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7D supersymmetric Yang-Mills theory on toric and hypertoric manifolds

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Abstract
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This thesis consists of an introduction and three research papers in the general area of geometry and physics. In particular we study 7D supersymmetric Yang-Mills theory and related topics in toric and hypertoric geometry. Yang-Mills theory is used to describe particle interactions in physics but it also plays an important role in mathematics. For example, Yang-Mills theory can be used to formulate topological invariants in differential geometry. In Paper I we formulate 7D maximally supersymmetric Yang-Mills theory on curved manifolds that admit positive Killing spinors. For the case of Sasaki-Einstein manifolds we perform a localisation calculation and find the perturbative partition function of the theory. For toric Sasaki-Einstein manifolds we can write the answer in terms of a special function that count integer lattice points inside a cone determined by the toric action. In Papers II and III we consider 7D maximally supersymmetric Yang-Mills theory on hypertoric 3-Sasakian manifolds. We show that the perturbative partition function can again be formulated in terms of a special function counting integer lattice points in a cone, similar to the toric case. We also present a factorisation result for these functions.

Keywords: Yang-Mills theory, supersymmetry, toric geometry, hypertoric geometry, Sasaki-Einstein, 3-Sasaki, localisation, special functions

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Dedicated to my parents

List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I K. Polydorou, A. Rocén and M. Zabzine, *7D supersymmetric Yang-Mills on curved manifolds*, JHEP **1712** (2017) 152. [arXiv: 1710.09653]

II A. Rocén, *7D supersymmetric Yang-Mills on a 3-Sasakian manifold*, JHEP **1811** (2018) 24. [arXiv: 1808.06917]

III N. Iakovidis, J. Qiu, A. Rocén and M. Zabzine, *7D supersymmetric Yang-Mills on hypertoric 3-Sasakian manifolds*, Manuscript.

Contents

1. Introduction

The main part of this thesis consists of the three appended papers. In this introduction we briefly review some background material and survey some of the key concepts discussed in the thesis. The aim of this introduction is to give a general idea of the topics studied in the thesis and we refer the interested reader to the references provided for more thorough treatments.

This thesis considers supersymmetric quantum field theory and related geometrical structures. Quantum field theory was originally developed by physicists in the last century to describe the forces of nature, culminating in the ‘Standard model’ of particle physics. However, in the past fifty years or so, there has been a wide range of applications of field-theoretic ideas to mathematics. One famous example is Donaldson’s [1] construction of topological invariants for four-manifolds, based on the ideas of gauge theories. By studying the space of anti-self-dual connections (instantons) he constructed topological invariants sensitive to smooth structures. This resulted in the surprising result that there are ‘exotic’ $\mathbb{R}^4$’s, i.e. spaces homeomorphic but not diffeomorphic to $\mathbb{R}^4$. These ideas also inspired Floer’s work on 3-manifolds. After developing an infinite-dimensional version of Morse theory, based on earlier work of Witten [2], Floer [3] considered the Chern-Simons functional on the space of connections of a principal SU(2)-bundle over the 3-manifold M_3. The critical points are the flat connections and their flow lines can be interpreted as instantons on $M_3 \times \mathbb{R}$. Witten [4] showed that these ideas of Donaldson and Floer could be formulated in terms of twisted $N = 2$ supersymmetric Yang-Mills theory.

Going up in dimensions there exist similar invariants introduced by Donaldson-Thomas [5]. These can also be interpreted physically in terms of supersymmetric string and gauge theories, see e.g. [6].

Supersymmetric field theories thus play a key role in several topics of modern mathematics. In this thesis we consider 7D supersymmetric Yang-Mills theory and its connection to toric and hypertoric geometry. The theory considered here can be regarded as a 7D lift of the 6D Hermitian-Yang-Mills theory and might thus be of relevance to enumerative geometry in higher dimensions.

2. Yang-Mills theory

The concept of gauge theory was central to the development of theoretical physics in the past century. At the same time, mathematical concepts such as fibre bundles and characteristic classes lead to new results in differential geometry and topology. In the 1970's people started realising that these two topics were very much related - gauge theories could be stated in terms of principal bundles, gauge fields in terms of connections on them, etc.

A very important class of gauge theories are the so called Yang-Mills theories. In this section we aim to give a pedagogical introduction to Yang-Mills theory and related concepts such as instantons. The approach we take here is to introduce Yang-Mills theory as a non-abelian generalisation of electromagnetism and hence we begin the discussion there.

2.1 Electromagnetism

In classical electromagnetism the source free Maxwell's equations are

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \qquad \nabla \cdot \vec{B} = 0, \tag{2.1}$$

$$\nabla \times \vec{B} = \frac{\partial \vec{E}}{\partial t}, \qquad \nabla \cdot \vec{E} = 0, \tag{2.2}$$

where $\vec{E} = (E_x, E_y, E_z)$ and $\vec{B} = (B_x, B_y, B_z)$ are the electric and magnetic fields.

In Lorentzian spacetime $\mathbb{R}^{1,3}$, with metric $\eta = diag(+ - - -)$ we can re-package these fields into the electro-magnetic tensor

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}. \tag{2.3}$$

Electro-magnetic duality states that Maxwell's equations are unchanged under

$$\vec{E} \to -\vec{B}, \quad \text{and} \quad \vec{B} \to \vec{E}\,. \tag{2.4}$$

This corresponds to taking the dual of the electro-magnetic tensor:

$$*F_{\mu\nu} := \tfrac{1}{2}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & -E_z & E_y \\ B_y & E_z & 0 & -E_x \\ B_z & -E_y & E_x & 0 \end{pmatrix} . \tag{2.5}$$

We can then re-state the source-free Maxwell's equations as

$$\partial_\mu F^{\mu\nu} = 0, \quad \partial_\mu \left(*F^{\mu\nu}\right) = 0 . \tag{2.6}$$

These equations have a natural formulation in terms of differential forms, where we consider the 2-form

$$F = \tfrac{1}{2}F_{\mu\nu}\, dx^\mu \wedge dx^\nu . \tag{2.7}$$

The electro-magnetic dual $*F$ is then the Hodge-dual

$$*F = \tfrac{1}{4}\varepsilon_{\mu\nu\alpha\beta}F^{\alpha\beta}dx^\mu \wedge dx^\nu . \tag{2.8}$$

The source-free Maxwell's equations can then be stated as

$$dF = 0, \quad d * F = 0 , \tag{2.9}$$

where d denotes the deRham differential and $*$ the Hodge-star.

By the Poincaré Lemma $dF = 0$ means that we can write $F = dA$, i.e.

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu . \tag{2.10}$$

This A is called the 4-potential and physically $A_\mu = (A_0, \vec{A})$, where A_0 is the electric potential and $\vec{A}$ the magnetic potential:

$$\vec{E} = -\nabla A_0 - \frac{\partial \vec{A}}{\partial t}, \quad \vec{B} = \nabla \times \vec{A} . \tag{2.11}$$

This leads us to the Lagrangian formulation of electromagnetism. Here A_μ is the main object of study, and the action is given by

$$S = \int d^4x\, \mathcal{L} = \int d^4x \, \left(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}\right) \quad \leftrightarrow \quad S = -\tfrac{1}{2}\int F \wedge *F . \tag{2.12}$$

The Euler-Lagrange equations (equations of motion)

$$\delta S = 0 \iff \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)}\right) - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0 , \tag{2.13}$$

are given by

$$\partial_\mu F^{\mu\nu} = 0 \quad \leftrightarrow \quad d * F = 0 . \tag{2.14}$$

From this viewpoint the Maxwell's equation $dF = 0$ is automatically satisfied since $dF = d^2A = 0$ and $d * F = 0$ corresponds to the equation of motion.

Apart from standard global symmetries, such as translations, this theory is also invariant under the following transformation

$$A_\mu \to A_\mu + \partial_\mu f\,, \tag{2.15}$$

where f is any scalar function. Since the curl of a gradient is zero we see that the fields (2.11) are invariant under this transformation. This is an example of a gauge symmetry, i.e. a local symmetry where the parameter $f(x)$ depends on the spacetime coordinate x. In Yang-Mills theory, which we introduce shortly, we will see how to generalise this concept.

2.2 Yang-Mills theory

Yang-Mills theories can be considered to be non-abelian generalisations of electromagnetism. They were introduced by Yang and Mills in 1954 [7] and have since become cornerstones of the 'Standard model' of particle physics where the electromagnetic, weak and strong forces are described by Yang-Mills theories. Yang-Mills theory has also played a central role in mathematics, as we will discuss later in this section.

Let G be a compact semi-simple Lie group with Lie algebra $\mathfrak{g}$. For concreteness, let us focus on $G = \mathrm{SU}(2)$. We take the generators of the Lie algebra to be $T^a = -\frac{i}{2}\sigma^a$, $a = 1, 2, 3$, where σ^a are the Pauli spin matrices

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.16}$$

Then

$$[T^a, T^b] = \varepsilon^{abc} T^c\,, \tag{2.17}$$

i.e. the structure constants are given by $f^{abc} = \varepsilon^{abc}$ and

$$\mathrm{Tr}(T^a T^b) = -\tfrac{1}{2}\delta^{ab}\,. \tag{2.18}$$

Let us describe SU(2) Yang-Mills theory in $\mathbb{R}^4$. Note that we now consider Euclidean signature (as opposed to Lorentzian in the previous section). Here the gauge potential $A_\mu(x)$ is a matrix-valued vector field $A_\mu(x) = A^a_\mu(x) T^a$, where T^a form the representation of $\mathfrak{su}(2)$ discussed above. The field-strength tensor $F_{\mu\nu}$ is given by

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]\,. \tag{2.19}$$

In terms of the covariant derivative

$$D_\mu = \partial_\mu + [A_\mu, \cdot] \quad \leftrightarrow \quad d_A = d + [A, \cdot] \tag{2.20}$$

we can write

$$F_{\mu\nu} = [D_\mu, D_\nu] \, . \tag{2.21}$$

A gauge transformation is a function $g(x) : \mathbb{R}^4 \to \mathrm{SU}(2)$. The gauge transformations of A_μ and $F_{\mu\nu}$ are given by

$$A_\mu \to g^{-1} A_\mu g + g^{-1} \partial_\mu g \, , \tag{2.22}$$

$$F_{\mu\nu} \to g^{-1} F_{\mu\nu} g \, . \tag{2.23}$$

We could also write $g(x) = \exp(\lambda^a(x) T^a)$, for some functions λ^a and re-state the above gauge transformations in their infinitesimal forms.

The Yang-Mills action is given by

$$S = \int d^4x \ \mathrm{Tr}\left(-\tfrac{1}{2} F_{\mu\nu} F^{\mu\nu}\right) \quad \leftrightarrow \quad S = -\int \mathrm{Tr}\left(F \wedge *F\right) \tag{2.24}$$

which leads to the Euler-Lagrange equation

$$D_\mu F^{\mu\nu} = 0 \quad \leftrightarrow \quad d_A * F = 0 \, . \tag{2.25}$$

This equation is called the Yang-Mills equation.

Note that $d_A^2 \neq 0$, but $d_A F = 0$ still holds by the Jacobi identity.

The question arises how to find (finite-action) solutions to the Yang-Mills equation. To do so, one can use a 'Bogomolni trick' as in [8]. On 2-forms the Hodge star squares to one and we can define the projectors

$$P^\pm = \tfrac{1}{2}(1 \pm *) \, . \tag{2.26}$$

This leads to an orthogonal decomposition of 2-forms:

$$F = F^+ + F^- \, , \tag{2.27}$$

where

$$*F^\pm = \pm F^\pm \, . \tag{2.28}$$

We can then write the action as

$$S = -\left(\int \mathrm{Tr}\left(F^+ \wedge *F^+\right) + \int \mathrm{Tr}\left(F^- \wedge *F^-\right) \right) \tag{2.29}$$

and obtain the bound

$$S \geq \left| \int \mathrm{Tr}\left(F^+ \wedge *F^+\right) - \int \mathrm{Tr}\left(F^- \wedge *F^-\right) \right| = \left| \int \mathrm{Tr}\left(F \wedge F\right) \right| \, . \tag{2.30}$$

We get equality when $F = F^+$ or $F = F^-$, i.e. when $F = \pm * F$.

To summarise, the Yang-Mills action is given by

$$S = \int d^4x \ \mathrm{Tr}\left(-\tfrac{1}{2} F_{\mu\nu} F^{\mu\nu}\right) \quad \leftrightarrow \quad S = -\int \mathrm{Tr}\left(F \wedge *F\right) \, . \tag{2.31}$$

The equations of motion are given by the Yang-Mills equation

$$D_\mu F^{\mu\nu} = 0 \quad \leftrightarrow \quad d_A * F = 0 \tag{2.32}$$

and there is a bound on the action that is saturated when

$$F = \pm * F\,. \tag{2.33}$$

The above equation is called the instanton equation and its solutions are called instantons (for the + sign) or anti-instantons (for the − sign). Moreover, such solutions automatically solve the Yang-Mills equation since

$$d_A * F = \pm d_A F = 0. \tag{2.34}$$

The point of the 'Bogomolni trick' is thus that we can find solutions to a second order equation (the Yang-Mills equation) by instead solving a first order equation (the instanton equation).

Another interesting aspect is that the lower bound of the action has a topological interpretation. In fact

$$S \geq 8\pi^2 \left| N \right| ,$$

where

$$N = -\frac{1}{8\pi^2} \int \mathrm{Tr}\,(F \wedge F)$$

is an integer called the 'instanton number' and is the integral of the second Chern class. (Recall that we represented the Lie algebra by traceless matrices.)

We have a bound on the action for each N which is satisfied by (anti)-instantons $F = \pm * F$. The question arises how many solutions there are for each N (dimension of moduli space) and how to construct these solutions explicitly. In 1977 Schwarz [9] and Atiyah-Hitchin-Singer [10] used index theorems to show that the most general soultion has $8N$ parameters. A year later Atiyah-Drinfeld-Hitchin-Manin [11] provided an explicit solution on $\mathbb{R}^4$ with $8N$ parameters, know as the 'ADHM-construction'. The construction turns the analytic problem of finding solutions to the PDE to a purely algebraic problem in terms of a quaternionic matrix satisfying some conditions.

It is rather remarkable that already the simplest non-abelian gauge theory, SU(2) Yang-Mills, on the simplest possible 4-manifold, $\mathbb{R}^4$, leads to such interesting problems. Many generalisations of this set-up have been studied. With some modifications the ADHM-construction can be extended to the gauge groups SU(n), SO(n) and Sp(n). To generalise to other manifolds requires the mathematical formulation of gauge theories in terms of principal fibre bundles. In this picture the gauge field is formulated in terms of a connection and the field strength corresponds to its curvature, see e.g. [12] for a textbook discussion.

Donaldson [1] used instantons to define topological invariants on 4-manifolds. These invariants are sensitive to the smooth structure and one remarkable result is that there exist 'exotic' 4-spaces that homeomorphic but not diffeomorphic to $\mathbb{R}^4$. For this work Donaldson was awarded the Fields Medal in 1986.

The work of Donalson shows how mathematics and physics can interact in a very fruitful way, and there have been many examples of this since. This thesis aims to make a modest contribution to this tradition. On one hand we make use of mathematical results such as equivariant localisation and toric geometry to study physical theories. On the other hand we use physics arguments to find mathematical statements, such as factorisations and other properties of special functions.

In Paper I we consider a 7D generalisation of the instanton equation (2.33), similar to that previously considered in 5D [13]. We also obtain a 7D lift of the Hermitian-Yang-Mills equations studied in e.g. [6]. Although we do not pursue this point further in this thesis, the hope is that these equations could be mathematically interesting as well. It is plausible that the results presented in this thesis could have applications to Donaldson-Thomas theory [5].

2.3 Supersymmetric Yang-Mills theory

The work in this thesis deals with *supersymmetric* Yang-Mills theory. Supersymmetry, or SUSY for short, is a proposed symmetry that relate fermions and bosons. Although there is currently no experimental evidence that supersymmetry exists in nature, it has been widely studied by the theoretical physics community and there has also been interesting applications to mathematics. In fact, supersymmetry is a key component in many of the relations between mathematics and physics we have discussed previously. For example, Donaldson-Witten theory is formulated on the physics side in terms of supersymmetric Yang-Mills theory.

To make the Yang-Mills theory discussed in the previous section supersymmetric we need to introduce a fermionic partner to the bosonic field A_μ. We also need to formulate a supersymmetry transformation that mixes these two. Let Ψ be a massless spin-$\frac{1}{2}$ particle in the adjoint representation of the gauge group. Consider the D-dimensional action [14]

$$S_D = \int d^D x \, \mathrm{Tr} \left(-\tfrac{1}{2} F_{MN} F^{MN} + \bar{\Psi} \Gamma^M D_M \Psi \right) . \tag{2.35}$$

Whether or not this action can be supersymmetric (without adding additional fields) depends on the dimension D. It turns out that the possible dimensions are $D = 3, 4, 6, 10$ which is related to the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$, see e.g. [15] and references therein. Supersymmetric Yang-Mills theories in other dimensions can be obtained from dimensional reductions of these theories. Since this thesis deals with dimension seven, the $D = 10$ theory is the one relevant for us.

For the theory to be supersymmetric the bosonic and ferminonic degrees of freedom must match. In $D = 10$ we thus require the spinors to be both Majorana and Weyl. The 10-dimensional action

$$S_{10} = \int d^{10}x \ \mathrm{Tr}\left(\tfrac{1}{2}F_{MN}F^{MN} - \Psi\Gamma^M D_M\Psi\right) \tag{2.36}$$

can be shown to be invariant under the supersymmetry transformations

$$\delta A_\mu = \epsilon\Gamma_\mu\Psi\,, \tag{2.37}$$

$$\delta\Psi = \tfrac{1}{2}\Gamma^{MN}F_{MN}\epsilon\,, \tag{2.38}$$

where ϵ is a constant Majorana-Weyl spinor. The invariance relies on the existence of Fierz identities for Clifford algebras, and again this is related to list of possible dimensions $D = 3, 4, 6, 10$.

To obtain sypersymmetric Yang-Mills theories in other dimensions one need to include more fields. One approach to do this is to make dimensional reductions and this is the approach we take to obtain 7D supersymmetric Yang-Mills theory discussed in this thesis, see e.g. Paper I. The 7D supersymmetric Yang-Mills theory is unique and it is maximally supersymmetric with 16 supercharges.

The compactification gives rise to scalars ϕ_A, $A = 0, 8, 9$ and the action and supersymmetry transformations are modified to

$$\begin{aligned} S_{7D} = \frac{1}{g_{7D}^2}\int d^7x\sqrt{-g}\,\mathrm{Tr}\Big(\frac{1}{2}F^{MN}F_{MN} - \Psi\Gamma^M D_M\Psi + 8\phi^A\phi_A \qquad (2.39) \\ + \frac{3}{2}\Psi\Lambda\Psi - 2[\phi^A, \phi^B]\phi^C\varepsilon_{ABC}\Big) \end{aligned}$$

and

$$\begin{aligned} \delta_\epsilon A_M &= \epsilon\Gamma_M\Psi\,, \\ \delta_\epsilon\Psi &= \tfrac{1}{2}F_{MN}\Gamma^{MN}\epsilon + \tfrac{8}{7}\Gamma^{\mu B}\phi_B\nabla_\mu\epsilon\,, \end{aligned} \tag{2.40}$$

see [16] and Paper I for details.

This construction relies on the existence of a 10-dimensional Majorana-Weyl spinor ϵ satisfying the generalised Killing spinor equation

$$\nabla_\mu\epsilon = \frac{1}{2}\tilde{\Gamma}_\mu\Lambda\epsilon\,. \tag{2.41}$$

It turns out that the existence of such a spinor puts heavy constraints on which geometries we can consider. In Paper I we argue that such a

10D spinor ϵ can be constructed from positive Killing spinors on the 7D manifold. Manifolds admitting such spinors have been classified [17] and in 7D they are

- The seven-sphere S^7 (16 Killing spinors)
- 3-Sasakian manifolds (3 Killing spinors)
- Sasaki-Einstein manifolds (2 Killing spinors)
- Proper G_2-manifolds (1 Killing spinor)

In the next section we will describe these and other geometrical structures further.

Let us briefly remark that there may well be other 7D spaces on which one could place the theory. In view of the work of Festuccia and Seiberg [18] the equation (2.41) is just the first instance of a more general equation arising from supergravity considerations. However, very little is known about the existence of solutions to such an equation in higher dimensions and we do not pursue this further in this thesis.

3. Geometry

3.1 Symplectic geometry

In this section we give a very brief introduction to some topics in symplectic geometry that are of special relevance to this thesis. We refer the reader to e.g. [19] for a thorough introduction.

A smooth manifold M is said to be symplectic if it admits a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$, called a symplectic form. Such a manifold is necessarily of real dimension $2n$. Non-degeneracy means that ω^n never vanishes and thus provides a volume form, making M orientable. The simplest example of a symplectic manifold is $\mathbb{R}^{2n}$ with the standard symplectic form

$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i \,. \tag{3.1}$$

The Darboux theorem states that locally the symplectic form can always be put into this standard form. Thus there are no local invariants in symplectic geometry, only global.

3.1.1 Classical mechanics

Symplectic geometry arises naturally in classical mechanics in $\mathbb{R}^3$ when one considers the phase space $M = T^*\mathbb{R}^3 \cong \mathbb{R}^6$, where the coordinates (q_i, p_i), $i = 1, 2, 3$, represent position and momentum respectively. The standard symplectic form is given by

$$\omega = \sum_{i=1}^{3} dq_i \wedge dp_i \,. \tag{3.2}$$

A vector field X_H on M such that

$$\iota_{X_H}\omega = dH \tag{3.3}$$

for some function H is called a Hamiltonian vector field with Hamiltonian function H. In mechanics, the energy of the system is given by the Hamiltonian H. Since ω is non-degenerate we can solve

$$\iota_{X_H}\omega = dH \tag{3.4}$$

for the Hamiltonian vector field X_H. The time-evolution of the system then corresponds to flowing with this vector field

$$\dot{x} = X_H(x)\,, \tag{3.5}$$

where $x = (q_i, p_i)$ denotes the coordinate on M. Let us show that this corresponds to Hamilton's equations of classical mechanics. We have

$$\dot{x} = (\dot{q}, \dot{p}) = \dot{q}_i \frac{\partial}{\partial q_i} + \dot{p}_i \frac{\partial}{\partial p_i} \tag{3.6}$$

and so from (3.2) and (3.5)

$$\iota_{X_{H_t}} \omega = \dot{q}_i dp_i - \dot{p}_i dq_i\,. \tag{3.7}$$

But from (3.4) we also have that

$$\iota_{X_H} \omega = dH = \frac{\partial H}{\partial q_i} dq_i + \frac{\partial H}{\partial p_i} dp_i\,. \tag{3.8}$$

Combining these two expressions we obtain Hamilton's equations from classical mechanics

$$\dot{q}_i = \frac{\partial H}{\partial p_i}\,, \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}\,. \tag{3.9}$$

3.1.2 Lie group actions

Let G be a Lie group. Recall that this means that G is a smooth manifold with a smooth map $\cdot : G \times G \to G$ such that $(G, \cdot)$ is a group and taking the inverse is also smooth map. Let $\mathfrak{g}$ be the Lie algebra of G, which we can identify as $\mathfrak{g} \cong T_e G$, i.e. the tangent space at identity. We can map from $\mathfrak{g}$ to G via the exponential map $X \mapsto e^X$. For $g \in G$ we define the conjugation map

$$\phi_g(h) = ghg^{-1}\,, \tag{3.10}$$

where $h \in G$. The differential of this map at the identity is a map $\mathfrak{g} \to \mathfrak{g}$ called the adjoint action

$$\mathrm{Ad}_g = d_e \phi_g\,, \tag{3.11}$$

$$\mathrm{Ad}_g(X) = \left.\frac{d}{dt}\right|_{t=0} g e^{tX} g^{-1}\,. \tag{3.12}$$

Note that for abelian G the adjoint action is trivial. Let $\mathfrak{g}^*$ denote the dual Lie algebra and $\langle \cdot, \cdot \rangle : \mathfrak{g}^* \times \mathfrak{g} \to \mathbb{R}$ the pairing between the two. The co-adjoint action, Ad^*_g, on $\mathfrak{g}^*$ is defined by

$$\langle \mathrm{Ad}^*_g Y, X \rangle = \langle Y, \mathrm{Ad}_{g^{-1}} X \rangle\,. \tag{3.13}$$

A (left) group action of G on M is a differentiable map $\sigma : G \times M \to M$, which we write as $\sigma(g, x) = \sigma_g(x)$, such that

- $\sigma_e(x) = x, \forall x \in M$,
- $\sigma_{gh}(x) = \sigma_g(\sigma_h(x)), \forall x \in M, \forall g, h \in G$.

We say that such an action is effective if $\sigma_g(x) = x, \forall x \in M \implies g = e$, i.e. only the identity keeps all points fixed. If the action has no fixed points we say that it is free.

The differential of σ at identity is a map from the Lie algebra to the tangent space of M

$$d\sigma_e(x) : \mathfrak{g} \to T_x M \,. \tag{3.14}$$

For $X \in \mathfrak{g}$ the image of X under this map is a vector field on M called the fundamental vector field $X^{\#}$.

Now let us consider the action of a Lie group on a symplectic manifold M with symplectic form ω. We call the action symplectic if it preserves the symplectic form, i.e. if $\sigma_g^*(\omega) = \omega$.

As a first example of these concepts, we consider translations in classical mechanics. Conisder the phase space M with coordinates (q_i, p_i) and the standard symplectic form (3.2). We let $G = \mathbb{R}^3$ act on M by

$$a \in \mathbb{R}^3 \mapsto (q_i + a, p_i) \,, \tag{3.15}$$

i.e. $\sigma_a(q_i, p_i) = (q_i + a, p_i)$. Clearly this preserves the standard symplectic form. The Lie algebra $\mathfrak{g}$ is also $\mathbb{R}^3$ and the fundamental vector field generated by $X = (X_1, X_2, X_3) \in \mathbb{R}^3$ is

$$X^{\#} = X_i \frac{\partial}{\partial q_i} \,. \tag{3.16}$$

It follows that

$$\iota_{X^{\#}} \omega = X_i dp_i = d(p \cdot X) \tag{3.17}$$

and so the fundamental vector field $X^{\#}$ is Hamiltonian. The corresponding Hamiltonian function $H = p \cdot X$ is the linear momentum in the direction X.

Another example is that of rotations. Let $G = \mathrm{SO}(3)$ act on M by

$$\sigma_A(q, p) = (Aq, Ap) \,, \tag{3.18}$$

where $A \in \mathrm{SO}(3)$ is a 3×3-matrix with $\det(A) = 1$ and $AA^T = 1$. From $AA^T = 1$ it follows that this action preserves the symplectic form. We identify the Lie algebra $\mathfrak{g} = \mathrm{so}(3)$ with $\mathbb{R}^3$ via

$$\begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \leftrightarrow a = (a_1, a_2, a_3) \,. \tag{3.19}$$

Under this identification the Lie bracket corresponds to the cross product of vectors. The fundamental vector field generated by $X = (X_1, X_2, X_3)$ is given by

$$X^{\#} = (X \times q)_i \frac{\partial}{\partial q_i} + (X \times p)_i \frac{\partial}{\partial p_i} \,, \tag{3.20}$$

and we find that

$$\iota_{X^\#}\omega = d\left((q \times p) \cdot X\right) . \tag{3.21}$$

The Hamiltonian function $H = (q \times p) \cdot X$ for this action is thus the angular momentum in the direction X.

3.1.3 Moment maps

For the example of translations in mechanics we saw that

$$d(p \cdot X) = \iota_{X^\#}\omega . \tag{3.22}$$

We can define a 'linear momentum map' $\mu(q,p) = p$ and think of it as a map

$$\mu : M \to \mathfrak{g}^* \tag{3.23}$$

satisfying

$$d\langle \mu, X\rangle = \iota_{X^\#}\omega , \tag{3.24}$$

where $\langle \cdot, \cdot \rangle$ denotes the pairing between the Lie algebra and its dual.

Similarly, for rotations we can define the 'angular momentum map' $\mu(q,p) = q \times p$. We think of it as a map $M \to \mathfrak{g}^*$ and by (3.21) it also satisfies $d\langle \mu, X\rangle = \iota_{X^\#}\omega$.

We generalise this as follows: Let $\sigma : G \times M \to M$ be a symplectic group action. We call such an action Hamiltonian if there exists a map

$$\mu : M \to \mathfrak{g}^* \tag{3.25}$$

satisfying the following:

- For all $X \in \mathfrak{g}$ with fundamental vector field $X^\#$

$$d\langle \mu, X\rangle = \iota_{X^\#}\omega . \tag{3.26}$$

- μ is equivariant with respect to the action σ and the coadjoint action

$$\mu \circ \sigma_g = \mathrm{Ad}_g^* \circ \mu, \quad \forall g \in G . \tag{3.27}$$

Such a map μ is called a moment map.

Note that for abelian groups the (co)adjoint action is trivial and the second condition becomes just $\mu \circ \sigma_g = \mu$.

Let us consider the simple example of a U(1) action on $\mathbb{C}^n$. To make a connection with the previous discussion, note that we can identify $\mathbb{C}^n$ with $T^*\mathbb{R}^n$ via $z_i = q_i + ip_i$. The standard symplectic form on $\mathbb{C}^n$ is then

$$\omega = \frac{i}{2} \sum_i dz_i \wedge d\bar{z}_i . \tag{3.28}$$

The group $G = \mathrm{U}(1) = S^1$ is the unit circle which we can parametrise by e^{it}. The tangent space at identity is a line and thus $\mathfrak{g} \simeq \mathbb{R}$ which we may identify with the imaginary axis in the complex plane. The exponential map back to the Lie group is then $it \mapsto e^{it}$. The action $\sigma : G \times M \to M$ is

$$(e^{it}, z_i) \mapsto e^{it} z_i \,, \tag{3.29}$$

$$(e^{it}, \bar{z}_i) \mapsto e^{-it} \bar{z}_i \,, \tag{3.30}$$

and the fundamental vector field is then given by

$$X^{\#} = i \sum_i \left(z_i \frac{\partial}{\partial z_i} - \bar{z}_i \frac{\partial}{\partial \bar{z}_i} \right) . \tag{3.31}$$

To find the moment map we calculate

$$\iota_{X^{\#}} \omega = \iota_{X^{\#}} \left(\frac{i}{2} \sum_i dz_i \wedge d\bar{z}_i \right) \tag{3.32}$$

$$= -\frac{1}{2} \left(\sum_i z_i d\bar{z}_i + \bar{z}_i dz_i \right) \tag{3.33}$$

$$= -\frac{1}{2} d (\sum_i z_i \bar{z}_i) \tag{3.34}$$

$$= d \left(-\frac{1}{2} \sum_i |z_i|^2 \right) . \tag{3.35}$$

So in order for $d\langle \mu, X \rangle = \iota_{X^{\#}} \omega$ to hold, we can pick any

$$\mu(z) = -\frac{1}{2} \sum_i |z_i|^2 + c \,, \tag{3.36}$$

where $c \in \mathfrak{g}^*$ is any number. In general, moment maps are only determined by the action up to an additive constant. By construction, this map satisfies (3.26). Since the group is abelian the second condition (3.27) just becomes $\mu \circ \sigma = \mu$ and this also holds since

$$\mu(e^{it} z) = -\frac{1}{2} \sum_i |e^{it} z_i|^2 + c = -\frac{1}{2} \sum_i |z_i|^2 + c = \mu(z) \,. \tag{3.37}$$

For non-abelian actions this condition would restric the additive constant c to be fixed by the co-adjoint action, i.e. to be a central element.

3.1.4 Symplectic and Kähler quotients

In the example above, let us pick $c = \frac{1}{2}$. Then

$$\mu(z) = 0 \iff \sum_i |z_i|^2 = 1 \,, \tag{3.38}$$

so the zero level set of the moment map corresponds to the unit sphere

$$\mu^{-1}(0) = S^{2n-1}\,. \tag{3.39}$$

We have thus constructed a manifold, S^{2n-1}, from a U(1) action on the manifold $\mathbb{C}^n$. Moreover, this U(1) acts freely on S^{2n-1} and so the quotient $S^{2n-1}/\mathrm{U}(1)$ is also a manifold, namely $\mathbb{C}P^{n-1}$. This gives rise to a principal U(1)-bundle, which in this example is the standard Hopf-fibration:

$$\begin{array}{ccc} S^{2n-1} & \longleftarrow & S^1 \\ \downarrow & & \\ \mathbb{C}P^{n-1} & & \end{array}$$

Moreover, the symplectic form of the original manifold $\mathbb{C}^n$ descends to the quotient $\mathbb{C}P^{n-1}$, making it a symplectic manifold.

More generally, if we have a symplectic manifold M and a Hamiltonian action of a compact Lie group G that acts freely on $\mu^{-1}(0)$, then we get a new symplectic manifold via the 'symplectic quotient'

$$M//G := \mu^{-1}(0)/G\,. \tag{3.40}$$

We refer the reader to [19] for a textbook discussion.

The above quotient can also be extended to Kähler manifolds. Recall that a Kähler manifold is a symplectic manifold with a compatible integrable almost complex structure J. Compatibility means that

$$g(X,Y) = \omega(X,JY) \tag{3.41}$$

and the symplectic form ω is then called a Kähler form.

If the action of G discussed above also preserves the complex structure (or equivalently the metric), then the quotient $M//G$ is called a 'Kähler quotient' and gives rise to a new Kähler manifold.

The remarkable thing about such quotients is that already the simple case of tori acting on flat complex space can give highly non-trivial manifolds. For example, toric Sasaki-Einstein manifolds can be obtained using Kähler quotients, see e.g. [20]. We study such manifolds in Paper I.

3.1.5 Hyperkähler quotients

Just as complex manifolds are modelled on the complex numbers $\mathbb{C}$, hyperkähler manifolds are modelled on the quaternions $\mathbb{H}$. For complex manifolds the complex structure roughly corresponds to multiplication by i, while for hyperkähler manifolds we have three complex structures, roughly corresponding to multiplication by the quaternions i, j, k.

More formally, a hyperkähler manifold is a manifold admitting three complex structures I, J, K satisfying the quaternionic relations

$$I^2 = J^2 = K^2 = IJK = -1\,, \tag{3.42}$$

and has a metric that is Kähler with respect to each of these. Such a manifold is necessarily of real dimension $4n$.

If there is a group action of G on such a manifold that preserves all three complex structures I, J, K one can construct another quotient, similar to the ones discussed above, called the hyperkähler quotient [21]. One then has three moment maps, one for each complex structure, and one can view them as a single map

$$\vec{\mu} = (\mu_I, \mu_J, \mu_K) : M \to \mathfrak{g}^* \otimes \mathbb{R}^3\,. \tag{3.43}$$

Picking one of the complex structures, say I, the moment maps can be arranged into a real $\mu_{\mathbb{R}} = \mu_I$ and a complex $\mu_{\mathbb{C}} = \mu_J + i\mu_K$ moment map, the latter which is holomorphic w.r.t. to I. Taking the zero level set of the moment maps and quotienting by G one obtains a new hyperkähler manifold denoted

$$M///G = \vec{\mu}^{-1}(0)/G\,. \tag{3.44}$$

In particular, one can obtain interesting examples by taking hyperkähler quotients of flat quaternionic space by tori. This is discussed in Papers II and III where we use such quotients to construct hypertoric 3-Sasakian manifolds.

3.2 Contact geometry

Contact geometry can be regarded as an odd-dimensional analogue of symplectic geometry. Here we will review some of the key concepts that are used in the thesis. We refer the reader to [22, 23] for textbook discussions.

A contact manifold of dimension $2n+1$ is a manifold equipped with a 1-form κ such that

$$\kappa \wedge (d\kappa)^n \neq 0\,. \tag{3.45}$$

Such a 1-form is called a contact form. The hyperplane field $D = \ker(\kappa) \subset TM$ is called the 'contact structure'. Note that κ and $e^f\kappa$ give the same contact structure.

The canonical example of a contact manifold is $\mathbb{R}^{2n+1}$, with coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$, with the standard contact form

$$\kappa = dz + \sum_i x_i dy_i\,. \tag{3.46}$$

Just as in symplectic geometry there is a Darboux theorem that states that locally one can always put the contact structure in this standard form.

Associated to the contact form κ there exists a unique Reeb vector field R such that

$$\iota_R d\kappa = 0\,, \tag{3.47}$$

$$\iota_R \kappa = 1\,. \tag{3.48}$$

For the standard contact form (3.46) on $\mathbb{R}^{2n+1}$ the Reeb vector is $R = \partial_z$.

The tangent bundle can be decomposed as

$$TM = D \oplus L_R\,, \tag{3.49}$$

where L_R is the line tangent to the Reeb R and $D = \ker(\kappa)$ is the contact structure or 'horizontal space'.

Given a contact manifold one can find a Riemannian metric g and a $(1,1)$-tensor J satisfying

$$J^2 = -1 + \kappa \otimes R\,, \tag{3.50}$$

$$g(JX, JY) = g(X, Y) - \kappa(X)\kappa(Y)\,, \tag{3.51}$$

$$d\kappa(X, Y) = g(X, JY)\,. \tag{3.52}$$

Such a structure is called a contact metric structure. If we further have that R is Killing with respect to the metric, i.e.

$$\mathcal{L}_R g = 0\,, \tag{3.53}$$

then we have a K-contact structure.

Given a contact structure we can define the two projectors

$$P_V = \kappa \wedge \iota_R\,, \tag{3.54}$$

$$P_H = 1 - P_V = 1 - \kappa \wedge \iota_R\,, \tag{3.55}$$

which decompose the differential forms into 'vertical' and 'horizontal' parts

$$\Omega^k = \Omega^k_V \oplus \Omega^k_H\,. \tag{3.56}$$

From (3.49) and (3.50) it follows that the complexified tangent bundle can be decomposed as

$$T_{\mathbb{C}}M = (\mathbb{C} \otimes D)^{(1,0)} \oplus (\mathbb{C} \otimes D)^{(0,1)} \oplus (\mathbb{C} \otimes R)\,, \tag{3.57}$$

and similarly for 1-forms:

$$\Omega^1 = \Omega^{(1,0)}_H \oplus \Omega^{(0,1)}_H \oplus \Omega^0 \kappa\,, \tag{3.58}$$

where $\Omega^0\kappa$ denotes any function times the contact 1-form κ. This extends to general differential forms, which we decompose as

$$\Omega^k = \bigoplus_{p+q=k} \Omega_H^{(p,q)} \oplus \Omega_H^{k-1}\kappa\,. \tag{3.59}$$

Note that for K-contact manifolds (3.53) implies that $\mathcal{L}_R J = 0$ and thus $\mathcal{L}_R$ respects this decomposition.

3.2.1 Sasaki geometry

Note that $J_T := J|_D$ acts as an almost complex structure and $\omega_T := d\kappa|_D$ as a symplectic form on the horizontal space D. From the properties of the metric we see that there is an 'almost Kähler strucutre' on D. It is natural to ask when this is actually a Kähler structure, i.e. when J_T is integrable. This motivates the condition

$$N_J = -d\kappa \otimes R\,, \tag{3.60}$$

where N_J denotes the Nijenhuis tensor of J which in this case is zero on the horizontal space. If the condition (3.60) is satisfied we say that we have a Sasaki structure. The above condition can also be stated as

$$(\nabla_X J)Y = g(X,Y)R - \kappa(Y)X\,. \tag{3.61}$$

We can also go up one dimension and consider the metric cone over M defined via

$$C(M) = M \times \mathbb{R}^+\,, \tag{3.62}$$

with metric

$$g_C = dr^2 + r^2 g\,, \tag{3.63}$$

where r denotes the coordinate on $\mathbb{R}^+$. Then $\omega_C = d(r^2\kappa)$ is a symplectic form on $C(M)$ and we can define an almost complex structure on $C(M)$ via $J_C(X) = J(X) - \kappa(X)r\partial_r$ and $J_C(r\partial_r) = R$. Then M is Sasaki if and only if $C(M)$ is Kähler. In fact, we will use this as the definition in this thesis.

3.2.2 Sasaki-Einstein manifolds

Recall that a manifold is called Einstein if the Ricci tensor is proportional to the metric, i.e.

$$\mathrm{Ric}_g = \lambda g \tag{3.64}$$

for some constant λ. A Sasaki-Einstein manifold is a manifold that is both Sasaki and Einstein. In this case it follows that $\lambda = 2(n-1)$.

As a consequence, one can show that $\text{Ric}_{g_C} = 0$, i.e. the Kähler cone $C(M)$ is Ricci flat. The converse also holds: M is Sasaki-Einstein if and only if $C(M)$ is Kähler and Ricci flat (Calabi-Yau), and this provides an alternative definition.

3.2.3 Toric Sasaki-Einstein manifolds

A class of Sasaki-Einstein manifolds of particular relevance to this thesis are *toric* Sasaki-Einstein manifolds. A Sasaki-Einstein manifold M of dimension $2n-1$ is said to be toric if its Kähler cone $C(M)$ admits an effective, Hamiltonian action of the torus T^n that respects the complex structure. It is also required that the Reeb vector field lies in the Lie algebra of the torus action. Let us review some of the properties of toric Sasaki-Einstein manifolds, we refer the reader to [24, 19] for more details. A nice introduction to the more general area of toric varieties can be found in [25].

As a first example, consider the odd-dimensional sphere $M = S^{2n-1}$ with metric cone $C(M) = \mathbb{C}^n$. There is a natural action of the torus T^n on $\mathbb{C}^n$ given by a phase rotation of each complex coordinate:

$$z_i \mapsto e^{i\theta_i} z_i \,, \tag{3.65}$$

$$\bar{z}_i \mapsto e^{-i\theta_i} \bar{z}_i \,. \tag{3.66}$$

This action preserves the standard symplectic form (3.28) and in analogy to the example presented in section 3.1.3 one can check that it is effetive and Hamiltonian with moment map

$$\mu : \mathbb{C}^n \to \mathfrak{t}^{n*} \cong \mathbb{R}^n \,, \tag{3.67}$$

$$\mu(z_1, \ldots, z_n) = -\tfrac{1}{2}(|z_1|^2, \ldots, |z_n|^2) + c \,, \tag{3.68}$$

where $c = (c_1, \ldots, c_n)$ is a constant vector. The image of the moment map is a convex cone in $\mathbb{R}^n$ which is called the moment map cone. In the example above, we can do a change of basis and choose c such that the moment map cone is the positive orthant of $\mathbb{R}^n$.

The condition that the Reeb vector of the Sasaki-Einstein manifold lies in the Lie algebra of the torus action means that we can represent it by a vector R in $\mathbb{R}^n$. Viewing the Sasaki-Einstein manifold M as the hypersurface in $C(M)$ where $r = 1$, it can be shown that its image under the moment map is given by the intersection of the moment map cone and the hyperplane $\{y \cdot R = \frac{1}{2}\}$:

$$\mu(M) = \mu(\{r = 1\} \times M) = \mu(C(M)) \cap \{y \cdot R = \frac{1}{2}\} \,. \tag{3.69}$$

This slice, or 'base', of the moment map cone is a compact $(n-1)$-dimensional polytope. In fact it is a very special type of polytope known

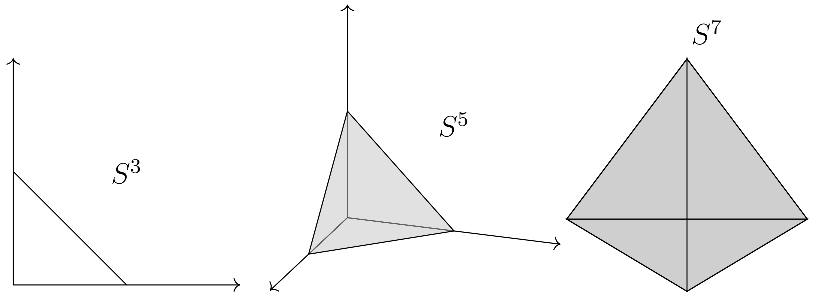

Figure 3.1. Moment map cones and polytopes for S^3, S^5 and S^7. For S^7 only the polytope is shown.

as a Delzant polytope, see e.g. [19] for a definition. For the example of S^{2n-1} we have $R = (1, \ldots, 1)$ and the Delzant polytope looks like an $(n-1)$-simplex, see figure 3.1.

A toric Sasaki-Einstein manifold can be viewed as a T^n fibration over its Delzant polytope, where one circle of the torus degenerate at each face. For example, S^3 can be viewed as a T^2 fibration over the interval illustrated in figure 3.1. At each end of the interval one of the two circles degenerate. In particular this means that there is a linear combination of the two circles that never degenerates and this gives rise to the Hopf fibration $S^1 \to S^3 \to \mathbb{C}P^1$. The stories are similar for S^5 and S^7.

There are plenty of examples of toric Sasaki-Einstein manifolds in 7D, for example those constructed by Martelli and Sparks [20]. These can be obtained via Kähler quotients and we give some examples in Paper I.

3.2.4 3-Sasakian manifolds

Let M be a Sasaki manifold with metric g and with R, κ, J as in previous sections. Then $\{R, \kappa, J\}$ is called a Sasaki-structure. A 3-Sasaki structure is a triplet of such Sasaki structures satisfying

$$\iota_{R_a} \kappa_b = \delta_{ab} \,, \tag{3.70}$$

$$[R_a, R_b] = \epsilon_{abc} R_c \,, \tag{3.71}$$

where ϵ_{abc} denotes the anti-symmetric symbol. The three contact structures give rise to three complex structures on the cone and the above relations mean that they satisfy the quaternion algebra and the cone is hyperkähler. We may use this as the definition, i.e. M is 3-Sasakian if and only if $C(M)$ is hyperkähler. 3-Sasakian manifolds are automatically Einstein (since hyperkähler manifolds are Ricci-flat) and so they are a special subcase of Sasaki-Einstein manifolds. Note that hyperkähler manifolds can only exist in dimensions $4n$, so 3-Sasakian manifolds can only exist in dimesions $4n - 1$. In particular, the spheres S^{4n-1} provide examples of 3-Sasakian manifolds. We refer the reader to [26] for further details, some of which are also discussed in Paper II.

Papers II and III deal with *hypertoric* 3-Sasakian manifolds. We say that a 3-Sasakian manifold M of dimension $4n-1$ is hypertoric if its hyperkähler cone $C(M)$ admits an effective, tri-Hamiltonian action of the torus T^n. That is, an action that is Hamiltonian with respect to each of the three Kähler structures and respects the three complex structures. Such cones $C(M)$ are examples of hypertoric varieties, first introduced by Bielawski and Dancer [27]. See also [28] for a review. In Papers II and III we review some of the key aspects of hypertoric geometry used in this thesis.

3.2.5 Killing spinors

As discussed in section 2.3, the formulation of supersymmetry used in this thesis requires the existence of Killing spinors.

Let M be a complete n-dimensional Riemannian spin manifold. A spinor ψ is said to be a Killing spinor with Killling constant $\alpha \in \mathbb{C}$ if

$$\nabla_X \psi = \alpha X \cdot \psi \,, \tag{3.72}$$

for all tangent vectors X, where $X \cdot \psi$ is the Clifford multiplication and ∇ the spin connection. The existence of such a Killing spinor implies that the manifold is Einstein with constant scalar curvature

$$R = -4n(n-1)\alpha^2 \,. \tag{3.73}$$

We will call a Killing spinor positive or negative depending on if it leads to positive or negative curvature respectively. If the curvature is zero, i.e. if $\nabla_X \psi = 0$, the spinor is said to be parallel. The manifolds admitting positive Killing spinors have been classified. The final piece was provided by Bär [17] by relating positive Killing spinors on M to parallel Killing spinors on the cone $C(M)$. Of particular interest to us is the following theorem [17]: A complete simply-connected Sasaki-Einstein manifold admits at least two linearly independent positive Killing spinors. Conversely, any odd-dimensional complete Riemannian spin manifold admitting two such spinors is Sasaki-Einsten.

For this reason, we shall focus on complete simply-connected Sasaki-Einstein manifolds in this thesis. Note that such manifolds are also spin. Simply connectedness also imply that the metric cone is Gorenstein. In particular this means that there exists a nowhere vanishing holomorphic $(n,0)$-form Ω on $C(M)$ such that

$$\mathcal{L}_{r\partial_r} \Omega = n\Omega \,, \tag{3.74}$$

where $2n = \dim_{\mathbb{R}}(C(M))$, see e.g. [29] for a discussion.

As mentioned in section 2.3, the seven-dimensional manifolds admitting positive Killing spinors are [17]:

- The seven-sphere S^7 (16 Killing spinors)
- 3-Sasakian manifolds (3 Killing spinors)
- Sasaki-Einstein manifolds (2 Killing spinors)
- Proper G_2-manifolds (1 Killing spinor)

Supersymmetric Yang-Mills on S^7 was studied in [16] and in this thesis we consider 7D Sasaki-Einstein manifolds (Paper I) and 3-Sasakian manifolds (Papers II and III). The last case, that of proper G_2-manifolds, is largely ignored in this thesis, so let us make some comments about it.

3.2.6 Proper G_2-manifolds

In dimension seven there exists a type of manifold that admit exactly one positive Killing spinor. Following [30] these are referred to as 'proper G_2-manifolds' in this thesis. Seven-dimensional manifolds admitting positive Killing spinors can also be characterised as manifolds admitting a 3-form Φ satisfying

$$d\Phi = -8\alpha(*\Phi)\,, \tag{3.75}$$

for some $\alpha \neq 0$. Such manifolds are called nearly parallel G_2-manifolds in [30]. If (3.75) holds for $\alpha = 0$ we say that we have a 'geometric G_2-structure'. Such manifolds are Ricci-flat, admit a parallel spinor, and are typically referred to as just 'G_2-manifolds' in the literature. Their Ricci-flatness makes them suitable for 'realistic' compactifications of M-theory. One can also formulate supersymmetric Yang-Mills theory on such manifolds by making use of the parallel spinor. As discussed in e.g. [31] one can take the supersymmetry off-shell and find a cohomological complex where the differential squares to a gauge transformation. It can then be argued that action is minimised by 'G_2-instantons' satisfying

$$*F = \Phi \wedge F\,. \tag{3.76}$$

In Paper I we argue that supersymmetric Yang-Mills can be formulated on proper G_2-manifolds but we do not pursue this further. The reason for this is that we need a contact structure to proceed with our localisation arguments. The supersymmetry can be taken off-shell using the same pure-spinor formalism as in Paper I, see [32] for a discussion. We believe that it should be possible to find a cohomological complex also for the proper G_2-case, with a differential that squares to a gauge transformation. We expect such a theory to give rise to some form of generalised G_2-instanton equations. This would be an interesting topic for further study.

4. Localisation

The results of this thesis rely on a framework of mathematical and physical results know as 'localisation formulas' that allow us to evaluate certain integrals exactly. What typically happens in localisation is that an integral over some 'large' domain reduces to an evaluation over a much smaller space, such as a discrete set of points. One may say that the integral 'localises' there. For example, Cauchy's residue formula in complex analysis could be considered to be a localisation formula. There the line integral of an analytic function is expressed as a sum of residues at a discrete set of points.

The localisation results used in quantum field theories typically rely on group actions. One of the first such results is by Duistermaat and Heckman [33]: Let M be a compact symplectic manifold of real dimension $2n$ with symplectic form ω. Assume there is a Hamiltonian U(1)-action, whose moment map we denote μ, that has a discrete set of fixed-points x_i. Then

$$\int_M \frac{1}{n!}\omega^n e^{-\mu} = \sum_{f.p.} \frac{e^{-\mu(x_i)}}{e(x_i)}\,, \tag{4.1}$$

where $e(x_i)$ is the product of the weights of the U(1) action at the tangent space at x_i.

This formula is a special case of a more general equivariant localisation formula, discovered independently by Atiyah-Bott [34] and Berline-Vergne [35].

In the next section we will discuss the Berline-Vergne-Atiyah-Bott formula, as this example generalises nicely to the infinite dimensional setting of path integrals in quantum field theories.

4.1 The Berline-Vergne-Atiyah-Bott localisation formula

Let M be a compact n-dimensinoal manifold with a U(1) action. Let $V(x)$ be the vector field associated to the U(1) action and consider the equivariant differential

$$d_V = d + \iota_V\,, \tag{4.2}$$

where d is the ordinary deRham differential and ι_V denotes contraction with the vector field. A form α is said to be equivariantly closed if

$$d_V\alpha = 0\,. \tag{4.3}$$

Note that α is of mixed degree and that (4.3) relates the various degrees.

The Berline-Vergne-Atiyah-Bott formula states that for an equivariantly closed form α

$$\int_M \alpha = \sum_{f.p.} \frac{(2\pi)^{n/2}\alpha_0(x_i)}{\sqrt{\det \partial_\nu V^\mu(x_i)}} \,, \tag{4.4}$$

where the sum is taken over the (isolated) fixed-points x_i and α_0 denotes the zero-form component of α.

Let us derive this formula using the language of supergeometry as this mimics the quantum field theoretic version nicely.

Conisder the odd tangent bundle ΠTM with coordinates (x^μ, ψ^μ). The x^μ are ordinary commuting (bosonic) coordinates on the base M, while ψ are anti-commuting (fermionic) coordinates on the fiber. The ψ^μ are sometimes called Grassmann coordinates and satisfy

$$\psi^\mu \psi^\nu = -\psi^\nu \psi^\mu \,. \tag{4.5}$$

The ψ^μ can be identified with the one-forms dx^μ and the wedge product of forms just becomes ordinary multiplication of the ψ^μ subject to (4.5). One can define integration for Grassman variables with

$$\int d\psi\, \psi = 1 \,, \qquad \int d\psi\, 1 = 0 \,, \tag{4.6}$$

see e.g. [36] for a discussion.

We can think of mixed-degree differential forms as smooth functions on ΠTM and write the integral of such a form as

$$\int_M \alpha = \int d^n x\, d^n \psi\, \alpha(x, \psi) \,, \tag{4.7}$$

where $d^n x = dx^1 \wedge \cdots \wedge dx^n$ and $d^n \psi = d\psi^1 \wedge \cdots \wedge d\psi^n$ form the canonical measure on ΠTM.

Consider a U(1) action on M with vector field V^μ and define the equivariant differential as in (4.2). Note that $d_V^2 = \mathcal{L}_V$, i.e. d_V squares to a Lie derivative by Cartan's formula.

Note that on our coordinates

$$d_V x^\mu = dx^\mu \equiv \psi^\mu \,, \tag{4.8}$$
$$d_V \psi^\mu = V^\mu(x) \,, \tag{4.9}$$

which resembles a supersymmetry transformation.

Let α be an equivariantly closed form, i.e. $d_V \alpha(x, \psi) = 0$. Our goal is to calculate

$$\int d^n x\, d^n \psi\, \alpha(x, \psi) \,. \tag{4.10}$$

To do this we introduce the auxiliary object

$$Z[t] = \int d^n x \, d^n \psi \, \alpha(x,\psi) e^{-t d_V W} \, . \tag{4.11}$$

The main idea of the localisation argument is to show that this expression is independent of t and then the integral in (4.10), which corresponds to $Z[0]$, can be computed for any convenient value of t. Typically one takes $t \to \infty$.

It can be shown that (4.11) is independent of t if $d_V^2 W = \mathcal{L}_V W = 0$. This follows from α being equivariantly closed, the Leibniz rule, and Stokes' theorem.

The next task is to find such a W. If $\mathcal{L}_V g = 0$, i.e. the vector field V is Killing, then

$$W = V^\mu g_{\mu\nu} \psi^\nu \tag{4.12}$$

satisfies $d_V^2 W = 0$. This can be checked by a direct computation. For this choice of W we have

$$d_V W = g_{\mu\nu} V^\mu V^\nu + (\partial_\mu V^\lambda g_{\lambda\nu}) \psi^\mu \psi^\nu \, , \tag{4.13}$$

and thus we see that in the $t \to \infty$ limit the fixed points where $V(x_i) = 0$ will dominate the integral. For simplicity, let us assume that there is a single isolated fixed-point at $x_i = 0$. In order to compute

$$Z[0] = \lim_{t\to\infty} \int d^n x \, d^n \psi \, \alpha(x,\psi) e^{-t g_{\mu\nu} V^\mu V^\nu - t(\partial_\mu V^\lambda g_{\lambda\nu}) \psi^\mu \psi^\nu} \, , \tag{4.14}$$

we make the following change of coordinates around the fixed point:

$$\tilde{x} = \sqrt{t} x \, , \quad \tilde{\psi} = \sqrt{t} \psi \, . \tag{4.15}$$

Note that this change of coordinates leaves the measure invariant. In the limit $t \to \infty$ only the quadratic terms in the exponent, which are independent of t, will contribute. Explicitly, the t-independent terms are

$$\begin{aligned} S_{\mu\nu} \tilde{x}^\mu \tilde{x}^\nu + A_{\mu\nu} \tilde{\psi}^\mu \tilde{\psi}^\nu :=& \\ & g_{\lambda\sigma} (\partial_\mu V^\lambda(0)) (\partial_\nu V^\sigma(0)) \tilde{x}^\mu \tilde{x}^\nu + (g_{\lambda\mu} \partial_\nu V^\lambda(0)) \tilde{\psi}^\mu \tilde{\psi}^\nu \, , \end{aligned} \tag{4.16}$$

where $S_{\mu\nu}$ and $A_{\mu\nu}$ are symmetric and anti-symmetric matrices respectively. We can thus evaluate the integral in terms of Gaussian integrals in even and odd coordinates:

$$Z[0] = \lim_{t\to\infty} \int d^n x\, d^n\psi\, \alpha(x,\psi) e^{-tg_{\mu\nu}V^\mu V^\nu - t(\partial_\mu V^\lambda g_{\lambda\nu})\psi^\mu\psi^\nu} \tag{4.17}$$

$$= \lim_{t\to\infty} \int d^n\tilde{x}\, d^n\tilde{\psi}\, \alpha\left(\frac{\tilde{x}}{\sqrt{t}}, \frac{\tilde{\psi}}{\sqrt{t}}\right) e^{-S_{\mu\nu}\tilde{x}^\mu\tilde{x}^\nu - A_{\mu\nu}\tilde{\psi}^\mu\tilde{\psi}^\nu + \mathcal{O}\left(\frac{1}{t}\right)} \tag{4.18}$$

$$= (2\pi)^{n/2}\alpha\,(0,0)\, \frac{\mathrm{Pf}(A)}{\sqrt{\det S}} \tag{4.19}$$

$$= \frac{(2\pi)^{n/2}\alpha\,(0,0)}{\sqrt{\det \partial_\nu V^\mu(0)}}\,. \tag{4.20}$$

Summing up all such contributions from each of the fixed points we obtain the Berline-Vergne-Atiyah-Bott formula (4.4).

4.2 Localisation of supersymmetric gauge theories

The argument in the previous section can be generalised in many ways. The corresponding argument for infinite-dimensional path integrals was first considered by Witten in [2]. Since then there have been many applications and generalisations of localisation techniques.

For example, Gromov-Witten invariants can be computed using localisation techniques, at least in the toric case, see e.g. [37] for a review. Another famous example is Nekrasov's localisation results for the partition function of $N = 2$ supersymmetric theories in the Ω-deformed background [38, 39] which derives the Seiberg-Witten prepotential from first principles.

A major break-through was made by Pestun [40] who used localisation techniques to find the partition function of supersymmetric Yang-Mills theories on the four-sphere. He also calculated certain observables, Wilson loops, and verified a conjecture by Erickson-Semenoff-Zarembo [41] and Drukker-Gross [42] relating these to matrix models. The method of Pestun has since been generalised to many other theories and geometries in various dimensions, see [43] for a review.

4.3 Localisation of 7D supersymmetric Yang-Mills theory

The approach we take to localise 7D supersymmetric Yang-Mills theory in this thesis is based on that of [16] for S^7 which in turn generalises Pestun's arguments for S^4.

The fields of the theory are mapped to differential forms and the supersymmetry and BRST-operators form the equivariant differential. This differential squares to a Lie derivative along the Reeb vector field and a gauge transformation. The localisation calculation results in the following expression for the perturbative partition function

$$Z^{pert} = \int_g d\sigma \, e^{-\frac{24}{g_{7D}^2} V_7 \, \mathrm{Tr}(\sigma^2)} det'_{adj} \, sdet_{\Omega_H^{(0,\bullet)}}(-\mathcal{L}_R + iG_\sigma) \,. \tag{4.21}$$

In the expression above, the integral is taken over the Lie algebra g of the gauge group, g_{7D}^2 is the coupling constant, and V_7 denotes the volume of the 7D manifold. The first determinant is taken over the adjoint representation (excluding zero modes) and the superdeterminant is taken over horizontal $(0,p)$-forms. Here $\mathcal{L}_R$ is the Lie derivative along the Reeb vector field R and G_σ is a gauge transformation. The next step is then to evaluate the superdeterminant

$$sdet_{\Omega_H^{(0,\bullet)}}(-\mathcal{L}_R + x) \,. \tag{4.22}$$

As shown by Schmude [44] this superdeterminant can be expressed in terms of holomorphic functions on the cone $C(M)$. Let us briefly outline the reason for this. As is typical for supersymmetric theories, there are huge cancellations between bosonic and fermionic contributions, i.e. between the numerator and denominator in the superdeterminant (4.22). The result of these cancellations in our case is that we only get contributions from the Kohn-Rossi cohomology groups. These are the cohomology groups corresponding to the horizontal Cauchy-Riemann operator[1]

$$\bar{\partial}_H : \Omega_H^{(p,q)} \to \Omega_H^{(p,q+1)} \,. \tag{4.23}$$

For the 7D manifolds studied in this thesis, we only get contributions from $H_{KR}^{(0,0)}$ and $H_{KR}^{(0,3)}$. Moreover, the Calabi-Yau property of the cone provides an explicit isomorphism between these two cohomologies. The whole superdeterminant (4.22) can thus be expressed in terms of $H_{KR}^{(0,0)}$ but

$$H_{KR}^{(0,0)}(M) \cong H^0(\mathcal{O}_{C(M)}) \,, \tag{4.24}$$

and the calculation boils down to 'counting' the holomorphic functions on the cone. There are infinitely many holomorphic functions on the cone, so in order to 'count' them one needs to put a grading on them (corresponding to some action) such that each graded component is finite-dimensional. Then one looks at the dimension of each graded piece.

For toric Sasaki-Einstein manifolds we can grade the holomorphic functions by their charges under the U(1)'s of the toric action. These are then

[1]To be more precise, our $\bar{\partial}_H$ is the *restriction* of the horizontal Cauchy-Riemann operator to horizontal forms. However, since any non-horizontal part will have zero charge under $\mathcal{L}_R$ we need not worry about this subtlety here.

in one-to-one correspondence with integer lattice points inside a cone determined by the toric action. We discuss this case in Paper I.

In Paper II and III we consider the case of hypertoric 3-Sasakian manifolds. We show that when one grade the holmomorphic functions in terms of the U(1)'s of the hypertoric action and the Reeb one gets a similar picture. The holomorphic functions correspond to integer lattice points inside a cone determined by the hypertoric action. This correspondence is no longer one-to-one but the multiplicities can be expressed in terms of the distance from the lattice points to the boundary of the cone.

The contribution to the superdeterminant from the holomorphic functions, when combined with the shifted contribution from $H^{(0,3)}_{KR}$, can be expressed in terms of a special function. We can then write the perturbative partition function (4.21) in terms of a matrix model involving this special function. For toric Sasaki-Einstein manifolds this special function is the generalised quadruple sine function that has been studied in e.g. [45]. For hypertoric 3-Sasakian manifolds we get a new type of special functions. Future work includes studying these functions from a mathematical point of view. For example, it would be interesting to derive their integral representations, asymptotic behaviours and factorisation properties.

5. Summary of papers

5.1 Paper I

In Paper I we study 7D maximally supersymmetric Yang-Mills theory on curved manifolds. We argue that the construction of such a theory on the seven-sphere by Minahan and Zabzine [16] can be generalised to any 7D manifold admitting positive Killing spinors. Such manifolds have been classified in the maths literature and are given by the seven-sphere S^7 (16 Killing spinors), 3-Sasakian manifolds (3 Killing spinors), Sasaki-Einstein manifolds (2 Killing spinors) and proper G_2-manifolds (1 Killing spinor). For manifolds admitting at least two Killing spinors we map the theory to a cohomological complex and perform a localisation calculation. For toric Sasaki-Einstein manifolds we find that the perturbative partition function can be expressed in terms of a generalised quadruple sine function that count integer lattice points inside a cone determined by toric data. Studying factorisation properties of this function allows us to speculate about the non-perturbative part of the partition function. We also provide some heuristic arguments for an alternative factorisation for S^7 based on its 3-Sasaki structure. We also discuss a generalised form of instanton equations and make some remarks about observables.

5.2 Paper II

In Paper II we extend the results of Paper I to a specific example of a 3-Sasakian manifold. The manifold we consider is not toric but its hyperkähler cone has hypertoric symmetry. This hypertoric symmetry is used to enumerate the holomorphic functions on the cone and this allows us to find the perturbative partition function. The result is also verified by an index calculation which also provides a factorisation result.

5.3 Paper III

Paper III generalises the 'proof-of-concept' calculation in Paper II to arbitrary hypertoric 3-Sasakian manifolds. The perturbative partition function is expressed in terms of a special function that count integer lattice points inside a cone determined by hypertoric data. We also present a factorisation result for this function.

6. Summary in Swedish

Kvantfältteori är ett av de viktigaste teoretiska ramverken inom modern fysik. Den mest berömda kvantfältteorin är Standardmodellen som beskriver tre av de fyra fundamentala krafterna i naturen: elektromagnetismen samt den svaga och starka växelverkan. Dess förutsägelser har bekräftats experimentellt med extrem noggrannhet, t.ex. vid CERNs partikelacceleratorer. En viktig kategori av kvantfältteorier är så kallade gaugeteorier. Kännetecknande för dessa teorier är att de är invarianta under vissa lokala symmetrier, d.v.s. under symmetritransformationer som beror på rumstidskoordinaten. Elektromagnetismen är ett exempel på en så kallad abelsk gaugeteori. De andra två teorierna i Standardmodellen, den svaga och starka växelverkan, är exempel på icke-abelska gaugeteorier, eller Yang-Mills teorier.

Yang-Mills teorier är således fundamentala byggstenar för Standardmodellen men har även lett till många intressanta resultat inom matematiken. Matematiskt sett formulerar man gaugeteorier via en viss typ av fiberbuntar över mångfalder. Gaugefält motsvarar då förbindelser på dessa fiberbuntar. Donaldson [1] använde dessa idéer för att definiera nya topologiska invarianter på fyrdimensionella mångfalder. Dessa invarianter kan skilja mellan olika glatta strukturer och Donaldson kunde bland annat visa att det finns 'exotiska' fyrdimensionella rum som är homeomorfa men inte diffeomorfa med $\mathbb{R}^4$.

Donaldsons arbete har generaliserats i många riktningar. Ett berömt exempel är Floerhomologi [3] som ger invarianter för tredimensionella mångfalder. Det finns även generaliseringar i högre dimensioner, t.ex. Donaldson-Thomas-teori [5].

Supersymmetri är en symmetri som relaterar fermioner och bosoner. Det finns inget experimentellt stöd för att supersymmetri skulle finnas i naturen men supersymmetriska gaugeteorier spelar ändå en viktig roll inom modern teoretisk fysik och matematik. Bland annat visade Witten [4] att Donaldsons och Floers invarianter kunde formuleras via vridna $N = 2$ supersymmetriska Yang-Mills teorier.

I denna avhandling studerar vi en specifik supersymmetrisk gaugeteori och använder oss av matematiska lokaliseringsresultat för att göra exakta beräkningar. Vi studerar maximalt supersymmetrisk Yang-Mills teori i sju dimensioner på toriska och hypertoriska mångfalder - en typ av geometriska rum med mycket symmetri.

I artikel I generaliserar vi Minahan och Zabzines [16] konstruktion av maximalt supersymmetrisk Yang-Mills-teori på sju-sfären till andra sjudimensionella mångfalder som tillåter positiva Killingspinorer. För Sasaki-Einsteinmångfalder konstruerar vi ett kohomologiskt komplex och använder oss av lokalisering för att hitta den perturbativa tillståndssumman. Denna tillståndssumma kan beskrivas via en superdeterminant. För toriska Sasaki-Einsteinmångfalder relaterar vi denna superdeterminant till holomorfa funktioner på den metriska konen över vår mångfald. Dessa funktioner kan tack vare den toriska symmetrin beskrivas som punkterna i ett heltalsgitter inuti en konisk polytop. Resultatet blir att tillståndssumman kan beskrivas med hjälp av en speciell funktion som i det toriska fallet är en generaliserad kvadrupel sinusfunktion. Faktoriseringsegenskaper hos dessa funktioner gör att vi även kan spekulera kring den fulla tillståndssummans struktur.

I artikel II studerar vi återigen sjudimensionell supersymmetrisk Yang-Mills-teori, denna gång på en specifik 3-Sasakimångfald. Eftersom 3-Sasakimångfalder även är Sasaki-Einstein kan vi använda samma lokaliseringsargument som i artikel I. Mångfalden vi studerar i artikel II är inte torisk och vi behöver därför en annan teknik för att beräkna superdeterminanten i tillståndssumman. Mångfalden vi studerar är dock hypertorisk och detta gör att vi kan beskriva de holomorfa funktionerna på mångfaldens kon och därmed beräkna den perturbativa tillståndssumman. Vi verifierar även resultatet via en index-beräkning samt påvisar ett faktoriseringsresultat.

I artikel III generaliserar vi metoden i artikel II till godtyckliga sjudimensionella 3-Sasakimångfalder med hypertorisk symmetri. Den perturbativa tillståndssumman ges nu av en ny speciell funktion som beskrivs via heltalsgitterpunkter i en polytop som bestäms av den hypertoriska symmetrin. Vi presenterar även ett faktoriseringsresultat för denna funktion.

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