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Cardy-like asymptotics of the 4d $\mathcal{N} = 4$ index and AdS_5 blackholes

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Abstract: Choi, Kim, Kim, and Nahmgoong have recently pioneered analyzing a Cardy-like limit of the superconformal index of the 4d $\mathcal{N} = 4$ theory with complexified fugacities which encodes the entropy of the dual supersymmetric AdS_5 blackholes. Here we study the Cardy-like asymptotics of the index within the rigorous framework of elliptic hypergeometric integrals, thereby filling a gap in their derivation of the blackhole entropy function, finding a new blackhole saddle-point, and demonstrating novel bifurcation phenomena in the asymptotics of the index as a function of fugacity phases. We also comment on the relevance of the supersymmetric Casimir energy to the blackhole entropy function in the present context.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory, Supersymmetric Gauge Theory

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Contents

1 Introduction

It has been a long-standing challenge in AdS$_5$/CFT$_4$ to reproduce the entropy of the charged, rotating, BPS, asymptotically AdS$_5$ blackholes of [1–5] from a microscopic counting of BPS states in the 4d $\mathcal{N}=4$ CFT. Several attempts in this direction were made in the past fifteen years or so (e.g. [6–10]), leading to various new lessons for holography and superconformal field theory (SCFT), but not to the desired microscopic count.

In particular, an index was devised in [6, 11] for counting the BPS states of general 4d SCFTs. The index counts all of the states — in the radial quantization of the SCFT — that are annihilated by a chosen supercharge. We adopt conventions in which such states satisfy the "BPS condition" $\Delta - J_1 - J_2 - \frac{3}{2}r = 0$, where Δ, J_1, J_2, r are the quantum numbers of the 4d $\mathcal{N}=1$ superconformal group SU$(2,2|1)$. The index

$$\mathcal{I}(p,q,u_k) := \mathrm{Tr}\left[(-1)^F e^{\hat{\beta}(\Delta - J_1 - J_2 - \frac{3}{2}r)} p^{J_1+\frac{r}{2}} q^{J_2+\frac{r}{2}} \prod_k u_k^{q_k}\right], \tag{1.1}$$

is thus independent of $\hat{\beta}$, but it does depend on the *spacetime fugacities* p,q, as well as the *flavor fugacities* u_k associated with flavor quantum numbers q_k commuting with the supercharge. In the case of the $\mathcal{N}=4$ theory, the SU(4) R-symmetry of the $\mathcal{N}=4$ superconformal algebra decomposes into SU(3)$\times$U$(1)_r$, so there is an SU(3) "flavor" symmetry group commuting with the chosen supercharge; hence there are three q_k with $\sum_{k=1}^{3} q_k = 0$, and three u_k satisfying $\prod_{k=1}^{3} u_k = 1$. It is customary to define $y_k := (pq)^{1/3} u_k$ and $Q_k := q_k + r/2$. Then, dismissing $\hat{\beta}$, we can rewrite the index of the $\mathcal{N}=4$ theory as

$$\mathcal{I}(p,q,y_{1,2,3}) = \mathrm{Tr}\left[(-1)^F p^{J_1} q^{J_2} y_1^{Q_1} y_2^{Q_2} y_3^{Q_3}\right]. \tag{1.2}$$

This index was computed at finite rank for the U(N) $\mathcal{N}=4$ theory in the original paper [6]. Then, in an initial attempt to make contact with holography, the large-N limit of the index

JHEP06(2019)134

was evaluated for *real-valued fugacities* and was seen to be $O(N^0)$; the result perfectly matched the index of the KK supergravity multi-particle states in the dual AdS_5 theory, but clearly could not account for the $O(N^2)$ entropy of the bulk supersymmetric AdS_5 blackholes [6]. For some time this negative result was interpreted as an indication that the index does not encode the bulk blackhole microstates.

Very recently it has been discovered by Choi, Kim, Kim, and Nahmgoong (CKKN) [12], and independently by Benini and Milan [13], that allowing the five fugacities in the index to take complex values one can achieve the desired $O(e^{N^2})$ behavior in the large-N limit of the index. Benini and Milan have succeeded in directly obtaining the AdS_5 blackhole entropy function in the large-N limit of the index [13], while CKKN took a different route and derived the entropy function in a double-scaling — Cardy-like as well as large-N — limit [12, 14]. In the present paper we derive the entropy function in a Cardy-like limit of the index at finite rank; although our analysis is closely related to that of CKKN [14], ours is more analogous to the Cardy-formula [15] derivations of blackhole entropy in AdS_3/CFT_2 (e.g. [16–18]) where the central charge is kept fixed.

The study of the Cardy-like asymptotics of 4d superconformal indices had some history prior to [14], but was again mostly limited to real-valued fugacities (e.g. [19–25]). The idea that blackhole microstate counting requires complex-valued fugacities in the $\mathcal{N} = 4$ index was not properly appreciated until the recent work of Hosseini, Hristov, and Zaffaroni (HHZ) [26]. This work provided the impetus for the later investigations of CKKN [12, 14] and Benini-Milan [13]. HHZ started from the supergravity side and bridged half-way towards the CFT by presenting a "grand-canonical" function — henceforth the HHZ function — from which a Legendre transform gives the micro-canonical entropy of the AdS_5 blackholes; the remaining challenge was to extract the HHZ function in an appropriate asymptotic regime from the index. In particular, it was understood by HHZ [26] (based on recent lessons from AdS_4/CFT_3 [27, 28]) that complexified fugacities are needed in the index in order to make contact with the grand-canonical function of the AdS_5 blackholes. As alluded to above, CKKN [14] and independently Benini and Milan [13] have recently completed the bridge between the CFT and the bulk by deriving the HHZ function through asymptotic analysis of the $\mathcal{N} = 4$ theory index, the first group in a double-scaling limit and the second group in a large-N limit.

In the present paper we analyze the Cardy-like asymptotics of the $\mathcal{N} = 4$ theory index with complexified fugacities using the rigorous machinery of elliptic hypergeometric integrals [29–32] in various Cardy-like regimes of parameters where the flavor fugacities approach the unit circle and the spacetime fugacities approach 1. In particular, we fill a gap in the CKKN derivation of the HHZ function in this limit by showing that the eigenvalue configuration they chose in their asymptotic analysis of the matrix-integral expression for the index is indeed the dominant configuration in the regime of parameters pertaining to the blackhole saddle-point they considered (though we demonstrate that it in fact fails to be the dominant configuration in essentially half of the parameter-space, where a more intricate analysis is called for). Moreover, we discover a new blackhole saddle-point in a different regime of parameters, corresponding to fugacities that are complex conjugate to those at the CKKN saddle-point. We present intuitive arguments suggesting that no other blackhole saddle-points with such large entropies exist in the Cardy-like limit. We also illustrate interesting dependence of the qualitative behavior of the Cardy-like asymptotics of the index on the complex phases of the fugacities.

In the rest of this introduction we give a sketchy account of the asymptotic analysis extracting the blackhole entropy function from the appropriate Cardy-like limit of the superconformal index of the $\mathcal{N}=4$ theory. The main body of the paper starts in section 2 where we elaborate on the sketchy derivation of the present section; we study the Cardy-like asymptotics of the $\mathcal{N}=4$ theory index with all its fugacities complexified, clarifying — and addressing a gap in — the CKKN derivation of the HHZ function. A thorough enough understanding of the asymptotics of the index in different Cardy-like regimes of parameters results in that section which reveals a second blackhole saddle-point in a regime complementary to that of CKKN, and moreover allows us to argue intuitively that no further saddle-points with such large entropies exist. In section 3 we keep the spacetime fugacities real-valued, and demonstrate novel bifurcation phenomena in the Cardy-like asymptotics of the index as a function of the flavor-fugacity phases. Section 4 discusses the relation between the Hamiltonian superconformal index and the Lagrangian index computed through path-integration; the two differ by a Casimir-energy factor which is argued to be irrelevant to the blackhole entropy function in the present context. Finally, section 5 discusses the important open ends of the present work.

1.1 Outline of the CKKN derivation in the elliptic hypergeometric language

We now present an outline of the CKKN derivation [14] of the HHZ function [26], translated to the language of elliptic hypergeometric integrals. More precisely, the problem we consider differs from that of [14] in two respects:

- while [14] considered the $\mathcal{N}=4$ theory with $\mathrm{U}(N)$ gauge group, we consider the $\mathrm{SU}(N)$ theory — the details are rather similar and the end results are related via $N^2 \to N^2-1$ shifts;
- while in [14] a double-scaling — Cardy-like as well as large-N — limit is taken to simplify the analysis, here in analogy with the Cardy-formula derivations of blackhole entropy in $\mathrm{AdS}_3/\mathrm{CFT}_2$ we keep N finite and only take a Cardy-like limit.

The special function as the starting point

The superconformal index of the $\mathrm{SU}(N)$ $\mathcal{N}=4$ theory is given by the following elliptic hypergeometric integral (see e.g. [33]):

$$\begin{aligned}\mathcal{I}(p,q,y_{1,2,3}) &:= \mathrm{Tr}\left[(-1)^F p^{J_1} q^{J_2} y_1^{Q_1} y_2^{Q_2} y_3^{Q_3}\right] \\ &= \frac{\big((p;p)(q;q)\big)^{N-1}}{N!}\prod_{k=1}^{3}\Gamma^{N-1}\big(y_k\big)\oint\prod_{j=1}^{N-1}\frac{\mathrm{d}z_j}{2\pi i z_j}\prod_{1\le i<j\le N}\frac{\prod_{k=1}^3\Gamma\big(y_k(z_i/z_j)^{\pm1}\big)}{\Gamma\big((z_i/z_j)^{\pm1}\big)},\end{aligned} \tag{1.3}$$

with the unit-circle contour for the $z_j=e^{2\pi i x_j}$ while $\prod_{j=1}^N z_j=1$, and with p,q,y_k strictly inside the unit circle such that $\prod_{k=1}^3 y_k=pq$. The two special functions $(\cdot;\cdot)$ and $\Gamma(\cdot)$ are respectively the *Pochhammer symbol* and the *elliptic gamma function* [34]:

$$(a;q) := \prod_{k=0}^{\infty}(1-aq^k), \tag{1.4}$$

$$\Gamma(z) := \prod_{j,k\geq 0}\frac{1-z^{-1}p^{j+1}q^{k+1}}{1-zp^jq^k}, \tag{1.5}$$

and $\Gamma(z^{\pm 1})$ stands for $\Gamma(z)\Gamma(z^{-1})$.

The integral expression gives the index as a meromorphic function of p, q, y_k in the domain $0 < |p|, |q|, |y_k| < 1$. A contour deformation can presumably allow meromorphic continuation of the index to $0 < |p|, |q| < 1$, $y_k \in \mathbb{C}^*$ (cf. [35]).

Asymptotic analysis in the limit encoding blackholes

The Cardy-type limit analyzed prior to the work of CKKN [14] was of the form $p, q, y_k \to 1$; more precisely, it was what in the mathematics literature is referred to as the *hyperbolic limit* of the elliptic hypergeometric integral [32, 36]. CKKN considered instead limits of the type $p, q \to 1$, $y_i \to e^{i\theta_i}$, with $\theta_i \notin 2\pi\mathbb{Z}$: they correctly recognized that giving finite (non-vanishing) phases to the flavor fugacities can obstruct the bose-fermi cancelations[1] occurring in the hyperbolic limit. For future reference we define σ, τ, Δ_k through $p = e^{2\pi i\sigma}, q = e^{2\pi i\tau}, y_k = e^{2\pi i\Delta_k}$, and write the appropriate limit explicitly as

$$\textit{the CKKN limit:}\quad |\sigma|, |\tau|, \mathrm{Im}\Delta_k \to 0,\ \text{with } \frac{\tau}{\sigma} \in \mathbb{R}_{>0}, \mathrm{Re}\Delta_k \text{ fixed, and } \mathrm{Im}\tau, \mathrm{Im}\sigma > 0.$$

Note that the "balancing condition" $\prod_{k=1}^{3} y_k = pq$ implies $\sum_{k=1}^{3}\Delta_k - \sigma - \tau \in \mathbb{Z}$, and that the restriction $\mathrm{Im}\tau, \mathrm{Im}\sigma > 0$ keeps us in the domain of meromorphy of the index.

The asymptotic analysis of the integral (1.3) now proceeds as follows. As will be explained in section 2, the leading asymptotics comes from the elliptic gamma functions $\Gamma(\cdot)$, so the Pochhammer symbols $(\cdot;\cdot)$ and the $N!$ in the pre-factor can be neglected. The required estimate, reviewed in section 2, follows from Proposition 2.11 of Rains [32]:

$$\Gamma(e^{2\pi i x}) = e^{-2\pi i\frac{\kappa(x)}{12\tau\sigma}+\mathcal{O}(\frac{\tau+\sigma}{\tau\sigma})}, \tag{1.6}$$

for $|\tau|, |\sigma| \to 0$, with $\mathrm{Im}\tau, \mathrm{Im}\sigma > 0$, and $\frac{\tau}{\sigma} \in \mathbb{R}_{>0}$, $x \in \mathbb{R}/\mathbb{Z}$. Here $\kappa(\cdot)$ is the continuous, odd, piecewise cubic,[2] periodic function

$$\begin{aligned}\kappa(x) &:= \{x\}(1-\{x\})(1-2\{x\}) \\ &\quad \big(= 2x^3 - 3x|x| + x \qquad \text{for } x \in [-1,1]\big),\end{aligned} \tag{1.7}$$

with $\{x\} = x - \lfloor x \rfloor$.

[1] A similar obstruction mechanism is at work in the AdS_3/CFT_2 context, where the entropy of the AdS_3 blackholes is derived from a Cardy-like limit of the CFT_2 elliptic genus $\chi(q, y)$: the limit $q, y \to 1$ does not encode the bulk blackholes, but the limit $q \to 1$, $y \to e^{i\theta}$ with $\theta \notin 2\pi\mathbb{Z}$ does. However, note that while in the AdS_3/CFT_2 context q can be kept real, in AdS_5/CFT_4 the spacetime fugacities p, q should take off the real line to meet the blackhole saddle-points. See [13, 27, 28] for related discussions of "$\mathcal{I}$-extremization" in the large-N analysis.

[2] Hence the "k"appa symbol introduced for it in [22].

In order to apply the estimate (1.6) to the gamma functions in (1.3) we have to identify the phase of the arguments with $2\pi x$; then, for instance, we can apply (1.6) to the gamma function in the numerator of the integrand of (1.3) by identifying x with $\mathrm{Re}\Delta_k \pm (x_i - x_j)$. This way we can simplify (1.3) to

$$\mathcal{I}(p,q,y_{1,2,3}) \xrightarrow{\text{in the CKKN limit}} \int_{-1/2}^{1/2} \mathrm{d}^{N-1}\boldsymbol{x}\ e^{-2\pi i \frac{Q_h(\boldsymbol{x};\mathrm{Re}\Delta_k)}{\tau\sigma}}, \tag{1.8}$$

with Q_h given by[3]

$$Q_h(\boldsymbol{x};\mathrm{Re}\Delta_k) := \frac{1}{12}\sum_{k=1}^{3}\left((N-1)\kappa(\mathrm{Re}\Delta_k) + \sum_{1\le i<j\le N}\kappa(\mathrm{Re}\Delta_k \pm (x_i - x_j))\right), \tag{1.9}$$

where $\kappa(A\pm B)$ stands for $\kappa(A+B)+\kappa(A-B)$. It only remains to evaluate the asymptotics of the integral (1.8).

Note that we are assuming $\mathrm{Im}(\tau\sigma)\neq 0$; this corresponds to complexifying the "temperature" as explained below. When $\mathrm{Im}(\tau\sigma)=0$ the integrand of (1.8) — or already the r.h.s. of (1.6) — would be a pure phase, and not sufficient to describe the exponential growth of the blackhole microstates. The $\mathrm{Im}(\tau\sigma)=0$ case is therefore not directly relevant to the AdS_5 blackhole physics, but it exhibits some interesting asymptotic bifurcation phenomena that are discussed in section 3.

The last step of the asymptotic analysis of the index involves arguing that in the appropriate range of parameters the dominant small-$|\tau|,|\sigma|$ configurations in (1.8) have $x_i - x_j = 0$ (implying $x_j = \text{const}$, which in our $SU(N)$ case would mean that all the holonomies are equal to $\frac{n}{N}$ for some $n\in\{0,1,\dots,N-1\}$). CKKN simply assumed [14] this to be the case. In section 2 we will prove that for the range of parameters relevant to the AdS_5 blackholes (e.g. for $\mathrm{Im}(\tau\sigma)>0$ & $\mathrm{Re}\Delta_{1,2}, -1-\mathrm{Re}\Delta_1-\mathrm{Re}\Delta_2 \in]-1,0[$) their assumption is correct. Hence the asymptotics of the index becomes[4]

$$\log\mathcal{I}(p,q,y_{1,2,3}) \approx -\frac{2\pi i}{\tau\sigma}Q_h(0;\mathrm{Re}\Delta_k) = -2\pi i\frac{N^2-1}{12\tau\sigma}\sum_{k=1}^{3}\kappa(\mathrm{Re}\Delta_k). \tag{1.10}$$

The right-hand-side is a nonanalytic function of the $\mathrm{Re}\Delta_k$, manifestly invariant under $\mathrm{Re}\Delta_k \to \mathrm{Re}\Delta_k + 1$ as it should be.

To match the grand-canonical function of HHZ [26] we now pick a particular chamber in the parameter-space so that an analytic expression can be written down. Specifically, assuming $\mathrm{Im}(\tau\sigma)>0$, going into the chamber $-1<\mathrm{Re}\Delta_{1,2,3}<0$ with $\mathrm{Re}\Delta_3 = -1-\mathrm{Re}\Delta_1-\mathrm{Re}\Delta_2$, we can simplify $\sum_{k=1}^{3}\kappa(\mathrm{Re}\Delta_k)$ to $6\mathrm{Re}\Delta_1\mathrm{Re}\Delta_2\mathrm{Re}\Delta_3$, and arrive at

$$\log\mathcal{I}(p,q,y_{1,2,3}) \approx -2\pi i\frac{N^2-1}{2\tau\sigma}\mathrm{Re}\Delta_1\mathrm{Re}\Delta_2\mathrm{Re}\Delta_3. \tag{1.11}$$

[3] Compare with the Q_h function defined in [22]; gauge anomaly cancelation implies that both are piecewise "Q"uadratic as a function of $\boldsymbol{x}$. The subscript h is used because the CKKN limit is a variant of the "h"yperbolic limit of the elliptic hypergeometric integral.

[4] Compare with eq. (2.34) of CKKN [14]; note that $2\pi i\Delta_k^{\text{here}} = -\Delta_k^{\text{CKKN}}$, while $2\pi i\{\sigma,\tau\} = -\omega_{\{1,2\}}^{\text{CKKN}}$.

JHEP06(2019)134

Analytic continuation of (1.11) to complex Δ_k (i.e. replacing every $\text{Re}\Delta_k$ with Δ_k) allows recovering the subleading terms in the CKKN limit and connecting with the complex HHZ function [26] (see the end of section 2 for more details):

$$\log \mathcal{I}(p,q,y_{1,2,3}) \approx -2\pi i \frac{N^2-1}{2\tau\sigma}\Delta_1\Delta_2\Delta_3, \tag{1.12}$$

with $\Delta_1+\Delta_2+\Delta_3-\tau-\sigma=-1$.

So far in this subsection we have been essentially rephrasing the developments due to CKKN [14]. One of the novel contributions of the present paper is to demonstrate in section 2 that when $\text{Im}(\tau\sigma)<0$ another chamber with $0<\text{Re}\Delta_{1,2,3}<1$ and $\text{Re}\Delta_3 = 1-\text{Re}\Delta_1-\text{Re}\Delta_2$ yields the asymptotics (1.12) in the CKKN limit, this time with $\Delta_1+\Delta_2+\Delta_3-\tau-\sigma=+1$.

Legendre transform and blackhole entropy

Thinking of the index (1.2) as the generating function of the degeneracies $d(J_{1,2},Q_{1,2,3})$ of the BPS states[5] in the $\mathcal{N}=4$ theory, methods of elementary analytic combinatorics can be used to extract the large-$J_{1,2},Q_{1,2,3}$ asymptotics of $d(J_{1,2},Q_{1,2,3})$ from the Cardy-like asymptotics of the index. The CKKN limit of the index encodes the degeneracy of the BPS states as $Q_{1,2,3}\sim\Lambda,\ J_{1,2}\sim\Lambda^{3/2},\ \Lambda\to\infty$, such that the charge relation [14, 37]

$$Q_1Q_2Q_3+\frac{N^2-1}{2}J_1J_2 = \left(Q_1+Q_2+Q_3+\frac{N^2-1}{2}\right) \times\left(Q_1Q_2+Q_2Q_3+Q_3Q_1-\frac{N^2-1}{2}(J_1+J_2)\right), \tag{1.13}$$

of the bulk AdS_5 blackholes is satisfied [14].

The degeneracies can be obtained from the generating function through

$$d(J_{1,2},Q_{1,2,3}) = \oint \mathcal{I}(p,q,y_{1,2,3})p^{-J_1}q^{-J_2}\left(\prod_{k=1}^{3} y_k^{-Q_k}\right)\frac{\mathrm{d}p}{2\pi i p}\frac{\mathrm{d}q}{2\pi i q}\left(\prod_{k=1}^{2}\frac{\mathrm{d}y_k}{2\pi i y_k}\right), \tag{1.14}$$

with all of the contours slightly inside the unit circle; note that y_3 is not independent, so is not integrated over on the r.h.s. (cf. section 5 of [13]). The asymptotic degeneracy can be obtained using a saddle-point evaluation of the integral on the right-hand side. Using the Cardy-like asymptotics in (1.12), the result for the asymptotic entropy $S(J_{1,2},Q_{1,2,3}) = \log d(J_{1,2},Q_{1,2,3})$ becomes

$$S(J_{1,2},Q_{1,2,3}) \approx \left(-2\pi i\frac{N^2-1}{2\tau\sigma}\Delta_1\Delta_2\Delta_3 - 2\pi i\sigma J_1 - 2\pi i\tau J_2 - \sum_{k=1}^{3} 2\pi i \Delta_k Q_k\right)_{\text{ext}}, \tag{1.15}$$

with $\sum\Delta_k-\tau-\sigma=-1$, as well as $-1<\text{Re}\Delta_{1,2,3}<0$, $\text{Im}\tau\sigma>0$. The subscript "ext" on the r.h.s. means picking its extremized value on the saddle-point.

[5] Although at first glance it appears that because of the $(-1)^F$ factor in it the index (1.2) counts the number of bosonic states minus the number of fermionic states, as argued in [13], on the blackhole saddle-points essentially all the states are expected to be bosonic, so the index counts a degeneracy.

The extremization problem was addressed for the $\mathrm{Im}\tau\sigma > 0$ case by HHZ [26], but was made completely explicit and analytic by CKKN [14] (and independently in appendix B of [37] by Cabo-Bizet, Cassani, Martelli, and Murthy), who found the blackhole saddle-point at

$$\Delta_k = -\frac{\frac{1}{S-2\pi i Q_k}}{\sum_{j=1}^{3}\frac{1}{S-2\pi i Q_j}-\sum_{l=1}^{2}\frac{1}{S+2\pi i J_l}}, \quad \{\sigma,\tau\} = -\frac{\frac{1}{S+2\pi i J_{\{1,2\}}}}{\sum_{j=1}^{3}\frac{1}{S-2\pi i Q_j}-\sum_{l=1}^{2}\frac{1}{S+2\pi i J_l}}, \tag{1.16}$$

satisfying $\Delta_1+\Delta_2+\Delta_3-\tau-\sigma=-1$, and giving the entropy

$$S \approx 2\pi\sqrt{Q_1Q_2+Q_2Q_3+Q_3Q_1-\frac{N^2-1}{2}(J_1+J_2)}, \tag{1.17}$$

which thanks to the charge relation (1.13) can be written in the alternative form

$$S \approx 2\pi\sqrt{\frac{Q_1Q_2Q_3+\frac{N^2-1}{2}J_1J_2}{Q_1+Q_2+Q_3+\frac{N^2-1}{2}}}. \tag{1.18}$$

Both of the relations (1.17), (1.18) correctly reproduce the Bekenstein-Hawking entropy of the BPS AdS_5 blackholes of [1–5], upon using the AdS/CFT dictionary $N^2-1=\frac{\pi\ell_{\mathrm{AdS}_5}}{2G_{\mathrm{AdS}_5}}$, with $\ell_{\mathrm{AdS}_5}, G_{\mathrm{AdS}_5}$ respectively the radius and the Newton constant of the bulk AdS_5.

Note that the term $\frac{N^2-1}{2}(J_1+J_2)$ in (1.17) and the term $\frac{N^2-1}{2}$ in (1.18) are subleading in the CKKN scaling limit $J^2\sim Q^3\to\infty$; capturing them was the sole purpose of keeping the subleading terms in (1.12) (compared to (1.11)), as well as in (1.13), (1.15), and (1.16).

In section 2 we show that (1.12) is valid in the CKKN limit also when $0<\mathrm{Re}\Delta_{1,2,3}<1$, $\mathrm{Im}\tau\sigma<0$, though this time with $\Delta_1+\Delta_2+\Delta_3-\tau-\sigma=+1$, and find a new blackhole saddle-point at

$$\Delta_k = \frac{\frac{1}{S+2\pi i Q_k}}{\sum_{j=1}^{3}\frac{1}{S+2\pi i Q_j}-\sum_{l=1}^{2}\frac{1}{S-2\pi i J_l}}, \quad \{\sigma,\tau\} = \frac{\frac{1}{S-2\pi i J_{\{1,2\}}}}{\sum_{j=1}^{3}\frac{1}{S+2\pi i Q_j}-\sum_{l=1}^{2}\frac{1}{S-2\pi i J_l}}, \tag{1.19}$$

with the same entropy S as that of the CKKN saddle-point.[6] We moreover argue that besides the two just described — having complex conjugate fugacities $p, q, y_{1,2,3}$ — no other blackhole saddle-points with such large entropies exist in the Cardy-like asymptotics of the $\mathcal{N}=4$ theory index.

Final remarks

A remaining gap for unequal Q_k. A rather serious gap in the above derivation is revealed upon closer inspection of the critical Δ_k in (1.16) and (1.19): while our asymptotic analysis is valid only in the limit $\mathrm{Im}\Delta_k\to 0$, the blackhole saddle-points have nonzero $\mathrm{Im}\Delta_k$

[6] The observation that the HHZ function with $\Delta_1+\Delta_2+\Delta_3-\tau-\sigma=+1$ also happens to give the right entropy has been made previously in appendix B of [37]. Our contribution here is to clarify the physics of this observation by showing that the HHZ function with $\Delta_1+\Delta_2+\Delta_3-\tau-\sigma=+1$ actually arises as the asymptotics of the index in a regime of parameters separate from that considered by CKKN.

unless $Q_1 = Q_2 = Q_3$. It is therefore only in the special case with equal — or approximately equal — charges that the above derivation (augmented with the refinements of section 2) is satisfactory. CKKN assumed in a leap of faith [14] that the asymptotics (1.12) remains valid away from the limit $\text{Im}\Delta_k \to 0$, and thus the blackhole entropy derivation can be extended to the general case with unequal charges. In section 2 we present a partial justification for this extrapolation; the complete justification is beyond the scope of the present paper, and its absence constitutes the most important open end of this work.

Cardy-like versus large-N. The above derivation extracts the AdS_5 blackhole entropy from a "high-temperature" (Cardy-like) limit of the 4d superconformal index at finite N. This is analogous to how the classic papers of Strominger-Vafa [16], BMPV [17], and Strominger [18] derived the Bekenstein-Hawking entropy of certain blackholes in what nowadays might be called an $\text{AdS}_3/\text{CFT}_2$ context.

From the holographic perspective, a more conceptually satisfying derivation would involve the large-N limit of the index. In $\text{AdS}_3/\text{CFT}_2$ such conceptually satisfactory derivations can be found in [38, 39]. In the $\text{AdS}_5/\text{CFT}_4$ context this was achieved very recently by Benini and Milan [13], leveraging the Bethe Ansatz formula of Closset, Kim, and Willett [40]. Curiously, although the derivation in [13] is not limited to the equal-charge blackholes, because of certain technical obstacles it so far applies only to the case with equal angular momenta $J_1 = J_2$ and the general case with $J_1 \neq J_2$ is still open. The more general Bethe Ansatz formula of [41] seems promising in that direction.

2 Complexified temperature and AdS_5 blackholes ($|\beta| \to 0,\ 0 < |\arg\beta| < \frac{\pi}{2}$)

In this section we fill in the gaps of subsection 1.1. In particular, we give a rigorous derivation of the estimate (1.6) for the elliptic gamma function, fill the gap in the CKKN asymptotic analysis in the region $\text{Im}\tau\sigma > 0$, extend the analysis to the region $\text{Im}\tau\sigma < 0$ where we find a new blackhole saddle-point, and argue that no further blackhole saddle-points with such large entropies exist.

The elliptic gamma function estimate (1.6)

Let us define the parameters b, β through $\tau = \frac{i\beta b^{-1}}{2\pi}$, $\sigma = \frac{i\beta b}{2\pi}$. For $p, q \in \mathbb{R}$, the parameter β defined as such was referred to as the *inverse-temperature* in [21, 22]; here we similarly refer to β as the *complexified inverse-temperature*. Throughout the present work we assume $b \in \mathbb{R}_{>0}$ (i.e. $\tau/\sigma \in \mathbb{R}_{>0}$); this simplifies the analysis and suffices for making contact with blackhole physics in the Cardy-like limit. We also take $\text{Re}\beta > 0$ (i.e. $|\arg\beta| < \frac{\pi}{2}$) to stay within the domain of meromorphy of the index (1.3). In terms of b, β we have

$$\textit{the CKKN limit:} \quad |\beta|, \text{Im}\Delta_k \to 0, \ \text{with } b \in \mathbb{R}_{>0}, \text{Re}\Delta_k \text{ fixed}, \ \text{and } \text{Re}\beta > 0.$$

The starting point for deriving the estimate (1.6) is the following identity, essentially due to Narukawa [42]:

$$\Gamma(e^{2\pi ix}) = e^{2\pi i Q_+(x;\sigma,\tau)}\psi_b\left(-\frac{2\pi ix}{\beta} - \frac{b+b^{-1}}{2}\right)\prod_{n=1}^{\infty}\frac{\psi_b(-\frac{2\pi in}{\beta} - \frac{2\pi ix}{\beta} - \frac{b+b^{-1}}{2})}{\psi_b(-\frac{2\pi in}{\beta} + \frac{2\pi ix}{\beta} + \frac{b+b^{-1}}{2})}, \tag{2.1}$$

where

$$\begin{aligned} Q_+(x;\sigma,\tau) = &-\frac{x^3}{6\tau\sigma} + \frac{\tau+\sigma+1}{4\tau\sigma}x^2 - \frac{\tau^2+\sigma^2+3\tau\sigma+3\tau+3\sigma+1}{12\tau\sigma}x \\ &+ \frac{1}{24}(\tau+\sigma+1)(1+\tau^{-1}+\sigma^{-1}), \end{aligned} \tag{2.2}$$

and $\psi_b(x)$ a function [see appendix A of [22] for its definition in terms of the hyperbolic gamma function] with the important property that for $\arg x$ inside compact subsets of $(-\pi, 0)$, and fixed $b > 0$

$$\log\psi_b(x) \sim 0, \qquad (\text{as } |x| \to \infty) \tag{2.3}$$

with an exponentially small error, of the type $e^{-|x|}$ — see Corollary 2.3 of Rains [32] for the precise statement and see appendix B of [43] for an earlier analysis in a different notation. This property of ψ_b guarantees that the infinite product in (2.1) is convergent when $\mathrm{Re}\beta > 0$.

For x strictly inside the strip

$$S^+: \quad 0 < \mathrm{Re}(xe^{-i\arg\beta}) < \mathrm{Re}(e^{-i\arg\beta}), \tag{2.4}$$

as $|\beta| \to 0$ with $|\arg\beta| < \pi/2$ and $b > 0$ fixed, all the ψ_b functions on the r.h.s. of (2.1) approach unity exponentially fast. Moreover, the dominant piece of Q_+ in the limit is of order $\frac{1}{\tau\sigma}$ and gives

$$\Gamma(e^{2\pi ix}) = e^{-2\pi i\frac{2x^3-3x^2+x}{12\tau\sigma} + \mathcal{O}(\frac{\tau+\sigma}{\tau\sigma})} \quad (\text{for } x \in S^+). \tag{2.5}$$

Since the l.h.s. of the above relation is periodic in $x \to x+1$, we can extend it beyond $x \in S^+$ by replacing every x on the r.h.s. with its horizontal shift $\{x\} := x - \lfloor \mathrm{Re}x + \mathrm{Im}x \cdot \tan(\arg\beta)\rfloor$ to inside S^+. For $x \in \mathbb{R}$ we have $\{x\} = x - \lfloor x \rfloor$; this yields our desired estimate (1.6).

Equivalently, we could use Proposition 2.11 of Rains [32], after identifying v_{there} with $|\beta|/2\pi$, and $\omega_{1,2\ \text{there}}$ with $ib^{\pm 1}e^{i\arg\beta}$.

A somewhat subtle point is that the estimate (1.6) is not uniform with respect to x when applied to the ("vector multiplet") gamma functions in the denominator of the r.h.s. of (1.3) — or more generally (2.5) is not uniform when x approaches the boundaries of the strip S^+. We need a uniform estimate because we want to apply the estimate in the integrand of the index; cf. the paragraph of eq. (3.15) in [22]. We expect though that an argument similar to that in the paragraph below eq. (3.30) of [22] can be given implying that the non-uniform estimate introduces a negligible error on the leading asymptotics of the index; we postpone the rigorous analysis of this point to the future.

JHEP06(2019)134

Cardy-like asymptotics of the index (1.10)

It follows from the relation between the Pochhammer symbol and the Dedekind eta function

$$\eta(\tau) = e^{2\pi i\tau/24}(e^{2\pi i\tau}; e^{2\pi i\tau}), \tag{2.6}$$

and the modular property $\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$ of the eta function that in the Cardy-like limit the Pochhammer symbols on the r.h.s. of (1.3) contribute an exponential growth of

$$(p;p)^{N-1}(q;q)^{N-1} = e^{\mathcal{O}(\frac{\tau+\sigma}{\tau\sigma})}.$$

They can hence be neglected, along with the $N!$ in the denominator of (1.3), in the Cardy-like limit when $0 < |\mathrm{arg}\beta| < \pi/2$. We thus end up with (1.8) as promised.[7]

We remind the reader that if $\beta \in \mathbb{R}_{>0}$ the integrand of (1.8) becomes a pure phase, and the more precise asymptotic analysis of section 3 has to be performed.

For $0 < |\mathrm{arg}\beta| < \frac{\pi}{2}$, in the small-$|\beta|$ limit the integral (1.8) is localized around the minima of $-\sin(2\mathrm{arg}\beta) \cdot Q_h(\boldsymbol{x}; \mathrm{Re}\Delta_k)$, whose $\boldsymbol{x}$-dependent part can be read from (1.9) to be

$$-\frac{\sin(2\mathrm{arg}\beta)}{12} \sum_{1\leq i<j\leq N} \sum_{k=1}^{3} \kappa(\mathrm{Re}\Delta_k \pm x_{ij}), \tag{2.7}$$

with $x_{ij} = x_i - x_j$. The expression

$$V^Q(x_{ij}; \mathrm{arg}\beta, \mathrm{Re}\Delta_k) = -\sin(2\mathrm{arg}\beta) \cdot \sum_{k=1}^{3} \kappa(\mathrm{Re}\Delta_k \pm x_{ij}), \tag{2.8}$$

is thus roughly a pair-wise potential for the "holonomies" x_i.

We take $\mathrm{arg}\beta$ and $\mathrm{Re}\Delta_{1,2}$ to be our control-parameters; $\mathrm{Re}\Delta_3$ is determined (mod $\mathbb{Z}$ to be precise, which is enough) by the balancing condition. We take the fundamental region of $\mathrm{Re}\Delta_{1,2}$ to be $[-1/2, 1/2]$. The two qualitatively different behaviors that the function V^Q can exhibit in various regions of the space of the control-parameters $\mathrm{Re}\Delta_{1,2}$ are shown in figure 1 for $-\pi/2 < \mathrm{arg}\beta < 0$. This figure can be deduced either by numerically scanning (using Mathematica for instance) the fundamental region $\mathrm{Re}\Delta_{1,2} \in [-1/2, 1/2]$, for some fixed $\mathrm{arg}\beta \in (-\pi/2, 0)$, or by analytically investigating the function $\sum_{k=1}^{3} \kappa(\mathrm{Re}\Delta_k \pm x_{ij})$ in its various regions of analyticity. Note that an M-type potential means $x_{ij} = 0$ is preferred in the small-$|\beta|$ limit, while a W-type potential means some $x_{ij} \neq 0$ (always a neighborhood of $x_{ij} = \pm 1/2$ it turns out) is preferred. Since figure 1 is a bit too featureful, we use the equivalence $\mathrm{Re}\Delta_{1,2} \to \mathrm{Re}\Delta_{1,2} \pm 1$ to shift its triangular regions so that the equivalent figure 2 is obtained, which is one of the main results of the present paper. It should be clear from the $\sin(2\mathrm{arg}\beta)$ factor in (2.8) that the M and W wings in figure 2 switch places if $\mathrm{arg}\beta$ is taken to be inside $(0, \pi/2)$ instead.

[7] More precisely, we also have to show that the complex phase of the integrand, arising from the subleading terms that we have ignored, does not cause a *completely destructive interference* modifying the leading asymptotics; cf. the paragraph below that of eq. (3.15) in [22]. For non-chiral SCFTs without flavor fugacities, the absence of such cancelations was established in [23]. For the case at hand, we expect such "unnatural" completely destructive interferences to be absent at least for generic $\mathrm{Re}\Delta_{1,2}$; we postpone the rigorous analysis of this point to the future.

JHEP06(2019)134

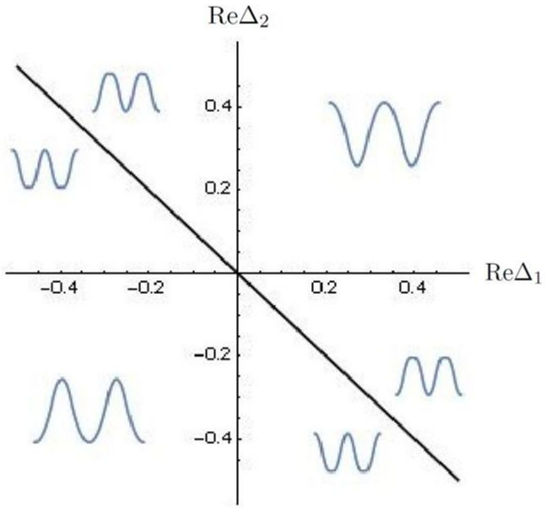

Figure 1. The qualitative behavior of $V^Q(x; \arg\beta, \mathrm{Re}\Delta_k)$ as a function of x for fixed $\mathrm{Re}\Delta_{1,2}$ and fixed $\arg\beta \in (-\pi/2, 0)$, in various regions of the space of the control-parameters $\mathrm{Re}\Delta_{1,2}$.

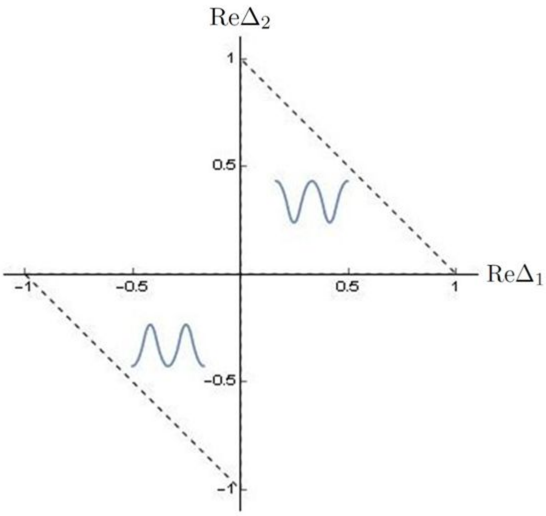

Figure 2. The qualitative behavior of $V^Q(x; \arg\beta, \mathrm{Re}\Delta_k)$, as a function of x for fixed $\mathrm{Re}\Delta_{1,2}$ and fixed $\arg\beta \in (-\pi/2, 0)$, in the two complementary regions $-1 < \mathrm{Re}\Delta_1, \mathrm{Re}\Delta_2, -1-\mathrm{Re}\Delta_1-\mathrm{Re}\Delta_2 < 0$ (lower-left) and $0 < \mathrm{Re}\Delta_1, \mathrm{Re}\Delta_2, 1-\mathrm{Re}\Delta_1-\mathrm{Re}\Delta_2 < 1$ (upper-right) of the space of the control-parameters $\mathrm{Re}\Delta_{1,2}$. The M and W wings switch places if $\arg\beta$ is taken to be inside $(0, \pi/2)$ instead.

To be specific, let us continue with the arg$\beta \in (-\pi/2, 0)$ case for the moment. Then on the M wing of figure 2 the minimum value of V^Q occurs at $x = 0$. Moreover, since V^Q is stationary at $x = 0$, the phase of the integral (1.8) is stationary there. We conclude that the $x_{ij} = 0$ configurations — which in our SU(N) case correspond to $x_j = \frac{n}{N}$ for some $n \in \{0, 1, \dots, N-1\}$ independent of j — indeed dominate the leading small-β asymptotics of the index for $-\frac{\pi}{2} < \arg\beta < 0$ and $-1 < \text{Re}\Delta_1, \text{Re}\Delta_2, -1-\text{Re}\Delta_1-\text{Re}\Delta_2 < 0$; this proves that CKKN's conjecture in [14] is valid in the range of parameters just mentioned, and fills the gap in their derivation of the HHZ function in this region of the parameter-space. (Note that in our SU(N) case each choice of n completely breaks the $\mathbb{Z}_N$ center symmetry generated by $x_j \to x_j + \frac{1}{N}$, thus meeting deconfinement expectations.)

On the bifurcation set, indicated by the dashed lines in figure 2, the functions V^Q and Q_h vanish; a more precise analysis using the techniques of section 3 is then required, but in any case it is clear that the asymptotic growth of the index is much slower (with $\text{Re}\log\mathcal{I} = \mathcal{O}(\frac{1}{|\beta|})$) there, so we do not discuss this set any further.

The question we would like to address now is: do we have a faster or a slower asymptotic growth on the W wing? Here by scanning (using Mathematica for instance) the whole range $-1/2 < \text{Re}\Delta_{1,2} < 1/2$ we realize that the minima $\sum\kappa(+1/3 \pm 1/2) = -5/9$ at $\text{Re}\Delta_{1,2} = +1/3$ and $x_{ij} = \pm 1/2$ are lower than the minimum $\sum\kappa(-1/3 \pm 0) = -4/9$ at $\text{Re}\Delta_{1,2} = -1/3$ and $x_{ij} = 0$. Does this mean that the index exhibits a faster asymptotic growth at $\text{Re}\Delta_{1,2} = +1/3$? The answer turns out to be no for $N = 2$, and seems to be no for all $N > 2$ as well.

For $N = 2$ the reason is that the $\boldsymbol{x}$-independent piece of Q_h in (1.9) moves $\sum\kappa(-1/3 \pm 0) = -4/9$ further down by $-2/9$, while it moves $\sum\kappa(+1/3 \pm 1/2) = -5/9$ further up by $+2/9$. Thus in the CKKN limit with $-\pi/2 < \arg\beta < 0$ we have

$$\mathcal{I}_{N=2}(p, q, y_{1,2,3}) \xrightarrow{\text{Re}\Delta_k = -\frac{1}{3}} e^{\frac{2\pi i}{12\tau\sigma}\cdot\frac{2}{3}}, \tag{2.9}$$

while

$$\mathcal{I}_{N=2}(p, q, y_{1,2,3}) \xrightarrow{\text{Re}\Delta_k = +\frac{1}{3}} e^{\frac{2\pi i}{12\tau\sigma}\cdot\frac{1}{3}}. \tag{2.10}$$

In short, for $N = 2$ the fastest asymptotic growth in the CKKN limit with $-\pi/2 < \arg\beta < 0$ occurs on the M wing of figure 2.

For higher ranks there is a more important reason why points on the W wing can not compete with the fastest asymptotic growth on the M wing. That is because for $N > 2$ it is impossible to distribute N holonomies x_i on the fundamental region $[-1/2, 1/2]$ (with $-1/2$ and $1/2$ identified of course, and with x_N determined from the rest via $\sum_{i=1}^N x_i \in \mathbb{Z}$) and have all of them at equal distance $|x_{ij}| = 1/2$ from each other. Colloquially speaking, it is not possible to capitalize on the minima of V^Q on the W wing at $|x_{ij}| = 1/2$ with all the holonomies, whereas it is possible to do so on the minima at $|x_{ij}| = 0$ on the M wing; hence as we increase the rank it becomes more and more intuitively likely that the fastest asymptotic growth of the index should occur on the M wing, and so we expect that only this region potentially bears entropy functions of the AdS_5 blackholes.

Let us recapitulate our findings so far. We have demonstrated that the $|x_{ij}| = 0$ points are preferred in the CKKN limit on the M wing of the space of the control-parameters $\text{Re}\Delta_{1,2}$, and thus the asymptotic result (1.10) is valid there. We have also argued intuitively that the W wing yields slower asymptotic growth and is not expected to bear entropy functions as large as those of the AdS_5 blackholes.

It is straightforward to deduce the analogous statements for $0 < \arg\beta < \frac{\pi}{2}$. In that case the M and W wings of figure 2 are swapped. Hence this time it is on the upper-right wing that the $|x_{ij}| = 0$ configurations are preferred in the CKKN limit, and the asymptotic result (1.10) is valid, though this time with $\text{Re}\Delta_3 = 1 - \text{Re}\Delta_1 - \text{Re}\Delta_2$. We also know that for $N = 2$ the fastest asymptotic growth of the index occurs on the upper-right wing, and we expect the same to be true as N increases.

The blackhole saddle-points (1.16), (1.19)

Now we ask: in the case $-\pi/2 < \arg\beta < 0$ does the lower-left wing, and in the case $0 < \arg\beta < \frac{\pi}{2}$ does the upper-right wing contain blackhole saddle-points? To make contact with the AdS_5 blackholes we have to find the critical points of the Legendre transform of $\log \mathcal{I}$ in the CKKN limit. In both cases it turns out that one blackhole saddle-point exists. The latter saddle-point seems to have been overlooked in [14], but can be obtained with minor modification of the computations in their section 2.3 as we now outline. Recall that when $0 < \arg\beta < \frac{\pi}{2}$ we impose $\sum_k \Delta_k = \tau + \sigma + 1$ rather than $\sum_k \Delta_k = \tau + \sigma - 1$; while CKKN [14] (following HHZ [26]) impose the latter relation via

$$\Delta_k = \frac{-z_k}{1+z_1+z_2+z_3+z_4}, \quad \sigma = \frac{z_4}{1+z_1+z_2+z_3+z_4}, \quad \tau = \frac{1}{1+z_1+z_2+z_3+z_4}, \tag{2.11}$$

the former relation can be simply imposed by putting

$$\Delta_k = \frac{z_k^*}{1+z_1^*+z_2^*+z_3^*+z_4^*}, \quad \sigma = \frac{-z_4^*}{1+z_1^*+z_2^*+z_3^*+z_4^*}, \quad \tau = \frac{-1}{1+z_1^*+z_2^*+z_3^*+z_4^*}. \tag{2.12}$$

We now would like to argue that the $z_{1,2,3,4}^*$ which solve the extremization problem for $0 < \arg\beta < \frac{\pi}{2}$ are indeed the complex conjugates of the $z_{1,2,3,4}$ that CKKN found solving the extremization problem for $-\pi/2 < \arg\beta < 0$. To demonstrate this, we present some of the details of the extremization problem, in parallel with section 2.3 of CKKN [14]. Setting the derivatives of (1.15) with respect to $z_{1,2,3,4}^*$ to zero, we get

$$Q_k + J_1 = -\frac{N^2-1}{2}\frac{z_1^* z_2^* z_3^*}{z_4^*}\left(\frac{1}{z_k^*} + \frac{1}{z_4^*}\right), \quad J_2 - J_1 = \frac{N^2-1}{2}\frac{z_1^* z_2^* z_3^*}{z_4^*}\left(\frac{1}{z_4^*} - 1\right), \tag{2.13}$$

similar to the CKKN case — cf. their eq. (2.75). However, extremization with respect to z_4^* yields

$$S = 2\pi i \left(\frac{N^2-1}{2}\frac{z_1^* z_2^* z_3^*}{z_4^*} + J_2\right), \tag{2.14}$$

with a different sign on the r.h.s. compared to the CKKN case — cf. their eq. (2.79). As a result, the equations for $z_{1,2,3,4}^*$ following from the above relations read

$$z_k^* = -\frac{-S + 2\pi i J_2}{-S - 2\pi i Q_k}, \quad z_4^* = \frac{-S + 2\pi i J_2}{-S + 2\pi i J_1}, \tag{2.15}$$

with only an $S \to -S$ difference compared to the CKKN case — cf. their eq. (2.88). In particular, since S is real, our z^*s are complex conjugates of their zs, as claimed. The relations (1.16) and (1.19) follow rather effortlessly from (2.11) and (2.12) respectively.

To obtain S, we can follow CKKN and write things in terms of $f^* := \frac{N^2-1}{2}\frac{z_1^* z_2^* z_3^*}{z_4^*}$, so that eq. (2.13) becomes

$$\frac{1}{z_4^*} = \frac{J_2 - J_1}{f^*} + 1, \qquad \frac{1}{z_k^*} = -\frac{Q_k + J_2}{f^*} - 1, \tag{2.16}$$

and then use the definition of f^* to obtain

$$f^* = -\frac{N^2-1}{2}\frac{f^{*2}(J_2 - J_1 + f^*)}{(Q_1 + J_2 + f^*)(Q_2 + J_2 + f^*)(Q_3 + J_2 + f^*)}, \tag{2.17}$$

which is a cubic relation for f^*. The cubic equation that follows for $S = 2\pi i(f^* + J_2)$ will then have the entropy functions (1.17) and (1.18) as its solutions.

To demonstrate the self-consistency of our computations we need to show that the CKKN/HHZ saddle-point (1.16) is indeed on the lower-left wing of figure 2 and has $-\pi/2 < \arg\beta < 0$, while the new saddle-point lies on the upper-right wing and has $0 < \arg\beta < \frac{\pi}{2}$. We show only the second statement, as the first follows using the fact that the two saddle-points have their Δ_k, τ, σ negative complex conjugate of each other.

A quick way to the desired result is to note that $S + 2\pi i Q_k$ are on a straight line in the complex plane, so that their reciprocals are on a circle. This observation motivates the change of variables $\frac{1}{S+2\pi i Q_k} = \frac{1}{2S}(1 + e^{-i\phi_k})$, with $\phi_k \in (0, \pi)$. Then the desired ranges of $\mathrm{Re}\Delta_k$ and $\arg\beta$ follow easily from (1.19) and the vector representation of the complex numbers $1 + e^{-i\phi_k}$, after neglecting the subleading terms in the CKKN scaling limit $J^2 \sim Q^3 \to \infty$.

In summary, we have shown that when $0 < \arg\beta < \frac{\pi}{2}$ a blackhole saddle-point exists on the upper-right wing of figure 2; as comparison of eqs. (2.11) and (2.12) shows, the new saddle-point has fugacities p^*, q^*, y_k^* that are complex conjugates of the fugacities at the CKKN/HHZ saddle-point. Moreover, we have argued that besides this and the CKKN/HHZ saddle-point no other saddle-points with such large entropies exist in the Cardy-like limit.

Moving the flavor fugacities away from the unit circle

As we noted at the end of subsection 1.1, unless $Q_1 = Q_2 = Q_3$, the critical Δ_k have nonzero imaginary parts, and thus the critical fugacities u_k (and also y_k) lie away from the unit circle. Hence to complete the blackhole entropy derivation for the general case with unequal Q_k, we need to be able to justify the Cardy-like asymptotics (1.12) when $\mathrm{Im}\Delta_k$ are not sent to zero.

A partial justification is as follows. Let us assume that $\mathrm{Im}\Delta_k$ are small enough so that the integral (1.3) still represents the index, albeit with a slightly deformed contour of

JHEP06(2019)134

integration.[8] We can then use (2.5) to arrive at the following variant of (1.8):

$$\mathcal{I}(p,q,y_{1,2,3}) \to \int_{-1/2}^{1/2} \mathrm{d}^{N-1}\boldsymbol{x}\; e^{-2\pi i \frac{Q_h(\boldsymbol{x};\arg\beta,\Delta_k)}{\tau\sigma}}, \tag{2.18}$$

with

$$Q_h(\boldsymbol{x};\arg\beta,\Delta_k) := \frac{1}{12}\sum_{k=1}^{3}\left((N-1)\kappa(\Delta_k) + \sum_{1\le i<j\le N}\kappa(\Delta_k \pm (x_i - x_j))\right), \tag{2.19}$$

where $\kappa(x)$ is still defined as in (1.7), but with $\{x\} := x - \lfloor \mathrm{Re}x + \mathrm{Im}x \cdot \tan(\arg\beta)\rfloor$ as discussed around (2.5).

We expect that for fixed $\arg\beta$ (either in $(-\pi/2,0)$ or in $(0,\pi/2)$), and for small enough $\mathrm{Im}\Delta_k$, the catastrophic behavior of the pair-wise potential for the holonomies to remain similar to that discussed above, with the two complementary "wings" $\Delta_{1,2}, 1-\Delta_1-\Delta_2 \in S^+$ and $\Delta_{1,2}, -1-\Delta_1-\Delta_2 \in S^+ - 1$ being associated to M- or W-type behaviors, with one or the other having $x=0$ as its preferred configuration depending on the sign of $\arg\beta$. Then for $\arg\beta \in (0,\pi/2)$ one can use (2.5) on the wing $\Delta_{1,2}, 1-\Delta_1-\Delta_2 \in S^+$ to arrive at (1.12) with $\sum_k \Delta_k = \tau+\sigma+1$, while for $\arg\beta \in (-\pi/2,0)$ one can use (2.5) with $x \to x+1$ on the wing $\Delta_{1,2}, -1-\Delta_1-\Delta_2 \in S^+ - 1$ to arrive at (1.12) this time with $\sum_k \Delta_k = \tau+\sigma-1$.

Beyond a small neighborhood of $\mathrm{Im}\Delta_k = 0$ the methods of the present paper do not seem powerful enough to demonstrate (1.12). Whether the fascinating formalism of [40, 41] can help addressing the general case with nonzero $\mathrm{Im}\Delta_k$ is currently being investigated.

3 Real-valued temperature ($\beta \to 0$, $\beta \in \mathbb{R}_{>0}$)

In this section we keep the spacetime fugacities p,q real-valued and define $b,\beta \in \mathbb{R}_{>0}$ through $p = e^{-\beta b}, q = e^{-\beta b^{-1}}$. We also keep the flavor fugacities $u_k = e^{2\pi i T_k}$ on the unit circle (hence $T_k \in \mathbb{R}$), and study the effect of finite nonzero T_k on the small-β asymptotics of the index.

In order to provide some conceptual context for the somewhat technical analysis in the rest of this section we now briefly discuss the path-integral interpretation of the index with real-valued p,q. We will still be analyzing the Hamiltonian index $\mathcal{I}$, and only importing intuition from the path-integral picture — until the next section where the path-integral partition function is analyzed.

The superconformal index with real p,q can be obtained via the path-integral SUSY partition function of the theory on $S^3_b \times S^1_\beta$, where S^3_b is the squashed three-sphere with unit radius and squashing parameter b, while S^1_β is the circle with circumference β [44]. The integration variables z_i in the index (1.3) correspond to the eigenvalues of the holonomy matrix $P\exp(i\oint_{S^1_\beta} A_0)$, with A_0 the component along S^1_β of the SU(N) gauge field. The u_k correspond to the eigenvalues of the background holonomy matrix $P\exp(i\oint_{S^1_\beta} A^u_0)$, with A^u_0 the component along S^1_β of the background gauge field A^u associated to the "flavor" SU(3)

[8] It appears like we might only need the contour-deformation to be small near $x_j = \frac{n}{N}$, which give the dominant eigenvalue configuration in the regime of parameters pertaining to the blackhole saddle-points.

of the $\mathcal{N} = 4$ theory. The path-integral partition function is actually a Casimir-energy factor different from the index; this factor is irrelevant for the present analysis and we postpone its discussion to the next section. Interpreting the S^3_b as the spatial manifold and the S^1_β as the Euclidean time circle, we refer to β as the inverse-temperature in analogy with thermal quantum physics — even though our fermions have supersymmetric (i.e. periodic) boundary conditions around S^1_β.

Next, we note that while large-N QFTs ($N \to \infty$) on compact spatial manifolds can have finite-temperature phases associated to large-N saddle-points, in the present work we are considering a finite-N QFT on a compact spatial manifold (namely S^3_b), which can not be assigned a phase at any finite temperature. In the high-temperature limit ($\beta \to 0$), however, infinite-temperature phases can be associated to the small-β saddle-points. In particular, we will say that the infinite-temperature phase of the index is *Higgsed* if the dominant small-β saddle-point(s) of its matrix-integral lie away from the "origin" $\boldsymbol{x} = 0$. For example, the infinite-temperature phase of the index of the SU(2) ISS model is Higgsed, but that of the $\mathcal{N} = 1$ SU(N) SQCD (say in the conformal window) is not [22].

Moreover, we will say that the infinite-temperature phase of the index is *exponentially growing* if for the leading small-β asymptotics we have $\mathrm{Re}\log\mathcal{I} \approx A/\beta$ with $A > 0$; in other words if the index exhibits exponential *growth* in the high-temperature limit.

Below we will see that for generic non-zero $T_k \in \mathbb{R}$ the infinite-temperature phase of the index of the SU(N) $\mathcal{N} = 4$ theory is Higgsed, and in the $N = 2$ case for some specific range of T_k also exponentially growing.

Taking p, q to be real means taking τ, σ to be pure imaginary. Then we have $\mathrm{Im}(\tau\sigma) = 0$, so that the estimate (1.6) gives only a pure phase; we thus have to consider the subleading terms in the exponent of its r.h.s. to get information about the modulus of the index. The improved estimate is [22]

$$\log\Gamma\left((pq)^{r/2}\, e^{2\pi i x}\right) = 2\pi i\left(-\frac{\kappa(x)}{12\tau\sigma} + (r-1)\,\frac{\tau+\sigma}{4\tau\sigma}\vartheta(x) - (r-1)\frac{\tau+\sigma}{24\tau\sigma}\right) + O(\tau^0,\sigma^0) \qquad (\text{for } r\in(0,2) \text{ and } x\in\mathbb{R}), \tag{3.1}$$

where the continuous, positive, even, periodic function

$$\vartheta(x) := \{x\}(1-\{x\}) \qquad \left(= |x| - x^2 \qquad \text{for } x \in [-1,1]\right), \tag{3.2}$$

is defined after Rains [32].

In order to apply the estimate (3.1) to the gamma functions in (1.3) we have to interpret the modulus of the arguments of the gamma functions as $(pq)^{r/2}$, and interpret the phase of the arguments as $2\pi x$; then, for instance, we can apply (3.1) to the gamma function in the numerator of the integrand of (1.3) by identifying r, x as $r = 2/3$, $x = T_k \pm (x_i - x_j)$; note that the balancing condition $\prod_{k=1}^3 y_k = pq$ implies $\sum_{k=1}^3 T_k \in \mathbb{Z}$. Since the Pochhammer symbols in (1.3) yield asymptotics that cancel the contribution of the gamma functions from the third term on the r.h.s. of (3.1) [20, 22], applying (3.1) to (1.3) we get

$$\mathcal{I}(p,q,u_{1,2,3}) \xrightarrow{p,q \text{ real},\ |u_k|=1} \int_{-1/2}^{1/2} \mathrm{d}^{N-1}\boldsymbol{x}\; e^{-\frac{(2\pi)^2}{\beta}\left(\frac{b+b^{-1}}{2}\right)L_h(\boldsymbol{x}, r_k=2/3; T_k) + i\frac{(2\pi)^3}{\beta^2}Q_h(\boldsymbol{x};T_k)}, \tag{3.3}$$

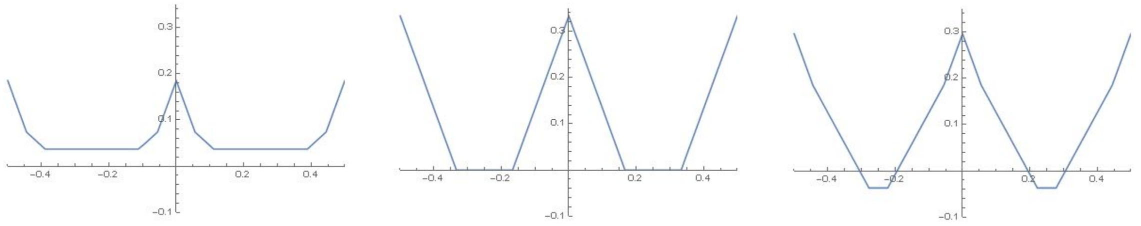

Figure 3. The L_h function (3.4) in the $N=2$ case, as a function of x_1, for $T_{1,2}=-1/9$ (left), $T_{1,2}=-1/3$ (middle), and $T_{1,2}=-4/9$ (right).

where we have used $\tau = i\beta b^{-1}/2\pi$ and $\sigma = i\beta b/2\pi$. The functions L_h and Q_h are the natural generalizations of those defined in [22] for $T_k=0$, and are explicitly given by[9]

$$\begin{aligned} L_h(\boldsymbol{x}, r_k = 2/3; T_k) = (N-1)\cdot\frac{1}{6}\big(\vartheta(T_1)+\vartheta(T_2)+\vartheta(T_3)\big) \\ + \sum_{1\le i<j\le N} \frac{1}{6}[\vartheta(x_i-x_j+T_1)+\vartheta(x_i-x_j-T_1) \\ +\vartheta(x_i-x_j+T_2)+\vartheta(x_i-x_j-T_2) \\ +\vartheta(x_i-x_j+T_3)+\vartheta(x_i-x_j-T_3)]-\vartheta(x_i-x_j). \end{aligned} \tag{3.4}$$

$$\begin{aligned} Q_h(\boldsymbol{x}; T_k) = (N-1)\cdot\frac{1}{12}\big(\kappa(T_1)+\kappa(T_2)+\kappa(T_3)\big) \\ + \sum_{1\le i<j\le N} \frac{1}{12}\big(\kappa(x_i-x_j+T_1)-\kappa(x_i-x_j-T_1) \\ +\kappa(x_i-x_j+T_2)-\kappa(x_i-x_j-T_2) \\ +\kappa(x_i-x_j+T_3)-\kappa(x_i-x_j-T_3)\big). \end{aligned} \tag{3.5}$$

Note that for $T_k=0$ both functions identically vanish, as in [22].

Since the $1/\beta^2$ term in the exponent of the r.h.s. of (3.3) gives a pure phase, the dominant contribution to the integral presumably comes from the locus of minima of $L_h(\boldsymbol{x}, r_k = 2/3; T_k)$. One has to make sure that $Q_h(\boldsymbol{x}; T_k)$ is stationary at that locus though, otherwise a more careful analysis is required.

The SU(2) case

Take for example the $N=2$ case. Figure 3 shows the L_h function of the SU(2) $\mathcal{N}=4$ theory for sample values of T_k. As the picture clearly shows, at the point $x_1=0$ the integrand is maximally suppressed.

It is easy to check that the dominant configuration for $N=2$ is $|x_1|=1/4$ (figure 3 is suggestive of this also); not only L_h is minimized there, but also Q_h is stationary as desired. Moreover, we see from figure 3 that depending on T_k the minimum of L_h can be positive, negative, or zero. Only when the minimum is negative the infinite-temperature phase is exponentially growing. The contours of $L_h(x_1=\pm 1/4, r_k=2/3; T_k)$ are shown

[9] Because of the ABJ $\mathrm{U}(1)_R$ anomaly cancelation L_h is a piecewise "L"inear function of $\boldsymbol{x}$ (cf. [22]). See figure 3.

JHEP06(2019)134

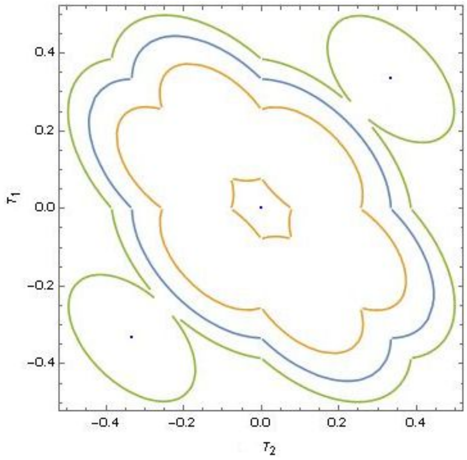

Figure 4. Contours of $L_h(x_1 = \pm 1/4, r_k = 2/3; T_k)$. The blue curve and dots correspond to zero value, inside the blue curve except at the origin corresponds to positive values, while outside the blue curve and away from the blue dots corresponds to negative values.

in figure 4: outside the blue contour we have $L_h(x_1 = \pm 1/4, r_k = 2/3; T_k) < 0$, so the index is exponentially growing, except on the blue dots at $T_1 = T_2 = \pm 1/3$ where $L_h(x_1 = \pm 1/4, r_k = 2/3; T_k)$ vanishes.

Let us review what we have observed. While for $T_k = 0$ both functions L_h and Q_h are zero and the index has a power-law asymptotics (more precisely an $\mathcal{I} \approx 1/\beta$ behavior as $\beta \to 0$ [22]), finite nonzero T_k can induce Mexican-hat potentials for the holonomies in the high-temperature limit, triggering an infinite-temperature exponential growth in the index.

Higher ranks

We now show that the integrand of the index is maximally suppressed at $\boldsymbol{x} = 0$ in fact for arbitrary $N \geq 2$ and $T_k \in \mathbb{R}/\mathbb{Z}$.

Let us study the behavior of the L_h function in (3.4) with respect to x_i. For this purpose, we use the following equality derived in [22] (cf. eq. (3.51) there), valid for $-1/2 \leq u_i \leq 1/2$:

$$(2M-2) \sum_{1\leq l\leq M} \vartheta(u_l) - \sum_{1\leq l<m\leq M} \vartheta(u_l+u_m) - \sum_{1\leq l<m\leq M} \vartheta(u_l-u_m) = 2 \sum_{1\leq l<m\leq M} \min(|u_l|,|u_m|). \tag{3.6}$$

We will use the above identity with $M = 4$ and $u_{1,2,3} = T_{1,2,3}$; we would moreover like to take $u_4 = x_i - x_j$, but this is not allowed since the range $-1 < x_i - x_j < 1$ is incompatible with $-1/2 \leq u_4 \leq 1/2$; to fix that we put instead $u_4 = \{x_i - x_j + 1/2\} - 1/2$. Using (3.6) we can now rewrite the L_h function in (3.4) such that its only $\boldsymbol{x}$-dependent piece is

$$-\frac{1}{3} \sum_{1\leq i<j\leq N} \sum_{k=1}^{3} \min\bigl(|T_k|, |\{x_i - x_j + 1/2\} - 1/2|\bigr) \tag{3.7}$$

The above expression is obviously negative-semi-definite as a function of x_i, and it is maximized when $x_i - x_j = 0$. So the index is Higgsed for any $T_k \in \mathbb{R}/\mathbb{Z}$ at infinite temperature.

Whether (or for which range of $T_k \in \mathbb{R}$) the infinite-temperature phase of the index can be exponentially growing when $N > 2$, is an interesting problem which seems to require an intricate analysis.

4 Supersymmetric Casimir energy with complex chemical potentials

When all the fugacities p, q, u_k are real-valued, the index $\mathcal{I}(p, q, u_k)$ is related to the path-integral SUSY partition function $Z(\beta, b, m_k)$ of the theory on $S^3_b \times S^1_\beta$ via [45]

$$Z(\beta, b, m_k) = e^{-\beta E_{\text{SUSY}}(b, m_k)} \mathcal{I}(p, q, u_k), \tag{4.1}$$

where $E_{\text{SUSY}}(b, m_k)$ is known as the supersymmetric Casimir energy, β, b, m_k are defined through

$$p = e^{-\beta b}, \quad q = e^{-\beta b^{-1}}, \quad u_k = e^{-\beta m_k}, \tag{4.2}$$

and S^3_b is the squashed three-sphere with unit radius and squashing parameter b, while S^1_β is the circle with circumference β. (The special case of (4.1) with $m_k = 0$ was understood already in [21, 46], based on earlier slightly contrasting computations of [44].)

As made clear by HHZ [26] (and further elucidated in [12–14, 37]) making contact with the AdS_5 BPS blackholes requires considering complex fugacities p, q, u_k in the index. With the goal of understanding the role of the supersymmetric Casimir energy in the blackhole entropy discussion, in this section we study the relation between Z and $\mathcal{I}$ for complex fugacities such that $b \in \mathbb{R}_{>0}$ and $\beta \in \mathbb{C}$ with $\text{Re}\beta > 0$ as in section 2, while u_k are on the unit circle as in section 3. Rather than modifying the background geometry to achieve such complexified β (cf. [44]), we simply analytically continue the results obtained for real p, q.

Let us consider a free chiral multiplet to begin with; as in [21, 44], we expect that solving this case leads to the solution of the interacting non-abelian case as well.

Following appendix A of [21], we start with the one-loop determinant of the nth KK mode on $S^3 \times S^1$. Eq. (A.15) in [21] now generalizes to

$$\log Z^{(n)} = \ell_b \left(-(R-1)\frac{b + b^{-1}}{2} + \frac{2\pi i}{\beta}(n + T_k) \right), \tag{4.3}$$

where ℓ_b is the special function discussed in [21], the R-charge of the multiplet is denoted by R, and $T_k := \frac{i\beta m_k}{2\pi} \in \mathbb{R}$, with m_k the only chemical potential the chiral multiplet couples to.

Define $X := (R-1)\frac{b+b^{-1}}{2}$ for notational convenience. Following [21] step by step, we now rewrite $\log Z^{(n)}$ in terms of ψ_b which has a simple asymptotic behavior. Eq. (A.2) of [21] implies that in terms of ψ_b:

$$\log Z^{(n)} = \log \psi_b \left(X - \frac{2\pi i}{\beta}(n + T_k) \right) + \frac{i\pi}{2} \left(X - \frac{2\pi i}{\beta}(n + T_k) \right)^2 - i\pi \left(\frac{b^2 + b^{-2}}{24} \right). \tag{4.4}$$

Now, using the fact that $\ell_b(-x) = -\ell_b(x)$, we can rewrite

$$\begin{aligned} \log Z^{(n)} = \log \psi_b \left(\left[X - \frac{2\pi i}{\beta}(n + T_k) \right] \text{sgn}(n + T_k) \right)^{\text{sgn}(n+T_k)} - \frac{i\pi}{2} \frac{4\pi^2}{\beta^2} \text{sgn}(n + T_k)(n + T_k)^2 \\ + \frac{4\pi^2}{2\beta} \text{sgn}(n + T_k)(n + T_k) X + \left[\frac{i\pi}{2} X^2 - i\pi \left(\frac{b^2 + b^{-2}}{24} \right) \right] \text{sgn}(n + T_k). \end{aligned} \tag{4.5}$$

One way to check the above equation is to check it separately for $\mathrm{sgn}(n+T_k) = +1$ and $\mathrm{sgn}(n+T_k) = -1$, using $\ell_b(-x) = -\ell_b(x)$ and eq. (A.2) in [21]. The reason for this rewriting is to divide ψ_bs into the numerator and denominator of Z, so we can eventually relate Z to $\mathcal{I}$ using expressions such as (2.1).

Finally, we sum (4.5) over $n \in \mathbb{Z}$. In doing so, we use the relations Di Pietro and Honda used [24] for analyzing the high-temperature asymptotics of the index:

$$\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n+T_k) = 1 - 2\{T_k\}, \tag{4.6}$$

$$\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n+T_k)\big(n+T_k\big) = \vartheta(T_k) - \frac{1}{6}, \tag{4.7}$$

$$\sum_{n\in\mathbb{Z}} \mathrm{sgn}(n+T_k)\big(n+T_k\big)^2 = -\frac{1}{3}\kappa(T_k). \tag{4.8}$$

The definitions are $\{x\} := x - \lfloor x \rfloor$, $\vartheta(x) := \{x\}(1-\{x\})$, $\kappa(x) := \{x\}(1-\{x\})(1-2\{x\})$.

With this regularization — combining techniques from [21] and [24] — we obtain

$$\begin{aligned} \log Z = \sum_{n\in\mathbb{Z}} \log \psi_b &\left(\left[X - \frac{2\pi i}{\beta}(n+T_k)\right]\mathrm{sgn}(n+T_k)\right)^{\mathrm{sgn}(n+T_k)} \\ &+ \frac{i(2\pi)^3}{12\beta^2}\kappa(T_k) + \frac{(2\pi)^2}{2\beta}(R-1)\frac{b+b^{-1}}{2}\left(\vartheta(T_k) - \frac{1}{6}\right) \\ &+ \left[\frac{i\pi}{2}\left((R-1)\frac{b+b^{-1}}{2}\right)^2 - i\pi\left(\frac{b^2+b^{-2}}{24}\right)\right](1-2\{T_k\}). \end{aligned} \tag{4.9}$$

Putting $T_k = 0$ we can compare with eq. (A.16) in [21], noting that $\kappa(0) = \vartheta(0) = 0$, so that the only surviving term on the second line of the r.h.s. of the above relation gives the Di Pietro-Komargodski asymptotics [20] as $\beta \to 0$; the first and the third lines combine to give the first and the third terms on the r.h.s. of eq. (A.16) in [21].

We are done with our regularization. We believe our method of regularization is correct because we have been careful with the convergence of the infinite product appearing in Z — or equivalently the convergence of the infinite sum appearing in $\log Z$ — after regularization, and because we have used well-established tools of analytic continuation[10] for evaluating the sums (4.6)–(4.8). As a byproduct, from the second line on the r.h.s. of (4.9) we can read off the high-temperature asymptotics of the partition function of a chiral multiplet with a flavor fugacity on the unit circle.

We now would like to relate Z as obtained in (4.9) to the index $\mathcal{I}$. We use (2.1) and the fact that the index of the chiral multiplet is $\Gamma((pq)^{R/2}u_k)$. For simplicity we assume $0 < T_k < 1$, and replace all $\{T_k\}$ in (4.9) with T_k. Then we set (4.9) equal to

$$\log \mathcal{I} - \beta E_{\mathrm{SUSY}}(b, T_k) = 2\pi i Q_+(x = R(\tau+\sigma)/2 + T_k; \sigma, \tau) + \sum \log \psi_b - \beta E_{\mathrm{SUSY}}(b, T_k). \tag{4.10}$$

The end result is that E_{SUSY} comes out just as in [21, 46]: there is no dependence on T_k!

[10] See chapter VII of [47] for some context.

In other words, for $u_k = e^{2\pi i T_k}$ on the unit circle (which is relevant to the equal-charge AdS_5 blackholes) we have $E_{\rm SUSY}(b, m_k) = E_{\rm SUSY}(b, 0)$. Since in the small-$|\beta|$ limit with $b > 0$ fixed we have $\beta E_{\rm SUSY}(b, 0) \to 0$, we conclude that on the saddle-point associated to the equal-charge blackholes the supersymmetric Casimir energy has no significance in the leading Cardy-like asymptotics of the partition function Z. In particular, the Casimir-energy factor relating Z and $\mathcal{I}$ is irrelevant to the blackhole entropy function arising in the Cardy-like limit of either.

The relation between the above discussion and the interesting proposal of [37] which seems to involve analytic continuation of Z with respect to τ and σ is currently under study.

5 Summary and open problems

We have presented a careful analysis of the asymptotics of the SU(N) $\mathcal{N} = 4$ theory index (1.3) in the CKKN limit where the flavor fugacities approach the unit circle and the spacetime fugacities approach 1. We emphasize that compared to the previous work [22] the Cardy-like limit studied here is more general in two respects: *i*) instead of sending the flavor fugacities to 1 as in [22], following CKKN here we allowed the flavor fugacities to approach the unit circle; *ii*) although in section 3 we kept the "inverse-temperature" β on the positive real axis as in [22], in section 2 we complexified β and let $|\arg\beta| \in (0, \pi/2)$ which was necessary for making contact with the HHZ function and the AdS_5 blackholes.

For complexified temperature (with $0 < |\arg\beta| < \pi/2$), we have demonstrated that in the CKKN limit, the dominant holonomy configuration in the index is dictated by the pairwise potential (2.8). We explained that depending on the sign of $\arg\beta$, the pairwise potential has M- or W-type behavior on complementary wings of the space of the control-parameters $\mathrm{Re}\Delta_{1,2}$ — see figure 2. On the M wings the potential is minimized at the origin, so the holonomies condense (at either of N possibilities $x_j = 0, \frac{1}{N}, \ldots, \frac{N-1}{N}$ breaking the $\mathbb{Z}_N$ center), thereby giving rise to the HHZ function (1.12) (with $\Delta_3 = \tau + \sigma - \Delta_1 - \Delta_2 + \mathrm{sgn}(\arg\beta)$) in the leading asymptotics of the index, from which we extracted two blackhole saddle-points (one for each sign of $\arg\beta$). On the other hand, on the W wings, except for the $N = 2$ case, the quantitative analysis seems difficult; we only presented intuitive arguments suggesting that for $N > 2$ the index has a slower asymptotic growth there and therefore no blackhole saddle-points with entropies as large as the two just mentioned are expected in those regions.

Problem 1) In the CKKN limit

$$|\sigma|, |\tau|, \mathrm{Im}\Delta_k \to 0, \text{ with } \frac{\tau}{\sigma} \in \mathbb{R}_{>0}, \mathrm{Re}\Delta_k \text{ fixed, and } \mathrm{Im}\tau, \mathrm{Im}\sigma > 0, \tag{5.1}$$

find the asymptotics of the SU(N) $\mathcal{N} = 4$ theory index for $N > 2$ on the W wings; that is, for τ, σ inside the 2nd quadrant and $\mathrm{Re}\Delta_{1,2}$ on the lower-left wing of figure 2, or for τ, σ inside the 1st quadrant and $\mathrm{Re}\Delta_{1,2}$ on the upper-right wing of figure 2. In particular, show that the fastest asymptotic growth in those regions is slower than the fastest growth on the complementary M wings.

JHEP06(2019)134

Even without addressing the above problem, we have successfully derived two blackhole saddle-points in the CKKN limit. However, the saddle-points have flavor fugacities that are away from the unit circle unless the three charges Q_k are (approximately) equal [in which case the critical $\Delta_{1,2,3}$ are in fact (approximately) simply $\mathrm{sgn}(\arg\beta) \times \frac{1}{3}$]. Therefore our derivation of the blackhole entropy function is incomplete for the general blackholes with unequal Q_k. To complete the analysis for the general case we have to derive the asymptotic relation (1.12) when $\mathrm{Im}\Delta_k$ are not sent to zero.

Problem 2) For complexified temperature ($0 < |\arg\beta| < \pi/2$) perform the asymptotic analysis of the index in the limit

$$|\sigma|, |\tau| \to 0, \text{ with } \frac{\tau}{\sigma} \in \mathbb{R}_{>0}, \Delta_k \in \mathbb{C} \text{ fixed, and } \mathrm{Im}\tau, \mathrm{Im}\sigma > 0. \tag{5.2}$$

In particular, derive the HHZ function, once with $\sum_k \Delta_k = \tau + \sigma - 1$ and once with $\sum_k \Delta_k = \tau + \sigma + 1$, in two separate regimes of parameters.

We would like to emphasize that although we have not given a complete derivation of the entropy function for the general case with unequal charges, our analysis in the equal-charge case already allows addressing various conceptual issues in the derivation. One such conceptual issue has been the significance of the rather special relation $\sum_k \Delta_k = \tau + \sigma - 1$ in the HHZ function [26]. In the present paper we have shown that a similar asymptotics arises with $\sum_k \Delta_k = \tau + \sigma + 1$ in a separate region of parameters, leading to a second blackhole saddle-point with fugacities that are complex conjugate to those of the HHZ saddle-point. (See [13] for related statements in the large-N analysis.)

Another conceptual point that we were able to clarify in the special case with equal charges was the insignificance of the supersymmetric Casimir energy to the blackhole entropy function in the Cardy-like limit. Generalizing that discussion to the case with the flavor fugacities away from the unit circle constitutes another important open problem related to the present work.

Problem 3) Study the supersymmetric Casimir energy of the $\mathcal{N} = 4$ theory with flavor fugacities away from the unit circle. In particular, investigate its relevance to the blackhole entropy function in the Cardy-like limit.

Note added. While this work was nearing completion the preprint [48] appeared on arXiv which has some overlap with our section 2. Reference [48] seems to suggest that extra hairy-blackhole [49, 50] saddle-points might reside on the W wings of the parameter-space. As discussed in section 2, we find it more likely that no such *extra* blackhole saddle-points with entropies as large as the ones discussed here (as expected to be the case for the hairy blackholes of [49, 50]) exist in the Cardy-like limit of the index. The existence/interpretation of extra saddle-points in the large-N analysis [13] is of course a separate issue.

Also, note that [48] gives around its eq. (2.11) a neat analytic proof for the M-type behavior of the pairwise potential in a specific subset of the parameter-space for $\arg\beta > 0$ (i.e. $\mathrm{Re}(i/\tau\sigma) < 0$). However, the argument below (2.9) there that *the periodicity of the* *pairwise potential can be used to extend the analytic proof of the M-type behavior to the whole parameter-space* does not seem applicable; as illustrated in figure 2, in essentially half of the parameter-space the pairwise potential is in fact maximized at the origin (this can actually be seen from Honda's analytic proof in the appropriate region of the parameter-space, together with the oddity of the potential under $\Delta_k \to -\Delta_k$). Consequently, the W-type behavior of the pairwise potential in half of the parameter-space when $\arg\beta > 0$, as well as the M-type behavior in half of the parameter-space when $\arg\beta < 0$ (and thus also the second blackhole saddle-point) seem to have been overlooked in [48].

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JHEP06(2019)134

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JHEP06(2019)134