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Gromov-Witten invariants for $\mathbb{C}P^n$

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Abstract

In this master thesis we define a symplectic manifold (M, ω) and study the Gromov-Witten invariants which in some sense counts the number of pseudoholomorphic curves in M. The main goal is to study these curves and define the Gromov-Witten genus zero three point invariant. We will also do an explicit calculation of the invariant for $\mathbb{C}P^n$ with the Fubini-Study form.

Contents

1 Introduction

The main goal of this thesis is to define the Gromov-Witten genus zero three point invariant and calculate it for symplectic manifold $\mathbb{C}P^n$ with the Fubini-Study form.

A smooth symplectic manifold (M, ω) is a smooth manifold M with the additional structure of a smooth alternating 2-form $\omega \in \Omega^2(M)$ satisfying

1. $d\omega = 0$ (ω is closed),

2. If $\omega_x(v, w) = 0$ for all $w \in T_xM$ then $v = 0$ (ω is non nondegenerate).

For such a symplectic manifold we can add even more structure by choosing a compatible almost complex structure, that is a smoothly varying linear map $J_x : T_xM \to T_xM$ satisfying

1. $J_x^2 = -\mathbb{1}$ (J_x is a complex structure),

2. $J_x^*\omega_x = \omega_x$ (J_x is a symplectomorphism),

3. $\omega_x(v, J_xv) > 0$ for all $v \neq 0$ (J_x is ω_x-tame).

Compatible almost complex structures always exists but they are not unique, however in the end the Gromov-Witten invariant is independent of the choice of almost complex structure J.

When we have chosen a compatible almost complex structure J we can introduce the concept of a parameterized pseudoholomorphic or J-holomorphic sphere in M as a smooth map $u : S^2 \to M$ satisfying the non-linear Cauchy-Riemann equation

$$\overline{\partial}_J(u) := \frac{1}{2}(du + J \circ du \circ j) = 0$$

where j is the complex structure on S^2.

We consider the space of all J-holomorphic spheres in M of a specific homotopy class $[u_0]$ and denote it by $\mathcal{M}([u_0], J)$. Under good conditions this space turns out to be a smooth finite dimensional manifold, however $\mathcal{M}([u_0], J)$ in usually not compact and since we want to integrate over it we consider the Deligne-Mumford compactification $\overline{\mathcal{M}}_{0,3}([u_0], J)$.

We define the three point evaluation map as

$$\begin{aligned} \mathrm{ev} : \mathcal{M}([u_0], J) &\to M^3 \\ u &\mapsto u(0) \times u(1) \times u(\infty). \end{aligned}$$

The Gromov-Witten invariant for the data of a homotopy class $[u_0]$ and $[\beta] \in H^d(M^3)$ where $d = \dim \overline{\mathcal{M}}_{0,3}([u_0], J)$ is defined as

$$\mathrm{Gr}([u_0], [\beta]) = \int_{\overline{\mathcal{M}}_{0,3}([u_0],J)} \mathrm{ev}^*\beta.$$

We will prove that when $[\beta]$ have a Poincaré dual submanifold $S \subseteq M^3$ this is essentially the number (counted with sign) of the J-holomorphic spheres in M such that the evaluation map lands in S.

At the end of this thesis we will do complete calculations of the Gromov-Witten genus zero three point invariant for the symplectic manifold $\mathbb{C}P^n$ with the Fubini-Study form ω_{FS} defined in section 7. We will use the complex structure induced from $\mathbb{C}^{n+1}$ as the compatible almost complex structure J. To do that calculation or counting of the J-holomorphic spheres we explicitly determine the J-holomorphic spheres in $\mathbb{C}P^n$, we show that all J-holomorphic spheres $u : S^2 \to \mathbb{C}P^n$ are given by

$$u([z, w]) = [P_0(z, w), \ldots , P_n(z, w)]$$

where $P_1, \ldots , P_n$ are homogeneous polynomials all of the same degree m. We say that the degree of the J-holomorphic sphere $u : S^2 \to \mathbb{C}P^n$ is the same as the degree of the homogeneous polynomials that define it, then we prove that all J-holomorphic spheres $u : S^2 \to \mathbb{C}P^n$ of the same degree are in the same homotopy class. From the formula above it follows that

$$d = \dim \overline{\mathcal{M}}_{0,3}([u_0], J) = 2(n + 1)(m + 1) - 2.$$

Since the cohomology classes for $\mathbb{C}P^n$ are

$$H^k(\mathbb{C}P^n) \cong \begin{cases} \mathbb{R} & k \text{ even},\ 0 \le k \le 2n \\ 0 & \text{otherwise} \end{cases}$$

and the ring structure is given by $H^*(\mathbb{C}P^n) \cong \mathbb{R}[\omega_{FS}]/\omega_{FS}^n$, it follows from Künneth theorem that a basis for $H^d(\mathbb{C}P^n)^3$ is given by elements of the form

$$[\omega_{FS}]^a \otimes [\omega_{FS}]^b \otimes [\omega_{FS}]^c, \quad \begin{cases} 0 \le a, b, c \le n \\ a + b + c = d/2. \end{cases}$$

The result we end up with is the following. Let $[u_0]$ be the homotopy class of J-holomorphic spheres in $\mathbb{C}P^n$ of degree m. Then when $[\beta] \in H^d(\mathbb{C}P^n)^3$ is of the form $[\beta] = [\omega_{FS}]^a \otimes [\omega_{FS}]^b \otimes [\omega_{FS}]^c$, the Gromov-Witten genus zero three point invariant is

$$\mathrm{Gr}([u_0], [\beta]) = \begin{cases} 1 & \text{if } 0 \le m \le 1 \\ 0 & \text{if } m \ge 2. \end{cases}$$

2 Symplectic vector spaces

In this section we discuss vector spaces equipped with a symplectic form and basic properties of these. This will be the structure that we put on the tangent spaces when we define symplectic manifolds. All vector spaces in this section are assumed to be real and have finite dimension.

Definition 2.1 (Symplectic vector space)**.** *Let V be a vector space. An alternating bilinear 2-form $\omega : V \times V \to \mathbb{R}$ is called symplectic if it is nondegenerate which means that if $\omega(v, w) = 0$ for all $w \in V$ then $v = 0$. The pair (V, ω) is then called a symplectic vector space.*

Definition 2.2 (Symplectic subspace)**.** *Let (V, ω) be a symplectic vector space. A linear subspace $W \subseteq V$ is symplectic if $(W, \omega|_W)$ is a symplectic vector space on its own, this is equivalent to $\omega|_W$ being nondegenerate.*

We will prove later that all symplectic vector spaces have even dimension and is isomorphic (symplectomorphic) to the following.

Example 2.3. Let $V = \mathbb{R}^n \oplus \mathbb{R}^n$ with the basis $e_1, \ldots, e_n, f_1, \ldots f_n$. Then the bilinear form ω_0 defined on basis vectors via

$$\begin{cases} \omega_0(e_i, e_j) = \omega_0(f_i, f_j) = 0 & 1 \leq i, j \leq n \\ \omega_0(e_i, f_j) = -\omega_0(f_j, e_i) = \delta_{ij} & 1 \leq i, j \leq n \end{cases}$$

is symplectic. ★

Proof. That ω_0 is alternating follows since it is alternating on basis vectors. To see nondegeneracy, assume that $v = \sum_{k=1}^n x_k e_k + \sum_{k=1}^n y_k f_k$ and that $\omega_0(v, w) = 0$ for all $w \in V$. This implies that $-y_k = \omega_0(v, e_k) = 0$ and that $x_k = \omega_0(v, f_k) = 0$, therefore $v = 0$. □

Remark 2.4.

1. *The form ω_0 in example 2.3 can also be defined by*

$$\omega_0 = \sum_{k=1}^n e_k^* \wedge f_k^*$$

where e_k^*, f_k^* are the dual vectors in the basis $e_1, \ldots, e_n, f_1, \ldots, f_n$, or in matrix form

$$\omega_0(v, w) = w^T \begin{pmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} v.$$

2. *A basis* $e_1, \dots, e_n, f_1, \dots f_n$ *in a symplectic vector space* (V, ω) *such that* $\omega = \sum_{k=1}^n e_k^* \wedge f_k^*$ *is called a symplectic basis.*

As we in inner product spaces can define the orthogonal complement for subspaces, we can in symplectic vector spaces define the symplectic complement for subspaces.

Definition 2.5 (Symplectic complement)**.** *Let* (V, ω) *be a symplectic vector space and let* $W \subseteq V$ *be a subspace. We define a subspace called the symplectic complement* W^ω *by*

$$W^\omega = \{v \in V \mid \omega(v, w) = 0 \text{ for all } w \in W\}.$$

Unlike as we for the orthogonal complement in inner product spaces always have that $W \oplus W^\perp = V$, it is not in general true in symplectic vector spaces that $W \oplus W^\omega = V$. This is demonstrated in the following example.

Example 2.6. Let $(\mathbb{R}^n \oplus \mathbb{R}^n, \omega_0)$ be the symplectic vector space from example 2.3 and let $W = \mathbb{R}\langle e_1, \dots, e_k, f_1, \dots, f_l\rangle \subseteq \mathbb{R}^n \oplus \mathbb{R}^n$, $0 \leq k, l \leq n$ then the symplectic complement is $W^{\omega_0} = \mathbb{R}\langle e_{l+1}, \dots e_n, f_{k+1}, \dots, f_n\rangle$. ★

Proof. Let $v = \sum_{i=1}^n x_i e_i + \sum_{i=1}^n y_i f_i \in W^{\omega_0}$, then $-y_i = \omega_0(v, e_i) = 0$ for all $1 \leq i \leq k$ and $x_i = \omega_0(v, f_i) = 0$ for all $1 \leq i \leq l$ and therefore $v \in \mathbb{R}\langle e_{l+1}, \dots e_n, f_{k+1}, \dots, f_n\rangle$.

Now assume that $v \in \mathbb{R}\langle e_{l+1}, \dots e_n, f_{k+1}, \dots, f_n\rangle$ then $\omega_0(v, e_i) = 0$ for all $1 \leq i \leq k$ and $\omega_0(v, f_i) = 0$ for all $1 \leq i \leq l$ the conclusion is that $\omega_0(v, w) = 0$ for all $w \in W$ and hence $v \in W^{\omega_0}$. □

Now we will prove that every symplectic vector space have even dimension and have a symplectic basis, see remark 2.4, the proof starts with the following lemma.

Lemma 2.7. *Let* (V, ω) *be a symplectic vector space. For a subspaces* $W \subseteq V$ *we have that* $\dim W + \dim W^\omega = \dim V$ *and* $W^{\omega\omega} = W$*. If the subspace* W *is symplectic then* $V = W \oplus W^\omega$ *and* W^ω *is also symplectic.*

Proof. Define the linear map

$$\begin{aligned} \iota_\omega : & V \to V^* \\ & v \mapsto \omega(v, -) \end{aligned}$$

which is a bijective since ω is nondegenerate. Also define the surjective restriction

$$\begin{aligned} \pi : & V^* \to W^* \\ & v^* \mapsto v^*|_W. \end{aligned}$$

By construction $\mathrm{im}(\pi \circ \iota_\omega) = W^*$ and $\ker(\pi \circ \iota_\omega) = W^\omega$ and hence the dimension theorem gives us that

$$\dim V = \dim(\mathrm{im}(\pi \circ \iota_\omega)) + \dim(\ker(\pi \circ \iota_\omega)) = \dim W + \dim W^\omega.$$

To show that $W^{\omega\omega} = W$ assume that $w \in W$ then $\omega(w, v) = 0$ for all $v \in W^\omega$ and hence $w \in W^{\omega\omega}$. This shows that $W \subseteq W^{\omega\omega}$ and since $\dim(W) = \dim(W^{\omega\omega})$ we conclude that $W^{\omega\omega} = W$.

If W is symplectic then $\omega|_W$ is nondegenerate and hence $W \cap W^\omega = \{0\}$ and since $\dim W + \dim W^\omega = \dim V$ we get $V = W \oplus W^\omega = W^{\omega\omega} \oplus W^\omega$. Since $W^\omega \cap W^{\omega\omega} = \{0\}$, $\omega|_{W^\omega}$ is nondegenerate and therefore W^ω is symplectic. □

Theorem 2.8 (Existence of symplectic basis)**.** *Let (V, ω) be a symplectic vector space of dimension n, then n is even and V has a symplectic basis.*

Proof. The proof is by induction on $2n$, with base case $n = 0$ and $n = 1$. Suppose the result hold for all symplectic vector space of dimension $n - 2$. Choose any $e_1 \in V$, $e_1 \neq 0$ and then since ω is nondegenerate we can find $f_1 \in V$ such that $\omega(e_1, f_1) = 1$. Then $W := \mathbb{R}\langle e_1, f_1 \rangle$ is symplectic and by lemma 2.7 $V = W \oplus W^\omega$ with W^ω also symplectic. By the induction hypothesis it follows $\dim W^\omega = n - 2$ is even and that we have a symplectic basis $e_2, \ldots, e_n, f_2, \ldots, f_n$ of W^ω. Hence n is even and $e_1, \ldots, e_n, f_1, \ldots, f_n$ is a symplectic basis of V. □

Corollary 2.9. *Let V be a $2n$-dimensional vector space and $\omega : V \times V \to \mathbb{R}$ an alternating bilinear 2-form. Then ω is nondegenerate if and only if $\omega^n \neq 0$.*

Proof. First assume that ω is nondegenerate and let $e_1, \ldots, e_n, f_1, \ldots, f_n$ be a symplectic basis then

$$\omega^n = \left(\sum_{k=1}^{n} e_k^* \wedge f_k^* \right)^n = n! \bigwedge_{k=1}^{n} e_k^* \wedge f_k^* \neq 0.$$

Now assume that ω is degenerate and choose $v \neq 0$ such that $\omega(v, w) = 0$ for all $w \in V$. Now choose a basis $v_1, \ldots, v_{2n}$ such that $v_1 = v$. This implies that $\omega^n(v_1, \ldots, v_{2n}) = 0$ and hence $\omega^n = 0$. □

Definition 2.10. *A linear map $\Phi : V \to W$ between the symplectic vector spaces (V, ω_1) and (W, ω_2) is called a symplectomorphism if it preserves the symplectic structure i.e. $\Phi^*\omega_2 = \omega_1$ or $\omega_2(\Phi(v), \Phi(w)) = \omega_1(v, w)$ for all $v, w \in V$. If Φ is a bijection (V, ω_1) and (W, ω_2) are said to be symplectomorphic.*

Theorem 2.11. *Let (V, ω) be a symplectic vector space of dimension $2n$. Then there exists a bijective symplectomorphism between $(\mathbb{R}^n \oplus \mathbb{R}^n, \omega_0)$ with the standard symplectic form from example 2.3 and (V, ω). Therefore all symplectic vector spaces of the same dimension are symplectomorphic.*

Proof. Let $e_1, \dots, e_n, f_1, \dots f_n$ be the standard basis in $\mathbb{R}^n \oplus \mathbb{R}^n$ and pick a symplectic basis $e'_1, \dots, e'_n, f'_1, \dots, f'_n$ of V. The linear map $\Phi : \mathbb{R}^n \oplus \mathbb{R}^n \to V$ defined by $\Phi(e_k) = e'_k,\ \Phi(f_k) = f'_k$, $1 \le k \le n$ on basis vectors is a bijective symplectomorphism. □

3 Complex structures

In this section we discuss another structure to have on vector spaces namely complex structures.

Definition 3.1 (Complex structure)**.** *Let V be a real vector space. A complex structure on V is linear map $J : V \to V$ such that $J^2 = -\mathbb{1}$.*

Example 3.2. Let $V = \mathbb{R}^n \oplus \mathbb{R}^n$ with basis $e_1, \dots, e_n, f_1, \dots, f_n$. Then $J_0 : V \to V$ with matrix $\left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$ is a complex structure. ★

Note that real vector spaces with complex structures and complex vector spaces are the same thing.

If V is a real vector space with complex structure J then V is a complex vector space with scalar multiplication defined as $(a + bi)v = av + bJv$.

Conversely if V is a complex vector space then V is a real vector space with the complex structure defined as $Jv = iv$. In this case if we choose a complex basis $b_1, \dots, b_n$ for V we also have the real basis $e_1 = b_1, \dots, e_n = b_n$ and $f_1 = ib_1, \dots, f_n = ib_n$. A vector $v \in V$ can then be written either in complex coordinates or real coordinates

$$v = \sum_{k=1}^{n} z_k b_k = \sum_{k=1}^{n} x_k e_k + y_k f_k$$

where $z = x + iy,\ x, y \in \mathbb{R}^n$. The matrix of $J = i$ in the real basis is $[J] = \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$.

The above discussion shows that every real vector space V with a complex structure J have even dimension and is isomorphic to the vector space with complex structure in example 3.2, we formulate this as a theorem.

Theorem 3.3. *Let V be a vector space of dimension $2n$ with a complex structure J, then there exists a linear bijective map $\Phi : \mathbb{R}^n \oplus \mathbb{R}^n \to V$ such that $[\Phi^{-1} J \Phi] = J_0 = \left(\begin{smallmatrix} 0 & -\mathbb{1} \\ \mathbb{1} & 0 \end{smallmatrix}\right)$.*

Proof. Define the complex multiplication $(a+bi)v = av + bJv,\ v \in V$ so that V becomes a complex vector space and choose a complex basis $b_1, \ldots, b_n$ of V. Let $e_1, \ldots, e_n, f_1, \ldots f_n$ be the standard basis of $\mathbb{R}^n \oplus \mathbb{R}^n$, then the map $\Phi(e_k) = b_k$, $\Phi(f_k) = Jb_k$, $1 \le k \le n$ satisfies that $[\Phi^{-1} J \Phi] = J_0$. □

Now we define when a complex structure is compatible with a symplectic form, when that's the case this will result in an inner product.

Definition 3.4 (Compatible complex structure)**.** *Let (V, ω) be a symplectic vector space, then a complex structure J is said to be ω-compatible if the following holds:*

1. *$J^*\omega = \omega$ (J is a symplectomorphism),*
2. *$\omega(v, Jv) > 0$ for all $v \neq 0$ (J is ω-tame).*

Lemma 3.5. *Let (V, ω) be a symplectic vector space and J a compatible complex structure. Then $\langle v, w \rangle := \omega(v, Jw)$ defines an inner product on V.*

Proof.

1. $\langle -, - \rangle$ is symmetric: $\langle v, w \rangle = \omega(v, Jw) = \omega(w, Jv) = \langle w, v \rangle$.
2. $\langle -, - \rangle$ is bilinear since J is linear and ω is bilinear.
3. $\langle -, - \rangle$ is positive definite by the taming condition.

□

Example 3.6. Let $V = \mathbb{C}^n$ with the standard complex basis $b_1, \ldots, b_n$ or by the discussion above the real basis $e_1 = b_1, \ldots, e_n = b_n, f_1 = ib_1, \ldots, f_n = ib_n$. Then the standard symplectic form $\omega_0 : V \times V \to \mathbb{R}$ defined by

$$\omega_0 = \frac{i}{2} \sum_{k=1}^{n} b_k^* \wedge \overline{b_k^*} = \sum_{k=1}^{n} e_k^* \wedge f_k^*$$

where $b^* = e^* + if^*$, and $\overline{b^*}$ denotes complex conjugation is compatible with complex structure $J_0 v = iv$. The inner product defined in lemma 3.5 is the same as the standard real inner product $\omega_0(z, iw) = \langle z, w \rangle_{\mathbb{R}}$. ★

Proof. The equality of the representations of ω_0 follows by

$$\frac{i}{2} \sum_{k=1}^{n} b_k^* \wedge \overline{b_k^*} = \frac{i}{2} \sum_{k=1}^{n} (e_k^* + if_k^*) \wedge (e_k^* - if_k^*) = \sum_{k=1}^{n} e_k^* \wedge f_k^*.$$

That the complex structure is ω_0-compatible and that $\omega_0(z, iw) = \langle z, w\rangle_{\mathbb{R}}$ follows from

$$\begin{aligned}\omega_0(iz, iw) &= \frac{i}{2}\sum_{k=1}^{n} b_k^*(iz)\overline{b_k^*}(iw) - b_k^*(iw)\overline{b_k^*}(iz) \\ &= \frac{i}{2}\sum_{k=1}^{n} b_k^*(z)\overline{b_k^*}(w) - b_k^*(w)\overline{b_k^*}(z) \\ &= \omega_0(z, w)\end{aligned}$$

and

$$\begin{aligned}\omega_0(z, iw) &= \sum_{k=1}^{n} e_k^*(z)f_k^*(iw) - e_k^*(iw)f_k^*(z) \\ &= \sum_{k=1}^{n} e_k^*(z)e_k^*(w) + f_k^*(w)f_k^*(z) \\ &= \langle z, w\rangle_{\mathbb{R}}\end{aligned}$$

□

The standard symplectic form relates to the complex inner product in the following way.

Let $(\mathbb{C}^n, \omega_0)$ be the standard symplectic vector space from example 3.6.

Definition 3.7 (Complex inner product)**.** *We define the complex inner product in* $\mathbb{C}^n$ *as*

$$\langle z, w\rangle_{\mathbb{C}} = \sum_{k=1}^{n} z_k\overline{w_k}$$

where z_k, w_k *are the coordinates for* $z, w \in \mathbb{C}^n$ *in the standard basis.*

Definition 3.8 (Complex orthogonal complement)**.** *Let* $W \subseteq \mathbb{C}^n$ *be a real subspace then we define the complex orthogonal complement as*

$$W^{\perp} = \{z \in \mathbb{C} \mid \langle z, w\rangle_{\mathbb{C}} = 0 \text{ for all } w \in W\}.$$

Theorem 3.9. *We have the following useful relation for* $z, w \in \mathbb{C}^n$

$$\langle z, w\rangle_{\mathbb{C}} = \langle z, w\rangle_{\mathbb{R}} - i\omega_0(z, w).$$

Proof. It follows by decomposing z and w into real and imaginary part, let $z = x + iy$, $x, y \in \mathbb{R}^n$ and $w = u + vi$, $u, v \in \mathbb{R}^n$

$$\begin{aligned}\langle z, w\rangle_{\mathbb{C}} &= \sum_{k=1}^{n}(x_k + iy_k)(u_k - iv_k)\\ &= \sum_{k=1}^{n}(x_k u_k + y_k v_k)\\ &\quad - i\sum_{k=1}^{n}(x_k v_k - y_k u_k)\\ &= \langle z, w\rangle_{\mathbb{R}} - i\omega_0(z, w).\end{aligned}$$

□

Corollary 3.10. *The standard symplectic form* $\omega_0 : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{R}$ *is invariant under* S^1*-action i.e.* $\omega_0(az, aw) = \omega_0(z, w)$ *for* $a \in S^1 = \{z \in \mathbb{C},\ |z| = 1\}$.

Proof. $\omega_0(az, aw) = -\mathrm{Im}\langle az, aw\rangle_{\mathbb{C}} = -\mathrm{Im}\langle z, w\rangle_{\mathbb{C}} = \omega_0(z, w)$. □

Theorem 3.11. *Any complex subspace* $W \subseteq \mathbb{C}^n$ *is symplectic and the symplectic complement* W^{ω_0} *is the same as the complex orthogonal complement* $W^{\perp}$.

Proof. We show that $\omega_0|_W$ is nondegenerate. Assume that $z \in W$ and that $\omega_0(z, w) = 0$ for all $w \in W$ then specifically $\omega_0(z, iz) = \langle z, z\rangle_{\mathbb{R}} = 0$, and hence $z = 0$.

By theorem 3.9 we have that $W^{\perp} \subseteq W^{\omega_0}$. Now assume that $z \in W^{\omega_0}$. If $w \in W$ then $\langle z, w\rangle_{\mathbb{C}} = \langle z, w\rangle_{\mathbb{R}} - i\omega_0(z, w) = \omega_0(z, iw) = 0$. This shows that $z \in W^{\perp}$, and hence $W^{\omega_0} = W^{\perp}$. □

4 Symplectic manifolds

In this section we define and give examples of symplectic manifolds which is our main object of study.

Definition 4.1 (Symplectic manifold)**.** *A smooth symplectic manifold is a pair* (M, ω) *where* M *is a smooth finite dimensional manifold and* $\omega \in \Omega^2(M)$ *is a closed symplectic 2-form.*

Remark 4.2. *Since* $\omega_x \in \Omega^2(T_xM)$ *is symplectic it means that* T_xM *is even dimensional and hence the dimension of* M *is also even.*

Remark 4.3. *If (M, ω) is a symplectic manifold then by corollary 2.9 ω^n is an orientation and therefore M is orientable.*

The following theorem due to Jean Gaston Darboux tell us that there are no local invariants for symplectic manifolds. The proof of Darboux theorem can be found in [MS17] chapter 3.

Theorem 4.4 (Darboux)**.** *Let (M, ω) be a symplectic manifold of dimension $2n$ and $x \in M$ any point. Then there exist a chart $(U, \varphi : U \to \mathbb{R}^n \oplus \mathbb{R}^n)$ such that $x \in U$ and $(\varphi^{-1})^*(\omega) = \omega_0 = \sum_{k=1}^n dx_k \wedge dy_k$.*

Now we give some examples of symplectic manifolds.

Example 4.5. The Euclidean space $\mathbb{R}^n \oplus \mathbb{R}^n \ni (x, y)$ with the standard symplectic form $\omega_0 = \sum_{k=1}^n dx_k \wedge dy_k$ is a symplectic manifold. ★

Proof. We already showed in example 2.3 that ω_0 is nondegenerate. The form ω_0 is closed since it is constant. □

Example 4.6. Any orientable 2-dimensional manifold M has a symplectic structure. ★

Proof. Since M is orientable we can let ω be a nowhere vanishing 2-form. Since $\omega_x \neq 0$ by corollary 2.9 ω_x is nondegenerate. We also have that $d\omega = 0$ since $\dim(M) = 2$. □

Example 4.7. The spheres S^{2n} for $2n \geq 4$ does not have a symplectic structure. ★

Proof. Recall that the de Rham cohomology groups for S^{2n} are

$$H^k(S^{2n}) = \begin{cases} \mathbb{R} & \text{when } k = 0 \text{ or } k = 2n \\ 0 & \text{otherwise.} \end{cases}$$

Now assume that ω is a symplectic 2-form and since $d\omega = 0$ we have that $[\omega] \in H^2(S^{2n}) = \{0\}$, this implies that $[\omega^n] = 0$ and hence $\omega^n = d\eta$ for some $\eta \in \Omega^{2n-1}(S^{2n})$. By stokes theorem $\int_{S^{2n}} \omega^n = \int_{\partial S^{2n}} \eta = 0$ which contradicts that ω^n is an orientation, remark 4.3. □

Example 4.8. The real projective plane $\mathbb{R}P^{2n}$ does not have a symplectic structure. ★

Proof. The real projective plane $\mathbb{R}P^{2n}$ is not orientable so by remark 4.3 $\mathbb{R}P^{2n}$ can't have a symplectic structure. To see that $\mathbb{R}P^{2n}$ not is orientable, assume that it is and that $\omega \in \Omega^{2n}(\mathbb{R}P^{2n})$ is an orientation. Consider the projection $\pi : S^{2n} \to \mathbb{R}P^{2n}$ and the reflection $-\mathbb{1} : S^{2n} \to S^{2n}$. Since ω is a volume form on $\mathbb{R}P^{2n}$, $\eta := \pi^*(\omega)$ is a volume form on S^{2n} and since $\pi = \pi \circ -\mathbb{1}$ we have that $\eta = (-\mathbb{1})^*\eta$. But $-\mathbb{1}$ is orientation reversing for even spheres so we get $\int_{S^{2n}} \eta = \int_{S^{2n}} (-\mathbb{1})^*\eta = -\int_{S^{2n}} \eta$ and since $\int_{S^{2n}} \eta \neq 0$ we get a contradiction. □

5 Connections on S^1-bundles

A smooth S^1-bundle is a tuple (E, B, π) usually denoted as $S^1 \to E \xrightarrow{\pi} B$, where E, B are smooth manifolds and $\pi : E \to B$ is a surjective smooth map such that for each point $x \in B$ there exists an open neighborhood $U \subseteq B$ at x such that $\pi^{-1}(U)$ is diffeomorphic to the product $U \times S^1$ where $S^1 = \{a \in \mathbb{C},\ |a| = 1\}$, we should have a commutative diagram

$$\begin{array}{ccc} \pi^{-1}(U) & \xrightarrow{\varphi} & U \times S^1 \\ {\scriptstyle\pi}\downarrow & \swarrow{\scriptstyle\pi_1} & \\ U & & \end{array}$$

where π_1 is projection onto the first coordinate. If we in addition have a smooth fibre preserving right S^1-action $E \times S^1 \to E$ such that

1. If $y \in E$ and $a, b \in S^1$ such that $ya = yb$, then $a = b$ (S^1 acts freely),
2. If $y_1, y_2 \in E_x$ there is a $a \in S^1$ such that $y_1 a = y_2$ (S^1 acts transitivity),

then we call the S^1-bundle principal.

For an S^1-bundle $S^1 \to E \xrightarrow{\pi} B$ it is useful to split the tangent space of the total space into vertical and horizontal vectors $T_yE = H_y \oplus V_y$, that is called a connection. In this section we discuss connections on S^1-bundles in general and some properties of those.

Definition 5.1 (Vertical distribution)**.** *Let $S^1 \to E \xrightarrow{\pi} B$ be an S^1-bundle. The vertical distribution is the subbundle $V := \ker \pi_* \subseteq TE$ only containing vectors tangent to the fibres.*

Remark 5.2. *If the S^1-bundle in definition 5.1 is principal we have a canonical choice of basis in V_y for $y \in E$, namely we can define $\frac{d}{d\theta} \in V_y \subseteq T_yE$ as $\frac{d}{d\theta} = \gamma'(0)$ for $\gamma(\theta) = ye^{i\theta}$.*

Lemma 5.3. *The vertical distribution V for a principal S^1-bundle is trivial $V \cong E \times \mathbb{R}$.*

Proof. Since the S^1-bundle is principal we can define the never-zero section $s : E \to V$ as $s(y) = (y, \frac{d}{d\theta})$. □

Example 5.4. For the canonical principal S^1-bundle

$$S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n,$$

the vertical distribution is $V_z = \mathbb{R}\langle iz \rangle$. ★

Proof. It follows that $V_z = \ker \pi_{z,*} = \mathbb{R}\langle iz \rangle$ since iz is tangent to S^1

$$\frac{d}{d\theta} z e^{i\theta}\Big|_{\theta=0} = iz.$$

□

Definition 5.5 (Connection for S^1-bundles)**.** *Let $S^1 \to E \xrightarrow{\pi} B$ be a principal S^1-bundle. A connection is a smooth choice of horizontal complements H_y such that $T_yE = H_y \oplus V_y$, where we require that $a_* H_y = H_{ya}$ for $a \in S^1$. Tangent vectors in H_y are called horizontal and tangent vectors in V_y are called vertical.*

Theorem 5.6 (Connection 1-form)**.** *Let $S^1 \to E \xrightarrow{\pi} B$ be a principal bundle and $T_yE = H_y \oplus V_y$ a connection. Then there exists a unique 1-form $\alpha \in \Omega^1(E)$ such that $\ker \alpha_y = H_y$ and $\alpha_y(2\pi \frac{d}{d\theta}) = 1$. It follows that $a^*\alpha = \alpha$ for $a \in S^1$.*

Proof. For each point $y \in E$ the unique $\alpha_y : T_yE = H_y \oplus V_y \to \mathbb{R}$ satisfying $\ker \alpha_y = H_y$ and $\alpha_y(2\pi \frac{d}{d\theta}) = 1$ is

$$\alpha_y \left(h + c \cdot 2\pi \frac{d}{d\theta} \right) = c, \quad h \in H_y, c \in \mathbb{R}.$$

Now since $a_* H_y = H_{ya}$ for $a \in S^1$ it follows that $a^*\alpha = \alpha$ since

$$(a^*\alpha)_y \left(h + c \cdot 2\pi \frac{d}{d\theta} \right) = \alpha_{ya} \left(a_* h + c \cdot a_* 2\pi \frac{d}{d\theta} \right) = c = \alpha_y \left(h + c \cdot 2\pi \frac{d}{d\theta} \right)$$

for $h \in H_y$ and $c \in \mathbb{R}$. □

Remark 5.7. *The converse is also true given a 1-form $\alpha \in \Omega^1(E)$ such that $\alpha_y(2\pi \frac{d}{d\theta}) = 1$ and $a^*\alpha = \alpha$ for $a \in S^1$ we get a connection $T_yE = \ker \alpha_y \oplus V_y$.*

Lemma 5.8. *Let $S^1 \to E \xrightarrow{\pi} B$ be a principal S^1-bundle. If $\alpha \in \Omega^1(E)$ and $\beta \in \Omega^1(E)$ are connection 1-forms then $\gamma := (1-t)\alpha + t\beta,\ t \in [0,1]$ is also a connection 1-form.*

Proof. It is easy to check that $\gamma(2\pi \frac{d}{d\theta}) = 1$ and $a^*\gamma = \gamma,\ a \in S^1$. □

Theorem 5.9. *Let $S^1 \to E \xrightarrow{\pi} B$ be a principal S^1-bundle. Then there exists a global connection $T_yE = H_y \oplus V_y$.*

Proof. Let $(U_\alpha)_{\alpha \in A} \subseteq B$ be a covering with the trivializations

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\varphi_\alpha} & U_\alpha \times S^1 \\ {\scriptstyle \pi}\downarrow & \swarrow {\scriptstyle \pi_1} & \\ U_\alpha. & & \end{array}$$

Locally in $\pi^{-1}(U_\alpha) \cong U_\alpha \times S^1$ we have the trivial connection

$$T(\pi^{-1}(U_\alpha)) = TU_\alpha \oplus TS^1$$

with the connection 1-form $\beta_\alpha \in \Omega^1(\pi^{-1}(U_\alpha))$. Let $\rho_\alpha : U_\alpha \to \mathbb{R}$ be a partition of unity subordinate to the cover, then we can define a global connection

$$\beta_y = \sum_{\substack{\alpha \in A \\ \pi(y) \in U_\alpha}} \rho_\alpha(\pi(y))\beta_{\alpha,y}.$$

□

Example 5.10. Let $S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$ be the complex projective space then $T_zS^{2n+1} = z^\perp \oplus \mathbb{R}\langle iz \rangle$ where $z^\perp$ is the complex orthogonal complement is a connection and the corresponding 1-form $\alpha \in \Omega^1(S^{2n+1})$ is

$$\alpha_z = \frac{1}{2\pi}\omega_0(z, -) = \frac{1}{2\pi}\sum_{k=1}^{n+1}(x_k dy_k - y_k dx_k) = \frac{i}{4\pi}\sum_{k=1}^{n+1}(z_k d\overline{z_k} - \overline{z_k} dz_k)$$

where $z = x + iy$ and $dz = dx + idy$. ★

Proof. To see that $T_zS^{2n+1} = z^\perp \oplus \mathbb{R}\langle iz \rangle$ actually is a connection note that $a_*(z^\perp) = a(z^\perp) = (az)^\perp$ for $a \in S^1$. The equality of the expressions follows by

$$\begin{aligned}\frac{1}{2\pi}\omega_0(z,-) &= \frac{1}{2\pi}\sum_{k=1}^{n+1} dx_k \wedge dy_k(z,-)\\ &= \frac{1}{2\pi}\sum_{k=1}^{n+1}(dx_k(z)dy_k - dy_k(z)dx_k)\\ &= \frac{1}{2\pi}\sum_{k=1}^{n+1}(x_k dy_k - y_k dx_k)\end{aligned}$$

and

$$\begin{aligned}&\frac{i}{4\pi}\sum_{k=1}^{n+1}(z_k d\overline{z_k} - \overline{z_k}dz_k)\\ =&\frac{i}{4\pi}\sum_{k=1}^{n+1}((x_k+iy_k)(dx_k - idy_k) - (x_k - iy_k)(dx_k + idy_k))\\ =&\frac{i}{4\pi}\sum_{k=1}^{n+1}(-2ix_k dy_k + 2iy_k dx_k)\\ =&\frac{1}{2\pi}\sum_{k=1}^{n+1}(x_k dy_k - y_k dx_k)\,.\end{aligned}$$

We see that $\ker \alpha_z = z^\perp$ since if $w \in z^\perp$ we have

$$\alpha_z(w) = \frac{1}{2\pi}\omega_0(z,w) = -\frac{1}{2\pi}\mathrm{Im}\langle z,w\rangle_{\mathbb{C}} = 0.$$

Since $\frac{d}{d\theta} = iz$ it follows that

$$\alpha\left(2\pi\frac{d}{d\theta}\right) = \omega_0(z,iz) = \langle z,z\rangle_{\mathbb{R}} = 1.$$

By theorem 5.6 α is S^1 invariant i.e. $a^*\alpha = \alpha$ for $a \in S^1$. □

6 The first Chern class for S^1-bundles

Let $S^1 \to E \xrightarrow{\pi} B$ be a principal S^1-bundle, the first Chern class of E is defined as $c_1(E) = [s^* d\alpha] \in H^2(B)$ where $s : B \supseteq U \to E$ are local sections (local sections since a global sections might not exist) and $\alpha \in \Omega^1(E)$ is a connection 1-form. In this section we show that this construction independent of s and α.

Theorem 6.1. *Let $S^1 \to E \xrightarrow{\pi} B$ be a principal S^1-bundle and $\omega \in \Omega^k(E)$ a differential k-form such that $\omega(v_1, \dots, v_k) = 0$ if one of the v_i's is in $V = \ker \pi_*$ and S^1-invariant i.e. $a^*\omega = \omega$ or equivalently*

$$\omega_{ya}(a_* v_1, \dots, a_* v_k) = \omega_y(v_1, \dots, v_k)$$

for $a \in S^1$. Then the pullback of ω through a local section $s : B \supseteq U \to E$ is independent of s.

Proof. Let $x \in B$ and let $s, t : B \supseteq U \to E$ be local sections such that $x \in U$ and let $a \in S^1$ be such that $t(x) = s(x)a$. We have a commutative diagram

$$\begin{array}{ccc} T_{s(x)}E & \xrightarrow{\pi_{s(x),*}} & T_xB \\ {\scriptstyle a_*}\downarrow & \nearrow_{\pi_{t(x),*}} & \\ T_{t(x)}E & & \end{array}$$

which implies that $t_{x,*}(v) - a_* s_{x,*}(v) \in V_{t(x)}$ for all $v \in T_xB$ that is since

$$\begin{aligned} \pi_{t(x),*}(t_{x,*}(v) - a_* s_{x,*}(v)) &= \pi_{t(x),*} t_{x,*}(v) - \pi_{t(x),*} a_* s_{x,*}(v) \\ &= \pi_{t(x),*} t_{x,*}(v) - \pi_{s(x),*} s_{x,*}(v) \\ &= \mathbb{1}v - \mathbb{1}v = 0. \end{aligned}$$

Therefore

$$\begin{aligned} (t^*\omega)_x(v_1, \dots, v_k) &= \omega_{t(x)}(t_{x,*}(v_1), \dots, t_{x,*}(v_k)) \\ &= \omega_{s(x)a}(a_* s_{x,*}(v_1), \dots, a_* s_{x,*}(v_k)) \\ &= \omega_{s(x)}(s_{x,*}(v_1), \dots, s_{x,*}(v_k)) \\ &= (s^*\omega)_x(v_1, \dots, v_k) \end{aligned}$$

and hence $t^*\omega = s^*\omega$. $\square$

If we in theorem 6.1 take a family of local sections $s_\alpha : B \supseteq U_\alpha \to E$ such that $\cup U_\alpha = B$, then the pullback of ω is independent of local sections and hence $s_\alpha^*(\omega) \in \Omega^k(B)$ is globally defined.

Theorem 6.2. *Let $S^1 \to E \xrightarrow{\pi} B$ be a principal S^1-bundle with a connection $T_yE = H_y \oplus V_y$ represented by the 1-form $\alpha \in \Omega^1(E)$. Then the pullback of $d\alpha$ through a local section $s : B \supseteq U \to E$ is independent of s.*

Proof. First note that $d\alpha$ is S^1-invariant since $a^* d\alpha = d(a^*\alpha) = d\alpha$ for $a \in S^1$. Let $y \in E$ and take a trivialisation $(U \subseteq B, \varphi : U \to \mathbb{R}^n)$ where

$n = \dim(B)$, $\pi(y) \in U$ and $\pi^{-1}(U) \cong U \times S^1$. Since $\alpha(2\pi \frac{d}{d\theta}) = 1$ we can in the trivialisation write

$$\alpha = \sum_{k=1}^{n} f_k(x,\theta)dx_k + \frac{1}{2\pi}d\theta.$$

Taking the exterior derivative gives

$$\begin{aligned} d\alpha &= \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial f_k}{\partial x_l}(x,\theta)dx_l \wedge dx_k + \sum_{k=1}^{n} \frac{\partial f_k}{\partial \theta}(x,\theta)d\theta \wedge dx_k + \frac{1}{2\pi}dd\theta \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n} \frac{\partial f_k}{\partial x_l}(x,\theta)dx_l \wedge dx_k \end{aligned}$$

where the second term vanish since α is S^1-invariant, therefore $d\alpha(v,w) = 0$ if $v \in V$ or $w \in V$. Using theorem 6.1 the pullback of $d\alpha$ is independent of the local section s. □

Definition 6.3 (First Chern class for S^1-bundles)**.** *Let $S^1 \to E \xrightarrow{\pi} B$ be a principal S^1-bundle. Let $T_yE = H_y \oplus V_y$ be a connection represented by $\alpha \in \Omega^1(E)$ and let $s : B \supseteq U \to E$ be local sections. Then by theorem 6.2 the pullback $s^*d\alpha \in \Omega^2(B)$ is independent of the local sections and hence globally defined. Also $ds^*d\alpha = s^*dd\alpha = 0$. We define the first Chern class as $c_1(E) = [s^*d\alpha] \in H^2(B)$.*

Theorem 6.4. *The first Chern class $c_1(E)$ of an S^1-bundle is independent of the connection.*

Proof. Let $T_yE = H_y \oplus V_y$ and $T_yE = H'_y \oplus V_y$ be two connections represented by $\alpha \in \Omega^1(E)$ and $\beta \in \Omega^1(E)$.

Since $\alpha - \beta$ is S^1 invariant and $(\alpha - \beta)(v) = 0$ for $v \in V$ by theorem 6.1 the pullback $s^*(\alpha - \beta) \in \Omega^1(B)$ through local sections $s : B \supseteq U \to E$ is globally defined and independent of the local sections. This implies that

$$ds^*(\alpha - \beta) = s^*d\alpha - s^*d\beta$$

and hence $c_1(E) = [s^*d\alpha] = [s^*d\beta]$. □

7 The Fubini-Study form on $\mathbb{C}P^n$

The symplectic manifold that we are going to calculate the Gromov-Witten invariant for is $(\mathbb{C}P^n, \omega_{FS})$. In this section we define the Fubini-Study form $\omega_{FS} \in \Omega^2(\mathbb{C}P^n)$.

Let $S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$ be the canonical principal S^1-bundle with base the complex projective space and let

$$\alpha = \frac{1}{2\pi} \sum_{k=1}^{n+1} (x_k dy_k - y_k dx_k)$$

be the connection 1-form defined in example 5.10, taking the exterior derivative gives $d\alpha = \frac{1}{\pi}\omega_0$.

If $s : \mathbb{C}P^n \supseteq U \to S^{2n+1}$ are local sections then we define the Fubini-Study form as

$$\omega_{FS} := s^* d\alpha = \frac{1}{\pi} s^* \omega_0 \in \Omega^2(\mathbb{C}P^n)$$

which by theorem 6.2 is globally defined independent of s. Note that the cohomology class of Fubini-Study form is the same as the first Chern class $c_1(S^{2n+1}) = [\omega_{FS}] \in H^2(\mathbb{C}P^n)$.

To see that $(\mathbb{C}P^n, \omega_{FS})$ is a symplectic manifold we also have to show that ω_{FS} is nondegenerate. To do so we let $[z] \in \mathbb{C}P^n$, we can assume that $s([z]) = z$. Define the linear map

$$\Phi_{[z]} := \pi_{z^\perp} \circ s_{[z],*} : T_{[z]}\mathbb{C}P^n \to z^\perp \subseteq z^\perp \oplus \mathbb{R}\langle iz \rangle = T_z S^{2n+1}.$$

The map $\Phi_{[z]}$ is bijective since $\dim(T_{[z]}\mathbb{C}P^n) = 2n = \dim z^\perp$ and we have a left inverse $\pi_{z,*}$

$$\pi_{z,*} \circ (\pi_{z^\perp} \circ s_{[z],*}) = (\pi_{z,*} \circ \pi_{z^\perp}) \circ s_{[z],*} = \mathrm{id}.$$

We have the equality

$$\omega_{FS,[z]}(v, w) = \frac{1}{\pi}\omega_{0,z}(s_{[z],*}v, s_{[z],*}w) = \frac{1}{\pi}\omega_{0,z}(\Phi_{[z]}v, \Phi_{[z]}w)$$

and since $z^\perp$ is a complex and hence by lemma 3.11 a symplectic subspace we see that if $\omega_{FS,[z]}(v, w) = 0$ for all $w \in T_{[z]}\mathbb{C}P^n$ then $v = 0$.

8 Hilbert Manifolds

In this section we will define and give examples of Hilbert manifolds which is a type of an infinite dimensional manifolds suited for generalizing the implicit function theorem. It will turn out that the space of smooth functions between smooth manifolds completed in a certain Hilbert norm will have the structure of an infinite dimensional Hilbert manifold. First we define Hilbert spaces.

Definition 8.1. *A Hilbert space is a vector space H equipped with an inner product $\langle -, - \rangle : H \times H \to \mathbb{R}$ such that the induced metric $\|f\| := \langle f, f \rangle^{1/2}$ is complete.*

Example 8.2. The space

$$\mathcal{L}^2(\mathbb{R}^n, \mathbb{R}^m) = \left\{ f : \mathbb{R}^n \to \mathbb{R}^m, \ \int_{\mathbb{R}^n} \|f\|^2 d\mu < \infty \right\} / \sim$$

of square integrable functions where we mod out with the equivalence relation where $f \sim g$ if f and g are equal almost everywhere, equipped with the inner product $\langle f, g \rangle := \int_{\mathbb{R}^n} \langle f(x), g(x) \rangle d\mu$ is a Hilbert space. ★

Example 8.3. Let M and N be two smooth Riemannian manifolds and $f : M \to N$ a fixed smooth map. Then the set of $W^{k,2}$ sections

$$W^{k,2}(M, f^*TN) = \{s : M \to f^*TN, \ \sum_{i=0}^{k} \int_M \|s^{(i)}\| d\mu < \infty\} / \sim$$

where k is large enough so that all maps in $W^{k,2}(M, f^*TN)$ are continuous ($k = 2$ is good enough when $\dim M = 2$), $\|s^{(i)}\|$ is defined using the Levi-Civita connection on TM and TN, we mod out with the equivalence relation where $s \sim t$ if s and t are equal almost everywhere, with the inner product

$$\langle s, t \rangle := \sum_{i=0}^{k} \int_M \langle s^{(i)}(x), t^{(i)}(x) \rangle d\mu$$

is a Hilbert space. ★

Next we define what it means for a map between Hilbert spaces to be smooth.

Definition 8.4. *Let H_1, H_2 be Hilbert spaces and $\Phi : H_1 \supseteq U \to H_2$ be a map and U open. The Gateaux derivative of Φ with respect to $f \in H_1$ is defined by*

$$\frac{\partial \Phi}{\partial f} : U \to H_2$$
$$g \mapsto \lim_{t \to 0} \frac{\Phi(g + tf) - \Phi(g)}{t}$$

if the limit exists. If we for any sequence $f_1, \dots, f_k \in H_1$ have that the derivative $\frac{\partial}{\partial f_k} \cdots \frac{\partial}{\partial f_1} \Phi$ exists and is continuous, we say that Φ is smooth.

Now we are in position to define a Hilbert manifold which locally is just a Hilbert space. The definition is similar to the definition of finite dimensional smooth manifolds.

Definition 8.5. *A Hilbert manifold modelled over the Hilbert space H is separable Hausdorff topological space M with an atlas*

$$(U_\alpha, \varphi_\alpha : U_\alpha \to \varphi_\alpha(U_\alpha) \subseteq H)_{\alpha \in A}$$

where U_α are open, $\cup_{\alpha \in A} U_\alpha = M$ and φ_α are homeomorphisms. We also require the transition functions

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$

to be smooth.

The type of Hilbert manifolds that we are specially interested in are function spaces. The construction is as follows. Let M and N be smooth compact Riemann manifolds. We will show that the space

$$W^{k,2}(M, N) \cap [f_0]$$

where k is large enough so that $W^{k,2}(M, N)$ only consists of continuous maps and $[f_0]$ is a homotopy class, is a Hilbert manifold modelled over the Hilbert space $W^{k,2}(M, f_0^*TN)$ of sections. By Whitney embedding theorem the manifold N can be embedded as a smooth submanifold in $\mathbb{R}^d$ for some d. Take a tubular neighborhood $N \subseteq T \subseteq \mathbb{R}^d$. Since T is diffeomorphic to the normal bundle of N we get a projection map $\pi : T \to N$. Now for a fixed $y \in N$ we have the inclusion $\iota_y : T_yN \to \mathbb{R}^d$, $\iota_y(0) = y$ and there exists an $\epsilon > 0$ such that the map

$$e_y := \pi \circ \iota_y : T_yN \supseteq B_\epsilon \to N$$

is a well defined and injective.

Let $f \in W^{k,2}(M, N) \cap [f_0]$ where we want to define a chart, take a global $\epsilon > 0$ such that all the maps $e_{f(x)} : T_{f(x)}N \supseteq B_\epsilon \to N$ defined. By example 8.3 the space $W^{k,2}(M, f^*TN)$ of sections is a Hilbert space and since f and f_0 are homotopic we have that $W^{k,2}(M, f^*TN) \cong W^{k,2}(M, f_0^*TN)$ are smoothly isomorphic, see [BT13] chapter 6.

Let $U_f \subseteq W^{k,2}(M, f^*TN)$ be the subset of sections such that $\|s(x)\| < \epsilon$. Then U_f is open and we can define a chart at f by

$$\varphi^{-1} : U_f \to W^{k,2}(M, N)$$
$$\varphi^{-1}(s)(x) = e_{f(x)}(s(x)).$$

It can be shown that the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are smooth and that $W^{k,2}(M, N) \cap [f_0]$ is complete.

Definition 8.6. *Let $\Phi : M \to N$ be a map between Hilbert manifolds, we say that Φ is smooth at the point $f \in M$ if there exists a chart $(U \subseteq M, \varphi)$ at f and a chart $(V \subseteq N, \psi)$ at $\Phi(f)$ such that*

$$\psi \circ \Phi \circ \varphi^{-1} : \varphi(U \cap \Phi^{-1}(V)) \to \psi(V)$$

is smooth. The map Φ is called smooth if it is smooth at all points $f \in M$.

9 Inverse and implicit function theorems

The inverse and implicit function theorem for functions $f : \mathbb{R}^m \to \mathbb{R}^n$ can be generalised to arbitrary smooth finite dimensional manifolds and in that setting the theorems becomes very natural and easy to formulate. The theorems can also be generalized even further to infinite dimensional Hilbert manifolds. First the finite dimensional case.

Let M and N be smooth manifolds of dimensions m and n and $f : M \to N$ be a smooth map.

Definition 9.1.

1. *A point $x \in M$ is called a regular point if $f_{x,*} : T_x M \to T_{f(x)} N$ is surjective.*

2. *A point $y \in N$ is called a regular value if all $x \in f^{-1}(y)$ are regular points.*

Theorem 9.2 (Inverse function theorem)**.** *Let M and N both have the same dimension and $x \in M$ be a regular point. Then there exists an open neighborhood $U \subseteq M$ of x such that $f|_U : U \to f(U)$ is a diffeomorphism.*

Theorem 9.3 (Implicit function theorem)**.** *Let $y \in N$ a regular value, then $f^{-1}(y) \subseteq M$ is a smooth submanifold of dimension $m - n$.*

Proof. Let us prove this using the inverse function theorem. Let $x \in f^{-1}(y)$. Take a chart $(U \subseteq M, \varphi : U \to \mathbb{R}^m)$ at x and a chart $(V \subseteq N, \psi : V \to \mathbb{R}^n)$ at y such that $\varphi(U) \subseteq V$. We can assume that $\varphi(x) = 0$ and $\psi(y) = 0$. Introduce the notation $\widetilde{f} = \psi \circ f \circ \varphi^{-1}$ and notice that $\widetilde{f}(0) = 0$. Define the function

$$\Phi : \varphi(U) \to \psi(V) \oplus \ker \widetilde{f}_{0,*}$$
$$\Phi = \widetilde{f} \oplus \pi_{\ker \widetilde{f}_{0,*}}$$

where π is the orthogonal projection. Since x is a regular point have that $\dim(\ker \widetilde{f}_{0,*}) = m - n$. We get that $\Phi_{0,*} = \widetilde{f}_{0,*} \oplus \pi_{\ker \widetilde{f}_{0,*}}$ is an isomorphism since it is injective and the dimensions match up. By the inverse function theorem there exists an open neighborhood $\varphi(W) \subseteq \varphi(U) \subseteq \mathbb{R}^m$ with $0 \in \varphi(W)$ such that $\Phi : \varphi(W) \to \Phi(\varphi(W))$ is an diffeomorphism. Now we can define a chart $(W \cap f^{-1}(y), \theta)$ at $x \in f^{-1}(y)$ by

$$\begin{aligned}
&\theta : W \cap f^{-1}(y) \to \{0\} \oplus \ker \widetilde{f}_{0,*} \\
&\theta = \Phi \circ \varphi
\end{aligned}$$

and since Φ and φ are diffeomorphisms θ is also a diffeomorphism, everything is smooth so the transition functions $\theta_\beta \circ \theta_\alpha^{-1} = \Phi_\beta \circ \varphi_\beta \circ \varphi_\alpha \circ \Phi_\alpha^{-1}$ are smooth, this shows that $f^{-1}(y)$ is a smooth manifold.

Take the chart $(W \cap f^{-1}(y), \theta)$ of $f^{-1}(y)$ at x and the chart $(W, \Phi \circ \varphi)$ of M at x, then the inclusion $\iota : f^{-1}(y) \hookrightarrow M$ in those charts is just the identity so the derivative is obviously injective, this shows that $f^{-1}(y)$ is a smooth submanifold. □

If M and N are orientable we can give an orientation to $f^{-1}(y)$ in the following way. Let $x \in f^{-1}(y)$ and let $w_1, \dots, w_n \in T_x M$ be tangent vectors such that $f_{x,*}(w_1), \dots, f_{x,*}(w_n) \in T_y N$ form an positively oriented basis. We define a basis $v_1, \dots, v_{m-n} \in T_x f^{-1}(y) \cong \ker f_{x,*} \subseteq T_x M$ to be positively oriented if and only if $w_1, \dots, w_n, v_1, \dots, v_{m-n} \in T_x M$ is positively oriented.

Corollary 9.4. *Assume that M and N are orientable and that $S \subseteq N$ is an oriented smooth submanifold of dimension k that is intersected transversally by f. Then $f^{-1}(S)$ is an oriented smooth submanifold of dimension $m-n+k$.*

Proof. Take a tubular neighborhood $S \subseteq T$ diffeomorphic to the normal bundle $T \cong NS$. Let $x_0 \in f^{-1}(S)$ be arbitrarily and take an open neighborhood $U \subseteq S$ such that $f(x_0) \in U$ and $\pi^{-1}(U) \cong U \times \mathbb{R}^{n-k}$ is trivial. Let $\pi_2 : U \times \mathbb{R}^{n-k} \to \mathbb{R}^{n-k}$ be the projection and define

$$\begin{aligned}
g :& f^{-1}(U \times \mathbb{R}^{n-k}) \to \mathbb{R}^{n-k} \\
& x \mapsto \pi_2(f(x))
\end{aligned}$$

Since f is transversal to S we have for all $x \in f^{-1}(U) = g^{-1}(0)$ that $f_*(T_x M) + T_{f(x)} S = \mathbb{R}^n$ and $T_{f(x)} S = \ker \pi_{2,*}$ and hence 0 is a regular value for g and therefore $f^{-1}(U) = g^{-1}(0) \subseteq M$ is an oriented smooth submanifold of dimension $m - n + k$. This is true for all $x_0 \in f^{-1}(S)$ and hence $f^{-1}(S)$ is an oriented smooth submanifold of dimension $m - n + k$. □

Corollary 9.5. *Let $\mathbb{R}^n \to E^{m+n} \xrightarrow{\pi} B^m$ be an oriented vector bundle over an oriented base, then the zero-section is an oriented smooth submanifold in E of dimension m. Let $s : B \to E$ be a section transversal to the zero section. Then $s^{-1}(0)$ is an oriented smooth submanifold of E of dimension $m - n$.*

Now the infinite dimensional case. Let M and N be Hilbert manifolds and $\Phi : M \to N$ be a smooth map.

Theorem 9.6 (Inverse function theorem)**.** *Assume that $f \in M$ is a point such that $\Phi_{f,*}$ is an isomorphism. Then there exists an open neighborhood $U \subseteq M$ of f such that $\Phi|_U : U \to \Phi(U)$ is a diffeomorphism.*

Theorem 9.7 (Implicit function theorem)**.** *Assume that $g \in N$ is a point such that $\Phi_{f,*}$ is a surjective Fredholm operator (Fredholm meaning that $\Phi_{f,*}$ is bounded, have a closed image and finite dimensional kernel and cokernel) for all $f \in \Phi^{-1}(g)$. Then $\Phi^{-1}(g) \subseteq M$ is a smooth submanifold where the dimension of each component might vary, the dimension of the component containing $f \in \Phi^{-1}(g)$ is the same as the dimension as the kernel of $\Phi_{f,*}$.*

10 J-holomorphic spheres

Let (S^2, j) be the complex manifold $S^2 \cong \mathbb{C}P^1$ with the complex structure $j : TS^2 \to TS^2$ and let (M, ω, J) be a smooth symplectic manifold of dimension $2m$ with a compatible almost complex structure $J : TM \to TM$.

Definition 10.1. *A parameterized J-holomorphic sphere u in M is a smooth map $u : S^2 \to M$ such that the following diagram commutes*

$$\begin{array}{ccc} T_zS^2 & \xrightarrow{du_z} & T_{u(z)}M \\ {\scriptstyle j_z}\downarrow & & \downarrow{\scriptstyle J_{u(z)}} \\ T_zS^2 & \xrightarrow{du_z} & T_{u(z)}M, \end{array}$$

i.e. $J_{u(z)} \circ du_z = du_z \circ j_z$, or du_z is $J_{u(z)}$-linear for all $z \in S^2$.

Now we define an infinite dimensional vector bundle $\varepsilon \xrightarrow{\pi} W^{2,2}(S^2, M)$ where the fibre ε_u over u is the space of J-antilinear 1-forms

$$\varepsilon_u := W^{1,2}\Omega^{0,1}(S^2, u^*TM) := \{\alpha \in W^{1,2}\Omega^1(S^2, u^*TM) \mid \alpha \circ j = -J \circ \alpha\}.$$

Consider the operator $\overline{\partial}_J$ which we define as

$$\begin{aligned} \overline{\partial}_J &: W^{2,2}(S^2, M) \to \varepsilon_u \\ u &\mapsto \frac{1}{2}(du + J \circ du \circ j) \end{aligned}$$

and indeed $\overline{\partial}_J(u)$ is J-antilinear since

$$\overline{\partial}_J(u) \circ j = \frac{1}{2}(du \circ j - J \circ du) = -J \circ \overline{\partial}_J(u).$$

The operator $\overline{\partial}_J$ measures in some sense how far du is from being J-linear, namely $\overline{\partial}_J(u)_z = 0$ if and only if du_z is J-linear, or in other words $\overline{\partial}_J(u) = 0$ exactly when u is J-holomorphic.

Now we define the section

$$\begin{aligned} \mathcal{S} :& W^{2,2}(S^2, M) \to \varepsilon \\ & u \mapsto (u, \overline{\partial}_J(u)), \end{aligned}$$

it follows that $\mathcal{S}(u) = 0$ if and only if u is J-holomorphic.

The space $W^{2,2}(S^2, M)$ is not connected so in what follows we have to choose a homotopy class. Let $[u_0]$ denote the space of all functions of the same homotopy class as $u_0 : S^2 \to M$.

Definition 10.2. *We define the space of all parameterized J-holomorphic spheres in M of homotopy class $[u_0]$ as $\mathcal{M}([u_0], J) := \mathcal{S}^{-1}(0) \cap [u_0]$.*

In what follows next we are going to take a closer look at the section $\mathcal{S}$ and the space $\mathcal{M}([u_0], J)$. We would like $\mathcal{M}([u_0], J)$ to be a compact orientable finite dimensional smooth manifold but in general $\mathcal{M}([u_0], J)$ is neither compact or a manifold.

Let us start with studying the linearization $(d\mathcal{S})_u$ of $\mathcal{S}$ at points where $\mathcal{S}(u) = 0$ with the goal of finding some formulas in local coordinates. In that case we have that

$$(d\mathcal{S})_u : T_u W^{2,2}(S^2, M) \to T_{(u,0)}\varepsilon$$

where $T_u W^{2,2}(S^2, M) \cong W^{2,2}\Omega^0(S^2, u^*TM)$ is the space of vector fields along u and $T_{(u,0)}\varepsilon$ splits as

$$T_{(u,0)}\varepsilon \cong T_u W^{2,2}(S^2, M) \oplus T_0\varepsilon_u \cong W^{2,2}\Omega^0(S^2, u^*TM) \oplus \varepsilon_u.$$

The first component of $(d\mathcal{S})_u$ is just the identity so we only need to study the second component i.e. $(d\mathcal{S})_u$ composed with the projection $\pi_u : T_{(u,0)}\varepsilon \to \varepsilon_u$, that is called the vertical differential of the section $\mathcal{S}$ at u with notation

$$D_u := \pi_u \circ (d\mathcal{S})_u.$$

In local conformal coordinates $(U_\alpha \subseteq S^2, \varphi_\alpha : U_\alpha \to \mathbb{C} = \{s+ti \mid s,t \in \mathbb{R}\})$ on S^2 i.e. $\varphi_{\alpha,*} \circ j = i \circ \varphi_{\alpha,*}$, and local coordinates $(V_\alpha \subseteq M, \psi_\alpha : V_\alpha \to \mathbb{R}^{2n})$

on M. We can arrange so that $u(U_\alpha) \subseteq V_\alpha$ for all α and we denote the local version of u with $u_\alpha := \psi_\alpha \circ u \circ \varphi_\alpha^{-1}$ and the local version of J with $J_\alpha := \psi_{\alpha,*} \circ J \circ \psi_{\alpha,*}^{-1}$. The formula for the local version

$$\overline{\partial}_{J_\alpha}(u_\alpha) \in W^{1,2}\Omega^{0,1}(\varphi(U_\alpha), u_\alpha^* T\psi(V_\alpha)) \subseteq W^{1,2}(\varphi(U_\alpha) \times \mathbb{C}, \mathbb{R}^{2n})$$

of $\overline{\partial}_J(u)$ is given by

$$\begin{aligned}
\overline{\partial}_{J_\alpha}(u_\alpha) &= \psi_{\alpha,*} \circ \overline{\partial}_J(u) \circ \varphi_{\alpha,*}^{-1} \\
&= \psi_{\alpha,*} \circ \frac{1}{2}(d(\psi_\alpha^{-1} \circ u_\alpha \circ \varphi_\alpha) + J(u) \circ d(\psi_\alpha^{-1} \circ u_\alpha \circ \varphi_\alpha) \circ j) \circ \varphi_{\alpha,*}^{-1} \\
&= \frac{1}{2}(du_\alpha + J_\alpha(u_\alpha) \circ du_\alpha \circ i) \\
&= \frac{1}{2}(\partial_s u_\alpha ds + \partial_t u_\alpha dt + J_\alpha(u_\alpha) \circ \partial_s u_\alpha ds \circ i + J_\alpha(u_\alpha) \circ \partial_t u_\alpha dt \circ i) \\
&= \frac{1}{2}(\partial_s u_\alpha + J_\alpha(u_\alpha) \circ \partial_t u_\alpha) ds + \frac{1}{2}(\partial_t u_\alpha - J_\alpha(u_\alpha) \circ \partial_s u_\alpha) dt.
\end{aligned}$$

Let $\xi \in W^{2,2}\Omega^0(S^2, u^*TM) = T_u W^{2,2}(S^2, M)$ be a tangent vector at the point u, locally

$$\xi_\alpha \in W^{2,2}\Omega^0(\varphi(U_\alpha), u_\alpha^* T\psi(V_\alpha)) \cong W^{2,2}(\varphi(U_\alpha), \mathbb{R}^{2n}).$$

The vertical differential $D_u\xi \in \varepsilon_u \cong W^{1,2}\Omega^{0,1}(S^2, u^*TM)$, locally

$$D_{u_\alpha}\xi_\alpha \in W^{1,2}\Omega^{0,1}(\varphi(U_\alpha), u_\alpha^* T\psi(V_\alpha)) \subseteq W^{1,2}(\varphi(U_\alpha) \times \mathbb{C}, \mathbb{R}^{2n}).$$

A calculation in the local coordinates and dropping the index α for readability yields the formula

$$\begin{aligned}
D_u\xi &= \lim_{t \to 0} \frac{\overline{\partial}_J(u + t\xi) - \overline{\partial}_J(u)}{t} \\
&= \lim_{t \to 0} \frac{1}{2t}(\partial_s(u + t\xi) + J(u + t\xi) \circ \partial_t(u + t\xi) - \partial_s u - J(u) \circ \partial_t u) ds \\
&\quad + \lim_{t \to 0} \frac{1}{2t}(\partial_t(u + t\xi) - J(u + t\xi) \circ \partial_s(u + t\xi) - \partial_t u + J(u) \circ \partial_s u) dt \\
&= \lim_{t \to 0} \frac{1}{2}\left(\partial_s\xi + J(u + t\xi) \circ \partial_t\xi + \frac{J(u + t\xi) - J(u)}{t} \circ \partial_t u\right) ds \\
&\quad + \lim_{t \to 0} \frac{1}{2}\left(\partial_t\xi - J(u + t\xi) \circ \partial_s\xi - \frac{J(u + t\xi) - J(u)}{t} \circ \partial_s u\right) dt \\
&= \frac{1}{2}(\partial_s\xi + J(u) \circ \partial_t\xi + \partial_\xi J(u) \circ \partial_t u) ds \\
&\quad + \frac{1}{2}(\partial_t\xi - J(u) \circ \partial_s\xi - \partial_\xi J(u) \circ \partial_s u) dt.
\end{aligned}$$

If we use that u is J-holomorphic and the fact that J anticommutes with $\partial_\xi J$ since $(\partial_\xi J)J + J(\partial_\xi J) = \partial_\xi J^2 = 0$, we get the global formula

$$D_u\xi = \overline{\partial}_J\xi - \frac{1}{2}(J(u)\circ\partial_\xi J(u))\circ\partial_J(u)$$

where $\partial_J(u) = \frac{1}{2}(du - J(u)\circ du\circ j)$.

It can be shown that the linearization D_u is a Fredholm operator with Fredholm index $2\int_{S^2} c_1(u^*TM) + 2m$, note that D_u when $\overline{\partial}_J(u) = 0$ is just Fredholm and not always surjective so the section $\mathcal{S}$ might not intersect the zero-section transversally. However by Sard's theorem it is possible to make an arbitrarily small perturbation $\overline{\partial}_{J,\epsilon} = \overline{\partial}_J - \epsilon$ where $\epsilon : W^{2,2}(S^2, M) \to \varepsilon_u$ such the $\mathcal{S}_\epsilon = (\mathbb{1}, \overline{\partial}_{J,\epsilon})$ is intersecting the zero-section transversally. If the perturbation is assumed to be compact the Fredholm index is unchanged.

The following is one of the main theorems. A version of this in the finite dimensional case is given in section 9. However in this situation both the base space and the fibres are infinite dimensional. A full proof of this can be found in [MS12].

Theorem 10.3. *If $\mathcal{S} : W^{2,2}(S^2, M)\cap[u_0] \to \varepsilon$ intersects the zero-section transversally, then $\mathcal{M}([u_0], J) \subseteq W^{2,2}(S^2, M)$ is a smooth oriented submanifold of dimension $2\int_{S^2} c_1(u_0^*TM) + 2m$.*

Lemma 10.4. *If we assume that (M, ω, J) is a complex manifold then an orientation on $\mathcal{M}([u_0], J)$ can be given by the following almost complex structure. Let $\xi \in T_u(\mathcal{M}([u_0], J))$, that is $\xi \in W^{2,2}\Omega^0(S^2, u^*TM)$ and $D_u\xi = 0$, then we define $J\xi = J\circ\xi \in T_u(\mathcal{M}([u_0], J))$.*

Proof. We have to check that $D_u(J\xi) = 0$. To do so take conformal coordinates on S^2 and M and by the formulas above we get

$$\begin{aligned} D_u(J\xi) &= \frac{1}{2}(\partial_s(J\xi) + J\partial_t(J\xi))ds + \frac{1}{2}(\partial_t(J\xi) - J\partial_s(J\xi))dt \\ &= \frac{1}{2}(J\partial_s\xi - \partial_t\xi)ds + \frac{1}{2}(J\partial_t\xi + \partial_s\xi)dt \\ &= JD_u(\xi) = 0. \end{aligned}$$

□

11 Bubbling trees

Let (M, ω, J) be a smooth symplectic manifold with a compatible almost complex structure J and let $\mathcal{M}([u_0], J) \subseteq W^{2,2}(S^2, M)$ be the space of parameterized J-holomorphic spheres in M defined in section 10. That space is usually not compact which for example can give problems if we want to integrate over it. In this section we discuss the Deligne-Mumford compactification of $\mathcal{M}([u_0], J)$.

We first define the energy of a J-holomorphic sphere $u : S^2 \to M$ as

$$E(u) := \int_{S^2} u^*\omega.$$

In local conformal coordinates $\varphi : S^2 \supseteq U \to \mathbb{C} = \{s + ti \mid s, t \in \mathbb{R}\}$ on S^2 where we denote the local version of u by $\widetilde{u} := u \circ \varphi^{-1}$ we have that

$$\widetilde{u}^*\omega\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right) = \omega\left(\widetilde{u}_*\left(\frac{\partial}{\partial s}\right), \widetilde{u}_*\left(\frac{\partial}{\partial t}\right)\right) = \left|\widetilde{u}_*\left(\frac{\partial}{\partial s}\right)\right|^2 = |\partial_s(\widetilde{u})|^2$$

and therefore $\widetilde{u}^*\omega = |\partial_s(\widetilde{u})|^2 ds \wedge dt$. This implies that $E(u) \geq 0$ and that $E(u) = 0$ if and only if u is constant. The energy of a J-holomorphic sphere defined on a subsets of S^2, $u : S^2 \supseteq \Omega \to M$, is defined the same way, $E(u, \Omega) := \int_U u^*\omega$.

The following theorem plays a central role in the compactification.

Theorem 11.1 (Removal of singularities)**.**

1. *Let $u : S^2 \backslash \{\infty\} \to M$ be a J-holomorphic curve and assume that $E(u) < \infty$, then u extends to a J-holomorphic sphere $u : S^2 \to M$.*
2. *If $u^\nu : S^2 \backslash \{\infty\} \to M$ is a sequence of J-holomorphic curves such that $\sup_{\nu,z} \|du^\nu(z)\|_{op} < \infty$. Then u^ν has a subsequence converging in the C^∞-topology to a J-holomorphic curve $u : S^2 \backslash \{\infty\} \to M$. Furthermore if there exists a uniform bound on the energy $\sup_\nu E(u^\nu) \leq c$ then the energy of limit curve u is also bounded by c and by (1) u extends to a J-holomorphic sphere $u : S^2 \to M$.*

Take a sequence $u^\nu : S^2 \to M$ of J-holomorphic spheres with unbounded derivatives, but uniformly bounded energy

$$\sup_{\nu,z} \|du^\nu(z)\|_{op} = \infty, \quad \sup_\nu E(u^\nu) \leq c.$$

Such sequences exists and since the derivatives are unbounded this sequence can not have a convergent subsequence in the C^∞-topology and therefore demonstrates that the space $\mathcal{M}([u_0], J)$ is non compact. However the energy will concentrate at points where the derivative blows up, at those points a new (or several new) J-holomorphic spheres will *bubble off*. This can be seen by the following rescaling argument.

Take a sequence $z^\nu \in S^2$ of points where $\|du^\nu\|$ attains its maximum, then $\|du^\nu(z^\nu)\|_{op} \to \infty$ and since S^2 is compact the sequence z^ν can be chosen to be convergent $z^\nu \to z_0$ for some $z_0 \in S^2$. Define $c^\nu := \|du^\nu(z^\nu)\|_{op}$. Use local conformal coordinates $\varphi : S^2 \supseteq U \to \mathbb{C}$ on S^2, $z_0 \in U$, without loss of generality we can assume that $\frac{1}{2} \leq \|d\varphi(z)\|_{op} \leq 2$ for all $z \in U$. Let $\epsilon > 0$ be such that $B_\epsilon(\varphi(z^\nu)) \subseteq \varphi(U)$ for all $\nu \in \mathbb{N}$ and consider the reparameterized sequence $v^\nu : B_{\epsilon c^\nu} \to M$ defined by

$$v^\nu(z) = u^\nu \circ \varphi^{-1}\left(z^\nu + \frac{z}{c^\nu}\right).$$

Then the derivative of v^ν are uniformly bounded

$$\|dv^\nu(z)\|_{op} = \|du(\varphi^{-1}(z^\nu + z/c^\nu))\|_{op}\|d\varphi^{-1}(z^\nu + z/c^\nu)\|_{op}\frac{1}{c^\nu} \leq 2$$

also the energy is uniformly bounded

$$E(v^\nu) = \int_{B_{\epsilon c^\nu}} v^{\nu,*}\omega \leq \int_{S^2} u^{\nu,*}\omega \leq c.$$

Therefore by theorem 11.1 v^ν has a convergent subsequence converging to a J-holomorphic curve $v : S^2\backslash\{0\} \to M$ that can be extended to a J-holomorphic sphere $v : S^2 \to M$. The same calculation as above also shows that $\|dv^\nu(0)\|_{op} \geq \frac{1}{2}$ so the limit sphere v is not constant. The J-holomorphic sphere $v : S^2 \to M$ is called a bubble.

To solve the problem that $\mathcal{M}([u_0], J)$ not is compact we add extra points (stable maps) to allow for bubbling i.e. compactifying $\mathcal{M}([u_0], J)$. The definition of stable maps is based on the intuition of the bubbling behavior described above, but the details are somewhat technical. Now following [Sal99] section 4.

A tree is a connected graph without cycles defined in the following way.

Definition 11.2. *A tree is a pair (T, E) where T is a finite set and E is a relation $E \subseteq T \times T$ satisfying:*

1. *(Symmetry) If $\alpha E \beta$ then $\beta E \alpha$ for $\alpha, \beta \in T$.*
2. *(Anti-reflexivity) $\alpha \not{E} \alpha$ for $\alpha \in T$.*
3. *(Connectivity) If $\alpha \neq \beta$, $\alpha, \beta \in T$ then there exists $\gamma_1, \ldots, \gamma_n \in T$ such that $\gamma_1 = \alpha$, $\gamma_n = \beta$ and $\gamma_i E \gamma_{i+1}$.*
4. *(No cycles) If $\gamma_1, \ldots, \gamma_n \in T$ with $\gamma_i E \gamma_{i+1}$ and $\gamma_i \neq \gamma_{i+2}$ then $\gamma_0 \neq \gamma_n$.*

Definition 11.3. *A tree homomorphism between* (T,E) *and* $(\widetilde{T},\widetilde{E})$ *is a map* $f : T \to \widetilde{T}$ *such that:*

1. $f^{-1}(\widetilde{\alpha})$ *is connected for* $\widetilde{\alpha} \in \widetilde{E}$.
2. *If* $\alpha, \beta \in T$ *and* $\alpha E \beta$ *then* $f(\alpha) = f(\beta)$ *or* $f(\alpha) E f(\beta)$.

If f *is bijective we call* f *a tree isomorphism.*

A parameterized stable map modelled on a tree (T,E) is a collection of J-holomorphic spheres $u_{\alpha\in T} : S^2 \to M$ each corresponding to a bubble.

Definition 11.4 (Stable maps)**.** *Let* (M,ω,J) *be a symplectic manifold with a compatible almost complex structure* J. *A parameterized stable map in* M *with* k *marked points modelled over a tree* (T,E) *is a tuple*

$$(\mathbf{u},\mathbf{z}) = (\{u_\alpha\}_{\alpha\in T}, \{z_{\alpha\beta}\}_{\alpha E\beta}, \{\alpha_i, z_i\}_{1\le i\le k})$$

where $u_\alpha : S^2 \to M$ *is* J*-holomorphic,* $z_{\alpha\beta} \in S^2$ *and* $(\alpha_i, z_i) \in T \times S^2$ *satisfying:*

1. *If* $\alpha E \beta$ *then* $u_\alpha(z_{\alpha\beta}) = u_\beta(z_{\beta\alpha})$.
2. *If* $\alpha E\beta$ *and* $\alpha E \gamma$ *with* $\beta \neq \gamma$*, then* $z_{\alpha\beta} \neq z_{\alpha\gamma}$.
3. *If* $\alpha_i = \alpha_j$ *and* $i \neq j$ *then* $z_i \neq z_j$.
4. *If* $\alpha E \beta$ *and* $\alpha_i = \alpha$ *then* $z_i \neq z_{\alpha\beta}$.
5. *If* u_α *is constant then the set* $Z_\alpha = \{z_{\alpha\beta}\} \cup \{z_i \mid \alpha_i = \alpha\}$ *consists of at least three elements.*

We define the following equivalence relation on the set of parameterized stable maps. Two parameterized stable maps are equivalent if they only differ by precompositions of a collection of Möbius transformations.

Definition 11.5 (Equivalence)**.** *Two parameterized stable maps* $(\mathbf{u},\mathbf{z})$ *and* $(\widetilde{\mathbf{u}},\widetilde{\mathbf{z}})$ *with* k *marked points modelled over the trees* T *and* $\widetilde{T}$ *are called equivalent if there exists a tree isomorphism* $f : T \to \widetilde{T}$ *and a collection of Möbius transformations* $\{\varphi_\alpha\}_{\alpha\in T}$ *such that:*

1. *If* $\alpha \in T$ *then* $\widetilde{u}_{f(\alpha)} = u_\alpha \circ \varphi_\alpha^{-1}$.
2. *If* $\alpha E \beta$ *then* $\widetilde{z}_{f(\alpha)f(\beta)} = \varphi(z_{\alpha\beta})$.
3. *If* $1 \le i \le k$ *then* $\widetilde{\alpha}_i = f(\alpha_i)$ *and* $\widetilde{z}_i = \varphi_{\alpha_i}(z_i)$.

To define convergence of parameterized stable maps, we need to introduce the following notations for a parameterized stable map $(\mathbf{u}, \mathbf{z})$ modelled over the tree T.

The energy of a single bubble γ is notated as

$$m_\gamma(\mathbf{u}) = \int_{S^2} u_\gamma^* \omega.$$

The total energy is notated as

$$E(\mathbf{u}) = \sum_{\gamma \in T} m_\gamma(\mathbf{u}).$$

If we cut the tree T at the edge $\alpha E \beta$ the tree will be divided into two components. Let $T_{\alpha\beta}$ be the component containing β, the energy of that component is notated as

$$m_{\alpha\beta}(\mathbf{u}) = \sum_{\gamma \in T_{\alpha\beta}} m_\gamma(\mathbf{u}).$$

Let $\alpha \in T$ and $\Omega \subseteq S^2$, we define the energy of the bubble α inside Ω (including the complete bubbling system that is connected to α inside Ω) as

$$E_\alpha(\mathbf{u}, \Omega) = \int_\Omega u_\alpha^* \omega + \sum_{\substack{\alpha E \beta \\ z_{\alpha\beta} \in \Omega}} m_{\alpha\beta}(\mathbf{u}).$$

Definition 11.6 (Gromov convergence)**.** *A sequence $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ of parameterized stable maps with k marked points modelled over the trees T^ν is said to Gromov converge to the parameterized stable map $(\mathbf{u}, \mathbf{z})$ with k marked points modelled over the tree T, if there for sufficiently large ν exists surjective tree homomorphisms $f^\nu : T \to T^\nu$ and a collection of Möbius transformations $\{\varphi_\alpha^\nu\}_{\alpha \in T}$ such that:*

1. *If $\alpha \in T$, the sequence $u^\nu_{f^\nu(\alpha)} \circ \varphi^\nu_\alpha : S^2 \to M$ converges to u_α uniformly with all derivatives on compact subsets of $S^2 \backslash Z_\alpha$. Moreover if $\alpha E \beta$ then*

$$m_{\alpha\beta}(\mathbf{u}) = \lim_{\epsilon \to 0} \lim_{\nu \to \infty} E_{f^\nu(\alpha)}(\mathbf{u}^\nu, \varphi^\nu_\alpha(B_\epsilon(z_{\alpha\beta}))).$$

2. *Let $\alpha E \beta$ and ν_j is some subsequence. If $f^{\nu_j}(\alpha) = f^{\nu_j}(\beta)$ for all j then $(\varphi^{\nu_j}_\alpha)^{-1} \circ \varphi^{\nu_j}_\beta$ converges to the constant function $z_{\alpha\beta}$ uniformly on compact subsets of $S^2 \backslash \{z_{\beta\alpha}\}$. If $f^{\nu_j}(\alpha) \neq f^{\nu_j}(\beta)$ for all j then $z_{\alpha\beta} = \lim_{j \to \infty} (\varphi^{\nu_j}_\alpha)^{-1}(z^{\nu_j}_{f^{\nu_j}(\alpha) f^{\nu_j}(\beta)})$.*

3. If $1 \leq i \leq k$ *then* $\alpha_i^\nu = f^\nu(\alpha_i)$ *and* $z_i = \lim_{\nu\to\infty}(\varphi_{\alpha_i}^\nu)^{-1}(z_i^\nu)$*.*

Now we can define the moduli space of unparameterized stable maps with k marked points. Let $\widetilde{\mathcal{M}}_{g=0,k}([u_0], J)$ denote the set of all parameterized stable maps in M with k-marked points of homotopy class $[u_0]$ (the 0 in the subscripts stands for the genus of the curves, in this thesis we are only considering spheres and therefore $g = 0$). The quotient with the equivalence relation in definition 11.5 is denoted by

$$\overline{\mathcal{M}}_{0,k}([u_0], J) = \widetilde{\mathcal{M}}_{0,k}([u_0], J)/\sim$$

and is called the moduli space of unparameterized stable maps with k marked points. The moduli space $\overline{\mathcal{M}}_{0,k}([u_0], J)$ can under good conditions be shown to be an orbifold which is a generalization of a manifold to allow for certain types of singular points.

For the moduli space we have the following result. If

$$(\mathbf{u}^\nu, \mathbf{z}^\nu) \in \overline{\mathcal{M}}_{0,k}([u_0], J)$$

is a sequence where $\sup_\nu E(\mathbf{u}^\nu) < \infty$, then $(\mathbf{u}^\nu, \mathbf{z}^\nu)$ have a Gromov convergent subsequence, this result is called Gromov compactness.

Note that the compactification of the space $\mathcal{M}([u_0], J)$ is $\overline{\mathcal{M}}_{0,3}([u_0], J)$ with three marked points. That is since there is an injection

$$\begin{aligned}\Phi : \mathcal{M}([u_0], J) &\to \overline{\mathcal{M}}_{0,3}([u_0], J)\\ u &\mapsto [(u, \emptyset, (0, 1, \infty))]\end{aligned}$$

from the space of parameterized J-holomorphic spheres to the space of unparameterized stable maps, this is actually a bijection if we restrict to stable maps modelled on a tree with a single vertex.

We end this section by giving explicit examples that demonstrates when bubbling occurs, both with and without marked points.

Example 11.7. The constant sequence $u^\nu : S^2 \to S^2$ defined by $u^\nu(z) = z$ with three marked points $(\frac{1}{\nu},\ \frac{1}{\nu^2},\ 0)$ is Gromov converging to the stable map modelled over the tree

$$\alpha — \beta — \gamma$$

defined by

$$\begin{aligned}&u_\alpha(z) = z, \quad z_{\alpha\beta} = 0,\\ &u_\beta(z) = 0, \quad z_{\beta\alpha} = \infty,\ z_{\beta\gamma} = 0,\\ &u_\gamma(z) = 0, \quad z_{\gamma\beta} = \infty\end{aligned}$$

with three marked points $((\beta,1),\ (\gamma,1),\ (\gamma,0))$. To see this we precompose with the Möbius transformations

$$\varphi_\alpha^\nu(z) = z, \quad \varphi_\beta^\nu(z) = \frac{z}{\nu}, \quad \varphi_\gamma(z) = \frac{z}{\nu^2}.$$

★

Example 11.8. The sequence $u^\nu : S^2 \to \mathbb{C}P^3$ without marked points defined by

$$\begin{aligned} u^\nu(z) &= \left[z\left(z-\frac{1}{\nu}\right), z^2\left(z-\frac{1}{\nu}\right), \frac{1}{\nu^2}\left(z-\frac{1}{\nu}\right), \frac{z}{\nu^2}\right], \\ u^\nu(\infty) &= [0,1,0,0] \end{aligned}$$

is Gromov converging to the stable map without marked points modelled over the tree

$$\begin{array}{ccc} \alpha \,—\, & \beta & \,—\, \gamma \\ & | & \\ & \delta & \end{array}$$

defined by

$$\begin{aligned} u_\alpha(z) &= [1,z,0,0], \quad && z_{\alpha\beta} = 0, \\ u_\beta(z) &= [1,0,0,0], \quad && z_{\beta\alpha} = \infty,\ z_{\beta\gamma} = 0,\ z_{\beta\delta} = 1, \\ u_\gamma(z) &= [z,0,1,0], \quad && z_{\gamma\beta} = \infty, \\ u_\delta(z) &= [z,0,0,1], \quad && z_{\delta\beta} = \infty. \end{aligned}$$

To see this we precompose with the Möbius transformations

$$\varphi_\alpha^\nu(z) = z, \quad \varphi_\beta^\nu(z) = \frac{z}{\nu}, \quad \varphi_\gamma(z) = \frac{z}{\nu^2}, \quad \varphi_\delta^\nu(z) = \frac{1}{\nu}\left(1+\frac{z}{\nu}\right).$$

★

12 Gromov-Witten invariants

In this section we are going to define the genus zero Gromov-Witten three point invariant.

Let (M,ω) be a smooth symplectic manifold of dimension $2m$ and choose an ω-compatible almost complex structure $J : TM \to TM$, we will later in section 13 show that the Gromov-Witten invariant that we define is independent of the choice of almost complex structure J.

First we define the three point evaluation map, let $[u_0]$ be a homotopy class in the space $W^{2,2}(S^2,M)$.

Definition 12.1. *The three point evaluation map is defined as*

$$\begin{aligned} \mathrm{ev} : \mathcal{M}([u_0], J) &\to M^3 \\ u &\mapsto u(0) \times u(1) \times u(\infty). \end{aligned}$$

Lemma 12.2. *If we assume that (M, ω, J) is a complex manifold, then the three point evaluation map $\mathrm{ev} : \mathcal{M}([u_0], J) \to M^3$ intertwines the almost complex structure on $\mathcal{M}([u_0], J)$ given by lemma 10.4 and the almost complex structure on M^3, i.e. $\mathrm{ev}_* \circ J = J \circ \mathrm{ev}_*$.*

Proof. If $\xi \in T_u(\mathcal{M}([u_0], J))$ then

$$\mathrm{ev}_{u,*}(J\xi) = (J\xi(0), J\xi(1), J\xi(\infty)) = J\mathrm{ev}_{u,*}(\xi).$$

□

Lemma 12.3. *The three point evaluation map can be extended to the compactified moduli space $\overline{\mathcal{M}}_{0,3}([u_0], J)$ by*

$$\begin{aligned} \mathrm{ev} : \overline{\mathcal{M}}_{0,3}([u_0], J) &\to M^3 \\ (\mathbf{u}, \mathbf{z}, ((\alpha, z_1), (\beta, z_2), (\gamma, z_3))) &\mapsto u_\alpha(z_1) \times u_\beta(z_2) \times u_\gamma(z_3). \end{aligned}$$

Proof. This is well defined since if $(\mathbf{u}, \mathbf{z})$ is equivalent to $(\widetilde{\mathbf{u}}, \widetilde{\mathbf{z}})$ it is clear that $\mathrm{ev}(\mathbf{u}, \mathbf{z}) = \mathrm{ev}(\widetilde{\mathbf{u}}, \widetilde{\mathbf{z}})$. Also for the embedding

$$\begin{aligned} \Phi : \mathcal{M}([u_0], J) &\to \overline{\mathcal{M}}_{0,3}([u_0], J) \\ u &\mapsto [(u, \emptyset, (0, 1, \infty))] \end{aligned}$$

it is clear that $\mathrm{ev}(\Phi(u)) = \mathrm{ev}(u)$ for $u \in \Phi : \mathcal{M}([u_0], J)$. □

Now we have everything that we need to define the genus zero Gromov-Witten three point invariant.

Definition 12.4 (Genus zero Gromov-Witten three point invariant)**.** *We let (M, ω, J) be a symplectic manifold with a ω-compatible almost complex structure J and let $[u_0]$ be a connected component of $W^{2,2}(S^2, M)$. Assume that the section $\mathcal{S} : W^{2,2}(S^2, M) \cap [u_0] \to \varepsilon$ intersects the zero-section transversally. By theorem 10.3, $\mathcal{M}([u_0], J)$ is then a smooth oriented submanifold of dimension $d = 2\int_{S^2} c_1(u_0^* TM) + 2m$. For a closed $[\beta] \in H^d(M^3)$. We define the genus zero Gromov-Witten three point invariant as*

$$\mathrm{Gr}_J([u_0], [\beta]) = \int_{\overline{\mathcal{M}}_{0,3}([u_0], J)} \mathrm{ev}^* \beta.$$

13 Oriented cobordism

When we defined the Gromov-Witten invariants in section 12 we studied the solution space of the non-linear Cauchy-Riemann equation $\overline{\partial}_J(u) = 0$. In the definition of the Gromov-Witten invariant we had to make a choice of a compatible almost complex structure J, in this section we will see that the invariant is independent of that choice and therefore we can drop the index J in $\mathrm{Gr}_J([u_0], [\beta])$ and just write

$$\mathrm{Gr}([u_0], [\beta]).$$

Let us first define how to orient the boundary of manifolds. A smooth manifold M of dimension m with boundary is a space that is locally diffeomorphic to an open region of the upper half space of dimension m, we have charts

$$\varphi : M \supseteq U \to \mathbb{H}^m = \{(x_1, \dots, x_m) \in \mathbb{R}^m \mid x_1 \geq 0\}.$$

The boundary of M that is defined as $\partial M = \{x \in M \mid \varphi(x) \in \partial \mathbb{H}^m\}$ is then a smooth submanifold of dimension $m-1$ where the charts

$$\varphi : U \cap \partial M \to \mathbb{R}^{m-1} \cong \{(0, x_2, \dots, x_m) \in \mathbb{R}^m\}$$

are induced from the charts on M.

An orientation on M induces an orientation on ∂M in the following way, a basis of tangent vectors $(v_2, \dots, v_m) \in T_x^{m-1}\partial M$ is defined to be positively oriented if and only if $(v_1, \iota_* v_2, \dots, \iota_* v_m) \in T_x^m M$ is positively oriented where $v_1 \in T_x M$ is an outwards pointing tangent vector.

Definition 13.1. *Let M and N be two m-dimensional oriented closed smooth manifolds. An oriented cobordism between M and N is a $m+1$ dimensional oriented manifold W with $\partial W \cong M \sqcup \overline{N}$ where $\overline{N}$ is N with the reversed orientation. In that case we say that M is oriented cobordent to N.*

We will now show that the Gromov-Witten invariant is independent of J, let $\mathcal{J}(M, \omega)$ denotes the space of all ω-compatible almost complex structures.

Theorem 13.2. *Let (M, ω) be a symplectic manifold and $J_0, J_1 \in \mathcal{J}(M, \omega)$. Chose a connected component $[u_0]$ of $W^{2,2}(S^2, M)$. We assume that both $\mathcal{S}_{J_0}, \mathcal{S}_{J_1} : W^{2,2}(S^2, M) \cap [u_0] \to \varepsilon$ intersects the zero-section transversally. For a closed $[\beta] \in H^d(M^3)$ where $d = \dim \mathcal{M}([u_0], J_0) = \dim \mathcal{M}([u_0], J_1)$ we have that*

$$\mathrm{Gr}_{J_0}([u_0], [\beta]) = \mathrm{Gr}_{J_1}([u_0], [\beta]).$$

Proof. Since the space $\mathcal{J}(M,\omega)$ is connected see [MS17] chapter 4, we can define a path

$$\gamma : [0,1] \to \mathcal{J}(M,\omega)$$

such that $\gamma(0) = J_0$ and $\gamma(1) = J_1$. Define a family of maps

$$F : [0,1] \times W^{2,2}(S^2, M) \supseteq [0,1] \times [u_0] \to \varepsilon$$
$$F(t,u) = (u, \overline{\partial}_{J_t}(u)).$$

If F is transversal to the zero-section everything is fine otherwise we can make a small perturbation $F_\delta = F - \delta$ such that $F_\delta(t,-) = F(t,-)$ for t in a neighborhood of 0 and 1 and that F_δ is transversal to the zero-section.

By assumption

$$W := F_\delta^{-1}(0) = \{(t, \overline{\mathcal{M}}([u_0], J_t)),\ t \in [0,1]\} \subseteq [0,1] \times [u_0]$$

is an oriented smooth manifold by the transversality of F_δ. It follows that

$$\partial W = \mathcal{M}([u_0], J_0) \sqcup \overline{\mathcal{M}([u_0], J_1)} \quad \text{or} \quad \partial W = \overline{\mathcal{M}([u_0], J_0)} \sqcup \mathcal{M}([u_0], J_1),$$

the reversed orientations are since $\frac{\partial}{\partial t}$ is pointing inwards when $t = 0$ and outwards when $t = 1$, so W is a cobordism. Since $[\beta] \in H^d(M^3)$ is closed, Stokes theorem gives us that

$$\begin{aligned}
\mathrm{Gr}_{J_0}([u_0],[\beta]) - \mathrm{Gr}_{J_1}([u_0],[\beta]) &= \int_{\overline{\mathcal{M}}_{0,3}([u_0],J_0)} \mathrm{ev}^*\beta - \int_{\overline{\mathcal{M}}_{0,3}([u_0],J_1)} \mathrm{ev}^*\beta \\
&= \pm \int_{\partial W} \mathrm{ev}^*\beta \\
&= \pm \int_W d(\mathrm{ev}^*\beta) = 0.
\end{aligned}$$

□

Remark 13.3. *The technique of defining a cobordism used in the proof of theorem 13.2 can also be used in the case when we study the solution space of the perturbed equation $\overline{\partial}_{J,\epsilon} = 0$ to see that the Gromov-Witten invariant is independent of the perturbation ϵ.*

14 Intersection theory

In the section we show a way of computing the Gromov-Witten invariant defined in section 12 by counting the intersections of the evaluation map with certain submanifolds in M^3. To do this we have to first define the closed Poincaré dual.

Definition 14.1 (Closed Poincaré dual). *Let M be an oriented smooth manifold of dimension m and $S \subseteq M$ a closed oriented smooth submanifold of dimension k. The closed Poincaré dual of S is the unique cohomology class $[\beta] \in H^{m-k}(M)$ satisfying that for any $[\omega] \in H_c^k(M)$ we have that*

$$\int_S \iota^*\omega = \int_M \omega \wedge \beta.$$

Note that it is always possible to construct a closed Poincaré dual to S in the following way. First take a tubular neighborhood T of S, that is diffeomorphic to the normal bundle $T \cong NS$, then take the Thom class $[\Phi] \in H_{cv}^{m-k}(NS) = H_{cv}^{m-k}(T)$ where H_{cv}^* denotes the cohomology with compact vertical support, then extend $[\Phi]$ by zero to get the closed Poincaré dual $[\beta] := j_*[\Phi] \in H^{m-k}(M)$. In this way we can make β supported in an arbitrary small tubular neighborhood of S.

Lemma 14.2. *Let $M = M_1 \times M_2$ be the product of oriented smooth manifolds of dimensions m_1 and m_2 and let $S_1 \subseteq M_1$, $S_2 \subseteq M_2$ be closed oriented smooth submanifolds of dimensions k_1 and k_2. Let*

$$[\beta_1] \in H^{m_1-k_1}(M_1), \quad [\beta_2] \in H^{m_2-k_2}(M_2)$$

be the closed Poincaré duals of S_1 and S_2. Then the closed Poincaré dual of $S = S_1 \times S_2 \subseteq M$ is

$$[\beta] = (-1)^{(m_1-k_1)k_2}[\beta_1] \otimes [\beta_2] \in H^{m_1+m_2-k_1-k_2}(M).$$

Proof. Let $\pi_1 : M_1 \times M_2 \to M_1$, $\pi_2 : M_1 \times M_2 \to M_2$ be the projections and $\iota_1 : S_1 \to M_1$, $\iota_2 : S_2 \to M_2$, $\iota : S_1 \times S_2 \to M_1 \times M_2$ the inclusions. It is enough to prove the result for $[\omega]$ of the form

$$[\omega] = [\omega_1] \otimes [\omega_2] \in H_c^\alpha(M_1) \otimes H_c^\beta(M_2) \subseteq H_c^{k_1+k_2}(M), \ \alpha + \beta = k_1 + k_2,$$

that is since we can extend the result linearly to hold for all $[\omega] \in H_c^{k_1+k_2}(M)$. The following shows that $[\beta]$ is indeed the closed Poincaré dual of S

$$\begin{aligned}
\int_S \iota^*\omega &= \int_{S_1\times S_2} \iota^*(\pi_1^*\omega_1 \wedge \pi_2^*\omega_2) \\
&= \int_{S_1} \iota_1^*\omega_1 \int_{S_2} \iota_2^*\omega_2 \\
&= \int_{M_1} \omega_1 \wedge \beta_1 \int_{M_2} \omega_2 \wedge \beta_2 \\
&= (-1)^{(m_1-k_1)k_2} \int_M (\pi_1^*\omega_1 \wedge \pi_2^*\omega_2) \wedge (\pi_1^*\beta_1 \wedge \pi_2^*\beta_2) \\
&= \int_M \omega \wedge \beta.
\end{aligned}$$

□

The following is the main result of this section and will let us compute the Gromov-Witten invariant by counting intersections.

Theorem 14.3. *Let M^m be a compact oriented smooth manifold with boundary, N^n an oriented smooth manifold, and $f : M \to N$ a smooth map. Assume that $m \leq n$. Let $S \subseteq N$ be a closed oriented smooth submanifold of dimension $n - m$ intersected transversally by f, also assume that*

$$f(\partial M) \cap S = \emptyset.$$

Let $[\beta] \in H^m(N)$ be the closed Poincaré dual of S and β a representative supported in a small neighborhood of S away from $f(\partial M)$. Then

$$\int_M f^*\beta = \int_{f^{-1}(S)} 1$$

and since $f^{-1}(S)$ is discrete and finite this is just counting the points in $f^{-1}(S)$ and taking their orientations into account.

Proof. Let T be the tubular neighborhood of S described in the theorem. We have the commutative diagram

$$\begin{array}{ccccc} H^0(S) & \xleftarrow{\pi_*} & H^m_{cv}(T) & \xrightarrow{j_*} & H^m(N) \\ \downarrow f^* & & \downarrow f^* & & \downarrow f^* \\ H^0(f^{-1}(S)) & \xleftarrow{\pi_*} & H^m_{cv}(f^{-1}(T)) & \xrightarrow{j_*} & H^m(M) \end{array}$$

where j_* is extension by zero and π_* is integration along the fibres. Commutativity of the right square is clear. To see the commutativity of the left square let $[\omega] \in H^m_{cv}(T)$ and $x \in f^{-1}(S)$, denote the fibre of T at $f(x)$ by $T_{f(x)}$ and the image under f of the fibre of $f^{-1}(T)$ at x by $f(f^{-1}(T)_x)$, then $T_{f(x)}$ and $f(f^{-1}(T)_x)$ are homotopic, their boundaries are outside the support of ω, and ω is closed so

$$(f^*\pi_*\omega)_x - (\pi_* f^*\omega)_x = \int_{T_{f(x)}} \omega - \int_{f(f^{-1}(T)_x)} \omega = 0.$$

Let $[\Phi] \in H^m_{cv}(T)$ be the Thom class of T, then $[\beta] := j_*[\Phi] \in H^m(N)$ is the closed Poincaré dual of S. By the commutativity of the diagram and Fubini's theorem and we have that

$$\int_M f^*\beta = \int_{f^{-1}(T)} f^*\Phi = \int_{f^{-1}(S)} \pi_* f^*\Phi = \int_{f^{-1}(S)} 1.$$

□

Remark 14.4. *In theorem 14.3, if in addition M and N have almost complex structures and f intertwines those structures, i.e. $f_* \circ J_M = J_N \circ f_*$, and that $S \subseteq N$ is an almost complex submanifold. Then all orientations are positive and hence*

$$\int_M f^*\beta = \int_{f^{-1}(S)} 1 = \# f^{-1}(S).$$

Now the Gromov-Witten invariant can be calculated by the following.

Corollary 14.5. *Let (M, ω, J) be a smooth symplectic manifold with a compatible almost complex structure. Let $[u_0]$ be a homotopy class. Assume that $\overline{\mathcal{M}}_{0,3}([u_0], J)$ is a smooth d-dimensional orbifold. Let $S \subseteq M^3$ be a closed submanifold of codimension d with closed Poincaré dual $[\beta]$ intersected transversally by the evaluation map. Assume that the evaluation map does not intersect the S at the bubbles*

$$\operatorname{ev}(\partial\overline{\mathcal{M}}_{0,3}([u_0], J)) \cap S = \emptyset.$$

Then

$$\operatorname{Gr}([u_0], [\beta]) = \int_{\operatorname{ev}^{-1}(S)} 1,$$

if further assume that (M, ω, J) is a complex manifold and that S is a complex submanifold then

$$\operatorname{Gr}([u_0], [\beta]) = \#\operatorname{ev}^{-1}(S).$$

15 Gromov-Witten for $\mathbb{C}P^n$

In this final section we do a complete computation of all the Gromov-Witten genus zero three point invariant for $\mathbb{C}P^n$ with the Fubini-Study form ω_{FS} defined in section 7. For all computations we choose the standard compatible almost complex structure J induced from the complex structure on $\mathbb{C}^{n+1}$, for this J the section $u \mapsto (u, \overline{\partial}_J u)$ is transversal to the zero-section so we don't need to do any perturbation. Since this standard J is actually a complex structure on $\mathbb{C}P^n$ we call the J-holomorphic spheres just holomorphic.

Since the computation in this section is for $\mathbb{C}P^n$ the following example regarding the closed Poincaré dual to powers of $[\omega_{FS}]$ is very important.

Example 15.1. Let $\mathbb{C}P^{n-k} = \{[z_0, \ldots, z_{n-k}, 0, \ldots, 0]\} \subseteq \mathbb{C}P^n$, then the closed Poincare dual of $\mathbb{C}P^{n-k}$ is $[\beta] := [\omega_{FS}]^k \in H^{2k}(\mathbb{C}P^n)$. ★

Proof. First we calculate the integral $\int_{\mathbb{C}P^n} \omega_{FS}^n$. To do this we use Fubini on the fibre bundle $S^1 \to S^{2n+1} \xrightarrow{\pi} \mathbb{C}P^n$, let $\alpha = \frac{1}{2\pi}\sum_{k=1}^{n+1}(x_k dy_k - y_k dx_k)$, then

$$\int_{S^{2n+1}} \pi^*\omega_{FS}^n \wedge \alpha = \int_{\mathbb{C}P^n} \omega_{FS}^n \int_{S^1} \alpha.$$

If we parameterize $\varphi : [0, 2\pi] \to S^1$ as $\varphi(\theta) = e^{i\theta}z$ the integral over S^1 can be calculated as

$$\int_{S^1} \alpha = \int_0^{2\pi} \alpha_{e^{i\theta}z}(ie^{i\theta}z)d\theta = \int_0^{2\pi} \frac{1}{2\pi}\sum_{k=1}^{n+1}(x_k^2 + y_k^2)d\theta = 1$$

where $x = \mathrm{Re}(e^{i\theta}z)$ and $y = \mathrm{Im}(e^{i\theta}z)$. For the integral over S^{2n+1} we note that

$$\begin{aligned}\pi^*\omega_{FS}^n \wedge \alpha &= \frac{1}{2\pi^{n+1}}\left(\sum_{k=1}^{n+1}(dx_k \wedge dy_k)\right)^n \wedge \sum_{k=1}^{n+1}(x_k dy_k - y_k dx_k)\\ &= \frac{n!}{2\pi^{n+1}}\sum_{k=1}^{n+1}\bigwedge_{\substack{l=1\\ l\neq k}}^{n+1}(dx_l \wedge dy_l) \wedge \sum_{k=1}^{n+1}(x_k dy_k - y_k dx_k)\\ &= \frac{n!}{2\pi^{n+1}}\sum_{k=1}^{n+1}(x_k dy_k - y_k dx_k) \wedge \bigwedge_{\substack{l=1\\ l\neq k}}^{n+1}(dx_l \wedge dy_l)\end{aligned}$$

which is a multiple of the standard volume form on S^{2n+1} and therefore

$$\int_{S^{2n+1}} \pi^*\omega_{FS}^n \wedge \alpha = \frac{n!}{2\pi^{n+1}}\mathrm{vol}(S^{2n+1}) = 1.$$

The conclusion is that $\int_{\mathbb{C}P^n} \omega_{FS}^n = 1$.

Recall that the cohomology ring of $\mathbb{C}P^n$ is the quotient polynomial ring $H^*(\mathbb{C}P^n) \cong \mathbb{R}[\omega_{FS}]/\omega_{FS}^{n+1}$, therefore any $[\omega] \in H_c^{2n-2k}(\mathbb{C}P^n)$ can be written as $[\omega] = \alpha[\omega_{FS}]^{n-k}$ for some $\alpha \in \mathbb{R}$, for such $[\omega]$ we have that

$$\int_{\mathbb{C}P^{n-k}} \iota^*\alpha\omega_{FS}^{n-k} = \alpha = \int_{\mathbb{C}P^n} \alpha\omega_{FS}^{n-k} \wedge \omega_{FS}^k.$$

∎

Remark 15.2. *In example 15.1, even if $\mathbb{C}P^{n-k} \overset{\iota}{\hookrightarrow} \mathbb{C}P^n$ is embedded as in the diagram*

$$\begin{array}{ccc} \mathbb{C}^{n-k+1}\backslash\{0\} & \xhookrightarrow{I} & \mathbb{C}^{n+1}\backslash\{0\} \\ \pi\big\downarrow & & \big\downarrow\pi \\ \mathbb{C}P^{n-k} & \xhookrightarrow{\quad\iota\quad} & \mathbb{C}P^{n} \end{array}$$

where I is complex linear and injective, we will still get the same closed Poincaré dual. That is since the space of all complex linear injective maps $\mathbb{C}^{n-k+1} \to \mathbb{C}^{n+1}$ is connected and the integral over a closed manifold of the pullback of closed forms is homotopy invariant.

Also note that the submanifold $\mathbb{C}P^{n-k} \overset{\iota}{\hookrightarrow} \mathbb{C}P^n$ is a complex submanifold.

Now we will study the space of holomorphic spheres in $\mathbb{C}P^n$ and prove that all holomorphic spheres are given by homogeneous polynomials.

Theorem 15.3. *Any holomorphic sphere $u : S^2 \to \mathbb{C}P^n$ takes the form*

$$u([z, w]) = [P_0(z, w), \dots, P_n(z, w)]$$

where $P_0, \dots, P_n$ are homogeneous polynomials of the same degree m of two variables without any common zeros in $\mathbb{C}^2\backslash\{0\}$, i.e.

$$(P_0(z, w), \dots P_n(z, w)) \neq (0, \dots, 0).$$

Proof. Obviously the result is true for holomorphic spheres in $\mathbb{C}P^0$.

For $\mathbb{C}P^1$ assume that $u : S^2 \to \mathbb{C}P^1$ is a non constant holomorphic function (meromorphic as a function $u : \mathbb{C} \to \mathbb{C}$). Then u has finitely many poles and finitely many zeros, lets say that the poles are $a_1, \dots, a_{m_1}$ and the zeros are $b_1, \dots, b_{m_2}$ (each pole and zero occurring in the list the same number of times as its multiplicity). Then the function $v : S^2 \to \mathbb{C}P^1$ defined as

$$v(z) = \prod_{k=1}^{m_1}(z - a_k)\prod_{k=1}^{m_2}\frac{1}{(z - b_k)}u(z)$$

have neither poles or zeros. If $v(\infty)$ is finite then v is entire and bounded and hence by Liouville's theorem v is a constant, on the other hand if $v(\infty) = \infty$ then $1/v$ is entire and bounded and again by Liouville's theorem v is also in this case constant. The conclusion is that

$$u(z) = A\prod_{k=1}^{m_1}\frac{1}{z - a_k}\prod_{k=1}^{m_2}(z - b_k)$$

for a constant $A \in \mathbb{C}$. Let $m = \max(m_1, m_2)$, in homogeneous coordinates in both the domain and range u can when $z \neq 0$ be written as

$$\begin{aligned} u([z,w]) &= \left[1, A\prod_{k=1}^{m_1}\frac{1}{w/z - a_k}\prod_{k=1}^{m_2}(w/z - b_k)\right] \\ &= \left[z^{m-m_1}\prod_{k=1}^{m_1}(w - a_k z), Az^{m-m_2}\prod_{k=1}^{m_2}(w - b_k z)\right] \end{aligned}$$

and u has the required form.

Now we use induction for the case $n \geq 2$. Let $u : S^2 \to \mathbb{C}P^n$ be holomorphic, by Sard's theorem there exists a regular value $q \in \mathbb{C}P^n$ or in other words $q \notin \operatorname{im} u$. Take a complex linear bijection $l : \mathbb{C}P^n \to \mathbb{C}P^n$ such that $l(q) = [0, \ldots, 0, 1]$ and define the projection

$$\begin{aligned} &\pi : \mathbb{C}P^n \backslash \{[0, \ldots, 0, 1]\} \to \mathbb{C}P^{n-1} \\ &[z_0, \ldots, z_{n-1}, z_n] \mapsto [z_0, \ldots, z_{n-1}]. \end{aligned}$$

The map $\pi \circ l \circ u$ is holomorphic and defined everywhere so by the induction hypothesis it is given by homogeneous polynomials of degree m. Therefore the map $l \circ u$ is given by

$$l \circ u([z,w]) = [P_0(z,w), \ldots, P_{n-1}(z,w), f(z,w)].$$

for some homogeneous polynomials $P_0, \ldots, P_{n-1}$ of the same degree m and $f : \mathbb{C}^2 \backslash \{0\} \to \mathbb{C}$ is also homogeneous of degree m. Restricted to $S^2 \backslash \{\infty\}$ the map is given by

$$l \circ u([1,w]) = [P_0(1,w), \ldots, P_{n-1}(1,w), f(1,w)]$$

and since we for every $w_0 \in \mathbb{C}$ have a $0 \leq k \leq n-1$ such that $P_k(1, w_0) \neq 0$, we can study $l \circ u$ in the coordinate chart

$$\begin{aligned} &\varphi_k : \mathbb{C}P^n \to \mathbb{C}^n \\ &[z_0, \ldots, z_k, \ldots, z_n] \mapsto \left(\frac{z_0}{z_k}, \ldots, 1, \ldots, \frac{z_n}{z_k}\right) \end{aligned}$$

defined in a neighborhood of $l \circ u([1, w_0])$ to see that $f(1,w) : \mathbb{C} \to \mathbb{C}$ is holomorphic at w_0. Take the limit $w \to \infty$

$$\lim_{w\to\infty} l \circ u([1,w]) = \lim_{w\to\infty} \frac{1}{w^m}[P_0(1,w), \ldots, P_{n-1}(1,w), f(1,w)]$$

and since $\frac{1}{w^m}(P_0(1,w),\ldots,P_n(1,w)) \to (a_0,\ldots,a_{n-1}) \in \mathbb{C}^n\backslash\{0\}$ for some $a_k \in \mathbb{C}$ and that $\lim_{w\to\infty} l \circ u([1,w]) = l \circ u([0,1]) \neq (0,\ldots,0,1)$, the limit

$$\lim_{w\to\infty} \frac{1}{w^m} f(1,w)$$

must exist and be finite and therefore by Liouville's theorem $f(1,w)$ is a polynomial of degree at most m. By the homogeneity of f, $f(z,w) = z^m f(1,w/z)$ is a homogeneous polynomial of degree m, so $l \circ u$ is of the correct form. Now compose $l \circ u$ with l^{-1} to see that u is given by homogeneous polynomials.

□

Definition 15.4. *The degree of a holomorphic sphere $u : S^2 \to \mathbb{C}P^n$ given by $u([z,w]) = [P_0(z,w),\ldots,P_n(z,w)]$ where $P_0,\ldots,P_n$ are homogeneous polynomials of the same degree is defined to be the degree of the polynomials.*

Theorem 15.5. *Let $u : S^2 \to \mathbb{C}P^n$, $u([z,w]) = [P_0(z,w),\ldots,P_n(z,w)]$ where $P_0,\ldots,P_n$ are homogeneous polynomials of two variables of degree m without any common zeros in $\mathbb{C}^2\backslash\{0\}$. Then u is homotopic to the map*

$$u_0 : S^2 \to \mathbb{C}P^n$$
$$u_0([z,w]) = [z^m, w^m, w^m, \ldots, w^m].$$

Proof. In the case where $m = 0$ then u is a constant and the homotopy $u \to u_0$ can be defined by

$$u_t([z,w]) = [P_{0,t},\ldots,P_{n,t}],\ t \in [0,1]$$

where $(P_{0,t},\ldots,P_{n,t})$ is a path from $(P_0(z,w),\ldots,P_n(z,w))$ to $(1,\ldots,1)$ in $\mathbb{C}^{n+1}\backslash\{0\}$.

When $m \geq 1$ we divide the homotopy into two steps. First we get rid of the identically zero components by the homotopy $u \to v$ defined by

$$u_t([z,w]) = \left[\begin{Bmatrix} P_0(z,w) & \text{if } P_0 \not\equiv 0 \\ tz^m & \text{if } P_0 \equiv 0 \end{Bmatrix},\ldots,\begin{Bmatrix} P_n(z,w) & \text{if } P_n \not\equiv 0 \\ tz^m & \text{if } P_n \equiv 0 \end{Bmatrix}\right], t \in [0,1]$$

and let

$$v([z,w]) = u_1([z,w]) = [Q_0(z,w),\ldots,Q_n(z,w)].$$

We factorize the polynomials $Q_0,\ldots,Q_n$

$$Q_k(z,w) = \prod_{i=1}^{m}(a_{k,i}z + b_{k,i}w), \quad (a_{k,i},b_{k,i}) \in \mathbb{C}^2\backslash\{0\}.$$

Note that the factors $(az+bw)$ and $(cz+dw)$, $(a,b),(c,d) \in \mathbb{C}^2\backslash\{0\}$ have the same zeros in $(z,w) \in \mathbb{C}^2\backslash\{0\}$ if and only if $[a,b]=[c,d]$ as points on $\mathbb{C}P^1$. Now continuously move

$$\begin{cases} [a_{0,i}, b_{0,i}] \to [1,0] \\ [a_{k,i}, b_{k,i}] \to [0,1], \; 1 \leq k \leq n \end{cases}$$

on the sphere $\mathbb{C}P^1$ in such a way that there will never exists a sequence $(i_k)_{k=0}^n$, $1 \leq i_k \leq m$ such that

$$[a_{0,i_0}, b_{0,i_0}] = [a_{1,i_1}, b_{1,i_1}] = \cdots = [a_{n,i_n}, b_{n,i_n}].$$

Denote the path of the points by $(a_{k,i,t}, b_{k,i,t}) \in \mathbb{C}^{n+1}\backslash\{0\}$, $t \in [0,1]$ and assume that

$$\begin{cases} (a_{k,i,0}, b_{k,i,0}) = (a_{k,i}, b_{k,i}) \\ (a_{0,i,1}, b_{0,i,1}) = (1,0) \\ (a_{k,i,1}, b_{k,i,1}) = (0,1), \; 1 \leq k \leq n. \end{cases}$$

We define the homotopy $v \to u_0$ by

$$v_t([z,w]) = \left[\prod_{i=1}^m (a_{0,i,t}z + b_{0,i,t}w), \ldots, \prod_{i=1}^m (a_{n,i,t}z + b_{n,i,t}w) \right], \; t \in [0,1]$$

then $v_1([z,w]) = u_0([z,w]) = [z^m, w^m, w^m, \ldots, w^m]$ and we are done. □

The above lemma show that the space of parameterized holomorphic spheres $u : S^2 \to \mathbb{C}P^n$ can be divided into homotopy classes where each homotopy class contains all holomorphic spheres of a given degree $m \geq 0$.

As discussed above holomorphic spheres $u : S^2 \to \mathbb{C}P^n$ of degree m can be written as

$$u([z,w]) = \left[\sum_{i=0}^m a_{0,i} z^{m-i} w^i, \ldots, \sum_{i=0}^m a_{n,i} z^{m-i} w^i \right]$$

where we have the $(n+1)(m+1)$ complex parameters $a_{k,i} \in \mathbb{C}$, $0 \leq k \leq n$, $0 \leq i \leq m$, but if we simultaneously rescale all the parameters with any non-zero complex number the map u will not change and therefore the dimension of the connected component of holomorphic spheres $u : S^2 \to \mathbb{C}P^n$ of degree m will be

$$\dim \overline{\mathcal{M}}_{0,3}([u_0], J) = 2(n+1)(m+1) - 2$$

where $\deg u_0 = m$.

Now we can start the work with computing the Gromov-Witten genus zero three point invariants for $\mathbb{C}P^n$. Recall that invariant is defined by

$$\mathrm{Gr}([u_0],[\beta]) = \int_{\overline{\mathcal{M}}_{0,3}([u_0],J)]} \mathrm{ev}^*\beta$$

for closed $[\beta] \in H^d(M^3)$ where $d = \dim \overline{\mathcal{M}}_{0,3}([u_0],J)$. If $S \subseteq M$ is a complex submanifold with Poincaré dual $[\beta]$ intersected transversally by the evaluation map at a finite number of points and without intersections at the bubbles, we get from corollary 14.5 that

$$\mathrm{Gr}([u_0],[\beta]) = \# f^{-1}(S).$$

Let $M = \mathbb{C}P^n$ be the complex projective space and divide the calculation into the following cases.

1. If $\deg(u_0) = m = 0$, $\dim \overline{\mathcal{M}}_{0,3}([u_0],J) = 2n$ and $\dim(\mathbb{C}P^n)^3 = 6n$. All holomorphic spheres $u \in \mathcal{M}([u_0],J)$ are constants so

$$u([z,w]) = [A_0,\dots,A_n],\ A_k \in \mathbb{C}$$

and

$$\mathrm{ev}(u) = [A_0,\dots,A_n] \times [A_0,\dots,A_n] \times [A_0,\dots,A_n].$$

A basis in $H^{2n}(\mathbb{C}P^n)^3$ is given by

$$[\omega_{FS}]^a \otimes [\omega_{FS}]^b \otimes [\omega_{FS}]^c, \quad \begin{cases} 0 \le a,b,c \le n \\ a+b+c = n \end{cases}$$

with Poincaré duals

$$\begin{aligned} &\mathbb{C}P^{n-a} \times \mathbb{C}P^{n-b} \times \mathbb{C}P^{n-c} \\ =&\{[z_0,\dots,z_n],\ z_0 = \dots = z_{a-1} = 0\} \\ &\times \{[z_0,\dots,z_n],\ z_a = \dots = z_{a+b-1} = 0\} \\ &\times \{[z_0,\dots,z_n],\ z_{a+b} = \dots = z_{a+b+c-1} = 0\}. \end{aligned}$$

When $[\beta]$ is of the form $[\beta] = [\omega_{FS}]^a \otimes [\omega_{FS}]^b \otimes [\omega_{FS}]^c$, the evaluation map intersects this Poincaré dual one time with positive orientation when $u([z,w]) = [0,\dots,0,1]$. Since all spheres in $\mathcal{M}([u_0],J)$ are constants and we only have three marked points no bubbling can occur, in fact $\mathcal{M}([u_0],J) = \overline{\mathcal{M}}_{0,3}([u_0],J)$ and therefore

$$\mathrm{Gr}([u_0],[\beta]) = 1.$$

2. If $\deg(u_0) = m = 1$, $\dim \overline{\mathcal{M}}_{0,3}([u_0], J) = 4n + 2$ and $\dim(\mathbb{C}P^n)^3 = 6n$. All holomorphic spheres $u \in \mathcal{M}([u_0], J)$ takes the form

$$u([z,w]) = [A_0 z + B_0 w, \dots, A_n z + B_n w], \;\; A_k, B_k \in \mathbb{C}$$

and

$$\mathrm{ev}(u) = [A_0, \dots, A_n] \times [A_0 + B_0, \dots, A_n + B_n] \times [B_0, \dots, B_n].$$

A basis in $H^{4n+2}(\mathbb{C}P^n)^3$ is given by

$$[\omega_{FS}]^a \otimes [\omega_{FS}]^b \otimes [\omega_{FS}]^c, \quad \begin{cases} 0 \leq a, b, c \leq n \\ a + b + c = 2n + 1 \end{cases}$$

with Poincaré duals

$$\begin{aligned} &\mathbb{C}P^{n-a} \times \mathbb{C}P^{n-b} \times \mathbb{C}P^{n-c} \\ =&\{[z_0, \dots, z_n], \; z_{n-a+1} = \dots = z_n = 0\} \\ &\times \{[z_0, \dots, z_n], \; z_0 = z_1 = \dots = z_{n-a} = z_c = z_{c+1} = \dots = z_n\} \\ &\times \{[z_0, \dots, z_n], \; z_0 = \dots = z_{c-1} = 0\}. \end{aligned}$$

When $[\beta]$ is of the form $[\beta] = [\omega_{FS}]^a \otimes [\omega_{FS}]^b \otimes [\omega_{FS}]^c$, the evaluation map intersects this Poincaré dual one time with positive orientation when

$$u([z,w]) = [\underbrace{z, z, \dots, z}_{n-a+1}, \underbrace{0, 0, \dots, 0}_{n-b}, \underbrace{w, w, \dots, w}_{n-c+1}].$$

Since the moduli space here of degree one maps is not compact we also need to make sure that no intersections occur at the bubbles. But when the moduli space bubbles of we will get a constant bubble with at least two marked points, but our Poincaré duals $\mathbb{C}P^{n-a}$, $\mathbb{C}P^{n-b}$ and $\mathbb{C}P^{n-c}$ are pairwise disjoint and therefore the evaluation map will never intersect the submanifold $\mathbb{C}P^{n-a} \times \mathbb{C}P^{n-b} \times \mathbb{C}P^{n-c}$ at the bubbles and therefore

$$\mathrm{Gr}([u_0], [\beta]) = 1.$$

3. Lastly if $\deg(u_0) = m \geq 2$, $d = \dim \overline{\mathcal{M}}_{0,3}([u_0], J) \geq 6n + 4$ and $\dim(\mathbb{C}P^n)^3 = 6n$. So for any $[\beta] \in H^d(\mathbb{C}P^n)^3 = \{0\}$ we have that

$$\mathrm{Gr}([u_0], [\beta]) = 0.$$

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