Embedding inflation in string theory

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August 29, 2019

Abstract

We introduce slow-roll inflation in string theory on both a conceptual level and a detailed one. In order to do this we first briefly review important concepts of inflation and string theory. We then reconstruct models of string inflation in the so-called Racetrack scenario for two different cases where the difference being the number of Kähler moduli used as inflaton. Furthermore, we briefly relate our results to the more recent discussion on whether AdS/dS solutions actually exist in string theory. In this instance our results seem to indicate that uplifting is a crucial component to obtain AdS/dS solutions.

Sammanfattning


När man har alla matematiska objekt tillgängliga så kan man börja skapa modeller där man böddar in inflation i strängteori. Vi visar två sådana modeller i detta projekt där skillnaden mellan modellerna är hur många nya fält vi har i vår effektiva fältteori.

Vi spenderar också lite tid på att undersöka den kritik som finns mot dessa modeller samt mot denna process. Vi kollar speciellt på en förmodan som förbjuder dessa effektiva teorier från att existera i strängteori.
Part I
Understanding the undertaking

1 Introduction

The modern understanding of cosmology as well string theory are two ideas that are consistent and work within their own respective frameworks. They are different from one another to be sure. The study of cosmology is the study of the universe and its history as a whole and as such it is possible to conduct observations to strengthen or disprove claims. String theory on the other hand concern itself with the fundamental structure and interactions of the universe and is, as of right now, not experimentally verifiable. Even so, a general goal of theoretical physicists is to come up with a overarching theory from which all physics materialize. A stepping stone in this endeavour is to unite the theories that already exist, such as modern cosmology and string theory. A part of one such unification is what we are going to show in this very report.

We are going to realize inflation, a process that is thought to have occurred early on in the history of the universe, with type II B string theory, which is a variant of string theory. We utilize both analytical and numerical calculations for this task. The numerical calculations and graphs are constructed using the software Matlab.

The report is divided into three parts along with an appendix. The purpose of the first part is to introduce inflation and string theory on a conceptional level and also try to motivate their significance to physics. The second part introduces mathematical expressions and objects that are necessary to properly define inflation, string theory and later the unification models. Part three is where we put everything together and construct concrete models where inflation is realized using tools from string theory. We also scrutinise the results in this part. Lastly, the appendix contain some of the more cumbersome derivations as well as the Matlab scripts and functions used for the numerical calculations and plots.

This report is in large part based on the review [1] by Daniel Baumann and Liam McAllister. We also reconstruct the string inflation models by J.J. Blanco-Pillado et al. in [2] and [3].

2 Introducing inflation

We already stated that inflation is a process that occurred early on in the history of the universe, allow us to elaborate. Inflation is defined as a period of rapid expansion of space [4]. To give a example of the time scales we are discussing, inflation is thought to have occurred around $10^{-34}$ seconds after the start of universe[5]. The amount of time the universe spent expanding is thought to

\[ t \approx 10^{-34} \text{ seconds} \]

1It might be too bold to speak for every theoretical physicist. Some might nonetheless have this goal.

2What exactly is considered to be the start of the universe is up to debate. The universe might not even have ‘started’, at least not using our intuition of the word. In this report we treat the start, i.e. time $t = 0$, as the time where our understanding of physics starts being applicable.
be 60 $e$-folds where 1 $e$-fold is the time period needed for exponential expansion of space to be increased by a factor of $e$. We will properly describe what we exactly mean by "expand" in part II.

One might wonder why one would bother with such an idea. "So what?" one might ask. It turns out that without inflation our understanding of the evolution of the universe is flawed. Inflation solves three famous problems in cosmology, the flatness problem, the horizon problem and the monopole problem \[4, 5\]. We will outline these problems in the sections below and show how inflation solves one of them in part II, the horizon problem.

To understand the horizon problem one first need to understand the concept of the particle horizon. This horizon tells us the maximum distance a photon can travel through space starting from a chosen initial time until present time. The initial time can be, and usually is, the start of the universe. This distance tells us something interesting, it tells us the maximum possible distance where two patches of space are in causal connection. This is because no information can travel faster than the speed of light and thus the speed of light act as a limit for how far apart objects can be and still interact.

This all seems fine and well, but there is an issue. The problem is that too much of space seem to be in casual connection. More specifically, patches of space that are separated by a distance larger than the particle horizon seem to be in contact. We say this because of how homogeneous the universe looks on large scales\[3\]; everything looks almost exactly the same. To get this appearance without allowing for interaction between certain patches to interact is tricky, this would mean that the universe started in such a way. This is of course not disallowed by physics, yet one would want a proper explanation to why the universe looks the way it looks rather than stating "it just happened to be this way". We will come back to this problem in part II.

Moving on, according to Einstein's theory of general relativity, spacetime can be curved. For instance, gravity can be thought of as the curvature of space due to to the presence of a mass. One could then wonder about the curvature of the entire universe. There are three possibilities: the universe is positively curved, negatively curved or not curved, also known as flat. To visualize these three versions one could consider parallel lines. In a positively curve universe two parallel line would eventually converge, in a negatively curved universe they would diverge and in flat space they would remain parallel. Astronomers have tried (using data from the CMB and type 1a supernovae \[4\]) and succeeded in figuring out the curvature of our universe and have concluded that it is very close to flat.

Again, this result is somewhat problematic in the right context. To explain this problem it is easiest to show what we mean by flatness mathematically

$$|1 - \Omega(a)| = \frac{k}{(aH)^2}. \quad (1)$$

To not digress from the point we will only explain this expression briefly. The LHS, $|1 - \Omega(a)|$, is the called the curvature parameter. The closer it is to zero, i.e. $\Omega \sim 1$, the more flat we consider space to be. Moving on, $k$ measures the curvature of space. If $k = +1$ we have positively curved space, if $k = -1$ we

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\[3\] In fact, the cosmological principle states that the universe is homogeneous and isotropic on large scales \[1\].
have negatively curved space and if $k = 0$ there is no curvature. Lastly, $(aH)^{-1}$ is called the \emph{comoving Hubble horizon} and we will talk more about it in part II. For now, it is sufficient to know that it grows with time.

The problem is as follows. If $|1 - \Omega(a)| \approx 0$ at present time and $(aH)^{-1}$ grows with time then $|1 - \Omega(a)|$ had to be even closer to zero in the past. For instance, during Big Bang Nucleosynthesis we would have $|1 - \Omega(a_{BBN})| \leq \mathcal{O}(10^{-16})$ and before that it would have to be even smaller \cite{1}. What would cause the universe to act in this way? Why is the universe so extremely flat? Again one could state that the initial conditions of the universe just happened to be such that this appearance came to be. But again one would like an explanation that does not simply rely on coincidence.

The inflationary solution to this enigma is that inflation naturally makes the universe very flat during the accelerated expansion. Thus the universe is allowed to start off with various initial curvatures yet we always end up with a flat universe after inflation.

The monopole problem could be the easiest to understand conceptually. It is thought that the fundamental forces of the universe except gravity at one point were unified as one \cite{4}. One often talk about \emph{grand unification theory} (GUT). At some point in the evolution of the universe these forces split into the ones we experience today. It also thought that this split would generate magnetic monopoles. This is due to topological defects that come about through the loss of symmetry in the transition \cite{4}.

The problem in this case is the apparent absence of such monopoles. The inflationary solution is quite simple in this instance. If the forces were separated some time before inflation then the accelerated expansion would dilute the universe of emerging monopoles to the point where the possibility of encountering one would be minuscule.

Aside from these cosmological problems there is also another argument available for the credibility of inflation. Inflation combined with quantum effects can actually explain why there exists inhomogeneities in the universe, inhomogeneities such as galaxies, stars and planets. Quantum fluctuations during the inflationary era gets blown up to huge proportions during inflation and as such perturb the universe to the point where large scale structures are able to form \cite{5}. One can even see traces of these fluctuation in the \emph{cosmic microwave background} (CMB) as the CMB contain slight temperature fluctuations. These small fluctuations during the distant past are the seeds that eventually lead to the existence of human supporting cosmic structures, we owe them our lives.

3 Introducing string theory

Let us leave inflation behind for a moment and consider the other theory relevant to this report. As the name suggests, \emph{string theory} is a theory about strings. In the theory, these one dimensional objects assumes the roles of fundamental building blocks in the universe. In short, every conceivable physical thing is made up of strings in some way \cite{6}. This scenario is different from the, perhaps, more intuitive idea that every physical object is made up of point particles which are zero dimensional objects.

There are, however, parallels one can draw between point-like objects and string-objects that could grant some basic understanding of the string-scenario.
Consider, for example, the worldline that a particle moving through a two-dimensional universe (one spatial dimension and one temporal dimension) would trace out, as seen in figure 1. This scenario makes sense in that time will tick along at a constant rate and at each time step we can observe the particle’s changing spatial position, thus granting us the notion of movement of the particle.

Figure 1: The worldline of a particle whose position changes with time.

A similar situation can be considered in the string case. At each time step the string will now occupy more than one point on the spatial axis, it will occupy
all points within a certain interval. The result of this construction can be seen in figure 2; the worldline is replaced with a "worldsheet". We can imagine further a more complicated scenario in which the string can be extended in more than one spatial dimension, thus producing a worldsheet that can be twisted and turned and even connected to itself in a tube-like fashion. Imagine, if you will, the trajectory of a squiggling, one dimensional worm swimming in the ocean.

An important distinction to emphasise when it comes to string theory compared to observed reality is the energy scale in which string theory takes place. Indeed, we do not actually observe one dimensional strings in the natural world, neither by eye nor by instruments. Energy scales of string theory are much larger than that of observed reality, they are scales that are unattainable by modern day instruments. To actually be a verified theory, this is a problem that string theory needs to overcome. We will come back to energy scales in part II.

Similar to our introduction to inflation a question comes to mind: why bother? Why would we want to construct a theory with objects such as these? Different physicists might give different answers to this question but we are going to following the reasoning presented by Joseph Polchinski in [6].

The reasoning starts with quantum field theory (QFT), a highly successful theoretical framework. Using this theory physicists have constructed the well known Standard Model, the model that define the fundamental particles of the universe as well as their interactions. Together with general relativity this model describes all physics very accurately down to the scale of quarks and have been experimentally confirmed.

There is, however, a problem with QFTs describing gravity interactions. These theories are not renormizable and are therefore affected by short-distance divergences. The problem is easiest to illustrate using a picture, see figure 3. Figure 3 shows a Feynman diagram. Diagrams such as these illustrates the interaction of two or more particles, in this particular picture we see two massive particles interacting with one another by exchanging a graviton. Translated into mathematics diagrams such as these contain much information and can be used, for instance, to calculate scattering amplitudes. The problem comes from the points of interaction, the vertices. In the limit where the vertices approach each other, i.e. short-distance interaction, we obtain a divergence for large energy scales. This problem is amplified the more exchanging particles with add to the diagrams as the corresponding additional vertices also can coincide.

Figure 3: Two massive particles exchanging a graviton in a Feynman diagram.
The string theory solution to this problem is fairly simple. In string theory there are no point of interaction, i.e. no vertices, instead the interaction is smeared out across a two-dimensional region. This is only doable because the interacting objects in string theory, strings, are one dimensional objects compared to zero dimensional particles. Using strings, the Feynman diagram presented in figure 3 instead takes the form illustrated by figure 4.

There are other benefits to string theory as is presented in [6], and some would also argue that the theory contain mathematical beauty, but this is far as we will go in this report. The main part of the report is not to motivate string theory but rather to utilize it.

4 Inflation in string theory

It could beneficial to take a moment to consider what it would mean to combine inflation and string theory. By combining these theories we do not prove either of them as neither of them have been confirmed as of yet. The theories are, however, widely accepted and have a lot going for them, as we have hopefully conveyed. By combining these two theories we would be able confirm that they are consistent with each other and thus we could conclude that the most popular version of modern cosmology is compatible with one of the most promising candidates for quantum gravity. This would strengthen the credibility of each theory separately. If one of the theories at one point would be experimentally verified then that would also increase the credibility for its counterpart. Conversely, if one of the theories would be experimentally verified as false then the counterpart could face issues. In short, a union of string theory and inflation is a useful theoretical tool going forward.
Part II
Defining the necessary tools

5 Mathematical description of inflation

Simply stating that inflation is a period of rapid expansion is a good way to intuitively understand the process, but if we are going to work with inflationary models we need a more mathematical definition. Such a definition is what we will give in this section and we will also show that this definition naturally solve one of the cosmological problems introduced above, namely the horizon problem.

5.1 Comoving Hubble radius

Let us introduce the following object, \( d_{\text{horizon}} \),

\[
d_{\text{horizon}} = \int_0^t \frac{dt'}{a(t')} = \int_0^a \frac{da}{Ha^2} = \int_0^a d(\ln a) \frac{1}{aH}.
\]

This is called the "comoving particle horizon" [5]. Comoving scales are scales that remain constant with respect to the expansion of the universe. The particle horizon defines the maximum distance a photon could travel since a time \( t = 0 \), where \( t = 0 \) is usually chosen to be the beginning of the universe. The comoving horizon is defined in terms of the scale factor, \( a(t) \), and the Hubble parameter,

\[
H(t) \equiv \frac{\dot{a}}{a(t)} \frac{1}{a(t)}.
\]

The scale factor is a function of time and it measures the expansion of the universe. For instance, a distance, let us call it \( \tilde{D} \), at present time is related the same distance in the past, \( \tilde{D} \), in the following way

\[
D(t) = a(t)\tilde{D}.
\]

Since the universe is expanding this means that \( a(t) \) is a positive function in time. The fact that we have introduced the scale factor in the denominator of the first equality of equation (2) is what makes the particle horizon comoving, i.e. it is invariant under a change of scales.

In the last equality we find the denominator \( \frac{1}{aH} \) which is called the *comoving Hubble radius*. According to conventional Big Bang evolution, this factor increases with time and thus \( d_{\text{horizon}} \) increases with time (see more details in the Appendix). However, in the definition of inflation we are going to use we reject this notion.

5.2 Defining inflation in terms of a decreasing radius

We define inflation as the time period where the following holds true [5]

\[
\frac{d}{dt} (a(t)H(t))^{-1} < 0.
\]
We restrict the time, \( t \), to be \( t_{\text{start}} < t < t_{\text{end}} \) where \( t_{\text{start}} \) and \( t_{\text{end}} \) mark the start and end for inflation. Let us now discuss the consequences of this definition.

If one would insert equation (5) into equation (2) we immediately see that \( d_{\text{horizon}} \) decreases during inflation. However, as \( t \) grows we eventually encounter the situation where \( t > t_{\text{end}} \) and the comoving Hubble radius starts to increase again. This actually make up the inflation solution to the horizon problem. For instance, let us assume that two patches of the universe were in casual contact with each other BEFORE inflation, this would naturally explain the likeness of these patches. During inflation \( d_{\text{horizon}} \) decreases and these patches could be separated by a distance larger than \( d_{\text{horizon}} \) and thus cease to be in casual contact. Then at a later time, let us say present time, these patches could be observable in the sky and confuse the observing humans in their likeness even though they are not in causal contact. The point of this example is that even if patches of the sky are not in casual contact today they might have been in the past.

Moreover, we can see additional consequences from our definition for inflation. Let us compute the derivative for starters utilizing the definition of \( H \) presented in equation (3)

\[
\frac{d}{dt} (aH)^{-1} = -(aH)^{-2} \frac{d}{dt} (aH) = -(aH)^{-2} \frac{d^2}{dt^2} a < 0. \tag{6}
\]

From this we see that

\[
\frac{d^2 a}{dt^2} > 0, \tag{7}
\]

which indicates that during inflation there would be accelerated expansion of space.

Accelerated expansion has in turn another consequence. If we look at the following Friedmann equation\(^4\)

\[
\frac{1}{a} \frac{d^2 a}{dt^2} = -\frac{1}{6} (\rho + 3p), \tag{8}
\]

we see that we can plug in equation (7) to obtain

\[
\frac{d^2 a}{dt^2} = -\frac{a}{6} (\rho + 3p) > 0
\]

\[
\Rightarrow \frac{1}{3} \rho > p. \tag{9}
\]

This indicates that we also have negative pressure during inflation. In the section we will investigate what, exactly, exerts this pressure.

5.3 The inflaton field

Most inflation models include a scalar field\(^5\) \( \phi \), called the inflaton field.\(^5\)\(^6\). This field parametrizes the evolution of the energy density of the universe during

\(^4\)The Friedmann equations are derived from Einstein’s famous field equation by Alexander Friedmann using a slightly modified Minkowski metric ansatz. This modified metric is commonly known as the FRW metric and we will properly introduce it later on in the report.

\(^5\)Some models include more than one.
inflation. The corresponding action, \( S_\phi \), for the inflaton is

\[
S_\phi = \int d^4x \sqrt{-\text{det} g_{\mu\nu}} \left( \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right),
\]

where \( x \) are the coordinates for a four dimensional spacetime and \( g_{\mu\nu} \) is the spacetime metric with \( \mu, \nu = 0, 1, 2, 3 \). In this case we use the FRW metric \([4], [5]\)

\[
g_{00} = \eta_{00}, \\
g_{ij} = a^2 \eta_{ij},
\]

where \( i, j = 1, 2, 3 \) and \( \eta_{\mu\nu} \) is the flat Minkowski metric. In the action we also see a potential as a function of the inflaton, \( V(\phi) \). The shape of this potential varies depending on the model of inflation you use and its significance will become apparent further down in the report. For now it is sufficient to know that inflation corresponds to the inflaton "falling" down along the potential towards a minimum. The existence of a minimum is important for a process that takes place after inflation, namely reheating. We will not mention this process further but more information can be found in \([7]\).

From the action we can define the stress energy tensor \([5], [8]\)

\[
T^{(\phi)}_{\mu\nu} \equiv -\frac{2}{\sqrt{-\text{det} g_{\mu\nu}}} \frac{\delta S_\phi}{\delta g^\mu_{\nu}} = \partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \left( \frac{1}{2} g^{\lambda\sigma} \partial_\lambda \phi \partial_\sigma \phi + V(\phi) \right),
\]

where we have used

\[
\frac{\delta}{\delta g^\mu_{\nu}} \sqrt{-\text{det} g_{\mu\nu}} = -\frac{\sqrt{-\text{det} g_{\mu\nu}}}{2} g_{\mu\nu}.
\]

From now on we make use of an assumption and a choice. The assumption is that the inflaton is homogeneous across the spatial universe and thus only depend on time, \( \phi = \phi(t) \). Secondly, we choose to use a frame that is comoving with the field. This allows us to identify \([5]\)

\[
T_{00} = \rho, \\
T_{ii} = -a^2 p,
\]

where all other elements are zero and \( i = 1, 2, 3 \). Using equation (12) and (14) as well as the assumption \( \phi = \phi(t) \) we write

\[
\rho^{(\phi)} = \left( \frac{d\phi}{dt} \right)^2 + \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + V = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + V,
\]

\[
a^2 p^{(\phi)} = -a^2 \left( \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + V \right) = a^2 \left( \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 - V \right).
\]

Using these we can write the equation of state (defined in Appendix A)

\[
\omega^{(\phi)} = \frac{p^{(\phi)}}{\rho^{(\phi)}} = \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 - V \left( \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + V \right).
\]

This shows that the inflaton can lead to negative pressure and accelerated expansion if the potential term dominates over the kinetic term.
5.4 Slow-roll conditions

Using the action for the inflaton field we can obtain the equation of motion

$$\delta S_\phi = \frac{1}{\sqrt{-\det g_{\mu\nu}}} \partial_\mu (\sqrt{-\det g_{\mu\nu}} \partial^\mu \phi) + \frac{dV}{d\phi} = 0.$$ \hspace{1cm} (18)

This can be rewritten if we still assume $\phi = \phi(t)$

$$\frac{1}{a^3} \frac{d}{dt} \left(a^3 \frac{d\phi}{dt}\right) + \frac{dV}{d\phi} = 3H \frac{d\phi}{dt} + \frac{d^2\phi}{dt^2} + \frac{dV}{d\phi} = 0,$$ \hspace{1cm} (19)

where we have used $\sqrt{-\det g_{\mu\nu}} = \sqrt{a^3} = a^3$.

We can also obtain a new expression for $H$ by utilizing equation (103) and (15)

$$H^2 = \frac{1}{3} \left( \frac{1}{2} \left( \frac{d\phi}{dt} \right)^2 + V \right).$$ \hspace{1cm} (20)

Equation (19) and (20) are both going to be important for constructing inflationary conditions as we will now illustrate.

As we have shown above, inflation corresponds to accelerated expansion as well as negative pressure. We also argued that this correspond to a situation where

$$\left( \frac{d\phi}{dt} \right)^2 \ll V(\phi).$$ \hspace{1cm} (21)

Furthermore, if we want inflation to be sustained for an extended period we also require [5]

$$\left| \frac{d^2\phi}{dt^2} \right| \ll \left| 3H \frac{d\phi}{dt} \right| \quad \left| \frac{dV}{d\phi} \right|.$$ \hspace{1cm} (22)

These conditions make equation (19) and (20) take the following forms

$$3H \frac{d\phi}{dt} + \frac{dV}{d\phi} = 0,$$ \hspace{1cm} (23)

$$H^2 = \frac{V}{3}.$$ \hspace{1cm} (24)

Using equation (23) and (24) we can reiterate the conditions given in equation (21) and (22) by introducing so-called slow-roll parameters, $\varepsilon$ and $\eta$, and enforcing their smallness [3]

$$\varepsilon \equiv \frac{1}{2H^2} \left( \frac{d\phi}{dt} \right)^2 = \frac{1}{2} \left( \frac{1}{V} \frac{dV}{d\phi} \right)^2,$$ \hspace{1cm} (25)

$$\eta \equiv -\frac{1}{H} \frac{d^2\phi}{dt^2} \left( \frac{d\phi}{dt} \right)^{-1} + \frac{1}{2H^2} \left( \frac{d\phi}{dt} \right)^2 = \frac{1}{V} \frac{d^2V}{d\phi^2}.$$ \hspace{1cm} (26)
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where $|\eta|, \epsilon \ll 1$. Let us also, for good measure, write these parameters without natural units

$$
\epsilon = \frac{M_{\text{pl}}^2}{2} \left( \frac{\partial_\phi V}{V} \right)^2 \ll 1,
$$

$$
|\eta| = \left| \frac{M_{\text{pl}}^2}{V} \partial_\phi^2 V \right| \ll 1,
$$

where $\partial_\phi = \frac{d}{d\phi}$ and $M_{\text{pl}}$ is the Planck mass.

To conclude, these two conditions must be fulfilled in order for inflation to occur. We see that shape of the potential will be the deciding factor in this regard. Due to the fractions of the scalar potential and its derivatives, the inflationary region has to be very flat in order to fulfill these conditions.

### 6 String theory

In section 3 we introduced string theory in a very informal way. In this section we will give a somewhat more rigorous introduction and, more relevantly, define the objects we are going need from the theory in order to build models of inflation. This section is primarily based on the review [1] by Daniel Baumann and Liam McAllister.

To start things of, let us analyse the worldsheet introduced in section 3 embedded in a $D$ dimensional spacetime. A good way to describe this system is by introducing the corresponding action with no external forces [1],[6]

$$
S_P = -\frac{1}{4\pi \alpha'} \int_M d\tau d\sigma \sqrt{-\det \gamma^{ab}} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu,
$$

(28)

where $\alpha'$ is a constant related to the tension of the string (the tension is given by $T = \frac{1}{2\pi \alpha'}$), $M$ denotes the two dimensional worldsheet, $\tau$ and $\sigma$ are the coordinates of said worldsheet, $\gamma^{ab}$ is the metric on the worldsheet and $X^\mu$ denotes a scalar field which corresponds to the coordinates in spacetime. This action is called the Polyakov action, hence the subscript $P$. Notice that this action is similar to that of a free particle moving through flat spacetime.

The inflationary process we want to study in this report is, however, more complicated than a string moving through flat spacetime. As such, this action will not be sufficient for our purposes, but it does provide a good starting point.

#### 6.1 Energy scales in string theory

Before we move on further with our introduction of string theory it would be useful to consider the relevant energy scales. The energy scale in which string theory is developed on a fundamental level is the string scale which, as we mentioned, is different to the classical scale. This is the scale for which the Polyakov action is defined. We denote the scale with $M_s = \frac{1}{\sqrt{\alpha'}}$, which is proportional to the constant $\alpha'$. By considering energies below $M_s$, massive string states will not become excited and can therefore be integrated out, leaving only massless states in the theory. The theory reduces to a ten dimensional effective supergravity (see section 6.2 for more details on supersymmetry and supergravity).
Realizing inflation in string theory requires us to compactify the ten dimensional geometry on a six dimensional manifold (we discuss this in more details further down) which, consequently, introduces another scale below the string scale, $M_{KK}$, called the Kaluza-Klein scale. If the volume of the six dimensional space is given by $V$, then the Kaluza-Klein scale is proportional to $M_{KK} \approx \frac{M_s}{\sqrt[6]{V}}$.

We often have $M_{KK} \ll M_s$ which tells us that the energy scale between $M_{KK}$ and $M_s$ corresponds to a ten dimensional effective supergravity (SUGRA) according to the reasoning in the paragraph above.

Using $M_{KK}$ and $M_s$ as well as the string coupling $g_s$ it is possible to schematically derive the four dimensional Planck scale, $M_{pl}$,

$$M_{pl} \sim g_s^{-1} \frac{M_s^4}{M_{KK}}.$$  \hspace{1cm} (29)

The value of $g_s$ is determined by the vacuum expectation value of a scalar field called the dilaton, we will introduce it in a later section. $M_{pl}$ is much larger than $M_s$.

For our purposes it is helpful to introduce yet another scale, $M_{SUSY}$, the scale in which supersymmetry (SUSY) breaks. The relation between $M_{SUSY}$ and the other scales are dependent on the situation one considers. For instance, if one would consider a situation without supersymmetric compactification we have $M_{SUSY} \geq M_{KK}$ since supersymmetry is broken at compactification scales or higher.

The energy scale at inflation, $H$, (probed by the cosmic microwave background) is related to $M_{SUSY}$ according to $M_{SUSY} < H$. This hierarchy is due to the desire to only have supersymmetry breaking during inflation, not before. This is essential for most models of string inflation.

On the other side we prefer $H < M_{KK}$. This indicate that the scale of $H$ is that of four dimensional effective theories. Note that this does not necessarily mean that the supersymmetry from $M_{KK}$ is lost. The energy scale hierarchy can thus be presented as follows

$$M_{SUSY} < H < M_{KK} < M_s < M_{pl}.$$  \hspace{1cm} (30)

### 6.2 Superstrings and supergravity

One limitation of equation (28) is the fact that $S_P$ only considers bosonic strings whereas we are going to need fermionic ones as well. In order to obtain an action that is sufficient for our purposes we need to enter the realm of supersymmetry, or more specifically, $N = 1$ supersymmetry in four dimensions.

The simplest and most straightforward way to introduce fermions is by adding a fermion term, $S_F$, to equation (28),

$$S \equiv S_P + S_F = -\frac{1}{4\pi\alpha'} \int_M d^4\sigma \sqrt{-\det g} \gamma^{ab} \partial_a X^\mu \partial_b X_\mu - i \bar{\psi}^\mu \rho^a \partial_a \psi_\mu.$$  \hspace{1cm} (31)

Here $\psi^\mu$ is a Dirac spinor and $\rho^a$ are two dimensional matrices that satisfy the so-called Clifford algebra. The algebra in question is given by

$$\{\rho^a, \rho^b\} = -2\eta^{ab}.$$  \hspace{1cm} (32)
The \( \psi^\mu \) act as the fermions of the theory. They can be written in terms of their two components

\[
\psi^\mu = \begin{bmatrix} \psi^\mu_- \\ \psi^\mu_+ \end{bmatrix}.
\]

Putting equation (33) inside equation (31) together with \( \partial^\pm \equiv \frac{1}{2}(\partial_\tau \pm \partial_\sigma) \) we obtain

\[
S_F = \frac{i}{2\pi\alpha'} \int d\tau d\sigma \left[ \psi^\mu_+ \partial_\mu \psi^\nu_- + \psi^\mu_- \partial_\mu \psi^\nu_+ \right] \eta_{\mu\nu},
\]

where we have assumed conformal gauge, \( \gamma_{ab} \to \eta_{ab} \). In this way the action is separated into left and right going fermions as is indicated by the + or - subscript. To get an intuitive understanding of the situation one can think of the fermions as excitations that travels either to the left or to the right along the string.

There is one particular case we need to consider in more detail, the case where the string is closed. In this instance we have to specify the periodicity of the fermions as they travel across the string worldsheet. We define sectors for this purpose: the Ramond sector (R) with periodicity \( \psi^\mu_\pm(\sigma + \pi) = \pm \psi^\mu_\pm(\sigma) \) and the Neveu-Schwarz sector (NS) with periodicity \( \psi^\mu_\pm(\sigma + \pi) = -\psi^\mu_\pm(\sigma) \). The choice for left moving fermions is independent from right moving fermions and as such there are in total four sectors: R-R, R-NS, NS-R and NS-NS.

There is one more condition we have to enforce in order for equation (34) to be a consistent theory with closed string fermions, the so-called GSO projection \([1],[11]\). These projections lead to two different types of string theories, type IIA and IIB. In IIA one performs opposite projections in the R-NS and NS-R sectors while in IIB one performs identical projections in these sectors.

There are more types of consistent string theories in addition to those mentioned in the paragraph above: type I, \( SO(32) \) heterotic and \( E_8 \times E_8 \) heterotic \([1]\). In this report we are only interested in type IIB string theory so we will not define the other types. However, every type are related to each other through dualities in an overarching theory called \( M\)-theory.

### 6.2.1 Type IIB supergravity

Let us now investigate SUSY string theories at energies below the fundamental string scales, \( E < M_s \), and we start off by considering the non-SUSY perspective.

First of all, we eventually want to quantize the action for the string in order to incorporate quantum field theory into our theory. However, we require that this quantum field theory is Weyl invariant. Weyl invariance, on the other hand, is obtained when the trace of the stress-energy tensor corresponding to the considered action is zero \([6]\).

Let us look at a concrete example. Equation (28) can be generalized by including backgrounds of other massless string states to give \([1],[6]\)

\[
S_\sigma = -\frac{1}{4\pi\alpha'} \int d\sigma d\tau \sqrt{-\det \gamma_{ab}} \left[ \gamma^{ab} G_{MN}(X) + \epsilon^{ab} B_{MN}(X) \right]
\cdot \partial_a X^M \partial_b X^N + \alpha' \Phi(X) R(\gamma),
\]

\[(35)\]

\(\text{6}\)There are several reasons for why this projection is necessary. For example, the theory would otherwise contain an unwanted tachyon and also have anticommuting operators that map bosons to bosons \([10]\).
where $G_{MN}$ is the metric, $\Phi$ is a scalar called the dilaton, $\varepsilon^{ab}$ is the Levi-Civita symbol, $B_{MN}$ is an antisymmetric tensor and $R$ is the Ricci scalar constructed from $\gamma$. If one expands the background fields around a point one obtains interaction terms. As such this action describes an interacting quantum field theory \[1\].

The trace of the stress-energy tensor is then \[6\]

\[T^a_a = -\frac{1}{2\alpha'} \beta^G_{\mu\nu} \gamma^{ab} \partial_a X^\mu \partial_b X^\nu - \frac{i}{2\alpha'} \beta^B_{\mu\nu} \varepsilon^{ab} \partial_a X^\mu \partial_b X^\nu - \frac{1}{2} \beta^\Phi R,\]

where

\[\beta^G_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha' \nabla_{(\mu} H_{\nu)\lambda} + \mathcal{O}(\alpha'^2),\]

\[\beta^B_{\mu\nu} = -\frac{\alpha'}{2} \nabla^2 H_{\mu\nu} + \alpha' \nabla^a \Phi H_{a\mu\nu} + \mathcal{O}(\alpha'^2),\]

\[\beta^\Phi = \frac{D-26}{6} - \frac{\alpha'}{2} \nabla^2 \Phi + \alpha' \nabla^a \Phi \nabla^a \Phi - \frac{\alpha'}{24} H_{\mu\nu\lambda} H^{\mu\nu\lambda} + \mathcal{O}(\alpha'^2),\]

where the three-form $H$ is the field strength of $B_{MN}$, i.e., $H_3 = dB_2$, and $D$ is the dimension. Following the argument we require that these $\beta$ to vanish for Weyl invariance. The resulting field equations can be derived from the following spacetime action \[6\]

\[S_B = \frac{1}{2\kappa^2_D} \int d^D X \sqrt{-\det G_{MN}} e^{-2\Phi} \left( R + 4(\Phi)'^2 - \frac{1}{2} |H_3|^2 - \frac{2(D-26)}{3\alpha'} + \mathcal{O}(\alpha') \right).\]

The constant $\kappa_D$ is a coupling constant related to the tension of the string. This action parametrizes massless interactions of bosonic strings.

Notice the special case where $D = 26$. Not only does this case make the last term in the action vanish but it is also the only case where the action enjoy non-anomalous symmetries \[12\]. These symmetries being Weyl invariance and two-dimensional diffeomorphisms.

When we consider SUSY, and consequently consider fermions, the action given by equation \[38\] ceases to be accurate. We require additional bosonic and fermionic fields, the former of which arises from the NS-NS and R-R sectors while the latter arises from NS-R and R-NS sectors \[1\].

Now, considering type IIA and type IIB string theory in the NS-NS sector in ten dimensions we again find $G_{MN}$, $\Phi$ and $B_2$ whose spacetime action is

\[S_{NS} = \frac{1}{2\kappa^2} \int d^{10} X \sqrt{-\det G_{MN}} e^{-2\Phi} \left( R + 4(\Phi)'^2 - \frac{1}{2} |H_3|^2 \right),\]

which is similar to equation \[38\]. In this case we can explicitly write the constant $2\kappa^2$ as $2\kappa^2 = (2\pi)^7 (\alpha')^4$ \[6\]. However, as stated above these fields are not sufficient when considering SUSY. In type IIB string theory, which is the type
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we focus on from now on, we have three additional fields: a zero-form $C_0$, a two-form $C_2$ and a four-form $C_4$. The bosonic part of the total low energy action for type IIB string theory is instead [1]

$$S_{IIB} = S_{NS} - \frac{1}{4\kappa^2} \left( \int d^{10}X \sqrt{-\det G_{MN}} \left[ |F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2} |\tilde{F}_5|^2 \right] + \int C_4 \wedge H_3 \wedge F_3 \right),$$

(40)

where $F_1 = dC_{i-1}$, $\tilde{F}_3 = F_3 - C_0 \wedge H_3$ and $\tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3$. We also impose a self duality condition on $\tilde{F}_5$

$$\tilde{F}_5 = \ast \tilde{F}_5.$$  

(41)

Note that this condition does not naturally follow from the action $S_{IIB}$ [11], in this sense this action could be considered a "pseudo-action". Also note the gauge invariance $C \rightarrow C + dL$. The action (40) is the one we will use going forward.

7 Building a four dimensional EFT from string theory

The supersymmetric string theory is a theory that only permits a ten dimensional spacetime [7]. This is not a problem until one tries to use the theory to describe our perceived reality in which we only experience four dimensions. In tackling cosmological subjects, such as inflation, one does so in four dimensions and in this report we follow this trend. Thus we have to create an effective field theory (EFT) in four dimension from the ten dimensional SUSY string theory. In order to construct this EFT one needs to somehow deal with the extra six dimensions.

7.1 Compactification

We now introduce the procedure called compactification. When one compactifies one separate the ten dimensional geometry into a four dimensional, observable part and a six dimensional non-observable part. This non-observable part is considered "small" in the sense that any translation in this direction is completely undetectable at low energies. Conveniently one writes the ten dimensional geometry used in SUSY string theory in terms of the four dimensional perceived part and a six dimensional compact part in the following way [11]

$$\mathcal{M}_{10} = \mathcal{M}_4 \times \mathcal{M}_6,$$

(42)

where $\mathcal{M}_4$ is the usual four dimensional spacetime and $\mathcal{M}_6$ is a compact six dimensional manifold. We say that one compactifies string theory on $\mathcal{M}_6$.

---

[7] This is due to charges. In SUSY string theory the central charge given by scalars and fermions is given by $c = \frac{3}{2} D$ where $D$ is the dimension. The ghost central charge, on the other hand, is $-15$. In order for the total central charge to vanish we then have $c = 0 = \frac{3}{2} D - 15$ which gives us the dimension $D = 10$. For more details on why the charge must vanish see [5] and [11].

[8] "Our" as in humankind.
A reasonable ansatz for the line element for the geometry in equation (42) would be [1], [13]

$$ds^2_{10} = G_{MN}dX^M dX^N = e^{2A(y)}g_{\mu\nu}dx^\mu dx^\nu + e^{-2A(y)}\tilde{g}_{mn}dy^m dy^n,$$

where $N,M = 0,\ldots,9$, $\mu,\nu = 0,\ldots,3$ and $m,n = 4,\ldots,9$. The function $A(y)$ is a function on the compact space and is called the warp factor and as the name suggests it measures warping of the total manifold. There could be many reasons for this warping such as the charge or stress energy of fields in the action apart from the metric or the stress energy of other objects entirely such as the so-called branes or orientifold planes. A totally flat solution would have $A(y) = 0$ and $g_{\mu\nu} = \eta_{\mu\nu}$.

### 7.1.1 Moduli

A consequence of doing compactification is that your resulting theory will obtain a certain amount of extra scalar fields called moduli [1]. Mathematically, these moduli parametrize geometrical deformations and the amount of moduli obtained is dependant on the choice of $M_6$. The moduli appear in the action of the four dimensional EFT and we will illustrate this following an example given in [1].

We start from the following geometry

$$G_{MN}dX^M dX^N = e^{-6u(x)}g_{\mu\nu}dx^\mu dx^\nu + e^{2u(x)}\tilde{g}_{mn}dy^m dy^n. \, (44)$$

The exponential factors are not warp factors in this case as can be seen from their $x$ dependence compared to the previous $y$ dependence. Instead they represent fluctuation of the internal metric that changes the volume of $M_6$. Now, consider the ten dimensional Einstein-Hilbert action

$$S_{EH} = \frac{1}{2\kappa^2} \int d^{10}X \sqrt{-\det G_{MN}} e^{-2\Phi} R_{10}, \, (45)$$

where $R_{10}$ is the Ricci scalar constructed from $G_{MN}$. Using equation (44) we could rewrite $R_{10}$ in terms of $R_4$ and $R_6$, i.e. the Ricci scalars constructed from $g_{\mu\nu}$ and $\tilde{g}_{mn}$. This would result in the following action

$$S_{EH} = \frac{1}{2\kappa^2} \int d^4x \sqrt{-\det g_{\mu\nu}} \int d^6y \sqrt{\det \tilde{g}_{mn}} e^{-2\Phi} (R_4 + e^{-8u}R_6 + 12\partial_{\mu}u \partial^\mu u). \, (46)$$

Note that we obtained an additional term in the integrand dependant on $u$. This corresponds to the kinetic term for the new scalar field $u$, which is what we call a modulus. This is one example of how one can obtain moduli in one’s theory through compactification.

### 7.2 Calabi-Yau compactification

It is very common to use a Calabi-Yau manifold as $M_6$ for compactification [1], [3], [13], [14], [15], [16], [17]. The defining property of a Calabi-Yau manifold is that it is Ricci flat in addition to the properties belonging to so-called Kähler manifolds [18]. In this section we go through some useful aspects of Calabi-Yau
compactification and how some different Calabi-Yau differ from one another in this context.

As illustrated in the previous section, we obtain moduli when doing a compactification. The moduli that appear in Calabi-Yau compactifications are called Kähler moduli and complex structure moduli [19].

The Kähler moduli appear as deformations of the Kähler form,

\[ J = i\tilde{g}_{ab} dy^a \wedge dy^b, \]  

where \( \tilde{g}_{ab} \) is the metric on \( M_6 \) and \( y \) its coordinates. By choosing an appropriate basis for \( J \) we can rewrite it in terms of the Kähler moduli

\[ J = t^I(x) \omega_I, \]  

where \( t^I \) are the moduli and \( \omega_I \) is an introduced set of harmonic \((1,1)\)-forms. The index \( I \) consists of integers, \( I = 0, \ldots, h^{1,1} \), where the Hodge number, \( h^{1,1} \), indicate the number of Kähler moduli. The Kähler form can also be used to express the compactification volume

\[ V = \frac{1}{6} \int_{M_6} J \wedge J \wedge J = \frac{1}{6} c_{abc} t^a t^b t^c, \]  

where \( c_{abc} \) denotes the intersection numbers of \( M_6 \) and is therefore dependent on the particular choice of \( M_6 \). Only the real part of the Kähler moduli contribute to the volume (will see concrete examples of this later on in the report), the imaginary part are called axions.

Axions are defined by their Peccei-Quinn shift symmetry [1]:

\[ \text{Im}(t) \rightarrow \text{Im}(t) + \text{constant}. \]  

The models we are going to consider does not make use of the shift symmetry of the imaginary part of the Kähler moduli but there are other models of string inflation that does. This makes axions relevant and interesting for cosmology, specifically inflation. We discuss axions a bit more in section 11.

Moving on, a similar constructions to the Kähler moduli can be made for the complex structure moduli, which are deformation on the complex structure on \( M_6 \),

\[ \delta g_{qr} = -\frac{3i}{\Omega_{abc} \Omega_{def}} \zeta^A(x)(\chi_A)_{qbc} \Omega^{def}, \]  

where \( \zeta^A \) are the complex structure moduli, \( \chi_A \) is a set of harmonic \((1,2)\)-forms and \( \Omega \) is a characteristic \((3,0)\)-form that is dependent on the Calabi-Yau in question. Similarly to \( I \), the index \( A \) consist of integers, \( A = 0, \ldots, h^{1,2} \), where \( h^{1,2} \) is the number of complex structure moduli.

Both \( h^{1,1} \) and \( h^{1,2} \) are dependent on the choice of the particular Calabi-Yau chosen for compactification. As will be shown below, these numbers affect the resulting EFT.

### 7.3 Super-, Kähler and scalar potentials

As was discussed in earlier sections, the scalar potential should fulfil the slow-roll conditions for inflation to take place. Therefore we require a scalar potential
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in our EFT that we can test for this purpose. This scalar potential will be obtained by using the F-term potential formula \[20\]

\[V_F = e^K \left( K^{I\bar{J}} D_I W D_{\bar{J}} W - 3|W|^2 \right), \tag{52}\]

where \(K\) is called the Kähler potential, \(K^{I\bar{J}}\) is the inverse of the Kähler metric \((K^{I\bar{J}} = \partial_I \partial_{\bar{J}} K)\) and \(W\) is the superpotential. The derivative is defined as

\[D_I W = \partial_I W + W \partial_I K, \tag{53}\]

and the indices run over all moduli as well the axio-dilaton \(\tau\), given by \[1], \[13\]

\[\tau \equiv C_0 + i e^{-\Phi}. \tag{54}\]

If the derivatives in equation \(52\) vanish for all \(I\) then there is a SUSY-preserving vacuum solution of the theory. Note that the expressions above are only valid for four dimensional, \(\mathcal{N} = 1\) theories.

As we can see, we require two objects in order to obtain a scalar potential: \(W\) and \(K\) as functions of the inflaton.

### 7.3.1 Kähler potential

We will find and motivate the expression for \(W\) that we are going to use in the next subsection, the expression for \(K\), however, is something we present here without further derivation. It was derived by E. Witten in \[21\] when he compactified a ten dimensional SUGRA on a general Calabi-Yau. It takes the following form:

\[K = -2 \ln(V) - \ln(-i(\tau - \bar{\tau})) - \ln \left( -i \int \Omega \wedge \bar{\Omega} \right), \tag{55}\]

where \(V\) is the compactification volume, \(\Omega\) is the characteristic \((3,0)\)-form of the compact space. An important aspect of this \(K\) is that it has a no-scale structure. This means that

\[\sum_{I,J} K^{I\bar{J}} D_I W D_{\bar{J}} \bar{W} = 3|W|^2, \tag{56}\]

where \(T_i\) are the Kähler moduli. This, in turn, means that vanishing derivatives in the scalar potential results in \(V_F = 0\). Furthermore, since in this case the potential is positive semi-definite, \(V_F = 0\) corresponds to a minimum. Also, the vanishing derivatives does not automatically preserve SUSY.

### 7.3.2 Superpotential

In this section we will find an appropriate expression for \(W\). To carry out this task we again take a look at the type IIB action for the SUSY string given by equation \[40\]. By rewriting and adding a term corresponding to local stress-energy sources, \(S_{loc}\), the action can be expressed as \[8\]

\[S_{IIB} = \frac{1}{2\kappa^2} \int d^{10}X \sqrt{-\det \hat{G}_{MN}} \left[ \hat{R} - \frac{\partial_M \tau \partial^M \tau}{2(\text{Im}(\tau))^2} - \frac{G_3 \cdot \bar{G}_3}{12 \text{Im}(\tau)} - \frac{\tilde{F}_5^2}{4 \cdot 5!} \right] + \frac{1}{8i\kappa^2} \int \frac{C_4 \wedge G_3 \wedge \bar{G}_3}{\text{Im}(\tau)} + S_{loc}, \tag{57}\]
where we have introduced
\[ \hat{G}_{MN} \equiv e^{\frac{\Phi}{2}} G_{MN}, \]
\[ G_3 \equiv F_3 - \tau H_3. \]  
(58)

We restrict \( \tau \) and thus \( G_3 \) to only be dependent on the compact space.

Furthermore we need an explicit metric. We will choose the ten dimensional metric given by equation (43) with a flat four dimensional component \( (g_{\mu \nu} = \eta_{\mu \nu}) \).

Additionally, the self-duality of \( \tilde{F}_5 \) allows us to assume the simple ansatz
\[ \tilde{F}_5 = (1 + \tau ) [d \alpha \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3], \]  
(59)

where \( \alpha \) is a function of \( y \).

With the action and the metric at hand we can calculate additional objects. First and foremost we obtain the stress-energy tensor in the same way as in equation (12), i.e.
\[ T_{MN} = -\frac{2}{\sqrt{-\det G_{MN}}} \delta S_{IIB} / \delta G_{MN}. \]  
(60)

The stress-energy tensor can in turn be used in the trace reversed Einstein equation to obtain the corresponding Ricci tensor
\[ R_{MN} = \kappa^2 \left( T_{MN} - \frac{1}{8} G_{MN} T_M^M \right). \]  
(61)

Inserting equation (57) into equation (60) and putting the result into equation (61) with the restriction that we only are interested in the non-compact component \( (M,N \rightarrow \mu, \nu) \) gives us (see the Appendix for details)
\[ R_{\mu \nu} = e^{2A} \eta_{\mu \nu} \left( G_{mnp} \bar{G}^{mnp} \frac{12i \tau}{8} \delta S_{IIB} / \delta G_{MN} + \frac{e^{-8A}}{4} \partial_m \alpha \partial^m \alpha \right) + R_{\mu \nu}^{loc}. \]  
(62)

The Ricci tensor can, of course, also be calculated directly from the Riemann tensor using the chosen geometry (see the Appendix for details)
\[ R_{\mu \nu} = -\eta_{\mu \nu} e^{4A} \nabla^2 A = -\frac{1}{4} \eta_{\mu \nu} \left( \nabla^2 e^{4A} - e^{-6A} \partial_m e^{4A} \partial^m e^{4A} \right), \]  
(63)

where the ’\( \sim \)’ denotes the use of the metric of the compact space, see equation (43). Putting the two different expressions for the Ricci tensor equal to each other and solving for \( \nabla^2 e^{4A} \) gives us
\[ \nabla^2 e^{4A} = e^{2A} \frac{G_{mnp} \bar{G}^{mnp}}{12i \tau} + e^{-6A} \left[ \partial_m \alpha \partial^m \alpha + \partial_m e^{4A} \partial^m e^{4A} \right] \]
\[ + \frac{\kappa^2 e^{2A}}{2} (T_m - T_\mu)_{\mu \nu}^{loc}. \]  
(64)

Let us leave the Ricci tensor for a little while and take a deeper look into \( \tilde{F}_5 \). We first defined \( \tilde{F}_5 \) in equation (40) as
\[ \tilde{F}_5 = F_5 - \frac{1}{2} C_2 \wedge H_3 + \frac{1}{2} B_2 \wedge F_3 \]
\[ = dC_4 - \frac{1}{2} C_2 \wedge dB_2 + \frac{1}{2} B_2 \wedge dC_2, \]  
(65)
Applying an exterior derivative to this expression and adding a source term, $2\kappa^2 U^\text{loc}$, which is a six-form, gives (remember that $dd = 0$)

$$d\hat{F}_5 = -\frac{1}{2} dC_2 \wedge H_3 + \frac{1}{2} dB_2 \wedge dC_2 + 2\kappa^2 U^\text{loc}$$

or

$$= \frac{1}{2} (H_3 \wedge F_3 + \frac{1}{2} H_3 \wedge F_3 + 2\kappa^2 U^\text{loc}) = H_3 \wedge F_3 + 2\kappa^2 U^\text{loc}.$$  \hspace{1cm} (66)

Replacing $\hat{F}_5$ with equation (59) and solving for $\hat{\nabla}^2 (e^{4A} - \alpha)$ gives (see the Appendix for details)

$$\hat{\nabla}^2 (e^{4A} - \alpha) = \frac{e^{2A}}{6i\kappa^2} (iG_3 - *_6 G_3) + e^{-6A} \partial_m \alpha \partial^m e^{4A} + 2\kappa^2 e^{2A} U^\text{loc},$$

where $*_6$ is the six-dimensional Hodge star.

Now, at last we subtract equation (67) from equation (64)

$$\hat{\nabla}^2 (e^{4A} - \alpha) = \frac{e^{2A}}{6i\kappa^2} (iG_3 - *_6 G_3) + e^{-6A} \partial_m \alpha \partial^m e^{4A} + 2\kappa^2 e^{2A} U^\text{loc},$$

(68)

We can draw several conclusions from this result. Assuming that $\frac{1}{4} (T^m_m - T^\mu_\mu)^\text{loc} \geq U^\text{loc}$ we see that the RHS is non-negative. We also see that the LHS integrates to zero due to the divergence theorem. Together, these observations forces each term in the RHS to be zero independently from one another. Thus we obtain the following conditions given the mentioned inequality

$$iG_3 = *_6 G_3,$$

$$e^{4A} = \alpha,$$

$$U^\text{loc} = \frac{1}{4} (T^m_m - T^\mu_\mu)^\text{loc}.$$  \hspace{1cm} (69)

A configuration that obeys these conditions is called a imaginary self-dual (ISD) solution since $G_3$ is imaginary self-dual [1].

The ISD conditions can be obtained from the following, so-called, Gukov-Vafa-Witten superpotential [22]

$$W_0 = \frac{c}{4} \int G_3 \wedge \Omega,$$  \hspace{1cm} (70)

where $c$ is a constant. This is not quite the superpotential we want to use in equation (52), we also want correction terms.

7.3.3 Corrections to $K$ and $W$

When taking quantum mechanics into account we (potentially) receive both perturbative and non-perturbative corrections to the Kähler potential and the superpotential. As we will see in Part III, corrections to the superpotential are beneficial in order to construct inflation scenarios.

We start with briefly mentioning the corrections to the Kähler potential. The Kähler potential will receive perturbative corrections from quantum effect,
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however these corrections will be ignored when considering inflation. The resulting Kähler potential in a four dimensional EFT is \[ K = -2 \ln \left[ V - \frac{\chi(M_6)\zeta(3)}{4g_s^2(2\pi)^3} + O \left( \frac{1}{V} \right) + O \left( \frac{1}{g_s} \right) \right], \tag{71} \]

where \( \zeta(3) \) is Apéry’s constant and \( \chi(M_6) \) corresponds to the Euler characteristic of \( M_6 \). Based on this expression we see in which situation the corrections can be ignored, the case where \( V \) is sufficiently large. However, the strength of the string coupling could make the situation more complicated in the higher order terms. For instance strong coupling could make higher orders large enough to contribute, i.e. make them non-perturbative. We will not go into the complexity of the situation but rather settle with stating that henceforth the volume \( V \) will be sufficiently large and the string coupling will be sufficiently weak, for more details see \[23\].

More relevant to this report are the corrections to the superpotential. \( W_0 \) and its corrections will together constitute \( W \). However, \( W \) receives no perturbative corrections but rather non-perturbative ones \[24\]. This is because \( W \) is a holomorphic function and perturbative corrections would give a non-holomorphic contribution.

It can be shown \[23, 26, 27, 28, 29, 30\] that the non-perturbative part with only one Kähler modulus can be generated through gaugino condensation in a gauge group of rank \( N \) (such as \( SU(N) \)) following a certain EFT

\[ W_{np} = Ae^{-\frac{2\pi T}{N}} = Ae^{-aT}, \tag{72} \]

where the prefactor \( A \) generally depend on all moduli except the Kähler one and \( T \) is the Kähler modulus. Superpotentials that makes use of this type of corrections will henceforth be called \( KKLT \), named after the authors (Kachru, Kallosh, Linde and Trivedi) who proposed that these kind of non-perturbative corrections could be used to construct dS vacua in string theory \[31\]. If we would change the gauge group to a product of gauge groups the perturbation could instead take the form

\[ W_{np} = Ae^{-\frac{2\pi T}{N}} + Be^{-\frac{2\pi M}{M}} = Ae^{-aT} + Be^{-bT}, \tag{73} \]

where \( N, M \) are the ranks of the individual gauge groups in the product.

The restriction to only one Kähler modulus is not general, non-perturbative correction(s) can be obtained for Calabi-Yau with \( h^{1,1} > 1 \). Similar to the expressions above, these correction could look like\[7\]

\[ W_{np} = \sum_{i=1}^{h^{1,1}} A_i e^{-\frac{2\pi x_i}{N}}. \tag{74} \]

### 7.4 Using proper energy scales

There is one more ingredient we require before we start building inflation models. If we were to insert the \( K \) presented above along with \( W_0 \) and its corrections into

\[ \text{There are some considerations regarding the assumption that there exist a corrections term for each Kähler modulus. It is unknown whether this is valid in general.} \]
the scalar potential we would encounter technical problems. There are many terms and many moduli, each of which could represent inflaton fields. It would be preferable to streamline the problem in order to avoid heavy calculations. Fortunately, by constraining the calculation to a certain energy scale things simplify, as we will explain below.

To start things off we again take a closer look at the superpotential given by equation (70). This superpotential is dependent on both \( \tau \) and the complex structure moduli through the definition of \( G_3 \) and \( \Omega \) respectively, but Kähler moduli do not appear. This means that both the complex structure moduli and \( \tau \) receive supersymmetric masses whilst the Kähler moduli do not. Let us call the energy scale of these masses \( m_{\text{class}} \). If we restrict our calculations to scales \( E \ll m_{\text{class}} \) then there is no way to excite the modes of \( \tau \) or the complex structure moduli beyond the ground level. Effectively, these objects become constants and can be integrated out. Consequently, \( W_0 \) also becomes a constant.

This in turn would mean that the only non-zero contribution from the derivatives of \( W \) in the scalar potential, \( V_F \), defined in equation (52) would come from the non-perturbative correction terms, hence their importance.

Furthermore, this choice of scales allows us to make alterations to equation (55). The complex structure moduli as well as \( \tau \) are fixed at \( D_I \tau = 0 \) meaning that the only dynamics fields that are left are the Kähler moduli. We are only interested in this part of moduli space and can therefore omit the constant terms.
Part III
Constructing inflation models

In the following sections we will make use of the expressions we introduced in the previous part in order to construct viable string inflation models. In particular we will use the F-term potential introduced in equation (52) along with all mentioned assumptions and observe whether or not it contains regions which fulfill the conditions presented in equation (27). We will also utilize several numerical calculations and graphs in this endeavour. All of these come from scripts and functions constructed in Matlab. These scripts and functions can all be found in the Appendix.

The models we will look at in part were originally constructed by J.J. Blanco-Pillado et al. in [2] and [3], we reproduce their results. As such we will use the same numerical values for our parameters as they did.

8 Inflation in the KKLT and Racetrack scenario

An explicit example of inflation with Kähler moduli as inflaton can be seen in a modified KKLT scenario, namely the so-called Racetrack scenario. For comparison we will also consider the unmodified KKLT scenario. To make things as simple as possible we only focus on Calabi-Yau manifolds with a single Kähler modulus in this section. This modulus will behave as an inflaton in the four dimensional theory.

We consider the following scalar potential [2]

\[ V = V_F + V_{\text{uplift}}, \]  

(75)

where \( V_F \) is the F-term potential and \( V_{\text{uplift}} \) is an uplift term, this term comes about by including anti-D3 branes. More specifically, the tension of said branes generate the positive valued \( V_{\text{uplift}} \). The purpose if this inclusion is to "uplift" the negative valued minimum of \( V_F \) to a positive value, thus making it so that \( V \) obtains a positive minimum. This corresponds to uplifting a AdS vacuum to a dS vacuum. A consequence of including \( V_F \) is that it necessarily breaks supersymmetry, cf. with section 7.3.

It should be noted that we are making an assumption by including this uplifting term. We are assuming that they work as intended, which is something that has been criticised lately [32], [33], [34], [35], [36], [37], [38], [39], [40]. The critique are levered against the stability of the anti-D3 brane as well as the backreaction of moduli after SUSY breaking. We will not partake in this discussion in this report but instead use anti-D3 branes without prejudice for simplicity. We are aware, however, that this might not be physically acceptable.

As stated before, there are two ingredients needed to attain an explicit scalar potential, which are \( W \) and \( K \), and now we require another one, \( V_{\text{uplift}} \). The last two of these are identical in both the KKLT and Racetrack scenarios, the first one differs somewhat.

The Kähler potential used is the first term of equation (55). Since we are only considering a single Kähler modulus we know the explicit expression for
The compact volume \[ V = (T + \bar{T})^2, \] where \( T \) is the Kähler modulus. This volume leads to the following Kähler potential

\[ K = -3 \ln(T + \bar{T}). \] (77)

The Kähler modulus can be split into a real and imaginary part, \( T = X + iY \), which makes the Kähler potential take the following form

\[ K = -3 \log(2X). \] (78)

We see that \( K \) is independent of the imaginary part of \( T \).

Moving on, \( V_{\text{uplift}} \) is given by \[ V_{\text{uplift}} = \frac{E}{X^\alpha}, \] (79)

where \( E \) is dependent on the warp factor as well as the tension of the anti-D3 brane. The exponent \( \alpha \) is either 2 or 3 depending on the placement of the brane. In the case where the brane is located at the end of the Calabi-Yau throat we have \( \alpha = 2 \) whereas \( \alpha = 3 \) correspond to the unwarped region. We will use \( \alpha = 2 \) as this is energetically preferable when using anti-D3 branes as uplifting objects \[.\]

Finally we are left with \( W \). The expression for \( W \) varies depending on whether or not we are considering unmodified KKLT or Racetrack, as stated above. The expression for KKLT is

\[ W_{\text{KKLT}} = W_0 + Ae^{-aT} \] (80)

and for Racetrack we have

\[ W_{\text{RT}} = W_0 + Ae^{-aT} + Be^{-bT}, \] (81)

By combining equation (75), (77) or (78), (79) as well as (80) or (81) we obtain the scalar potential as a function of \( T \) (or equivalently: \( X \) and \( Y \)) along with a whole host of parameters. This ordeal is straightforward but somewhat tedious, we do it in the Appendix. A notable assumption we make here is that the parameters are real, this is not true in general. We make this assumption for simplicity but later on in the report we will make use of complex parameters and thus alter the potential accordingly. The result is as follows for the KKLT potential

\[ V_{\text{KKLT}} = \frac{aAe^{-aX}}{6X^2} [Ae^{-aX}(aX + 3) + 3W_0 \cos(aY)]. \] (82)

The result for the Racetrack potential is

\[ V_{\text{RT}} = \frac{E}{X^2} + \frac{e^{-aX}}{6X^2} [aA^2e^{-aX}(aX + 3) + 3W_0aA \cos(aY)] + \frac{e^{-bX}}{6X^2} [bB^2e^{-bX}(bX + 3) + 3W_0bB \cos(bY)] + \frac{e^{-(a+b)X}}{6X^2} [AB \cos((a - b)Y)(2abX + 3a + 3b)]. \] (83)
8.1 Identifying viable regions for slow-roll inflation

With the scalar potentials at hand we are able to identify whether or not they are viable for inflation. We check this by observing if there are some values of $T$ that make the potentials fulfil the slow-roll conditions presented in equation (27). Remember that the regions of the potentials need to be very flat for these conditions to be fulfilled. By choosing certain parameters and plotting these potentials against $T$ we are able to see if such regions exists, see figure 5 and 6 for these graphs.

![Figure 5: The scalar potential in the KKLT scenario plotted against the real and imaginary part of $T$. The following values are used for the parameters: $A = \frac{1}{50}$, $a = \frac{2\pi}{100}$, $B = \frac{-35}{1000}$, $b = \frac{2\pi}{90}$, $W_0 = \frac{-1}{25000}$, $E = 4,14668 \cdot 10^{-12}$, $\alpha = 2$. The $V$-axis is scaled with a factor of $10^{14}$.](image)

We immediately see a difference in shape between the graphs, the KKLT potential simply "falls" as the real part of $T$ increases whereas the Racetrack potential contains a saddle point. This saddle point is located at $X = 123.22$, $Y = 0$ and at this point the value of the potential is $V_{\text{RT,saddle}} = 1.6549 \cdot 10^{-16}$. Furthermore, the Racetrack potential contains two local minima at $Y \sim -20$ and $Y \sim 20$. These two facts point towards the Racetrack potential being suitable for inflation while the KKLT potential is not since the Racetrack potential is very flat near the saddle point and contain at least one local minima. In the KKLT case the inflaton will continuously fall along the potential to large values of $T$ which in turn will decompactify the system. The lack of a local minimum also prevents reheating.

One should mention that this is the case for one particular choice of parameters. One could also imagine that there could be other choices of parameters that might not provide a saddle point in the Racetrack scenario or maybe even
Figure 6: The scalar potential in the Racetrack scenario plotted against the real and imaginary part of $T$. The following values are used for the parameters:\n\[ A = 1.5 \times 10^{-3}, \quad a = 2 \times \pi \times 10^{-3}, \quad B = -35 \times 10^{-3}, \quad b = 2 \times \pi \times 90, \quad W_0 = -1.25 \times 10^{-12}, \quad E = 4.14668 \times 10^{-12}, \quad \alpha = 2. \]

A saddle point is placed at $\text{Im}(T) = 0$, $\text{Re}(T) = 123.22$ and at this point the value of the potential is $V_{\text{saddle}} = 1.6549 \times 10^{-16}$. The $V$-axis is scaled with a factor of $10^{16}$.

A choice of parameters that make it so that the KKLT potential can be used for inflation instead. It turns out that the shape of the potentials are both somewhat sensitive to the choice of parameters and more often than not one is left with the waterfall-like shape of the KKLT potential presented above.

Ultimately, the test that the Racetrack potential above need to pass in order to be viable for inflation is that it satisfies the slow-roll conditions. In order to calculate the slow-roll parameters for the Racetrack potential near the saddle point we require the derivatives of $V_{\text{RT}}$. We also need to find the direction in the XY plane that corresponds to the inflationary trajectory. Fortunately, the direction is easily found by realizing that due to the nature of saddle points we know that the function is at its maximum in one direction while it is in its minimum in another. In our case we see that $X$ is at its minimum at the saddle point while $Y$ is at its maximum, thus the inflaton (the Kähler modulus) will effectively only move in the $Y$ direction near the saddle point, making $X$ a constant. As such, the expressions for the relevant derivatives are as follows

\[
\frac{\partial V_{\text{RT}}}{\partial Y} = -3W_0a^2Ae^{-aX} \frac{\sin(aY)}{6X^2} - 3W_0b^2Be^{-bX} \frac{\sin(bY)}{6X^2} - \frac{ABe^{-aX}}{6X^2} (2abX + 3a + 3b)(a - b) \sin((a - b)Y),
\]

(84)
\begin{equation}
\frac{\partial^2 V_{\text{RT}}}{\partial Y^2} = - \frac{3W_0a^3Ae^{-aX}}{6X^2} \cos(aY) - \frac{3W_0b^3Be^{-bX}}{6X^2} \cos(bY) \\
\quad - \frac{ABe^{-aX}}{6X^2}(2abX + 3a + 3b)(a - b)^2 \cos((a - b)Y). \tag{85}
\end{equation}

As stated, these expression can be used instead of the full derivative for \( V_{\text{RT}} \) near the saddle point only. We could, of course, calculate the \( X \) derivatives also but they are not necessary.

Furthermore, the kinetic terms for \( X \) and \( Y \) are not canonical

\begin{equation}
L_{\text{kin}} = \frac{3M_{\text{pl}}^2}{4X^2} (\partial_\mu X \partial^\mu X + \partial_\mu Y \partial^\mu Y). \tag{86}
\end{equation}

Due to the rescaling by \( X^2 \) this enables us to write the slow-roll conditions as

\begin{align}
\varepsilon_{\text{saddle}} &= \frac{X^2}{3} \left( \frac{\partial_Y V}{V} \right)^2 \ll 1, \\
\eta_{\text{saddle}} &= \frac{2X^2}{3} \frac{\partial_Y^2 V}{V} \ll 1. \tag{87}
\end{align}

Using the same parameters as in figure 5 and 6 as well as \( X = 123.22 \) and \( Y = 0 \) (approximately the position of the saddle point) we get the following values for the slow-roll parameters:

\begin{align}
\varepsilon_{\text{saddle}} &= 0, \\
\eta_{\text{saddle}} &= -0.002166. \tag{88}
\end{align}

These values fulfill the conditions set by equation (87) and therefore we can conclude that the Racetrack scalar potential allows for inflation in a region near its saddle point.

8.1.1 A more thorough look at the inflationary direction

We stated above that the inflaton only roll in the \( Y \) direction at the saddle point in the Racetrack case. While this statement is seemingly true by comparing to the corresponding figure, one should be thorough and actually check if this is the case. We check this by calculating the mass matrix at the saddle point

\begin{equation}
M = \begin{bmatrix}
\partial_X \partial_X V_{\text{RT}} & \partial_X \partial_Y V_{\text{RT}} \\
\partial_Y \partial_X V_{\text{RT}} & \partial_Y \partial_Y V_{\text{RT}}
\end{bmatrix}_{\text{saddle}} = 10^{-19} \begin{bmatrix}
0.5027 & 0 \\
0 & -0.0004
\end{bmatrix}. \tag{89}
\end{equation}

From this matrix we can read of the eigenvalues and we can confirm that the inflationary direction is indeed in the \( Y \) direction.

9 Racetrack inflation with two Kähler moduli

As we saw in the previous section we can have inflation using a superpotential and Kähler potential dependant on only one Kähler modulus. However, it has been argued in [3] that there is a drawback with this scenario, there is no explicit construction which produces the used superpotential. On the other hand, an
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explicit construction exists for the case where we have two Kähler moduli [14]. In this case the authors of [14] compactify on the explicit Calabi-Yau denoted by \( \mathbb{P}^3_{(1,1,6,9)} \). As such it is natural to investigate whether or not two Kähler moduli, \( T_1 \) and \( T_2 \), work for Racetrack inflation.

As in the previous we again require a superpotential and a Kähler potential. Starting things off we split the Kähler moduli into their real and imaginary parts, \( T_i = X_i + iY_i \), with \( i = 1, 2 \). The Kähler potential will thus look as follows [3]

\[
K = -2 \ln \left( -2 \ln \frac{\sqrt{2}}{18} \left( X_2^3 - X_1^3 \right) \right).
\] (90)

Following equation (74) the superpotential will take the following form

\[
W = W_0 + Ae^{-aT_1} + Be^{-bT_2}.
\] (91)

Notice the similarity to the previous case with only one modulus, the difference is that the exponents depend on different moduli in this case.

Using this \( K \) and \( W \) we can compute the F-term potential. This is done similarly to what we did in the previous section

\[
V_F = \frac{216}{(X_2^3 - X_1^3)^2} \left[ B^2 b(bX_2^2 + 2bX_1^3X_2^4 + 3X_2)e^{-2bX_2}ight. \\
\quad + A^2 a(aX_1^2 + 2aX_1^3X_2^4 + 3X_1)e^{-2aX_1} \\
\quad + 3BBW_0X_2e^{-bX_2}\cos(bY_2) + 3AAW_0X_2e^{-aX_2}\cos(aY_1) \\
\left. \quad + 3ABe^{-aX_1-bX_2}(aX_1 + bX_2 + 2abX_1X_2)\cos(-aY_1 + bY_2) \right].
\] (92)

Moreover, just as in the case with one modulus we require an uplifting term

\[
V_{\text{uplift}} = \frac{D}{(X_2^3 - X_1^3)},
\] (93)

where \( D \) is a parameter similar to \( E \) in the previous case.

9.1 Identifying viable regions for slow-roll inflation

Plotting this function is more difficult compared to the previous case as the scalar potential is now a function of four variables. We can, however, make helpful observations beforehand. If we restrict ourselves to the \((Y_1, Y_2)\) plane we see that we have periodic minima due to the trigonometric functions in the scalar potential. More concretely, for small \( W_0 \) we have a minima when \( \cos(-aY_1 + bY_2) = -1 \) meaning that

\[
-aY_1 + bY_2 = \pi + 2\pi n,
\] (94)

where \( n \) is an integer. For instance, we could have \( aY_1 = 0 \) and \( bY_2 = \pi \). Using this we enable ourselves to plot \( V \) against \( X_i \) while fixing \( Y_i \) to a minimum. Furthermore, if we fix \( Y_1 \) to their minimum we could then find the minimum for
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$X_i$ given this constraint. Then we could plot $V$ against $Y_i$ while fixing $X_i$ as well.

Using the parameters in figure 7 and 8, as well as $Y_1 = 0$, $Y_2 = \frac{\pi}{6}$ as a minimum for $Y_i$, we numerically find the minimum for $X_i$ to be located at $X_1 = 98.7584, X_2 = 171.0613$. We then produce the plots mentioned above. The plots $V$ against $Y_i$ and $V$ against $X_i$ can be seen in figure 7 and 8 respectively. Figure 9 is a zoomed in plot of 8.

![Plot of scalar potential versus Im(T1), Im(T2)](image)

Figure 7: The scalar potential plotted against the imaginary part of $T_i$ whilst fixing $X_i$ to be $X_1 = 98.7584, X_2 = 171.0613$. The following values are used for the parameters: $A = 0.56, a = \frac{2\pi}{3}, B = 7.46666 \cdot 10^{-3}, b = \frac{2\pi}{25}, W_0 = 5.22666 \cdot 10^{-6}, D = 6.21019 \cdot 10^{-9}$. The local minima are the Y-plane representation of the four dimensional minima. The V-axis is scaled with a factor of $10^{14}$.

Because of the periodic nature of the potential in the $Y$ plane there exists multiple saddle points as a consequence of the multiple minima. This can be seen in figure 7 the saddle points are located in between minima. For instance, if we have a minimum located at $Y_1 = 0, Y_2 = \frac{\pi}{6}$ we have a saddle point at $Y_1 = \frac{\pi}{3}, Y_2 = \frac{\pi}{3}$. Fixing the potential at this point and solving for $X_i$ as we did above gives $X_1 = 108.9618, X_2 = 217.6893$. At this point the value of the potential is $V_{\text{saddle}} = 3.352 \cdot 10^{-16}$. The plot $V$ against $X$ whilst fixing $Y$ to a maximum can be seen in figure 10 figure 11 is a zoomed in plot of 10.

The importance of saddle points was seen in the previous section, i.e. the region near them could be flat enough to enable inflation. As such we again want to calculate $\varepsilon$ and $\eta$ near the saddle point. As the potential is a function of four variables we need to express the derivatives in equation (27) properly. We use the same approach as the authors of 3 which means that the slow-roll...
Figure 8: The scalar potential plotted against the real part of $T_i$ whilst fixing $Y_i$ to be $Y_1 = 0$, $Y_2 = \frac{\pi}{6}$. The following values are used for the parameters: $A = 0.56$, $a = \frac{2\pi}{36}$, $B = 7.46666 \cdot 10^{-5}$, $b = \frac{2\pi}{258}$, $W_0 = 5.22666 \cdot 10^{-6}$, $D = 6.21019 \cdot 10^{-9}$. The local minimum is the $X$-plane representation of the four dimensional minima. The $V$-axis is scaled with a factor of $10^{14}$.

Figure 9: A zoomed in plot of the one given in figure 8. It is zoomed around the minimum located at $X_1 = 98.7584$, $X_2 = 171.0613$. 

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Figure 10: The scalar potential plotted against the real part of $T_i$ whilst fixing $Y_i$ to be $Y_1 = 20$, $Y_2 = \frac{\pi}{5}$. The following values are used for the parameters: $A = 0.56$, $a = \frac{2\pi}{35}$, $B = 7.46666 \cdot 10^{-5}$, $b = \frac{2\pi}{255}$, $W_0 = 5.22666 \cdot 10^{-6}$, $D = 6.21019 \cdot 10^{-9}$. The local minima is the X-plane representation of the four dimensional saddle point. The V-axis is scaled with a factor of $10^{14}$.

Figure 11: A zoomed in plot of the one given in figure 10. It is zoomed around the minimum located at $X_1 = 108.9618$, $X_2 = 217.6893$. 

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parameters are given by
\[ \varepsilon = \left( \frac{K^{ij} \partial_i V \partial_j V}{V^2} \right) \]
\[ \eta = \min \left( \text{eig} \left[ \frac{K^{ij} \partial_i V}{V} - \frac{K^{ij} (\partial_i \partial_j V - K^{kl} \partial_i \partial_j K \partial_k V)}{V} \right] \right) \]

With these expressions and the parameters given in figure 7, 8, and 9 the slow roll parameters are
\[ \varepsilon_{\text{saddle}} = 8.4579 \cdot 10^{-30}, \]
\[ \eta_{\text{saddle}} = 0.0108, \] (96)

near the saddle point. These values are small enough to permit inflation.

10 Investigating Racetrack saddle points further

We have now seen two cases where the conditions for slow-roll inflation are fulfilled using the Racetrack scenario and Kähler moduli as inflaton. It would appear that the purpose of the report has been fulfilled, we have embedded inflation in string theory. However, all of our results so far would fall apart if it would turn out that no de Sitter solutions exist in string theory [41] or if the saddle points we have discovered and worked with turned out to be non-physical. In fact, it has been argued that this could be the case.

In [42] a conjecture has been put forward that any scalar potential consistent with quantum gravity should fulfill the following condition
\[ |\nabla V| \geq cV, \] (97)

where \( c \) is a dimensionless constant close to one. The main takeaway from this conjecture is that it does not permit dS critical points. Notice that this conjecture is comparable to the first slow-roll condition presented earlier in the report. Notice also that the slow-roll condition and this conjecture are in strict contradiction to one another. If this conjecture is true then our results, and slow-roll inflation, faces a serious issue.

This conjecture have faced opposition, see for instance [15]. The authors of [15] argues that the conjecture is, in fact, too restrictive. They argue for their case by constructing several Racetrack potentials that contained dS or AdS (which could in theory be uplifted to dS) critical points whilst still being physical. They listed several conditions that their potentials had to fulfill in order to be considered physically acceptable. The list is as follows

• large enough internal volume,
Table 1: The values chosen in case 0 and 1 are identical to the ones chosen in case 0 and 1 in [15]. The values in case 2 are the same as in section 9.

<table>
<thead>
<tr>
<th>Case</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>$-1/100$</td>
<td>$0.26050 - 0.30090 i$</td>
<td>$0.56$</td>
</tr>
<tr>
<td>$a$</td>
<td>$2\pi/100$</td>
<td>$2\pi/300$</td>
<td>$2\pi/40$</td>
</tr>
<tr>
<td>$B$</td>
<td>$1$</td>
<td>$-0.65453 + 0.75603 i$</td>
<td>$7.46666 \cdot 10^{-5}$</td>
</tr>
<tr>
<td>$b$</td>
<td>$2\pi/50$</td>
<td>$2\pi/150$</td>
<td>$2\pi/258$</td>
</tr>
<tr>
<td>$W_0$</td>
<td>$0$</td>
<td>$-0.025920 + 0.022994 i$</td>
<td>$5.22666 \cdot 10^{-6}$</td>
</tr>
<tr>
<td>$D$</td>
<td>$0$</td>
<td>$0$</td>
<td>$6.21019 \cdot 10^{-9}$</td>
</tr>
</tbody>
</table>

- weak coupling,
- large complex structure values,
- positivity of all the kinetic terms at the points of interest,
- small periodicity of the axionic fields,
- sub-planckian energy densities,
- discrete values of the fluxes and the superpotential.

As all of their potentials fulfilled these conditions they concluded that potentials with AdS/dS critical points are indeed quite general whilst also being physical. However, the authors made some simplifications in their procedure. Most notably they restricted themselves to using one Kähler modulus only and they did not include any uplifting terms.

In the following section we are going to comment on the argument in [15] by searching for saddle points for potentials with two Kähler moduli. Due to the time limitations of this project, we restrict our treatment to a case study using the two sets of Racetrack parameters presented in [15], and the parameters used in section 9. The values for these parameters can be see in table 1.

### 10.1 Method and results

Since we are using two Kähler moduli the $K$ and $W$ from section 9 can be reused and thus the scalar potential will be identical to section 9 as long as the parameters are real. However, as mentioned, we will omit the uplift term. The uplifting term only affects the $X$ plane which means that the periodic behaviour in the $Y$ plane will remain. This ensures that we can find the saddle points in the same way as we did in section 9 whilst using the parameters from case 0 and 1, assuming that saddle points exists in the first place.

It should be noted that case 1 uses complex parameters. This forces us to slightly rewrite the scalar potential in order to keep it real. The rewritten
potential is as follows

\[ V_{\text{case1}} = \frac{216}{\left( X_2^2 - X_1^2 \right)^2} \left[ |B|^2 b e^{-2b X_2} \left( 2b X_1^2 X_2 + 3X_1 + X_2 b \right) + |A|^2 a e^{-2a X_1} \left( 2a X_2^2 X_1^2 + 3X_1 + X_1 a \right) + \frac{3}{2} e^{-\left( aX_1 + bX_2 \right)} \left( X_1 a + X_2 b + 2X_1 X_2 a b \right) + \left( e^{-\left( -aY_1 + bY_2 \right)} A^* B + e^{-\left( aY_1 - bY_2 \right)} A^* B \right) + \frac{3}{2} X_1 a e^{-a X_1} \left( W_0 A^* e^{aY_1} + W_0^* A e^{-aY_1} \right) + \frac{3}{2} X_2 b e^{-b X_2} \left( W_0 B^* e^{bY_2} + W_0^* B e^{-bY_2} \right) \right]. \] (98)

We failed to find any saddle points for case 0 and 1 when using this method. We will discuss this result in the following section.

10.2 Comments on the results

In this section we try to figure out why we did not find any saddle point in the previous section. There are two logical explanations: the set of parameters do not give saddle points when using potentials with two Kähler moduli and/or the Matlab script that does the computation is in some way faulty.

The first explanation seem reasonable. The expressions for the scalar potentials used for one and two Kähler moduli are different. As such there are no guarantees that a set of parameters that provide saddle points in the one modulus case would also provide a saddle point in the two moduli case. Therefore, if this explanation is the right one, we cannot draw any conclusions regarding the generality of AdS/dS saddle points for scalar potentials using two Kähler moduli. We would have to redo the entire procedure for other, saddle point allowing, sets of parameters (should they exist) in order to comment on the generality.

It is somewhat more difficult to confirm or deny the other explanation. A good way of going about it is to run the script for a known case where a saddle point exist and see whether or not the Matlab script gives the same result. In essence we have such a case, namely the case presented in section 9. We most definitely found a saddle point in section 9 but we used a scalar potential with an uplifting term which, as mentioned, is something we wanted to avoid during this investigation. However, the fact that the script can find a saddle point indicates that it works as intended.

To investigate further we removed the uplifting term from the potential and ran the script. Interestingly, we cannot find a saddle point in this case. Thus, according to the script, the saddle point vanish when we remove uplifting. More specifically, when removing around 40 % of the uplift parameter \( D \) the script no longer finds a physical saddle point. If we instead would increase \( D \) we notice that already at a two percent increase the position of the saddle point increase with about 1000 steps in the \( \text{Re}(T_2) \) direction. Later, at around a 60 % increase the script starts to break down as it no longer can find a local minimum.
However, a saddle point can be re-obtained when $D$ is at its lower limit or slightly below if $W_0$ is also changed to counteract the alteration in $D$. For example, if $D$ is at its lower limit and $W_0$ is reduced by 20% a saddle point is again obtained. This way of re-obtaining a saddle point does not work for any change in $D$. If the $D = 0$ no change in $W_0$ will result in a saddle point.

Assuming that the script is working as intended we could draw an interesting conclusion: uplifting is a crucial component for the existence of saddle points in scalar potentials with two Kähler moduli. This would be in contradiction to the result in [15] since the authors never used uplifting to begin with and still got positive results. Another explanation would be that the removal of the uplifting term alters the expression to the point where the set of parameters is no longer valid. This explanation is somewhat supported by our investigation when changing $W_0$ as a change in $W_0$ did reintroduce a saddle point in one particular case. However, as we noted in the paragraph above this method is not viable when $D = 0$, which again indicates that uplifting is necessary. In either case, one would have to run the script for several sets of parameters (that are physically viable as is indicated by the list above) to draw a less speculative conclusion.

11 Discussion

In this report we have successfully constructed two different putative string inflation models. We started of by deriving and motivating the slow-roll conditions presented in equation (27). These conditions were used as a check to see if the string models did or did not allow for inflation. These conditions indicated that a flat region of the inflaton potential was required to enable inflation.

We also introduced the F-term potential in equation (52) which we used as the potential to be inserted into the slow-roll conditions. The motivation for the use of the F-term potential can be boiled down to the motivations for the terms inside the expression, namely the Kähler potential and the superpotential along with their corrections. The motivation for these objects had us go deeper into the field of string theory.

To represent the inflaton we used the scalar fields that emerge from string compactification on a Calabi-Yau three-fold. More specifically, we used the Kähler moduli. The Kähler moduli, as well as the complex structure moduli and the axio-dilaton, made their appearance in the Kähler potential presented in equation (55), although by an appropriate choice of energy scales we eventually omitted the non-dynamic fields. Furthermore, the corrections to the Kähler potential could be ignored if the compact volume was large enough.

In order to motivate the superpotential we introduced type IIB string theory and presented its bosonic action in equation (40). From this action we also brought forward the ISD conditions, presented in equation (69), which in turn were used to argue for the GKW superpotential, presented in equation (70). The corrections to the GKW superpotential were introduced on the KKL-T form in equation (72) and are generated by gaugino condensation. Again, by an appropriate choice of energy scales the GKW superpotential was made a constant, only its corrections contained dynamic fields due to their Kähler moduli dependence.

With the groundwork laid out we were enabled to reconstruct two models of...
string inflation based on previous work. The first one of these, seen in section 8, utilized a Calabi Yau with hodge number $h^{(1,1)} = 1$ which gave us only one Kähler modulus to be used as inflation. We identified a saddle points in its potential and argued that this region was flat enough to allow for inflation. This argument was backed up by a numerical calculation that indeed confirmed that the slow-roll conditions were fulfilled.

The second string inflation model, presented in section 9, had two Kähler moduli as inflation. This feature made the system more complex and as such the procedure to identify a flat enough region to allow for inflation was more in-depth. Nevertheless, this model too fulfilled the slow-roll conditions. A benefit with this second model is that there are known Calabi Yau three-folds with two Kähler moduli that allow the perturbative corrections needed. This is not the case for one Kähler modulus.

Finally we briefly discussed the credibility of the results. We used the conjecture presented in [42] as well as a recent critique in [15] as basis for our discussion. In particular, we made a case study in which we adopted the procedure in [15] to find physical saddle points for our string inflation model with two Kähler moduli. Interestingly enough, we could only find a saddle points with the inclusion of an uplifting term, indicating that uplifting is a crucial component for the existence of physical AdS/dS saddle points. A more thorough/systematic scan over different parameter values is needed to settle this question.

11.1 Moving forward

We had to make use of several constraint in order to get tangible expressions for our scalar potentials. For summary we will list them here:

- we only consider slow-roll inflation,
- we restricted ourself to type IIB string theory,
- we compactify on Calabi-Yau manifolds only,
- we only consider ISD configurations,
- we restrict the energy scales to $E \ll m_{\text{class}}$,
- we assume that uplifting works as intended (as was discussed earlier),
- we only consider Racetrack scenarios.

By relaxing or changing some of these conditions one might get other, more general, results. We briefly discuss the implication of this below.

The majority of the report is based on the fact that we only consider slow-roll inflation. If one would want to consider other types of inflation then some other procedure would be required to obtain proper regions of the scalar potentials or perhaps other potentials would be needed altogether. For instance, if one do not want slow-roll then the need for a flat, inflationary region would disappear.

Moving on, the restriction to type IIB string theory is not a restriction in the usual sense since all types of string theory are related through dualities. As such all types of string theory would be applicable for the procedure outlined in this report in some sense. However, a more general approach would be to not
use the perspective of any one type but instead use the overarching $M$-theory. How such an approach would work out is largely unknown at the moment.

The choice to compactify on Calabi-Yau is one done for convenience. This choice enabled us to write the Kähler potential in the form presented in equation (55) and also simplified some calculations in the Appendix. By changing the compact manifold these handy aids would be disappear or be altered. However, since the scalar potential is dependent on $K$ the consequence of altering equation (55) could lead to interesting results if done cleverly enough.

On the topic of what contributes to the scalar potential, $W$ is a major contributor. The expression for $W$, or rather $W_0$, used in this report is in turn dependent on the ISD conditions. By violating these conditions we are unable to use $W_0$, which can have an effect on the scalar potential.

The decision to restrict the energy scales to $E \ll m_{\text{class}}$ is more to combat technical difficulties rather than make statements regarding how the physics work. Even so it affects the result. By instead considering $E \ll m_{\text{class}}$ we reintroduce other moduli as variables rather than constants. In this case one has to figure out the role they play. If they are additional variables, for example, they would act as inflaton beside the Kähler moduli. If this is undesired in some way one would need another way to make them fixed. This could potentially tie in to the choice of compact manifold as this choice affect the moduli as well.

So far we have only considered Racetrack and KKLT scenarios. There are several more ways to embed inflation in string theory as is outlined in [1]. One could just as well chooses another such approach if one is unsatisfied with the constraints of Racetrack inflation.

To conclude, if one wants to investigate the Racetrack approach further including all constraints this is also doable. The most apparent way to do this is to increase the Hodge number $h^{1,1}$ to three or more, but this would significantly make the situation more complex. Using the Matlab scripts developed in this project, this seems out of reach. Other, more efficient, algorithms would be needed to make progress.
Appendix

Appendix A

In this section we show why the comoving Hubble radius increases with time. We start from the so-called continuity equation

\[
\frac{d\rho}{dt} + 3H(\rho + p) = 0,
\]

(99)

where \( \rho \) is the energy density of the universe and \( p \) is the pressure. The energy density and the pressure can be related to each other through the definition of the equation of state, \( \omega \),

\[
\omega = \frac{p}{\rho}.
\]

(100)

Inserting this into the continuity equation gives us

\[
\frac{d\rho}{dt} + 3H\rho(1 + \omega) = \frac{d\rho}{dt} + 3\frac{da}{dt} \frac{\rho(1 + \omega)}{a} = 0,
\]

\[
\Rightarrow \frac{d\rho}{dt} \frac{1}{\rho} + 3\frac{da}{dt} \frac{1 + \omega}{a} = 0.
\]

(101)

We use that \( \frac{d\rho}{dt} \frac{1}{\rho} = \frac{d\ln \rho}{dt} \) to obtain

\[
\frac{d\ln \rho}{dt} = -3(1 + \omega) \frac{d\ln a}{dt},
\]

\[
\Rightarrow \frac{d\ln \rho}{d\ln a} = -3(1 + \omega),
\]

\[
\Rightarrow \rho = \rho_0 a^{-3(1+\omega)},
\]

(102)

where \( \rho_0 \) is a constant if integration.

Let us now introduce the so-called Friedmann equation in natural coordinates and for flat space

\[
H^2 = \frac{\rho}{3},
\]

\[
\Rightarrow \frac{1}{aH} = \sqrt{\frac{3}{\rho a^2}}.
\]

(103)

Inserting equation (102) into the expression above gives

\[
(aH)^{-1} = \sqrt{\frac{3}{\rho_0}} a^{\frac{3}{2}(1+\omega)-1} = \sqrt{\frac{3}{\rho_0}} a^{\frac{1}{2}(1+3\omega)}.
\]

(104)

From this we see that the behaviour of the comoving Hubble radius depends on \( \omega \). For conventional Big Bang evolution we have \( \omega \geq 0 \), which in turn gives a growing \((aH)^{-1}\).
Appendix B

In this section we prove equation (62). We insert equation (57) into equation (60) while restricting to the external coordinates. Since there are many terms in $S_{\text{IIB}}$ we treat each separately. The $\partial_M \tau$ term will disappear since we are only concerned with the non-compact coordinates. We start treating the $G_3$ term

$$T_{MN}^{(G_3)} = - \frac{2}{\sqrt{-\det G_{MN}}} \frac{1}{2\kappa^2} (-1) \frac{\delta}{\delta G_{MN}} \frac{\sqrt{-\det \hat{G}_{MN}} \hat{G}_{mnp} \hat{G}^{mnp}}{12 \text{Im}(\tau)}$$

$$= \frac{1}{12\sqrt{-\det G_{MN}} \kappa^2 \text{Im}(\tau)} \left( - \frac{\sqrt{-\det G_{MN}} \hat{G}_{MN} \hat{G}_{mnp} \hat{G}^{mnp}}{2} + \frac{\delta}{\delta G_{MN}} \hat{G}^{mn'p'} \hat{G}^{mnp} \hat{G}_{mnp} \hat{G}_{m'n'p'} \right)$$

$$= \frac{1}{12\sqrt{-\det G_{MN}} \kappa^2 \text{Im}(\tau)} \left( - \frac{\sqrt{-\det G_{MN}} \hat{G}_{MN} \hat{G}_{mnp} \hat{G}^{mnp}}{2} + 3 \hat{G}^{mn'p'} \hat{G}^{mnp} \hat{G}_{mnmN} \hat{G}_{m'n'N} \right)$$

$$= \frac{1}{12\kappa^2 \text{Im}(\tau)} \left( - \frac{1}{2} \hat{G}_{MN} \hat{G}_{mnp} \hat{G}^{mnp} + 3 \hat{G}_{mnM} \hat{G}^{mn} \right).$$

The corresponding Ricci term then becomes

$$R_{MN}^{(G_3)} = \kappa^2 \left( \frac{1}{12\kappa^2 \text{Im}(\tau)} \left( - \frac{1}{2} \hat{G}_{MN} \hat{G}_{mnp} \hat{G}^{mnp} + 3 \hat{G}_{mnM} \hat{G}^{mn} \right) \right)$$

$$= \frac{1}{8} \hat{G}_{MN} \left( - \frac{1}{12\kappa^2 \text{Im}(\tau)} \left( -5 \hat{G}_{mnp} \hat{G}^{mnp} + 3 \hat{G}_{mnM} \hat{G}^{mnM} \right) \right)$$

$$= \frac{1}{96 \text{Im}(\tau)} \left( -4 + 5 - 3 \right) \hat{G}_{MN} \hat{G}_{mnp} \hat{G}^{mnp} + \frac{3}{12 \text{Im}(\tau)} \hat{G}_{mnM} \hat{G}^{mn}.$$  

We restrict the result to non-compact coordinates $(M, N \to \mu \nu)$

$$R_{\mu\nu}^{(G_3)} = - \frac{1}{48 \text{Im}(\tau)} \hat{g}_{\mu\nu} \hat{G}_{mnp} \hat{G}^{mnp},$$

where we used $\hat{G}_{mn\mu} = 0$.

We move on to the $\hat{F}_5$ term. Before we calculate the stress-energy tensor
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term we rewrite the factor $\tilde{F}_5^2$ on a more convenient form

$$\tilde{F}_5^2 = (1 + \ast) \left[ da \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \right]$$

$$= 2 \left[ \partial_m a dx^m \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \right]$$

$$\wedge \ast \left[ \partial_m a dx^m \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \right]$$ (108)

$$= 2 \cdot 5! \partial_m a \partial^m a e^{-8A} \sqrt{-\det \hat{G}_{mn} dx^0 \wedge \cdots \wedge dx^9},$$

where we used

$$\left[ \frac{1}{r!} \omega_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \right] \wedge \ast \left[ \frac{1}{r!} \eta_{\mu_1 \ldots \mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \right]$$

$$= \frac{1}{r!} \omega_{\mu_1 \ldots \mu_r} \eta^{\mu_1 \ldots \mu_r} \sqrt{-\det \hat{G}_{MN} dx^1 \wedge \cdots \wedge dx^9}. \quad (109)$$

Now we calculate the stress energy term in the same way as before

$$T_{MN}^{(F_5)} = -\frac{2}{\sqrt{-\det G_{MN}}} \frac{1}{2\kappa^2} \frac{1}{(-1)} \frac{\delta}{\delta G_{MN}} \sqrt{-\det \hat{G}_{MN} \tilde{F}_5^2}$$

$$= e^{-8A} \frac{\delta}{\delta G_{MN}} \det \hat{G}_{MN} \partial_m a \partial^m a dx^0 \wedge \cdots \wedge dx^9 \delta \frac{\delta}{\delta G_{MN}} \left( -\det \hat{G}_{MN} \right)$$

$$= \frac{e^{-8A}}{2\kappa^2} \partial_m a \partial^m a dx^0 \wedge \cdots \wedge dx^9 \quad (110)$$

$$= \frac{e^{-8A}}{2\kappa^2} \sqrt{-\det \hat{G}_{MN} \tilde{G}_{MN} \partial_m a \partial^m a dx^0 \wedge \cdots \wedge dx^9}$$

$$= -\frac{e^{-8A}}{2\kappa^2} \tilde{G}_{MN} \partial_m a \partial^m a,$$

where we used

$$1 = \sqrt{-\det \hat{G}_{MN} dx^0 \wedge \cdots \wedge dx^9}. \quad (111)$$

The corresponding Ricci term restricted to the non-compact coordinates becomes

$$R_{\mu \nu}^{(F_5)} = \kappa^2 \left( T_{\mu \nu}^{(F_5)} - \frac{1}{8} g_{\mu \nu} g^{\rho \sigma} T_{\rho \sigma}^{(F_5)} \right)$$

$$= \frac{\kappa^2}{2} T_{\mu \nu}^{(F_5)} = -\frac{e^{-8A}}{4} g_{\mu \nu} \partial_m a \partial^m a. \quad (112)$$
Thus the total expression for $R_{\mu\nu}$ is

$$R_{\mu\nu} = -g_{\mu\nu} \left( \frac{G_{mnlp} G_{mnp}}{48 \Im(\tau)} + \frac{e^{-8A}}{4} \partial_m \alpha \partial^m \alpha \right) + R_{\mu\nu}^{\text{loc}}$$

(113)

$$= -e^{2A} \eta_{\mu\nu} \left( \frac{G_{mnlp} G_{mnp}}{48 \Im(\tau)} + \frac{e^{-8A}}{4} \partial_m \alpha \partial^m \alpha \right) + R_{\mu\nu}^{\text{loc}}$$

**Appendix C**

In this section we prove equation (113). The four-dimensional Ricci tensor is obtained from the Riemann tensor in the following way assuming the geometry presented in equation (113)

$$R_{\mu\nu} = \hat{G}^{MN} R_{MN} = \hat{G}^{\rho\sigma} R_{\rho\mu\sigma\nu} + \hat{G}^{\mu\nu} R_{\mu\nu} = R_{\mu\nu}^p + R_{\mu\nu}^n. \quad (114)$$

Before we continue we should specify that Greek index always refer to the external geometry while Latin letters refers to the internal and capital Latin letters refers to the ten dimensional geometry. We calculate $R_{\mu\nu}^p$ and $R_{\mu\nu}^n$ separately starting with $R_{\mu\nu}^p$

$$R_{\mu\nu}^p = \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\nu\sigma} \Gamma^\sigma_{\rho\mu} - \Gamma^\rho_{\sigma\mu} \Gamma_{\rho\nu\sigma}$$

(115)

$$= \partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\rho\mu} + \Gamma^\rho_{\nu\sigma} \Gamma_{\rho\mu\sigma} - \Gamma^\rho_{\sigma\mu} \Gamma_{\rho\nu\sigma} - \Gamma^\rho_{\sigma\nu} \Gamma_{\rho\mu\sigma}$$

where the Christoffel symbol, $\Gamma^\rho_{\mu\nu}$, is given by

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} \hat{G}^{\rho\sigma} \left( \partial_\mu \hat{G}_{\nu\sigma} + \partial_\nu \hat{G}_{\mu\sigma} - \partial_\sigma \hat{G}_{\mu\nu} \right). \quad (116)$$
Since there are a lot of Christoffel symbols in equation (115) we write their expressions one by one

\[ \Gamma^\rho_{\nu\mu} = \frac{1}{2} \hat{G}^{\rho Q} \left( \partial_\mu \hat{G}_{Q\nu} + \partial_\nu \hat{G}_{\mu Q} - \partial_Q \hat{G}_{\nu\mu} \right) \]
\[ = \frac{1}{2} \hat{G}^{\rho \lambda} \left( \partial_\mu \hat{G}_{\lambda \nu} + \partial_\nu \hat{G}_{\mu \lambda} - \partial_\lambda \hat{G}_{\nu\mu} \right) \]
\[ = \frac{1}{2} \eta^{\rho \lambda} (\partial_\mu \eta_{\lambda \nu} + \partial_\nu \eta_{\mu \lambda} - \partial_\lambda \eta_{\nu\mu}) = 0 \]

\[ \Gamma^\rho_{\nu\mu} = \frac{1}{2} \hat{G}^{\rho Q} \left( \partial_\mu \hat{G}_{Q\rho} + \partial_\rho \hat{G}_{\mu Q} - \partial_Q \hat{G}_{\rho\nu} \right) \]
\[ = \frac{1}{2} \hat{G}^{\rho \lambda} \left( \partial_\mu \hat{G}_{\lambda \rho} + \partial_\rho \hat{G}_{\mu \lambda} - \partial_\lambda \hat{G}_{\rho\nu} \right) \]
\[ = \frac{1}{2} \eta^{\rho \lambda} (\partial_\mu \eta_{\lambda \rho} + \partial_\rho \eta_{\mu \lambda} - \partial_\lambda \eta_{\rho\nu}) = 0 \]

\[ \Gamma^\rho_{\nu\mu} = \frac{1}{2} \hat{G}^{\rho Q} \left( \partial_\mu \hat{G}_{Q\nu} + \partial_\nu \hat{G}_{\mu Q} - \partial_Q \hat{G}_{\nu\mu} \right) \]
\[ = \frac{1}{2} \hat{G}^{\rho \lambda} \left( \partial_\mu \hat{G}_{\lambda \nu} + \partial_\nu \hat{G}_{\mu \lambda} - \partial_\lambda \hat{G}_{\nu\mu} \right) \]
\[ = \frac{1}{2} \eta^{\rho \lambda} (\partial_\mu \eta_{\lambda \nu} + \partial_\nu \eta_{\mu \lambda} - \partial_\lambda \eta_{\nu\mu}) = 0 \]

\[ \Gamma^\rho_{\nu\mu} = \frac{1}{2} \hat{G}^{\rho Q} \left( \partial_\mu \hat{G}_{Q\rho} + \partial_\rho \hat{G}_{\mu Q} - \partial_Q \hat{G}_{\rho\nu} \right) \]
\[ = \frac{1}{2} \hat{G}^{\rho \lambda} \left( \partial_\mu \hat{G}_{\lambda \rho} + \partial_\rho \hat{G}_{\mu \lambda} - \partial_\lambda \hat{G}_{\rho\nu} \right) \]
\[ = \frac{1}{2} \eta^{\rho \lambda} (\partial_\mu \eta_{\lambda \rho} + \partial_\rho \eta_{\mu \lambda} - \partial_\lambda \eta_{\rho\nu}) = 0 \]

\[ \Gamma^\rho_{\nu\mu} = \frac{1}{2} \hat{G}^{\rho Q} \left( \partial_\mu \hat{G}_{Q\rho} + \partial_\rho \hat{G}_{Q\nu} - \partial_Q \hat{G}_{\rho\nu} \right) \]
\[ = \frac{1}{2} \hat{G}^{\rho \lambda} \left( \partial_\mu \hat{G}_{\lambda \rho} + \partial_\rho \hat{G}_{\nu \lambda} - \partial_\lambda \hat{G}_{\rho\nu} \right) \]
\[ = \frac{1}{2} \hat{G}^{\rho \lambda} \partial_\rho \hat{G}_{\lambda \nu} = \eta^{\rho \lambda} (\partial_\lambda \partial_\rho A - \eta_{\lambda \rho} \partial_\rho A) = 4 \eta^{\rho \lambda} \partial_\lambda A \]

\[ \Gamma^\rho_{\nu\mu} = \frac{1}{2} \hat{G}^{\rho Q} \left( \partial_\mu \hat{G}_{Q\nu} + \partial_\nu \hat{G}_{\mu Q} - \partial_Q \hat{G}_{\nu\mu} \right) \]
\[ = \frac{1}{2} \hat{G}^{\rho \lambda} \left( \partial_\mu \hat{G}_{\nu \lambda} + \partial_\nu \hat{G}_{\mu \lambda} - \partial_\lambda \hat{G}_{\nu\mu} \right) \]
\[ = -\frac{1}{2} \hat{G}^{\rho \lambda} \partial_\rho \hat{G}_{\nu\mu} = -e^{\lambda A} \eta_{\nu\mu} \partial_\rho A \]
\[\Gamma^\rho_{\nu\sigma} = \frac{1}{2} \hat{G}^{\rho Q} \left( \partial_x \hat{G}_{Q\nu} + \partial_y \hat{G}_{Q\sigma} - \partial_Q \hat{G}_{\nu\sigma} \right) \]
\[= \frac{1}{2} \hat{G}^{\rho\lambda} \left( \partial_x \hat{G}_{\lambda\nu} + \partial_y \hat{G}_{\lambda\sigma} - \partial_{\lambda} \hat{G}_{\nu\sigma} \right) \]
\[= \frac{1}{2} \eta^{\rho\lambda} \left( \partial_x \eta_{\lambda\nu} + \partial_y \eta_{\lambda\sigma} - \partial_{\lambda} \eta_{\nu\sigma} \right) \]
\[= 0\]
\[\Gamma^\rho_{\mu\nu} = \frac{1}{2} \hat{G}^{\rho Q} \left( \partial_x \hat{G}_{Q\rho} + \partial_y \hat{G}_{Q\mu} - \partial_Q \hat{G}_{\rho\mu} \right) \]
\[= \frac{1}{2} \hat{G}^{\rho\lambda} \left( \partial_x \hat{G}_{\lambda\rho} + \partial_y \hat{G}_{\lambda\mu} - \partial_{\lambda} \hat{G}_{\rho\mu} \right) \]
\[= \frac{1}{2} \hat{G}^{\rho\lambda} \partial_x \hat{G}_{\lambda\mu} = \eta^{\rho\lambda} \eta_{\lambda\nu} \partial_x A = \delta_{\rho}^{\nu} \partial_x A \]
\[\Gamma^\rho_{\nu\mu} = \frac{1}{2} \hat{G}^{\rho Q} \left( \partial_x \hat{G}_{Q\rho} + \partial_y \hat{G}_{Q\mu} - \partial_Q \hat{G}_{\rho\mu} \right) \]
\[= \frac{1}{2} \hat{G}^{\rho\lambda} \left( \partial_x \hat{G}_{\lambda\rho} + \partial_y \hat{G}_{\lambda\mu} - \partial_{\lambda} \hat{G}_{\rho\mu} \right) \]
\[= \frac{1}{2} \hat{G}^{\rho\lambda} \partial_x \hat{G}_{\lambda\mu} = \eta^{\rho\lambda} \eta_{\lambda\nu} \partial_x A = \delta_{\rho}^{\nu} \partial_x A \]
\[\Gamma^\rho_{\mu\nu} = \frac{1}{2} \hat{G}^{\rho Q} \left( \partial_x \hat{G}_{Q\rho} + \partial_y \hat{G}_{Q\mu} - \partial_Q \hat{G}_{\rho\mu} \right) \]
\[= \frac{1}{2} \hat{G}^{\rho\lambda} \left( \partial_x \hat{G}_{\lambda\rho} + \partial_y \hat{G}_{\lambda\mu} - \partial_{\lambda} \hat{G}_{\rho\mu} \right) \]
\[= \frac{1}{2} \hat{G}^{\rho\lambda} \partial_x \hat{G}_{\lambda\mu} = \eta^{\rho\lambda} \eta_{\lambda\nu} \partial_x A = \delta_{\rho}^{\nu} \partial_x A \]

We know that the Christoffel symbol for a flat metric is zero. Thus \( R^\rho_{\mu\rho\nu} \) is

\[ R^\rho_{\mu\rho\nu} = -4e^{4A} \eta_{\mu\nu} \partial^\rho A \eta_{\rho\mu} A + e^{4A} \delta_{\nu}^{\rho} \eta_{\rho\mu} \partial_y A \partial^\rho A = -3e^{4A} \eta_{\nu\mu} \partial_y A \partial^\rho A. \] (117)

We do the same for \( R^n_{\mu\nu} \)

\[ R^n_{\mu\nu} = \partial_{\mu} \Gamma^n_{\nu\mu} - \partial_{\nu} \Gamma^n_{\mu\nu} + \Gamma^n_{\mu\sigma} \Gamma^n_{\rho\sigma} + \Gamma^n_{\nu\sigma} \Gamma^n_{\mu\sigma} - \Gamma^n_{\mu\nu} \Gamma^n_{\rho\mu} - \Gamma^n_{\nu\sigma} \Gamma^n_{\rho\mu}. \] (118)
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The Cristoffel symbols are as follows

\[ \Gamma^\alpha_{\nu\mu} = \frac{1}{2} \tilde{G}^{\alpha\mu} \left( \partial_\mu \tilde{G}_{\alpha\nu} + \partial_\nu \tilde{G}_{\alpha\mu} - \partial_\alpha \tilde{G}_{\nu\mu} \right) \]

\[ = \frac{1}{2} \tilde{G}^{\alpha\mu} \left( \partial_\mu \tilde{G}_{\nu\alpha} + \partial_\nu \tilde{G}_{\mu\alpha} - \partial_\alpha \tilde{G}_{\nu\mu} \right) \]

\[ = -\frac{1}{2} \tilde{G}^{\alpha\mu} \partial_\alpha \tilde{G}_{\nu\mu} = -e^A \eta_{\nu\mu} \partial^\alpha A \]

\[ \Gamma^\alpha_{\nu\mu} = \frac{1}{2} \tilde{G}^{\alpha\mu} \left( \partial_\mu \tilde{G}_{\nu\alpha} + \partial_\alpha \tilde{G}_{\nu\mu} - \partial_\nu \tilde{G}_{\alpha\mu} \right) \]

\[ = \frac{1}{2} \tilde{G}^{\alpha\mu} \left( \partial_\mu \tilde{G}_{\nu\alpha} + \partial_\alpha \tilde{G}_{\nu\mu} - \partial_\nu \tilde{G}_{\alpha\mu} \right) \]

\[ = \frac{1}{2} \tilde{G}^{\alpha\mu} \partial_\alpha \tilde{G}_{\nu\mu} = 0 \]

\[ \Gamma^\sigma_{\nu\mu} = \frac{1}{2} \tilde{G}^{\sigma\mu} \left( \partial_\mu \tilde{G}_{\sigma\nu} + \partial_\nu \tilde{G}_{\sigma\mu} - \partial_\sigma \tilde{G}_{\nu\mu} \right) \]

\[ = \frac{1}{2} \tilde{G}^{\sigma\mu} \left( \partial_\mu \tilde{G}_{\sigma\nu} + \partial_\nu \tilde{G}_{\sigma\mu} - \partial_\sigma \tilde{G}_{\nu\mu} \right) \]

\[ = \frac{1}{2} \tilde{G}^{\sigma\mu} \partial_\sigma \tilde{G}_{\nu\mu} = 0 \]

\[ \Gamma^\nu_{\mu\sigma} = \frac{1}{2} \tilde{G}^{\nu\sigma} \left( \partial_\sigma \tilde{G}_{\mu\nu} + \partial_\nu \tilde{G}_{\mu\sigma} - \partial_\mu \tilde{G}_{\sigma\nu} \right) \]

\[ = \frac{1}{2} \tilde{G}^{\nu\sigma} \left( \partial_\sigma \tilde{G}_{\mu\nu} + \partial_\nu \tilde{G}_{\mu\sigma} - \partial_\mu \tilde{G}_{\sigma\nu} \right) \]

\[ = \frac{1}{2} \eta^{\nu\sigma} \left( \partial_\mu \eta_{\lambda\nu} + \partial_{\nu} \eta_{\lambda\mu} - \partial_{\lambda} \eta_{\mu\nu} \right) = 0 \]

\[ \Gamma^\nu_{\mu\sigma} = \frac{1}{2} \tilde{G}^{\nu\sigma} \left( \partial_\sigma \tilde{G}_{\mu\nu} + \partial_\nu \tilde{G}_{\mu\sigma} - \partial_\mu \tilde{G}_{\sigma\nu} \right) \]

\[ = \frac{1}{2} \tilde{G}^{\nu\sigma} \left( \partial_\sigma \tilde{G}_{\mu\nu} + \partial_\nu \tilde{G}_{\mu\sigma} - \partial_\mu \tilde{G}_{\sigma\nu} \right) \]

\[ = \frac{1}{2} \tilde{G}^{\nu\sigma} \left( \partial_\sigma \tilde{G}_{\mu\
\mu} + \partial_\nu \tilde{G}_{\mu\sigma} - \partial_\mu \tilde{G}_{\sigma\nu} \right) \]

\[ = \frac{1}{2} \tilde{G}^{\nu\sigma} \partial_\sigma \tilde{G}_{\mu\nu} = -\frac{1}{2} \tilde{g}^{\nu\sigma} \partial_\sigma \tilde{G}_{\mu\nu} \]

\[ \Gamma^\nu_{\nu\mu} = \frac{1}{2} \tilde{G}^{\nu\mu} \left( \partial_\mu \tilde{G}_{\nu\nu} + \partial_\nu \tilde{G}_{\mu\nu} - \partial_\nu \tilde{G}_{\nu\mu} \right) \]

\[ = \frac{1}{2} \tilde{G}^{\nu\mu} \left( \partial_\mu \tilde{G}_{\nu\nu} + \partial_\nu \tilde{G}_{\mu\nu} - \partial_\nu \tilde{G}_{\nu\mu} \right) \]

\[ = \frac{1}{2} \tilde{G}^{\nu\mu} \partial_\nu \tilde{G}_{\nu\mu} = -e^A \eta_{\nu\mu} \partial^\nu A \]
With these results equation (114) becomes another gives ways as is indicated by equation (59) and (66). Putting these equal to one
\[ \tilde{G}_{\nu\sigma} = e^{A} \theta_{\nu\sigma} \partial^{\nu} A \]
\[ \Gamma^n_{\nu\sigma} = \frac{1}{2} \tilde{G}^{nq} \left( \partial_q \tilde{G}_{Qq} + \partial_r \tilde{G}_{Qr} - \partial_q \tilde{G}_{r\nu} \right) \]
\[ = \frac{1}{2} \tilde{G}^{nq} \left( \partial_q \tilde{G}_{Qq} + \partial_r \tilde{G}_{Qr} - \partial_q \tilde{G}_{r\nu} \right) \]
\[ = -\frac{1}{2} \tilde{G}^{nq} \partial_q \tilde{G}_{r\nu} = -e^{A} \theta_{\nu\sigma} \partial^{\nu} A \]
\[ \Gamma^n_{\mu\nu} = \frac{1}{2} \tilde{G}^{nq} \left( \partial_q \tilde{G}_{Qq} + \partial_r \tilde{G}_{Qr} - \partial_q \tilde{G}_{r\nu} \right) \]
\[ = \frac{1}{2} \tilde{G}^{nq} \left( \partial_q \tilde{G}_{Qq} + \partial_r \tilde{G}_{Qr} - \partial_q \tilde{G}_{r\nu} \right) \]
\[ = \frac{1}{2} \tilde{G}^{nq} \partial_q \tilde{G}_{r\nu} = 0 \]
\[ \Gamma^n_{\nu\nu} = \frac{1}{2} \tilde{G}^{nq} \left( \partial_q \tilde{G}_{Qq} + \partial_r \tilde{G}_{Qr} - \partial_q \tilde{G}_{r\nu} \right) \]
\[ = \frac{1}{2} \tilde{G}^{nq} \left( \partial_q \tilde{G}_{Qq} + \partial_r \tilde{G}_{Qr} - \partial_q \tilde{G}_{r\nu} \right) \]
\[ = \frac{1}{2} \tilde{G}^{nq} \partial_q \tilde{G}_{r\nu} = 0. \]
A Christoffel symbol using a Calabi-Yau metric vanishes as Calabi-Yau are Ricci flat. We find \( R^n_{\mu\nu} \) to be
\[ R^n_{\mu\nu} = \partial_{\nu} \left( -e^{A} \eta_{\mu\nu} \partial^{\nu} A \right) + 6 e^{A} \eta_{\mu\nu} \partial_{\nu} A \partial^{\nu} A + e^{A} \eta_{\mu\nu} \delta^{\nu}_{\mu} \partial_{\nu} A \]
\[ = -\eta_{\mu\nu} e^{A} \left( 4 \partial_{\nu} A \partial^{\nu} A + \partial_{\nu} \partial^{\nu} A \right) + 7 e^{A} \eta_{\mu\nu} \partial_{\nu} A \partial^{\nu} A \quad (119) \]
\[ = -\eta_{\mu\nu} e^{A} \partial_{\nu} A \partial^{\nu} A + 3 e^{A} \eta_{\mu\nu} \partial_{\nu} A \partial^{\nu} A. \]
With these results equation (114) becomes
\[ R_{\mu\nu} = -3 e^{A} \eta_{\mu\nu} \partial_{\nu} A \partial^{\nu} A + \eta_{\mu\nu} e^{A} \partial_{\nu} A \partial^{\nu} A + 3 e^{A} \eta_{\mu\nu} \partial_{\nu} A \partial^{\nu} A \]
\[ = -\eta_{\mu\nu} e^{A} \partial_{\nu} A \partial^{\nu} A = -\eta_{\mu\nu} e^{A} \tilde{F}^2 A. \quad (120) \]
This entire process can also be used to prove equation (46).

Appendix D

Here we motivate equation (67). We know that \( d\tilde{F}_3 \) can be expressed in two ways as is indicated by equation (49) and (60). Putting these equal to one another gives
\[ d(1 + \ast) \left[ da \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \right] = H_3 \wedge F_3 + 2\kappa^2 U^{loc} \]
\[ \rightarrow \ast_6 d(1 + \ast) \left[ da \wedge dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \right] = \ast_6 (H_3 \wedge F_3 + 2\kappa^2 U^{loc}). \quad (121) \]
The LHS can be simplified

\[ \ast_{6} d(1 + \ast) \left[ d\alpha \wedge dx^{0} \wedge dx^{1} \wedge dx^{2} \wedge dx^{3} \right] \]
\[ = \ast_{6} d \ast \left[ d\alpha \wedge dx^{0} \wedge dx^{1} \wedge dx^{2} \wedge dx^{3} \right] \]
\[ = \ast_{6} d \ast \left[ (\partial_{\alpha})dx^{m} \wedge dx^{0} \wedge dx^{1} \wedge dx^{2} \wedge dx^{3} \right] \]
\[ = \ast_{6} d \left[ \frac{\sqrt{\det G^{MN}}}{5!} (\partial_{\alpha})e_{\alpha}^{\mu_{1} \mu_{2} \mu_{3} \mu_{4}} e^{\mu_{1}} e^{-8A} e^{\mu_{2}} e^{\mu_{3}} e^{\mu_{4}} \right] \]
\[ = \ast_{6} d \left[ \frac{\sqrt{\det G^{MN}}}{5!} (\partial_{\alpha})e_{\mu_{1} \mu_{2} \mu_{3} \mu_{4} m_{1} m_{2} m_{3} m_{4} m_{5}} dy^{m_{1}} \wedge dy^{m_{2}} \wedge dy^{m_{3}} \wedge dy^{m_{4}} \wedge dy^{m_{5}} \right] \]
\[ = \ast_{6} d \left[ \frac{\sqrt{\det g_{mn}}}{5!} \left( \partial_{\alpha} e \right) e^{-2A} e^{8A} x_{\mu_{1} \mu_{2} \mu_{3} \mu_{4}} dy^{m_{1}} \wedge dy^{m_{2}} \wedge dy^{m_{3}} \wedge dy^{m_{4}} \wedge dy^{m_{5}} \right] \]
\[ = \ast_{6} d \left[ \frac{\sqrt{\det g_{mn}}}{6!} (\partial_{\alpha}) e^{-2A} e^{8A} x_{m_{1} \ldots m_{5}} dy^{m_{1}} \wedge \ldots \wedge dy^{m_{5}} \right] \]
\[ = \ast_{6} \partial \left[ \left( \partial_{\alpha} e \right) e^{-2A} e^{8A} x_{m_{1} \ldots m_{5}} dy^{m_{1}} \wedge dy^{m_{2}} \wedge \ldots \wedge dy^{m_{5}} \right] \]
\[ = \partial_{\alpha} \left[ \left( \partial_{\alpha} e \right) e^{-2A} e^{8A} x_{m_{1} \ldots m_{5}} dy^{m_{1}} \wedge dy^{m_{2}} \wedge \ldots \wedge dy^{m_{5}} \right] \]
\[ = \partial_{\alpha} \left[ \left( \partial_{\alpha} e \right) e^{-2A} e^{8A} x_{m_{1} \ldots m_{5}} \right] \ast_{6} \ast_{6} 1 + (\partial_{\alpha} e) e^{-2A} e^{8A} \ast_{6} \partial_{\alpha} \ast_{6} 1 \]

49
\begin{align}
\epsilon_{\mu_1\ldots\mu_4m_1\ldots m_5} &= \frac{5!}{4!}(-1)^4\epsilon_{\mu_1\ldots\mu_4}e_{mn1\ldots m_5}, \\
1 &= \frac{\text{det}\tilde{g}_{mn}}{6!}\epsilon_{nm1\ldots m_5}dy^m \wedge dy^{m_2} \wedge \cdots \wedge dy^{m_5}, \\
\tilde{\nabla}^2 \alpha &= \frac{1}{2}(\partial_m \alpha)g^{mn}g^{qr}\partial_n \tilde{g}_{qr} + \partial_n (g^{mn}\partial_m \alpha),
\end{align}

and that $G_{MN}$ is diagonal.

The RHS of equation (121) can be written as
\begin{equation}
\kappa_6(H_3 \wedge F_3 + 2\kappa_6U^{loc}) = -\frac{\kappa_6(G_3 \wedge \bar{G}_3)}{2\text{Im}(\tau)} + 2\kappa_6U^{loc}.
\end{equation}

Let us take a closer look at $G_3 \wedge \bar{G}_3$. It can be written as
\begin{equation}
G_3 \wedge \bar{G}_3 = \frac{1}{(3!)^2}G_{mnp}\bar{G}_{qrs}dy^m \wedge dy^n \wedge dy^p \wedge dy^q \wedge dy^r \wedge dy^s.
\end{equation}

Applying a Hodge star to this expression gives us
\begin{equation}
\kappa_6(G_3 \wedge \bar{G}_3) = \frac{1}{(3!)^2}G_{mnp}\bar{G}_{qrs}\sqrt{\text{det}\tilde{g}_{mn}}e_{mnqpqrs} = \frac{1}{3!}G_{mnp}(\kappa_6\bar{G}^{mnp}).
\end{equation}

This allows us to rewrite equation (124) as
\begin{equation}
-\frac{\kappa_6(G_3 \wedge \bar{G}_3)}{2\text{Im}(\tau)} + 2\kappa_6U^{loc} = \frac{G_{mnp}(\kappa_6\bar{G}^{mnp})}{12\text{Im}(\tau)} + 2\kappa_6U^{loc}.
\end{equation}

Putting the RHS and LHS into equation (121) and solving for $\tilde{\nabla}^2 \alpha$ gives us
\begin{equation}
\tilde{\nabla}^2 \alpha = i\kappa_6A^{-1}G_{mnp}(\kappa_6\bar{G}^{mnp}) + 2e^{-8A}[\partial_m \alpha \partial^m e^{4A} + 2\kappa_6^2e^{2A}U^{loc}].
\end{equation}

\section*{Appendix E}

Here we motivate equation (82) and (83). Starting from equation (75) we insert equation (76), (78) and (79).
\begin{equation}
V = e^{-3\log(2\xi)}\left(K^{ij}D_iW \bar{D}_jW - 3|W|^2\right) + \frac{E}{X^\alpha}
= \frac{1}{8\xi^3}\left(K^{ij}D_iW \bar{D}_jW - 3|W|^2\right) + \frac{E}{X^\alpha}.
\end{equation}
The only moduli that is not fixed is the Kähler modulus, thus we can simplify the sum to only run over this modulus. $K_{ij}$ also contain the Kähler potential but we calculate it separately

$$K^{TT} = (K_{TT})^{-1} = (-3\partial_T \partial_T \log(T + \bar{T}))^{-1} = -\frac{1}{3} \left( \frac{\partial_T}{T + \bar{T}} \right)^{-1}$$

$$= \frac{1}{3} \left( \frac{1}{(T + \bar{T})^2} \right)^{-1} = \frac{1}{3} \left( \frac{1}{(2X)^2} \right)^{-1} = \frac{4X^2}{3}$$ (130)

Inserting this result into (129) gives

$$V = \frac{1}{8X^3} \left( \frac{4X^2}{3} - D_T W \bar{D}_T W - 3|W|^2 \right) + \frac{E}{X^\alpha}. \quad (131)$$

Now we have to deal with $D_T W \bar{D}_T W$ since this factor also contain the Kähler potential due to the definition

$$D_T W = \partial_T W + W \partial_T K = \partial_T W + W \partial_T (-3 \log(T + \bar{T}))$$

$$= \partial_T W - 3W \frac{T + \bar{T}}{T + \bar{T}} = \partial_T W - 3W \frac{X}{2X}. \quad (132)$$

With this the scalar potential becomes

$$V = \frac{1}{8X^3} \left[ \frac{4X^2}{3} \left( \partial_T W - 3W \frac{X}{2X} \right) \left( \partial_T W - 3W \frac{X}{2X} \right) - 3|W|^2 \right] + \frac{E}{X^\alpha}$$

$$= \frac{1}{8X^3} \left[ \frac{1}{3} \left( 2X \partial_T W - 3W^2 - 3|W|^2 \right) \right] + \frac{E}{X^\alpha} \quad (133)$$

$$= \frac{1}{6X^3} \left( X^2 \partial_T W \partial_T \bar{W} - \frac{3}{2} X (W \partial_T W + W \partial_T \bar{W}) \right) + \frac{E}{X^\alpha}.$$ 

The last thing needed to obtain either equation (82) or (83) is to replace $W$ with a more concrete expression. At this point we need to choose whether we want to calculate the potential for the KKLT or Racetrack scenario as this choice will make us choose either equation (80) or (81) to replace $W$. After the choice has been made we simplify the resulting expression. We will calculate both cases starting with the KKLT scenario

$$V_{KKLT} = \frac{1}{6X^3} \left[ X^2 \partial_T (W_0 + Ae^{-aT}) \partial_T (W_0 + Ae^{-aT}) \right.$$

$$\left. - \frac{3}{2} X \{(W_0 + Ae^{-aT}) \partial_T (W_0 + Ae^{-aT}) \right.$$ 

$$\left. + (W_0 + Ae^{-aT}) \partial_T (W_0 + Ae^{-aT}) \} \right] + \frac{E}{X^\alpha}$$

$$= \frac{1}{6X^3} \left[ (aAX)^2 e^{-a(T + \bar{T})} + \frac{3aAX}{2} \left\{ e^{-aT} (W_0 + Ae^{-aT}) \right. \right.$$

$$\left. + e^{-aT} (W_0 + Ae^{-aT}) \} \right] + \frac{E}{X^\alpha} \quad (134)$$

$$= \frac{aAe^{-aX}}{6X^2} \left[ Ae^{-aX} (aX + 3) + 3W_0 \cos(aY) \right] + \frac{E}{X^\alpha}. \quad (135)$$
The Racetrack potential is obtained in a similar fashion

\[
V_{RT} = \frac{1}{6X^3} \left[ X^2 \partial_T (W_0 + Ae^{-aT} + B^{-bT}) \partial_T (W_0 + Ae^{-aT} + B^{-bT}) \right. \\
- \frac{3}{2} X ((W_0 + Ae^{-aT} + B^{-bT}) \partial_T (W_0 + Ae^{-aT} + B^{-bT}) \\
+ (W_0 + Ae^{-aT} + B^{-bT}) \partial_T (W_0 + Ae^{-aT} + B^{-bT})) \right] + \frac{E}{X^\alpha} \\
= \frac{1}{6X^3} \left[ X^2 (-aAe^{-aT} - bB^{-bT})(-aAe^{-aT} - bB^{-bT}) \right. \\
- \frac{3}{2} X \{ (W_0 + Ae^{-aT} + B^{-bT})(-aAe^{-aT} - bB^{-bT}) \\
+ (W_0 + Ae^{-aT} + B^{-bT})(-aAe^{-aT} - bB^{-bT}) \} \right] + \frac{E}{X^\alpha} \\
= \frac{E}{X^\alpha} + \frac{e^{-aX}}{6X^2} \left[ aA^2e^{-aX} (aX + 3) + 3W_0aA \cos(aY) \right] \\
+ \frac{e^{-bX}}{6X^2} \left[ bB^2e^{-bX} (bX + 3) + 3W_0bB \cos(bY) \right] \\
+ \frac{e^{-(a+b)X}}{6X^2} \left[ AB \cos((a - b)Y)(2abX + 3a + 3b) \right].
\]
Appendix F

The following script and functions was used to generate the results in section 8.

```matlab
% This script plots the KKLT and Racetrack scalar potentials against the
% inflaton, i.e. the Kähler modulus, for a certain set of parameters. Then
% it calculates the slow-roll parameters for the Racetrack potential at its
% saddle point. The scripts also calculates the mass matrix for the
% Racetrack potential.

% Line 8 and 9 removes variables from the Workspace and closes all Matlab windows.
clear
close all

% Line 12 to 26 defines the set of parameters and makes them global.

global A
global a
global B
global b
global W0
global E
global alpha

A = 1/50;
a = (2*pi)/100;
B = -35/1000;
b = (2*pi)/90;
W0 = -1/25000;
E = 4.14668*10^(-12);
alpha = 2;

[X, Y] = meshgrid(100:1:160, -30:1:30); % This line defines the meshgrid that we plot the Racetrack potential against. X is the real part of the Kähler potential, Y is the imaginary part.
V = potential(X, Y)*10^16; % We define "V" to be the Racetrack potential. The factor of 10^16 is there for scaling purposes.
figure
surf(X, Y, V) % In this line we make the actual surface plot for the Racetrack potential.
title('Racetrack potential')
ylabel('V')
xlabel('X')
```

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35 ylabel('Y')
36 V1=potential1(X, Y)*10^14; % We define "V1" to be the KKL T potential. The factor of 10^14 is there for scaling purposes.
38 figure
39 surf(X, Y, V1) % In this line we make the surface plot for the KKL T potential.
40 title('KKL T potential')
42 xlabel('X')
43 ylabel('Y')
44
50 VnderX=diff((E./X.^alpha) ./(6.*X.^2)) .(a .(A.*2) .(a.*X+3) .exp(-a.*X)+3.*W0.*a.*A.*cos(a.*Y))
51 + (exp(-b.*X) ./(6.*(X.^2))).*b.*(B.*2) .(b.*X+3).*exp(-b.*X)+3.*W0.*b.*B.*cos(b.*Y)+ (exp(-(a+b).*X)) ./(6.*X.^2).*A.*B.*(2.*a.*b.*X+3.*a+3.*b).*cos((a-b).*Y)), X);
52 VnderY=diff((E./X.^alpha) ./(6.*X.^2)) .(a .(A.*2) .(a.*X+3) .exp(-a.*X)+3.*W0.*a.*A.*cos(a.*Y))
53 + (exp(-b.*X) ./(6.*(X.^2))).*b.*(B.*2) .(b.*X+3).*exp(-b.*X)+3.*W0.*b.*B.*cos(b.*Y)+ (exp(-(a+b).*X)) ./(6.*X.^2).*A.*B.*(2.*a.*b.*X+3.*a+3.*b).*cos((a-b).*Y)), Y);
54 Vnder11X=diff(VnderX, X);
55 Vnder11Y=diff(VnderX, Y);
56 Vnder12X=diff(VnderY, X);
57 Vnder12Y=diff(VnderY, Y);
58
60 % In line 57 and 58 we set X and Y to be the position of the saddle point.
61 X=123.22;
63 Y=0;
68 % "Vnderder" is defined as the mass matrix of the Racetrack potential. In line 62 to 65 we define its elements at the saddle point.
69 Vnderder(1,1)=subs(Vnderder11X);
70 Vnderder(1,2)=subs(Vnderder11Y);
71 Vnderder(2,1)=subs(Vnderder12X);
72 Vnderder(2,2)=subs(Vnderder12Y);
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% In line 68 and 69 we express the mass matrix and its eigenvalues.
Q = double(Vderder)
eig(Q)

% In line 73 and 74 we express the value of the Racetrack potential at the saddle point.
derp = ['The value of the racetrack potential at the saddlepoint is ', num2str(potential(X, Y))];
disp(derp)

% In line 78 to 81 we calculate the slow-roll parameters at the saddle point and display their values.
epsilonRT = ((X^2)/3)*(subs(VderY)/potential(X, Y))^2;
etarT = ((2*(X^2))/3)*(Vderder(2,2)/potential(X, Y));
epsilon_saddle = vpa(epsilonRT)
etta_saddle = vpa(etaRT)

% This function contains the Racetrack potential used in "racetrack.m". The
% variables are the real and imaginary part of the Kähler modulus.

function V = potential(X, Y)
    global A
global a
global B
global b
global W0
global E
global alpha
 V = E./(X.^alpha) + (exp(-a.*X))./(6.*(X.^2)).*(a.*(A.^2)
    .*(a.*X+3).*exp(-a.*X)+3.*W0.*a.*A.*cos(a.*Y)) + (exp
    (-b.*X))./(6.*(X.^2)).*(b.*(B.^2)).*(b.*X+3).*exp(-b.*X)
    +3.*W0.*b.*B.*cos(b.*Y) + (exp(-(a+b).*X))./(6.*(X
    .^2)).*(A.*B.*((2.*a.*b.*X+3.*a+3.*b)).*cos((a-b).*Y));
end

% This function contains the KKLT potential used in "racetrack.m". The
% variables are the real and imaginary part of the Kähler modulus.

function V = potential1(X, Y)
    global A
global a
global W0
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8  global E
9  global alpha

10 \( V = \frac{E}{X^\alpha} + \left( (a \times A \times \exp(-a \times X)) \div (2 \times X^{\times 2}) \times (A \times \exp(-a \times X) \times (X \times a) \div 3 + 1) + W0 \times \cos(a \times Y) \right) \);
Appendix G

The following script and functions was used to generate the results in section 9 and 11.

1  % This script finds the X-plane coordinates of the
2  % minimum and saddle
3  % point of the Racetrack potential with two Kahler moduli
4  % as inflaton fields for a certain set of parameters and
5  % fixed Y-plane
6  % coordinates. X and Y are the real and imaginary part of
7  % the inflaton respectively.
8  % Then the script calculate the slow-roll parameters
9  % for a potential with four variables (the variables
10  % being the real and
11  % imaginary parts of the Kahler moduli). Then the script
12  % plot the
13  % potential against the imaginary part of the inflaton
14  % whilst fixing the
15  % real part to its minimum, plot the potential against
16  % the real part of the
17  % inflaton whilst fixing the imaginary part to one of its
18  % minimum and also
19  % plot the potential against the real part of the
20  % inflaton whilst fixing
21  % the imaginary part to one of its maximum.
22  % Line 15 and 16 removes variables from the Workspace and
23  % closes all Matlab windows.
24  clear
25  close all
26
27  % Here we define the set of parameters and make them
28  % global. We also
29  % introduce y1 and y2 as global constants. They will play
20  % the roles of
30  % fixed Y coordinates when necessary.
31  global A
32  global a
33  global B
34  global b
35  global W0
36  global D
37  global y1
38  global y2
39  A=0.56;
40  a=(2*pi)/40;
41  B=7.46666*10^-5;
42  b=(2*pi)/258;
43  W0=5.22666*10^-6;
Here we fix the Y coordinates to a minimum.
\[ y_1 = 0; \]
\[ y_2 = \pi/b; \]

Here we calculate and display the minimum in the X-plane for Y fixed at
\[ y_1 = 0; \]
\[ y_2 = \pi/b; \]

fun = @root2d; % fun contains the derivatives of the
potential.
x0 = [100,180]; % x0 acts as the starting guess for the
Matlab solver "fsolve".
x = fsolve(fun,x0); % This matlab solver finds the
minimum. It does so by finding where the derivatives
are zero.
resultat = [The four dimensional minimum is placed at X1=
num2str(x(1)), ', X2=', num2str(x(2)), ', Y1=0, Y2=
129.]

Here we fix the Y coordinates to a maximum.
\[ y_1 = \pi/a; \]
\[ y_2 = \pi/b; \]

Here we calculate and display the minimum in the X-plane for Y fixed at
\[ y_1 = \pi/a; \]
\[ y_2 = \pi/b; \]

fun = @root2d; % fun contains the derivatives of the
potential.
x0 = [100,180]; % x0 acts as the starting guess for the
Matlab solver "fsolve".
x = fsolve(fun,x0); % This matlab solver finds the
minimum. It does so by finding where the derivatives
are zero.
resultat = [The four dimensional saddle point is placed at
X1=', num2str(x(1)), ', X2=', num2str(x(2)), ', Y1=20,
Y2=129.]

In line 61 and 62 we define the Kahler potential.
syms T1 T2 barT1 barT2
K=log(1296)-2*log((T2+barT1)^3/2)-(T1+barT1)^3/2);

Here we define the elements of the Kahler metric.
barT1K=diff(K, barT1);
barT2K=diff(K, barT2);
K11 = ( \text{diff}(\text{barT}1K, T1))
K12 = ( \text{diff}(\text{barT}2K, T1))
K21 = ( \text{diff}(\text{barT}1K, T2))
K22 = ( \text{diff}(\text{barT}2K, T2))

Here we construct the Kahler metric and its inverse "\text{QQinv}"

\text{QQ}(1, 1) = K11;
\text{QQ}(1, 2) = K12;
\text{QQ}(2, 1) = K21;
\text{QQ}(2, 2) = K22;
\text{QQinv} = \text{inv}(\text{QQ})

Here we introduce the Racetrack potential and then we change its variables from \text{X}, \text{Y} to \text{barT}.

\text{V} = \frac{D}{((X2^* - (3/2)) - (X1^* - (3/2)))^2} + \frac{216}{((X2^* - (3/2)) - (X1^* - (3/2)))^2} + (B^2) \cdot b \cdot (b \cdot (X2 - 2) + 2 \cdot b \cdot (X1 - 2) + 3 \cdot X2) \cdot \exp(-2 \cdot b \cdot X2) + (A^2) \cdot a \cdot (a \cdot (X1 - 2) + 2 \cdot a \cdot (X2 - (3/2))) \cdot \exp((1/2) + 3 \cdot X1) \cdot \exp(-2 \cdot a \cdot X1) + 3 \cdot B \cdot b \cdot W_0 \cdot X2 \cdot \exp(-b \cdot X2) \cdot \cos(b \cdot X2) + 3 \cdot A \cdot a \cdot W_0 \cdot X1 \cdot \exp(-a \cdot X1) \cdot \cos(a \cdot X1) + 3 \cdot A \cdot b \cdot W_0 \cdot X1 \cdot \exp(-b \cdot X2) \cdot \cos(-a \cdot X1 - b \cdot X2)

\text{nyV} = \text{subs(V, [X1, X2, Y1, Y2], [(T1 + \text{barT}1) / 2, (T2 + \text{barT}2) / 2, (T1 - \text{barT}1) / (2 \cdot \sqrt{-1}), (T2 - \text{barT}2) / (2 \cdot \sqrt{-1})])}

Here we calculate \epsilon

\epsilon = (\text{QQinv}(1, 1) \cdot \text{diff}(\text{nyV}, T1) \cdot \text{diff}(\text{nyV}, \text{barT}1) + \text{QQinv}(1, 2) \cdot \text{diff}(\text{nyV}, T1) \cdot \text{diff}(\text{nyV}, \text{barT}2) + \text{QQinv}(2, 1) \cdot \text{diff}(\text{nyV}, T2) \cdot \text{diff}(\text{nyV}, \text{barT}1) + \text{QQinv}(2, 2) \cdot \text{diff}(\text{nyV}, T2) \cdot \text{diff}(\text{nyV}, \text{barT}2)) / (\text{nyV})^2;

In line 90 to 108 we calculate the elements in the matrix from which \eta is obtained.

\text{N11} = (\text{QQinv}(1, 1) \cdot \text{diff}(\text{diff}(\text{nyV}, T1), \text{barT}1) + \text{QQinv}(1, 2) \cdot \text{diff}(\text{diff}(\text{nyV}, T1), \text{barT}2)) / (\text{nyV})
\text{N12} = (\text{QQinv}(1, 1) \cdot \text{diff}(\text{diff}(\text{nyV}, T2), \text{barT}1) + \text{QQinv}(1, 2) \cdot \text{diff}(\text{diff}(\text{nyV}, T2), \text{barT}2)) / (\text{nyV})
\text{N21} = (\text{QQinv}(2, 1) \cdot \text{diff}(\text{diff}(\text{nyV}, T1), \text{barT}1) + \text{QQinv}(2, 2) \cdot \text{diff}(\text{diff}(\text{nyV}, T1), \text{barT}2)) / (\text{nyV})
\text{N22} = (\text{QQinv}(2, 1) \cdot \text{diff}(\text{diff}(\text{nyV}, T2), \text{barT}1) + \text{QQinv}(2, 2) \cdot \text{diff}(\text{diff}(\text{nyV}, T2), \text{barT}2)) / (\text{nyV})

\text{barN11} = (\text{QQinv}(1, 1) \cdot \text{diff}(\text{diff}(\text{nyV}, \text{barT}1), T1) + \text{QQinv}(2, 1) \cdot \text{diff}(\text{diff}(\text{nyV}, \text{barT}1), T2)) / (\text{nyV})
\[ \text{barN12} = (Q^{1,1} \text{diff}(nyV, \text{barT2}), \text{barT1}) + Q^{1,2} \text{diff}(nyV, \text{barT2}), \text{barT2})/nyV; \]
\[ \text{barN21} = (Q^{1,2} \text{diff}(nyV, \text{barT1}), \text{barT1}) + Q^{2,2} \text{diff}(nyV, \text{barT2}), \text{barT2})/nyV; \]
\[ \text{barN22} = (Q^{1,2} \text{diff}(nyV, \text{barT2}), \text{barT1}) + Q^{2,2} \text{diff}(nyV, \text{barT2}), \text{barT2})/nyV; \]
\[ \text{N1bar1} = (Q^{1,1} \text{diff}(nyV, \text{barT1}), \text{barT1}) + Q^{1,2} \text{diff}(nyV, \text{barT2}), \text{barT2})/nyV; \]
\[ \text{N1bar2} = (Q^{1,1} \text{diff}(nyV, \text{barT1}), \text{barT2}) + Q^{1,2} \text{diff}(nyV, \text{barT2}), \text{barT2})/nyV; \]
\[ \text{N2bar1} = (Q^{2,1} \text{diff}(nyV, \text{barT1}), \text{barT1}) + Q^{2,2} \text{diff}(nyV, \text{barT2}), \text{barT2})/nyV; \]
\[ N_2\bar{b}_1 = (Q Q_{inv} (1, 1) * \text{diff}(K, T1), \text{diff}(K, T1) * \text{diff}(\text{diff}(nyV, T1), T2)) \] 

\[ N_2\bar{b}_1 = (Q Q_{inv} (2, 2) * \text{diff}(\text{diff}(nyV, T1), T2)) \]
Here we change variables from $T$, $bar T$ to $X$, $Y$.

epsilon = \text{subs}(\epsilon, [T1, T2, bar T1, bar T2], [X1+\sqrt{1}, X2+\sqrt{1}]*Y1, X1-\sqrt{1}*Y1, X2-\sqrt{1}*Y2]);

bar N1 = \text{subs}(bar N1, [T1, T2, bar T1, bar T2], [X1+\sqrt{1}]*Y1, X2+\sqrt{1}]*Y2, X1-\sqrt{1}]*Y1, X2-\sqrt{1}]*Y2]);

bar N2 = \text{subs}(bar N2, [T1, T2, bar T1, bar T2], [X1+\sqrt{1}] * Y1, X1-\sqrt{1} * Y1, X2-\sqrt{1} * Y2]);

bar N11 = \text{subs}(bar N11, [T1, T2, bar T1, bar T2], [X1+\sqrt{1}]*Y1, X2+\sqrt{1}]*Y2, X1-\sqrt{1}]*Y1, X2-\sqrt{1}]*Y2]);

bar N12 = \text{subs}(bar N12, [T1, T2, bar T1, bar T2], [X1+\sqrt{1}]*Y1, X1-\sqrt{1}]*Y1, X2-\sqrt{1}]*Y2]);

bar N21 = \text{subs}(bar N21, [T1, T2, bar T1, bar T2], [X1+\sqrt{1}]*Y1, X1-\sqrt{1}]*Y1, X2-\sqrt{1}]*Y2]);

bar N22 = \text{subs}(bar N22, [T1, T2, bar T1, bar T2], [X1+\sqrt{1}]*Y1, X1-\sqrt{1}]*Y1, X2-\sqrt{1}]*Y2]);

N1 bar 1 = \text{subs}(N1 bar 1, [T1, T2, bar T1, bar T2], [X1+\sqrt{1}]*Y1, X1-\sqrt{1}]*Y1, X2-\sqrt{1}]*Y2]);

N1 bar 2 = \text{subs}(N1 bar 2, [T1, T2, bar T1, bar T2], [X1+\sqrt{1}]*Y1, X1-\sqrt{1}]*Y1, X2-\sqrt{1}]*Y2]);
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\[ N_{\text{bar}1} = \text{subs}(N_{\text{bar}1}, [T1, T2, T_{\text{bar}1}, T_{\text{bar}2}], [X1+\sqrt{-1}*Y1, X2+\sqrt{-1}*Y1, X1-\sqrt{-1}*Y1, X2-\sqrt{-1}*Y2]); \]

\[ N_{\text{bar}2} = \text{subs}(N_{\text{bar}2}, [T1, T2, T_{\text{bar}1}, T_{\text{bar}2}], [X1+\sqrt{-1}*Y1, X2+\sqrt{-1}*Y1, X1-\sqrt{-1}*Y1, X2-\sqrt{-1}*Y2]); \]

\[ \bar{N}_{\text{bar}1} = \text{subs}(\bar{N}_{\text{bar}1}, [T1, T2, T_{\text{bar}1}, T_{\text{bar}2}], [X1+\sqrt{-1}*Y1, X2+\sqrt{-1}*Y1, X1-\sqrt{-1}*Y1, X2-\sqrt{-1}*Y2]); \]

\[ \bar{N}_{\text{bar}2} = \text{subs}(\bar{N}_{\text{bar}2}, [T1, T2, T_{\text{bar}1}, T_{\text{bar}2}], [X1+\sqrt{-1}*Y1, X2+\sqrt{-1}*Y1, X1-\sqrt{-1}*Y1, X2-\sqrt{-1}*Y2]); \]

\% Here we prepare the matrix from which we obtain \( \eta, Q \), as well as fix the coordinates \( X_1, X_2, Y_1, Y_2 \) to correspond to the four dimensional saddle point.

\[ Q = \text{zeros}(4); \]
\[ X1 = x(1); \]
\[ X2 = x(2); \]
\[ Y1 = y; \]
\[ Y2 = y2; \]

\% Here we calculate and display the value of the slow-roll parameter \( \epsilon \) at the saddle point.
\[ \epsilon = \text{subs} (\epsilon); \]
\[ \epsilon = \text{double}(\epsilon); \]

\% Here we evaluate the matrix \( Q \) at the saddle point as well as calculate the slow-roll parameter \( \eta \) as the same point.
\[ Q(1,1) = \text{subs}(N_{11}); \]
\[ Q(1,2) = \text{subs}(N_{12}); \]
\[ Q(2,1) = \text{subs}(N_{21}); \]
\[ Q(2,2) = \text{subs}(N_{22}); \]
\[ Q(3,3) = \text{subs}(\bar{N}_{11}); \]
\[ Q(3,4) = \text{subs}(\bar{N}_{12}); \]
\[ Q(4,3) = \text{subs}(\bar{N}_{21}); \]
\[ Q(4,4) = \text{subs}(\bar{N}_{22}); \]
\[ Q(1,3) = \text{subs}(N_{1bar1}); \]
\[ Q(1,4) = \text{subs}(N_{1bar2}); \]
\[ Q(2,3) = \text{subs}(N_{2bar1}); \]
\[ Q(2,4) = \text{subs}(N_{2bar2}); \]
\[ Q(3,1) = \text{subs}(\bar{N}_{bar1}); \]
Q(3,2) = subs(bar N1 bar 2);
Q(4,1) = subs(bar N2 bar 1);
Q(4,2) = subs(bar N2 bar 2);
eig(Q);
eta = min(eig(Q));

% Here we display the value of the potential at the saddle point.
derp = ['The value of the racetrack potential at the saddle point is ', num2str(potential2(X1, X2, Y1, Y2))];
disp(derp);

% Here we define the meshgrid for which we plot V2 against. V2 is the potential with Y fixed to a minimum.
[X1, X2] = meshgrid(90:1:170, 150:1:350);
V2 = potential2(X1, X2, 0, 129)*10^-14; % The factor of 10^-14 is there for scaling purposes.

% Here we define the meshgrid for which we plot V1 against. V1 is the potential with X fixed to its minimum.
[Y1, Y2] = meshgrid(-40:2:40, 0:2:300);
V1 = potential2(98.75839, 171.06117, Y1, Y2)*10^-14; % The factor of 10^-14 is there for scaling purposes.

% Line 177 to 184 alters some elements in V2 in order to get a surface plot with better colour scaling.
[M, N] = size(V2);
for i = 1:1:M
    for j = 1:1:N
        if V2(i,j) > 0.51
            V2(i,j) = 0.51;
        end
    end
end
figure
surf(Y1, Y2, V1) % Here we make the V1 plot.
title('Plot of scalar potential versus Im(T1), Im(T2)')
xlabel('V')
ylabel('Y1')

figure
surf(X1, X2, V2) % Here we make the V2 plot.
axis([90 170 150 350 0 0.5]) % We compress the V-axis.
title('Plot of scalar potential versus Re(T1), Re(T2)')
xlabel('V')
ylabel('X1')
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```matlab
ylabel('X2')
figure
surf(X1, X2, V2) % Here we make a zoomed in version of the previous plot.
axis([90 130 150 250 0 0.05])
title('Plot of scalar potential versus Re(T1), Re(T2) zoomed in at the minimum')
zlabel('V')
xlabel('X1')
ylabel('X2')

% Here we define V2 as the potential with Y fixed to a maximum.
V2=potential2(X1, X2, 20, 129)*10^14;

% Line 214 to 221 alters some elements in V2 in order to get a surface plot
% with better colour scaling.
[M, N]=size(V2);
for i=1:1:M
    for j=1:1:N
        if V2(i,j) > 0.51
            V2(i,j) = 0.51;
        end
    end
end
figure
surf(X1, X2, V2) % Here we make the new V2 plot.
axis([90 130 150 250 0 0.05])
title('Plot of scalar potential versus Re(T1), Re(T2)')
zlabel('V')
xlabel('X1')
ylabel('X2')

figure
surf(X1, X2, V2) % Here we make a zoomed in version of the previous plot.
axis([100 120 150 250 0.03 0.05])
title('Plot of scalar potential versus Re(T1), Re(T2) zoomed in at the minimum')
zlabel('V')
xlabel('X1')
ylabel('X2')
```

1 % This function contains the first order X derivatives of the Racetrack potential
2 % used in "racetrack2.m". The Y coordinates are fixed to y1 and y2 respectively.

65
function F = root2d(x)
global A
global a
global B
global b
global W0
global D
global y1
global y2

% Here we calculate the derivatives of the potential.
syms X1 X2
dX1V = diff(D./(((X2.^(3/2))-(X1.^(3/2))).^2) + (216./(((X2.^(3/2))-(X1.^(3/2))).^2)) .* ((B.^(2)) .* b .* (b .* (X1.^(2)) + 2 .* b .* (X2.^(1/2)) + 3.*X2) .* exp(-2.*b.*X2) + (A.^(2)) .* a .* (a .* (X1.^(2)) + 2.*a .* (X2.^(1/2)) + 3.*X1) .* exp(-2.*a.*X1) + A.*B.*W0.*X2.*exp(-a.*X1-b.*X2).*exp(a.*X1+b.*X2+2.*a.*b.*X1.*X2).*exp(-a.*y1+b.*y2) + , X1);
dX2V = diff(D./(((X2.^(3/2))-(X1.^(3/2))).^2) + (216./(((X2.^(3/2))-(X1.^(3/2))).^2)) .* ((B.^(2)) .* b .* (b .* (X1.^(2)) + 2 .* b .* (X2.^(1/2)) + 3.*X2) .* exp(-2.*b.*X2) + (A.^(2)) .* a .* (a .* (X1.^(2)) + 2.*a .* (X2.^(1/2)) + 3.*X1) .* exp(-2.*a.*X1) + A.*B.*W0.*X2.*exp(-a.*X1-b.*X2).*exp(a.*X1+b.*X2+2.*a.*b.*X1.*X2).*exp(-a.*y1+b.*y2) + , X2);

% Here we replace the symbolic variables with the input variables.
f(1)=subs(dX1V, [X1, X2], [x(1), x(2)]);
f(2)=subs(dX2V, [X1, X2], [x(1), x(2)]);

% Here we express the result above as a numerical value.
F(1)=double(f(1));
F(2)=double(f(2));
F=F*10^54; % We rescale "F" in order to make it easier for "fsolve".

% This function contains the Racetrack potential used in "racetrack2.m". The
% variables are the real and imaginary part of the Kahler moduli.

function V = potential2(X1, X2, Y1, Y2)
global A
global a
\[ V = \begin{align*} 
&= \left( \frac{D}{\left( \left( x_2^{3/2} \right)^2 - \left( x_1^{3/2} \right)^2 \right)} \right)^2 + \left( 2^{16} \cdot \left( \left( x_2^{3/2} \right)^2 - \left( x_1^{3/2} \right)^2 \right) \right)^2 
&\quad \cdot \left( \left( b^2 \cdot b \cdot (b \cdot x_2^{1/2}) + 3 \cdot x_2 \right) \cdot \exp \left( -2 \cdot b \cdot x_2 \right) \right) 
&\quad \cdot \left( a \cdot a \cdot (a \cdot x_1^{1/2}) + 2 \cdot a \cdot (x_2^{3/2}) \right) \cdot \cos (a \cdot x_1 - b \cdot x_2) + 3 \cdot A 
&\quad \cdot \exp (-a \cdot x_1 - b \cdot x_2) \cdot (a \cdot x_1 - b \cdot x_2 + 2 \cdot a \cdot b \cdot x_2) \cdot \cos (-a \cdot x_1 - b \cdot x_2) \right) \end{align*} \]
References


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