Construction of the Higgs Mechanism
and the Lee-Quigg-Thacker-bound

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Abstract

In this paper the higgs mechanism for the standard model is constructed in steps. First by considering spontaneous breaking of discrete and continuous global gauge invariance. Then spontaneous breaking of local gauge invariance. These results are then used to construct the electroweak part of the standard model through application of the higgs mechanism. Finally, the LQT-upper bound of 1 TeV for the higgs mass is calculated through unitarity constraints.

1 Introduction

Due to the many, experimentally known, conserved quantities in particle physics, a suitable theory for it would most likely be a gauge theory. However, ordinary gauge theories lead to zero masses for gauge particles, while it is known experimentally that the masses of the weak gauge bosons ($W^\pm$, $Z^0$) are non-zero. To retain the symmetries in the theory, resulting in conserved quantities by Noether’s theorem, and still obtain non-zero masses for the gauge bosons spontaneous symmetry breaking and the higgs mechanism can be applied. This solves the problem but also necessitates the existence of a new particle with unknown mass, known as the higgs particle.

This paper will derive a Lagrangian for the electroweak sector of the standard model by applying the higgs mechanism. It will be done over several sections with increasing complexity. Starting with spontaneous symmetry breaking without any gauge fields present for theories that are rotationally invariant (or invariant under reflection in the 1-dimensional case) in sections 2 Spontaneous Breaking of Discrete Symmetry and 3 Spontaneous Breaking of Continuous Symmetry.

Gauge fields will then be added in section 4 Spontaneous Breaking of Gauge Symmetry where the higgs mechanism will be introduced as the process giving gauge particles masses. This will be done first for the Abelian U(1) case and then for the non-Abelian SU(2) case.

Then all prerequisites for constructing the electroweak standard model have been given and section 5 Higgs Mechanism in the Standard Model is about the above mentioned construction.

Finally, the paper ends on deriving the Lee-Quigg-Thacker upper bound for the higgs mass in section 6 Unitarity Constraints on the Higgs Mass, before giving a Summary in section 7.

2 Spontaneous Breaking of Discrete Symmetry

Consider the following Lagrangian

$$L = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \left(\frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4} \phi^4\right), \quad (1)$$

which is invariant under reflection. This can be seen by making the substitution $\phi \to -\phi$, in eqn. 1, which yields:

$$L = -\frac{1}{2} \partial_\mu (-\phi) \partial^\mu (-\phi) - \left(\frac{1}{2} \mu^2 (-\phi)^2 + \frac{\lambda}{4} (-\phi)^4\right) = -\frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \left(\frac{1}{2} \mu^2 \phi^2 + \frac{\lambda}{4} \phi^4\right)$$

The first term in the Lagrangian is recognized as the kinetic term and the expression in parenthesis as the potential $V(\phi)$. If $\mu^2 > 0$, then the potential is positive semi-definite, because $\lambda$ is defined to be real and positive while $\phi$ must be hermitian for the action to be hermitian. The unique minimum of $V(\phi)$ is then clearly given by $\phi = 0 \Rightarrow V(0) = 0$, which is called the ground/vacuum state.
Figure 1: In this graph $V(\phi) = \phi^4 + 50\phi^2$.

If $\mu^2 < 0$, then $V(\phi)$ is no longer positive semi-definite and the minima can be found by finding the zeroes of $V'(\phi)$:

$$0 = V'(\phi) = \mu^2 \phi + \lambda \phi^3 \Leftrightarrow \begin{cases} \phi_1 = 0 \\ \phi_{2,3} = \pm \sqrt{\frac{|\mu^2|}{\lambda}} = \pm v \end{cases}$$

Examining the graph of $V(\phi)$ given in figure 2 it is seen that the minima are given by $\phi_{2,3}$. 
The graph shows that $\phi_{2,3}$ are the minima while $\phi_1$ is a local maximum. In this plot $V(\phi) = \phi^4 - 50\phi^2$.

The Lagrangian can then be expanded around one of the minima, say $+v$, as follows:

\[
\phi(x) = v + \eta(x), \tag{3}
\]

resulting in:

\[
L = -\frac{1}{2} \partial_\mu \phi(x) \partial^\mu \phi(x) - \left( \frac{1}{2} \mu^2 \phi^2(x) + \frac{\lambda}{4} \phi^4(x) \right) = -\frac{1}{2} \partial_\mu (v + \eta(x)) \partial^\mu (v + \eta(x)) - \left( \frac{1}{2} \mu^2 (v + \eta(x))^2 + \frac{\lambda}{4} (v + \eta(x))^4 \right). \tag{4}
\]

Unlike the first Lagrangian this is not invariant under reflection, $\eta \rightarrow -\eta$, which can be seen for instance from the term

\[-\mu^2 v\eta\]

obtained by expanding the first term in the parenthesis. Upon making the substitution this term transforms into:

\[-\mu^2 v\eta \rightarrow \mu^2 v\eta \neq -\mu^2 v\eta\]

thus the Lagrangian in Eqn. (4) is not invariant under reflection and the symmetry has been spontaneously broken. Where the term spontaneously is used because there is no change in the physical system described by the Lagrangian, merely a change of how it is described by a translation of the coordinate system, that brings the origin to the vacuum state.
The mass corresponding to the field $\eta$ is identified as the constant $m_\eta$ in the term in the Lagrangian given on the following form:

$$-\frac{m_\eta^2}{2} \eta^2$$

The first term in the Lagrangian does not yield any term of this form. The second gives:

$$-\frac{\mu^2}{2} \eta^2$$

and the third produces (keeping only terms of the right order):

$$-\frac{\lambda}{4}(v + \eta)^2 = -\frac{\lambda}{4}(v^2 + 2v\eta + \eta^2) \Rightarrow -\frac{\lambda}{4}(2v^2\eta^2 + 4v^2\eta^2) = -\frac{3\lambda v^2}{2}$$

Collecting these terms and simplifying with Eqn. (2) yields the mass term:

$$-\frac{\mu^2}{2} + \frac{3\lambda v^2}{2} \eta^2 = -\frac{\mu^2}{2} \eta^2$$

The mass is then seen to be:

$$m_\eta = \sqrt{-2\mu^2},$$

which is real and positive since $\mu^2 < 0$.

Except for $\mu^2 = 0$, these are all possible cases, because $\mu^2$ must be real in order for the action to be real. The last case, $\mu^2 = 0$, does not yield anything new and is similar to the $\mu^2 > 0$ case.

3 Spontaneous Breaking of Continuous Symmetry

In the following Lagrangian, that is symmetric under rotations in the $\phi_1\phi_2$-plane,

$$\mathcal{L} = -\frac{1}{2}(\partial_\mu \phi_1)^2 - \frac{1}{2}(\partial_\mu \phi_2)^2 + \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2) - \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2,$$

the first two terms are recognized as the kinetic terms. Thus leaving

$$V(\phi_1, \phi_2) = \frac{\lambda}{4}(\phi_1^2 + \phi_2^2)^2 - \frac{1}{2}\mu^2(\phi_1^2 + \phi_2^2),$$

for the potential.

Extremizing the potential results in:

$$\left\{ \begin{array}{l}
0 = \frac{\partial V}{\partial \phi_1} = \lambda(\phi_1^2 + \phi_2^2)\phi_1 - \mu^2\phi_1 \\
0 = \frac{\partial V}{\partial \phi_2} = \lambda(\phi_1^2 + \phi_2^2)\phi_2 - \mu^2\phi_2
\end{array} \right. \Rightarrow \left\{ \begin{array}{l}
\phi_1 = 0 = \phi_2 \\
(\phi_1^2 + \phi_2^2) = \frac{4\lambda^2}{4\mu^2} =: v^2
\end{array} \right.$$ (9)

Inserting the solutions into the potential reveals that:

$$\left\{ \begin{array}{l}
V(0,0) = 0 \\
V(\phi_1^2 + \phi_2^2) = V(v^2) = \frac{\lambda}{4}v^4 - \frac{1}{2}\mu^2v^2 = \frac{\lambda}{4}\frac{\mu^2}{2}v^2 = -\frac{\mu^2}{4}v^2
\end{array} \right.$$ (10)

5
If \( \mu^2 < 0 \), then \( V = 0 \) is the minimum. However, if \( \mu^2 > 0 \), then the minimum is equal to \( V = -\frac{\mu^2}{4} v^2 \).

Henceforth it will be assumed that \( \mu^2 > 0 \). Then the potential will appear as in figure [3].

Figure 3: The graph shows that the circle \( \phi_1^2 + \phi_2^2 = v^2 \) is indeed the minima, while \( \phi_1 = 0 = \phi_2 \) is a local maximum. VEV stands for vacuum expectation value, which is the expectation value of the potential for the vacuum state. In this graph \( V(\phi_1, \phi_2) = -170(\phi_1^2 + \phi_2^2) + (\phi_1^2 + \phi_2^2)^2 \).

Now, let’s rewrite the Lagrangian in Eqn. [7] with the following substitution:

\[
\begin{align*}
\phi_1(x) &= \pi(x) \\
\phi_2(x) &= v + \sigma(x)
\end{align*}
\]

where the minimum in the \( \phi_2 \)-direction has been chosen to expand around. This yields:

\[
\begin{align*}
\mathcal{L} &= -\frac{1}{2} (\partial_\mu \phi_1)^2 - \frac{1}{2} (\partial_\mu \phi_2)^2 + \frac{1}{2} \mu^2 (\phi_1^2 + \phi_2^2) - \frac{\lambda}{4} (\phi_1^2 + \phi_2^2)^2 = -\frac{1}{2} (\partial_\mu \pi(x))^2 \\
&\quad - \frac{1}{2} (\partial_\mu (v + \sigma(x)))^2 + \frac{1}{2} \mu^2 (\pi^2(x) + (v + \sigma(x))^2) - \frac{\lambda}{4} (\pi^2(x) + (v + \sigma(x))^2)^2 \\
&= -\frac{1}{2} (\partial_\mu \pi)^2 - \frac{1}{2} (\partial_\mu \sigma)^2 + \frac{1}{2} \mu^2 (\pi^2 + (v + \sigma)^2) - \frac{\lambda}{4} (\pi^2 + (v + \sigma)^2)^2
\end{align*}
\]

The field \( \pi \) while then have a mass term of the form:

\[-\frac{m_\pi^2}{2} \pi^2\]
where \( m_\pi \) is the mass corresponding to the field. However, it turns out that this term is zero. The first two terms in the Lagrangian do not contribute to the mass term. The second term gives:

\[
\frac{\mu^2}{2} \pi^2
\]

and the last term yields:

\[
-\frac{\lambda}{4} \cdot 2v^2 \pi^2 = -\frac{\lambda \mu^2}{2} \pi^2 = -\frac{\mu^2}{2} \pi^2
\]

Adding these two terms then gives a mass, \( m_\pi = 0 \). The result is as expected. The \( \pi \)-field is just a different name for the \( \phi_1 \)-field, which can be seen from the transformation (Eqn. (11)). Since \( phi_1 \) was massless it makes sense for \( \pi \) to be massless as well.

The same process can be repeated to find the mass of the \( \sigma \)-field. The first two terms do not contribute in this case either, while the third gives:

\[
\frac{\mu^2}{2} \sigma^2
\]

and the fourth:

\[
-\lambda v^2 \sigma^2 = -\mu^2 \sigma^2
\]

which together give a mass of:

\[
-\frac{m_\sigma^2}{2} :\frac{\mu^2}{2} \sigma^2 - \mu^2 \sigma^2 = -\frac{\mu^2}{2} \sigma^2 \Rightarrow m_\sigma = \mu
\]

This can be further generalized to \( N \)-dimensions by replacing the Lagrangian in Eqn. (7) with:

\[
\mathcal{L} = -\frac{1}{2} \sum_{i=1}^{N} (\partial_\mu \phi_i)^2 + \frac{1}{2} \mu^2 \sum_{i=1}^{N} \phi_i^2 - \frac{\lambda}{4} \left( \sum_{i=1}^{N} \phi_i^2 \right)^2 , \tag{13}
\]

The potential is read off to be:

\[
V(\phi_1, ..., \phi_N) = \frac{\lambda}{4} \left( \sum_{i=1}^{N} \phi_i^2 \right)^2 - \frac{1}{2} \mu^2 \sum_{i} \phi_i^2 , \tag{14}
\]

Extremizing the potential then yields:

\[
0 = \frac{\partial V}{\partial \phi_k}
= \lambda \left( \sum_i \phi_i^2 \right) \phi_k - \mu^2 \phi_k
\equiv \phi_k = 0 \vee \sum_i \phi_i^2 = \frac{\mu^2}{\lambda} = : v^2
\tag{15}
\]

The result in Eqn. (15) is highly similar to the result in Eqn. (9). Previously it was shown that \( \phi_1 = 0 = \phi_2 \) does not yield the minimum (assuming \( \mu^2 > 0 \)).
is easily seen that this is true now as well. Just set all but two of the fields equal
to zero and the problem is reduced to the 2-dimensional one, but with different
labels for the fields. This can be done for any pair of fields in the Lagrangian.
Furthermore, the potential can be rewritten as a function of \( \sum \phi_i^2 \), which
means that the minimum of the \( N \)-dimensional potential can be written as a
sum of the 2-dimensional minima. Therefore \( \sum \phi_i^2 \) must equal \( v^2 \) to obtain the
minimum.

Then the transformation:

\[
\begin{align*}
\phi_1(x) &= \pi(x) + v \\
\phi_k(x) &= \sigma_k(x), \quad 2 \leq k \leq N
\end{align*}
\]

yields the following Lagrangian:

\[
\mathcal{L} = \frac{1}{2} (\partial_\mu (\pi + v))^2 - \frac{1}{2} \sum_{k=2}^{N} (\partial_\mu \sigma_k)^2 + \frac{1}{2} \mu^2 (\pi + v)^2 \\
&\quad + \frac{1}{2} \mu^2 \sum_{k=2}^{N} \sigma_k^2 - \frac{\lambda}{4} \left( (\pi + v)^2 + \sum_{k=2}^{N} \sigma_k^2 \right)^2 \\
&= \frac{1}{2} (\partial_\mu \pi)^2 - \frac{1}{2} \sum_{k=2}^{N} (\partial_\mu \sigma_k)^2 + \frac{1}{2} \mu^2 (\pi + v)^2 \\
&\quad + \frac{1}{2} \mu^2 \sum_{k=2}^{N} \sigma_k^2 - \frac{\lambda}{4} \left( (\pi + v)^2 + \sum_{k=2}^{N} \sigma_k^2 \right)^2
\]

The first three terms do not contribute to the mass term for the fields \( \sigma_k \). The
fourth yields a term:

\[
\frac{\mu^2}{2} \sigma_k^2
\]

The final term gives (keeping only terms of the right order at each step and
changing the summation index to avoid confusion):

\[
- \frac{\lambda}{4} \left( (\pi + v)^2 + \sum_{j=2}^{N} \sigma_j^2 \right)^2 \Rightarrow - \frac{\lambda}{4} (v^2 + \sigma_k^2)^2 \Rightarrow - \frac{\lambda}{2} v^2 \sigma_k^2 = - \frac{\lambda \mu^2}{2} \sigma_k^2 = - \frac{\mu^2}{2} \sigma_k^2
\]

Adding these two terms yields a mass of zero for all \( \sigma_k \)-fields.

4 Spontaneous Breaking of Gauge Symmetry

A theory is gauge invariant/symmetric if the action is invariant under a trans-
formation of the quantum fields. If that transformation also depends on the
coordinates of the system, then it is called a local gauge transformation. Quite
often, the Lagrangian will also be gauge invariant or nearly gauge invariant if
the action is.

A general (local) gauge transformation can be written on the form:

\[
U_{ij} = \delta_{ij} + \theta_{ij}(x) + \mathcal{O}(\theta^2)
\]
where $\delta_{ij}$ is the Kroenecker delta. In this thesis we will be interested in the $SU(N)$ and $SO(N)$ transformations, i.e. the Special Unitary and the Special Orthogonal transformations. Special means that the determinant of the transformation matrix, $\theta$, is $+1$, while unitary means that $UU^\dagger = 1$, which implies that $\theta$ must be hermitian. Finally, orthogonal means that $Ux \cdot Uy = x \cdot y$, i.e. that the transformation preserves the dot product. This requires $\theta$ to be an orthogonal matrix ($\theta^T \theta = 1$). $N$ stands for the number of dimensions.

The above imposes restrictions on $\theta_{ij}$. In the $SO(N)$ case $\theta$ must be real and antisymmetric, due to orthogonality. While in the $SU(N)$ case $\theta$ must be hermitian, as stated previously, and traceless, for the determinant to be $+1$. In both cases it must be an $N \times N$ matrix.

For an arbitrary $N \times N$ matrix there are $N^2$ parameters. In the $SO(N)$ case the restrictions leaves $\frac{1}{2}N(N-1)$ free parameters in $\theta$, because antisymmetry requires all $N$-elements on the diagonal to be zero and fixing the parameters on one side of the diagonal also fixes the $\frac{N^2-N}{2}$ parameters on the other side of the diagonal. Thus reducing the number of free parameters to:

$$N^2 - N - \frac{N^2 - N}{2} = \frac{1}{2}N(N-1)$$

as claimed. For the SU case the traceless condition only reduces the number of free parameters with 1, since $N - 1$ elements in the diagonal can be chosen arbitrarily as long as the last element is chosen so that the trace vanishes. This gives $N^2 - 1$ free parameters for the SU case.

Using this Eqn. (18) can be reexpressed as:

$$U_{ij} = \delta_{ij} - i\theta^a(x)(T^a)_{ij} + \mathcal{O}(\theta^2) \quad (19)$$

where $\theta^a$ are real scalar valued functions and $T^a$ are matrices s.t. they satisfy the same conditions put on $\theta$ as well as:

$$\text{Tr}(T^a T^b) = 2\delta^{ab} \quad (20)$$

The $T^a$'s are called generators and there number equals the number of free parameters.[1]

4.1 Abelian Case U(1)

In general the generators of a gauge group do not commute. However, in the special case of groups with only one generator, the generators are commuting trivially. Such is the case for $U(1)$ and $SO(2)$. If the generators commute the group is called Abelian.

The following Lagrangian:

$$\mathcal{L} = -(D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \frac{\lambda}{2} (\phi^\dagger \phi)^2 - \frac{1}{2} F^{\mu\nu} F_{\mu\nu} \quad (21)$$

where $F^{\mu\nu}$ is the electromagnetic field tensor ($F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$) and $D_\mu = \partial_\mu - i e A_\mu$, is invariant under a local $U(1)$ gauge transformation:

$$\phi(x) \rightarrow e^{i\alpha(x)} \phi(x), \quad A_\mu(x) \rightarrow A_\mu(x) - \frac{1}{e} \partial_\mu \alpha(x) \quad (22)$$
Here $e^{i\alpha}$ corresponds to the $1 \times 1$ matrix $\theta$, but unlike Eqn. (18) this is not restricted to first order in $\theta$.

The potential of the Lagrangian in Eqn. (21) is made up of the middle terms:

$$V(\phi, \phi^\dagger) = \frac{\lambda}{2}(\phi^\dagger \phi)^2 - \mu^2 \phi^\dagger \phi$$  \hspace{1cm} (23)

and its extrema can be found in the usual way:

$$\left\{ \begin{array}{l}
0 = \frac{\partial V}{\partial \phi} = \lambda \phi^\dagger \phi - \mu^2 \phi \\
0 = \frac{\partial V}{\partial \phi^\dagger} = \lambda \phi \phi^\dagger - \mu^2 \phi \\
\end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{l}
\phi = 0 = \phi^\dagger \\
\phi^\dagger \phi = \frac{\mu^2}{\lambda} =: v^2
\end{array} \right. \hspace{1cm} (24)

The second derivative then shows that the first alternative is a local maximum and the second a local minimum, assuming $\mu^2 > 0$.

The fields can then be expanded around the minima in the following way:

$$\left\{ \begin{array}{l}
\phi(x) = v + \frac{1}{\sqrt{2}}(\rho(x) + i\chi(x)) \\
\phi^\dagger(x) = v + \frac{1}{\sqrt{2}}(\rho(x) - i\chi(x))
\end{array} \right. \hspace{1cm} (25)

where $\rho$ and $\chi$ are real fields. Making this change in the Lagrangian (Eqn. (21)) yields:

$$\mathcal{L} = - (D_\mu (v^2 + \frac{1}{\sqrt{2}}(\rho(x) + i\chi(x))))^\dagger (D^\mu (v + \frac{1}{\sqrt{2}}(\rho(x) + i\chi(x))))$$

$$+ \mu^2 (v + \frac{1}{\sqrt{2}}(\rho(x) - i\chi(x)))(v + \frac{1}{\sqrt{2}}(\rho(x) + i\chi(x)))$$

$$- \frac{\lambda}{2} ((v + \frac{1}{\sqrt{2}}(\rho(x) - i\chi(x)))(v + \frac{1}{\sqrt{2}}(\rho(x) + i\chi(x))))^2 - \frac{1}{2} F_{\mu\nu} F^{\mu\nu} \hspace{1cm} (26)$$

The mass terms for $\rho$ and $\chi$ get contributions from the two middle terms. The first term does not contribute since the terms in it contain either derivatives or crossterms. The last term does not contain $\rho$ or $\chi$ at all. The second term gives the following contributions:

$$\frac{\mu^2}{2} \rho^2, \quad \frac{\mu^2}{2} \chi^2$$

while the third term yields (keeping only terms of the right order in each step):

$$- \frac{\lambda}{2} ((v + \frac{1}{\sqrt{2}}(\rho - i\chi))(v + \frac{1}{\sqrt{2}}(\rho + i\chi)))^2$$

$$= - \frac{\lambda}{2} ((v^2 + \frac{1}{\sqrt{2}}v \rho + \frac{1}{2}(\rho^2 + \chi^2))^2) \Rightarrow - \frac{\lambda}{2} ((2v^2 + v^2 \rho^2 + v^2 \chi^2)$$

$$= - \frac{3\lambda v^2}{2} \rho^2 - \frac{\lambda v^2}{2} \chi^2 = - \frac{3\mu^2}{2} \rho^2 - \frac{\mu^2}{2} \chi^2$$

So the mass terms for the two fields are:

$$\frac{2\mu^2}{2} \rho^2, \quad - \frac{\mu^2}{2} \chi^2$$
The masses are then seen to be $\sqrt{2}\mu$, for $\rho$, and 0, for $\chi$.

Finally, there is the mass term for the gauge field $A_\mu$. The last three terms in Eqn. (25) either do not contain $A_\mu$ or only derivatives of $A_\mu$, hence they do not contribute to the mass term. The first term on the other hand yields (in each step only keeping terms of the right order):

$$-(D_\mu v^2 + \frac{1}{\sqrt{2}}(\rho(x) + i\chi(x))))(D^\mu v + \frac{1}{\sqrt{2}}(\rho(x) + i\chi(x))))$$

$$\Rightarrow -(D_\mu v)^\dagger(D^\mu v) = -((\partial_\mu - i e A_\mu) v)^\dagger((\partial_\mu - i e A_\mu) v)$$

$$= -\frac{e^2 v^2 A^2}{2}$$

Thus the gauge field, $A_\mu$, has obtained a mass of $\sqrt{2}e v$.

SO(2) corresponds to a rotation in a plane, while U(1) corresponds to a change of phase in the complex plane. Since the complex plane can be viewed as a bijection from $\mathbb{R}^2$, these two cases are equivalent.

### 4.2 Non-Abelian Case SU(2)

The SU(2) group has three traceless hermitian matrices as generators. These could be for instance the three Pauli matrices, $\sigma^a$, that also satisfy Eqn. (20).

The Lagrangian in Eqn. (21) can be generalized to higher dimensions in the following way:

$$\mathcal{L} = \sum_{i=1}^{N} \left(-(D_\mu \phi_i)^\dagger(D^\mu \phi_i) + \mu^2 \phi_i^\dagger \phi_i - \frac{\lambda}{2} (\phi_i^\dagger \phi_i)^2 \right) - \frac{1}{2} \sum_{a=1}^{N^2-1} (F^a)^{\mu\nu} (F^a)_{\mu\nu}$$

(29)

where $(D_\mu \phi_i) = \partial_\mu \phi_i - ig \sum_{a=1}^{N^2-1} A^a_\mu (T^a)_{ij} \phi_j$, $(F^a)^{\mu\nu} = \partial^\mu (A^a)^\nu - \partial^\nu (A^a)^\mu$ and $g$ is a coupling constant replacing $e$ from the U(1) case. In the SU(2) case this reads:

$$\mathcal{L} = -(D_\mu \phi_1)^\dagger(D^\mu \phi_1) - (D_\mu \phi_2)^\dagger(D^\mu \phi_2) + \mu^2 \phi_1^\dagger \phi_1 + \mu^2 \phi_2^\dagger \phi_2 - \frac{\lambda}{2} (\phi_1^\dagger \phi_1)^2 - \frac{\lambda}{2} (\phi_2^\dagger \phi_2)^2 - \frac{1}{2} (F^1)^{\mu\nu} (F^1)_{\mu\nu} - \frac{1}{2} (F^2)^{\mu\nu} (F^2)_{\mu\nu} - \frac{1}{2} (F^3)^{\mu\nu} (F^3)_{\mu\nu}$$

(30)

which is invariant under the gauge transformation:

$$\phi_i(x) \to e^{i\alpha(x)} (\sigma^a)_{ij} \phi_j(x), \quad A_\mu(x) \to A_\mu^a(x) - \frac{1}{g} \sigma^a \partial_\mu \alpha(x)$$

(31)

The potential of this Lagrangian is then identified as:

$$V(\phi_1, \phi_1^\dagger, \phi_2, \phi_2^\dagger) = \frac{\lambda}{2} (\phi_1^\dagger \phi_1)^2 + \frac{\lambda}{2} (\phi_2^\dagger \phi_2)^2 - \mu^2 \phi_1^\dagger \phi_1 - \mu^2 \phi_2^\dagger \phi_2$$

(32)

There are no cross-terms between fields with different indices in the potential, hence derivatives wrt a field of a certain index will only retain terms with the
same index. Regarding terms with different indices separately, it is seen that they are the same as in Eqn. \((33)\). These two facts give that the minima are:

\[
\phi_1 \phi_2 = \phi_1 \phi_1 = \frac{\mu^2}{\lambda} =: v^2 \tag{33}
\]

The fields can be expanded around the minima.

\[
\begin{align*}
\phi_1(x) &= v + \frac{1}{\sqrt{2}}(\rho_1(x) + i\chi_1(x)) \\
\phi_2(x) &= v + \frac{1}{\sqrt{2}}(\rho_2(x) + i\chi_2(x))
\end{align*} \tag{34}
\]

where the \(\rho\)s and \(\chi\)s are real valued. Inserting this into the Lagrangian from Eqn. \((33)\) results in:

\[
\mathcal{L} = -(D_\mu \phi)_1 \dagger (D^\mu \phi)_1 - (D_\mu \phi)_2 \dagger (D^\mu \phi)_2 \\
+ \mu^2 \left( v + \frac{1}{\sqrt{2}}(\rho_1 - i\chi_1)(v + \frac{1}{\sqrt{2}}(\rho_1 + i\chi_1)) \\
+ \mu^2 (v + \frac{1}{\sqrt{2}}(\rho_2 - i\chi_2)(v + \frac{1}{\sqrt{2}}(\rho_2 + i\chi_2)) \\
- \frac{\lambda}{2} ((v + \frac{1}{\sqrt{2}}(\rho_1 - i\chi_1)(v + \frac{1}{\sqrt{2}}(\rho_1 + i\chi_1))^2 \\
- \frac{\lambda}{2} ((v + \frac{1}{\sqrt{2}}(\rho_2 - i\chi_2)(v + \frac{1}{\sqrt{2}}(\rho_2 + i\chi_2))^2 \\
- \frac{1}{2}(F^1)_{\mu\nu}(F^1)_{\mu\nu} - \frac{1}{2}(F^2)_{\mu\nu}(F^2)_{\mu\nu} - \frac{1}{2}(F^3)_{\mu\nu}(F^3)_{\mu\nu}
\right)
\]

where the \((D_\mu \phi)_1\)s have been kept to reduce messiness.

Following the same argument that lead to the minima it can be seen that, due to no cross-terms and similarities to the transformed Lagrangian in the U(1) case, the \(\chi\)-fields are massless and the masses of the \(\rho\)-fields are \(\sqrt{2}\mu\).

As for the gauge masses, only the terms containing \((D_\mu \phi)_i\) contribute to the mass terms, for the usual reasons. Expanding the first term and identifying \(T^a = \sigma^a\) yields (keeping only terms of the right order in each step and omitting the minus sign):

\[
(D_\mu \phi)_1 \dagger (D^\mu \phi)_1 = (D_\mu \phi_1 - ig \sum_{a=1}^{N^2-1} A^a_\mu(\sigma^a)_\mu \phi_j) \dagger (\partial^\mu \phi_1 \\
- ig \sum_{a=1}^{N^2-1} (A^a)^\mu(\sigma^a)_\mu \phi_j)
\]

\[
\Rightarrow ig \left( \sum_{a=1}^{3} A^a_\mu(\sigma^a)_\mu \phi_j \right) \dagger \left( -ig \sum_{a=1}^{3} (A^a)^\mu(\sigma^a)_\mu \phi_j \right) \tag{36}
\]

\[
= g^2 ((A^1)_\mu \phi_2 - i(A^2)_\mu \phi_2 \\
+ (A^3)_\mu \phi_1) \dagger ((A^1)^\mu \phi_2 - i(A^2)^\mu \phi_2 + (A^3)^\mu \phi_1) \\
\Rightarrow g^2 ((A^1)_\mu v - i(A^2)_\mu v \\
+ (A^3)_\mu v) \dagger ((A^1)^\mu v - i(A^2)^\mu v + (A^3)^\mu v) \\
\Rightarrow g^2 v^2 (A^1)^2 + g^2 v^2 (A^2)^2 + g^2 v^2 (A^3)^2
\]
The second term yields:

\[
((D_\mu \phi)_2) \cdot ((D^{\mu} \phi)_2) = (\partial_\mu \phi_2 - ig \sum_{a=1}^{N^2-1} A_a^\mu (\sigma^a)_2 \phi_j)^\dagger (\partial^{\mu} \phi_2 - ig \sum_{a=1}^{N^2-1} (A^a)^\mu (\sigma^a)_2 \phi_j)
\]

\[
\Rightarrow ig \left( \sum_{a=1}^{3} A_a^\mu (\sigma^a)_2 \phi_j \right)^\dagger \left( -ig \sum_{a=1}^{3} (A^a)^\mu (\sigma^a)_2 \phi_j \right)
\]

\[
= g^2 ( (A^1)_{\mu} \phi_1 + i(A^2)_{\mu} \phi_1 - (A^3)_{\mu} \phi_2 )^\dagger ( (A^1)^{\mu} \phi_1 + i(A^2)^{\mu} \phi_1 - (A^3)^{\mu} \phi_2 )
\]

\[
= g^2 ( (A^1)_{\mu} v + i(A^2)_{\mu} v - (A^3)_{\mu} v )^\dagger ( (A^1)^{\mu} v + i(A^2)^{\mu} v - (A^3)^{\mu} v )
\]

\[
= g^2 v^2 (A^1)^2 + g^2 v^2 (A^2)^2 + g^2 v^2 (A^3)^2
\]

Thus it is seen that all three gauge fields have obtained a mass of \(2gv\).

SU(2) is similar to SO(3). Consider the bijection between su(2) and \(R^3\):

\[su(2) \ni x = x^i \sigma^i \leftrightarrow (x_1, x_2, x_3) = \vec{x} \in R^3\] (38)

This shows that su(2) is isomorphic to \(R^3\). Furthermore, consider the most general form of a matrix \(U \in SU(2)\):

\[U(a, b) = \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}, \quad a, b \in C\] (39)

Now create the linear map \(x \rightarrow UxU^{-1}\). Since:

\[\text{Tr}(UxU^{-1}) = 0 \quad (UxU^{-1})^\dagger = UxU^{-1}\] (40)

\(UxU^{-1}\) is traceless and Hermitian, i.e. \(UxU^{-1} \in su(2)\) and according to Eqn. (38) it can be identified with \(R^3\). Lastly, notice that:

\[x \cdot y = \frac{1}{2} \text{Tr}(xy), \quad x, y \in su(2)\] (41)

and therefore

\[x \cdot y \rightarrow \text{Tr}(UxU^{-1}yU^{-1}) = \text{Tr}(UxyU^{-1}) = \text{Tr}(Ux_iy^iU^{-1}) = \text{Tr}(x_iy^iU^{-1}) = \text{Tr}(x_iy^i) = \frac{1}{2} \text{Tr}(xy) = \vec{x} \cdot \vec{y}\]

where the following relations have been used to get to the third line:

\[\sigma_i \sigma_j = \delta_{ij} + i\epsilon_{ijk} \sigma_k, \quad \text{Tr}(UxU^{-1} = 0)\] (43)
Eqn. [12] shows that this transformation preserves the dot product, which means that the transformation is orthogonal in three dimensions. Hence it belongs to SO(3). Thus for every \( U \in SU(2) \), \( x \rightarrow U x U^{-1} \) \( \in SO(3) \) and the mapping from SU(2) to SO(3) is surjective. It is not however isomorphic, since the map is surjective but not injective. In fact each element, except the zero element, in SO(3) gets mapped twice. This means that while SU(2) and SO(3) are not exactly the same, they are similar enough that several problems can be worked out in either language. [2]

What has been shown by spontaneous symmetry breaking in both the Abelian and non-Abelian cases are examples of Goldstone’s theorem:

If a quantum field has a non-zero vacuum expectation value (VEV) and the VEV of the field is invariant under the same transformation that leaves the Lagrangian invariant. Then, the vacuum states are degenerate and there exists massless fields.

5 Higgs Mechanism in the Standard Model

To construct the electroweak sector of the standard model we will begin from the following Lagrangian that is invariant under transformations of the SU(2) \( \times U(1) \)-group:

\[
\mathcal{L} = ((D_{\mu} \Phi_1)^{\dagger}((D_{\mu} \Phi_1) - ((D_{\mu} \Phi_2)^{\dagger}((D_{\mu} \Phi_2) - \frac{1}{4} \lambda(\Phi^{\dagger} \Phi - \frac{1}{2} v^2)^2
\]

\[
- \frac{1}{4} (F^{1})_{\mu \nu}^\mu (F^{1})_{\mu \nu} - \frac{1}{4} (F^{2})_{\mu \nu}^\mu (F^{2})_{\mu \nu} - \frac{1}{4} (F^{3})_{\mu \nu}^\mu (F^{3})_{\mu \nu} - \frac{1}{4} B^{\mu \nu} B_{\mu \nu}
\]

\[
= -((D_{\mu} \Phi_1)^{\dagger}((D_{\mu} \Phi_1) - ((D_{\mu} \Phi_2)^{\dagger}((D_{\mu} \Phi_2) - \frac{1}{4} \lambda(\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 - \frac{1}{2} v^2)^2
\]

\[
- \frac{1}{4} (F^{1})_{\mu \nu}^\mu (F^{1})_{\mu \nu} - \frac{1}{4} (F^{2})_{\mu \nu}^\mu (F^{2})_{\mu \nu} - \frac{1}{4} (F^{3})_{\mu \nu}^\mu (F^{3})_{\mu \nu} - \frac{1}{4} B^{\mu \nu} B_{\mu \nu}
\]

(44)

where:

\[
\Phi = \left( \begin{array}{c} \phi_1 \\ \phi_2 \end{array} \right)
\]

The \( F \):s and the \( B \) contain the gauge fields and are defined as:

\[
(F^{1})_{\mu \nu}^{\mu} = \partial_{\mu} (A^{1})_{\nu} - \partial_{\nu} (A^{1})_{\mu} + g_2 ((A^{2})_{\mu} (A^{3})_{\nu} - (A^{2})_{\nu} (A^{3})_{\mu})
\]

\[
(F^{2})_{\mu \nu}^{\mu} = \partial_{\mu} (A^{2})_{\nu} - \partial_{\nu} (A^{2})_{\mu} + g_2 ((A^{3})_{\mu} (A^{1})_{\nu} - (A^{3})_{\nu} (A^{1})_{\mu})
\]

\[
(F^{3})_{\mu \nu}^{\mu} = \partial_{\mu} (A^{3})_{\nu} - \partial_{\nu} (A^{3})_{\mu} + g_2 ((A^{1})_{\mu} (A^{2})_{\nu} - (A^{1})_{\nu} (A^{2})_{\mu})
\]

(45)

\[
B_{\mu \nu}^{\mu} = \partial_{\mu} B_{\nu} - \partial_{\nu} B_{\mu}
\]

where the \( A \):s and the \( B \) on the right side are the gauge fields, while \( g_2 \) is a coupling constant. The covariant derivative, \( (D_{\mu} \Phi)_i \), in Eqn. [14] is given by:

\[
(D_{\mu} \Phi)_i = \partial_{\mu} \Phi_i - i (g_2 (A^a)_{\mu} T^a + g_1 B_{\mu} Y)_{i} \Phi_j
\]

(46)

where \( g_1 \) is another coupling constant, \( T^a \) are the generators of SU(2) discussed in the previous section [3], but with a slightly different condition, namely:

\[
Tr(T^a T^b) = \frac{1}{2} \delta^{ab},
\]

(47)
Y is the generator for U(1). In this case Y = −\frac{1}{2}I. Y is called the hypercharge, which is a conserved quantity in particle physics.

The SU(2)×U(1)-group has been chosen specifically to yield the right conserved quantities in the electroweak theory.

Studying the Lagrangian in Eqn. (44), it is seen that the potential is:

\[ V(\phi_1, \phi_1^\dagger, \phi_2, \phi_2^\dagger) = \frac{1}{4} \lambda (\phi_1^\dagger \phi_1 + \phi_2^\dagger \phi_2 - \frac{1}{2} v^2)^2 \]  

(48)

Due to the positive semi-definiteness of this potential the minimum value is zero. Expanding around the real minima in the \(\phi_1\)-direction as follows:

\[
\begin{align*}
\phi_1(x) &= \frac{1}{\sqrt{2}} (v + H(x)) \\
\phi_2(x) &= 0
\end{align*}
\]

(49)

where H is real valued and the unitary gauge has been chosen to simplify further calculations. Then inserting this into Eqn. (44) yields:

\[
\mathcal{L} = -(\partial_\mu \frac{1}{\sqrt{2}} H - i (g_2 A^a)_{\mu} \sigma^a + g_1 B_\mu Y) \frac{1}{\sqrt{2}} (v + H)^{\dagger} (\partial^\mu \frac{1}{\sqrt{2}} H \\
- i (g_2 A^a)_{\mu} \sigma^a + g_1 B_\mu Y) \frac{1}{\sqrt{2}} (v + H)^{\dagger} (-i (g_2 A^a)_{\mu} \sigma^a + g_1 B_\mu Y) \frac{1}{\sqrt{2}} (v + H) - \frac{1}{4} \lambda (v + H)^2 - \frac{1}{2} v^2)^2 \\
- \frac{1}{4} (F^1)_{\mu\nu} (F^1)^{\mu\nu} - \frac{1}{4} (F^2)_{\mu\nu} (F^2)^{\mu\nu} - \frac{1}{4} (F^3)_{\mu\nu} (F^3)^{\mu\nu} - \frac{1}{4} B^{\mu\nu} B_{\mu\nu} \\
= -\frac{1}{2} \left( \partial_\mu H - i \frac{g_2}{2} (A^3)_{\mu} - \frac{1}{2} g_1 B_\mu \right) (v + H) \right)^{\dagger} (\partial^\mu H \\
- i \frac{g_2}{2} (A^3)_{\mu} - \frac{1}{2} g_1 B_\mu (v + H) \\
- \frac{1}{2} \left( (-i \frac{g_2}{2} (A^1)_{\mu} + \frac{g_2}{2} (A^2)_{\mu} (v + H) \right)^{\dagger} (-i \frac{g_2}{2} (A^1)_{\mu} + \frac{g_2}{2} (A^2)_{\mu} (v + H) \\
- \frac{1}{4} \lambda (v + H)^2 - \frac{1}{2} v^2)^2 \\
- \frac{1}{4} B^{\mu\nu} B_{\mu\nu}
\]

(50)

where Eqn. (46) has been used to expand \(D\).

Next, introduce the following gauge fields:

\[
\begin{align*}
W_\mu^\pm &:= \frac{1}{\sqrt{2}} ((A^1)_{\mu} \pm (A^2)_{\mu}) \\
Z_\mu &:= \cos(\theta_W) (A^3)_{\mu} - \sin(\theta_W) B_\mu \\
A_\mu &:= \sin(\theta_W) (A^3)_{\mu} + \cos(\theta_W) B_\mu
\end{align*}
\]

(51)

where \(\theta_W\) is called the weak mixing angle and is defined by:

\[
\theta_W := \tan^{-1} \left( \frac{g_1}{g_2} \right)
\]

(52)

Before substituting this into Eqn. (50) it is convenient to first look at the following combinations of gauge fields (the first equality is calculated explicitly,
while the rest can be calculated in a similar fashion:

\[
\frac{1}{\sqrt{2}}((F^1)_{\mu\nu} - i(F^2)_{\mu\nu}) = \frac{1}{\sqrt{2}}((\partial^\mu(A^1)^\nu - \partial^\nu(A^1)^\mu
+ g_2((A^2)^\mu(A^3)^\nu - (A^2)^\nu(A^3)^\mu) - i\partial^\mu(A^2)^\nu
- \partial^\nu(A^2)^\mu + g_2((A^3)^\mu(A^1)^\nu - (A^3)^\nu(A^1)^\mu))
\]
\[
= (\partial^\mu(1)_{\nu} - i(2)_{\nu}) - ig_2(3)_{\mu}(1)_{\nu} - i(2)_{\nu})
- \partial^\nu(1)_{\mu} - i(2)_{\mu})
+ ig_2(3)_{\nu}(1)_{\mu} - i(2)_{\mu})
\]
\[
= (\partial^\mu(W^+)^\nu - ig_2(3)_{\mu}(W^+)^\nu - \partial^\nu(W^+)^\mu
+ ig_2(3)_{\nu}(W^+)^\mu
= D^\mu(W^+)^\nu - D^\nu(W^+)^\mu
\]
\]

\[
\frac{1}{\sqrt{2}}((F^1)_{\mu\nu} + i(F^2)_{\mu\nu}) = (D^1)^\mu(W^-)^\nu - (D^1)^\nu(W^-)^\mu
\]
\[
(F^3)_{\mu\nu} = \sin(\theta_W)F_{\mu\nu} + \cos(\theta_W)Z_{\mu\nu} - ig_2((W^+)_{\mu}(W^-)^\nu - (W^+)_{\nu}(W^-)^\mu
\]
\[
B_{\mu\nu} = \cos(\theta_W)F_{\mu\nu} - \sin(\theta_W)Z_{\mu\nu}
\]

where \( D^\mu := \partial^\mu - ig_2(A^3)^\mu = \partial^\mu - ig_2(\sin(\theta_W)A^\mu + \cos(\theta_W)Z^\mu) \), \( F_{\mu\nu} := \partial^\mu A^\nu - \partial^\nu A^\mu \) and \( Z_{\mu\nu} = \partial^\mu Z^\nu - \partial^\nu Z^\mu \). Finally, substituting this into Eqn. (50) gives:

\[
\mathcal{L} = \left[ \frac{1}{2} (\partial_\mu H - i\frac{g_2}{2} \left( \frac{1}{\cos(\theta_W)} Z_\mu \right) (v + H) \right] (\partial^\mu H - i\frac{g_2}{2} \left( \frac{1}{\cos(\theta_W)} Z^\mu \right) (v + H))
- \frac{1}{2} (-i\frac{1}{\sqrt{2}} g_2(W^+)_{\mu}(v + H)) \right] (\frac{1}{\sqrt{2}} g_2(W^+)_{\mu}(v + H))
- \frac{1}{4} \lambda \left( \frac{1}{2} H^2 + vH \right)^2
- \frac{1}{8} (D^\mu(W^+)^\nu - D^\nu(W^+)^\mu + (D^1)^\mu(W^-)^\nu
- (D^1)^\nu(W^-)^\mu - D^\mu(W^-)^\nu + (D^1)^\nu(W^-)^\mu - D^\mu(W^+)^\nu + (D^1)^\nu(W^+)^\mu)
\]
\[
\mathcal{L} = \frac{1}{4} (\sin(\theta_W)F_{\mu\nu} + \cos(\theta_W)Z_{\mu\nu}
- ig_2((W^+)_{\mu}(W^-)^\nu - (W^+)_{\nu}(W^-)^\mu)(\sin(\theta_W)F_{\mu\nu}
+ \cos(\theta_W)Z_{\mu\nu} - ig_2((W^+)_{\mu}(W^-)^\nu - (W^+)_{\nu}(W^-)^\mu))
- \frac{1}{4} (\cos(\theta_W)F_{\mu\nu} - \sin(\theta_W)Z_{\mu\nu})(\cos(\theta_W)F_{\mu\nu} - \sin(\theta_W)Z_{\mu\nu})
\]

This is the Lagrangian for the electroweak part of the standard model. \( A \) is the electromagnetic gauge field whose gauge boson is the photon. There is no mass
term for $A$ in the Lagrangian, so its gauge boson is indeed massless just as the photon should be. There is however a mass term for $Z$ given by:

$$\frac{-g^2 v^2}{8 \cos^2 \theta_W} Z^\mu Z_\mu := -\frac{M_Z^2}{2} Z^\mu Z_\mu$$

(56)

which corresponds to the $Z^0$ gauge particle for weak interactions, that has a measured mass of $M_Z = 91.2$ GeV. There is also a mass term for $W^\pm$ given by:

$$\frac{-g^2 v^2}{4} (W^-)_\mu (W^+)\mu := -M_W^2 (W^-)_\mu (W^+)\mu$$

(57)

This corresponds to the $W^\pm$ gauge bosons for electroweak interactions, which have a measured mass of $M_W = 80.4$ GeV. Lastly, there is the higgs gauge boson connected to the higgs gauge field $H$ with the mass term:

$$\frac{-\lambda v^2}{4} H^2 := -\frac{M_H^2}{2} H^2$$

(58)

### 6 Unitarity Constraints on the Higgs Mass

The previous section illustrated the necessity of the higgs particle, but it did not provide a value for its mass ($\lambda$ is an unknown constant). This can be partially amended by studying constraints on the mass due to partial wave unitarity, which yields an upper bound for the mass.

A scattering amplitude in quantum field theory can be split up into partial waves in the following way:

$$\mathcal{T}(s,t) = 16\pi \sum_{j=0}^{\infty} (2j+1)a_j(s)P_j(\cos \theta)$$

(59)

where the $a_j$:s are called the partial wave amplitudes, the $P_j$:s are Legendre polynomials, $\theta$ is the scattering angle, while $s$ and $t$ are two of the three Mandelstam variables:

$$s := -(k_1 + k_2)^2 = -(k'_1 + k'_2)^2$$
$$t := -(k_1 - k'_1)^2 = -(k_2 - k'_2)^2$$
$$u := -(k_1 - k'_2)^2 = -(k_2 - k'_1)^2$$

(60)

where the $k$:s are four-momenta as defined in the Feynman diagrams below (figures 4-10).

Since, $|\mathcal{T}|^2$, is a probability for the scattering to occur, it cannot be larger than 1. Eqn. (59) then implies that the sum of the partial waves cannot exceed 1 as well. In particular $|a_0| \leq 1$. It could be argued that, since the $a_j$:s are in general complex, there might be negative cross-terms allowing for a larger value of $|a_0|$. This is not true, because the absolute value of the $a_j$:s decrease by increasing $j$. Thus, by this criteria it is possible to find an upper bound for the Higgs boson mass.

The upper bound is known as the Lee-Quigg-Thacker-bound (LQT-bound) and can be derived by considering only the tree level Feynman diagrams for scattering of the weak gauge bosons including the higgs boson. Diagrams of higher order can be neglected because either their contribution are some orders
of magnitude smaller, and hence neglect-able, or they are larger, but then perturbation theory fails and Feynman diagrams cannot be used at all. The latter case is not impossible, but is not covered by the standard model, hence it will not be considered here.\[4\]

In the following interactions exchanges of $\gamma$- and $Z^0$-particles are neglected, since their contribution to the amplitude is negligible, except in the last two processes where exchanges of $Z^0$-particles are taken into consideration. The Feynman diagrams and their amplitudes are as follows:

$$
\mathcal{T}(W^+W^- \rightarrow W^+W^-) = 
$$

$$
-\sqrt{2} G_F M_H^2 \left(1 + \frac{s-M_H^2}{s-M_H^2} + \frac{t-M_H^2}{t-M_H^2}\right)
$$

Figure 4: $W^+W^- \rightarrow W^+W^-$
\[ T(ZZ \rightarrow ZZ) = \]
\[ k_2' k_2 + k_1' k_1 \]
\[ H \]
\[ k_1' - k_1 \]
\[ H \]
\[ k_2' - k_1 \]
\[ H \]
\[ k_1' k_1 + k_2' k_2 \]
\[ H \]
\[ k_1' k_1 - k_1 \]
\[ H \]
\[ k_2' k_2 \]
\[ H \]
\[ -\sqrt{2} G_F M_H^2 \left( 1 + \frac{s}{s-M_H^2} + \frac{t}{t-M_H^2} + \frac{u}{u-M_H^2} \right) \]

Figure 5: ZZ \rightarrow ZZ

\[ T(HH \rightarrow HH) = \]
\[ k_2' k_2 + k_1' k_1 \]
\[ H \]
\[ k_1' - k_1 \]
\[ H \]
\[ k_2' - k_1 \]
\[ H \]
\[ k_1' k_1 + k_2' k_2 \]
\[ H \]
\[ k_1' k_1 - k_1 \]
\[ H \]
\[ k_2' k_2 \]
\[ H \]
\[ -3\sqrt{2} G_F M_H^2 \left( 1 + \frac{3M_H^2}{s-M_H^2} + \frac{3M_H^2}{t-M_H^2} + \frac{3M_H^2}{u-M_H^2} \right) \]

Figure 6: HH \rightarrow HH
\[ \mathcal{T}(HH \to W^+W^-) = \]

\[ \mathcal{T}(ZZ \to W^+W^-) = \]

Figure 7: $HH \to W^+W^-$

Figure 8: $ZZ \to W^+W^-$
\[
\mathcal{T}(HH \rightarrow ZZ) = -\sqrt{2}G_FM_H^2 \left( 1 + \frac{3M_H^2}{s-M_H^2} + \frac{M_H^2}{t-M_H^2} + \frac{M_H^2}{u-M_H^2} \right)
\]

Figure 9: HH \rightarrow ZZ

\[
\mathcal{T}(HZ \rightarrow HZ) = -\sqrt{2}G_FM_H^2 \left( 1 + \frac{3M_H^2}{s-M_H^2} + \frac{M_H^2}{t-M_H^2} + \frac{M_H^2}{u-M_H^2} \right)
\]

Figure 10: HZ \rightarrow HZ

where \( G_F \) is the Fermi coupling constant. All other scatterings, not shown in the Feynman diagrams above, are negligible for the amplitude.
Because we’re looking for an upper bound on the higgs mass it might as well be assumed that \( M_H \gg M_Z, M_W \). It can also be assumed that \( s \gg M_Z^2, M_W^2 \), i.e the energy of the system is very large. With these assumptions, and remembering that \( P_0 = 1 \), the zeroth partial wave amplitudes can be calculated through \([4, 5]\):

\[
    a_0 = \frac{1}{16\pi s} \int_{-s}^{0} T \, dt,
\]

(61) together with:

\[
    u = M_l^2 + M^2 + M_2^2 - s - t,
\]

(62) For example:

\[
    a_0(\ZZ \to \ZZ) = \frac{1}{16\pi s} \int_{-s}^{0} T(\ZZ \to \ZZ) \, dt
\]

\[
    = -\frac{G_F M^2_H}{8\sqrt{2}\pi s} \int_{-s}^{0} \left( 1 + \frac{s}{s - M^2_H} + \frac{t}{t - M^2_H} + \frac{u}{u - M^2_H} \right) \, dt
\]

\[
    = -\frac{G_F M^2_H}{8\sqrt{2}\pi} \left( 1 + \frac{s}{s - M^2_H} + 1 + \frac{1}{s} \int_{-s}^{0} \left( \frac{M^2_H}{t - M^2_H} + \frac{M^2_H}{u - M^2_H} \right) \, dt \right)
\]

\[
    \approx -\frac{G_F M^2_H}{8\sqrt{2}\pi} \left( 3 + \frac{s}{s - M^2_H} + \frac{M^2_H}{s} \log \left( \frac{M^2_H}{s + M^2_H} \right) \right)
\]

(63)

In the first approximation \( M_Z^2 \) has been neglected. In the second approximation \( s \) is set equal to \( M^2_H \) in the nominator of the second term, which is motivated
by Eqn. (60).
In a similar manner the other partial wave amplitudes can be calculated. Eqn. (61) shows the result.

\[
\begin{align*}
    a_0(W^+W^- \rightarrow W^+W^-) &= -\frac{G_F M_Y^2}{8\pi \sqrt{2}} \left( 2 + \frac{M_H^2}{s-M_H^2} - \frac{M_H^2}{s} \log \left( 1 + \frac{s}{M_H^2} \right) \right) \\
    a_0(ZZ \rightarrow ZZ) &= -\frac{G_F M_Y^2}{8\pi \sqrt{2}} \left( 3 + \frac{M_H^2}{s-M_H^2} - \frac{2M_H^2}{s} \log \left( 1 + \frac{s}{M_H^2} \right) \right) \\
    a_0(HH \rightarrow HH) &= -\frac{G_F M_Y^2}{8\pi \sqrt{2}} \left( 3 + \frac{9M_H^2}{s-M_H^2} - \frac{18M_H^2}{s-4M_H^2} \log \left( \frac{s}{M_H^2} - 3 \right) \right) \\
    a_0(HH \rightarrow W^+W^-) &= -\frac{G_F M_Y^2}{8\pi \sqrt{2}} \left( 1 + \frac{3M_H^2}{s-M_H^2} + \frac{4M_H^2}{\sqrt{s(s-4M_H^2)}} \log \left( \frac{s-2M_H^2-\sqrt{s(s-4M_H^2)}}{2M_H^2} \right) \right) \\
    a_0(ZZ \rightarrow W^+W^-) &= -\frac{G_F M_Y^2}{8\pi \sqrt{2}} \left( 1 + \frac{M_H^2}{s-M_H^2} \right) \\
    a_0(HH \rightarrow ZZ) &= -\frac{G_F M_Y^2}{8\pi \sqrt{2}} \left( 1 + \frac{3M_H^2}{s-M_H^2} + \frac{4M_H^2}{\sqrt{s(s-4M_H^2)}} \log \left( \frac{s-2M_H^2-\sqrt{s(s-4M_H^2)}}{2M_H^2} \right) \right) \\
    a_0(HZ \rightarrow HZ) &= -\frac{G_F M_Y^2}{8\pi \sqrt{2}} \left( 1 + \frac{M_H^2}{s-M_H^2} - \frac{3M_H^2 s}{(s-M_H^2)^2} \log \left( 1 + \frac{(s-M_H^2)^2}{s-M_H^2} \right) - \frac{M_H^2 s}{(s-M_H^2)^2} \log \left( \frac{s(2M_H^2-s)}{M_H^2} \right) \right)
\end{align*}
\]

If it is further assumed that the energy of the system is so large that \( s \gg M_H^2 \), then Eqn. (61) reduces to:

\[
\begin{align*}
    a_0(W^+W^- \rightarrow W^+W^-) &= -\frac{G_F M_Y^2}{4\pi \sqrt{2}} \\
    a_0(ZZ \rightarrow ZZ) &= -\frac{3G_F M_Y^2}{8\pi \sqrt{2}} \\
    a_0(HH \rightarrow HH) &= -\frac{3G_F M_Y^2}{8\pi \sqrt{2}} \\
    a_0(HH \rightarrow W^+W^-) &= -\frac{G_F M_Y^2}{8\pi \sqrt{2}} \\
    a_0(ZZ \rightarrow W^+W^-) &= -\frac{G_F M_Y^2}{8\pi \sqrt{2}} \\
    a_0(HH \rightarrow ZZ) &= -\frac{G_F M_Y^2}{8\pi \sqrt{2}} \\
    a_0(HZ \rightarrow HZ) &= -\frac{G_F M_Y^2}{8\pi \sqrt{2}} 
\end{align*}
\]

which on matrix form reads:

\[
\begin{pmatrix}
    1 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{8}} & 0 \\
    \frac{1}{\sqrt{2}} & 1 & \frac{1}{\sqrt{8}} & 0 \\
    \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & \frac{1}{3} & 0 \\
    0 & 0 & 0 & \frac{1}{2}
\end{pmatrix}
\]

in the basis \( \{ W^+W^-, \frac{1}{\sqrt{2}}ZZ, \frac{1}{\sqrt{2}}HH, HZ \} \). For \( a_0 \leq 1 \) the largest eigenvalue of this matrix has to be less than or equal to 1 as well. Calculating the eigenvalues
in units of $-\frac{G_F M_H^2}{4\pi \sqrt{2}}$ yields:

$$0 = \begin{vmatrix}
1 - \lambda & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 \\
\frac{1}{\sqrt{8}} & \frac{1}{4} & \frac{1}{4} - \lambda & 0 \\
\frac{1}{\sqrt{8}} & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{2} - \lambda
\end{vmatrix}$$

$$= \left(\frac{1}{2} - \lambda\right) \begin{vmatrix}
1 - \lambda & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 \\
\frac{1}{\sqrt{8}} & \frac{1}{4} & \frac{1}{4} - \lambda & 0 \\
\frac{1}{\sqrt{8}} & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{2} - \lambda
\end{vmatrix}$$

$$= \left(\frac{1}{2} - \lambda\right) \begin{vmatrix}
1 - \lambda & \frac{1}{\sqrt{8}} & \frac{1}{\sqrt{8}} & 0 \\
\frac{1}{\sqrt{8}} & \frac{1}{4} & \frac{1}{4} - \lambda & 0 \\
\frac{1}{\sqrt{8}} & \frac{1}{4} & \frac{1}{4} & 0 \\
0 & 0 & 0 & \frac{1}{2} - \lambda
\end{vmatrix}$$

$$= \left(\frac{1}{2} - \lambda\right)^2 \begin{vmatrix}
1 - \lambda & 1 & 1 & 0 \\
1 & \frac{1}{16} & \frac{1}{16} & 0 \\
1 & \frac{1}{16} & \frac{1}{16} & 0 \\
0 & 0 & 0 & 1 - \lambda
\end{vmatrix}$$

$$\Leftrightarrow \left\{ \begin{array}{l}
\lambda_1 = \lambda_2 = \frac{1}{2} \\
(1 - \lambda_3)^2 = \frac{1}{4} = 0 \Rightarrow 1 - \lambda_3, 4 = \pm \frac{1}{2} \Rightarrow \lambda_3, 4 = \frac{1}{2}, \frac{3}{2}
\end{array} \right.$$ 

Thus $-\frac{G_F M_H^2}{4\pi \sqrt{2}}$ is the largest eigenvalue and hence by unitarity the absolute value of this must be less than or equal to 1, which gives an upper bound for the higgs mass.

$$\frac{G_F M_H^2}{4\pi \sqrt{2}} \leq 1 \Rightarrow M_H^2 \leq \frac{8\pi \sqrt{2}}{3G_F} \approx 1(\text{TeV})^2$$

Which sets the upper bound on the higgs mass to 1 TeV.\[5\]

### 7 Summary

First it was shown in sections 2 and 3 that spontaneous symmetry breaking can give mass to previously massless fields through the higgs mechanism.

In section 4 gauge fields were added and spontaneous symmetry breaking of local gauge invariance was considered for both Abelian and non-Abelian cases. Here it was seen that the gauge fields, that were originally massless, would attain mass, and some of the originally massless scalar fields would gain mass as well.

Finally, the problem that was stated in the introduction could be solved in section 5. Applying spontaneous symmetry breaking and the higgs mechanism to a SU(2)×U(1)-symmetric Lagrangian yields masses to two of the gauge fields while the third remains massless. Thus corresponding to the electroweak gauge bosons known from experiment. A side effect of this solution was the existence
While the previous result necessitates the existence of the higgs particle it does not detail its mass. In section 6 an upper bound for the higgs mass, known as the LQT-bound, was derived by considering partial wave unitarity and the tree level Feynman diagrams contributing to the partial waves. It was shown that the mass cannot exceed 1 TeV without breaking the partial wave unitarity (or perturbation theory).

The method used in section 5 is not the only way for the gauge bosons to acquire mass, it is merely the easiest. There are other methods resulting in more types of higgs particles. The second simplest of these methods is known as the two higgs doublet model (2HDM), which utilises two doublets of complex scalar fields instead of just one as was done in section 5.

Unlike the simple model, 2HDM has the advantage of being compatible with supersymmetry, and could therefore be a better model. However, until experimental evidence for the existence of additional higgs particles has been made, the leading theory will be the easiest higgs model described in section 5.
References


