Published for SISSA by Springer

Received: January 21, 2019
Revised: May 13, 2019
Accepted: June 15, 2019
Published: June 27, 2019

T-duality in $(2, 1)$ superspace

M. Abou-Zeid,[a] C.M. Hull,[b] U. Lindström[b,c] and M. Roček[d]

[a] *SUB, Georg-August-Universität Göttingen, Platz der Göttinger Sieben 1, 37073 Göttingen, Germany*
[b] *Theory Group, The Blackett Laboratory, Imperial College London, Prince Consort Road, London SW7 2AZ, U.K.*
[c] *Department of Physics and Astronomy, Uppsala University, Box 516, SE-751 20 Uppsala, Sweden*
[d] *C.N. Yang Institute for Theoretical Physics, Stony Brook University, Stony Brook, NY 11794-3840,U.S.A.*

E-mail: bahom96@gmail.com, c.hull@imperial.ac.uk, lindoulf@gmail.com, martin.rocek@stonybrook.edu

Abstract: We find the T-duality transformation rules for 2-dimensional (2,1) supersymmetric sigma-models in (2,1) superspace. Our results clarify certain aspects of the (2,1) sigma model geometry relevant to the discussion of T-duality. The complexified duality transformations we find are equivalent to the usual Buscher duality transformations (including an important refinement) together with diffeomorphisms. We use the gauging of sigma-models in (2,1) superspace, which we review and develop, finding a manifestly real and geometric expression for the gauged action. We discuss the obstructions to gauging (2,1) sigma-models, and find that the obstructions to (2,1) T-duality are considerably weaker.

Keywords: Differential and Algebraic Geometry, Supersymmetry and Duality

ArXiv ePrint: 1901.00662

https://doi.org/10.1007/JHEP06(2019)138

Contents

1 Introduction

Supersymmetric nonlinear sigma models with D-dimensional target spaces have a rich structure, which makes them good tools for studying various geometries. The target-space geometries are constrained by the number of supersymmetries; in particular, there is a direct correspondence between target-space complex structures and world-volume supersymmetries. For two dimensional (p, q) supersymmetric models, the relationship between geometry and supersymmetry is particularly rich [1–11]. The $(2, 2)$ models of [2] have generalised Kähler geometry [12, 13]. A more general complex geometry with torsion arises for $(2, 0)$ supersymmetry [4], and for $(2, 1)$ supersymmetry [6], while the general geometry for (p, q) supersymmetric models for all p, q was found in [6]; see also [7, 8]. The $(2, 1)$ supersymmetric models [5] will be the focus of this paper and are relevant for supersymmetric compactifications of ten-dimensional superstring theories as well as for critical superstrings with $(2, 1)$ supersymmetry [14], which have interesting applications [15–18]. Their target space geometries include the generalised Kähler geometries of the $(2, 2)$ models as special cases. The reduction of $(2, 2)$ models to $(2, 1)$ superspace was discussed in ref. [19]. The $(2, 1)$ superspace formulation was first given in [20].

T-duality relates two-dimensional sigma-models that have different target space geometries but which define the same quantum field theory; for a review and references, see [21]. When the target space of a model has isometry group $\mathrm{U}(1)^d$, its T-dual is found by gauging the isometries and adding Lagrange multiplier terms (plus an important total derivative term) [22–24]. Integrating out the Lagrange multipliers constrains the (worldsheet) gauge fields to be trivial and so gives back the original model, while integrating out the gauge fields yields the T-dual theory, with the dual geometry given by the Buscher rules [22]. Various gaugings in and out of superspace have been described in [25–37].

The starting point for T-duality is the gauging of the sigma model, and extended supersymmetry imposes restrictions on the gauging. In particular, the isometries must be compatible with the supersymmetries, i.e. holomorphic with respect to all the associated complex structures [32, 38, 39]. For $(2, 2)$ supersymmetry, the gauging was discussed in [26–28, 32, 37, 38], while the gauging of $(2, 1)$ supersymmetric models was given in [34, 35] for the superspace formulation of [20] and in [32] for the formulation of [7, 8].

The supersymmetric T-duality transformations have an interesting geometric structure. For sigma models with Kähler target geometry, the T-duality changes the Kähler potential by a Legendre transformation [24, 40]. In general, duality can change the representation of the supersymmetry [40]. T-duality for $(2, 2)$ supersymmetric sigma models has been studied in [24, 41–44].

Here we will use the results of [34, 35] to analyse T-duality for $(2, 1)$ supersymmetric models in $(2, 1)$ superspace [20]. Adding Lagrange multiplier terms to the gauged theory and integrating out the gauge multiplets gives a dual geometry, with a $(2, 1)$ supersymmetric version of the T-duality transformation rules. The supersymmetric gauging involves a complexification of the action of the isometry group, resulting in a T-duality transformation that is a complexification of the usual T-duality rules. The complexified T-duality we find is equivalent to a real Buscher T-duality combined with a diffeomorphism; this is the same mechanism that was previously found for the $(2, 2)$ supersymmetric T-duality (see [22, 42]).

JHEP06(2019)138

For bosonic and $(1,1)$ sigma-models with Wess-Zumino term, there are geometric and topological obstructions to gauging in general [29, 30]. For T-duality, however, the obstructions are considerably milder [36, 45]. Here we will extend this discussion to $(2,1)$ models, analysing the obstructions to gauging and T-duality. Moreover, we will interpret our results for T-duality in terms of generalised moment maps and a generalised Kähler quotient.

The paper is organised as follows. In section 2, we first review the general gauged sigma model and the obstructions to its gauging. We then summarise the formulation of T-duality of [36] in terms of a lift to a higher-dimensional sigma model and show that the obstructions to T-duality are much milder than the obstructions to gauging — one can T-dualise an ungaugable sigma model. In particular, we recall and emphasize that the Buscher rules are modified when the sigma-model Lagrangian is invariant only up to a total derivative term under the isometry used for the T-duality [36, 45]. In section 3 we give the superspace description of the $(2,1)$ models. Section 4 discusses the isometries of $(2,1)$ models in superspace. In section 5, we review the superspace description of the $(2,1)$ Yang-Mills supermultiplet. In section 6 we review the results of [34, 35] on the gauging of the $(2,1)$ models. We discuss T-duality for the $(2,1)$ sigma models in section 7, and derive the duality transformations of the potentials for the $(2,1)$ geometries with torsion. We find the duality transformations for the metric and b-field, which give a complex version of the Buscher rules. In section 8, we explain how our complexified T-duality transformations give the real Buscher rules combined with diffeomorphisms, and illustrate this with some examples. In section 9, we adapt the general results of ref. [36] to the geometry and T-dualisation of $(2,1)$ models, including the cases for which there are obstructions to the gauging and for which the standard T-dualisation procedure fails. Section 10 contains a summary of our results. Some technical details are collected in four appendices.

2 The gauged sigma model and T-duality

2.1 The gauged bosonic sigma model

The two-dimensional sigma model with D-dimensional target space M is a theory of maps $\phi : \Sigma \to M$, where Σ is a 2-dimensional manifold. The action is the sum of a kinetic term S^0_{kin} and a Wess-Zumino term S^0_{WZ},

$$S^0 = S^0_{\text{kin}} + S^0_{\text{WZ}} \ . \tag{2.1}$$

Given a metric g on M and a metric h on Σ, the kinetic term can be written as

$$S^0_{\text{kin}} = \frac{1}{2}\int_\Sigma * \operatorname{tr}(h^{-1}\phi^* g)\,, \tag{2.2}$$

where the Hodge dual on Σ for the metric h is denoted by $*$ and $\phi^* g$ is the pull-back of g to Σ. If x^i $(i = 1, \ldots D)$ are coordinates on M and σ^a are coordinates on Σ, the map is given locally by functions $x^i(\sigma)$ and $\operatorname{tr}(h^{-1}\phi^* g) = h^{ab}(\phi^* g)_{ab} = h^{ab} g_{ij}\partial_a x^i \partial_b x^j$, so that the Lagrangian 2-form can be written locally as

$$L^0_{\text{kin}} = \frac{1}{2} g_{ij}(x(\sigma))\, dx^i \wedge *dx^j \ . \tag{2.3}$$

Here and in what follows, the pull-back $\phi^*(dx^i) = \partial_a x^i d\sigma^a$ will be written as dx^i, and it should be clear from the context whether a form on M or its pull-back is intended.

The Wess-Zumino term is constructed using a closed 3-form H on M. We write

$$S^0_{\text{WZ}} = \int_\Gamma \phi^* H \,, \tag{2.4}$$

where Γ is any 3-manifold with boundary Σ. This can be written in terms of local coordinates as

$$S^0_{\text{WZ}} = \frac{1}{3} \int_\Gamma H_{ijk}\, dx^i \wedge dx^j \wedge dx^k \,. \tag{2.5}$$

Locally, H is given in terms of a 2-form potential b with

$$H = db \,, \tag{2.6}$$

and the Wess-Zumino term can be written locally in terms of a 2-form Lagrangian on a patch in Σ

$$S^0_{\text{WZ}} = \frac{1}{2} \int_\Sigma b_{ij}(x(\sigma))\, dx^i \wedge dx^j . \tag{2.7}$$

The functional integral involving the Wess-Zumino term (2.4) is well-defined and independent of the choice of Γ provided $\frac{1}{2\pi} H$ represents an integral cohomology class[1] on M.

The conditions for gauging isometries of this model were derived in [29, 30] and will now be briefly reviewed. Suppose there are d Killing vectors ξ_K $(K = 1, \dots d)$ with $\mathcal{L}_K g = 0$, $\mathcal{L}_K H = 0$, where $\mathcal{L}_K$ is the Lie derivative with respect to ξ_K. The ξ_K generate an isometry group with structure constants $f_{KL}{}^M$, with

$$[\mathcal{L}_K, \mathcal{L}_L] = f_{KL}{}^M \mathcal{L}_M . \tag{2.8}$$

Then under the transformations

$$\delta x^i = \lambda^K \xi^i_K(x) \tag{2.9}$$

with constant parameters λ^K, the action (2.1) changes by a surface term if $\iota_K H$ is exact, so that the equation

$$\iota_K H = du_K \tag{2.10}$$

is satisfied for some (globally defined) 1-forms u_K. The u_K are defined by (2.10) up to the addition of exact forms. Thus the transformations (2.9) are global symmetries provided $\iota_K H$ is exact. When this is the case, the functions

$$c_{KL} \equiv \iota_K u_L \tag{2.11}$$

are globally defined. We note that in the special case in which the b-field is invariant,

$$\mathcal{L}_K b = 0 \,, \tag{2.12}$$

we have

$$u_K = \iota_K b \,, \tag{2.13}$$

but in general $u_K \neq \iota_K b$.

[1]When the third cohomology group $\mathbb{H}^3$ of M is nontrivial, this leads to a quantisation condition for H; if $\mathbb{H}^3$ is trivial, then (2.6) is globally defined and there is no quantisation condition.

JHEP06(2019)138

The gauging of the sigma-model [29–31] consists in promoting the symmetries (2.9) to local ones, with parameters that are now functions $\lambda^K(\sigma)$, by seeking a suitable coupling to connection 1-forms A^K on Σ transforming as

$$\delta A^M = d\lambda^K - f_{KL}{}^M A^K \lambda^L. \tag{2.14}$$

The conditions for gauging to be possible found in [29, 30] are that (i) $\iota_K H$ is exact, (ii) a 1-form $u_K = u_{Ki} dx^i$ satisfying (2.10) can be chosen that satisfies the equivariance condition

$$\mathcal{L}_K u_L = f_{KL}{}^M u_M \tag{2.15}$$

(so that $\iota_K H$ represents a trivial equivariant cohomology class [46]), and (iii)

$$\iota_K u_L = -\iota_L u_K \tag{2.16}$$

so that the globally defined functions (2.11) are skew,

$$c_{KL} = -c_{LK}. \tag{2.17}$$

Defining the covariant derivative of x^i by

$$D_a x^i \equiv \partial_a x^i - A_a^K \xi_K^i \tag{2.18}$$

and the field strength

$$F^M = dA^M - \frac{1}{2} f_{KL}{}^M A^K \wedge A^L, \tag{2.19}$$

the gauged action is [29]

$$S = S_{\text{kin}} + S_{WZW} \ . \tag{2.20}$$

The gauged metric term is minimally coupled:

$$S_{\text{kin}} = \frac{1}{2} \int_\Sigma g_{ij} Dx^i \wedge *Dx^j \,, \tag{2.21}$$

whereas the gauged Wess-Zumino-Witten term involves a non-minimal term:

$$S_{WZW} = \int_\Gamma \left(\frac{1}{3} H_{ijk} Dx^i \wedge Dx^j \wedge Dx^k + F^K \wedge u_{Ki} Dx^i \right) , \tag{2.22}$$

with $\partial\Gamma = \Sigma$. It was shown in [29, 30] that this is closed and locally can be written as

$$S_{WZW} = \int_\Sigma \left(\frac{1}{2} b_{ij} \, dx^i \wedge dx^j + A^K \wedge u_K + \frac{1}{2} c_{KL} A^K \wedge A^L \right) , \tag{2.23}$$

with $u_K = u_{Ki} dx^i$. If the gauge group G acts freely on M, then the gauged theory (2.20) gives a quotient sigma model with target space M/G (the space of gauge orbits) on fixing a gauge and eliminating the gauge fields using their equations of motion.

2.2 Dualisation

A general method of dualisation of the ungauged sigma model (2.1) on (M, g, H) is to gauge an isometry group G as above and add the Lagrange multiplier term $\int_\Sigma F^K \hat{x}_K$ involving d scalar fields $\hat{x}_K$. The Lagrange multiplier fields $\hat{x}_K$ impose the constraint that the gauge fields A are flat and so pure gauge locally, so that (at least locally) one recovers the ungauged model. (If Σ is simply connected, e.g. if $\Sigma = S^2$ or $\Sigma = \mathbb{R}^2$, then A is pure gauge and one recovers precisely the ungauged model.) Alternatively, fixing the gauge by a suitable constraint on the coordinates x^i and integrating out the gauge fields A gives a dual sigma model whose coordinates now include the fields $\hat{x}_K$. This method applies quite generally, including the cases of non-Abelian or non-compact G.

In general, the two dual sigma models are distinct in the quantum theory. However for special cases, the two dual sigma models can define the *same* quantum theory, in which case the two dual theories are said to be related by a T-duality. T-dual theories arise for isometry groups G that are compact and Abelian so that $G = \mathrm{U}(1)^d$ with the action of G defining a torus fibration on M for which the torus fibres are the orbits of G. There are also further restrictions on the torus fibration; see e.g. [36]. The classic example is that in which M is a torus T^d, with the natural action of $G = \mathrm{U}(1)^d$ on the torus.

String theory backgrounds require sigma models that define conformally invariant quantum theories. For a sigma model on (M, g, H) to define a conformal field theory in general requires the addition of a coupling to a dilaton field Φ on M through a Fradkin-Tseytlin term, and the T-duality then takes a sigma model on (M, g, H, Φ) to a dual one (M', g', H', Φ') on a manifold M' (which in general is different from M), with the two sigma models defining the same conformal field theory. A proof of the quantum equivalence of T-dual CFTs was given in [24].

For applications to T-duality, we focus on the case of Abelian isometries. We derive dual pairs of geometries for general Abelian isometry groups (including non-compact groups or ones that act with fixed points). It is convenient to refer to all of these as T-dualities, although not all lead to full quantum equivalence between dual theories, so not all are proper T-dualities in the strict sense. Our main interest will be in dual pairs that define equivalent quantum theories, but the same formulae apply to the more general class of dual theories.

For Abelian G, ${f_{KL}}^M = 0$, so that, assuming u satisfies the equivariance condition (2.15),

$$\mathcal{L}_K \xi^L = 0, \qquad \mathcal{L}_K u_L = 0, \qquad \mathcal{L}_K c_{LM} = 0. \tag{2.24}$$

Starting from (2.10) and (2.24), the identity

$$\iota_K \iota_L H = \mathcal{L}_K u_L - d\iota_K u_L \tag{2.25}$$

implies $\iota_K \iota_L H$ is exact, with

$$\iota_K \iota_L H = -dc_{KL}\,, \tag{2.26}$$

and

$$\iota_K \iota_L \iota_M H = 0\ . \tag{2.27}$$

To dualise the ungauged sigma model (2.1) on (M, g, H) with respect to d Abelian isometries, one gauges the isometries as above and adds the following Lagrange multiplier term involving d scalar Lagrange multiplier fields $\hat{x}_K$ [22–24, 47–49]

$$S_{LM} = \int_\Sigma A^K \wedge d\hat{x}_K \ . \tag{2.28}$$

This differs from the expression $\int_\Sigma F^K \hat{x}_K$ by a surface term that is crucial for quantum equivalence [24, 48]. The $\hat{x}_K$ impose the constraint that the gauge fields A are flat. For compact G, the holonomies $e^{i\oint A}$ around non-contractible loops on Σ are eliminated by requiring the $\hat{x}_K$ to be periodic coordinates of a torus T^d so that the winding modes of the $\hat{x}_K$ set the holonomies $e^{i\oint A}$ to the identity. Then the gauge field is trivial for any Σ and A can be absorbed by a gauge transformation, recovering the original ungauged model. Alternatively, fixing a gauge and integrating out the gauge fields A gives the T-dual sigma model. In adapted coordinates $x^i = (x^K, y^\mu)$ in which

$$\xi^i_K \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^K},$$

one can fix the gauge setting the x^K to constants and this gives a dual geometry with coordinates $(\hat{x}_K, y^\mu)$.

One of the conditions for gauging to be possible was that $c_{KL} = -c_{LK}$. If one relaxes this constraint, then (2.20) is no longer gauge invariant, with its gauge variation depending on the constants $c_{(KL)}$ and given by

$$\delta S = \int_\Sigma c_{(KL)} d\lambda^K \wedge A^L \ . \tag{2.29}$$

Remarkably, this variation can then be cancelled by the variation of (2.28) by requiring that

$$\delta \hat{x}_K = c_{(KL)} \lambda^L \tag{2.30}$$

so that $\hat{x}_K$ can be thought of as a compensator field, transforming as a shift under the gauge symmetry. This was first observed in [45] for the special case of a single isometry and extended to the general case in [36]. Furthermore, it was shown in [36] that introducing the fields $\hat{x}_K$ through (2.28) allows all three conditions for gauging listed above to be relaxed and replaced by one much milder condition. This allows the gauging and T-dualisation of ungaugable sigma models; we next review the construction of [36].

2.3 Duality as a quotient of a higher dimensional space

It is natural to seek to interpret the Lagrange multiplier fields $\hat{x}_K$ as d extra coordinates, so that we have a sigma model with $D + d$ dimensional target space $\hat{M}$ with coordinates $\hat{x}^\alpha = (x^i, \hat{x}_K)$, where $\alpha = 1, \ldots D + d$. Then the gauged action plus the Lagrange multipler term can be viewed as a gauge-invariant sigma model on $\hat{M}$, and this can be compared with the standard form of the gauged sigma model (2.20) reviewed above. In particular, the terms linear in A in the sum of the Wess-Zumino term (2.23) and the Lagrange multiplier term (2.28) are

$$S_{LM} = \int_\Sigma A^K \wedge (u_K + d\hat{x}_K), \tag{2.31}$$

which suggests introducing a modified 1-form

$$\hat{u}_K = u_K + d\hat{x}_K \tag{2.32}$$

on $\hat{M}$. If the condition that u_K is a globally defined one-form is dropped, the constraint that du_K is a globally defined closed 2-form suggests interpreting u_K as a connection one-form on a $\mathrm{U}(1)^d$ bundle over M. If $\hat{x}_K$ are taken as fibre coordinates, then $\hat{u}_K$ can be globally defined one-form on $\hat{M}$; this is the starting point for the construction of [36].

The space $\hat{M}$ with coordinates $\hat{x}^\alpha = (x^i, \hat{x}_K)$ is then a bundle over M with projection $\pi : \hat{M} \to M$ which acts as $\pi : (x^i, \hat{x}_K) \mapsto x^i$. A (degenerate) metric $\hat{g}$ and closed 3-form $\hat{H}$ can be chosen on $\hat{M}$ with no $\hat{x}_K$ components, i.e.

$$\hat{g} = \pi^* g, \qquad \hat{H} = \pi^* H, \tag{2.33}$$

where π^* is the pull-back of the projection. The pull-back will often be omitted in what follows, so that the above conditions will be abbreviated to $\hat{g} = g, \hat{H} = H$. Then the only non-vanishing components of $\hat{g}_{\alpha\beta}$ are g_{ij}, $\partial/\partial\hat{x}_K$ is a null Killing vector, and the only non-vanishing components of $\hat{H}_{\alpha\beta\gamma}$ are H_{ijk}.

We consider the general set-up with d commuting vector fields on M preserving H. This implies that there are local potentials u_K with $\iota_K H = du_K$, but they need not be global 1-forms, and need not satisfy (2.15) or (2.16).

We lift the Killing vectors ξ_K on M to vectors $\hat{\xi}_K$ on $\hat{M}$ with

$$\hat{\xi}_K = \xi_K + \Omega_{KL}\frac{\partial}{\partial\hat{x}_L}\,, \tag{2.34}$$

for some Ω_{KL} to be determined below. As the metric g and the torsion 3-form H are both independent of the coordinates $\hat{x}_K$, the $\hat{\xi}_K$ are Killing vectors on $\hat{M}$:

$$\hat{\mathcal{L}}_K \hat{g} = 0, \qquad \hat{\mathcal{L}}_K \hat{H} = 0. \tag{2.35}$$

As $du = d\hat{u}$, with $\hat{u}$ given in (2.32), it follows that

$$\hat{\iota}_K \hat{H} = d\hat{u}_K, \tag{2.36}$$

where $\hat{\iota}_K$ denotes the interior product with $\hat{\xi}_K$. From (2.32), we find

$$\hat{\iota}_K \hat{u}_L = \iota_K u_L + \Omega_{KL}. \tag{2.37}$$

If we now choose

$$\Omega_{KL} = -\frac{1}{2}(\iota_K u_L + \iota_L u_K)\,, \tag{2.38}$$

then

$$\hat{\iota}_K \hat{u}_L + \hat{\iota}_L \hat{u}_K = 0 \tag{2.39}$$

and the functions on $\hat{M}$ defined by

$$\hat{c}_{KL} \equiv \hat{\iota}_K \hat{u}_L \tag{2.40}$$

are found to satisfy

$$\hat{c}_{KL} = c_{[KL]}, \tag{2.41}$$

where the functions c_{KL} are defined in (2.11).

Next, the Lie derivative of the potentials $\hat{u}_K$ with respect to $\hat{\xi}$ is now zero:

$$\hat{\mathcal{L}}_K \hat{u}_L = 0, \tag{2.42}$$

so the $\hat{u}_K$ are equivariant. Finally, if

$$\iota_K \iota_L \iota_M H = 0, \tag{2.43}$$

then the isometry group generated by the $\hat{\xi}_K$ is Abelian,

$$[\hat{\mathcal{L}}_K, \hat{\mathcal{L}}_K] = 0. \tag{2.44}$$

Note that this condition implies that $\hat{\iota}_K \hat{\iota}_L \hat{\iota}_M \hat{H} = 0$.

The target space $\hat{M}$ has dimension $D + d$, where D is the dimension of M and d is the dimension of the Abelian gauge group G. The gauged model on $\hat{M}$ gives, on eliminating the gauge fields, a quotient sigma model with target space given by the space of orbits, $\hat{M}/G$, which is also of dimension D. The T-dual geometry is given by this quotient space.

In summary, if we start from a geometry (M, g, H) preserved by d commuting Killing vectors ξ_K, then on a patch U of M we can find local potentials u_K satisfying $du_K = \iota_K H$ and lift them to Killing vectors $\hat{\xi}_K$ and potentials $\hat{u}_K$ on a patch of $\hat{M}$. If the torsion 3-form H on M satisfies $\iota_K \iota_L \iota_M H = 0$, then there are no further local obstructions to gauging the isometries on $\hat{M}$ generated by $\hat{\xi}_K$, even when there are local obstructions to gauging the isometries on M generated by ξ_K. For the gauged action on $\hat{M}$ to be globally defined, one needs to specify the bundle over M by giving the transition functions for the coordinates $\hat{x}_K$, require that the $\hat{\xi}_K$ are globally defined vector fields on $\hat{M}$ and also that the $\hat{u}_K$ are globally defined 1-forms on $\hat{M}$. In the overlaps $U \cap U'$ of patches U, U' on M, the potentials u_K satisfying $du_K = \iota_K H$ are related by $u'_K = u_K + d\alpha_K$ for some transition functions α_K, so that the u_K are components of a connection on M with field strength given by $\iota_K H$. The $\hat{x}_K$ are then fibre coordinates with $\hat{u}_K = u_K + d\hat{x}_K$ globally defined on $\hat{M}$. If the Killing vectors ξ_K can be normalised so that $\frac{1}{2\pi}\iota_K H$ all represent integral cohomology classes, then the bundle can be taken to be a $\mathrm{U}(1)^d$ bundle with fibres $(S^1)^d$, while otherwise it is a line bundle with fibres $\mathbb{R}^d$. Details of the global structure are given in [36]. For T-duality, we require that the fibres be circles. Generalisations to cases in which the $\hat{\xi}_K$, the $\hat{u}_K$ or both are only locally defined, or in which $\iota_K \iota_L \iota_M H \neq 0$, were discussed in [36, 50–52]; such T-dualities, when they can be defined, typically lead to non-geometric backgrounds.

We remark on an important observation made in [45] for a single isometry and in [36] for the general case: when (2.12) is not satisfied, that is, when $\mathcal{L}_K b \neq 0$ and hence when

$u_K \neq \iota_K b$, *the Buscher rules* [22] *are modified.* For a single isometry in adapted coordinates $x^i = (x^0, y^\mu)$, $\xi = \partial/\partial x^0$, the dual geometry has coordinates $(\hat{x}^0, y^\mu)$ and the modified Buscher rules are:

$$\begin{aligned} g^D_{\hat{0}\hat{0}} &= \frac{1}{g_{00}} \ , \qquad g^D_{\hat{0}\mu} = \frac{u_\mu}{g_{00}} \ , \qquad g^D_{\mu\nu} = g_{\mu\nu} + \frac{1}{g_{00}} \left(u_\mu u_\nu - g_{0\mu} g_{0\nu} \right) \ , \\ b^D_{\hat{0}\mu} &= \frac{g_{0\mu}}{g_{00}} \ , \qquad b^D_{\mu\nu} = b_{\mu\nu} - \frac{1}{g_{00}} \left(u_\mu g_{0\nu} - g_{0\mu} u_\nu \right) \ . \end{aligned} \tag{2.45}$$

The usual Buscher rules are recovered when $u_\mu = b_{0\mu}$. Geometric formulae for the duality transformations for the tensors g, H (without using adapted coordinates) for arbitrary numbers of isometries are given in [36].

Finally, the global issues which may arise when T-dualising are dealt with in the standard way. Suppose the coordinate x^0 is periodic with $x^0 \sim x^0 + 2\pi$, and the metric contains the radii: $g_{00} = R^2$. Here, as throughout the paper, we have set the string tension $T = 1$, but to keep track of dimensions, we can introduce it by rescaling the metric $g_{ij} \to T g_{ij}$, so $g_{00} = TR^2$; then the radius in dimensionless units is $\sqrt{T}R$. After we gauge and introduce the dual coordinate $\hat{x}^0$, we can insure the holonomies of the gauge fields are trivial and hence the model is equivalent to the original ungauged model by insisting that $\hat{x}^0$ is periodic with $\hat{x}^0 \sim \hat{x}^0 + 2\pi$. Consider the functional integral given by

$$\int [Dx^i D\hat{x}^0 DA^K] \, e^{i(T\,S + S_{LM})}, \tag{2.46}$$

where S is the gauged sigma model action (2.20), and S_{LM} is the Lagrange multiplier term (2.28). Then (2.46) is invariant under large gauge transformations for compact worldsheets Σ of arbitrary topology, and using the Buscher rules we have

$$T\hat{R}^2 = g^D_{\hat{0}\hat{0}} = \frac{1}{g_{00}} = \frac{1}{TR^2} \quad \Rightarrow \quad \hat{R} = \frac{1}{TR} \ . \tag{2.47}$$

The analysis of the geometry, gauging and T-duality given in this section for bosonic sigma models readily extends to (1,1) supersymmetric sigma models formulated in (1,1) superspace: the geometry of the gauging is just as in the bosonic case. For such (1,1) models to have $(2,1)$ supersymmetry requires the existence of a complex structure with certain restrictions on the geometry. For the gauging to be possible with manifest $(2,1)$ supersymmetry requires the Killing vectors to be holomorphic. The geometry of the gauged $(2,1)$ sigma models and their application to T-duality will be analysed in the following sections.

3 The (2,1) sigma model in superspace

The $(2,1)$ superspace is parametrised by two Bose coordinates $\sigma^{\ddagger}, \sigma^{=}$, a complex Fermi chiral spinor coordinate $\theta^+, \bar{\theta}^+$, and a single real Fermi coordinate θ^- of the opposite chirality. It is natural to define the complex conjugate left-handed spinor derivatives

$$D_+ = \frac{\partial}{\partial \theta^+} + i\bar{\theta}^+ \frac{\partial}{\partial \sigma^{\ddagger}} \ , \quad \bar{D}_+ = \frac{\partial}{\partial \bar{\theta}^+} + i\theta^+ \frac{\partial}{\partial \sigma^{\ddagger}}, \tag{3.1}$$

as well as a real right-handed spinor derivative

$$D_- = \frac{\partial}{\partial \theta^-} + i\theta^- \frac{\partial}{\partial \sigma^=}. \tag{3.2}$$

These spinor derivatives satisfy the algebra

$$D_+^2 = 0\,, \quad \bar{D}_+^2 = 0\,, \quad D_-^2 = i\partial_=\,, \quad \{D_+, \bar{D}_+\} = 2i\partial_{\ddagger}. \tag{3.3}$$

We denote by M the D real dimensional target space manifold of the sigma model and pick local coordinates x^i, $i = 1, \ldots D$ in which the metric and torsion potential are g_{ij} and b_{ij}. It was shown in [2, 4, 6, 25] that invariance of the (1,1) supersymmetric sigma model action under a second (right-handed) chiral supersymmetry requires that

(i) D is even

(ii) M admits a complex structure[2] $J^i{}_j$

(iii) the metric is hermitian with respect to the complex structure and

(iv) the complex structure $J^i{}_j$ is covariantly constant with respect to the connection $\nabla^+ = \nabla + \frac{1}{2}g^{-1}H$ with torsion $\frac{1}{2}g^{-1}H$.

We assume that these conditions are satisfied so that the sigma model has $(2,1)$ supersymmetry. We choose a complex coordinate system z^α, $\bar{z}^{\bar{\beta}} = (z^\beta)^*$, $(\alpha, \bar{\beta} = 1 \ldots \frac{1}{2}D)$ in which the line element is $ds^2 = 2g_{\alpha\bar{\beta}}dz^\alpha d\bar{z}^{\bar{\beta}}$ and the complex structure is constant and diagonal,

$$J^i{}_j = i\begin{pmatrix} \delta^\beta{}_\alpha & 0 \\ 0 & -\delta^{\bar{\beta}}{}_{\bar{\alpha}} \end{pmatrix}. \tag{3.4}$$

The supersymmetric sigma model can then be formulated in $(2,1)$ superspace in terms of scalar superfields φ^α, $\bar{\varphi}^{\bar{\alpha}} = (\varphi^\alpha)^*$, which are constrained to satisfy the chirality conditions

$$\bar{D}_+\varphi^\alpha = 0 \;\;, \quad D_+\bar{\varphi}^{\bar{\alpha}} = 0. \tag{3.5}$$

The lowest components of the superfields $\varphi^\alpha|_{\theta=0} = z^\alpha$ are the bosonic complex coordinates of M. The most general renormalizable and Lorentz invariant $(2,1)$ superspace action written in terms of chiral scalar superfields is [20]

$$S = S_1 + S_2, \tag{3.6}$$

where

$$S_1 = i\int d^2\sigma d\theta^+ d\bar{\theta}^+ d\theta^- \left(k_\alpha D_- \varphi^\alpha - \bar{k}_{\bar{\alpha}} D_- \bar{\varphi}^{\bar{\alpha}}\right) \tag{3.7}$$

and

$$S_2 = i\int d^2\sigma d\theta^+ d\theta^- F(\varphi) + \text{complex conjugate}. \tag{3.8}$$

[2] Supersymmetric models with almost complex structures were considered in [53, 54]; they obey a modified supersymmetry algebra and are not considered here.

Here F is a holomorphic section, as it is defined only up to the addition of a complex constant. Since F depends only on chiral superfields, the integration in (3.8) is over θ^+ only (and not over $\bar{\theta}^+$). The term S_2 is the analogue of the F-term in four dimensional supersymmetric field theories. In particular, this term can spontaneously break supersymmetry, it is not generated in sigma model perturbation theory if it is not present at tree level, and it is subject to a nonrenormalisation theorem, so that it is not corrected from its tree level value (up to possible wave-function renormalisations).

The $(2,1)$ sigma model geometry is sometimes referred to as strong Kähler with torsion (or SKT for short). It is determined locally by the complex vector field $k_\alpha(z,\bar{z})$ with complex conjugate

$$(k_\alpha)^* = \bar{k}_{\bar{\alpha}}. \tag{3.9}$$

The metric, torsion potential and torsion are given by

$$\begin{aligned} g_{\alpha\bar{\beta}} &= \partial_\alpha \bar{k}_{\bar{\beta}} + \bar{\partial}_{\bar{\beta}} k_\alpha \\ b'_{\alpha\bar{\beta}} &= \partial_\alpha \bar{k}_{\bar{\beta}} - \bar{\partial}_{\bar{\beta}} k_\alpha \\ H_{\alpha\beta\bar{\gamma}} &= \frac{1}{2}\bar{\partial}_{\bar{\gamma}} \left(\partial_\alpha k_\beta - \partial_\beta k_\alpha\right) , \end{aligned} \tag{3.10}$$

where b' is the torsion potential in a gauge where it is purely $(1,1)$. If the torsion $H=0$, the manifold M is Kähler with $k_\alpha = \frac{1}{2}\frac{\partial}{\partial z^\alpha} K(z,\bar{z})$ where $K(z,\bar{z})$ is the Kähler potential, and the $(2,1)$ supersymmetric model actually has $(2,2)$ supersymmetry, while for $H \neq 0$, M is a hermitian manifold with torsion of the type introduced in [2, 4].

The torsion potential b_{ij} is only defined up to an antisymmetric tensor gauge transformation of the form

$$\delta b_{ij} = \partial_{[i}\lambda_{j]}. \tag{3.11}$$

The (1,1)-form potential

$$b' \equiv b'_{\alpha\bar{\beta}} d\bar{z}^{\bar{\beta}} \wedge dz^\alpha = (\bar{k}_{\bar{\beta},\alpha} - k_{\alpha,\bar{\beta}})\, d\bar{z}^{\bar{\beta}} \wedge dz^\alpha , \tag{3.12}$$

can be transformed to a (2,0)+(0,2) form by a gauge transformation

$$b' \to b' + d(\bar{k}_{\bar{\beta}} d\bar{z}^{\bar{\beta}} + k_\beta dz^\beta) = b^{(0,2)} + b^{(2,0)} \tag{3.13}$$

where

$$b^{(2,0)} = k_{\beta,\alpha} dz^\alpha \wedge dz^\beta, \qquad b^{(0,2)} = \bar{k}_{\bar{\beta},\bar{\alpha}} d\bar{z}^{\bar{\alpha}} \wedge d\bar{z}^{\bar{\beta}} . \tag{3.14}$$

The geometry (3.10) is preserved by the transformation

$$\delta k_\alpha = \tau_\alpha \tag{3.15}$$

provided τ_α satisfies

$$\bar{\partial}_{\bar{\beta}} \tau_\alpha = i \partial_\alpha \bar{\partial}_{\bar{\beta}} \chi \tag{3.16}$$

for some arbitrary real χ. This implies that τ is of the form

$$\tau_\alpha = i\partial_\alpha \chi + \vartheta_\alpha \ , \quad \bar{\partial}_{\bar{\beta}} \vartheta_\alpha = 0 \tag{3.17}$$

JHEP06(2019)138

for some holomorphic ϑ_α. The symmetry (3.15) is the analogue of the generalised Kähler transformation discussed in [2]. It leaves the metric and torsion invariant, but changes b_{ij} by an antisymmetric tensor gauge transformation of the form (3.11).

4 Isometries in the (2,1) sigma model

For the application to T-duality discussed in the following sections, we shall be interested in Abelian groups of isometries. For completeness, however, we discuss the general case of non-Abelian isometry groups.

Let G be a group of isometries of M generated by Killing vector fields ξ^i_K that preserve the metric and 3-form H, $\mathcal{L}_K g = 0$, $\mathcal{L}_K H = 0$, and satisfy the algebra (2.8). This symmetry will be consistent with (2,1) supersymmetry if

$$(\mathcal{L}_K J)^i{}_j = 0\ . \tag{4.1}$$

This allows us to write the symmetry of the $(2,1)$ supersymmetric model in (2,1) superspace as

$$\delta\varphi^i = \lambda^K \xi^i_K(\varphi)\ . \tag{4.2}$$

The constraint (4.1) is the condition that the ξ^i_K are holomorphic Killing vectors[3] with respect to the complex structure J_{ij}, giving

$$\partial_\alpha \bar{\xi}^{\bar{\beta}}_K = 0 \tag{4.3}$$

in complex coordinates. If the torsion vanishes, then M is Kähler, and the Kähler 2-form ω (with components $\omega_{ij} \equiv g_{ik}J^k{}_j$) is closed. For every holomorphic Killing vector ξ^i_K, the 1-form with components $\omega_{ij}\xi^j_K$ is closed and locally there are functions X_K such that $\omega_{ij}\xi^j_K = \partial_i X_K$; in complex coordinates, this equation becomes $\xi_{K\alpha} = i\partial_\alpha X_K$. The functions X_K are sometimes called Killing potentials and play a central role in gauging the supersymmetric sigma-models without torsion [25, 55]. When X_K are globally defined equivariant functions (i.e. $\mathcal{L}_K X_M = 0$), they are referred to as moment maps and the gauging implements the Kähler quotient construction.

When the torsion does not vanish, this generalises straightforwardly [32]. The locally defined 1-form u_K satisfies $\iota_K H = du_K$. If, in addition, (4.1) holds, then the 1-form with components $\nu_i \equiv \omega_{ij}(\xi^j_K + u^j_K)$ satisfies $\partial_{[\alpha}\nu_{\beta]} = 0$, so that there are generalised Killing potentials such that [32]

$$\xi_{\alpha K} + u_{\alpha K} = \partial_\alpha Y_K + i\partial_\alpha X_K. \tag{4.4}$$

The X_K and Y_K are locally defined functions on M; Y_K simply reflects the ambiguity in the definition of u_K in (2.10), and locally the $\partial_\alpha Y_K$ term can be absorbed into the definition of u_K.

Under the rigid symmetries (4.2), the variation of the action in (3.7) is

$$\delta S_1 = i\lambda^K \int d^2\sigma d\theta^+ d\theta^- \left((\mathcal{L}_K k_\alpha) D_-\varphi^\alpha - (\mathcal{L}_K \bar{k}_{\bar{\alpha}}) D_-\bar{\varphi}^{\bar{\alpha}}\right), \tag{4.5}$$

[3] A discussion of sigma models with non-holomorphic isometries can be found in ref. [39].

JHEP06(2019)138

where the Lie derivative of k_α is

$$\mathcal{L}_K k_\alpha = \xi_K^\beta \partial_\beta k_\alpha + \bar{\xi}_K^{\bar{\beta}} \bar{\partial}_{\bar{\beta}} k_\alpha + k_\beta \partial_\alpha \xi_K^\beta. \tag{4.6}$$

The variation of the superpotential term (3.8) in the action is

$$\delta S_2 = i\lambda^K \int d^2\sigma d\theta^+ d\theta^- \mathcal{L}_K F(\varphi) + \text{complex conjugate}, \tag{4.7}$$

so it will be left invariant by the isometries provided the holomorphic function $F(\varphi)$ is invariant up to constants, i.e. if the equations

$$\mathcal{L}_K F = e_K \tag{4.8}$$

are satisfied for some complex constants e_K.

In general, the isometry symmetries will not leave the potential k_α invariant, but will change it by a gauge transformation of the form (3.15)–(3.17), so that the action (3.7) is unchanged. The geometry and Killing potentials then determine the quantity $\mathcal{L}_K k_\alpha$ appearing in the variation (4.5) to take the form

$$\mathcal{L}_K k_\alpha = i\partial_\alpha \chi_K + \vartheta_{K\alpha}, \tag{4.9}$$

for some real functions χ_K and holomorphic 1-forms $\vartheta_{K\alpha}$,

$$\bar{\partial}_{\bar{\beta}} \vartheta_{K\alpha} = 0. \tag{4.10}$$

In ref. [34], the following explicit expressions for χ and ϑ were found:

$$\chi_K = X_K + i\left(\bar{\xi}_K^{\bar{\beta}} \bar{k}_{\bar{\beta}} - \xi_K^\beta k_\beta\right) \tag{4.11}$$

$$\vartheta_{K\alpha} = 2\xi_K^\gamma \partial_{[\gamma} k_{\alpha]} + \xi_{\alpha K} - i\partial_\alpha X_K. \tag{4.12}$$

Using (4.3), (4.4), and (4.6), it is straightforward to check that (4.11) and (4.12) satisfy (4.9) and (4.10) respectively. It follows that the action of the Lie bracket algebra on the vector potential k_α reduces to

$$[\mathcal{L}_K, \mathcal{L}_L] k_\alpha = f_{KL}{}^M \mathcal{L}_M k_\alpha, \tag{4.13}$$

as it must (cf. (2.8)). The obstructions to gauging of the supersymmetric sigma model (without superpotential) were analysed in [34, 35] following [29, 30, 32]. It was found that, in order for the gauging to be possible, the following two conditions must hold:

$$\begin{aligned} &(i) && \xi^\alpha_{(I} \vartheta_{J)\alpha} = 0 \\ &(ii) && \mathcal{L}_K X_L = f_{KL}{}^M X_M\ . \end{aligned} \tag{4.14}$$

Condition *(ii)* is the statement that the generalised Killing potentials must be equivariant. If they are also globally defined, then they are sometimes referred to as generalised moment maps.

Observe that, together with the relation (4.4), the expression (4.12) for $\vartheta_{J\alpha}$ implies

$$\xi^\alpha_{(I}\vartheta_{J)\alpha} = \xi^\alpha_{(I}u_{J)\alpha} \tag{4.15}$$

(as can be seen by contracting $\vartheta_{J\alpha}$ with ξ^α_I and symmetrising with respect to I and J), so that condition *(i)* above is equivalent to

$$c_{(IJ)} = 0, \tag{4.16}$$

where the functions c_{IJ} were defined in (2.11); compare eq. (2.17).

For the gauging of the superpotential term (3.8) to be possible, it is necessary that the constants e_K defined in (4.8) vanish, so that the holomorphic function $F(\varphi)$ is invariant under the isometry symmetries,

$$\mathcal{L}_K F = 0. \tag{4.17}$$

Consider the case of gauging one isometry that acts in adapted coordinates (φ^0, φ^μ) as a shift in $i(\varphi^0 - \bar{\varphi}^0)$, so that $\varphi^0 \to \varphi^0 + i\lambda$, $\bar{\varphi}^0 \to \bar{\varphi}^0 - i\lambda$. The Killing vector ξ then has components $(i, -i, 0, \dots)$, with

$$\xi^i \frac{\partial}{\partial x^i} = i\left(\frac{\partial}{\partial \varphi^0} - \frac{\partial}{\partial \bar{\varphi}^0}\right). \tag{4.18}$$

Then the condition (4.16) implies that

$$c = \xi^0 \vartheta_0 = 0 \quad \Rightarrow \quad \vartheta_0 = 0\,, \tag{4.19}$$

which, combined with (4.12) implies that

$$\partial_0 X = \partial_{\bar{0}} X = g_{0\bar{0}}. \tag{4.20}$$

5 The (2,1) gauge multiplet and gauge symmetries

We now promote the isometries (4.2) to local ones in which the constant parameters λ^K are replaced by $(2,1)$ superfields Λ^K,

$$\delta\varphi^\alpha = \Lambda^K \xi^\alpha_K \ , \quad \delta\bar{\varphi}^{\bar{\alpha}} = \bar{\Lambda}^K \bar{\xi}^{\bar{\alpha}}_K. \tag{5.1}$$

These transformations preserve the chirality constraints (3.5) only if the Λ^K are chiral,

$$\bar{D}_+ \Lambda^K = 0 \ , \quad D_+ \bar{\Lambda}^K = 0. \tag{5.2}$$

Under a finite transformation,

$$\varphi \to \varphi' = e^{L_{\Lambda\cdot\xi}}\varphi \ , \quad \bar{\varphi} \to \bar{\varphi}' = e^{L_{\bar{\Lambda}\cdot\bar{\xi}}}\bar{\varphi}, \tag{5.3}$$

where

$$L_{\Lambda\cdot\xi} \equiv \Lambda^K \xi^\alpha_K \frac{\partial}{\partial \varphi^\alpha} \tag{5.4}$$

is the generator of the infinitesimal diffeomorphism with parameter $\Lambda \cdot \xi$.

JHEP06(2019)138

The $(2,1)$ super Yang-Mills multiplet is given in $(2,1)$ superspace by a set of Lie-algebra valued super-connections $\mathcal{A}_{(2,1)} = (A_+, \bar{A}_+, A_-, A_{\ddagger}, A_=)$, with $A_\bullet = A_\bullet^K T_K$, where the Lie algebra generators T_K are hermitian and satisfy the algebra $[T_K, T_L] = if_{KL}{}^M T_M$. These connections can be used to define gauge covariant derivatives $\nabla_\bullet \equiv D_\bullet - iA_\bullet$, which are constrained by the conditions:

$$\left\{\nabla_+, \bar{\nabla}_+\right\} = 2i\nabla_{\ddagger}\,, \quad \{\nabla_-, \nabla_-\} = 2i\nabla_=\,, \quad \{\nabla_+, \nabla_-\} = \bar{W}\,, \quad \left\{\bar{\nabla}_+, \nabla_-\right\} = W\,, \qquad (5.5)$$

as well as $\nabla_+^2 = \bar{\nabla}_+^2 = 0$. The remaining relations among the derivatives follow from these conditions and the Bianchi identities, e.g.

$$[\nabla_+, \nabla_{\ddagger}] = \frac{1}{2i}\left[\nabla_+, \{\nabla_+, \bar{\nabla}_+\}\right] = 0\,, \qquad \bar{\nabla}_+ W = \left[\bar{\nabla}_+, \{\bar{\nabla}_+, \nabla_-\}\right] = 0\,, \qquad (5.6)$$

$$[\nabla_+, \nabla_=] = \frac{1}{2i}\left[\nabla_+, \{\nabla_-, \nabla_-\}\right] = i\left[\nabla_-, \{\nabla_-, \nabla_+\}\right] = i\nabla_- \bar{W}\,, \qquad (5.7)$$

$$\begin{aligned}[\nabla_-, \nabla_{\ddagger}] &= \frac{1}{2i}\left[\nabla_-, \{\nabla_+, \bar{\nabla}_+\}\right] = \frac{i}{2}\left[\nabla_+, \{\bar{\nabla}_+, \nabla_-\}\right] + \frac{i}{2}\left[\bar{\nabla}_+, \{\nabla_+, \nabla_-\}\right] \\ &= \frac{i}{2}\left(\nabla_+ W + \bar{\nabla}_+ \bar{W}\right)\,, \end{aligned} \qquad (5.8)$$

$$[\nabla_{\ddagger}, \nabla_=] = \frac{1}{2i}\left[\nabla_{\ddagger}, \{\nabla_-, \nabla_-\}\right] = i\left[\nabla_-, [\nabla_-, \nabla_{\ddagger}]\right] = -\frac{1}{2}\nabla_- \left(\nabla_+ W + \bar{\nabla}_+ \bar{W}\right)\,. \qquad (5.9)$$

The conditions (5.5) were introduced in [34, 35]. Their consequences (5.6)-(5.9) correct statements in [34, 35].

The constraints (5.5) can be solved to give all connections in terms of a scalar pre-potential V and the spinorial connection A_-. In the chiral representation, the spinorial derivatives that appear in the algebra (5.5) are given by

$$\bar{\nabla}_+ = \bar{D}_+\,, \quad \nabla_+ = e^{-V} D_+ e^V\,, \quad \nabla_- \equiv D_- - iA_-\,, \qquad (5.10)$$

where $V = \bar{V}$ is hermitian, and the spinor connection A_- is hermitian up to a similarity transformation because we are in chiral representation:[4] $\bar{A}_- = e^V(A_- + iD_-)e^{-V}$. We then find

$$\begin{aligned}\nabla_{\ddagger} &\equiv -\frac{i}{2}\left\{\bar{D}_+, e^{-V} D_+ e^V\right\} = \partial_{\ddagger} - \frac{i}{2}\bar{D}_+ D_+ V + O(V^2)\,, \\ \nabla_= &\equiv -i\nabla_-^2 = \partial_= - (D_- A_-) + iA_-^2\,, \end{aligned} \qquad (5.11)$$

so that

$$A_{\ddagger} = \frac{1}{2}\bar{D}_+ D_+ V + O(V^2)\,, \quad A_= = -iD_- A_- + O(A_-^2)\,. \qquad (5.12)$$

The field strengths are obtained from (5.5) and (5.10),

$$\bar{W} \equiv \left\{e^{-V} D_+ e^V, D_- - iA_-\right\} = -iD_+(A_- - iD_- V) + O(VA_-, V^2)\,, \qquad (5.13)$$

$$W \equiv \left\{\bar{D}_+, D_- - iA_-\right\} = -i\bar{D}_+ A_-\,. \qquad (5.14)$$

[4] Real representations are reviewed in appendix A.

Again, these are not complex conjugates because we are in a chiral representation. Note that if instead we used the anti-chiral representation, we would have $\bar{W} = -iD_+A_-$, and $W = \{e^V \bar{D}_+ e^{-V}, D_- - iA_-\}$.

We now turn to the gauge transformations of the $(2,1)$ Yang-Mills supermultiplet. Under a finite gauge transformation, the hermitian superfield prepotential V transforms as

$$e^V \to e^{V'} = e^{i\bar{\Lambda}} e^V e^{-i\Lambda} \ . \tag{5.15}$$

For infinitesimal Λ, this yields

$$\delta V = i(\bar{\Lambda} - \Lambda) - \frac{i}{2}\left[V, \Lambda + \bar{\Lambda}\right] + O(V^2) \ . \tag{5.16}$$

In chiral representation, the superconnection A_- transforms as

$$\nabla'_- = e^{i\Lambda} \nabla_- e^{-i\Lambda} \quad \Rightarrow \quad \delta A_- = \nabla_- \Lambda \ . \tag{5.17}$$

The antichiral representation would be reached from this by a similarity transformation with e^V, giving the antichiral representation covariant derivative

$$\nabla_\bullet^{(AC)} = e^V \nabla_\bullet e^{-V} \ . \tag{5.18}$$

The spinor covariant derivative ∇_- is real in the sense that after taking the adjoint one is in the antichiral representation: $\bar{\nabla}_- = e^V \nabla_- e^{-V}$. In particular,

$$\delta \bar{A}_- = \nabla_- \bar{\Lambda} \ . \tag{5.19}$$

For fields in a linear representation of the gauge group, the Lie algebra generators act in that representation. For the superfields $\varphi, \bar{\varphi}$, the symmetry is realised non-linearly, with the Lie algebra element T_K generating the transformation $\varphi \to \varphi + \lambda^K \xi_K(\varphi)$.

Covariant derivatives can act on different representations of the group. This action is encoded in the matrix used to represent the generators of the Lie algebra; they can also act nonlinearly on the superfields $\varphi, \bar{\varphi}$. In this case, the covariant derivative uses this non-linear realisation:

$$\nabla_\bullet \varphi^\alpha \equiv D_\bullet \varphi^\alpha - A_\bullet^K \xi_K^\alpha \, , \quad \bar{\nabla}_\bullet \bar{\varphi}^{\bar{\alpha}} \equiv \bar{D}_\bullet \bar{\varphi}^{\bar{\alpha}} - \bar{A}_\bullet^K \bar{\xi}_K^{\bar{\alpha}} \ . \tag{5.20}$$

The scalar superfields $\varphi, \bar{\varphi}$ transform under the local isometry symmetries as in (5.1). Following [25], we define the chiral-representation version of $\bar{\varphi}$ as

$$\tilde{\varphi} = e^{L_{V\cdot\bar{\xi}}} \bar{\varphi} \, , \tag{5.21}$$

where

$$L_{V\cdot\bar{\xi}} \equiv iV^K \bar{\xi}_K^{\bar{\alpha}} \frac{\partial}{\partial \bar{\varphi}^{\bar{\alpha}}} \ . \tag{5.22}$$

Then the superfields $\varphi, \tilde{\varphi}$ satisfy the covariant chirality constraints

$$\bar{\nabla}_+ \varphi^\alpha = 0 \ , \quad \nabla_+ \tilde{\varphi}^{\bar{\alpha}} = 0 \, , \tag{5.23}$$

and transform under the isometry symmetries as

$$\delta\varphi^\alpha = \Lambda^K \xi_K^\alpha \ , \quad \delta\tilde{\varphi}^{\bar\alpha} = \Lambda^K \bar\xi_K^{\bar\alpha}(\tilde\varphi)\ . \tag{5.24}$$

Here $\bar\xi_K^{\bar\alpha}(\tilde\varphi)$ is obtained from $\bar\xi_K^{\bar\alpha}(\bar\varphi)$ by replacing $\bar\varphi$ with $\tilde\varphi$. Note that the transformation of $\tilde\varphi$ involves the parameter Λ, while that for $\bar\varphi$ involves $\bar\Lambda$. The covariant derivatives of $\tilde\varphi$ are in chiral representation, and hence are given by:

$$\nabla_\bullet \tilde\varphi^{\bar\alpha} = D_\bullet \tilde\varphi^{\bar\alpha} - A_\bullet^K \bar\xi_K^{\bar\alpha}(\tilde\varphi)\ . \tag{5.25}$$

When gauging one translational isometry, we again choose adapted coordinates $(\varphi^0, \bar\varphi^{\bar 0}, \dots)$ in which the Killing vector has components $(i, -i, 0, \dots)$ and acts as in (4.18). Then the above relations simplify: the only fields that transform are

$$\begin{aligned} \delta\varphi^0 &= i\Lambda\,, \qquad & \delta\bar\varphi^{\bar 0} &= -i\bar\Lambda\,, \qquad & \delta\tilde\varphi^{\bar 0} &= -i\Lambda\,, \\ \delta V &= i\left(\bar\Lambda - \Lambda\right)\,, \qquad & \delta A_- &= D_-\Lambda\,, \end{aligned} \tag{5.26}$$

and minimal coupling is simply given by

$$\tilde\varphi^{\bar 0} = \bar\varphi^{\bar 0} + V\ . \tag{5.27}$$

As explained above, since the transformation $\delta A_- = D_-\Lambda$ involves the chiral parameter Λ, it is necessarily complex, and A_- is not real; however, the combination

$$A_- - \frac{i}{2} D_- V \tag{5.28}$$

has the real transformation

$$\delta\left(A_- - \frac{1}{2} D_- V\right) = \frac{1}{2} D_-(\Lambda + \bar\Lambda)\,, \tag{5.29}$$

and is real — see appendix A for details.

6 The gauged (2,1) sigma model

The gauged $(2,1)$ sigma model in superspace was studied in [34, 35] and the full nonpolynomial gauged action was constructed in [35] using the methods of ref. [25]. We now briefly summarise the main results of the analysis, referring the reader to these papers for the derivations and further details of the construction.

Under the infinitesimal rigid transformations (4.2), the variation of the $(2,1)$ full superspace Lagrangian

$$L_1 = i\left(k_\alpha D_- \varphi^\alpha - \bar k_{\bar\alpha} D_- \bar\varphi^{\bar\alpha}\right) \tag{6.1}$$

is given by (4.5)

$$\delta L_1 = i\lambda^K\big((\mathcal{L}_K k_\alpha) D_-\varphi^\alpha - (\mathcal{L}_K \bar k_{\bar\alpha}) D_- \bar\varphi^{\bar\alpha}\big)\,. \tag{6.2}$$

Invariance of the action requires (4.9):

$$\mathcal{L}_K k_\alpha = i\partial_\alpha \chi_K + \vartheta_{\alpha K}\,, \tag{6.3}$$

with χ_K a real function and $\vartheta_{K\alpha}$ a holomorphic 1-form which were shown in ref. [34] to take the explicit forms (4.11) and (4.12) respectively. The variation of the superpotential term (3.8) is given in (4.7), which vanishes provided the function F satisfies (4.17), i.e. if it is invariant under the rigid isometries (4.2).

Now consider promoting the rigid isometries to local symmetries (5.1). The variation of the $(2,1)$ superpotential term (3.8) is given by

$$\delta S_2 = i \int d^2\sigma d\theta^+ d\theta^- \Lambda^K \mathcal{L}_K F(\varphi) + \text{complex conjugate} \tag{6.4}$$

and this will vanish provided the function $F(\varphi)$ is itself invariant under the local isometries, i.e. (4.17) holds for such isometries; in the following we will assume that this is the case and concentrate on the gauging of the full superspace term (3.7) in the action.

The main result of [35] is that the $(2,1)$ superspace action (3.7) can be gauged provided the geometric condition (4.14) holds, in which case the gauge invariant superspace Lagrangian for the gauged $(2,1)$ sigma model is (to all orders in the gauge coupling, which we have absorbed into the gauge fields)

$$L_{1g} = \left[i\left(k_\alpha D_- \varphi^\alpha - \bar{k}_{\bar{\alpha}} D_- \tilde{\varphi}^{\bar{\alpha}}\right) - A_-^K X_K\right](\varphi, \tilde{\varphi}) - \frac{e^L - 1}{L} V^K \bar{\vartheta}_{\bar{\alpha}K} D_- \bar{\varphi}^{\bar{\alpha}} \ . \tag{6.5}$$

The operator $L \equiv i V^K \bar{\xi}_K^{\bar{\alpha}} \frac{\partial}{\partial \bar{\varphi}^{\bar{\alpha}}}$ is the one defined in (5.22), and the expression $\frac{1}{L}(e^L - 1)$ in the Lagrangian can be defined by its Taylor series expansion in L or equivalently by $\int_0^1 dt\, e^{tL}$. The gauge invariance of the action obtained from integrating the Lagrangian (6.5) over superspace is proven in appendix B for the case of a single isometry.

The full gauged sigma model $(2,1)$ superspace action is then

$$S_{\text{tot}} = \int d^2\sigma\, d^2\theta^+ d\theta^- L_{1g} + \left(\int d^2\sigma\, d\theta^+ d\theta^- F(\varphi) + \text{c.c.}\right) \tag{6.6}$$

for an invariant superpotential $F(\varphi)$: $\mathcal{L}_K F = 0$.

This form of the gauged action was given in [35], but is not immediately comparable to the more geometric gauged action given for the bosonic model in (2.21),(2.23). As shown in appendix B, using the relations

$$i(k_{\alpha,\bar{0}} + \bar{k}_{\bar{0},\alpha}) = u_\alpha - i X_{,\alpha}\,, \quad \vartheta_\alpha = i(k_{\alpha,0} - k_{0,\alpha}) - u_\alpha \tag{6.7}$$

we can rewrite the gauged Lagrangian as (for the case of a single isometry — the general case is similar):

$$\begin{aligned} L_{1g} &= i\left(k_\alpha D_- \varphi^\alpha - \bar{k}_{\bar{\alpha}} D_- \bar{\varphi}^{\bar{\alpha}}\right)(\varphi, \bar{\varphi}) - \left(A_- - \frac{i}{2} D_- V\right) X(\varphi, \tilde{\varphi}) \\ &\quad + V \frac{e^L - 1}{L}\left[\left(u_\alpha - \frac{i}{2} X_{,\alpha}\right) D_- \varphi^\alpha + \left(\bar{u}_{\bar{\alpha}} + \frac{i}{2} X_{,\bar{\alpha}}\right) D_- \bar{\varphi}^{\bar{\alpha}}\right] \ . \end{aligned} \tag{6.8}$$

Because this Lagrangian is geometric, some properties that are hard to see in (6.5) are more transparent in this form. For example, the hermiticity of the action follows directly:

JHEP06(2019)138

the combination $A_- - \frac{i}{2}D_-V$ (5.28) is real, with the real transformation (5.29)

$$\delta\left(A_- - \frac{i}{2}D_-V\right) = \frac{1}{2}D_-(\Lambda + \bar{\Lambda}) \ . \tag{6.9}$$

Since (2.24) and (4.14) imply that u_α and X are invariant under (rigid) gauge transformations, we have, for any real function f, $f(\mathcal{L}_{\xi+\bar{\xi}})X = 0$. In particular, this implies

$$f(L)X = f([L - \bar{L}] + \bar{L})X = f([iV\mathcal{L}_{\bar{\xi}+\xi}] + \bar{L})X \quad \Rightarrow \quad f(L)X = f(\bar{L})X \, , \tag{6.10}$$

where $\bar{L} = -iV\xi^\alpha \frac{\partial}{\partial\varphi^\alpha}$ and the holomorphy of ξ implies $[L, \bar{L}] = 0$. Similarly

$$f(L)i[u_\alpha D_-\varphi^\alpha - \bar{u}_{\bar{\alpha}}D_-\bar{\varphi}^{\bar{\alpha}}] = f(\bar{L})i[u_\alpha D_-\varphi^\alpha - \bar{u}_{\bar{\alpha}}D_-\bar{\varphi}^{\bar{\alpha}}] \ . \tag{6.11}$$

The hermiticity of the action then follows.

7 T-duality of (2,1) supersymmetric theories

7.1 Generalities

The generalisation of T-duality [21–23, 47–49, 56, 57] to conformally invariant sigma models which admit isometries, and its explicit form in $(2, 2)$ superspace, were elucidated in ref. [24]. In this section we generalise the construction to the superspace formulation of the $(2, 1)$ supersymmetric sigma models reviewed above. The general procedure which defines the dual pairs locally is the same as in refs. [21–24, 47–49, 57]. First, gauge the sigma model isometries and add a Lagrange multiplier term constraining the gauge multiplet to be flat. Second, eliminate the gauge fields by solving their field equations. Classically, this ensures the equivalence of the dual models, modulo global issues that arise in the case of compact isometries if the gauge fields have nontrivial holonomies along noncontractible loops. However, as already explained at the end of section 2, these issues are taken care of by giving the Lagrange multipliers suitable periodicities and adding a total derivative term, so that the holonomies are constrained to be trivial [24]; we will assume that this can also be done in the $(2, 1)$ supersymmetric case at hand. Quantum mechanically, the duality in the $(2, 2)$ case receives corrections from the Jacobian obtained upon integrating out the gauge fields, which at one loop leads to a simple shift of the dilaton [22].

For an Abelian gauging, the field strengths $W, \bar{W}$ in chiral representation given in (5.13), (5.14) are:

$$\bar{W}^K = -iD_+(A_-^K - iD_-V^K) \, , \quad W^K = -i\bar{D}_+A_-^K \ . \tag{7.1}$$

The condition that the gauge multiplet is pure gauge can be imposed by constraining $W, \bar{W}$ to vanish by adding to the Lagrangian (6.8) a term

$$L_\Theta = -\Psi_{K-}W^K - \bar{\Psi}_{K-}\bar{W}^K \ . \tag{7.2}$$

To this we add a total derivative term, which is important for constraining the holonomies of the flat connections correctly, to obtain

$$L_\Theta = -\left(\Theta_K + \bar{\Theta}_K\right)A_-^K - iD_-\bar{\Theta}_K V^K , \tag{7.3}$$

where $\Theta = -i\bar{D}_+\Psi_-, \bar{\Theta} = -iD_+\bar{\Psi}_-$ are chiral (respectively antichiral) Lagrange multiplier superfields. The full action to consider is then

$$S_{1g} + S_\Theta, \qquad S_\Theta \equiv \int d^2\sigma d^2\theta^+ d\theta^- L_\Theta. \tag{7.4}$$

Integrating out the Lagrange multipliers Θ or Ψ_- gives

$$W^K = 0, \qquad \bar{W}^K = 0\,, \tag{7.5}$$

which implies that V and A_- are pure gauge (with the boundary terms constraining the holonomies):

$$A_-^K = D_-\Lambda^K \quad , \qquad V^K = i\left(\bar{\Lambda}^K - \Lambda^K\right). \tag{7.6}$$

The term S_Θ then vanishes, and we recover the original sigma model with action (3.6)–(3.8). Alternatively, integrating out the gauge fields gives the T-dual theory.

In the special case of one isometry, we can choose local complex coordinates $\{z^\alpha, \bar{z}^{\bar{\alpha}}\} = \{z^0; \bar{z}^{\bar{0}}; z^\mu; \bar{z}^{\bar{\mu}}\}$, with $\mu, \bar{\mu} = 1, \ldots \frac{D}{2} - 1$, such that the isometry acts by a translation leaving $(z^0 + \bar{z}^0)$ invariant. Moreover we can use a diffeomorphism combined with a b field gauge transformation to arrange for the metric and b field to depend only on $(z^0 + \bar{z}^0)$ and on the set of coordinates $\{z^\mu; \bar{z}^{\bar{\mu}}\}$, but to be independent of $i(z^0 - \bar{z}^0)$; however, if we use a geometric formulation, there is no need to do so. The indices $\mu, \bar{\mu}$ now run over the 'spectator' coordinates transverse to $z^0, \bar{z}^{\bar{0}}$. As reviewed in the previous section, the requirement of $(2,1)$ supersymmetry restricts the admissible isometries to those that act holomorphically on chiral superfields.

7.2 Computations

Recall that the geometry constrains $\mathcal{L}_K k_\alpha$ to take the form $\mathcal{L}_K k_\alpha = i\partial_\alpha \chi_K + \vartheta_{K\alpha}$ with

$$\bar{\partial}_{\bar{\beta}} \vartheta_{K\alpha} = 0 \tag{7.7}$$

$$\chi_K = X_K + i\left(\bar{\xi}_K^{\bar{\beta}} \bar{k}_{\bar{\beta}} - \xi_K^\beta k_\beta\right)$$

$$\vartheta_{K\alpha} = 2\xi_K^\gamma \partial_{[\gamma} k_{\alpha]} + \xi_{\alpha K} - i\partial_\alpha X_K \tag{7.8}$$

(as follows from (4.9), (4.10), (4.11) and (4.12)) .

We can perform the T-duality starting from either of the two forms of the gauged Lagrangian, (6.5) or (6.8); for completeness, we consider both.

7.2.1 T-duality from the gauged Lagrangian (6.5)

In addition to the X_K, we define new potentials

$$Z_K = X_K + 2i\bar{\xi}_K^{\bar{\beta}} k_{\bar{\beta}}\ . \tag{7.9}$$

Since the Abelian isometries are independent, we focus on only one of them for the sake of clarity and henceforth drop the index K. Splitting the indices as $\alpha = (0, \mu)$, and using coordinates adapted to the isometry $\xi^0 = i$ and $\xi^\mu = 0$, from (7.8), we find

$$\begin{aligned}
X &= -(k_0 + \bar{k}_{\bar{0}}) + \chi \\
Z &\equiv X + 2\bar{k}_{\bar{0}} = -k_0 + \bar{k}_{\bar{0}} + \chi \\
\vartheta_0 &= -i(g_{0\bar{0}} + \partial_0 X) = 0 \\
\vartheta_\mu &= -i\left[(\partial_\mu k_0 - \partial_0 k_\mu) + g_{\mu\bar{0}} + \partial_\mu X\right] \ ;
\end{aligned} \tag{7.10}$$

it can be checked that ϑ_α is holomorphic, $\partial_{\bar{\beta}}\vartheta_\alpha = 0$. This is the set-up in the case where the obstructions to gauging vanish (cf. eq. (4.14)). However, as we shall see in section 9, this is not the most general situation in which T-duality is possible.

Now consider the general gauged Lagrangian in (6.5) with a single translational isometry. In adapted coordinates we have

$$\tilde{\varphi}^{\bar{\mu}} = \bar{\varphi}^{\bar{\mu}} \,, \quad \tilde{\varphi}^{\bar{0}} = \bar{\varphi}^{\bar{0}} + V \,, \tag{7.11}$$

and the gauge invariance of the Lagrangian is shown in appendix B.

For later purposes, we may rewrite (6.5) as

$$\begin{aligned}
L_g &= i(k_\alpha D_-\varphi^\alpha - \bar{k}_{\bar{\alpha}} D_-\tilde{\varphi}^{\bar{\alpha}}) - X\left(A_- - \frac{i}{2}D_-V\right) - \frac{i}{2}ZD_-V \\
&\quad - \left(\frac{e^L - 1}{L}\bar{\vartheta}_{\bar{\mu}}(\bar{\varphi})\right) V D_-\bar{\varphi}^{\bar{\mu}} \ .
\end{aligned} \tag{7.12}$$

Here $k_\alpha, \bar{k}_{\bar{\alpha}}, X, Z$ are all functions of $\varphi, \tilde{\varphi} \equiv e^L\bar{\varphi}$, whereas $\bar{\vartheta}$ is a function of $\bar{\varphi}$.

We add to the general Lagrangian (7.12) the invariant L_Θ (7.3) and consider $L_T = L_g + L_\Theta$ where Θ and $\bar{\Theta}$ are chiral and antichiral superfield Lagrange multipliers. As discussed above, integrating out Θ and $\bar{\Theta}$ sets the field strengths (7.1) to zero (modulo boundary terms):

$$\bar{W} \equiv -\bar{D}_+ A_- = 0, \qquad W \equiv D_+\left(A_- - iD_-V\right) = 0 \,, \tag{7.13}$$

so that the gauge multiplet is pure gauge: $A_- = D_-\Lambda$, $V = i(\bar{\Lambda} - \Lambda)$ (7.6). Shifting $\varphi \to \varphi + i\Lambda$, we recover the original ungauged action (3.7).

To find the dual action, we integrate out the gauge fields instead. Specialising once again to one isometry, the variation of V gives an expression for A_-; however, since A_- enters as a Lagrange multiplier, it drops out of the final Lagrangian.[5] The variation of A_- implies

$$X(\varphi, \tilde{\varphi}) + \Theta + \bar{\Theta} = 0 \,, \tag{7.14}$$

which should be solved for $V = V(\Theta + \bar{\Theta}, \varphi, \bar{\varphi})$. We can eliminate all dependence on $\varphi^0, \bar{\varphi}^{\bar{0}}$ by choosing the gauge $\varphi^0 = 0$; in this gauge, $\tilde{\varphi}^{\bar{0}} = V$, and

$$V\frac{e^L - 1}{L}\bar{\vartheta}(\bar{\varphi}) \equiv \int_0^1 dt\, e^{tL} V\bar{\vartheta}_{\bar{\mu}}(\bar{\varphi}^{\bar{0}}, \bar{\varphi}^{\bar{\nu}}) \ \ \to \ \ \int_0^1 dt\, V\bar{\vartheta}_{\bar{\mu}}(tV, \bar{\varphi}^{\bar{\nu}}) \ . \tag{7.15}$$

[5] For completeness, we give the calculation of A_- in appendix C.

Then (7.14) implies

$$\frac{dX}{d\Theta} = \frac{dX}{d\bar{\Theta}} = -1 \iff V_{,\Theta} = -\frac{1}{X_{,\bar{0}}}, \quad V_{,\bar{\Theta}} = -\frac{1}{X_{,\bar{0}}},$$
$$\frac{dX}{d\varphi^{\mu}} = \frac{dX}{d\bar{\varphi}^{\bar{\mu}}} = 0 \iff V_{,\mu} = -\frac{X_{,\mu}}{X_{,\bar{0}}}, \quad V_{,\bar{\mu}} = -\frac{X_{,\bar{\mu}}}{X_{,\bar{0}}}. \tag{7.16}$$

We also need the following expression for D_-V, which we find by differentiating (7.14) and using the last equation in (7.8) (which gives $X_{,0} = -g_{0\bar{0}}$):

$$D_-V = \frac{1}{g_{0\bar{0}}}\left(D_-(\Theta + \bar{\Theta}) + X_{,\mu}D_-\varphi^{\mu} + X_{,\bar{\mu}}D_-\bar{\varphi}^{\bar{\mu}}\right). \tag{7.17}$$

Using these results, we now evaluate $L_g + L_\Theta$ from (7.12) and (7.3) to find the dual Lagrangian:

$$L^{(D)} = i\left(\frac{1}{2}\left[V - \frac{Z}{g_{0\bar{0}}}\right]D_-\Theta - \frac{1}{2}\left[V + \frac{Z}{g_{0\bar{0}}}\right]D_-\bar{\Theta} + \left[k_\mu - \frac{1}{2}\frac{ZX_{,\mu}}{g_{0\bar{0}}}\right]D_-\varphi^{\mu}\right.$$
$$\left. - \left[\bar{k}_{\bar{\mu}} - i\int_0^1 dt\, V\bar{\vartheta}_{\bar{\mu}}(tV, \bar{\varphi}^{\bar{\nu}}) + \frac{1}{2}\frac{ZX_{,\bar{\mu}}}{g_{0\bar{0}}}\right]D_-\bar{\varphi}^{\bar{\mu}}\right), \tag{7.18}$$

where $V(\varphi, \bar{\varphi}, \Theta + \bar{\Theta})$ is found by solving (7.14).

7.2.2 T-duality from the geometric form (6.8) of the gauged Lagrangian

Using the equivalent geometric form of the gauged Lagrangian (6.8) together with the Lagrange multiplier term (7.3) instead, the dual Lagrangian reads

$$\hat{L}^{(D)} = i\left(\frac{1}{2}VD_-\Theta - \frac{1}{2}VD_-\bar{\Theta} + \left[k_\mu - iV\frac{e^L - 1}{L}\left(u_\mu - \frac{i}{2}X_{,\mu}\right)\right]D_-\varphi^{\mu}\right.$$
$$\left. - \left[\bar{k}_{\bar{\mu}} + iV\frac{e^L - 1}{L}\left(\bar{u}_{\bar{\mu}} + \frac{i}{2}X_{,\bar{\mu}}\right)\right]D_-\bar{\varphi}^{\bar{\mu}}\right). \tag{7.19}$$

7.3 The dual geometry

From (7.18), we can identify the components of the dual vector potential k^D as follows

$$k^D_\Theta = \frac{1}{2}\left[V - \frac{Z}{g_{0\bar{0}}}\right]$$
$$\bar{k}^D_{\bar{\Theta}} = \frac{1}{2}\left[V + \frac{Z}{g_{0\bar{0}}}\right]$$
$$k^D_\mu = \left[k_\mu - \frac{1}{2}\frac{ZX_{,\mu}}{g_{0\bar{0}}}\right]$$
$$\bar{k}^D_{\bar{\mu}} = \left[\bar{k}_{\bar{\mu}} - i\int_0^1 dt\, V\bar{\vartheta}_{\bar{\mu}}(tV, \bar{\varphi}^{\bar{\nu}}) + \frac{1}{2}\frac{ZX_{,\bar{\mu}}}{g_{0\bar{0}}}\right]. \tag{7.20}$$

Note that $\bar{k}^D_{\bar{\mu}}$ differs from the complex conjugate of k^D_μ by a a complex transformation of the form (3.15)-(3.17), so k^D differs from a real vector by such a transformation. Likewise, from (7.19) we read off

$$\begin{aligned}
\hat{k}^D_\Theta &= \frac{1}{2}V \\
\bar{\hat{k}}^D_{\bar{\Theta}} &= \frac{1}{2}V \\
\hat{k}^D_\mu &= [k_\mu - iV\frac{e^L-1}{L}(u_\mu - \frac{i}{2}X_{,\mu})] \\
\bar{\hat{k}}^D_{\bar{\mu}} &= [\bar{k}_{\bar{\mu}} + iV\frac{e^L-1}{L}(\bar{u}_{\bar{\mu}} + \frac{i}{2}X_{,\bar{\mu}})]\ .
\end{aligned} \tag{7.21}$$

Here $\bar{\hat{k}}^D_{\bar{\mu}}$ is the complex conjugate of $\hat{k}^D_\mu$ so $\hat{k}^D$ is a real vector. Formulae (7.20) and (7.21) only differ by terms that do not affect the metric and b field. We can calculate the components of the dual metric g^D and of the dual b-field b^D. Using (3.10), we find

$$\begin{aligned}
g^D_{\Theta\bar{\Theta}} &= \frac{1}{g_{0\bar{0}}} \\
g^D_{\mu\bar{\Theta}} &= \frac{1}{g_{0\bar{0}}}[b_{\mu 0} + i\vartheta_\mu] = \frac{-iu_\mu}{g_{0\bar{0}}} \\
g^D_{\bar{\mu}\Theta} &= \frac{1}{g_{0\bar{0}}}[b_{\bar{\mu}\bar{0}} - i\bar{\vartheta}_{\bar{\mu}}] = \frac{i\bar{u}_\mu}{g_{0\bar{0}}} \\
g^D_{\mu\bar{\mu}} &= g_{\mu\bar{\mu}} - \frac{1}{g_{0\bar{0}}}\left[g_{\mu\bar{0}}g_{\bar{\mu}0} - (b_{\mu 0} + i\vartheta_\mu)(b_{\bar{\mu}\bar{0}} - i\bar{\vartheta}_{\bar{\mu}})\right] = g_{\mu\bar{\mu}} - \frac{1}{g_{0\bar{0}}}\left[g_{\mu\bar{0}}g_{\bar{\mu}0} - u_\mu\bar{u}_{\bar{\mu}}\right]
\end{aligned} \tag{7.22}$$

and

$$\begin{aligned}
b^D_{\Theta\mu} &= \frac{g_{\bar{0}\mu}}{g_{0\bar{0}}} \\
b^D_{\bar{\Theta}\bar{\mu}} &= \frac{g_{0\bar{\mu}}}{g_{0\bar{0}}} \\
b^D_{\mu\nu} &= b_{\mu\nu} - \frac{2}{g_{0\bar{0}}}g_{\bar{0}[\mu}(b_{\nu]0} + i\vartheta_{\nu]}) = b_{\mu\nu} + \frac{2i}{g_{0\bar{0}}}g_{\bar{0}[\mu}u_{\nu]} \\
b^D_{\bar{\mu}\bar{\nu}} &= b_{\bar{\mu}\bar{\nu}} - \frac{2}{g_{0\bar{0}}}g_{0[\bar{\mu}}(b_{\bar{\nu}]\bar{0}} - i\bar{\vartheta}_{\bar{\nu}]}) = b_{\bar{\mu}\bar{\nu}} - \frac{2i}{g_{0\bar{0}}}g_{0[\bar{\mu}}\bar{u}_{\bar{\nu}]}\quad .
\end{aligned} \tag{7.23}$$

In the case of N Abelian isometries the expressions for the dual geometry involve $N \times N$ matrices replacing some entries, for example $g_{0\bar{0}} \to (g+b)_{mn}$ as in the bosonic case [48].

8 Comparison to the Buscher rules

The results (7.22),(7.23) for the (2,1) duality transformations are similar but not identical to the Buscher transformations in the modified form (2.45). In the Buscher duality (2.45), a coordinate x^0 is replaced by a dual coordinate $\hat{x}^0$ (e.g. if x^0 is a coordinate on a circle of radius R, $\hat{x}^0$ is a coordinate on the dual circle of radius $2\pi/RT$, again reinstating the string tension to keep track of dimensions), whereas in the (2,1) duality transformations, a *complex* coordinate $z^0 = \varphi^0|_{\theta=0}$ is replaced by a dual *complex* coordinate $\hat{z}^0 = \Theta|_{\theta=0}$. This arises because the $(2,1)$ gauging involves the action of the *complexification* of the isometry group. As was explained in [22, 42] for the $(2,2)$ case, the complex duality transformation consists of a T-duality and a diffeomorphism: it gives a T-duality transformation of the imaginary part of the coordinate z^0 and a coordinate transformation of the real part. Writing $z^0 = y^0 + ix^0$, $\hat{z}^0 = \hat{y}^0 + i\hat{x}^0$, the (2,1) duality transformation consists of a T-duality transformation in which the coordinate x^0 is replaced by a dual coordinate $\hat{x}^0$ (so that if x^0 is a coordinate on a circle of radius R, $\hat{x}^0$ is a coordinate on the dual circle of radius $2\pi/RT$), while $\hat{y}^0$ is related to y^0 by a coordinate transformation (so that if y^0 is a coordinate on a circle of radius R, $\hat{y}^0$ is a different coordinate on the same circle of radius R).

To see this, we start from the constraint (7.14):

$$X(\varphi^0 + \bar{\tilde{\varphi}}^{\bar{0}}, \varphi^\mu, \tilde{\varphi}^{\bar{\mu}}) + \Theta + \bar{\Theta} = 0\ . \tag{8.1}$$

Setting $\theta = 0$ and choosing the Wess-Zumino gauge in which $V|_{\theta=0} = 0$, this implies

$$X(z^0 + \bar{z}^{\bar{0}}, z^\mu, \bar{z}^{\bar{\mu}}) + \hat{z}^0 + \bar{\hat{z}}^{\bar{0}} = 0\,, \tag{8.2}$$

which gives

$$X(2y^0, z^\mu, \bar{z}^{\bar{\mu}}) + 2\hat{y}^0 = 0\ . \tag{8.3}$$

The solution of this equation gives $\hat{y}^0$ as a function of $y^0, z^\mu, \bar{z}^{\bar{\mu}}$, so that the complex duality transformation gives the coordinate transformation

$$y^0 \to \hat{y}^0(y^0, z^\mu, \bar{z}^{\bar{\mu}}) \tag{8.4}$$

together with the T-duality transformation replacing x^0 with the dual coordinate $\hat{x}^0$.

This can also be understood by comparing our (2,1) superspace analysis with the corresponding computation in (1,1) superspace which gives the Buscher duality (2.45): the equivalence of the two calculations is guaranteed, and so will relate the (2,1) duality transformations to the Buscher ones. The explicit calculations are carried out in appendix D. We now illustrate this discussion with two simple and instructive examples.

8.1 T-duality on the complex plane

Our first simple example is the complex plane dualised with respect to the isometry given by a rotation about the origin[6]

$$z \to e^{i\lambda} z \tag{8.5}$$

for real λ. The adapted coordinates are $\varphi = \ln z$, transforming under the isometry by an imaginary shift $\varphi \to \varphi + i\lambda$. The metric is given by

$$ds^2 = dz d\bar{z} = e^{\varphi + \bar{\varphi}} d\varphi d\bar{\varphi}\,, \tag{8.6}$$

[6]This example is interesting in that it shows that the flat plane, which, when regarded as a string background, has no winding modes, is formally dual to a singular geometry with no normalizable (radial) momentum modes and only winding modes.

for which the potential can be taken to be

$$k_0 = \bar{k}_{\bar{0}} = \frac{1}{2} e^{\varphi + \bar{\varphi}} \ . \tag{8.7}$$

In this case, the Lagrangian is invariant, and $\vartheta = \chi = 0$, so (7.10) gives

$$X = -e^{\varphi + \bar{\varphi}} \ . \tag{8.8}$$

On gauging, this becomes $X = -e^{\varphi + \bar{\varphi} + V}$ and, on choosing the gauge $\varphi = 0$, this reduces to

$$X = -e^{V} \,, \tag{8.9}$$

so that (8.1) implies

$$V = \ln(\Theta + \bar{\Theta}) \ . \tag{8.10}$$

Then using (7.21), we have

$$\hat{k}^D_\Theta = \bar{\hat{k}}^D_{\bar{\Theta}} = \frac{1}{2} V = \frac{1}{2} \ln(\Theta + \bar{\Theta}) \,, \tag{8.11}$$

and

$$g_{\Theta\bar{\Theta}} = \frac{1}{\Theta + \bar{\Theta}} \quad \Rightarrow \quad d\hat{s}^2 = \frac{1}{\Theta + \bar{\Theta}} \, d\Theta \, d\bar{\Theta} \ . \tag{8.12}$$

In real coordinates $\varphi = y + ix$, the line element (8.6) is

$$ds^2 = e^{2y}(dx^2 + dy^2) \,, \tag{8.13}$$

and the isometry is generated by ∂_x. Dualizing gives

$$d\hat{s}^2 = e^{2y} dy^2 + e^{-2y} d\hat{x}^2 \ . \tag{8.14}$$

To compare this line element to (8.12), we write $\Theta = \hat{y} + i\hat{x}$, and use the coordinate transformation (8.1):

$$e^{2y} = (\Theta + \bar{\Theta}) =: 2\hat{y} \ . \tag{8.15}$$

Then (8.12) becomes:

$$d\hat{s}^2 = \frac{1}{2\hat{y}} (d\hat{y}^2 + d\hat{x}^2) = e^{-2y}(e^{4y} dy^2 + d\hat{x}^2) \,, \tag{8.16}$$

which does indeed match (8.14).

8.2 T-duality on a torus

Consider a flat torus $S^1 \times S^1$ parametrised by a single complex coordinate z and let φ be the $(2,1)$ superfield such that $\varphi| \equiv z$. For simplicity, we consider the case of a single holomorphic isometry and we suppress all spectator fields. We take the flat metric on the torus to be

$$ds^2 = R^2(dx^2 + dy^2) = R^2 dz d\bar{z} \tag{8.17}$$

with

$$z = y + ix \ . \tag{8.18}$$

The coordinate x that we are dualizing is scaled so that its periodicity is

$$x \sim x + 2\pi\,, \tag{8.19}$$

so it parametrises a circle of circumference $2\pi R$ and radius R; the coordinate y can have any periodicity:

$$y \sim y + \tau\,, \tag{8.20}$$

so the circumference of the corresponding circle is τR.

We consider the $(2,1)$ sigma model whose target space has the above geometry (with zero b-field). This is defined by the potential

$$k_\varphi = \frac{1}{2}R^2(\varphi + \bar{\varphi}) = 2R^2 y. \tag{8.21}$$

The isometry is generated by

$$\xi = \frac{\partial}{\partial x} = -2i\frac{\partial}{\partial(\varphi - \bar{\varphi})} \tag{8.22}$$

and the Killing potential is

$$X = R^2(\varphi + \bar{\varphi}). \tag{8.23}$$

The Lie derivative of k is zero, so we are in the simple case with $\chi = \vartheta_\alpha = 0$. The T-dual metric is then

$$d\hat{s}^2 = \frac{1}{R^2}(d\hat{x}^2 + d\hat{y}^2) = \frac{1}{R^2}d\hat{z}d\bar{\hat{z}}, \tag{8.24}$$

where

$$\hat{z} = \Theta| = \hat{y} + i\hat{x}, \tag{8.25}$$

and the dual b-field is zero. Eq. (8.24) looks like the metric that would result from T-dualising on both circles, but to see whether this is the case, we need to be careful with the periodicities. From the T-duality, we know that

$$\hat{x} \sim \hat{x} + 2\pi\,, \tag{8.26}$$

so the circumference of the $\hat{x}$ circle is $\frac{2\pi}{R}$ and we find the dual radius $\hat{R} = \frac{1}{R}$ as expected. The constraint (8.1), together with (8.23) and (8.25), gives

$$\hat{y} = -R^2 y\,, \tag{8.27}$$

so the periodicity of $\hat{y}$ is $\hat{y} \sim \hat{y} + R^2\tau$. Using the dual metric (8.24), the circumference of the circle parameterised by $\hat{y}$ is $R^{-1}R^2\tau = R\tau$ which is the same as that of the original circle parameterised by y. The T-duality has implemented the change of variables (8.27) from y to $\hat{y}$ and this diffeomorphism preserves the circumference of the circle. Rewriting (8.24) in terms of $\hat{x}$ and the original coordinate y, we find

$$d\hat{s}^2 = \frac{1}{R^2}d\hat{x}^2 + R^2 dy^2 \tag{8.28}$$

which is the result of the standard Buscher rules for T-duality in the x-circle. Thus we see that the $(2,1)$ T-duality, which appears to give a T-duality in two directions, in fact gives a T-duality in just one direction, combined with a diffeomophism whose role is to maintain the extra supersymmetry and the complex geometry.

9 Geometry and obstructions for (2,1) T-duality

We start by recalling the results of [36] reviewed in section 2. For a sigma model with Abelian isometries generated by Killing vectors ξ_K, the conditions for gauging are that the u_K are globally defined 1-forms that are invariant, $\mathcal{L}_K u_L = 0$, and satisfy $\iota_K u_L = -\iota_L u_K$. For T-duality, we require none of these conditions but only that $\iota_K \iota_L \iota_M H = 0$, and we introduce a bundle $\hat{M}$ over M with fibre coordinates $\hat{x}_K$. The metric and H-flux are defined by (2.33) and we take

$$\hat{u}_K = u_K + d\hat{x}_K. \tag{9.1}$$

We lift the Killing vectors ξ_K on M to Killing vectors $\hat{\xi}_K$ on $\hat{M}$ satisfying (2.34) and (2.38). The space $\hat{M}$ can be chosen so that $\hat{u}$ is invariant and globally defined on $\hat{M}$ with (2.39) satisfied, so that the only condition necessary for gauging and hence for T-duality is (2.43). The T-dual space is then $\hat{M}/G$ where G is the Abelian gauge group generated by the $\hat{\xi}_K$.

For the $(2,1)$ supersymmetric sigma model to be defined on M, M has to be complex with the geometry reviewed in section 3 and the Killing vectors must be holomorphic. Then there are generalised Killing potentials $Y_K + iX_K$ satisfying (4.4). This can be written as

$$\xi_K + u_K = dY_K + i(\partial - \bar{\partial})X_K, \tag{9.2}$$

with real 1-forms $u_K = u_{iK}dx^i$, $\xi_K = g_{ij}\xi^j_K dx^i$. Locally, we can absorb Y_K into a redefinition of u_K as discussed below (4.4). For gauging of the sigma model on M to be possible, the final form of u_K that arises after absorbing all the dY_K terms should be a globally defined one-form; for T-duality, this is not necessary as the u_K do not need to be globally defined. If the $(2,1)$ sigma-model on M allows a (1,1) gauging,[7] then the gauging will be $(2,1)$ supersymmetric provided the Killing vectors are holomorphic and the potentials X_K are globally defined scalars which are invariant: $\mathcal{L}_K X_L = 0$. The $(2,1)$ gauging is defined by restricting to the subspace $X = 0$ and taking a quotient by G to give $X^{-1}(0)/G$.

For $(2,1)$ T-duality, introducing n extra coordinates $\hat{x}_K$ $(K = 1,\dots,n)$ would in general be inconsistent with supersymmetry; for example, if n is odd, $\hat{M}$ would have odd dimension and so cannot be complex. Instead, we introduce n complex coordinates Θ_K corresponding to the chiral Lagrange multiplier fields introduced in section 5. This leads to a complex manifold $\check{M}$ with holomorphic coordinates $z^a = (\varphi^\alpha, \Theta_K)$ that is a bundle over M with projection $\check{\pi} : \check{M} \to M$ with $\check{\pi} : (\varphi^\alpha, \Theta_K) \mapsto \varphi^\alpha$. A metric $\check{g}$ and closed 3-form $\check{H}$ can be chosen on $\check{M}$ with no Θ_K components, i.e.

$$\check{g} = \check{\pi}^* g, \qquad \check{H} = \check{\pi}^* H, \tag{9.3}$$

where $\check{\pi}^*$ is the pull-back of the projection.

Writing

$$\Theta_K = (\hat{y}_K + i\hat{x}_K), \tag{9.4}$$

we identify the coordinates $\hat{x}_K$ with the extra coordinates needed for the (1,1) T-duality. Then

$$\hat{u}_K = u_K + 2d\hat{x}_K = u_K + id(\bar{\Theta}_K - \Theta_K) \tag{9.5}$$

[7] A (1,1) gauging is the same as the bosonic gauging discussed in section 2.

and we take the Killing vectors on $\check{M}$ to be the $\hat{\xi}_K$ given by (2.34) and (2.38). Now if (9.2) holds on M (with Y_K absorbed into u_K), then on $\check{M}$ we have

$$\hat{\xi}_K + \hat{u}_K = i(\partial - \bar{\partial})X_K + id(\bar{\Theta}_K - \Theta_K) \tag{9.6}$$

and, since $\partial\bar{\Theta} = \bar{\partial}\Theta = 0$, this can be rewritten as

$$\hat{\xi}_K + \hat{u}_K = +i(\partial - \bar{\partial})\check{X}_K, \tag{9.7}$$

where

$$\check{X}_K = X_K + \Theta_K + \bar{\Theta}_K = X_K + 2\hat{y}_K. \tag{9.8}$$

It follows that $\hat{u}$ can be chosen so that it is globally defined on $\check{M}$ and invariant. Then $d\check{X}_K$ will be invariant under the action of the Killing vectors $\hat{\xi}$, so that

$$\hat{\mathcal{L}}_L\check{X}_K = C_{LK} \tag{9.9}$$

for some constants C_{LK}. Introducing the Killing vectors on $\check{M}$ given by

$$\check{\xi}_K = \hat{\xi}_K - C_{KL}\frac{\partial}{\partial\hat{y}_L}, \tag{9.10}$$

we have that the $\check{X}_K$ are invariant:

$$\check{\mathcal{L}}_L\check{X}_K = 0. \tag{9.11}$$

Then the bundle $\check{M}$ can be defined so that the $\hat{u}_K$ and $\check{X}_K$ are globally defined (so that the transition functions for u_K on M determine those of $\hat{x}_K$ and the transition functions for X_K on M determine those of $\hat{y}_K$). As $\hat{u}_K$ and $\check{X}_K$ are globally defined and invariant under the action of $\check{\xi}$, the isometries generated by $\check{\xi}$ can be gauged provided (2.43) holds, giving a $(2,1)$ supersymmetric gauged sigma model on $\check{M}$.

The gauging imposes the generalised moment map constraint

$$\check{X}_K = 0, \tag{9.12}$$

which is the condition (7.14) obtained previously for the case of one isometry. This defines a $D+n$ dimensional subspace $\check{X}^{-1}(0)$ of the $D+2n$ dimensional space $\check{M}$. The gauging then gives the quotient $\check{X}^{-1}(0)/G$, which is of dimension D; this is the T-dual target space.

10 Summary

We now summarise the geometry of $(2,1)$ T-duality. The examples in section 8 and appendix D provide explicit illustrations of many of the points discussed.

The $(2,1)$ supersymmetric sigma models are $(1,1)$ supersymmetric sigma-models in which extra geometric structure leads to an extra supersymmetry. T-duality of any $(1,1)$ sigma-model is well understood: if the target space has an isometry, the T-dual geometry is given by the standard Buscher rules if the b-field is invariant under the isometry. If the b-field is only invariant up to a gauge transformation, so that only $H = db$ is invariant, then the T-dual geometry is given by the modified Buscher rules, given in (2.45) for adapted coordinates and in [36] for the general case.

The (p, q) supersymmetric sigma models (with $p \geq 1, q \geq 1$) are a special class of $(1, 1)$ models which have extra geometric structure arising from the $(p - 1) + (q - 1)$ complex structures of the target space. As such, the T-duality of any such model can be taken to be the standard $(1, 1)$ T-duality. However, if the complex structures are preserved by the isometry, then the T-duality is consistent with the supersymmetry, taking a (p, q) supersymmetric sigma model to a (p, q) supersymmetric sigma model and then considerable insight arises from performing the T-duality in (p, q) superspace. In particular, the metric and b-field for supersymmetric sigma-models are typically given in terms of a potential and T-duality can then be understood in terms of transformation rules for the potential instead of for the metric and b-field. For example, for $(2, 2)$ supersymmetric sigma models with Kähler target space, the b-field is trivial and the metric is given in terms of a scalar Kähler potential K. Then T-duality for $(2, 2)$ supersymmetric sigma models is realised as a Legendre transformation of the Kähler potential [24, 40].

Our interest here is in the case of $(2, 1)$ supersymmetry. The metric and b-field of the $(2, 1)$ supersymmetric sigma model are given locally in terms of a 1-form potential k. We find the T-duality transformation of the 1-form potential k to a dual potential k^D given in (7.21) that can be viewed as a generalisation of the Legendre transformation of the Kähler potential K arising in the $(2, 2)$ case.[8] This dual potential k^D gives the dual metric g^D and the dual torsion potential b^D given by eqs. (7.22)–(7.23). These are not the dual metric and b-field arising from the Buscher rules, but look like a complex form of these. The relation between the two expressions for the T-duality transformations was discussed in section 8: just as for the $(2, 2)$ supersymmetric case, the $(2, 1)$ T-duality transformations arising in this way in fact correspond to the usual Buscher rules together with diffeomorphisms. That is, the target space geometry specified by the $(2, 1)$ duals g^D and b^D is diffeomorphic to the T-dual geometry arising from the Buscher rules.

T-duality can be analysed by gauging isometries and adding Lagrange multiplier terms that trivialise the world-sheet gauge fields. The general $(1, 1)$ sigma model can be formulated in $(1, 1)$ superspace, and thus the T-duality of such models can be analysed by gauging isometries and adding Lagrange multiplier terms in $(1, 1)$ superspace. There are obstructions to gauging isometries for sigma-models with Wess-Zumino terms that were reviewed in subsection 2.1. However, the obstructions are much milder if the Lagrange multiplier term is added - the freedom to choose a gauge variation for the Lagrange multiplier fields gives a wider class of gauge-invariant actions. The only constraint the geometry must satisfy for T-duality to be possible is (2.27).

[8]The case in which the $(2, 1)$ model is a reduction of a $(2, 2)$ model, discussed in appendix D1, illustrates this point. The Legendre transformation of the scalar potential $K(\phi + \bar{\phi})$ is given by $\tilde{K}(G) = K(V) + VG$ where V is unconstrained and G is a real linear superfield. This is to be evaluated at $K_{,V} = -G$ (solving for $V(G)$) and results in K and $\tilde{K}$ being dual. The corresponding dual vector potentials in the $(2, 1)$ reduction are given by $k_\alpha = K_{,\alpha}$ and $\tilde{k}_\alpha = \tilde{K}_{,\alpha}$. It is a property of the Legendre transformation that $\tilde{K}_{,G} = V$. This should be compared to the expressions for dual vector potentials in (7.21), keeping in mind that V there is a solution to (7.14).

For (p,q) supersymmetric sigma models, we T-dualise in (p,q) superspace by coupling to (p,q) vector multiplets and adding (p,q) supersymmetric Lagrange multiplier terms. Here we used the $(2,1)$ superspace formulation of [20] and the gauging found in [34, 35] to find the T-dual $(2,1)$ geometry. As for the $(2,2)$ case, the supersymmetric gauging requires a gauging of a complexification of an isometry group. If the group G of real isometries to be gauged is generated by Killing vectors ξ^i_K, the group which is actually gauged in the $(2,1)$ supersymmetric gauging is the complexification G^C of G generated by the ξ^i_K together with the vector fields $J^i{}_j\xi^j_K$. The vector fields $J^i{}_j\xi^j_K$ are in general not Killing vectors, so that they generate diffeomorphisms that are not isometries. This gives a good local picture of the gauging. Global issues are discussed in [58].

The geometry of the T-duality was discussed in sections 2 and 9. The $(1,1)$ T-duality for n commuting isometries is understood through the construction of a 'doubled' manifold $\hat{M}$ with n extra coordinates $\hat{x}_K$ arising from the Lagrange multipliers, so that the coordinates x^K of the torus fibres generated by the n Killing vectors ξ_K are doubled to give a 'doubled torus' with $2n$ coordinates $x^K, \hat{x}_K$. The action of the isometry group G is lifted to $\hat{M}$, and the T-duality space is the quotient $\hat{M}/G$.

The $(2,1)$ duality instead introduces n extra *complex* coordinates Θ_K corresponding to each Killing vector, giving a space $\check{M}$ with an extra $2n$ real dimensions. The T-dual space is now the symplectic quotient of $\check{M}$ given by taking the quotient of the subspace $\check{X} = 0$, giving a generalised moment map and a generalisation of the Kähler quotient construction. For generalised Kähler spaces, this reduces to the generalised moment map of [59], which was constructed from a $(2,2)$ gauging in [26].

Acknowledgments

We are indebted to Rikard von Unge for collaboration in the early stages of this project. M.A. wishes to thank Nathan Berkovits, Reimundo Heluani, Kentaro Hori, Nikita Nekrasov and Warren Siegel for discussions and helpful comments. We thank NORDITA, the GGI, the ICTP-SAIFR, the IMPA, Imperial College London, the Mainz ITP, and the SCGP for hospitality while this work was in progress. We have benefited from participation in several Simons Summer Workshops at the SCGP. This work was supported by the EPSRC programme grant "New Geometric Structures from String Theory" EP/K034456/1 and by STFC grant ST/L00044X/1. MR was supported in part by NSF grant # PHY1620628.

A Review of chiral and vector representations

The superspace constraints (5.5) can be solved in terms of unconstrained superfields. The chiral representation solution is given in (5.10) in terms of V and A_-; in this representation, gauge covariant derivatives transform with the chiral parameter Λ. A real representation [60] can be found by writing

$$e^V = e^{\bar{\Omega}} e^{\Omega}, \tag{A.1}$$

with gauge transformations that depend on both Λ and on new, hermitian parameters K:

$$e^{\Omega'} = e^{iK} e^{\Omega} e^{-i\Lambda} \quad , \quad e^{\bar{\Omega}'} = e^{i\bar{\Lambda}} e^{\bar{\Omega}} e^{-iK} \ . \tag{A.2}$$

Note that this is compatible with (5.15). Performing a similarity transformation $\nabla_\bullet \to e^{\Omega}\nabla_\bullet e^{-\Omega}$ on the chiral representation derivatives (5.10) gives the hermitian vector representation derivatives

$$\begin{aligned} \bar{\nabla}_+ &= e^{\Omega}\bar{D}_+ e^{-\Omega} \ , \quad \nabla_+ = e^{-\bar{\Omega}}D_+ e^{\bar{\Omega}}\,, \\ \nabla_- &= e^{\Omega}(D_- - iA_-)e^{-\Omega} = D_- - i(A_- - iD_-\Omega) + O(\Omega^2)\ . \end{aligned} \tag{A.3}$$

We can use the K transformation to make Ω real; in that case, $\Omega = \bar{\Omega} = \frac{1}{2}V$ and we find that

$$A_- - \frac{i}{2}D_-V + O(V^2) \tag{A.4}$$

is hermitian. The higher order terms are absent in the Abelian case (compare (5.28)).

B Gauge invariance and hermiticity of the action

We now derive the following useful identities for the operator $\frac{e^L-1}{L} \equiv \int_0^1 dt\, e^{tL}$:

$$\frac{\partial}{\partial V}\left(V\frac{e^L-1}{L}\right) = e^L\,, \tag{B.1}$$

which immediately implies the corollary

$$D_-\left(V\frac{e^L-1}{L}\right) \equiv (D_-V)\frac{\partial}{\partial V}\left(V\frac{e^L-1}{L}\right) + V\frac{e^L-1}{L}D_- = (D_-V)e^L + V\frac{e^L-1}{L}D_-\ . \tag{B.2}$$

We prove (B.1) as an operator relation by applying it to a test function $f(\varphi^i)$, where φ^i represents $\varphi^\alpha, \bar{\varphi}^{\bar{\alpha}}$:

$$\begin{aligned} \frac{\partial}{\partial V}\left(V\frac{e^L-1}{L}\right)f(\varphi^i) &\equiv \frac{\partial}{\partial V}\int_0^1 dt\, V e^{tL} f(\varphi^i) = \frac{\partial}{\partial V}\int_0^1 dt\, V f(\bar{\varphi}^{\bar{0}} + tV, \varphi^\alpha, \bar{\varphi}^{\bar{\mu}}) \\ &= \int_0^1 dt [f(\bar{\varphi}^{\bar{0}} + tV) + Vtf(\bar{\varphi}^{\bar{0}} + tV)_{,\bar{0}}] \\ &= \int_0^1 dt \frac{\partial}{\partial t}[tf(\bar{\varphi}^{\bar{0}} + tV)] = f(\bar{\varphi}^{\bar{0}} + V)\,, \end{aligned} \tag{B.3}$$

where the dependence on the spectator fields is suppressed after the first line.

These identities help us prove several important relations. We start with the proof of gauge invariance of (6.5). Its gauge variation may be written as

$$\begin{aligned} i\Lambda\left((\mathcal{L}k_\alpha)D_-\varphi^\alpha - (\mathcal{L}\bar{k}_{\bar{\alpha}})D_-\bar{\varphi}^{\bar{\alpha}}\right) - (k_0 + \bar{k}_{\bar{0}})D_-\Lambda - D_-\Lambda X - A_-\delta X \\ -\delta\left[\left(\frac{e^L-1}{L}\bar{\vartheta}_{\bar{\mu}}(\bar{\varphi})\right)VD_-\bar{\varphi}^{\bar{\mu}}\right]\ . \end{aligned} \tag{B.4}$$

The first term in (B.4) is

$$i\Lambda(\mathcal{L}k_\alpha)D_-\varphi^\alpha = i\Lambda(\vartheta_\alpha(\varphi) + i\chi_{,\alpha}(\varphi,\tilde{\varphi}))D_-\varphi^\alpha = -\Lambda\chi_{,\alpha}(\varphi,\tilde{\varphi})D_-\varphi^\alpha\,, \tag{B.5}$$

where a chiral term has been dropped since the measure $\int d^2\theta^+ d\theta^-$ annihilates it. The second term is

$$\begin{aligned} -i\overline{(\Lambda\mathcal{L}\bar{k}_{\bar\alpha})}D_-\tilde{\varphi}^{\bar\alpha} &= -i\Lambda(\bar\vartheta_{\bar\alpha}(\tilde\varphi) - i\chi_{,\bar\alpha}(\varphi,\tilde\varphi))D_-\tilde\varphi^{\bar\alpha} \\ &= -\Lambda\left(i\bar\vartheta_{\bar\mu}(\tilde\varphi)D_-\tilde\varphi^{\bar\mu} + \chi_{,\bar\alpha}(\varphi,\tilde\varphi)D_-\tilde\varphi^{\bar\alpha}\right), \end{aligned} \tag{B.6}$$

where we used that $\bar\vartheta_{\bar 0} = 0$, cf. (4.19). After partially integrating the χ terms in (B.5) and (B.6) we add them to the third term in (B.4) and find

$$-(k_0 + \bar{k}_{\bar 0} - \chi)D_-\Lambda = XD_-\Lambda\,, \tag{B.7}$$

which cancels the fourth term in (B.4). The fifth term is zero since X is equivariant. The sixth term is

$$-i(\bar\Lambda - \Lambda)e^L\bar\vartheta_{\bar\mu}(\bar\varphi)D_-\bar\varphi^{\bar\mu} + iV\frac{e^L-1}{L}\bar\vartheta_{\bar\mu,\bar 0}(\bar\varphi)\bar\Lambda D_-\bar\varphi^{\bar\mu}\,, \tag{B.8}$$

where we used (B.1). The Λ-term in (B.8) cancels the $\bar\vartheta$-term in (B.6), leaving

$$-i\bar\Lambda e^L\bar\vartheta_{\bar\mu}(\bar\varphi)D_-\bar\varphi^{\bar\mu} + iV\frac{e^L-1}{L}\bar\vartheta_{\bar\mu,\bar 0}(\bar\varphi)\bar\Lambda D_-\bar\varphi^{\bar\mu}\,. \tag{B.9}$$

From the definition (5.22) of L we see that

$$V\bar\vartheta_{\bar\mu,\bar 0} = L\bar\vartheta_{\bar\mu}\,, \tag{B.10}$$

and we have

$$i\bar\Lambda\left(-e^L + \frac{e^L-1}{L}L\right)\bar\vartheta_{\bar\mu}(\bar\varphi)D_-\bar\varphi^{\bar\mu} = -\left(i\bar\Lambda\bar\vartheta_{\bar\mu}(\bar\varphi)D_-\bar\varphi^{\bar\mu}\right), \tag{B.11}$$

which is antichiral and hence annihilated by the measure.

As discussed at the end of section 6, hermiticity of the gauged action follows immediately from the expression (6.8). We now prove that (6.5) *is* (6.8) (modulo total derivatives). We use the relations (6.7), which we rewrite here for convenience:

$$u_\alpha = i(k_{\alpha,\bar 0} + \bar{k}_{\bar 0,\alpha} + X_{,\alpha}) = i(g_{\alpha\bar 0} + X_{,\alpha})\,, \tag{B.12}$$

$$\bar\vartheta_{\bar\alpha} = -i(\bar{k}_{\bar\alpha,\bar 0} - \bar{k}_{\bar 0,\bar\alpha}) - \bar{u}_{\bar\alpha}\,. \tag{B.13}$$

We write out (6.5), using the definitions $\tilde\varphi = e^L\bar\varphi$, $e^L = 1 + \frac{e^L-1}{L}L$, and (B.13)

$$\begin{aligned} L_{1g} &= \left[i\left(k_\alpha D_-\varphi^\alpha - \bar{k}_{\bar\alpha}D_-\tilde\varphi^{\bar\alpha}\right) - A_-X\right](\varphi,\tilde\varphi) - V\left(\frac{e^L-1}{L}\bar\vartheta_{\bar\alpha}\right)D_-\bar\varphi^{\bar\alpha} \\ &= i(k_\alpha D_-\varphi^\alpha - \bar{k}_{\bar\alpha}D_-\bar\varphi^{\bar\alpha})(\varphi,\bar\varphi) - (i\bar{k}_{\bar 0}D_-V + A_-X)(\varphi,\tilde\varphi) \\ &\quad + iV\frac{e^L-1}{L}(k_{\alpha,\bar 0}D_-\varphi^\alpha - \bar{k}_{\bar\alpha,\bar 0}D_-\bar\varphi^{\bar\alpha}) - V\frac{e^L-1}{L}(-i(\bar{k}_{\bar\alpha,\bar 0} - \bar{k}_{\bar 0,\bar\alpha}) - \bar{u}_{\bar\alpha})D_-\bar\varphi^{\bar\alpha} \\ &= i(k_\alpha D_-\varphi^\alpha - \bar{k}_{\bar\alpha}D_-\bar\varphi^{\bar\alpha})(\varphi,\bar\varphi) - (i\bar{k}_{\bar 0}D_-V + A_-X)(\varphi,\tilde\varphi) \\ &\quad + iV\frac{e^L-1}{L}(k_{\alpha,\bar 0}D_-\varphi^\alpha - \bar{k}_{\bar 0,\bar\alpha}D_-\bar\varphi^{\bar\alpha}) + V\frac{e^L-1}{L}(\bar{u}_{\bar\alpha})D_-\bar\varphi^{\bar\alpha}\,. \end{aligned} \tag{B.14}$$

Subtracting (6.8), we have:

$$0 \stackrel{?}{=} -iD_-V\bar{k}_{\bar{0}}(\varphi,\tilde{\varphi}) + iV\frac{e^L-1}{L}(k_{\alpha,\bar{0}}D_-\varphi^\alpha - \bar{k}_{\bar{0},\bar{\alpha}}D_-\bar{\varphi}^{\bar{\alpha}})$$
$$-\frac{i}{2}D_-VX(\varphi,\tilde{\varphi}) - V\frac{e^L-1}{L}\left[\left(u_\alpha - \frac{i}{2}X_{,\alpha}\right)D_-\varphi^\alpha + \frac{i}{2}X_{,\bar{\alpha}}D_-\bar{\varphi}^{\bar{\alpha}}\right] . \quad \text{(B.15)}$$

Applying the identity (B.2) on the two D_-V terms in (B.15), we find, up to total derivatives

$$\begin{aligned}
0 &\stackrel{?}{=} iV\frac{e^L-1}{L}D_-\bar{k}_{\bar{0}} + iV\frac{e^L-1}{L}(k_{\alpha,\bar{0}}D_-\varphi^\alpha - \bar{k}_{\bar{0},\bar{\alpha}}D_-\bar{\varphi}^{\bar{\alpha}}) \\
&\quad + \frac{i}{2}V\frac{e^L-1}{L}D_-X - V\frac{e^L-1}{L}\left[\left(u_\alpha - \frac{i}{2}X_{,\alpha}\right)D_-\varphi^\alpha + \frac{i}{2}X_{,\bar{\alpha}}D_-\bar{\varphi}^{\bar{\alpha}}\right] \\
&= V\frac{e^L-1}{L}\left[i(\bar{k}_{\bar{0},\alpha}D_-\varphi^\alpha + \bar{k}_{\bar{0},\bar{\alpha}}D_-\bar{\varphi}^{\bar{\alpha}}) + i(k_{\alpha,\bar{0}}D_-\varphi^\alpha - \bar{k}_{\bar{0},\bar{\alpha}}D_-\bar{\varphi}^{\bar{\alpha}})\right] \\
&\quad + V\frac{e^L-1}{L}\left[\frac{i}{2}(X_{,\alpha}D_-\varphi^\alpha + X_{,\bar{\alpha}}D_-\bar{\varphi}^{\bar{\alpha}}) - \left(u_\alpha - \frac{i}{2}X_{,\alpha}\right)D_-\varphi^\alpha - \frac{i}{2}X_{,\bar{\alpha}}D_-\bar{\varphi}^{\bar{\alpha}}\right] \\
&= V\frac{e^L-1}{L}\left[i(\bar{k}_{\bar{0},\alpha} + \bar{k}_{\alpha,\bar{0}})D_-\varphi^\alpha + (iX_{,\alpha} - u_\alpha)D_-\varphi^\alpha\right] . \qquad \text{(B.16)}
\end{aligned}$$

This vanishes because of (B.12).

If we keep all the total derivative terms, we find that the difference between the two Lagrangians (6.5) and (6.8) is the total derivative

$$L_{1g} - L'_{1g} = -\frac{i}{2}D_-\left(V\frac{e^L-1}{L}Z(\varphi,\bar{\varphi})\right), \qquad \text{(B.17)}$$

where $Z \equiv X + 2\bar{k}_{\bar{0}} = -k_0 + \bar{k}_{\bar{0}} + \chi$ is defined in (7.9)–(7.10).

C Calculation of $A_-(\varphi)$

As observed in section 7.2.1, the spinor connection A_- enters the action that we use for T-duality — the sum of (6.8) and (7.3) — as a Lagrange multiplier, and hence is not needed; for completeness we present its calculation here.

The expression for the potential A_- is found from the V field equation. The variation of V in the sum of (6.8) and (7.3) is

$$\begin{aligned}
\delta V\frac{\partial}{\partial V}(L_{1g} + L_\Theta) &= \delta V\frac{\partial}{\partial V}\left(V\frac{e^L-1}{L}\left[\left(u_\alpha - \frac{i}{2}X_{,\alpha}\right)D_-\varphi^\alpha + \left(\bar{u}_{\bar{\alpha}} + \frac{i}{2}X_{,\bar{\alpha}}\right)D_-\bar{\varphi}^{\bar{\alpha}}\right]\right. \\
&\qquad \left. - \left(A_- - \frac{i}{2}D_-V\right)X(\varphi,\tilde{\varphi}) - iVD_-\bar{\Theta}\right) \\
&= \delta V\left(e^L\left[\left(u_\alpha - \frac{i}{2}X_{,\alpha}\right)D_-\varphi^\alpha + \left(\bar{u}_{\bar{\alpha}} + \frac{i}{2}X_{,\bar{\alpha}}\right)D_-\bar{\varphi}^{\bar{\alpha}}\right]\right. \\
&\qquad \left. - \left(A_- - \frac{i}{2}D_-V\right)X_{\bar{0}}(\varphi,\tilde{\varphi}) - iD_-\bar{\Theta}\right) + \frac{i}{2}XD_-\delta V \\
&= \delta V\left(u_\alpha D_-\varphi^\alpha + \bar{u}_{\bar{\alpha}}D_-\bar{\varphi}^{\bar{\alpha}} - iX_{,\alpha}D_-\varphi^\alpha - A_-X_{,\bar{0}} - iD_-\bar{\Theta}\right), \qquad \text{(C.1)}
\end{aligned}$$

where $u, \bar{u}$ and X now all depend on φ and $\tilde{\varphi}$. The V field equation results from setting the expression multiplying δV to zero, and determines A_- to be

$$A_- = \frac{1}{X_{,\bar{0}}}\Big(u_\alpha D_- \varphi^\alpha + \bar{u}_{\bar{\alpha}} D_- \bar{\varphi}^{\bar{\alpha}} - i X_{,\alpha} D_- \varphi^\alpha - i D_- \bar{\Theta} \Big). \tag{C.2}$$

We note that, using the A_- field equation (7.14), we have

$$\begin{aligned} A_- - iD_- V &= \frac{1}{X_{,\bar{0}}}\Big(u_\alpha D_- \varphi^\alpha + \bar{u}_{\bar{\alpha}} D_- \bar{\varphi}^{\bar{\alpha}} - i X_{,\alpha} D_- \varphi^\alpha - i D_- \bar{\Theta} \Big) \\ &\quad + \frac{i}{X_{,\bar{0}}}\Big(X_{,\alpha} D_- \varphi^\alpha + X_{,\bar{\alpha}} D_- \bar{\varphi}^{\bar{\alpha}} + D_- \Theta + D_- \bar{\Theta} \Big) \\ &= \frac{1}{X_{,\bar{0}}}\Big(u_\alpha D_- \varphi^\alpha + \bar{u}_{\bar{\alpha}} D_- \bar{\varphi}^{\bar{\alpha}} + i X_{,\bar{\alpha}} D_- \bar{\varphi}^{\bar{\alpha}} + i D_- \Theta \Big) \,. \end{aligned} \tag{C.3}$$

Since X is real and equivariant, $X_0 = X_{\bar{0}}$, and (C.3) is indeed the complex conjugate of (C.2). Hence the combination $A_- - \frac{i}{2} D_- V$ in (5.28), which is the average of (C.2) and (C.3), is manifestly real.

D Reduction

In this appendix we reduce $(2,2)$ models to $(2,1)$ superspace and $(2,1)$ models to $(1,1)$ superspace.

D.1 Reduction of a Kähler (2,2) sigma model to (2,1) superspace

The gauged $(2,2)$ action for chiral superfields ϕ reads [25]

$$\int d^2x d^2\theta d^2\bar{\theta} \left(K(\phi, \bar{\phi}) - \frac{1}{2} V \frac{e^L - 1}{L} X \right), \tag{D.1}$$

where $-\frac{1}{2}X$ is the Killing potential for the isometry. To reduce to $(2,1)$ we write

$$\mathbb{D}_- = D_- - iQ_- \quad \Rightarrow \quad D_- = \frac{1}{2}(\mathbb{D}_- + \bar{\mathbb{D}}_-) \ , \ \ Q_- = \frac{i}{2}(\mathbb{D}_- - \bar{\mathbb{D}}_-) \,, \tag{D.2}$$

and act with Q_- on K, splitting the fermionic measure as follows:

$$d^2\theta d^2\bar{\theta} \sim \mathbb{D}_+ \bar{\mathbb{D}}_+ D_- Q_- \ . \tag{D.3}$$

We also write the gauge covariant derivative as

$$\nabla_- \equiv D_- - iA_- = \frac{1}{2}(\nabla_- + \bar{\nabla}_-)_| \equiv \frac{1}{2}(\mathbb{D}_- + \mathbb{D}_- V + \bar{\mathbb{D}}_-)_| = D_- + \frac{1}{2}(D_- V - iQ_- V)_| \ , \tag{D.4}$$

which implies

$$Q_- V_| = 2\Big(A_- - \frac{i}{2} D_- V \Big) \,, \tag{D.5}$$

where a vertical bar denotes the reduction to $(2,1)$ superspace. Note that this is proportional to the real combination (5.28). We use the chiral constaint

$$\bar{\mathbb{D}}_- \phi = 0 \quad \Rightarrow \quad Q_- \phi_| = i D_- \phi \,, \tag{D.6}$$

and keep the $(2,2)$ notation (ϕ, V) for the $(2,1)$ superfields (that is, we don't bother writing $\varphi := \phi_|$, etc.).

This leads to the $(2,1)$ action

$$\int d^2x \mathbb{D}_+ \bar{\mathbb{D}}_+ D_- \Big(i(K_{,\alpha} D_- \phi^\alpha - K_{,\bar{\alpha}} D_- \bar{\phi}^{\bar{\alpha}}) \\ -2\Big(A_- - \frac{i}{2} D_- V\Big)\frac{1}{2} X - V \frac{e^L - 1}{L} \frac{i}{2}(X_{,\alpha} D_- \phi^\alpha - X_{,\bar{\alpha}} D_- \bar{\phi}^{\bar{\alpha}})\Big). \tag{D.7}$$

As expected, this is the $(2,1)$ action (6.8) with $k_\alpha = K_{,\alpha}$ and $u_\alpha = 0$ because the geometry is Kähler.

D.2 Reduction of a general (2,1) sigma model to (1,1) superspace

The general gauged $(2,1)$ model (6.8) is reduced to $(1,1)$ superspace using similar techniques to those in the previous subsection. However, the reduction of the gauge multiplet is somewhat different.

As above, we define

$$\mathbb{D}_+ = D_+ - iQ_+ \tag{D.8}$$

and write the measure as

$$\mathbb{D}_+ \bar{\mathbb{D}}_+ D_- \sim D_+ D_- Q_+ \ . \tag{D.9}$$

We can define gauge covariant $(1,1)$ derivatives from the $(2,1)$ derivatives in section 5; however, since we need to distinguish them, we will write ∇ for the $(2,1)$ gauge covariant derivatives and ∇ for the $(1,1)$ derivatives. In $(1,1)$ superspace, the natural group has a real superfield gauge parameter, and does *not* involve a complexification. Instead, the real part of the $(2,1)$ complex gauge parameter Λ is used to gauge away $V_|$. This is the $(1,1)$ version of Wess-Zumino gauge; note that $V_| = D_\pm V_| = 0$, but $(Q_+ V)_| \neq 0$. In Wess-Zumino gauge, the $(1,1)$ objects that we find are independent of whether we are in chiral, anti-chiral, or real representation.

Similarly to (D.4), we define the $(1,1)$ gauge covariant derivative:

$$\nabla_+ \equiv D_+ - iA_+ = \frac{1}{2}(\nabla_+ + \bar{\nabla}_+)_| \equiv \frac{1}{2}(\mathbb{D}_+ + \mathbb{D}_+ V + \bar{\mathbb{D}}_+)_| = D_+ + \frac{1}{2}(D_+ V - iQ_+ V)_| \, , \tag{D.10}$$

where a vertical bar denotes the reduction to $(1,1)$ superspace. Here we have used (5.10). Because in Wess-Zumino gauge $(D_+ V)_| = 0$, we find

$$Q_+ V_| = 2A_+ \ . \tag{D.11}$$

We also reduce the field strengths W, $\bar{W}$ in (5.14),(5.13):

$$W = -i\bar{\mathbb{D}}_+ A_- \ , \quad \bar{W} = -i\mathbb{D}_+(A_- - iD_- V) \ . \tag{D.12}$$

Since in Wess-Zumino gauge $(D_- V)_| = 0$, we find

$$W_| \equiv (-iD_+ A_- + Q_+ A_-)_| \ , \quad \bar{W}_| \equiv (-iD_+ A_- - Q_+ A_-)_| - 2iD_- A_+ \ . \tag{D.13}$$

Then the real part of W is just the $(1,1)$ gauge field strength:

$$\frac{1}{2}(W+\bar{W})_| = -i(D_+A_- + D_-A_+) = \{\nabla_+, \nabla_-\}\,, \tag{D.14}$$

and the imaginary part is a real scalar field:

$$\frac{i}{2}(\bar{W}-W)_| = (-iQ_+A_-)_| + D_-A_+ =: -\frac{1}{2}s \quad \Rightarrow \quad (Q_+A_-)_| = -i\left(\frac{1}{2}s + D_-A_+\right)\,. \tag{D.15}$$

We will use the following identity below:

$$Q_+\left(A_- - \frac{i}{2}D_-V\right)_| = -\frac{i}{2}s\,. \tag{D.16}$$

We now push in the generator of the nonmanifest left supersymmetry Q_+:

$$\begin{aligned}\int d^2x d^2\theta^+ d\theta^- L_{1g} = &\int d^2x D_+D_-Q_+\Big(i\left(k_\alpha D_-\varphi^\alpha - \bar{k}_{\bar{\alpha}} D_-\bar{\varphi}^{\bar{\alpha}}\right)(\varphi,\bar{\varphi})\\ &\qquad -\left(A_- - \frac{i}{2}D_-V\right)X(\varphi,\bar{\varphi})\\ &\qquad +V\frac{e^L-1}{L}\left[\left(u_\alpha - \frac{i}{2}X_{,\alpha}\right)D_-\varphi^\alpha + \left(\bar{u}_{\bar{\alpha}} + \frac{i}{2}X_{,\bar{\alpha}}\right)D_-\bar{\varphi}^{\bar{\alpha}}\right]\Big)\\ = &\int d^2x D_+D_-\Big(2k_{\alpha,\bar{\beta}}\, D_+\bar{\varphi}^{\bar{\beta}} D_-\varphi^\alpha + 2\bar{k}_{\bar{\alpha},\beta}\, D_+\varphi^\beta D_-\bar{\varphi}^{\bar{\alpha}} + \frac{i}{2}sX\\ &\qquad +A_-\left[iX_{,\alpha}\, D_+\varphi^\alpha - iX_{,\bar{\alpha}}\, D_+\bar{\varphi}^{\bar{\alpha}} + 2X_{,0}\, A_+\right]\\ &\qquad +2A_+\left[v_\alpha D_-\varphi^\alpha + \bar{v}_{\bar{\alpha}} D_-\bar{\varphi}^{\bar{\alpha}}\right]\Big)\,,\end{aligned} \tag{D.17}$$

where we have integrated by parts to eliminate the $D_+D_-\varphi$ and $D_+D_-\bar{\varphi}$ terms, and use the shorthand notation

$$v_\alpha = u_\alpha - \frac{i}{2}X_{,\alpha}\,. \tag{D.18}$$

This is not yet manifestly left-right symmetric. We use (6.7):

$$u_\alpha = i(k_{\alpha,\bar{0}} + \bar{k}_{\bar{0},\alpha}) + iX_{,\alpha} = i(g_{\alpha\bar{0}} + X_{,\alpha}) \tag{D.19}$$

to rewrite this as

$$\begin{aligned}\int d^2x D_+D_-\Big(&2k_{\alpha,\bar{\beta}}\, D_+\bar{\varphi}^{\bar{\beta}} D_-\varphi^\alpha + 2\bar{k}_{\bar{\alpha},\beta}\, D_+\varphi^\beta D_-\bar{\varphi}^{\bar{\alpha}} + \frac{i}{2}sX + 2X_{,0}\, A_-A_+\\ &+ A_-\left[(u_\alpha - ig_{\alpha\bar{0}})D_+\varphi^\alpha + (\bar{u}_{\bar{\alpha}} + ig_{0\bar{\alpha}})D_+\bar{\varphi}^{\bar{\alpha}}\right]\\ &+ A_+\left[(u_\alpha + ig_{\alpha\bar{0}})D_-\varphi^\alpha + (\bar{u}_{\bar{\alpha}} - ig_{0\bar{\alpha}})D_-\bar{\varphi}^{\bar{\alpha}}\right]\Big)\,.\end{aligned} \tag{D.20}$$

Using the definition of gauge covariant derivatives:

$$\begin{aligned}\nabla_\pm\varphi^0 &= D_\pm\varphi^0 - \xi^0 A_\pm = D_\pm\varphi^0 - iA_\pm\\ \nabla_\pm\bar{\varphi}^{\bar{0}} &= D_\pm\bar{\varphi}^{\bar{0}} - \xi^{\bar{0}} A_\pm = D_\pm\bar{\varphi}^{\bar{0}} + iA_\pm\end{aligned} \tag{D.21}$$

as well as (3.10) in the form

$$2k_{\alpha,\bar\beta} = g_{\alpha\bar\beta} - b_{\alpha\bar\beta} \ , \quad 2k_{\bar\beta,\alpha} = g_{\alpha\bar\beta} + b_{\alpha\bar\beta} \, , \tag{D.22}$$

we rewrite (D.20) as

$$\int d^2x D_+D_-\Big(g_{\alpha\bar\beta}\left[\nabla_+\bar\varphi^{\bar\beta}\nabla_-\varphi^\alpha + \nabla_+\varphi^\alpha\nabla_-\bar\varphi^{\bar\beta}\right] + b_{\alpha\bar\beta}\left[D_+\varphi^\alpha D_-\bar\varphi^{\bar\beta} - D_+\bar\varphi^{\bar\beta}D_-\varphi^\alpha\right] \\ + \frac{i}{2}sX + A_-\left[u_\alpha D_+\varphi^\alpha + \bar u_{\bar\alpha}D_+\bar\varphi^{\bar\alpha}\right] + A_+\left[u_\alpha D_-\varphi^\alpha + \bar u_{\bar\alpha}D_-\bar\varphi^{\bar\alpha}\right]\Big) \, . \tag{D.23}$$

Except for the sX term discussed below, this is precisely the general gauged sigma model (2.21),(2.23) in $(1,1)$ superspace.

Next, we reduce the Lagrange multiplier term L_Θ (7.3) to $(1,1)$ superspace:

$$\begin{aligned} L_\Theta &= -\int d^2x D_+D_-Q_+\left[(\Theta+\bar\Theta)A_- + iD_-\bar\Theta V\right] \\ &= -\int d^2x D_+D_-Q_+\left[(\Theta+\bar\Theta)(A_- - \frac{i}{2}D_-V) + \frac{i}{2}D_-(\bar\Theta-\Theta)V\right] \\ &= -\int d^2x D_+D_-\left[iD_+(\Theta-\bar\Theta)A_- - \frac{i}{2}s(\Theta+\bar\Theta) + iA_+D_-(\bar\Theta-\Theta)\right] \\ &= \int d^2x D_+D_-\left[-i(A_-D_+ + A_+D_-)(\bar\Theta-\Theta) + \frac{i}{2}s(\Theta+\bar\Theta)\right] \, . \end{aligned} \tag{D.24}$$

Combining (D.23) and (D.24), we find the usual real T-duality with in addition the $(1,1)$ superfield s acting as a Lagrange multiplier to impose the condition

$$(\Theta+\bar\Theta) + X(\varphi,\bar\varphi) = 0 \, . \tag{D.25}$$

This is a diffeomorphism that expresses $\Theta+\bar\Theta$ in terms of $\varphi,\bar\varphi$; because of the isometry in the original model, $\Theta+\bar\Theta$ does *not* depend on $x^0 := \frac{i}{2}(\bar\varphi^{\bar 0} - \varphi^0)$, which is the coordinate dual to $\hat x^0 := \frac{i}{2}(\bar\Theta - \Theta)$.

References

[1] L. Álvarez-Gaumé and D.Z. Freedman, *Geometrical structure and ultraviolet finiteness in the supersymmetric σ-model*, *Commun. Math. Phys.* **80** (1981) 443 [INSPIRE].

[2] S.J. Gates, Jr., C.M. Hull and M. Roček, *Twisted multiplets and new supersymmetric nonlinear σ-models*, *Nucl. Phys.* **B 248** (1984) 157 [INSPIRE].

[3] P.S. Howe and G. Sierra, *Two-dimensional supersymmetric nonlinear σ-models with torsion*, *Phys. Lett.* **148B** (1984) 451 [INSPIRE].

[4] C.M. Hull and E. Witten, *Supersymmetric σ-models and the heterotic string*, Phys. Lett. **B 160** (1985) 398 [INSPIRE].

[5] C.M. Hull, *σ Model β-functions and String Compactifications*, Nucl. Phys. **B 267** (1986) 266 [INSPIRE].

[6] C.M. Hull, *Lectures on nonlinear sigma models and strings*, lectures given at the *Vancouver Advanced Research Workshop*, published in *Super Field Theories*, H. Lee and G. Kunstatter, Plenum, New York U.S.A. (1988).

[7] P.S. Howe and G. Papadopoulos, *Ultraviolet behavior of two-dimensional supersymmetric nonlinear σ models*, Nucl. Phys. **B 289** (1987) 264 [INSPIRE].

[8] P.S. Howe and G. Papadopoulos, *Further remarks on the geometry of two-dimensional nonlinear σ models*, Class. Quant. Grav. **5** (1988) 1647 [INSPIRE].

[9] C. Hull and U. Lindström, *All* $(4,1)$*: σ-models with* $(4,q)$ *off-shell supersymmetry*, JHEP **03** (2017) 042 [arXiv:1611.09884] [INSPIRE].

[10] C. Hull and U. Lindström, *All* $(4,0)$*: σ-models with* $(4,0)$ *off-shell supersymmetry*, JHEP **08** (2017) 129 [arXiv:1707.01918] [INSPIRE].

[11] C. Hull and U. Lindström, *The generalised complex geometry of* (p,q) *Hermitian geometries*, arXiv:1810.06489 [INSPIRE].

[12] N. Hitchin, *Generalized Calabi-Yau manifolds*, *Quart. J. Math. Oxford Ser.* **54** (2003) 281 [math/0209099].

[13] M. Gualtieri, *Generalized complex geometry*, math/0401221.

[14] H. Ooguri and C. Vafa, $N = 2$ *heterotic strings*, Nucl. Phys. **B 367** (1991) 83 [INSPIRE].

[15] D. Kutasov and E.J. Martinec, *New principles for string/membrane unification*, Nucl. Phys. **B 477** (1996) 652 [hep-th/9602049] [INSPIRE].

[16] D. Kutasov, E.J. Martinec and M. O'Loughlin, *Vacua of M-theory and* $N = 2$ *strings*, Nucl. Phys. **B 477** (1996) 675 [hep-th/9603116] [INSPIRE].

[17] D. Kutasov and E.J. Martinec, *M-branes and* $N = 2$ *strings*, Class. Quant. Grav. **14** (1997) 2483 [hep-th/9612102] [INSPIRE].

[18] M.B. Green, *World sheets for world sheets*, Nucl. Phys. **B 293** (1987) 593 [INSPIRE].

[19] C. Hull et al., *Generalized Kähler geometry in* $(2,1)$ *superspace*, JHEP **06** (2012) 013 [arXiv:1202.5624] [INSPIRE].

[20] M. Dine and N. Seiberg, $(2,0)$ *superspace*, Phys. Lett. **B 180** (1986) 364 [INSPIRE].

[21] A. Giveon, M. Porrati and E. Rabinovici, *Target space duality in string theory*, Phys. Rept. **244** (1994) 77 [hep-th/9401139] [INSPIRE].

[22] T.H. Buscher, *A symmetry of the string background field equations*, Phys. Lett. **B 194** (1987) 59 [INSPIRE].

[23] T.H. Buscher, *Path integral derivation of quantum duality in nonlinear σ-models*, Phys. Lett. **B 201** (1988) 466 [INSPIRE].

[24] M. Roček and E.P. Verlinde, *Duality, quotients and currents*, Nucl. Phys. **B 373** (1992) 630 [hep-th/9110053] [INSPIRE].

[25] C.M. Hull, A. Karlhede, U. Lindström and M. Roček, *Nonlinear σ models and their gauging in and out of superspace*, Nucl. Phys. **B 266** (1986) 1 [INSPIRE].

[26] A. Kapustin and A. Tomasiello, *The general* (2, 2) *gauged σ-model with three-form flux*, JHEP **11** (2007) 053 [hep-th/0610210] [INSPIRE].

[27] W. Merrell, L.A. Pando Zayas and D. Vaman, *Gauged* (2, 2) *σ-models and generalized Kähler geometry*, JHEP **12** (2007) 039 [hep-th/0610116] [INSPIRE].

[28] U. Lindström et al., *New $N = (2, 2)$ vector multiplets*, JHEP **08** (2007) 008 [arXiv:0705.3201] [INSPIRE].

[29] C.M. Hull and B.J. Spence, *The gauged nonlinear σ model with Wess-Zumino term*, Phys. Lett. **B 232** (1989) 204 [INSPIRE].

[30] C.M. Hull and B.J. Spence, *The geometry of the gauged σ-model with Wess-Zumino term*, Nucl. Phys. **B 353** (1991) 379 [INSPIRE].

[31] I. Jack, D.R.T. Jones, N. Mohammedi and H. Osborn, *Gauging the general σ model with a Wess-Zumino term*, Nucl. Phys. **B 332** (1990) 359 [INSPIRE].

[32] C.M. Hull, G. Papadopoulos and B.J. Spence, *Gauge symmetries for (p, q) supersymmetric σ-models*, Nucl. Phys. **B 363** (1991) 593 [INSPIRE].

[33] C.M. Hull, G. Papadopoulos and P.K. Townsend, *Potentials for $(p, 0)$ and* (1, 1) *supersymmetric σ-models with torsion*, Phys. Lett. **B 316** (1993) 291 [hep-th/9307013] [INSPIRE].

[34] M. Abou Zeid and C.M. Hull, *Geometry, isometries and gauging of* (2, 1) *heterotic σ-models*, Phys. Lett. **B 398** (1997) 291 [hep-th/9612208] [INSPIRE].

[35] M. Abou Zeid and C.M. Hull, *The gauged* (2, 1) *heterotic σ-model*, Nucl. Phys. **B 513** (1998) 490 [hep-th/9708047] [INSPIRE].

[36] C.M. Hull, *Global aspects of T-duality, gauged σ-models and T-folds*, JHEP **10** (2007) 057 [hep-th/0604178] [INSPIRE].

[37] P.M. Crichigno and M. Roček, *On gauged linear σ-models with torsion*, JHEP **09** (2015) 207 [arXiv:1506.00335] [INSPIRE].

[38] L. Álvarez-Gaumé and D.Z. Freedman, *Potentials for the supersymmetric nonlinear σ-model*, Commun. Math. Phys. **91** (1983) 87 [INSPIRE].

[39] C.M. Hull, *Complex structures and isometries in the* (2, 0) *supersymmetric nonlinear σ-model*, Mod. Phys. Lett. **A 5** (1990) 1793 [INSPIRE].

[40] U. Lindström and M. Roček, *Scalar tensor duality and $N = 1$, $N = 2$ nonlinear σ-models*, Nucl. Phys. **B 222** (1983) 285 [INSPIRE].

[41] M.T. Grisaru, M. Massar, A. Sevrin and J. Troost, *Some aspects of $N = (2, 2)$, $D = 2$ supersymmetry*, Fortsch. Phys. **47** (1999) 301 [hep-th/9801080] [INSPIRE].

[42] U. Lindström et al., *T-duality and generalized Kähler geometry*, JHEP **02** (2008) 056 [arXiv:0707.1696] [INSPIRE].

[43] W. Merrell and D. Vaman, *T-duality, quotients and generalized Kähler geometry*, Phys. Lett. **B 665** (2008) 401 [arXiv:0707.1697] [INSPIRE].

[44] P.M. Crichigno, *The semi-chiral quotient, hyper-Kähler manifolds and T-duality*, JHEP **10** (2012) 046 [arXiv:1112.1952] [INSPIRE].

JHEP06(2019)138

[45] E. Alvarez, L. Álvarez-Gaumé, J.L.F. Barbon and Y. Lozano, *Some global aspects of duality in string theory*, *Nucl. Phys.* **B 415** (1994) 71 [hep-th/9309039] [INSPIRE].

[46] J.M. Figueroa-O'Farrill and S. Stanciu, *Gauged Wess-Zumino terms and equivariant cohomology*, *Phys. Lett.* **B 341** (1994) 153 [hep-th/9407196] [INSPIRE].

[47] E.S. Fradkin and A.A. Tseytlin, *Quantum equivalence of dual field theories*, *Annals Phys.* **162** (1985) 31 [INSPIRE].

[48] A. Giveon and M. Roček, *Introduction to duality*, hep-th/9406178 [INSPIRE].

[49] B.E. Fridling and A. Jevicki, *Dual representations and ultraviolet divergences in nonlinear σ models*, *Phys. Lett.* **134B** (1984) 70 [INSPIRE].

[50] C.M. Hull, *Doubled geometry and T-folds*, *JHEP* **07** (2007) 080 [hep-th/0605149] [INSPIRE].

[51] D.M. Belov, C.M. Hull and R. Minasian, *T-duality, gerbes and loop spaces*, arXiv:0710.5151 [INSPIRE].

[52] C.M. Hull et al., *Generalized Kähler geometry and gerbes*, *JHEP* **10** (2009) 062 [arXiv:0811.3615] [INSPIRE].

[53] B. de Wit and P. van Nieuwenhuizen, *Rigidly and locally supersymmetric two-dimensional nonlinear σ models with torsion*, *Nucl. Phys.* **B 312** (1989) 58 [INSPIRE].

[54] G.W. Delius, M. Roček, A. Sevrin and P. van Nieuwenhuizen, *Supersymmetric σ models with nonvanishing Nijenhuis tensor and their operator product expansion*, *Nucl. Phys.* **B 324** (1989) 523 [INSPIRE].

[55] J. Bagger and E. Witten, *The gauge invariant supersymmetric nonlinear sigma model*, *Phys. Lett.* **B 118** (1982) 103.

[56] K. Kikkawa and M. Yamasaki, *Casimir effects in superstring theories*, *Phys. Lett.* **149B** (1984) 357 [INSPIRE].

[57] W. Siegel, *Unusual representations of local groups*, *Phys. Lett.* **134B** (1984) 318 [INSPIRE].

[58] N.J. Hitchin, A. Karlhede, U. Lindström and M. Roček, *Hyper-Kähler metrics and supersymmetry*, *Commun. Math. Phys.* **108** (1987) 535 [INSPIRE].

[59] Y. Lin and S. Tolman, *Reduction of twisted generalized Kähler structure*, math.DG/0510010.

[60] J. Wess, *supersymmetry-supergravity*, in *Topics in quantum field theory and gauge theories, VIII GIFT Int. Seminar on Theoretical Physics, Salamanca, Spain, June 13–19*, J.A. de Azcarraga ed., Springer, Germany (1978) [PRINT-77-0885].