_1} = \sum_i p_i |\psi_i\rangle \langle \psi_i|
\] (146)

Now create an another copy of \( A_1 \) which we denote \( A_2 \), and consider the composite system \( A = A_1 \otimes A_2 \). We are in particular interested in the state vector

\[
|A\rangle = \sum_i \sqrt{p_i} |\psi_i\rangle_1 \otimes |\psi_i\rangle_2
\] (147)

Here the subscripts on the state vectors denote from which vector space it “originates” from. The corresponding density matrix is

\[
\rho_A = |A\rangle \langle A|
\] (148)

To see that this state vector contains all information of the original mixed state, Equation (146), we define the operation that extracts this information. We define the partial trace to be the linear operator that acts on the density matrix of composite systems of the form \( X \otimes Y \) defined by

\[
tr_Y(|x_1\rangle \langle x_2| \otimes |y_1\rangle \langle y_2|) = |x_1\rangle \langle x_2| tr(|y_1\rangle \langle y_2|)
\] (149)

Once again \( tr() \) denotes the ordinary trace. If we apply the partial trace to Equation (148) we get

\[
tr_{A_2}(|A\rangle \langle A|) = \sum_{ij} \sqrt{p_j} |\psi_i\rangle_1 \langle \psi_j|_1 \delta_{ij}
\] (151)

\[
= \sum_i p_j |\psi_i\rangle_1 \langle \psi_j|_1
\] (152)

\[
= \rho_{A_1}
\] (153)

So we now recover the original mixed state as we wanted. We have the tools to move on to the more recent paper by Gao, Jafferis and Wall [4] which combines the BTZ shock wave and what we learned about smeared non-local sources.

### XII. THE GJW CONSTRUCTION

#### A. Introducing AdS/CFT correspondence and Thermofield double

A more recent contribution to the field of traversable wormholes was made by P. Gao, D. L. Jafferis, and A. C. Wall (GJW) in 2017 [4]. At the heart of their paper lies a result put forward by J. Maldacena [12] that the space-time of an BTZ black-hole is dual to particular pure state called the thermofield double. The thermofield double can be thought of as two copies of a conformal quantum-field theory (CFT), each copy residing on one of the two borders of the AdS\(_5\) Carter-Penrose diagram. To be more precise, the BTZ geometry corresponds to the states one gets by performing the purification procedure, described in the above section, to the (mixed) states of the CFTs\(^7\). This duality between lower dimensional field

\(^7\) The fact that the behavior of entire 3 dimensional space-time can be determined by a 2-dimensional field theory is highly non-trivial and one of the more prominent instances of a deep conjecture in quantum gravity known as “The holographic principle”.
theories and the black hole/worm hole geometry is known as the AdS/CFT correspondence. The above mentioned purified states are known as a thermofield double state and given by

\[ |\text{TFD}\rangle = \frac{1}{\sqrt{Z(\beta)}} \sum_n e^{-\beta E_n} |\text{CFT}_L\rangle |\text{CFT}_R\rangle \quad (154) \]

The factor \( Z(\beta) \) is known as the partition function and serves to normalise the probabilities of the state. \( \beta \) in turn is known as the Boltzmann factor and is proportional to the inverse of the temperature. The inverse temperature of the BTZ black hole is defined

\[ \beta = \frac{2\pi \ell^2}{r_h} \quad (155) \]

B. Adding a Non-Local Coupling to the Thermofield Double

Having covered what the thermofield double is the layout of GJW construction is qualitatively very similar to the example we saw with the scalar field. GJW also introduces a perturbation to the action of the form

\[ \delta S = -\int dt d\phi h(t, \phi) \mathcal{O}(-t, \phi) \mathcal{O}(t, \phi) \quad (156) \]

Here \( \mathcal{O}(t,x)_{R/L} \) is operators corresponding again to a scalar field that residing on the right and left edges of the BTZ geometry respectively. \( h(t, \phi) \) is a perturbation factor that “switches on”, takes on a non-zero value, at some given time \( t_0 \). The fact that perturbation is integral tells us that the the perturbation is smeared over some region in space and time; the \(-t\) in the argument of operator is to compensate the fact that coordinate runs in opposite directions in the two asymptotically-Ads regions of the BTZ space-time. The state we are perturbing are no longer the same ground state that was used in the scalar field example but the \( |\text{TFD}\rangle \) state which now describes our system. Aside from this most of our discussion from the scalar field perturbation stays true. This new perturbation is non-local, connecting the causally disconnected borders of the space-time and we are still interested in expectation value of stress tensor to see if it violates the energy conditions and produces negative energy densities. We leave out the field theoretical calculations done by the authors, the final result of the paper was obtained numerically and we present the result in the below figure, figure 16.

![Figure 16](image1)

In figure 16 we see the expectation value of the UU-component, in Kruskal light cone coordinates, along a light-like path were \( V \) was held constant. The interaction was switched on at some given time and then remains on for all later time. The different lines represents different parameter choices for the interaction; different numerical values for the scaling dimension. The value of the scaling dimension determiners how the field behaves under coordinate changes corresponding to spacial dilations. We can see clearly that negative energy densities are present for all parameter values.

![Figure 17](image2)

Fig.17 is similar but shows the value for \( T_{UU} \) for an interaction that only lasts a finite time. The spike in the graph corresponds to the switch of point and we can see the expectation values once again turn positive. GJW was interested in finding out if the later case violated the average null energy condition (ANEC). They therefore integrated the expectation value of \( T_{UU} \) along for different switch-on points \( U_0 \) and switch of points \( U_f \).
FIG. 18. $T_{UU}$ integrated along a null path, both for infinitely and finitely lasting interactions from [4].

It is clear from this figure that for all values of $\Delta$, and no matter if there is a cut off for the interaction, the integral remains negative. ANEC is clearly violated and one can quite confidently say that this non-local interaction is a promising candidate as source for negative energy in the BTZ chock wave construction. GJW note that the wormhole should open up by:

$$\Delta V \propto \frac{h G_N}{\ell} \quad (157)$$

Here, again, $h$ is the interaction parameter which was assumed to be a very small number, $G_N$ is Newton’s constant and $\ell$ is the radius of curvature of AdS-space. This corresponds to a wormhole that is open for a very short time and then closes, mainly available for highly boosted observers travelling from the border at very early times. Since this could be described as the most complete construction of a traversable wormhole in this report it seems fitting to make a comment about its usefulness. Aside from the obvious facts that this constructions takes place in AdS-space and is using non-local interactions there are some more limitations placed on the usefulness, as discussed by GJW. They argue that this sort of wormhole can never be used to send a person or information to a location faster than the speed of light. This is of course something negative if one wishes to use the wormhole to send information but it could also be said to be a argument for the soundness of the construction. Superluminal information transfer opens up a large can of worms of pathological behaviors which would indicate that the original assumptions were them self flawed. The practicality of this wormhole is eloquently but by GJW as

“Hence traversable wormholes are like getting a bank loan: you can only get one if you are rich enough not to need it.”

As a final interesting note GJW notes the similarity between their non-local interaction and the topological Casimir effect; by linking the two sides of AdS space causally they have effectively identified them in a manner similar to process of creating a topological cylinder.

XIII. CONCLUDING REMARKS

We end this report with our look at the GJW construction hoping that the reader have gained some insight into and overview of the field of wormhole physics. Our list of constructions is far from comprehensive and there have been several other recent achievement in the field such as constructions in 3+1 Minkowski space relaying more explicitly on the Cassimir effect [13]. The examples we have presented in this report shows that wormholes are an active and rich field of study. The author hopes that the reader now shares his excitement about what discoveries that could be made in the near future.


