Elektrisk styrning av mekaniska vibrationer
Electric control of mechanical vibrations

Carolina Nilsson
Degree Project C in Physics, 1FA599, 15 hp. Uppsala University

July 18, 2019

Supervisor: Jonas Fransson, Department of Physics and Astronomy, Materials Theory, Uppsala University
Subject reader: Mattias Klintenberg, Department of Physics and Astronomy, Materials Theory, Uppsala University

UPPSALA
UNIVERSITET

Abstract

In this report a theoretical study have been made in the research field of Materials physics. The system under consideration is based on a similar system from an article. The system consists of a junction between a normal metal and a topological insulator, where one of the materials vibrates. Between the materials there can arise a tunneling current of electrons. The study is to see if an applied voltage over the system can make the vibrating material to exhibit a non-trivial dynamic. From the result we could conclude that the vibrating material were affected by the tunneling current. For different applied voltages we could see a clear difference in the phase and strength of the current when one of the materials vibrated. By studying the Logarithmic Fourier transformation of the current. It was concluded that the system can exhibit resonance, for lower voltages, and that the resonance frequency are dependent on the applied voltage. The calculations of the vibrating material's displacement were done both analytically and numerically. The computations were done using only arbitrary values of the coefficients.

Abstract

I den här rapporten har en teoretisk studie genomförts inom forsknings området för materialfysik. Systemet som studien utgått ifrån är baserat på ett liknande system ifrån en artikel. Systemet består av en normal metall och en topologisk isolator som är separerade från varandra, där ett av materialen vibrerar. Mellan materialen kan det uppstå en ström av tunnlande elektroner. Studien går ut på att se om en applicerad spänning över systemet kan få det vibrerande materialet att ge upphov till en dynamik som inte är uppenbar. Från resultatet kan vi konstatera att det vibrerande materialet påverkades av tunnelströmmen. För olika applicerade spänningar kunde vi se en tydlig skillnad i strömmens fas och styrka när ett av materialen vibrerade. Genom att studera den Logaritmiska Fourier transformen av strömmen. Kan vi dra slutsatsen att systemet kan uppvisa resonans för låga spänningar och att resonans frekvensen är beroende av spänningen. Beräkningarna för det vibrerande materialets avvikelse gjordes både analytiskt och numeriskt. Beräkningarna utfördes endast med godtyckliga värden på koefficienterna.

Contents

1 Introduction

The main study in this report is theoretical and in the research field of Materials Physics. The system under consideration is a classical Hamiltonian system.

The theory behind this project is based on a article, Vibrating Superconducting Island in a Josephson Junction [1]. In this article, the main study is the Josephson effect. This effect makes it possible for a supercurrent to flow across a junction of two superconductors in the absence of any electric field. The superconductors are separated with an insulating barrier and at low temperature the supercurrent flows without resistance. If an additional dc voltage is applied over the junction the tunneling current will be stronger and oscillatory [3]. In this project, a similar but simplified physical system of the one in the article is under consideration. The main difference is the materials of the conductors used for the junction. In this study the junction is between a normal metal and a topological insulator. One of the materials is attached to a spring and oscillates harmonically as a function of time when the applied voltage is zero, this can be described as a vibrating island.

In short the hypothesis is whether it is possible to find an electric current as a driving force in a classical Hamiltonian system. Since the two metals have very different material properties, the system might exhibit a non-trivial dynamic. The trivial dynamic would correspond to a situation where there is no interaction between the current of the tunneling electrons and the vibrating island.

The goal with the project is to study the changes in the electric current between the materials when one of the materials vibrates. The current is analyzed for different arbitrary distances and dc voltages. It is moreover, investigated if the system can exhibit resonance behaviour and if it is possible to control the resonance frequency by changing the voltage.

2 Review

The main theory is based on the article Vibrating Superconducting Island in a Josephson Junction [1]. In this article, the main study is the ac Josephson tunneling current. The system consists of a moving superconducting island between two superconducting leads. The superconducting material is special in the sense that there is a net attraction between electrons close to the Fermi surface. This attraction is the effect of screening by the ionic motion that overcomes the repulsive Coulomb interaction between the electrons. The net attraction gives rise to the possibility of forming bound electron pairs called Cooper pairs [3]. According to the Josephson effect at low temperatures, it is possible for Cooper pairs to cross a thin insulating barrier by quantum-mechanical tunneling. The tunneling Cooper pairs create a supercurrent across the junction which is a quantum-mechanical phenomena called the Josephson current. The dc current flows across the junction without resistance and in absence of an electrical field, this is called the dc Josephson effect. Then if a dc voltage is applied over the junction the current becomes stronger and oscillatory which give rise to the so called ac Josephson effect [3].

In this study, we consider a similar but simplified physical system. The system consists of a normal metal and a topological insulator separated with a thin insulating barrier. One of the materials oscillates harmonically when no voltage is applied to the system.

2.1 Material properties and the effects of a tunneling current

The main difference between this project and the study in the article, is the materials considered in the systems. In the article, the superconductive materials give rise to the possibility of tunneling Cooper pairs, which creates an ac Josephson current across the insulating barrier when a dc voltage is applied. The Josephson effects only occurs when both materials are superconductors. The effect has been experimentally verified and can be used to measure precise voltage standards. There have also been other studies of junctions with different materials that are separated with an insulating barrier at constant distance. When a voltage is applied over the junction it gives rise to a flowing current of tunneling unbounded electrons.

The result states that a tunneling current of electrons between two normal metals would obey Ohm's law for low voltages, and increase proportional to

the voltage applied. While if one of the metals is a superconductor and the other one is a normal metal, there is a critical voltage that is dependent on the superconductors energy gap. For voltages below this critical value the current is zero. When the critical voltage is exceeded, it can break up the Cooper pairs in the superconductor which leads to a current of tunneling electrons that increases fast when the voltage is increased [4].

2.2 Oscillatory behaviours

A system might exhibit an oscillatory behaviour if it is displaced from a position of stable equilibrium. The theory of oscillations is widespread throughout the society. It is commonly used in everyday systems and it has important physical properties that can be used in different physical applications and approximations. Historically, it could be used in a balance wheel for navigation. Today, we have devices as the atomic clock that uses the vibrations of an atom to regulate the time keep [5].

Depending on the physical system properties, different harmonical motions can arise. In our considered system, the oscillatory motion of the vibrating island is treated as simple harmonic when the applied voltage is zero. When a non zero voltage is considered, we treat the oscillations as a driven damped linear harmonic motion, derived from the effective Hamiltonan of the system.

In a simple harmonic motion, the restoring force of the system is conservative and proportional to the displacement. This motion is typically used for describing systems with small oscillations, for example, a mass on a spring, or a pendulum. The energy in such a system is conserved and the motion remains unchanged with time.

If the system is affected by a resistive force, the oscillations become damped with time. There are many applications of different forces that can act on the system, such as, sliding friction or friction from a medium in the system. Depending on the magnitude of the damping coefficients, the system can be underdamped or overdamped. For an underdamped system, the damping coefficient should be smaller then the system's natural frequency, and vice versa for an overdamped system [5]. In the underdamped system the frequency of the oscillations is less then the natural frequency of the system. The motion oscillates with time and the amplitude decays exponentially with

a dependence on the damping coefficients magnitude. A larger value on the damping coefficient means that the motion decreases faster with time. In a overdamped system the frequency of the oscillations is larger then the natural frequency of the system. The oscillatory motion of the system is so damped that it decays exponentially towards the systems equilibrium position with almost no oscillatory motion. A larger value on the damping coefficient means that the motion decreases slower with time.

A system can also be driven by an external force. The oscillatory motion can behave differently depending on how the external force affects the system with time. Since a damping factor decreases the system's energy and the oscillatory motion dies out with time. A driving force adds energy to the system. With a external force it is possible to maintain a oscillatory motion in a damped system. It is also possible for the system's motion to change dramatically and increase in amplitude.

If the force acting on the system is constant in time, the equilibrium position shifts, but the oscillatory behaviour does not change. If there is also a damping effecting the system, the oscillatory motion decreases with time towards the shifted equilibrium position. If the external force acting on the system is periodic instead of constant, it acts on the system with a driving frequency. The amplitude of the oscillatory motion then depends on both the natural frequency and the driving frequency of the system. Often in a physical systems driven by a periodic force, the amplitude reaches its largest value when the driving frequency is close to the systems natural frequency. Then if there is a damping acting on the system, this damping will also decrease the amplitude [5].

In the theory of oscillations there is a physical phenomena called resonance. This can occur when the damping in the system is small, or, non existing and when a periodic driving force is present. If the driving frequency is approximately the same as the systems natural frequency the amplitude of the oscillation increases drastically with time [5]. It is important to investigate if resonance can occur in a system, since the increase in energy in the system can have different consequences on the system, for example, it can change the structure of the system.

3 Theory

3.1 Hypothesis

If a voltage is applied to the system under consideration, it gives rise to an electric current of tunneling electrons between the materials. An effective Hamiltonian can then be constructed for the vibrating island in the system, using the Hamiltonian for a harmonic oscillator, and the Hamiltonian for the current. From the effective Hamiltonian, the equation of motions can be derived using Hamilton's equations. The current can, then, be computed with the vibrating island's displacement and momentum.

An analysis of the current is made, to study whether the current is effected by the displacement of the vibrating island, and how it changes due to different voltages. It is moreover, investigated if the system can exhibit resonance behaviour and if it is possible to control the resonance frequency by changing the voltage.

3.2 Model

The formula for the current of tunneling electrons and the Hamiltonian for the current were provided by Jonas Fransson [2].

3.2.1 The Effective Hamiltonian

The effective Hamiltonian for the vibrating material is described by the Hamiltonian for the classical harmonic oscillator and the Hamiltonian from the tunneling current of electrons.

$$H = H_{osc} + H_I \tag{1}$$

The Hamiltonian for the simple harmonic oscillator is described by

$$H_{osc} = \frac{p^2(t)}{2m} + \frac{ku^2(t)}{2} \tag{2}$$

Here $p(t)$ is the momentum of the oscillator, $u(t)$ is the displacement of the oscillator from its equilibrium position, m is the mass of the oscillator and k is the spring constant. The mass of the spring is neglected.

The Hamiltonian for the tunneling current of electrons are described by

$$H_I = \frac{\hbar}{2e}\left\{ \frac{J_0 V^3}{3} + \left[\Gamma_0 V + \Gamma_1 \frac{eV^2}{4}\left(\frac{3}{2} - \ln\frac{|eV/2|}{D_c}\right)\right]\partial_t \right\} \tau^2(t) \tag{3}$$

here the constants are defined as

$$J_0 = \frac{\pi e^3 N}{2\hbar {D_c}^2}$$

$$\Gamma_0 = \frac{2eND}{\hbar {D_c}^2}(1 + \ln \frac{D_c}{D})$$

$$\Gamma_1 = -\frac{2eN}{\hbar {D_c}^2} \tag{4}$$

where $\hbar$ is Planck's constant, e is the electron charge, N is the density of electron states, D is the electronic band width, D_c is a high-energy cut off, V is the voltage and $\tau(t)$ is the tunneling rate of the electrons.

The effective Hamiltonian can be expressed as

$$\begin{aligned} H =& \frac{p^2(t)}{2m} + \frac{ku^2(t)}{2} + \frac{\hbar}{2e}\Bigg\{ \frac{J_0 V^3}{3} \\ &+ \left[\Gamma_0 V + \Gamma_1 \frac{eV^2}{4}\left(\frac{3}{2} - \ln \frac{|eV/2|}{D_c}\right)\right]\partial_t \Bigg\} \tau^2(t) \end{aligned} \tag{5}$$

with the tunneling rate of the electrons described by

$$\tau(t) = \tau_0 e^{-\alpha u(t)} \tag{6}$$

Here α is the strength of the coupling between the electronic and mechanical degrees of freedom and τ_0 is the tunneling rate at $t = 0$. The tunneling rate is defined such that when $u(t) > 0$, the distance between the normal metal and topological insulator increases. This leads to a exponentially decreasing tunneling rate, and for $u(t) < 0$ it is exponentially increasing.

The effective Hamiltonian can be rewritten as

$$\begin{aligned} H =& \frac{p^2(t)}{2m} + \frac{ku^2(t)}{2} + \frac{\hbar}{2e}\Bigg\{ \frac{J_0 V^3}{3} \\ &- 2\alpha \frac{\partial u(t)}{\partial t}\left[\Gamma_0 V + \Gamma_1 \frac{eV^2}{4}\left(\frac{3}{2} - \ln \frac{|eV/2|}{D_c}\right)\right] \Bigg\} {\tau_0}^2 e^{-2\alpha u(t)} \end{aligned} \tag{7}$$

3.2.2 The tunneling current

The current of tunneling electrons between a normal metal and topological insulator can be calculated with following formula [2].

$$I(t) = \Big\{ J_0 V^2 + \Big[\Gamma_0 + \Gamma_1 \Big(\frac{eV}{2} (1 + \ln \frac{D_c}{|eV/2|}) + \frac{min\{\ 0, eV/2\}}{1 - (eV/2D)^2} \Big) \Big] \partial_t \Big\}\ \tau^2(t) \tag{8}$$

Here the first term in the current is directly dependent on τ, if the system were stationary only this term would contribute to the current. While the other terms are dependent on the time derivative of τ, such that the vibrating motion of the island gives rise to the contribution from these terms. The constants are defined in equation (4).

3.3 Derivation of equations of motions

To derive the equation of motions for the displacement and the momentum we use Hamilton's equations [6].

$$\begin{aligned} \frac{\partial u(t)}{\partial t} &= \frac{\partial H}{\partial p} \\ \frac{\partial p(t)}{\partial t} &= -\frac{\partial H}{\partial u} \end{aligned} \tag{9}$$

The equation of motion for the displacement becomes

$$\frac{\partial u(t)}{\partial t} = \frac{p(t)}{m} \tag{10}$$

and for the momentum

$$\frac{\partial p(t)}{\partial t} = -\,ku(t) + \frac{\hbar V^3 {\tau_0}^2 \alpha}{3e} J_0 e^{-2\alpha u(t)} - \frac{2\alpha^2 {\tau_0}^2 \hbar}{e} \Big[\Gamma_0 V + \Gamma_1 \frac{eV^2}{4} \Big(\frac{3}{2} - \ln \frac{|eV/2|}{D_c} \Big) \Big] e^{-2\alpha u(t)} \frac{\partial u(t)}{\partial t} \tag{11}$$

We can now construct two new constants

$$J_1 = \frac{\hbar V^3 {\tau_0}^2 \alpha}{3e} J_0$$

$$\Gamma_2 = -\frac{2\alpha^2 {\tau_0}^2 \hbar}{e}\left[\Gamma_0 V + \Gamma_1 \frac{eV^2}{4}\left(\frac{3}{2} - \ln\frac{|eV/2|}{D_c}\right)\right] \tag{12}$$

Combining equations (10) and (11), the time derivative of the momentum can be expressed as a function of $u(t)$ and $p(t)$

$$\frac{\partial p(t)}{\partial t} = -ku(t) + J_1 e^{-2\alpha u(t)} + \Gamma_2 e^{-2\alpha u(t)} \frac{p(t)}{m} \tag{13}$$

The system is now expressed as two coupled first order nonlinear differential equations

$$\begin{aligned} \frac{\partial u(t)}{\partial t} &= \frac{p(t)}{m} \\ \frac{\partial p(t)}{\partial t} &= -ku(t) + J_1 e^{-2\alpha u(t)} + \Gamma_2 e^{-2\alpha u(t)} \frac{p(t)}{m} \end{aligned} \tag{14}$$

3.4 Analytical Solution

3.4.1 Linearization of the coupled nonlinear system

A way to approach an analytical solution is through linearization. This can be done by using a Taylor expansion of $e^{-2\alpha u(t)}$ for small values of the exponent ($\alpha u(t) \ll 1$). The nonlinear terms can then be replaced with

$$e^{-2\alpha u(t)} \approx 1 - 2\alpha u(t) + \mathcal{O}((\alpha u)^2) \tag{15}$$

The system can, then, be expressed as two coupled first order linear differential equations.

$$\begin{aligned} \frac{\partial u(t)}{\partial t} &= \frac{p(t)}{m} \\ \frac{\partial p(t)}{\partial t} &= -ku(t) + J_1 - 2\alpha J_1 u(t) + \Gamma_2 \frac{p(t)}{m} \end{aligned} \tag{16}$$

3.4.2 Solving the system of linear equations

Taking the second derivative with respect to time of the displacement in equation (16), the time derivative of the momentum can be expressed in terms of the displacement, such that

$$\frac{\partial p(t)}{\partial t} = m\frac{\partial^2 u(t)}{\partial t^2} \tag{17}$$

The coupled system in equation (16) can now be expressed as a second order linear differential equation of $u(t)$ by combining it with (17). This gives

$$\frac{\partial^2 u(t)}{\partial t^2} - \frac{\Gamma_2}{m}\frac{\partial u(t)}{\partial t} + \frac{k + 2\alpha J_1}{m}u(t) = \frac{J_1}{m} \tag{18}$$

Equation (18) can be rewritten as a classical equation of motion

$$\frac{\partial^2 u(t)}{\partial t^2} + \Gamma\frac{\partial u(t)}{\partial t} + {w_0}^2 u(t) = F \tag{19}$$

where the constants are defined as

$$\Gamma = -\frac{\Gamma_2}{m}$$

$$w_0 = \sqrt{\frac{k + 2\alpha J_1}{m}}$$

$$F = \frac{J_1}{m} \tag{20}$$

Here, the damping factor of the system is defined as Γ, the natural frequency of the system is w_0 and the driving force is F, with the condition that $k + 2\alpha J_1 \geq 0$.

The solution for this differential equation can be expressed as a linear combination of a homogeneous solution and a particular solution. The homogeneous solution should satisfy the differential equation when F = 0.

The characteristic equation for the homogeneous differential equation can be expressed as

$$r^2 + \Gamma r + {w_0}^2 = 0 \tag{21}$$

which have the roots

$$r_{\pm} = -\frac{\Gamma}{2} \pm \sqrt{(\frac{\Gamma}{2})^2 - {w_0}^2} \tag{22}$$

The solution of interest arises when the value inside the square root is less then zero, since it will yield the oscillatory motion that is being studied. This solution will be described below. The other solutions for the over-damped motion can be found in the Appendix.

If $(\frac{\Gamma}{2})^2 - {w_0}^2 < 0$:

The roots of the characteristic equation can be expressed as

$$r_{\pm} = -\frac{\Gamma}{2} \pm i\sqrt{{w_0}^2 - (\frac{\Gamma}{2})^2} = -\frac{\Gamma}{2} \pm iw \tag{23}$$

The homogeneous solution is

$$u_h(t) = e^{-(\Gamma/2)t}\Big(A_1 cos(wt) + A_2 sin(wt)\Big) \tag{24}$$

The w in the roots of the characteristic equation is also referred to as the systems eigenfrequencies and have an important impact on the systems dynamics.

For the second part of the solution we need to find a particular solution. Let us make an ansatz that is dependent on the function on the right hand side of equation (19). In this case the function is constant from the linearization. Therefore we should try

$$u_p = At^2 + Bt + C \tag{25}$$

The ansatz plugged into equation (19) yields that $A = B = 0$, so this gives

$${w_0}^2 C = F \tag{26}$$

The constant should be equal to

$$C = \frac{F}{{w_0}^2} \tag{27}$$

Our particular solution becomes

$$u_p = \frac{J_1}{m{w_0}^2} \tag{28}$$

The linear combination of the homogeneous and particular solution is defined as

$$u(t) = u_h(t) + u_p \tag{29}$$

where the choice of $u_h(t)$ is described above.

3.5 Nummerical Solution

A numerical method was used on the nonlinear coupled system in equation (14). The alogrithm used to solve these first order differential equations were Runge-Kutta 4 for a coupled system. All calculations were done in MATLAB. The algorithm can be constructed as follows [7].

Consider the following system

$$\begin{aligned} \frac{\partial u(t)}{\partial t} &= f(u(t), p(t)) \\ \frac{\partial p(t)}{\partial t} &= g(u(t), p(t)) \end{aligned} \tag{30}$$

Each iteration is with time such that $t_{k+1} = t_k + k$. Here k is the time step. Then from setting initial conditions u_k and p_k it is possible to compute the next step in time with the following

$$\begin{aligned} u_{k+1} &= u_k + \frac{1}{6}(n_1 + 2n_2 + 2n_3 + n_4) \\ p_{k+1} &= p_k + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \end{aligned} \tag{31}$$

where

$$\begin{aligned} n_1 &= kf(u_k, p_k) \\ m_1 &= kg(u_k, p_k) \\ n_2 &= kf(u_k + \frac{n_1}{2}, p_k + \frac{m_1}{2}) \\ m_2 &= kf(u_k + \frac{n_1}{2}, p_k + \frac{m_1}{2}) \\ n_3 &= kf(u_k + \frac{n_2}{2}, p_k + \frac{m_2}{2}) \\ m_3 &= kf(u_k + \frac{n_2}{2}, p_k + \frac{m_2}{2}) \\ n_4 &= kf(u_k + n_3, p_k + m_3) \\ m_4 &= kf(u_k + n_3, p_k + m_3) \end{aligned} \tag{32}$$

4 Result

The result in this report will use only arbitrary values of the constants in equation (19) for studying the currents behaviour at different voltages. This is because the dimensional analysis of the coefficients in the model yields different results.

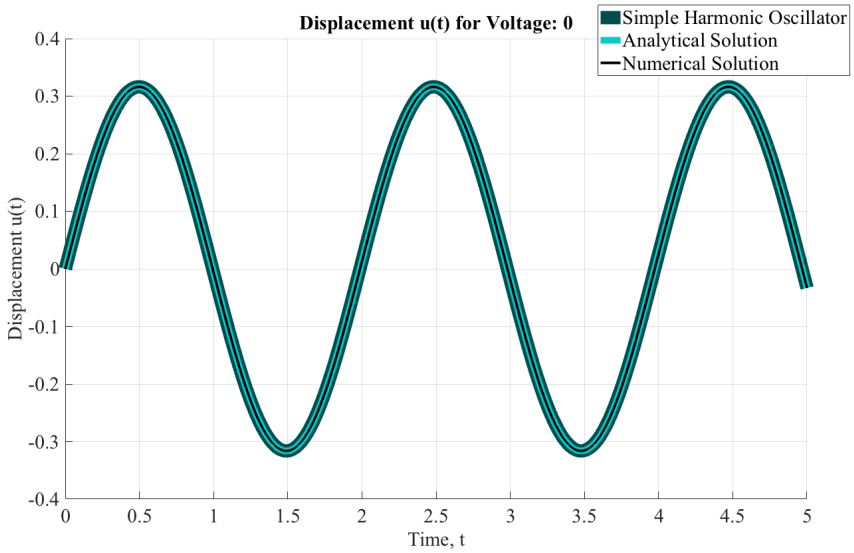

Figure 1: Displacement u(t) for the vibrating island, as a function of time, with zero applied voltage. The graph shows the solution for a theoretical simple harmonic oscillator unaffected by the tunneling current, the analytical solution affected by the tunneling current and the numerical solution affected by the tunneling current.

Figure 1, shows the displacement of the vibrating island, when the applied voltage is zero. When the applied voltage is zero, the effective Hamiltonian should become the same as the Hamiltonian for the simple harmonic oscillator. This is because all the terms in the Hamiltonian for the current are directly dependent of the applied voltage, such that the contribution becomes zero. A theoretical solution for a simple harmonic oscillator has also been computed and graphed. This is to ease the comparing with the computed solutions and to clearly show the differences with the solutions affected by the current.

Studying the difference in the displacement between the theoretical solution of a simple harmonic oscillator and the calculated solutions. The error between the analytical solution and the theoretical solution are equal to zero. While the error between the numerical solution and the theoretical solution is of order $\mathcal{O}(10^{-11})$. This is assumed to be the numerical error and can be neglected.

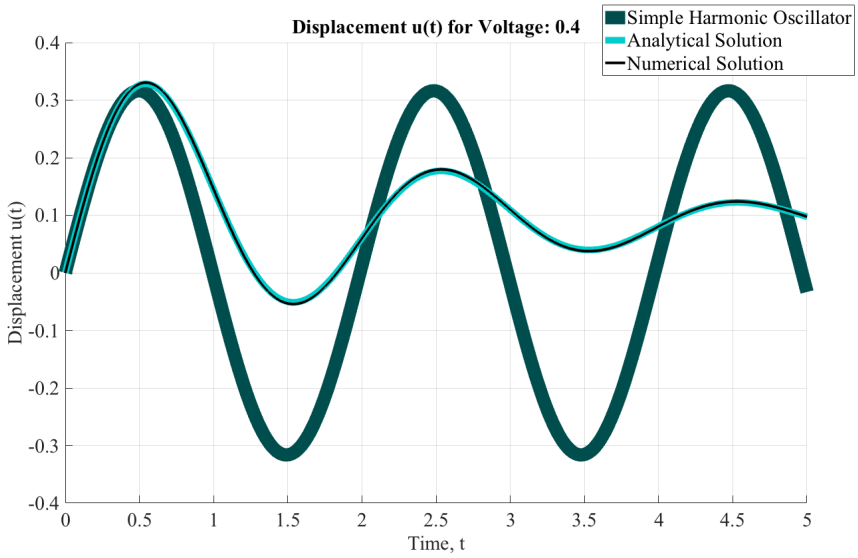

Figure 2: Displacement u(t) as a function of time for an applied voltage of 0.4

In Figure 2, we can study the behaviour of the displacement of the vibrating island for an applied voltage of 0.4. The theoretical solution for the simple harmonic oscillator are not dependent on the voltage and is still the same as in Figure 1. The analytical- and numerical solution of the displacement is being affected by the tunneling current. We can see that there is a small phase shift in the oscillatory motion and that there is a damping occurring. If we look closely we can also see that there is a shift upwards in the equilibrium position of the oscillatory motion. As time goes the oscillatory motion decreases and eventually the displacement becomes constant at the new equilibrium position. The analytical and numerical solutions are in good agreement with each other. The error between the solutions is of order $\mathcal{O}(10^{-3})$ and can be neglected.

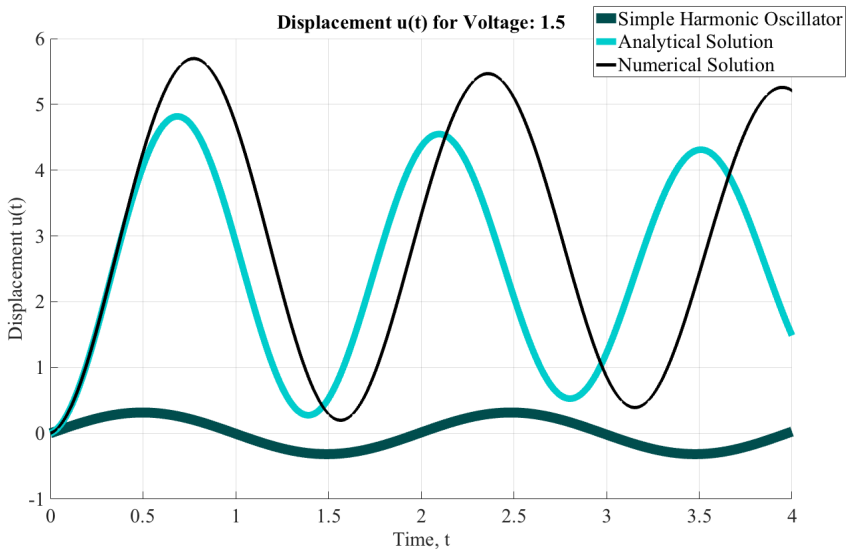

Figure 3: Displacement u(t) as a function of time for an applied voltage of 1.5

Figure 3, shows the displacement for a large applied voltage. We have seen in the previous figure 1, how the applied voltage is directly effecting the displacement. To large values of the voltage makes the linear approximation used for the analytical solution unstable. Here we clearly see a difference between the numerical and analytical solution when the applied voltage is 1.5.

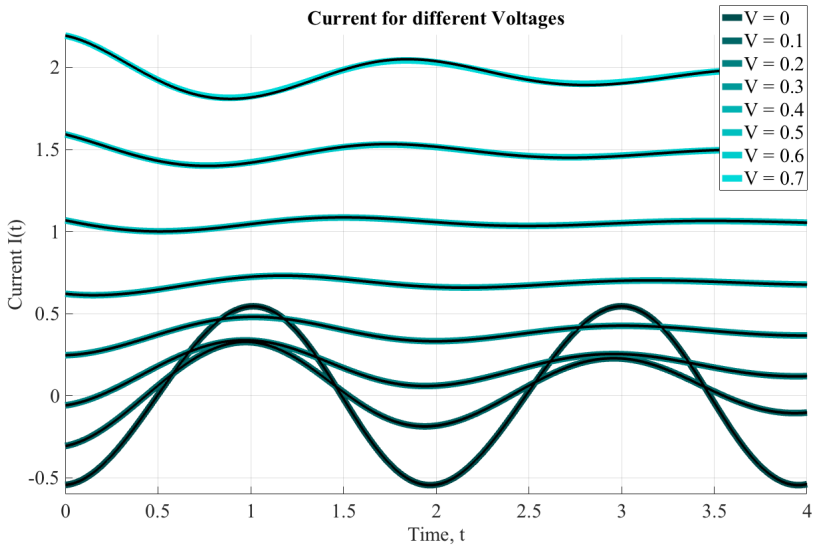

Figure 4: Analytical- and Numerical Current as a function of time

In Figure 4, the current have been computed with the displacement of the vibrating island. We have the analytical- and numerical solution of the current computed for different voltages. When the applied voltage is zero, we have a ac current that oscillates around zero and does not change its behaviour with time. When the voltage is increased the current becomes stronger and the equilibrium position is shifted towards higher values. Since the displacement is damped and becomes constant after a while, the current becomes constant when the island stops vibrating due to the damping. Here we also see that a higher applied voltage is effecting the phase of the current.

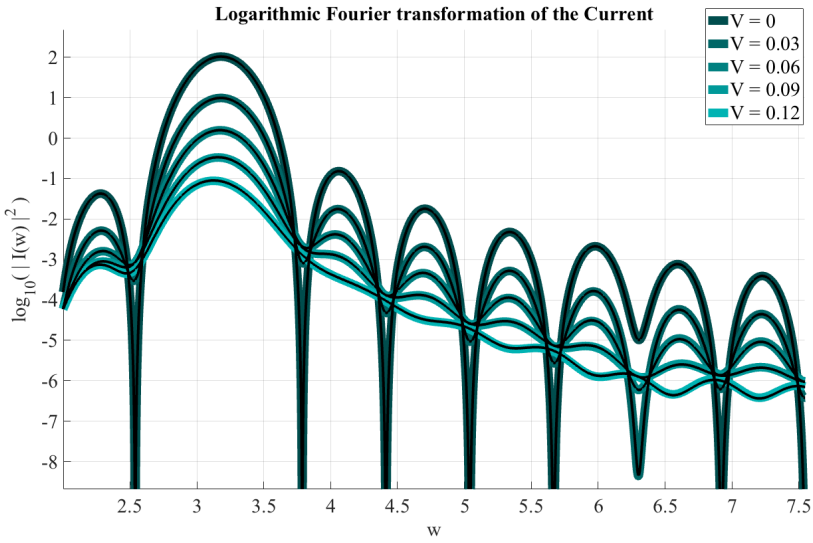

Figure 5: Logarithmic Fourier transformation of the current as a function of frequency. Here w_0 is fixed while the driving frequency w is varied.

Figure 5, shows the Logarithmic Fourier transformation of the current as a function of frequency. We clearly see that there is an increase in the energy of the current around $w \approx 3.2$, this is when the driving frequency approaches the systems natural frequency, such that $w \approx w_0$. We also see a small effect in the amplitude when the driving frequency approaches $2w_0$. When the applied voltage is increased there is still an effect in the energy around w_0 and $2w_0$. But for higher voltages the systems damping is also increasing which makes the resonance behaviour decreases.

5 Discussion

In this part of the report we will try to analyze the displacement of the vibrating island and the currents behaviour for different voltages.

In previous section, by studying figure 1,2 and 3. We analyzed the behaviour of the displacement of the vibrating island, for zero applied voltage, a small applied voltage and for a large applied voltage. If we vary the voltage in this interval, where the analytical- and numerical solutions are in an agreement with each other, the behaviour of the displacement is still the same. The equilibrium position is shifted upwards depending on the applied voltage, this is because the force is directly proportional V^3. This behaviour were predicted for when a constant force is acting on the system. Since the linear approximation for the analytical solution makes the force approximately constant, this is what we would expect. We could also see that the oscillatory motion decreases with time and for higher voltage the damping defined as Γ increases. The damping effecting the system is probably due to how the current affects the mechanical vibrations of the island, since it becomes zero when the applied voltage is zero. To see how the system reacts if we change the strength of the driving force or the damping we can set them to be equal to zero. When only the defined driving force is set to zero the oscillatory motion still decreases with time. But the equilibrium position is not shifted and the oscillations are symmetric around 0. Then if we restore the force and change the damping to zero. The oscillations remains with time and does not decrease. We can also see the shift upwards in the equilibrium position of the oscillations. This result is what we would expect for the driving force and the damping coefficient.

When we analyzed the current in figure 4, we saw that it was dependent on the vibrating islands displacement. When a applied voltage were affecting the system it damped the oscillatory behaviour of the current, such that the current became constant after a while. We also saw that for different applied voltage, there were different phase shifts in the oscillatory motion of the current. The strength of the tunneling current were also very dependent on the strength of the applied voltage. Then when the damping were set to be zero, to study how the system reacted, the current remained oscillatory and didn't become constant with time. But when the damping were restored and the force were set to zero, there were no visible difference in the currents behaviour.

Then we tried to study the Logarithmic Fourier transformation of the current as a function of the driving frequency. We could see that when the applied voltage were low, we clearly had a resonance behaviour when the driving frequency approached the systems natural frequency. Then when the voltage were increased the amplitude decreased. This is probably dependent on the damping coefficient of the system that is increasing with the applied voltage. For a resonance behaviour to occur the damping of the system needs to be small. If we continue to increase the voltage the peaks becomes more damped until they almost disappears. The natural frequency of the system, w_0 defined in equation (20), is directly dependent on the constant J_1, in equation (12). Since J_1 is proportional to V^3 the applied voltage affects the systems natural frequency. The resonance frequency of the system is approximately the same as the systems natural frequency, so it is also changing for different voltages.

6 Recommendation

If further research on this project would be of interest it might be possible, with access to the right material, to try verifying this behaviour experimentally. I believe it would be possible since there have been experiments to verify the Josephson current. Another thing that can be improved in this project is the coefficients for the current and Hamiltonian. Since only arbitrary values could be used for the coefficients. There might be possible to correct the coefficients with correct units and get a more complete theory statement.

7 Conclusion

In this report a study have been made on how a current of tunneling electrons can effect a system with a normal metal and a topological insulator, while one of the materials vibrates. The calculations and the result is theoretical in the research field of Materials physics. The provided model was solved both analytically and numerically and the solutions were in good agreement with each other. Only arbitrary values could be used in the calculations, since the dimensional analysis of the coefficients in the provided model yields different results. The behaviour of the current and displacement of the vibrating island were studied for different applied dc voltages. From the result it was concluded that the vibrating island were affected by the tunneling current. The displacement of the vibrating island showed a underdamped oscillatory motion and the equilibrium position were shifted upwards. We could also see a phase shift in the oscillatory motion that were dependent on the applied voltage. The currents behaviour seams to be very dependent on the vibrating materials displacement. For higher voltages the strength of the current were increased and there were also a visible phase shift in the oscillatory motion. From studying the Logarithmic Fourier transform of the Current as a function of the driving frequency we could clearly see resonance behaviour for low voltages. Higher applied voltage damped the resonance behaviour of the system. We also concluded that the resonance frequency should be affected by the applied voltage since the systems natural frequency is directly dependent on the voltage.

8 References

[1]: Fransson J, Jian-Xin Zhu, Balatsky A.V. Vibrating Superconducting Island in a Josephson Junction. Physical review letters. August 2008

[2]: Fransson J, private communication, 2019.

[3]: Ashcroft N.W, Mermin N.D. Solid State Physics. Philadelphia: Saunders College, cop; 1976.

[4]: Tipler P.A, Llewellyn R.A. Modern Physics. 5. uppl. New York: W.H Freeman; 2007.

[5]: Taylor J.R. Classical mechanics. Sausalito, California: University Science Books; 2004.

[6]: Goldstein H, Poole C, Safko J. Classical mechanics. 3. uppl. San Francisco: Addison Wesley; 2002.

[7]: Råde L, Westergren B. Mathematics Handbook. 5:7. uppl. Lund: Studentlitteratur AB; 2004.

9 Appendix

9.1 Homogeneous solutions for overdamped motions

If $(\frac{\Gamma}{2})^2 - {w_0}^2 > 0$:

The roots of the characteristic equation can be expressed as

$$r_{\pm} = -\frac{\Gamma}{2} \pm \sqrt{(\frac{\Gamma}{2})^2 - {w_0}^2} \tag{33}$$

The homogeneous solution is

$$u_h(t) = A_1 e^{r_+ t} + A_2 e^{r_- t} \tag{34}$$

If $(\frac{\Gamma}{2})^2 - {w_0}^2 = 0$:

The roots of the characteristic equation can be expressed as

$$r = r_+ = r_- = -\frac{\Gamma}{2} \tag{35}$$

The homogeneous solution is

$$u_h(t) = A_1 e^{rt} + A_2 t e^{rt} \tag{36}$$

9.2 MATLAB code

9.2.1 Skript

```
%% Elektrisk styrning av mekaniska vibrationer %%
%% Carolina Nilsson, Degree project C
clear all; close all;
tic;
global k alpha J1 Gamma2 m Gamma;
%% Time interval and stepsize %%
time = 10;
h = 0.001;
%% Initial conditions %%
v = [0 0.1 0.2 0.3 0.4 0.5 0.6 0.7]; % [Voltage]
u0 = 0;
p0 = 10^(-2);
%% Loop for different voltages %%
for j = 1:length(v)
V = v(j);
%% Arbitrary values %%
k = 0.1;
m = 0.01;
Tao = 1;
e = 1;
alpha = Tao/10;
Dc = 0.6;
D = 1;
h_bar = 1;
N = 1;
%% Constants %%
Tao2_N = N*(Tao^2);
J0 =  (pi*(e^3)*N)/(2*h_bar*(Dc^2));
Gamma1 =  -(2*e*N)/(h_bar*(Dc^2));
Gamma0 =  ((2*e*N*D)/(h_bar*(Dc^2)))*(1+log(Dc/D));

if V == 0
    C = 0;
else
C = log(abs((e*V)/2)/Dc);
end

J1 = ((h_bar*(V^3)*(Tao^2)*alpha)/(3*e))*J0;
Gamma2 = -((2*(alpha^2)*(Tao^2)*h_bar)/e)*( Gamma0*V
+ Gamma1*( ((e*(V^2))/4) * ( 3/2 - C) ));
Gamma = -(Gamma2/m);
```

```
%% Frequency %%
w0 = sqrt(k/m);
w1 = sqrt((k+2*alpha*J1)/m);
f0 = w0/(2*pi);
f1 = w1/(2*pi);
%% Driving force %%
F =  J1/m;

%% Check for which solution that is being used %%
Omega = (Gamma/2)^2-w1^2;
%% Initial conditions RK4 %%
Y_4(1,1) = u0 ; % u(t)
Y_4(1,2) = p0; %p(t)
%% Theoretical constants simple H.O %%
B1 = u0;
B2 = p0/(m*w0);
%% Iteration values RK4 %%
t = 0;
i = 1;
T(1,1) = 0;
%% RK4 calculation %%
while t<time
%% RK4 %%
    k1_4 = RungekuttaSolver(Y_4(i,:));
    k2_4 = RungekuttaSolver(Y_4(i,:)+(h/2).*k1_4);
    k3_4 = RungekuttaSolver(Y_4(i,:)+(h/2).*k2_4);
    k4_4 = RungekuttaSolver(Y_4(i,:)+h.*k3_4);
%% Computation %%
    Y_4(i+1,:) = Y_4(i,:) + (h/6).*(k1_4 + 2.*k2_4 + 2.*k3_4 + k4_4);
%% Iteration %%
    i = i+1;
    t = t+h;
    T(i) = t;
end
%% Analytical Solution %%
theo = AnalyticalSolutions(B1,B2,T,w0,w1,F,m,Gamma,Omega,p0);
%% Current computation %%
if ((e*V)/2) < 0
    beta = ((e*V)/2);
else
    beta = 0;
end

if V == 0
    C2 = 0;
else
C2 = log(Dc/(abs((e*V)/2)));
end
```

```
%% Current for oscillator %%
I_analytisk(j,:) = (sign(V)*J0*(V^2)*(Tao^2)).*exp(-2.*alpha.*theo(3,:))
+ (-2.*alpha.*(Tao^2) .* exp(-2.*alpha.*theo(3,:)) .* (theo(4,:)./m) .* ( Gamma0
+ Gamma1*(((e*V)/2) * (1+C2) + beta/(1-(((e*V)/(2*D))^2))) ) );

I_numerisk(j,:) = ((sign(V)*J0*(V^2)*(Tao^2)).*exp(-2.*alpha.*Y_4(:,1))
+ (-2.*alpha.*(Tao^2) .* exp(-2.*alpha.*Y_4(:,1)) .* (Y_4(:,2)./m) .* ( Gamma0
+ Gamma1*(((e*V)/2)*(1+C2) + beta/(1-(((e*V)/(2*D))^2)))) ) )';
%% Figures %%
%% Colors for figures %%
z = zeros(1,10);
color3 = [zeros(13,1),[0.3:0.05:0.9]',[0.3:0.05:0.9]'];
%% Figures Displacement %%
figure(j)
hold on
plot(T,theo(1,:),'Color',color3(1,:),'Linewidth',2)
plot(T,theo(3,:),'Color',color3(11,:), 'Linewidth',2)
plot(T,Y_4(:,1),'Color',[0 0 0],'Linewidth', 2)
hold off
xlabel('Time, t');ylabel('Displacement u(t)'); grid on;
title(['Displacement u(t) for Voltage: ' num2str(V)])
legend('Simple Harmonic Oscillator','Analytical Solution','Numerical Solution')
%% Error %%
 Error_hn = abs((theo(1,:)-Y_4(:,1)'));
 Error_ha = abs((theo(1,:)-theo(3,:)));
 Error_an = abs((theo(3,:)-Y_4(:,1)'));
 figure(length(v)+j)
 subplot(3,1,1)
 plot(T,Error_ha)
 title(['Error displacement H.O, Analytical. Voltage: ' num2str(V) ])
 grid on;
 subplot(3,1,2)
 plot(T,Error_hn)
 title(['Error displacement H.O, Numerical. Voltage: ' num2str(V) ])
 grid on;
 subplot(3,1,3)
 plot(T,Error_an)
 title(['Error displacement Analytical, Numerical. Voltage: ' num2str(V) ])
 grid on;
 %% Controll of approximation %%
disp(['u*alpha numerisk = ' num2str(Y_4(length(theo)-1,1)*alpha)])
disp(['u*alpha analytisk = ' num2str(theo(3,length(theo)-1)*alpha)])
end
```

```
%% Fouriertransform %%
N = 10000;
freq = linspace(0,4*w1,N);
for j = 1:1:length(v)
for x = 1:1:N
    funct1(j,x) = trapz(T,I_analytisk(j,:).*exp(-1i.*freq(x).*T));
    funct2(j,x) = trapz(T,I_numerisk(j,:).*exp(-1i.*freq(x).*T));
end
end

figure;
hold on
plot(freq,log(abs(funct1(1,:)).^2),'Color',color3(1,:),'Linewidth', 13)
plot(freq,log(abs(funct1(2,:)).^2),'Color',color3(3,:),'Linewidth', 13)
plot(freq,log(abs(funct1(3,:)).^2),'Color',color3(5,:),'Linewidth', 13)
plot(freq,log(abs(funct1(4,:)).^2),'Color',color3(7,:),'Linewidth', 13)
plot(freq,log(abs(funct1(5,:)).^2),'Color',color3(9,:),'Linewidth', 13)
plot(freq,log(abs(funct1(6,:)).^2),'Color',color3(10,:),'Linewidth', 13)
plot(freq,log(abs(funct1(7,:)).^2),'Color',color3(11,:),'Linewidth', 13)
plot(freq,log(abs(funct1(8,:)).^2),'Color',color3(12,:),'Linewidth', 13)
title('Logarithmic Fourier transformation of the Current')
xlabel('w');ylabel('log_{10}( | I(w) |^{2} )');
plot(freq,log(abs(funct2).^2),'k','Linewidth', 4)
legend('V = 0','V = 0.03','V = 0.06','V = 0.09','V = 0.12')
grid on;
hold off

%% Figures Current %%
figure;
n = [1 3 5 7 9 10 11 12];
for j = 1:length(v)
hold on
plot(T,I_analytisk(j,:),'Color', color3(n(j),:),'Linewidth',10)
xlabel('Time, t');ylabel('Current I(t)');
%axis([0 4 -0.6 2.2]);
grid on;
title('Current for different Voltages')
end
plot(T,I_numerisk,'k','Linewidth',4)
legend('V = 0', 'V = 0.1', 'V = 0.2', 'V = 0.3', 'V = 0.4', 'V = 0.5',
'V = 0.6', 'V = 0.7')
toc;
```

9.2.2 Analytical Solution

```
%% Analytical Solution for Elektrisk styrning av mekaniska vibrationer %%
%% Carolina Nilsson
function exact = AnalyticalSolutions(B1, B2, t, w0 ,w1,F,m,Gamma,Omega,p0)
exact = zeros(4,length(t));
Constant = (F/((w1^2)));
%% Simple Harmonical Oscillator
exact(1,:) = B1*cos(w0*t)+ B2*sin(w0*t); %u(t)
exact(2,:) = m*(-B1*w0*sin(w0*t)+B2*w0*cos(w0*t)); %p(t)
%% Analytical solution
if Omega > 0
    wp = -(Gamma/2)+sqrt(Omega);
    wm = -(Gamma/2)-sqrt(Omega);
    A2 = (1+((F*wp)/(w1^2)))/(m*(wm-wp));
    A1 = -A2-(F/(m*(w1^2)));
    uh = (A1*exp(wp*t))+(A2*exp(wm*t));
    ph = m*(A1*wp*exp(wp*t)+A2*wm*exp(wm*t));
elseif Omega < 0
    A1 = -Constant;
    A2 = (2*p0-m*Gamma*Constant)/(2*m*sqrt(-Omega));
    uh = exp(-(Gamma/2)*t).*(A1*cos(sqrt(-Omega)*t)+A2*sin(sqrt(-Omega)*t));
    ph = m*(-(Gamma/2).*uh + exp(-(Gamma/2)*t).*( -A1*sqrt(-Omega)*sin(sqrt(-Omega)*t)
    + A2*sqrt(-Omega)*cos(sqrt(-Omega)*t) ));
else
    A1 = -Constant;
    A2 = (2*p0-m*Gamma*Constant)/(2*m);
    uh = A1*exp(-(Gamma/2)*t)+A2*t*exp(-(Gamma/2)*t);
    ph = m*((-Gamma/2)*(uh)+A2*exp(-(Gamma/2)*t));
end
up = Constant;
exact(3,:) = uh+up; %u(t)
exact(4,:) = ph; %p(t)
end
```

9.2.3 RK4 Solver

```
%% RK4 for Elektrisk styrning av mekaniska vibrationer %%
%% Carolina Nilsson
function derivative = RungekuttaSolver(Y)
global k alpha J1 Gamma2 m;
%% Nonlinear coupled system of differential equations%%
derivative = [ Y(2)/m , (-k*Y(1))+J1*exp(-2*alpha*Y(1))
+Gamma2*exp(-2*alpha*Y(1))*(Y(2)/m)];
end
```