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The Nichols-Zoeller Theorem for Quasi-Hopf Algebras

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Abstract

We prove the quasi-Hopf algebra equivalent of the Nichols-Zoeller theorem as a corollary in the theory of tensor categories. Namely, a finite dimensional quasi-Hopf algebra is free over any of its quasi-Hopf subalgebras.

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1 Introduction

Freeness theorems are highly important in the field of algebra. A prime example of this is the Nichols-Zoeller theorem for Hopf algebras, which states that a finite dimensional Hopf algebra is a free (left or right) module over any of its Hopf subalgebras. This can be thought of as a Hopf algebra analogue of Lagrange's theorem for finite groups, which in fact follows immediately because of the strong connection between Hopf algebras and groups.

The original proof was done by showing that every left (H, B)-Hopf module is free as a left B-module (Nichols, Zoeller, 1989, [3]). Using a similar argument, the equivalent theorem for finite dimensional quasi-Hopf algebras and their quasi-Hopf subalgebras was proven true by Schauenburg (2004, [4]), which is the theorem that this paper is concerned with.

This does not hold in the infinite dimensional case, neither for quasi-Hopf nor for Hopf algebras, as shown by Oberst and Schneider (1974, [6]) who constructed a Hopf algebra counterexample.

This paper will prove the quasi-Hopf algebra equivalent of the Nichols-Zoeller theorem by first proving a freeness theorem for finite tensor categories, as done by Etingof el al. (2015, [1]).

We will start by defining abelian, monoidal and tensor categories, Frobenius-Perron dimensions, and Hopf and quasi-Hopf algebras themselves, which are needed for the proof. Familiarity with the basics of category theory is assumed.

2 Abelian categories

Abelian categories, like the name suggests, are categories that have properties similar to the category **Ab** of abelian groups. Before defining them, we will first introduce the concept of additive categories, which has a less strict definition.

There are also what's known as preadditive and preabelian categories, which we will not discuss in this paper.

2.1 Additive categories

Definition 2.1. An **additive category** is a category satisfying the following:

(A1) It is enriched over the category **Ab** of abelian groups, meaning that every hom-set has the structure of an abelian group (written additively) such that morphism composition is bilinear with respect to this structure.

(A2) It has a zero object (denoted 0).

(A3) Every pair of objects X, Y has a biproduct (denoted $X \bigoplus Y$).

Since the dual of every axiom is itself an axiom, if $\mathcal{C}$ is additive, then so is $\mathcal{C}^{op}$.

We denote the group operation in (A1) as $+$, which gives us:

$$f \circ (g + h) = (f \circ g) + (f \circ h) \text{ and } (f + g) \circ h = (f \circ h) + (g \circ h)$$

We also denote $f + (-g)$ as $f - g$.

By (A2) we mean an object that is both terminal and initial. We also denote the zero morphisms as simply 0, which are the hom-set identities in (A1).

Also note that (A3) is sometimes written as requiring coproducts, but for any category that is enriched over **Ab**[1], products and coproducts coincide (Mac Lane, 1998, p. 194, [2]).

Example 2.2. The category **Ab** of abelian groups is additive, with (A1) being pointwise addition of morphisms, (A2) being the trivial group $\{e\}$, and (A3) being the usual direct sum of groups.

1. Given abelian groups X, Y and $f(x), g(x) : X \longrightarrow Y$ we define:

   $$(f + g)(x) \coloneqq f(x) + g(x) = g(x) + f(x)$$

   and we get an abelian group structure with inverses $-f(x) = f(-x)$, the identity being the morphism $0(x)$ that maps all of X to 0_Y, and we can verify that the usual morphism composition:

   $$\circ : Hom(Y, Z) \times Hom(X, Y) \longrightarrow Hom(X, Z)$$

   is bilinear with respect to this structure.

2. Given an abelian group X, we have a unique $f(x) : X \longrightarrow \{e\}$, given by $f(x) = e$, so $\{e\}$ is terminal. We also have a unique $g(x) : \{e\} \longrightarrow X$, given by $g(e) = 0_X$, so $\{e\}$ is initial. By definition we have a zero object.

3. Given abelian groups X, Y, their direct sum $X \bigoplus Y$ with the addition operation defined as $(x, y) + (x', y') = (x + x', y + y')$ is also an abelian group and defines a categorical biproduct.

Remark 2.3. The category **Grp** of groups contains groups that aren't commutative, so we cannot define an abelian group structure its hom-sets.

The category **Set** of sets has $\emptyset$ as a terminal object and the singleton set as an initial object but no zero objects, so we quickly see that it is not additive.

The category **FinGrp** of finite groups doesn't have coproducts, which is harder to show and means that it is not additive.

Definition 2.4. Let $\mathcal{C}, \mathcal{D}$ be additive categories. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called an **additive functor** if the maps

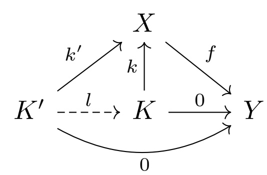
$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

are homomorphisms of abelian groups for any $X, Y \in \mathcal{C}$.

[1] More generally, enrichment over **CMod** of commutative monoids is sufficient.

2.2 Abelian categories

Definition 2.5. The kernel (if it exists) of a morphism $f : X \longrightarrow Y$ in an additive category is an object K and a morphism $k : K \longrightarrow X$ such that $fk = 0$ and for every other candidate (K', k') with $fk' = 0$, there is a unique morphism $l : K' \longrightarrow K$ such that $k' = kl$. We sometimes write (K, k) as K for simplicity.



Example 2.6. We know that the kernel in **Ab** of the canonical map $f : \mathbb{Z} \longrightarrow \mathbb{Z}/3\mathbb{Z}$ is $(3\mathbb{Z}, \pi)$. Consider the candidate $K = 0$. Since there is no $l : 3\mathbb{Z} \longrightarrow 0$ such that $\pi = 0l = 0$, we see that $\ker(f) \neq 0$ even though $f0 = 0$.

Proposition 2.7. *The kernel is monic.*

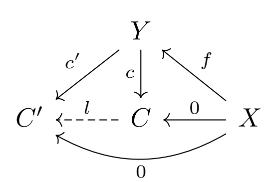
Proof. Given $K' \overset{l_1}{\underset{l_2}{\rightrightarrows}} K \overset{k}{\longrightarrow} X$ with $kl_1 = kl_2 = k'$, then $fk' = 0$ and (K', k') is a candidate. Thus $l_1 = l_2$ by uniqueness and k is monic.

Proposition 2.8. *The morphism f is monic if and only if* $\ker(f) = 0$.

Proof. If f is monic, then given the kernel (K, k) we get $fk = 0 = f0$, and because f is monic, $k = 0$. By definition we need a unique morphism from all candidates $(K', 0)$ to K, so K must be 0.

If $\ker(f) = 0$ then for any g such that $fg = 0$, we have a unique morphism l such that $g = kl = 0l = 0$, thus $fg = 0 \implies g = 0$. Now let $fg_1 = fg_2$, since our category is additive we can consider $f \circ (g_1 - g_2) = (fg_1) - (fg_2) = 0$. Then $f \circ (g_1 - g_2) = 0 \implies g_1 - g_2 = 0 \iff g_1 = g_2$ which shows f is monic.

Definition 2.9. The cokernel (if it exists) of a morphism $f : X \longrightarrow Y$ in an additive category is an object C and a morphism $c : Y \longrightarrow C$ such that $cf = 0$ and for every other candidate (C', c') with $c'f = 0$, there is a unique morphism $l : C \longrightarrow C'$ such that $c' = lc$. We sometimes write (C, c) as C for simplicity.



As can be seen, cokernels are duals of kernels, which means that a cokernel in a category $\mathcal{C}$ is a kernel in the opposite category $\mathcal{C}^{op}$. Since epimorphisms are duals of monomorphisms, we automatically get the following two propositions:

Proposition 2.10. *The cokernel is epic.*

Proposition 2.11. *The morphism f is epic if and only if* $\operatorname{coker}(f) = 0$.

In most of the common categories that we will consider, the cokernel of a morphism $f : X \longrightarrow Y$ will correspond to the quotient of Y by the image of f.

Definition 2.12. An **abelian category** is a category satisfying the following:

(A1-3) It is additive.

(A4) Every morphism has a kernel and a cokernel.

(A5) Every mono-/epimorphism is normal, meaning that it is a kernel/cokernel, respectively, of some morphism in the category.

Like additive categories, abelian categories are self-dual since the dual of every axiom is an axiom, so if $\mathcal{C}$ is additive then so is $\mathcal{C}^{op}$.

Example 2.13. The category $\mathbf{Vect}_{\Bbbk}$ of vector spaces over a field $\Bbbk$ is abelian, with (A1) pointwise addition of linear maps, (A2) the zero vector space $\{0\}$, and (A3) the usual direct sum of vector spaces. To verify (A1-3) see Example 2.2. Shown similarly, the category **R-Mod** of left-modules over a ring is abelian.

4. For $f : V \longrightarrow W$, $\ker(f)$ is the preimage of 0 and $\operatorname{coker}(f) = W/\operatorname{im}(f)$.

5. Given any $f : V \longrightarrow W$, we have the following canonical decomposition: $f : V \xrightarrow{i} \operatorname{im}(f) \xrightarrow{j} W$ where i is epic and j is monic. Looking at (A4), we also have $K \xrightarrow{k} V \xrightarrow{i} \operatorname{im}(f) \xrightarrow{j} W \xrightarrow{c} C$ where K is the kernel and C is the cokernel. Since $C = W/\operatorname{im}(f)$, we have $\ker(c) = \operatorname{im}(f) = (I, j)$. If f is monic, then $f = j = \ker(c)$, and f is a kernel.

   By the first isomorphism theorem, $\operatorname{im}(f) \cong X/\ker(f) = X/\operatorname{im}(k) = \operatorname{coker}(k) = (I, i)$. If f is epic, then $f = i = \operatorname{coker}(k)$ so f is a cokernel.

Remark 2.14. The category of even-dimensional vector spaces has morphisms without kernels, since the kernel of a linear map with odd rank is also odd.

The category of filtered modules over a ring has mono-/epimorphisms that aren't normal, so it is not abelian.

Definition 2.15. An additive category is called **$\Bbbk$-linear** if it is enriched over $\mathbf{Vect}_{\Bbbk}$, meaning that every hom-set has the structure of a vector space over $\Bbbk$ such that usual morphism composition is bilinear with respect to this structure.

Since vector spaces are also abelian groups, $\Bbbk$-linear implies **Ab**-enriched.

Definition 2.16. Let $\mathcal{C}, \mathcal{D}$ be $\Bbbk$-linear categories. A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called **$\Bbbk$-linear** if the maps

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) \longrightarrow \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))$$

are $\Bbbk$-linear homomorphisms of abelian groups for any $X, Y \in \mathcal{C}$.

2.3 Finite abelian categories

Note that by finite abelian categories, we don't mean finite in the sense of having finitely many morphisms or (locally) that every hom-set is finite, but rather in the sense of having finitely many simple objects.

Definition 2.17. A **subobject** of an object Y is an object X together with a monomorphism $i : X \longrightarrow Y$. We denote it $X \subseteq Y$ or $X \subset Y$ when i isn't epic.

An object that cannot be expressed as a non-trivial direct sum of its subobjects is called **indecomposable**. Note that 0 is not indecomposable.

In **Ab** we have subobjects such as the canonical map $i : G/N \longrightarrow N$ where N is a normal subgroup of G, but also the identity id_G and $f : 2\mathbb{Z} \longrightarrow \mathbb{Z}$, etc.

Definition 2.18. The dual notion of a subobject is a **quotient object**, which is an object Z together with an epimorphism $p : Y \longrightarrow Z$.

Given an object Y and a subobject $X \subseteq Y$ with $i : X \longrightarrow Y$, the cokernel of i is a quotient object of Y and we write $\mathrm{coker}(i) = Y/X$.

Definition 2.19. A **simple object** in an abelian category is a non-zero object that only has 0 and itself as subobjects.

The more common definition of simple objects is with quotient objects instead of subobjects, but since abelian categories are self-dual we get the same result.

Example 2.20. The simple objects in **Ab** are the simple groups, and the simple abelian groups are exactly the cyclic groups of prime order.

The simple objects in $\mathbf{Vect}_{\Bbbk}$ are the one-dimensional vector spaces, which are isomorphic to $\Bbbk$ itself.

The simple objects in the category $\mathbf{Rep}_{\Bbbk}(\mathbf{G})$ of $\Bbbk$-representations of a group G are its irreducible representations.

Definition 2.21. An object of **finite length** in an abelian category is an object X for which there exists a filtration:

$$0 = X_0 \subset X_1 \subset \cdots \subset X_{n-1} \subset X_n = X$$

Such that the cokernel X_i/X_{i-1} is simple for all $i > 0$.

Example 2.22. In $\mathbf{Vect}_{\Bbbk}$, for any $\Bbbk^n$ we have the filtration:

$$0 \subset \Bbbk \subset \Bbbk^2 \subset \cdots \subset \Bbbk^{n-1} \subset \Bbbk^n$$

Where every $\Bbbk^i/\Bbbk^{i-1} \cong \Bbbk$ which is simple. Similarly, for any any finite dimensional $\Bbbk$-vector space, one can consider the filtration over its basis elements. Thus every finite dimensional vector space in $\mathbf{Vect}_{\Bbbk}$ has finite length.

This leads us to a famous theorem in group theory which we can apply to abelian categories. We omit the proof for sake of brevity.

Theorem 2.23 (Krull-Schmidt). *Let X be an object of finite length in an abelian category. Then X has a unique decomposition into a direct sum of indecomposable objects, up to isomorphism.*

Definition 2.24. An **exact sequence** is a sequence of morphisms:

$$\cdots \longrightarrow X_{i-1} \xrightarrow{f_{i-1}} X_i \xrightarrow{f_i} X_{i+1} \longrightarrow \cdots$$

Where $\mathrm{im}(f_{i-1}) = \ker(f_i)$.

A **short exact sequence**, is an exact sequence of the form:

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{g} Z \longrightarrow 0$$

Since $\ker(f) = \mathrm{im}(k) = 0, f$ is monic, and since $\mathrm{im}(g) = \ker(c) = C, g$ is epic.

A functor $F : \mathcal{C} \longrightarrow \mathcal{D}$ is called exact if for any short exact sequence in $\mathcal{C}$ (written like above), the following sequence is exact in $\mathcal{D}$:

$$0 \longrightarrow F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z) \longrightarrow 0$$

Definition 2.25. A **projective object** in an abelian category $\mathcal{C}$ is an object P for which the hom-functor $\mathrm{Hom}_{\mathcal{C}}(P, -) : \mathcal{C} \longrightarrow \mathbf{Ab}$ is exact.

We say that an object X has a **projective cover** P if there exists a projective object together with an epimorphism $p : P \longrightarrow X$ such that for every other projective object N and epimorphism $g : N \longrightarrow X$ there exists an epimorphism $h : N \longrightarrow P$ such that $ph = g$.

Every projective object automatically has itself as a projective cover with the epimorphism being the identity morphism.

Example 2.26. Free modules are projective in **R-Mod**, in fact the projective objects are exactly the direct summands of the free modules.

Since all vector spaces are free $\mathbb{k}$-modules (every vector space has a basis), every object in $\mathbf{Vect}_{\mathbb{k}}$ is projective and therefore also has a projective cover.

Definition 2.27. A $\mathbb{k}$-linear abelian category is called **finite** if:

(F1) Every hom-set (which is a $\mathbb{k}$-vector space) has finite dimension.

(F2) Every object has finite length.

(F3) Every simple object has a projective cover.

(F4) There are only finitely many simple objects (up to isomorphism).

Example 2.28. The category $\mathbf{FinVect}_{\mathbb{k}}$ of finite dimensional vector spaces over a field $\mathbb{k}$ is a finite abelian category. That it is $\mathbb{k}$-linear and abelian can be verified as in Example 2.13 and conditions (F1-4) have been verified previously.

For a finite $\mathbb{k}$-linear abelian category $\mathcal{C}$, given any $X, Y \in \mathcal{C}$ where X is simple:

$$\dim_{\mathbb{k}} \mathrm{Hom}_{\mathcal{C}}(P(X), Y) = [Y : X]$$

Where $P(X)$ is a projective cover of X and $[Y : X]$ denotes the number of occurrences of X in the filtration of Y.

3 Tensor categories

There are a lot of criteria for a category to follow our definition of a tensor category. Most importantly though, it is equipped with a bifunctor called the tensor product, making it a monoidal category which we will define first.

3.1 Monoidal categories

Definition 3.1. A **monoidal category** is a category $\mathcal{C}$ satisfying the following:

(M1) It has bifunctor $\bigotimes : \mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, called the tensor product.

(M2) It has a natural isomorphism $a : (- \bigotimes -) \bigotimes - \xrightarrow{\sim} - \bigotimes (- \bigotimes -)$, called the associativity constraint, with components of the form $a_{X,Y,Z} : (X \bigotimes Y) \bigotimes Z \xrightarrow{\sim} X \bigotimes (Y \bigotimes Z)$.

(M3) It has an object called the unit object, denoted 1.

(M4) It has a natural isomorphism $l : 1 \bigotimes - \xrightarrow{\sim} -$ called the left unit constraint with components of the form $l_x : 1 \bigotimes X \xrightarrow{\sim} X$.

(M5) It has a natural isomorphism $r : - \bigotimes 1 \xrightarrow{\sim} -$ called the right unit constraint with components of the form $r_x : X \bigotimes 1 \xrightarrow{\sim} X$.

(M6) The following triangle diagram commutes for all objects X, Y:

$$\begin{array}{ccc} (X \otimes 1) \otimes Y & \xrightarrow{a_{X,1,Y}} & X \otimes (1 \otimes Y) \\ & {\scriptstyle r_X \otimes \mathrm{id}_Y} \searrow \qquad \swarrow {\scriptstyle \mathrm{id}_X \otimes l_Y} & \\ & X \otimes Y & \end{array}$$

(M7) The following pentagon diagram commutes for all objects X, Y, Z:

$$\begin{array}{ccc} & ((W \otimes X) \otimes Y) \otimes Z & \\ & {\scriptstyle a_{W,X,Y} \otimes \mathrm{id}_Z} \swarrow \qquad \searrow {\scriptstyle a_{W \otimes X,Y,Z}} & \\ (W \otimes (X \otimes Y)) \otimes Z & & (W \otimes X) \otimes (Y \otimes Z) \\ \downarrow {\scriptstyle a_{W,X \otimes Y,Z}} & & \downarrow {\scriptstyle a_{W,X,Y \otimes Z}} \\ W \otimes ((X \otimes Y) \otimes Z) & \xrightarrow{\mathrm{id}_W \otimes a_{X,Y,Z}} & W \otimes (X \otimes (Y \otimes Z)) \end{array}$$

One can quickly check that **Set** is monoidal with the cartesian product as the tensor product, the singleton set as the unit object, and with obvious choices for a, l, r. It also holds for the category **FinSet** of finite sets and can be generalized to categories of sets with some additional structure, such as the category **Grp**.

More generally, every category with a product and a terminal object is monoidal, in particular every additive category is monoidal with the direct sum and zero object as tensor product and unit object. However, this choice of tensor and unit is not unique and often other choices are more useful since they reveal more about the structure of the category.

A good example for illustrating this is $\mathbf{Vect}_{\Bbbk}$.

Definition 3.2. The standard **tensor product of vector spaces** U and V over a field $\Bbbk$, denoted $\bigotimes_{\Bbbk}$, is the quotient space $U \otimes V := Z/E$, where

$$Z := \text{span}\{(u, v) \mid u \in U, v \in V\}$$

And E is the subspace of Z generated by vectors of the form:

$$\begin{gathered}(u, v_1 + v_2) - (u, v_1) - (u, v_2),\\ (u_1 + u_2, v) - (u_1, v) - (u_2, v),\\ (\lambda u, v) - \lambda(u, v), \lambda \in \Bbbk\\ (u, \lambda v) - \lambda(u, v)\end{gathered}$$

The motivation is that bilinear maps $U \times V \longrightarrow W$ are equivalent to linear maps $U \otimes V \longrightarrow W$, which comes from the fact that we are taking a quotient of E, so vectors in E will be zero in Z/E. This means that vectors $u \otimes v$ in $U \otimes V$ have the following properties:

$$\begin{gathered}u \otimes v_1 + u \otimes v_2 = u \otimes (v_1 + v_2)\\ u_1 \otimes v + u_2 \otimes v = (u_1 + u_2) \otimes v\\ (\lambda u) \otimes v = \lambda(u \otimes v) = u \otimes (\lambda v)\end{gathered}$$

Given vectors $v = (v_1, \ldots, v_n), w = (w_1, \ldots, w_m)$ in U, V their tensor product $v \otimes w$ is equivalent to their outer product vw^T. Take for example $(2, 6), (4, 5, 3)$ in $\mathbb{R}^2, \mathbb{R}^3$ and we get $(2, 6) \otimes (4, 5, 3) = \left(\begin{smallmatrix} 8 & 10 & 6 \\ 24 & 30 & 18 \end{smallmatrix}\right)$.

Remark 3.3. If $(e_1, \ldots, e_n)$ is a basis for U and $(f_1, \ldots, f_m)$ a basis for V then $\{e_i \otimes f_j\}$ is a basis for $U \otimes V$ (where i, j go from 1 to n, m respectively).

Example 3.4. The category $\mathbf{Vect}_{\Bbbk}$ is monoidal with the one dimensional vector space $\Bbbk$ as the unit object, the standard tensor product of vector spaces $\bigotimes_{\Bbbk}$ and obvious choices for a, l, r.

This works since $\bigotimes_{\Bbbk}$ is associative, which is easily checked, and if V is a vector space over $\Bbbk$, then $\Bbbk \otimes V \cong V \cong \Bbbk \otimes V$.

To see this, consider that $\Bbbk$ is a one-dimensional vector space over itself and therefore has a basis $\{e\}$. If $(f_1, \ldots, f_m)$ is a basis for V then $\{e \otimes f_j\}$ is a basis for $\Bbbk \otimes V$ and $\{f_j \otimes e\}$ a basis for $V \otimes \Bbbk$, where $1 \leq j \leq m$.

We will hereafter consider this particular structure when dealing with $\mathbf{Vect}_{\Bbbk}$ as a monoidal category.

Definition 3.5. The **left dual** of an object X in a monoidal category is an object X^* for which there exists a pair of morphisms $\text{ev}_X : X^* \bigotimes X \longrightarrow 1$ and

$\mathrm{coev}_X : 1 \longrightarrow X \bigotimes X^*$, called the evaluation and coevaluation morphisms, such that the following two diagrams commute:

$$
\begin{array}{ccccc}
 & & X & & \\
 & \overset{l_X}{\nearrow} & & \overset{r_X}{\nwarrow} & \\
1 \otimes X & & & & X \otimes 1 \\
\downarrow{\scriptstyle \mathrm{coev}_X \otimes \mathrm{id}_X} & & & & \uparrow{\scriptstyle \mathrm{id}_X \otimes \mathrm{ev}_X} \\
(X \otimes X^*) \otimes X & & \xrightarrow{a_{X,X^*,X}} & & X \otimes (X^* \otimes X)
\end{array}
$$

$$
\begin{array}{ccccc}
 & & X^* & & \\
 & \overset{l_X}{\nearrow} & & \overset{r_X}{\nwarrow} & \\
1 \otimes X^* & & & & X^* \otimes 1 \\
\uparrow{\scriptstyle \mathrm{ev}_X \otimes \mathrm{id}_{X^*}} & & & & \downarrow{\scriptstyle \mathrm{id}_{X^*}\ \mathrm{coev}_X} \\
(X^* \otimes X) \otimes X^* & & \xleftarrow{a^{-1}_{X^*,X,X^*}} & & X^* \otimes (X \otimes X^*)
\end{array}
$$

Here X is the **right dual** of X^*. We denote a right dual of an object Y as ${}^*Y$. One can make a definition with similar diagrams using the morphisms $\mathrm{ev}'_X : X \bigotimes^* X \longrightarrow 1$ and $\mathrm{coev}'_X : 1 \longrightarrow^* X \bigotimes X$, which we will omit here.

An object that has both left and right duals is called rigid. This is equivalent to saying that a rigid object has a left dual and is itself a left dual to some object.

Example 3.6. In $\mathbf{Vect}_{\Bbbk}$, the left and right duals coincide and exist for vector spaces V that are finite-dimensional, which also means that they are the rigid objects. In that case the dual V^* is the standard notion of a dual vector space, which is the set of all linear maps $f : V \longrightarrow \Bbbk$ with pointwise addition and scalar multiplication with scalars from $\Bbbk$, sometimes written simply as $\mathrm{Hom}(V, \Bbbk)$.

3.2 Tensor categories

Definition 3.7. A **tensor category** is a category satisfying the following:

(T1-5) It is abelian.

(T6) It is $\Bbbk$-linear.

(T7) It is locally finite.

(T8-14) It is monoidal.

(T15) The tensor product $\bigotimes$ is bilinear on morphisms.

(T16) $\mathrm{End}(1) \cong \Bbbk$.

(T17) It is rigid, meaning that every object in the category is rigid.

The fact that a tensor category is a rigid, monoidal and abelian implies that the tensor product $\bigotimes$ is biexact (Etingof et al., 2015, p. 66, [1]).

Example 3.8. The category $\mathbf{FinVect}_{\Bbbk}$ of finite dimensional vector spaces is also a tensor category, with (T1-14) and (T17) being shown in Chapter 2, (T15) follows from the definition of $\bigotimes_{\Bbbk}$ and (T16) follows from the fact that $1 = \Bbbk$. We have also previously shown that it is finite, making it a finite tensor category.

The category $\mathbf{FinRep}_{\Bbbk}(\mathbf{G})$ of finite dimensional $\Bbbk$-representations of a group G is also a fintie tensor category. Note that $\mathbf{FinVect}_{\Bbbk} \cong \mathbf{FinRep}_{\Bbbk}(\{\mathbf{e}\})$.

Definition 3.9. Let $\mathcal{C}, \mathcal{D}$ be tensor categories and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an exact and faithful $\Bbbk$-linear functor. We call F a **quasi-tensor functor** if it maps $F(1_{\mathcal{C}}) = 1_{\mathcal{D}}$ and it is equipped with a functorial isomorphism

$$J : F(-) \otimes F(-) \xrightarrow{\sim} F(- \otimes -).$$

Remark 3.10. In other definitions of a quasi-tensor functor, the categories $\mathcal{C}, \mathcal{D}$ need not be rigid, but we would then need to make sure that their tensor products are still bilinear. Since we are not affected by this, we choose to make this simpler definition instead.

4 Frobenius-Perron dimensions

Frobenius-Perron dimensions provide us with a way to assign a value to objects in tensor categories. It is named after the following classical theorem in linear algebra, which we will state but not prove here.

4.1 Grothendieck rings

Theorem 4.1 (Frobenius-Perron). *Let A be a square matrix with non-negative real entries. Then the following holds:*

1. *A has a non-negative real eigenvalue. In particular, the spectral radius λ is an eigenvalue, meaning that $|\mu| \leq \lambda$ for all other eigenvalues μ. Additionally, the eigenvector corresponding to λ has non-negative entries.*

2. *If A has an eigenvector v with strictly positive entries, then the corresponding eigenvalue is λ.*

3. *If A has strictly positive entries, then λ is simple, meaning that $|\mu| < \lambda$ for all other eigenvalues μ and the corresponding eigenvector v has strictly positive entries.*

Definition 4.2. Let A be a ring that is free as a $\mathbb{Z}$-module. If A has a basis $B = \{b_i\}_{i \in I}$ such that $b_i b_j = \sum_{k \in I} c_{ij}^k b_k$, $c_{ij}^k \in \mathbb{Z}_+$, then A is called a $\mathbb{Z}_+$**-ring**.

If 1 is a basis element, then we call A a **unital** $\mathbb{Z}_+$-ring.

If for any X, Z in B, there exist elements Y_1, Y_2 such that XY_1 and Y_2X contain Z with a non-zero coefficient, then we call A a **transitive** $\mathbb{Z}_+$-ring.

A very important example of a transitive unital $\mathbb{Z}_+$-ring is the following:

Definition 4.3. Let $\mathcal{C}$ be a finite tensor category. The **Grothendieck ring**, $\mathrm{Gr}(\mathcal{C})$, is the free abelian group generated by isomorphism classes of simple objects X_i. The class of any object Y in $\mathcal{C}$ is given by

$$[Y] = \sum_i [Y : X_i] X_i$$

Where $[Y : X_i]$ denotes the number of occurrences of X_i in the filtration of Y.

Now we will show that it is in fact a transitive unital $\mathbb{Z}_+$-ring.

It is an abelian group with $[X] = [A] + [B]$ for every short exact sequence

$$0 \longrightarrow A \longrightarrow X \longrightarrow B \longrightarrow 0.$$

It has multiplicative unit [1], since 1 is simple, and multiplication is defined by

$$X_i X_j := [X_i \otimes X_j] = \sum_k [X_i \otimes X_j : X_k] X_k$$

Where $X_{i,j,k}$ are representatives of the simple objects in $\mathcal{C}$.

The exactness of the tensor product gives us the following two relations:

$$[(X_i \otimes X_j) \otimes X_m : X_l] = \sum_k [X_i \otimes X_j : X_k][X_k \otimes X_m : X_l],$$
$$[X_i \otimes (X_j \otimes X_m) : X_l] = \sum_k [X_j \otimes X_m : X_k][X_i \otimes X_k : X_l]$$

Which gives us an isomorphism $(X_i \otimes X_j) \otimes X_m \cong X_i \otimes (X_j \otimes X_m)$ and we see that the multiplication defined above is associative.

Lastly, transitivity comes from the fact that 1 is simple and $\mathcal{C}$ is rigid, so for any simple objects X, Z in $\mathcal{C}$, Z is contained in $X \otimes X^* \otimes Z$, since $X \otimes X^* = 1$. Thus one can always pick a simple object Y_1 contained in $X^* \otimes Z$ so that $X \otimes Y_1 = XY_1$ contains Z. One similarly shows the case for Y_2.

From this we can start to talk about Frobenius-Perron dimensions.

4.2 Frobenius-Perron dimensions

Definition 4.4. Let A be a transitive unital $\mathbb{Z}^+$-ring of finite rank and let B be its basis. We define the **Frobenius-Perron dimension,** FPdim, as the group homomorphism $\mathrm{FPdim} : A \longrightarrow \mathbb{C}$, so that for any X in A, $\mathrm{FPdim}(X)$ is the largest eigenvalue of the matrix N_X of left multiplication by X, given by

$$XX_i = \sum_j (N_X)_{ij} X_j, \text{for } X_i, X_j \in B$$

Since we only have strictly positive entries, by the Frobenius-Perron theorem the eigenvalue will correspond to the spectral radius of the matrix and one can also show that $\mathrm{FPdim}(X) \geq 1$.

$\mathrm{FPdim}(Y)$ for $Y \in A$ is extended from the basis elements by additivity.

For a finite tensor category $\mathcal{C}$ with simple objects X_i, the Frobenius-Perron dimension of any object X is given by its the Frobenius-Perron dimension of its corresponding object in $\mathrm{Gr}(\mathcal{C})$. The left multiplication matrix N_X of an object X would be given by $(N_X)_{ij} = [X \otimes X_i : X_j]$ for simple objects X_i, X_j and in the Grothendieck ring we get $XX_i = \sum_j (N_X)_{ij} X_j$.

Example 4.5. In the category $\mathbf{FinRep}_{\Bbbk}(\mathbf{G})$ of finite dimensional $\Bbbk$-representations of a group G, we have that $\mathrm{FPdim}(X) = \dim_{\Bbbk}(X)$ for all X.

Definition 4.6. Let $\mathcal{C}$ be a finite tensor category. We say that $\mathcal{C}$ is **integral** if the Frobenius-Perron dimensions of all elements in $\mathrm{Gr}(\mathcal{C})$ are integers.

5 Hopf algebras

Hopf algebras are structures with strong connections to groups. One may view them as an abstraction of the group algebra of a group over a field. They are common in representation theory, since they have the nice property that in the representation category of a Hopf algebra, the tensor product of two representations is a representation, as is the dual vector space of a representation. We will only consider representation categories of quasi-Hopf algebras, since they give us a connection to tensor categories.

5.1 Algebras

In field theory, an algebra over a field is a vector space equipped with a bilinear product, called multiplication, meaning that it is distributive over addition and compatible with scalar multiplication. One can extend this definition to algebras over commutative rings, but we will focus on algebras over fields and furthermore, algebras where the multiplication is associative.

Canonical examples of algebras are the real numbers and the complex numbers, with regular addition and multiplication. But in fact, every field is an algebra over itself. In categorical terms we will consider $\Bbbk$-algebras as objects in $\mathbf{Vect}_{\Bbbk}$, or more formally:

Definition 5.1. A **unital associative algebra** over a field $\Bbbk$ is a $\Bbbk$-vector space A together with two linear maps, the multiplication $\mu : A \bigotimes A \longrightarrow A$, and the unit $\iota : \Bbbk \longrightarrow A$ that satisfy both the associative law,

$$\mu \circ (\mathrm{id}_A \otimes \mu) = \mu \circ (\mu \otimes \mathrm{id}_A)$$

and the unit law:

$$\mu \circ (\mathrm{id}_A \otimes \iota) = \mu \circ (\iota \otimes \mathrm{id}_A) = \mathrm{id}_A$$

Dually, a **counital coassociative coalgebra** over a field $\Bbbk$ is a $\Bbbk$-vector space C together with two linear maps, the comultiplication $\Delta : C \longrightarrow C \bigotimes C$, and the counit $\epsilon : C \longrightarrow \Bbbk$ that satisfy both the coassociative law,

$$(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_c \otimes \Delta) \circ \Delta$$

and the counit law:

$$(\epsilon \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \epsilon) \circ \Delta = \mathrm{id}_C$$

We will from now on call them simply algebras and coalgebras. A $\Bbbk$-vector space B that is both an algebra and a coalgebra is called a **bialgebra**.

Example 5.2. The **group algebra** of a group G over a field $\Bbbk$, written $\Bbbk G$, which is the vector space over $\Bbbk$ that is freely generated by elements g in the underlying set of G, is a bialgebra. Denote the scalars $\lambda \in \Bbbk$.

1. It is a vector space by definition with elements of the form:

   $$v = \sum_{g \in G} \lambda_g \cdot g$$

   .

2. The multiplication is given on basis elements by the group operation:

   $$v \cdot u = \sum_{g_v \in G} \sum_{g_u \in G} (\lambda_{g_v} \lambda_{g_u})(g_v \cdot g_u)$$

3. We have the group identity as the unit, thus $\iota(\lambda) = e_G$.

4. Checking the associative and the unit laws are equivalent to checking that the following two diagrams commute (we write $\Bbbk G$ as B to save space):

   $$\begin{array}{ccc} B \otimes B \otimes B & \xrightarrow{\mathrm{id}_B \otimes \mu} & B \otimes B \\ {\scriptstyle \mu \otimes \mathrm{id}_B}\downarrow & & \downarrow{\scriptstyle \mu} \\ B \otimes B & \xrightarrow[\mu]{\qquad} & B \end{array} \quad , \quad \begin{array}{ccccc} B \otimes \Bbbk & \xrightarrow{\mathrm{id}_B \otimes \iota} & B \otimes B & \xleftarrow{\iota \otimes \mathrm{id}_B} & \Bbbk \otimes B \\ {\scriptstyle r_B}\downarrow & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle l_B} \\ B & = & B & = & B \end{array}$$

   Rewriting the diagrams with elements instead and writing the multiplication as $\mu(u, v) = uv$ gives us the following two diagrams:

   $$\begin{array}{ccc} < u, v, w > & \longmapsto & < u, vw > \\ \downarrow & & \downarrow \\ < uv, w > & \longmapsto & (uv)w = u(vw) \end{array}$$

   $$\begin{array}{ccccccc} < u, \lambda > & \longmapsto & < u, e_G > & & < e_G, u > & \longleftarrow & < \lambda, u > \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ u & = & u & = & u & = & u \end{array}$$

   These are fulfilled by (1.) and (2.), thus $\Bbbk G$ follows the categorical definition of an algebra.

5. The comultiplication is given by $\Delta(g) = g \otimes g$. One can see that the coassociative law holds by confirming that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\Delta} & B \otimes B \\ {\scriptstyle \Delta}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B \otimes \Delta} \\ B \otimes B & \xrightarrow[\Delta \otimes \mathrm{id}_B]{} & B \otimes B \otimes B \end{array}$$

6. The counit is given by $\epsilon(g) = 1_{\Bbbk}$. The counit law is equivalent to the following commuting diagram:

$$\begin{array}{ccccc} B \otimes B & \xleftarrow{\Delta} & B & \xrightarrow{\Delta} & B \otimes B \\ {\scriptstyle \epsilon \otimes \mathrm{id}_B}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_B} & & \downarrow{\scriptstyle \mathrm{id}_B \otimes \epsilon} \\ \Bbbk \otimes B & \xrightarrow{\sim} & B & \xleftarrow{\sim} & B \otimes \Bbbk \end{array}$$

   Thus $\Bbbk G$ is also a coalgebra, and therefore a bialgebra.

Note that our comultiplication is bilinear, so for any vector $\alpha g + \beta h \in B$ with $\alpha, \beta \in \Bbbk$ and $g, h \in G$, we get $\Delta(\alpha g + \beta h) = \alpha\Delta(g) + \beta\Delta(h) = \alpha(g \otimes g) + \beta(h \otimes h)$.

Remark 5.3. The definition of a bialgebra is self-dual, so if B is a bialgebra, then its dual B^* is also a bialgebra. Given the above example of group algebras, we get that the **dual of a group algebra** of a group G over a field $\Bbbk$, written $(\Bbbk G)^*$, is a bialgebra.

Definition 5.4. An **antipode** S on a bialgebra $(B, \mu, \iota, \Delta, \epsilon)$ is a linear map $S : B \longrightarrow B$ such that:

$$\mu \circ (\mathrm{id}_B \otimes S) \circ \Delta = \iota \circ \epsilon = \mu \circ (S \otimes \mathrm{id}_B) \circ \Delta$$

Antipodes are self dual, so if B is a finite-dimensional bialgebra with an antipode S_B, then the dual bialgebra B^* has an antipode $S_{B^*} = S_B^*$.

Example 5.5. In the example of group algebras, we want to find an antipode such that $S(g) \cdot g = g \cdot S(g) = e_G$, so our antipode is the inverse $S(g) = g^{-1}$.

Proposition 5.6. *If a bialgebra B has an antipode, then it is unique.*

Proof. Let S, S' be two antipodes, then using the properties of antipodes, the associativity of μ, and the coassociativity of Δ we get:

$$\begin{aligned} S &= \mu \circ (S \otimes (\mu \circ (\mathrm{id}_B \otimes S') \circ \Delta)) \circ \Delta \\ &= \mu \circ (\mathrm{id}_B \otimes \mu) \circ (S \otimes \mathrm{id}_B \otimes S') \circ (\mathrm{id}_B \otimes \Delta) \circ \Delta \\ &= \mu \circ (\mu \otimes \mathrm{id}_B) \circ (S \otimes \mathrm{id}_B \otimes S') \circ (\Delta \otimes \mathrm{id}_B) \circ \Delta \\ &= \mu \circ ((\mu \circ (S \otimes \mathrm{id}_B) \circ \Delta) \otimes S') \circ \Delta = S' \end{aligned}$$

Note that this does not hold for antipodes of quasi-bialgebras, whose antipodes need not be unique.

Definition 5.7. A **Hopf algebra** over $\Bbbk$ is a $\Bbbk$-bialgebra with an invertible antipode, meaning that there is a S^{-1} such that $SS^{-1} = \mathrm{id}_B = S^{-1}S$.

The group algebra is a Hopf algebra, with $S^{-1} = S$.

Since both bialgebras and antipodes are self dual, so are Hopf algebras, meaning that the dual of a group algebra is also a Hopf algebra.

Definition 5.8. For any Hopf algebra H, we call the elements g in H **grouplike** if $g \neq 0$ and $\Delta(g) = g \otimes g$.

The grouplike elements in a Hopf algebra H form a group $G(H)$ under the multiplication inherited from H. This is because $1 \otimes 1$ is always grouplike, the inherited multiplication is closed and associative and we get inverses from the antipode $S(g)$.

5.2 Quasi-Hopf algebras

If we relax the requirements of coassociativity and instead introduce an invertible element Φ that controls the non-coassociativity, we can generalize bialgebras and Hopf algebras to the following:

Definition 5.9. A **quasi-bialgebra** B over $\Bbbk$ is an algebra over $\Bbbk$ together with a comultiplication $\Delta : B \longrightarrow B \otimes B$, a counit $\epsilon : B \longrightarrow \Bbbk$ and an invertible element $\Phi \in B \otimes B \otimes B$ that satisfies the following identities:

$$(\mathrm{id}_B \otimes \Delta) \circ (\Delta(b)) = \Phi \circ (\Delta \otimes \mathrm{id}_B) \circ (\Delta(b)) \circ \Phi^{-1}, b \in B$$
$$(\mathrm{id}_B \otimes \mathrm{id}_B \otimes \Delta) \circ \Phi \circ (\Delta \otimes \mathrm{id}_B \otimes \mathrm{id}_B) \circ \Phi = (1 \otimes \Phi) \circ (\mathrm{id}_B \otimes \Delta \otimes \mathrm{id}_B) \circ \Phi \circ (\Phi \otimes 1)$$
$$(\epsilon \otimes \mathrm{id}_B) \circ (\Delta(b)) = b = (\mathrm{id}_B \otimes \epsilon) \circ (\Delta(b))$$
$$(\mathrm{id}_B \otimes \epsilon \otimes \mathrm{id}_B) \circ \Phi = 1 \otimes 1$$

Every bialgebra is also a quasi-bialgebra with $\Phi = 1 \otimes 1 \otimes 1$.

Definition 5.10. A **quasi-Hopf algebra** H over $\Bbbk$ is a quasi-bialgebra that has a bijective antipode $S : H \longrightarrow H$ and elements $\alpha, \beta \in H$ such that:

$$\text{For } \Phi = \sum_i \Phi_i^1 \otimes \Phi_i^2 \otimes \Phi_i^3, \text{ and } \Phi^{-1} = \sum_i \bar{\Phi}_i^1 \otimes \bar{\Phi}_i^2 \otimes \bar{\Phi}_i^3$$
$$\text{We have } \sum_i \Phi_i^1 \beta S(\Phi_i^2) \alpha \Phi_i^3 = 1, \text{ and } \sum_i S(\bar{\Phi}_i^1) \alpha \bar{\Phi}_i^2 \beta S(\bar{\Phi}_i^3) = 1$$
$$\text{And if } \Delta(a) = \sum b_i \otimes c_i, \text{ for } a \in H$$
$$\text{then } \sum_i S(a_i^1) \alpha a_i^2 = \epsilon(a)\alpha, \text{ and } \sum_i a_i^1 \beta S(a_i^2) = \epsilon(a)\beta$$

Every Hopf algebra is also a quasi-Hopf algebra with $\alpha = \beta = 1$.

Example 5.11. Let $\Bbbk$ be a field containing a primitive fourth root of unity ω, meaning that the smallest integer a such that $\omega^a = 1$ is $a = 4$. Let H denote the algebra over $\Bbbk$ generated by the elements x, y, z with the following relations:

$$x^2 = 1, y^4 = z^4 = 0, xy = -yx, xz = -zx, zy = \omega yz$$

Let $p = \frac{1}{2}(1 + x)$ and $q = \frac{1}{2}(1 - x)$. We then define comultiplication and the counit operation on H as follows:

$$\Delta(x) = x \otimes x, \epsilon(x) = 1$$
$$\Delta(y) = y \otimes (p + \omega q) + 1 \otimes py + x \otimes qy, \epsilon(y) = 0$$
$$\Delta(z) = z \otimes (p - \omega q) + 1 \otimes pz + x \otimes qz, \epsilon(y) = 0$$

Our antipode is given by:

$$S(x) = x, S(y) = -y(p + \omega q), \text{ and } S(z) = -z(p - \omega q)$$

And lastly $\alpha = x, \beta = 1$ and $\Phi = 1 \otimes 1 \otimes 1 - 2q \otimes q \otimes q$.

This does indeed define a quasi-Hopf algebra, and since the comultiplication is not coassociative, it is not a Hopf algebra (Bulacu et al., 2019, p. 124, [7]).

Definition 5.12. We call K a **quasi-Hopf subalgebra** of a quasi-Hopf algebra H if it is a subalgebra $K \subseteq H$ that is closed under the operations Δ, ϵ and S, and where $\Phi, \Phi^{-1} \in K \otimes K \otimes K$.

From the above one can also deduce the meaning of a **Hopf subalgebra**, namely a sub-bialgebra that is closed under the antipode.

Theorem 5.13. *A finite tensor category is integral if and only if it is equivalent to the category of representations of a finite-dimensional quasi-Hopf algebra.*

This is the link that connects tensor categories with quasi-Hopf algebras and is a key part in our proof of the Nichols-Zoeller theorem for quasi-Hopf algebras. The proof is omitted for sake of brevity (Etingof el al., 2015, p. 121, [1]).

6 Main results

Now we will begin to prove the Nichols-Zoeller theorem for quasi-Hopf algebras.

Definition 6.1. Let $\mathcal{C}$ be a finite $\Bbbk$-linear abelian category and let $K(\mathcal{C})$ be the free abelian group generated by the indecomposable projective objects of $\mathcal{C}$, up to isomorphism. Then elements of $K(\mathcal{C}) \otimes_{\mathbb{Z}} \Bbbk$ will be called the **virtual projective objects** of $\mathcal{C}$.

Definition 6.2. Let $\mathcal{C}$ be a finite tensor category. Then the **regular object** of $\mathcal{C}$ is the virtual projective object

$$R_{\mathcal{C}} = \sum_{i \in I} \mathrm{FPdim}(X_i) P_i$$

where $X_i, P_i \in \mathcal{C}$, X_i are simple and P_i are indecomposable projective objects.

Note that some authors define $\mathrm{FPdim}(\mathcal{C}) \coloneqq \mathrm{FPdim}(R_{\mathcal{C}})$.

Definition 6.3. Let $\mathcal{C}, \mathcal{D}$ be locally finite abelian categories and let $F : \mathcal{C} \longrightarrow \mathcal{D}$ be an additive functor. We say that F is **surjective** if every simple object in $\mathcal{D}$ is a subquotient of some object $F(X), X \in \mathcal{C}$.

This definition is not equivalent to the usual definition of an essentially surjective functor, which says that every object of $\mathcal{D}$ is isomorphic to some object $F(X)$.

Lemma 6.4. *Let $\mathcal{C}, \mathcal{D}$ be finite tensor categories, and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a surjective quasi-tensor functor. Then F maps projective objects to projective objects.*

This follows from the surjectivity and exactness of F (Etingof el al., 2015, p. 122, [1]). We will use it prove the following theorem:

Theorem 6.5. *Given finite tensor categories $\mathcal{C}, \mathcal{D}$ with regular objects $R_{\mathcal{C}}, R_{\mathcal{D}}$, and a surjective quasi-tensor functor $F : \mathcal{C} \longrightarrow \mathcal{D}$, we have $F(R_{\mathcal{C}}) = \frac{\mathrm{FPdim}(\mathcal{R}_{\mathcal{C}})}{\mathrm{FPdim}(\mathcal{R}_{\mathcal{D}})} R_{\mathcal{D}}$.*

Proof. By the above lemma, $F(R_{\mathcal{C}})$ is a virtually projective object. If we write $F(R_{\mathcal{C}})$ and $R_{\mathcal{D}}$ in the basis of P_i, then they are both eigenvectors of a multiplication matrix with strictly positive entries. Since $F(R_{\mathcal{C}})$ and $R_{\mathcal{D}}$ also have strictly positive entries, by the Frobenius-Perron theorem they both have the same eigenvalue, the spectral radius, which is simple for matrices with strictly positive entries. Therefore, they must be proportional, and we get the proportionality constant by computing their Frobenius-Perron dimensions.

Lemma 6.6. *Let $\mathcal{C}, \mathcal{D}$ be finite tensor categories, and $F : \mathcal{C} \longrightarrow \mathcal{D}$ a surjective quasi-tensor functor. If $\mathcal{C}$ is integral, then so is $\mathcal{D}$, and $F(R_{\mathcal{C}})$ is free of rank $\frac{\mathrm{FPdim}(\mathcal{R}_{\mathcal{C}})}{\mathrm{FPdim}(\mathcal{R}_{\mathcal{D}})}$, where $\mathrm{FPdim}(\mathcal{R}_{\mathcal{D}})$ divides $\mathrm{FPdim}(\mathcal{R}_{\mathcal{C}})$.*

Proof. Since $\mathcal{C}$ is integral, we have that $\mathrm{FPdim}(\mathcal{R}_{\mathcal{C}})$ is an integer, by assumption. Thus the multiplication matrix of $F(\mathcal{R}_{\mathcal{C}})$ has integer eigenvalue $\mathrm{FPdim}(\mathcal{R}_{\mathcal{C}})$. By the above theorem, $F(\mathcal{R}_{\mathcal{C}})$ is proportional to $\mathcal{R}_{\mathcal{D}}$ so the Frobenius-Perron dimensions of the simple objects of $\mathcal{D}$ correspond to the coordinates of the eigenvector with strictly positive entries given by the multiplication matrix of $F(\mathcal{R}_\mathcal{C})$. Therefore, all coordinates are rational numbers, but since they are also eigenvalues of some integer matrix, they are algebraic integers and the rational algebraic integers are integers.

This means that $\mathcal{D}$ is integral, which we know by Theorem 5.13 to be equivalent to the category of representations of a quasi-Hopf algebra B. Here $\mathcal{R}_\mathcal{D}$ corresponds to the free module of rank 1 over B, and so multiples of $\mathcal{R}_\mathcal{D}$ are free B-modules of finite rank. In particular, $F(R_\mathcal{C})$ is free of rank $\frac{\mathrm{FPdim}(\mathcal{R}_\mathcal{C})}{\mathrm{FPdim}(\mathcal{R}_\mathcal{D})}$, where $\mathrm{FPdim}(\mathcal{R}_\mathcal{D})$ divides $\mathrm{FPdim}(\mathcal{R}_\mathcal{C})$.

The above lemma together with Theorem 5.13 imply the following:

Theorem 6.7. *Let H be a finite-dimensional quasi-Hopf algebra over a field $\Bbbk$, with a quasi-Hopf subalgebra B. Then H is free as a left (and right) B-module, and* $\dim B$ *divides* $\dim H$.

This is the Nichols-Zoeller theorem for quasi-Hopf algebras, with the Hopf algebra version following immediately.

Worth noting is that in the Hopf algebra case, there exists a nice choice of the basis for H as a B-module (Schneider, 1992, [5]).

Corollary 6.8. *Let H be a quasi-Hopf algebra. If* $\dim(H) = p$*, where p is a prime, then H has no proper nontrivial quasi-Hopf subalgebras.*

Now consider the case when H is the group algebra $\Bbbk G$ of a finite group G. Any subgroup N of G gives rise to a Hopf subalgebra $\Bbbk N$ of H and so $\dim(\Bbbk N) = |N|$ divides $\dim(\Bbbk G) = |G|$. Thus, for any finite group G the order of its subgroups divide the order of G. This is the classic Lagrange's theorem.

7 References

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