Properties of generalized hooking networks

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This dissertation consists of two papers and an extended abstract, presented in the following order:

Paper 1:

Paper 2:

Extended abstract:

Introduction

A hooking network is a type of random network. At each step in the growth of the hooking network, a vertex $v$ called a latch is chosen from the network, and a graph $G_i$ is chosen from a collection of graphs called blocks, each with a labelled vertex $h_i$ called a hook. A copy of $G_i$ is attached to the hooking network by fusing together the latch $v$ with the hook $h_i$.

Arguably the most well known random network is the Erdős-Rényi random graph (see [6]). In this model, usually denoted $G(n,m)$, a graph is chosen uniformly at random amongst all graphs with $n$ vertices and $m$ edges. A closely related model (and often also called the Erdős-Rényi random graph) is $G(n,p)$ (see [7]); a graph on $n$ vertices is constructed where for each pair of vertices $v$ and $u$, the edge $e = \{v, u\}$ is added with probability $p \in [0, 1]$, independently of all other edges.

While random graphs have long been studied for their own interest (see for example the following books on random graphs [3, 13, 22]), the interest in random networks has grown with the emerging field of network science. One of the goals of network science is to model real-world networks with different types of randomly grown networks. The properties of the random networks are compared with those of real-world networks to see how well the model fits (see [1, 20] for overviews of network science). Two properties
often studied are the degrees of the vertices in the network, and the distances between vertices. These properties are studied for hooking networks in this dissertation.

The degree distribution $P$ of a graph is defined so that $P(k)$ is the fraction of vertices with degree $k$. Motivated by the degree distributions of real-world networks, Albert and Barabási [2] studied a random network model in 1999 that exhibits what they call preferential attachment. In the Barabási-Albert model, vertices are added one at a time. For a fixed number $m$, the $m$ neighbours of a newly added vertex are chosen amongst the already existing vertices with probability proportional to their degree, so that vertices with higher degree are more likely to be chosen. A similar and more mathematically precise model was studied in 2001 by Bollobas et. al [4].

The graph distance between two vertices is the length of the shortest path between the two vertices. The term small-world refers to the phenomenon by which distances in several real-world networks tend to be relatively small (see [23] and the famous example [19]). Mathematically speaking, a random network model is said to be small-world if the distance between two randomly chosen vertices is of the order $\log(n)$ as the number of vertices $n$ tends to infinity (see [1], or [22, definition 1.7] for a more precise definition).

Randomly grown trees are types of random networks that have also long been studied. A random recursive tree is a rooted tree constructed by starting with a single vertex $r$, which is the root of the tree, then adding vertices one at a time, at each step choosing a vertex uniformly at random amongst the existing vertices to be the parent of the new vertex. These types of trees have been studied since at least 1967 [21]. A plane-oriented random recursive tree is similar, except that the choice of the parent $v$ of each new vertex is made proportionally to the number of children of $v$. The preferential attachment tree generalizes both random tree models by making the choice of the parent $v$ at each step proportionally to $\chi \deg(v) + \rho$ for fixed parameters $\chi$ and $\rho$. The asymptotic degree distributions for all of these random tree models are well understood [17, 18, 12, 10], as are the distances in these networks (see for example [8, 14, 15]).

In the random tree models described above, the process of adding a child to a vertex $v$ can instead be thought of taking a single edge, $K_2$, and fusing one of the vertices of $K_2$ with $v$. In that sense, hooking networks generalize the random trees described above. To grow a hooking network, we fix a set $C = \{G_1, \ldots, G_m\}$ of graphs which we call blocks. Each block $G_i$ has a labelled vertex $h_i$ called a hook, and a weight $p_i$ such that $p_1 + \cdots + p_m = 1$. The network $G_0$ is initialized with a copy of one of the blocks. At every step, the network $G_n$ is constructed from $G_{n-1}$ by choosing a vertex $v$, called a latch, from the network. The choice of the latch is made proportionally to $\chi \deg(v) + \rho$, for real parameters $\chi$ and $\rho$. Next a block $G_i$ is chosen with probability $p_i$. Then, a copy of $G_i$ is attached to $G_{n-1}$ by fusing together the latch $v$ with the hook $h_i$.

These types of networks were first communicated to Cecilia Holmgren and me in January of 2018 by Hosam Mahmoud when he shared a preprint of his paper [16]. He considered what he calls self-similar hooking networks, which are hooking networks grown from a single block called a seed. When I shared an early draft of Paper 1 with Hosam, he replied with early results involving distances in self-similar hooking networks. These results were expanded upon and became the basis of Paper 2.
Paper 1 contains results on the degree distributions of hooking networks. A multivariate normal limit law is proved for the degree distributions of hooking networks as the number of blocks attached tends to infinity. The results are proved via generalized Pólya urns, defined as follows: there are \( q \) types of balls, each is assigned an activity \( a_i \) and a random vector \( \xi_i \). Define the random vector \( X_n = (X_{n,1}, \ldots, X_{n,q}) \), where \( X_{n,i} \) is the number of balls of type \( i \) at time \( n \) in the urn. At each step, a ball is chosen, with the probability of choosing ball \( i \) equal to \( a_i X_{n,i} / (\sum_{j=1}^{q} a_j X_{n,j}) \). If a ball of type \( i \) is chosen at time \( n \), then \( X_{n+1} = X_n + X_{n,i} \), where \( X_{n,i} \sim \xi_i \). The intensity matrix of the urn is defined to be \( A = (a_i \mathbb{E}[\xi_{j,i}])_{i,j=1}^{q} \). Janson [11] proved that under some technical conditions, if the intensity matrix \( A \) has a largest real eigenvalue \( \lambda_1 \) for which \( \lambda < \lambda_1/2 \) for all other eigenvalues \( \lambda \neq \lambda_1 \), then \( n^{-1/2}(X_n - n\mu) \overset{d}{\to} \mathcal{N}(0, \Sigma) \), for some vector \( \mu \) and some covariance matrix \( \Sigma \). In Paper 1, we describe how to view the vertices in the hooking network as balls in an urn. We show that the intensity matrix of such an urn satisfies the necessary conditions for asymptotic normality, and we also calculate the vector \( \mu \) in the normal limit law.

An extended abstract of paper 1, which was published in the conference proceedings of ANALCO19, is included after the papers in this dissertation. It contains a proof that is slightly corrected. I include it in this dissertation because I believe it offers a proof that is slightly easier to follow than the more general proofs given in paper 1.

The methods of paper 1 are also used to prove multivariate normal limit laws for the outdegrees of bipolar networks, introduced by Chen and Mahmoud [5]. These networks are directed graphs built from a set of blocks \( C = \{B_1, \ldots, B_m\} \). Each \( B_i \) has a single source \( N_i \) called the north pole, a single sink \( S_i \) called the south pole, and a weight \( p_i \) so that \( p_1 + \ldots + p_m = 1 \). The network \( B_0 \) is initialized with a copy of one of the blocks. At every step, a vertex \( v \) called a latch is chosen from the network. The choice of the latch is made proportionally to \( \chi \deg^+(v) + \rho \) for real parameters \( \chi \) and \( \rho \), where \( \deg^+(v) \) is the outdegree of \( v \). Next, an arc \( (v, u) \) leading out of \( v \) is chosen uniformly at random amongst all the arcs leading out of \( v \), and a block \( B_j \) is chosen with probability \( p_i \). The arc \( (v, u) \) is removed, then \( v \) is fused with the north pole \( N_i \) of \( B_i \), and \( v \) is fused with the south pole \( S_i \). Paper 1 contains a proof of a multivariate normal limit law for the outdegree distributions of bipolar networks as the number of blocks added tends to infinity.

In paper 2, a normal limit law is proved for the depth of a hooking network as the number of blocks tends to infinity. The depth \( D_n \) is defined to be the shortest distance of a randomly chosen vertex in the hooking network \( G_n \) to the master hook of the network (this is the hook of the first block chosen to initialize the networks). We prove that for hooking networks grown from blocks satisfying some technical assumptions, for a constant \( c \), the moment generating function of the random variable \( \log^{-1/2} n(D_n - c \log n) \) converges to the moment generating function of a normal distribution. It is well known...
that convergence of moment generating functions implies convergence in distribution (see for example [9, Theorem 5.9.5]). From this result, we can conclude that the distance of two randomly chosen vertices is on the order of log $n$ (since this is at most the sum of the distances of the two vertices to the master hook), and so hooking networks satisfying the conditions laid out in Paper 2 are examples of small-world networks.

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References


Normal limit laws for vertex degrees in randomly grown hooking networks and bipolar networks

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Abstract

We consider two types of random networks grown in blocks. Hooking networks are grown from a set of graphs as blocks, each with a labelled vertex called a hook. At each step in the growth of the network, a vertex called a latch is chosen from the hooking network and a copy of one of the blocks is attached by fusing its hook with the latch. Bipolar networks are grown from a set of directed graphs as blocks, each with a single source and a single sink. At each step in the growth of the network, an arc is chosen and is replaced with a copy of one of the blocks. Using Pólya urns, we prove normal limit laws for the degree distributions of both networks. We extend previous results by allowing for more than one block in the growth of the networks and by studying arbitrarily large degrees.

Keywords: Hooking networks, bipolar networks, central limit laws, Pólya urns, random trees, preferential attachment.

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1 Introduction

Several random tree models have been studied where at each step in the growth of the network, a vertex \( v \) is chosen amongst all the vertices of the tree, and a child is added to \( v \). When the choice of \( v \) is made uniformly at random, these trees are called random recursive trees. When the choice of \( v \) is made proportionally

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to its degree $\text{deg}(v)$, these trees are called *plane-oriented random recursive trees*. Both models are examples of *preferential attachment trees*, where the choice of $v$ is made proportionally to $\chi \text{deg}(v) + \rho$ for real parameters $\chi$ and $\rho$ (notice that a preferential attachment tree is a random recursive tree when $\chi = 0$ and is a plane-oriented random recursive tree when $\rho = 0$). Pólya urns were used to prove multivariate normal limit laws for the degree distributions in all of these random tree models [4, 6, 9, 10].

The process of adding a child to a vertex $v$ in a tree can instead be thought of as taking a single edge $K_2$ with a labelled vertex $h$, and fusing together the vertices $v$ and $h$. Hooking networks are grown in a similar manner from a set of graphs $\mathcal{C} = \{G_1, G_2, \ldots, G_m\}$, called *blocks*, where each block $G_i$ has a labelled vertex $h_i$ called a *hook*. At each step in the growth of the network, a vertex $v$ called a *latch* is chosen from the network, a block $G_i$ is chosen, and the hook $h_i$ and the vertex $v$ are fused together. A more precise formulation is laid out in Section 1.2.1.

Several graphs can be thought of as hooking networks. Any tree can be grown as a hooking network with a single edge $K_2$ as the only block. A block graph (or clique graph) is a hooking network whose blocks are complete graphs, and a cactus graph is a hooking network whose blocks are cycles (and that may include a single edge $K_2$ in the set of blocks).

We prove multivariate normal limit laws for the degree distributions of hooking networks as the number of blocks attached tends to infinity (see Theorem 1.3). We allow for a preferential attachment scheme for the choice of the latch (i.e., the latch $v$ is chosen proportionally to $\chi \text{deg}(v) + \rho$). We also assign to each block $G_i$ a value $p_i$ such that $p_1 + p_2 + \cdots + p_m = 1$, and choose the block $G_i$ to be attached with probability $p_i$.

Along with the results for degree distributions of the random tree models described above, Theorem 1.3 also generalizes other results on previously studied hooking networks. Gopaladesikan, Mahmoud, and Ward [3] introduced *blocks trees*, which can be thought of as hooking networks grown from a set of trees as blocks, where the root of each block has a single child and acts as the hook. In their model, the latch is chosen uniformly at random at each step, and the block to be attached is chosen according to an assigned probability value. They proved a normal limit law for the number of leaves (vertices with degree 1) in blocks trees. Mahmoud [8] proved multivariate normal limit laws for the number of vertices with small degrees in *self-similar hooking networks*, which are hooking networks grown from a single block called a *seed*. Both the case where the latch is chosen uniformly at random and the case where the latch is chosen proportionally to its degree were studied in [8]. In the extended abstract [2], we presented a proof of multivariate normal limit laws in the specific cases of hooking networks grown from several blocks when the choice of the latch as well as the choice of the block to be attached is made uniformly at random.

The methods used to prove our results for hooking networks also apply to proving multivariate normal limit laws for outdegree distributions of bipolar networks (see Theorem 1.6). Bipolar networks are grown from a set $\mathcal{C} = \{B_1, B_2, \ldots, B_m\}$ of directed graphs, each with a single source $N_i$: a vertex with zero indegree.
(deg^-(N_i) = 0), and a single sink S_i: a vertex with zero outdegree (deg^+(S_i) = 0). At each step in the growth of the network, an arc (v, u) is chosen and is replaced with one of the blocks B_i, by fusing N_i to v and S_i to u; see Section 1.2.2 for a more precise description. Previously, results were obtained for vertices of small outdegrees in bipolar networks grown from a single block, and where the arc (v, u) to be replaced is chosen uniformly at random [1]. We extend previous results by looking at bipolar networks grown from more than one block, by generalizing the choice of the arc to be replaced, and by studying arbitrarily large degrees.

1.1 Composition of the paper

The networks studied are described in more detail in Section 1.2. Alongside the descriptions of the networks, running examples of a hooking network and a bipolar network are described in Sections 1.2.1 and 1.2.2 respectively. Our main results are stated in Section 1.3. These include multivariate normal limit laws for the vectors of degrees of hooking networks and vectors of outdegrees of bipolar networks.

The theory of generalized Pólya urns developed by Janson in [5], which is the main tool used in the proofs, is summarized in Section 2.

The proofs of our main results are presented in Section 3. This is done in three steps. We start by describing how we study the vertices in our networks as balls in urns in Section 3.1. Properties of the intensity matrices for these urns are gathered in Section 3.2. In Section 3.3, we prove that the matrices studied in 3.2 are indeed the intensity matrices for the urns we are studying and, with the help of theorems proved in [5] and stated in Section 2, we finish the proofs of our main results.

1.2 The networks studied

In the growth of hooking networks and in the growth of bipolar networks, a vertex v is chosen at every step. The choice of the vertex v is made with probability proportional to \(\chi \deg(v) + \rho\) in the case of the hooking networks and proportional to \(\chi \deg^+(v) + \rho\) in the case of the bipolar networks, where \(\chi \geq 0\) and \(\rho \in \mathbb{R}\) so that \(\chi + \rho > 0\). Without loss of generality, we can limit the choice of \(\chi\) to 0 or 1. When \(\chi = 1\) we let \(\rho > -1\), while we let \(\rho\) be strictly positive when \(\chi = 0\) to avoid the degenerate case. For a positive integer \(k\), we let \(w_k := \chi k + \rho\).

1.2.1 Hooking networks

Let \(\mathcal{C} = \{G_1, G_2, \ldots, G_m\}\) be a set of connected graphs, each with at least 2 vertices, and each with a labelled vertex \(h_i\). We allow for the graphs to contain self-loops and multiple edges. The graph \(G_i\) is called a block, and the vertex \(h_i\) is called its hook. Each block \(G_i\) is also assigned a positive real number \(p_i\), called its weight, such that \(p_1 + p_2 + \cdots + p_m = 1\). For example, consider the set of blocks in Figure 1, with their hooks labelled and their weights written underneath.

Let \(\chi\) and \(\rho\) be real numbers satisfying the conditions set above. A sequence of hooking networks \(G_0, G_1, G_2, \ldots\) is constructed as follows: one of the blocks \(G_i\) is
chosen, and we set $G_0$ to be a copy of $G_i$ (the choice of the first block does not need to be done at random for our methods to work). The vertex $H$ that corresponds to the hook of this first block copied to make $G_0$ is called the master hook of the hooking networks constructed afterwards; when all the blocks are trees the master hook acts as the root of the network. Recursively for $n \geq 1$, the hooking network $G_n$ is constructed from $G_{n-1}$ by first choosing a latch $v$ at random proportionally to $\chi \deg(v) + \rho$ amongst all the vertices of $G_{n-1}$, then choosing a block $G_i$ according to its weight $p_i$. A copy of $G_i$ is attached to $G_{n-1}$ by fusing together the latch $v$ with the hook $h_i$ of the copy of $G_i$; that is, $h_i$ is deleted and edges are drawn from $v$ to the former neighbours of $h_i$. Figure 2 is a sequence of hooking networks constructed from the set of blocks in Figure 1 by taking a copy of $G_3$ and attaching copies of $G_4$, then $G_2$, and finally a copy of $G_1$. The master hook of the network is labelled $H$, and at each step the vertex chosen to be the latch is denoted by $\ast$.

Figure 1: A set of simple graphs as blocks

Figure 2: A sequence of hooking networks grown from the blocks $G_1, G_2, G_3$ and $G_4$ of Figure 1
1.2.2 Bipolar networks

For a vertex $v$ in a directed graph $B$, we denote by $\text{deg}^-(v)$ the indegree of $v$: the number of arcs leading into $v$, and by $\text{deg}^+(v)$ the outdegree of $v$: the number of arcs leading out of $v$. If $\text{deg}^-(v) = 0$ then $v$ is called a source, and if $\text{deg}^+(v) = 0$, $v$ is called a sink. A bipolar directed graph is a connected directed graph with at least 2 vertices, that contains a unique source $N$ which we call the north pole of $B$, and a unique sink $S$ which we call the south pole of $B$.

Let $C = \{B_1, B_2, \ldots, B_m\}$ be a set of bipolar directed graphs, each with their north pole $N_i$ and south pole $S_i$ identified. Each $B_i$ is called a block, and is assigned a weight $p_i$ such that $p_1 + p_2 + \cdots + p_m = 1$. For example, consider the set of blocks in Figure 3, with their north and south poles identified as well as their weights.

$$
\begin{align*}
N_1 & \quad S_1 \\
B_1 & \quad p_1 = 1/2 \\
N_2 & \quad S_2 \\
B_2 & \quad p_2 = 1/2
\end{align*}
$$

Figure 3: A set of bipolar directed graphs as blocks

Once again, we let $\chi$ and $\rho$ be real numbers satisfying the conditions set at the beginning of this section. We choose a block $B_i$ and set the bipolar network $B_0$ to be a copy of $B_i$ (once again, the choice of the first block need not be made at random). The vertices corresponding to the north and south poles of $B_0$ serve as the master source $N$ and master sink $S$ respectively of the bipolar networks constructed afterwards. For $n \geq 1$, the bipolar network $B_n$ is constructed from $B_{n-1}$ in a manner similar to that of hooking networks. First, a latch $v$ is chosen proportionally to $\chi \text{deg}^+(v) + \rho$ amongst all the vertices in $B_{n-1}$ that are not the master sink, then one of the arcs $(v,u)$ leading out of $v$ is chosen uniformly at random amongst all the arcs leading out of $v$, and finally a block $B_i$ is chosen according to its weight $p_i$. The arc $(v,u)$ is deleted, and a copy of the block $B_i$ is added by fusing the north pole $B_i$ with $v$, and fusing the south pole $S_i$ with $u$. We never allow the master sink to be chosen as a latch (since it has no arcs leading out of it). Figure 4 is a sequence of bipolar networks constructed from the blocks in Figure 3. The master source $N$ and the master sink $S$ are labelled, and at each step, the latch $v$ is denoted by $\ast$, and the arc $(v,u)$ to be removed is dashed.

Previously, Chen and Mahmoud [1] studied what they called self-similar bipolar networks. These are bipolar networks grown from a single bipolar directed graph as the only block. At each step in the growth of their networks, an arc $(v,u)$ is chosen uniformly at random amongst all the arcs to be deleted before being replaced with a copy of the block. This is equivalent to choosing $v$ proportionally to its outdegree $\text{deg}^+(v)$, and then choosing an arc $(v,u)$ uniformly at random amongst all the arcs leading out of $v$. Therefore, the model of bipolar networks introduced here extends...
1.3 Main results

Before we state the main results, we need a useful definition. In the interest of length, \textit{(out)degree} is used to denote either degree or outdegree in the following discussion, with the distinction being clear from the context.

Depending on the set of blocks that are used to grow the hooking networks or bipolar networks, it is possible for some positive integers to never appear as the \textit{(out)degree} of a vertex in the network, while some integers are only the \textit{(out)degree} of at most one vertex at some point in the growth of the network. By ignoring these so-called \textit{inadmissible (out)degrees}, formally defined below, the proofs using Pólya urns are simplified. We also show by a simple argument below (see Proposition 1.2) that only the master hook or master source may have an inadmissible \textit{(out)degree}. Excluding this single vertex from the \textit{(out)degree} distributions does not affect the asymptotic behaviour of these distributions.

**Definition 1.1.** Given a set $\mathcal{C}$ of blocks, a positive integer $k$ is called an \textit{admissible (out)degree} if with positive probability, there is some $n$ so that the $n$-th iteration of the network grown out of $\mathcal{C}$ has at least two vertices with \textit{(out)degree} $k$. A positive integer is called an \textit{inadmissible (out)degree} if it is not an admissible \textit{(out)degree}.

**Remark 1.1.** Our definition of admissible \textit{(out)degrees} differs slightly from that used in [1] and [8], where any \textit{(out)degree} that may appear in the network is considered an admissible \textit{(out)degree}.

In the example of hooking networks grown in Section 1.2.1 from the blocks in Figure 1, all of the hooks of the blocks have even degrees, and every other vertex in
the blocks has odd degrees. As a result, during the growth of the hooking networks, only the master hook has even degree, while every other vertex has odd degree (as is evidenced by the hooking networks in Figure 2). In that case, the even numbers are admissible degrees, and the odd numbers are inadmissible.

**Proposition 1.2.** The only vertex in a hooking network (or bipolar network) that can have an inadmissible (out)degree is the master hook (or master source) of the network.

**Proof.** We only prove the proposition for admissible degrees in hooking networks; the argument is similar for bipolar networks.

Suppose there is a positive probability that a vertex \( v \) which is not the master hook has degree \( k \) in the hooking network \( G_n \), and without loss of generality let \( n \) be the smallest number for which \( G_n \) has a vertex \( v \) with degree \( k \). We will show that with positive probability, another vertex that is not the master hook will have degree \( k \) in a later iteration of the hooking network.

The vertex \( v \) first appears in the network as a non-hook vertex with degree \( k_0 \) of a newly added block; say the block was \( G_{i_0} \) and \( v \) is a copy of the vertex \( v_0 \) in \( G_{i_0} \). If \( k_0 \neq k \), then that means hooks of other blocks were fused to \( v \), the first hook fused to \( v \) belonged to \( G_{i_1} \), the second belonged to \( G_{i_2} \), and so on until the last hook fused to \( v \) which belonged to \( G_{i_r} \) (which was the last block added to create \( G_n \)). With positive probability, a copy of the block \( G_{i_0} \) is joined to \( G_n \) by fusing the hook of \( G_{i_0} \) with a vertex that is not \( v \), say the master hook. Let \( u \) be the newly added vertex in the hooking network that is a copy of \( v_0 \) in \( G_{i_0} \).

For \( j = 1, \ldots, r \), there is a positive probability that the block \( G_{i_j} \) is added to the hooking network \( G_{n+j} \) by fusing the hook of \( G_{i_j} \) with \( u \). In this case, \( u \) has degree \( k \) in \( G_{n+r+1} \), and so there is a positive probability that 2 vertices (\( v \) and \( u \)) have degree \( k \) in \( G_{n+r+1} \). Therefore, \( k \) is an admissible degree.

Also note that in the case of bipolar networks, only the master sink of the network has outdegree 0, and we therefore ignore this vertex completely.

### 1.3.1 Main results for hooking networks

Let \( \mathcal{C} = \{G_1, G_2, \ldots, G_m\} \) be a set of blocks, each with an identified hook \( h_i \), and let \( (G, G_1, G_2, \ldots) \) be a sequence of hooking networks grown from \( C \), with the master hook of the network labelled \( H \). We allow for the latches and the blocks to be added at each step to be chosen in the manner laid out in Section 1.2 (that is, with linear preferential attachment with parameters \( \chi \) and \( \rho \), and weights \( p_i \) assigned to each block \( G_i \)). For a positive integer \( r \), let

\[
k_1 < k_2 < \cdots < k_r
\]

be the first \( r \) admissible degrees. For a positive integer \( k \), recall that \( w_k = \chi k + \rho \). For each block \( G_i \) in the set \( C \), let \( V(G_i) \) be its vertex set. For a positive integer \( k \), define

\[
f(k) := \sum_{G_i \in \mathcal{C}} p_i \cdot |\{v \in V(G_i) \setminus \{h_i\} : \deg(v) = k\}|
\]  

(1)
and
\[ g(k) := \sum_{G_i \in \mathcal{C}} p_i. \]  

(2)

The value \( f(k) \) is the expected number of new vertices of degree \( k \) (that are not hooks) added at any step, and \( g(k) \) is the probability that the degree of the latch chosen at any step is increased by \( k \) after fusing with the hook of the newly attached block. For example, for the blocks in Figure 1 we have that \( f(1) = 2 \) and \( f(3) = \frac{5}{3} \), while \( g(2) = \frac{1}{3} \) and \( g(4) = \frac{2}{3} \). Define
\[ \lambda_1 := \sum_{k \geq 1} (w_k f(k) + \chi k g(k)). \]  

(3)

Let \( \nu_1 := f(k_1)/(\lambda_1 + w_{k_1}) \), and define recursively for \( i = 2, \ldots, r \)
\[ \nu_i := \frac{1}{\lambda_1 + w_{k_i}} \left( f(k_i) + \sum_{j=1}^{i-1} w_{k_j} g(k_i - k_j) \nu_j \right). \]  

(4)

Let \( \nu \) be the vector
\[ \nu := (\nu_1, \nu_2, \ldots, \nu_r). \]  

(5)

For our running example of hooking networks grown from the blocks in Figure 1, if we let \( \chi = 1 \) and \( \rho = 0 \), then
\[ \lambda_1 = \frac{31}{3} \]  

(6)

and if we let \( r = 3 \), then the first 3 admissible degrees are 1, 3, 5 (recall that only odd numbers are admissible in this example), and
\[ \nu = \left( \frac{6}{34}, \frac{11}{85}, \frac{63}{3910} \right). \]  

(7)

We have the following multivariate normal limit law for the degrees of hooking networks:

**Theorem 1.3.** Let \( X_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,r}) \), where \( X_{n,i} \) is the number of vertices with admissible degree \( k_i \) in \( G_n \), where \( G_n \) is a hooking network grown from the set of blocks \( \mathcal{C} \) using linear preferential attachment with parameters \( \chi \) and \( \rho \). Let \( \lambda_1 \) be defined as in (3) and let \( \nu \) be the vector defined in (4) and (5). Then
\[ n^{-1/2}(X_n - n\lambda_1\nu) \xrightarrow{d} \mathcal{N}(0, \Sigma) \]  

for some covariance matrix \( \Sigma \).

In some special cases, we can say even more about the convergence in (8). For each block \( G_i \), let \( E(G_i) \) be the set of edges of \( G_i \), and let
\[ s_i := \sum_{u \in V(G_i)} (\chi \deg(u) + \rho) - \rho = 2\chi|E(G_i)| + \rho(|V(G_i)| - 1). \]  

(9)
Corollary 1.4. Let $X_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,r})$, where $X_{n,i}$ is the number of vertices with admissible degree $k_i$ in $G_n$, where $G_n$ is a hooking network grown from the set of blocks $C$ using linear preferential attachment with parameters $\chi$ and $\rho$. Let $\lambda_1$ be defined as in (3), let $\nu$ be the vector defined in (4) and (5), and let $s_i$ be defined as in (9) for each block $G_i$. Suppose that there exists a constant $s$ so that $s_i = s$ for all blocks $G_i$. Then the convergence (8) holds in all moments. In particular, $n^{-1/2}(\mathbb{E}X_n - n\lambda_1\nu) \to 0$, and so $n\lambda_1\nu$ in (8) can be replaced by $\mathbb{E}X_n$.

There are several cases where Corollary 1.4 applies. An obvious example is when there is only one block to choose from. Other examples include when $\chi = 0$ and all the graphs have the same number of vertices, or when $\rho = 0$ and all the graphs have the same number of edges.

To compare Theorem 1.3 with previous results on random recursive trees and preferential attachment trees, consider a hooking network grown from a single edge $K_2$ as the only block and where $\chi = 0$ and $\rho = 1$; as discussed earlier this produces random recursive trees. In this case, $f(1) = 1$, $g(1) = 1$, and $\lambda_1 = 1$, and so for any positive integer $r$ the vector $\nu = (\nu_1, \ldots, \nu_r)$ defined in (5) is given by

$$\nu = \left(\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2^r}\right).$$

We see that Theorem 1.3 includes previous results on random recursive trees [6, 9].

More generally, suppose that we look at a preferential attachment tree, where the latch $v$ is chosen with probability proportional to $\chi \deg v + \rho$. We once again have $f(1) = 1$ and $g(1) = 1$, and we have that $\lambda_1 = w_1 + \chi = w_2$. We see that $\nu_1 = 1/(w_2 + w_1)$ and by following the recursion of (4) we see that for any $i = 2, 3, \ldots$, then $\nu_i$ is given by

$$\nu_i = \frac{w_{i-1}}{w_2 + w_1} \left(\prod_{j=2}^{i-1} \frac{w_{j-1}}{w_2 + w_j}\right).$$

In particular when $\chi = 1$ and $\rho = 0$, then $n\lambda_1\nu_i = \frac{4n}{n(n+1)(n+2)}$, and so we see that Theorem 1.3 includes previous results on plane-oriented random recursive trees [6, 10], while (10) along with Theorem 1.3 is the result stated in [4, Theorem 12.2].

Remark 1.5. In the literature on random recursive trees and preferential attachment trees, the choice of the latch is usually made proportionally to $\chi \deg^+(v) + \rho'$, where $\deg^+(v)$ is the number of children of $v$. But we can simply let $\rho = \rho' - \chi$ to get the same model, and replace $w_k$ with $w'_{k-1} = \chi(k-1) + \rho'$ so that (10) resembles more the statements of the previous results [6, 4, 9, 10]. The only vertex where this does not translate is the root (or master hook) of the network, since $\deg(H) = \deg^+(H)$ in this case, but see Remarks 2.2 and 3.4 below for why this doesn’t affect the limiting distribution.

1.3.2 Main results for bipolar networks

Let $\mathcal{C} = \{B_1, B_2, \ldots, B_m\}$ be a set of blocks each with a north pole $N_i$ and a south pole $S_i$ identified, and let $B_0, B_1, B_2, \ldots$ be a sequence of bipolar networks grown
from $\mathcal{C}$, with the master source labelled $N$ and the master sink labelled $S$. The latches $v$, arcs $(v, u)$, and blocks $B_i$ are chosen in the manner laid out in Section 1.2 (by linear preferential attachment with parameters $\chi$ and $\rho$ for the latch, uniformly at random amongst arcs leading out of $v$ for $(u, v)$, and according to its weight $p_i$ for $B_i$). For a positive integer $r$, let

$$k_1 < k_2 < \cdots < k_r$$

be the first $r$ admissible outdegrees. We introduce similar notations as for the hooking network case. Again, recall that for a positive integer $k$, we let $w_k = \chi k + \rho$. For each block $B_i \in \mathcal{C}$, let $V(B_i)$ be its vertex set. For a positive integer $k$, define

$$f(k) := \sum_{B_i \in \mathcal{C}} p_i \cdot |\{v \in V(B_i) \setminus \{N_i, S_i\} : \deg^+(v) = k\}|$$  \hspace{1cm} (11)

and for a nonnegative integer $k$, define

$$g(k) := \sum_{\substack{B_i \in \mathcal{C} \\ \deg^+(N_i) = k+1}} p_i.$$  \hspace{1cm} (12)

The value $f(k)$ is the expected number of new vertices of outdegree $k$ added at any step, and $g(k)$ is the probability that the outdegree of a latch $v$ is increased by $k$ when $(u, v)$ is replaced with a block (note here that $g(0) \neq 0$ if there is a block whose north pole has outdegree 1). For the blocks of Figure 3, then $f(1) = 1$, $f(2) = 1$, and $f(3) = 1/2$, while $g(0) = 1/2$ and $g(1) = 1/2$. For a set of blocks $\mathcal{C}$, define

$$\lambda_1 := \sum_{k \geq 1} (w_k f(k) + \chi k g(k)).$$  \hspace{1cm} (13)

Let $\psi_1 := f(k_1)/ (\lambda_1 + w_{k_1} (1 - g(0)))$, and define recursively for $i = 2, \ldots, r$

$$\psi_i := \frac{1}{\lambda_1 + w_{k_i} (1 - g(0))} \left( f(k_i) + \sum_{j=1}^{i-1} w_{k_j} g(k_i - k_j) \psi_j \right).$$  \hspace{1cm} (14)

Define

$$\psi := (\psi_1, \psi_2, \ldots, \psi_r).$$  \hspace{1cm} (15)

For our running example of bipolar networks grown from the blocks in Figure 3, if we let $\chi = 0$ and $\rho = 1$, then

$$\lambda_1 = \frac{5}{2}$$  \hspace{1cm} (16)

and if we let $r = 3$, then the first 3 admissible outdegrees are 1, 2, 3, and

$$\psi = \left( \frac{1}{3}, \frac{7}{18}, \frac{25}{108} \right).$$  \hspace{1cm} (17)

We have the following multivariate normal limit law for the outdegrees in the growth of bipolar networks:
for each

If the drawn ball is of type a ball of type

At each time a random vector given (random or not) vector entry (or colours 1

A generalized Pólya urn process ( \( X_n \)) \(_{n=0}^\infty \) is defined as follows. There are \( q \) types (or colours) 1, 2, ..., \( q \) of balls and for each vector \( X_n = (X_{n,1},X_{n,2},\ldots,X_{n,q}) \), the entry \( X_{n,i} \geq 0 \) is the number of balls of type \( i \) in the urn at time \( n \), starting with a given (random or not) vector \( X_0 \). Each type \( i \) is assigned an activity \( a_i \in \mathbb{R}_{\geq 0} \) and a random vector \( \xi_i = (\xi_{i,1},\xi_{i,2},\ldots,\xi_{i,q}) \) satisfying \( \xi_{i,j} \geq 0 \) for \( i \neq j \) and \( \xi_{i,i} \geq -1 \). At each time \( n \geq 1 \), a ball is drawn at random so that the probability of choosing a ball of type \( i \) is

\[
\frac{a_i X_{n-1,i}}{\sum_{j=1}^q a_j X_{n-1,j}}.
\]

If the drawn ball is of type \( i \), then it is replaced along with \( \Delta X_{n,j} \) balls of type \( j \) for each \( j = 1,\ldots,q \), where the vector \( \Delta X_n = (\Delta X_{n,1},\Delta X_{n,2},\ldots,\Delta X_{n,q}) \) has the

\[
\text{Theorem 1.6. Let } Y_n = (Y_{n,1}, Y_{n,2}, \ldots, Y_{n,r}), \text{ where } Y_{n,i} \text{ is the number of vertices with outdegree } k_i \text{ in } B_n, \text{ where } B_n \text{ is a bipolar network grown from the set of blocks } C \text{ using linear preferential attachment with parameters } \chi \text{ and } \rho. \text{ Let } \lambda_1 \text{ be defined as in (13) and let } \psi \text{ be the vector defined in (14) and (15). Then }
\]

\[
n^{-1/2}(Y_n - n\psi) \overset{d}{\to} \mathcal{N}(0, \Sigma)
\]

for some covariance matrix \( \Sigma \).

Once again, we can say something more about the convergence in (18) in certain cases. For each block \( B_i \), let \( E(B_i) \) be the set of arcs of \( B_i \), and let

\[
s_i := \sum_{u \in V(B_i)} (\chi \deg^+(u) + \rho) - \chi - \rho = \chi(|E(B_i)| - 1) + \rho(|V(B_i)| - 1).
\]

\[
\text{Corollary 1.7. Let } Y_n = (Y_{n,1}, Y_{n,2}, \ldots, Y_{n,r}), \text{ where } Y_{n,i} \text{ is the number of vertices with admissible outdegree } k_i \text{ in } B_n, \text{ where } B_n \text{ is a bipolar network grown from the set of blocks } C \text{ using linear preferential attachment with parameters } \chi \text{ and } \rho. \text{ Let } \lambda_1 \text{ be defined as in (13), let } \psi \text{ be the vector defined in (14) and (15), and let } s_i \text{ be defined as in (19) for each block } B_i. \text{ Suppose that there exists a constant } s \text{ so that } s_i = s \text{ for all blocks } B_i. \text{ Then the convergence (18) holds in all moments. In particular, } n^{-1/2}(\mathbb{E}Y_n - n\lambda_1 s) \to 0, \text{ and so } n\lambda_1 \psi \text{ in (18) can be replaced by } \mathbb{E}Y_n.
\]

\[
\text{Remark 1.8. We could choose to study the indegrees of bipolar networks instead. Consider networks } B_0, B_1, B_2, \ldots \text{ grown from the blocks } C = \{B_1, \ldots, B_m\}. \text{ Now we choose the latch } v \text{ proportionally to } \chi \deg^-(v) + \rho \text{ (instead of } \chi \deg^+(v) + \rho), \text{ and the arc to be replaced with a block is chosen uniformly at random amongst the arcs leading into } v \text{ (instead of leading out of } v). \text{ The multivariate normal limit law for the indegree distribution of such networks is the same as that for the outdegree distribution of bipolar networks } B_0', B_1', B_2', \ldots \text{ grown in the manner laid out in Section 1.2.2 from the blocks } C = \{B_1', \ldots, B_m'\}, \text{ where the arcs of } B_i \text{ are reversed to make } B_i'.
\]

\section{Pólya urns}

A generalized Pólya urn process \((X_n)_{n=0}^\infty\) is defined as follows. There are \( q \) types (or colours) 1, 2, ..., \( q \) of balls and for each vector \( X_n = (X_{n,1},X_{n,2},\ldots,X_{n,q}) \), the entry \( X_{n,i} \geq 0 \) is the number of balls of type \( i \) in the urn at time \( n \), starting with a given (random or not) vector \( X_0 \). Each type \( i \) is assigned an activity \( a_i \in \mathbb{R}_{\geq 0} \) and a random vector \( \xi_i = (\xi_{i,1},\xi_{i,2},\ldots,\xi_{i,q}) \) satisfying \( \xi_{i,j} \geq 0 \) for \( i \neq j \) and \( \xi_{i,i} \geq -1 \). At each time \( n \geq 1 \), a ball is drawn at random so that the probability of choosing a ball of type \( i \) is

\[
\frac{a_i X_{n-1,i}}{\sum_{j=1}^q a_j X_{n-1,j}}.
\]

If the drawn ball is of type \( i \), then it is replaced along with \( \Delta X_{n,j} \) balls of type \( j \) for each \( j = 1,\ldots,q \), where the vector \( \Delta X_n = (\Delta X_{n,1},\Delta X_{n,2},\ldots,\Delta X_{n,q}) \) has the
same distribution as $\xi_i$ and is independent of everything else that has happened so far. We allow for $\Delta X_{n,i} = -1$, in which case the drawn ball is not replaced.

The intensity matrix of the Pólya urn is the $q \times q$ matrix

$$A := (a_j E\xi_{j,i})_{i,j=1}^q.$$  

By the choice of $\xi_{i,j}$, the matrix $\alpha I + A$ has non-negative entries for a large enough $\alpha$, and so by the standard Perron-Frobenius theory, $A$ has a real eigenvalue $\lambda_1$ such that all other eigenvalues $\lambda \neq \lambda_1$ satisfy $\text{Re} \lambda < \lambda_1$.

The following assumptions (A1)–(A7) are used in [5]. In the interpretation of balls in an urn, then the random vectors $\xi_i$ and $\Delta_n$ are integer-valued. However, for our applications, this is not necessarily the case, which is why our assumption (A1) below takes a slightly different form from the standard assumption (A1) in [5], taking instead the form discussed in [5, Remark 4.2] (note the indices of the variables in (A1) below). A type $i$ is called dominating if in an urn starting with a single ball of type $i$, there is a positive probability that a ball of type $j$ can be found in the urn at some time for every other type $j$. If every type is dominating, then the urn and its intensity matrix $A$ are irreducible.

(A1) For each $i$, either

(a) there is a real number $d_i > 0$ such that $X_{0,i}$ and $\xi_{1,i}, \xi_{2,i}, \ldots, \xi_{q,i}$ are multiplies of $d_i$ and $\xi_{i,i} \geq -d_i$, or

(b) $\xi_{i,i} \geq 0$.

(A2) $E(\xi_{i,j}^2) < \infty$ for all $i, j \in \{1, 2, \ldots, q\}$.

(A3) The largest eigenvalue $\lambda_1$ of $A$ is positive.

(A4) The largest eigenvalue $\lambda_1$ of $A$ is simple.

(A5) There exists a dominating type $i$ with $X_{0,i} > 0$.

(A6) $\lambda_1$ is an eigenvalue of the submatrix of $A$ given by the dominating types.

(A7) At each time $n \geq 1$, there exists a ball of dominating type.

In the Pólya urns we use, it is obvious that (A1) and (A2) hold. Our intensity matrices are also irreducible, and so (A5) and (A6) hold trivially, while the Perron-Frobenius theorem along with irreducibility guarantee that (A3) and (A4) hold. Our urns always have balls of positive activity, and so (A7) holds by the irreducibility of the urns.

Denote column vectors as $v$ with $v'$ as its transpose. The transpose of a matrix $A$ is also denoted as $A'$. Let $a = (a_1, \ldots, a_q)'$ denote the vector of activities, and let $u'_1$ and $v_1$ be the left and right eigenvectors of $A$ corresponding to the eigenvalue $\lambda_1$ normalized so that $a \cdot v_1 = a' v_1 = v'_1 a = 1$ and $u_1 \cdot v_1 = u'_1 v_1 = v'_1 u_1 = 1$. Define $P_{\lambda_1} = v_1 u'_1$ and $P_l = I_q - P_{\lambda_1}$. Define the matrices

$$B_i := E(\xi_i \xi'_i)$$  

for every $i = 1, \ldots, q$, denote $v_1 = (v_{1,1}, v_{1,2}, \ldots, v_{1,q})'$, and define the matrix

$$B := \sum_{i=1}^q v_{1,i} a_i B_i.$$  

(20)
In the case where \( \text{Re}\lambda < \lambda_1/2 \) for every eigenvalue \( \lambda \neq \lambda_1 \), define
\[
\Sigma_I := \int_0^\infty P_I e^{sA} B e^{sA'} P_I e^{-\lambda_1 s} ds,
\] (21)
where \( e^{-A} = \sum_{j=0}^\infty t^j A^j / j! \).

The result we use from [5] guarantees that if (A1)–(A7) hold and \( \text{Re}\lambda < \lambda_1/2 \) for all eigenvalues \( \lambda \neq \lambda_1 \), then
\[
n^{-1/2} (X_n - n\mu) \overset{d}{\to} N(0, \Sigma)
\] for some \( \mu = (\mu_1, \ldots, \mu_q) \) and \( \Sigma = (\sigma_{i,j})_{i,j=1}^q \). We state below results from [5] (and gathered in [4, Theorem 4.1]) which give the conditions for convergence to multivariate normal distributions as well as the values of \( \mu \) and \( \Sigma \).

**Theorem 2.1** ([5, Theorem 3.22 and Lemmas 5.4 and 5.3(i)]). Assume (A1)–(A7) and that the right and left eigenvectors corresponding to \( \lambda_1 \) are normalized as above.

(i) Then, as \( n \to \infty \),
\[
n^{-1/2} (X_n - n\mu) \overset{d}{\to} N(0, \Sigma)
\] (22)
with \( \mu = \lambda_1 v_1 \) and some covariance matrix \( \Sigma \).

(ii) Suppose further that, for some \( c > 0 \),
\[
a \cdot E(\xi_i) = c
\]
for every \( i = 1, \ldots, q \). Then the covariance matrix is given by \( \Sigma = c\Sigma_I \), where \( \Sigma_I \) is defined in (21).

(iii) Suppose that (ii) holds and that the matrix \( A \) is diagonalizable, and let \( \{u_i\}_{i=1}^q \) and \( \{v_i\}_{i=1}^q \) be dual basis of left and right eigenvectors respectively, i.e., \( u_i A = \lambda_i u_i \), \( Av_i = \lambda_i v_i \), and \( u_i \cdot v_j = \delta_{i,j} \). Then the covariance matrix \( \Sigma \) is given by
\[
\Sigma = c \sum_{j,k=2}^q \frac{u_j B u_k}{\lambda_1 - \lambda_j - \lambda_k} v_j v_k',
\] (23)
where \( B \) is defined in (20).

**Remark 2.2.** So long as (A5) is satisfied, the initial configuration \( X_0 \) of the urn does not have any effects on the limiting distribution.

**Remark 2.3.** By convergence in probability and dominated convergence (see for example [4, Remark 4.3]), we get that
\[
E X_n / n \to \mu.
\]

**Remark 2.4.** A recent result by Janson and Pouyanne [7] guarantees the convergence (22) to hold in all moments for certain balanced generalized Pólya urns; an urn is balanced if the change in total activity at every step is constant. Some of our urns satisfy the conditions of [7, Theorem 1.1] which implies in particular that \( n^{-1/2} (E X_n - n\mu) \to 0 \), and so \( n\mu \) in (22) can be replaced by \( E X_n \).
3 Proofs

We start by setting up the Pólya urns so that balls in the urn correspond to vertices in the growth of our network. Next, we prove important properties of the intensity matrices associated with these Pólya urns. Finally, the pieces are placed together to prove our main results.

3.1 Vertices as balls

In this section, we outline how we use the evolution of generalized Pólya urns to describe the evolutions of the degree distributions in the networks that we study. Throughout the section the notation \((\text{out})\text{degree}\) is used so that the discussion applies to both types of networks simultaneously. Recall that Theorem 1.3 and Corollary 1.4 apply to degrees of hooking networks, while Theorem 1.6 and Corollary 1.7 apply to outdegrees of bipolar networks.

We start by first looking at an urn with infinitely many types. We assign a type to each \((\text{out})\text{degree}\) in the network so that a ball of type \(k\) represents a vertex of \((\text{out})\text{degree}\) \(k\). We initiate each network by choosing a block from the list of blocks. This corresponds to starting a Pólya urn with a ball of the matching type for the \((\text{out})\text{degree}\) of each vertex in the block. In the evolution of the network, when a block is attached, this corresponds to choosing a ball in the urn of type corresponding to the \((\text{out})\text{degree}\) of the latch \(v\) and replacing it with a ball representing the new \((\text{out})\text{degree}\) of \(v\) along with balls representing the \((\text{out})\text{degrees}\) of the rest of the vertices of the block. Since a latch of \((\text{out})\text{degree}\) \(k\) is chosen at random proportionally to \(w_k = \chi k + \rho\), then all balls of type \(k\) have activity \(w_k\) in the Pólya urn so that a ball of type \(k\) is chosen at random proportionally to its activity \(w_k\).

The Pólya urn described above has infinitely many types, and so Theorem 2.1 does not apply. Therefore, we would like to instead use an urn with finitely many types in the same manner as is done in [4] and [6]. The urn is replaced with the following Pólya urn: let \(d\) be a positive integer corresponding to the largest \((\text{out})\text{degree}\) we wish to study in this instance of the model. A new ball of special type \(*\) with activity \(a_* = 1\) is introduced, and for every \(k > d\), each ball of type \(k\) is replaced with \(w_k\) balls of special type \(*\). In this way, the probability of choosing a ball of special type in the new urn is equal to the probability of choosing a ball of type \(k > d\) in the old urn. If a latch \(v\) with \((\text{out})\text{degree}\) \(k \leq d\) is chosen, and a block is attached so that \(v\) now has \((\text{out})\text{degree}\) \(k + j > d\), then the ball of type \(k\) is removed and \(w_{k+j}\) balls of special type are added. If instead \(v\) has \((\text{out})\text{degree}\) \(k > d\) and a block is attached so that the \((\text{out})\text{degree}\) of the vertex is now \(k + j\), then the ball of special type that was chosen is placed back in the urn, along with \(\chi j\) balls of special type.

The final change we will make to our urn is to represent the master hook of the hooking network or the master source of the bipolar network, say with \((\text{out})\text{degree}\) \(k\), with \(w_k\) balls of special type in our urn. This guarantees that all types of balls in the urn that are not special types correspond to \((\text{out})\text{degrees}\) that are admissible;
recall from Definition 1.1 that a positive integer \( k \) is an admissible (out)degree if there is a positive probability that at some point in the growth of the network at least two vertices have (out)degree \( k \), and recall from Proposition 1.2 that only the master hook of the hooking network or the master source of the bipolar network may have an inadmissible degree. For a positive integer \( d \), the possible types of balls present in the urn are exactly the admissible (out)degrees less than or equal to \( d \), together with a ball of special type \( \star \). In our intensity matrix, we can then omit the rows and columns corresponding to types that are never present in the urn. By restricting to admissible (out)degrees, it can be verified that now every ball in the urn is of dominating type. No matter the initial network (or initial configuration of the urn), there is a positive probability that a ball representing a vertex with the admissible (out)degree \( k \) will be present in the urn. Therefore the urn (and its intensity matrix) is irreducible. As discussed in Section 2, it is easy to verify that the assumptions (A1)–(A7) are satisfied for irreducible urns. To avoid confusion, we label the type of a ball with the (out)degree of the vertex it represents.

We illustrate how to calculate the intensity matrices for the urns associated with our running examples of a hooking network and a bipolar network given in Section 1.2.

### 3.1.1 A Pólya urn for our running example of a hooking network

Consider the blocks in Figure 1, and a sequence of hooking networks grown from these blocks. Let’s look at the instance of the model where the choice of a latch is made proportionally to its degree (i.e., when \( \chi = 1 \), \( \rho = 0 \) and so \( w_k = k \)). Suppose we look at vertices with degrees less than or equal to 5. As discussed after the definition of admissible degrees (Definition 1.1), the admissible degrees for these hooking networks are the odd numbers; and so 1, 3, 5 are the admissible degrees less than or equal to 5. The images in Figure 5 illustrates the possibilities of replacing a ball of type \( k \), corresponding to attaching a block to a latch with degree \( k \). The probabilities in the figure are the weights assigned to the blocks in Figure 1.

The intensity matrix for this urn has 4 rows and columns: one of each for balls of type 1, 3, 5, and the last row and column for balls of special type \( \star \). Let’s consider what happens when a block is attached to a latch with degree 1; this corresponds to choosing a ball of type 1. The probability that the block \( G_1 \) is attached is 1/6. The hook of \( G_1 \) has degree 2 and the two other vertices have degree 1. Then the ball of type 1 is removed and replaced with a ball of type 3 (the new degree of the latch \( v \)) along with two new balls of type 1. Performing similar calculations for the other blocks with the help of Figure 5, we get that

\[
E x_1 = \frac{1}{6} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 1 \\ 3 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} -1 \\ 4 \\ 1 \\ 0 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 6 \\ 12 \\ 4 \\ 0 \end{pmatrix}.
\]

Recall that the rows and columns for inadmissible degrees are removed, and so the
first row represents balls of type 1, the second row for balls of type 3, the third for balls of type 5, and the final row for balls of special type $\gamma$.

Now consider what happens when a ball of type 3 is chosen, i.e., if a vertex $v$ with degree 3 is chosen as a latch. If a hook with degree 4 is attached to $v$, the degree of $v$ is increased to 7. Recall that we instead place $w_\gamma = 7$ balls of special type when this happens. Performing similar calculations as above with the help of Figure 5 yields

$$E_{\xi_3} = \frac{1}{6} \begin{pmatrix} 2 \\ -1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 4 \\ -1 \\ 0 \\ 7 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 \\ 3 \\ 0 \\ 7 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 12 \\ 4 \\ 2 \\ 28 \end{pmatrix}. $$

Performing similar calculations when a ball of type 5 is chosen gives

$$E_{\xi_5} = \frac{1}{6} \begin{pmatrix} 2 \\ 0 \\ -1 \\ 7 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 4 \\ 0 \\ -1 \\ 9 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 2 \\ 2 \\ -1 \\ 7 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 0 \\ 4 \\ -1 \\ 9 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 12 \\ 10 \\ -6 \\ 50 \end{pmatrix}. $$

Finally let’s consider attaching a block to a vertex of degree greater than 5, or to the master hook of the network. In either case, this corresponds to choosing a ball
of special type. If the hook of the block \( G_i \) attached has degree two, then the ball of special type is replaced along with another \( 2 \chi = 2 \) balls of special type, while \( 4 \chi = 4 \) balls of special type are added if the hook has degree 4. Therefore, we calculate for the special type \(*\)

\[
\mathbb{E}\xi_* = \frac{1}{6} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 4 \\ 0 \\ 0 \\ 4 \end{pmatrix} + \frac{1}{6} \begin{pmatrix} 2 \\ 2 \\ 0 \\ 2 \end{pmatrix} + \frac{1}{3} \begin{pmatrix} 4 \\ 4 \\ 0 \\ 4 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 12 \\ 10 \\ 12 \\ 10 \end{pmatrix}.
\]

The activities for the types are \( w_1 = 1, w_3 = 3 \) and \( w_5 = 5 \) for types 1, 3, 5 respectively, while the special type \(*\) has activity 1. The intensity matrix \( A \) consists of \( \mathbb{E}\xi_1, 3\mathbb{E}\xi_3, 5\mathbb{E}\xi_5 \) for the first 3 columns, and \( \mathbb{E}\xi_* \) for the last column, thus we get

\[
A = \frac{1}{6} \begin{pmatrix} 6 & 36 & 60 & 12 \\ 12 & 12 & 50 & 10 \\ 4 & 6 & -30 & 0 \\ 0 & 84 & 250 & 20 \end{pmatrix}.
\]

One can verify that the eigenvalues of \( A \) are \( \lambda_1 = 31/3 \) and \(-1, -3, -5\) and we see that \( \lambda_1 \) is precisely the same that was defined in (6). By Theorem 2.1, we have a multivariate normal limit law. One can also verify that the right eigenvector \( v_1 \) of \( A \) associated with \( \lambda_1 \) satisfying \( a \cdot v_1 = 1 \), where \( a = (1, 3, 5, 1) \) is the vector of activities, is

\[
v_1 = \begin{pmatrix} 6 \\ 34 \\ 85 \\ 3913910 \\ 3910 \end{pmatrix}.
\]

Restricted to the first 3 entries, the vector \( v_1 \) is exactly the vector \( \nu \) defined in (7), and so by Theorem 2.1, Theorem 1.3 is true in this particular case.

### 3.1.2 A Pólya urn for our running example of a bipolar network

Now consider the blocks of Figure 3 and a sequence of bipolar networks grown from these blocks. Let’s look at the instance of the model where the choice of the latch is made uniformly at random (i.e., when \( \chi = 0, \rho = 1 \), and so \( w_k = 1 \)). All positive integers are admissible outdegrees. The images of Figure 6 illustrate the possibilities of replacing a ball of type \( k \), corresponding to choosing a latch \( v \) with outdegree \( k \) and one of the arcs leading out of \( v \) uniformly at random. The probabilities in the figure are the weights assigned to the blocks in Figure 3.

Suppose we look at vertices with outdegrees less than or equal to 3. We can calculate the intensity matrix in the same way as the intensity matrix for the hooking network example above. The main difference in this case is that there is a positive probability that the outdegree of a latch \( v \) is not changed. For example, if a ball of type 2 is chosen; that is, if a latch \( v \) with outdegree 2 is chosen, then with probability \( 1/2 \), the degree of \( v \) is not changed after the block \( B_1 \) is attached. In this case, the ball of type 2 is replaced in the urn, along with 2 balls of type 1 and
For the urn in this case, a vertex with outdegree greater than 3 is represented by a single ball of special type ${\ast}$. The intensity matrix is

$$\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$$

The eigenvalues for $A$ are $\lambda_1 = \frac{5}{2}$ and $-\frac{1}{2}$, and we see that $\lambda_1$ is precisely the same that was defined in (16). The right eigenvector $v_1$ of $A$ associated with $\lambda_1$ whose entries sum to 1 is

$$v_1 = \begin{pmatrix} 1 & 7 & 25 & 5 \\ 3 & 18 & 108 & 108 \end{pmatrix}.'$$

Restricted to the first 3 entries, the vector $v_1$ is exactly the vector $\psi$ defined in (17). The multivariate normal limit law claimed by Theorem 1.6 then holds by Theorem 2.1.

### 3.2 Properties of the intensity matrices

Recall that $w_k = \chi_k + \rho$. Let $k_1 < \cdots < k_r$ be positive integers. Let $f$ and $g$ be nonnegative functions. Let $A = (a_{ij})_{i,j=1}^{r+1}$ be the $(r+1) \times (r+1)$ matrix with
In the next section, we will prove that the intensity matrices of our urns are indeed of the same form as $A$, where $f(k)$ was introduced in (1) and (11), $g(k)$ was introduced in (2) and (12), and $k_1,\ldots,k_r$ are admissible degrees. But first, we prove properties of $A$ that are useful to the proofs of our main results. From Theorem 2.1, we see that to prove our main result, we need to prove properties of the eigenvalues and eigenvectors of $A$.

We need three assumptions on $f(k)$ and $g(k)$ to prove our lemmas. In the next section we show that these assumptions are satisfied for the values of $f(k)$ and $g(k)$ defined for our models:

(F) For all $k \leq k_r$ such that $k \neq k_i$ for all $i = 1,\ldots,r$, then $f(k) = 0$.

(G1) $\sum_{k \geq 0} g(k) = 1$.

(G2) For all $k \leq k_r$ such that $k \neq k_i$ for all $i = 1,\ldots,r$, then $g(k - k_j) = 0$ for all $j = 1,\ldots,r$.

Let

$$\lambda_1 = \sum_{k \geq 1} (w_k f(k) + \chi k g(k)).$$

We calculate the eigenvalues of $A$ in the following lemma.

**Lemma 3.1.** Assuming (F), (G1), and (G2), the matrix $A$ has eigenvalues

$$\lambda_1, w_{k_1} (g(0) - 1), w_{k_2} (g(0) - 1), \ldots, w_{k_r} (g(0) - 1).$$

**Proof.** We can calculate the eigenvalues of $A$ directly. For any $\lambda$, look at the matrix $A - \lambda I$. For each $i = 1,\ldots,r$, add $w_{k_i}$ times row $i$ to row $r + 1$ of $A - \lambda I$ to get the matrix $A'_\lambda$. Using assumptions (F) and (G2), along the $(r + 1)$-th row of $A'_\lambda$, the $j$-th entry for $j = 1,\ldots,r$ is

$$w_{k_j} \left( \sum_{k \geq 1} w_k f(k) + \sum_{k \geq k_j} w_k g(k - k_j) - w_{k_j} - \lambda \right)$$

while the $(r + 1)$-th entry is

$$\sum_{k \geq 1} (w_k f(k) + \chi k g(k)) - \lambda = \lambda_1 - \lambda.$$
Next, subtract \( w_k j \) times column \( r + 1 \) from column \( j \) in \( A'_\lambda \) for every \( j = 1, \ldots, r \) to get the matrix \( A''_\lambda \). Since

\[
w_k g(k - k_j) - \chi(k - k_j)g(k - k_j) = (\chi k_j + \rho)g(k - k_j) = w_k j g(k - k_j), \quad (28)
\]

the \( j \)-th entry for \( j = 1, \ldots, r \) of the \((r + 1)\)-th row is

\[
w_k j \left( \sum_{k \geq 1} w_k f(k) + \sum_{k \geq k_j} w_k g(k - k_j) - w_k j - \lambda \right) - w_k j (\lambda_1 - \lambda)
\]

\[
= w_k j \left( \sum_{k \geq 1} w_k f(k) + \sum_{k \geq k_j} w_k g(k - k_j) - w_k j - \lambda_1 \right) \quad (29)
\]

\[
= w_k j \left( \sum_{k \geq 1} w_k f(k) + \sum_{k \geq k_j} w_k g(k - k_j) - w_k j - \sum_{k \geq 1} (w_k f(k) + \chi k g(k)) \right)
\]

\[
= w_k j \sum_{k \geq k_j} w_k g(k - k_j) - w_k j \sum_{k \geq k_j} \chi(k - k_j)g(k - k_j) - w_k^2 j
\]

\[
= w_k j \sum_{k \geq k_j} w_k j g(k - k_j) - w_k^2 j \quad \text{(by (28))}
\]

\[
= w_k^2 j - w_k^2 j = 0. \quad \text{(by assumption (G1))}
\]

For every \( i, j \leq r \), then the \( i, j \)-th entry of \( A''_\lambda \) is simply \( a_{ij} = w_k j f(k_i) \) when \( i \neq j \) and \( a_{ii} - \lambda = w_k j f(k_i) \) on the diagonals, where \( a_{ij} \) is given in (25). Therefore, \( A''_\lambda \) is the following \((r + 1) \times (r + 1)\) matrix

\[
A''_\lambda = \begin{pmatrix}
  w_k j (g(0) - 1) - \lambda & 0 & \cdots & 0 & f(k_1) \\
  w_k j g(k_2 - k_1) & w_k j (g(0) - 1) - \lambda & \cdots & 0 & f(k_2) \\
  \vdots & \vdots & \ddots & \vdots & \vdots \\
  w_k j g(k_r - k_1) & w_k j g(k_r - k_2) & \cdots & w_k j (g(0) - 1) - \lambda & f(k_r) \\
  0 & 0 & \cdots & 0 & \lambda_1 - \lambda
\end{pmatrix}
\]

Since the determinant of a matrix is unchanged by adding one row to another or by subtracting a column from another, both \( A - \lambda I \) and \( A''_\lambda \) have the same determinant. We can calculate the determinant of \( A''_\lambda \) by expanding along the bottom row, and since the upper \( r \times r \) matrix of \( A''_\lambda \) is lower triangular, we see immediately that \( A \) has characteristic polynomial

\[
\pm(\lambda - \lambda_1) \prod_{i=1}^{r} (w_k i (g(0) - 1) - \lambda),
\]

from which we can read off the eigenvalues stated in the lemma. \( \square \)
We now calculate the right eigenvector of $A$ associated with $\lambda_1$. Let $v_{1,1} = f(k_1)/(\lambda_1 + w_{k_1}(1 - g(0)))$, and define recursively for $i = 2, \ldots, r$

$$v_{1,i} = \frac{1}{\lambda_1 + w_{k_i}(1 - g(0))} \left( f(k_i) + \sum_{j=1}^{i-1} w_{k_j} g(k_i - k_j) v_{1,j} \right). \quad (30)$$

Finally, define

$$v_{1,r+1} = 1 - \sum_{j=1}^{r} w_{k_j} v_{1,j} \quad (31)$$

and

$$v_1 = (v_{1,1}, v_{1,2}, \ldots, v_{1,r}, v_{1,r+1})'.$$

**Lemma 3.2.** Let $v_1$ be the vector defined above, and let $a = (w_{k_1}, w_{k_2}, \ldots, w_{k_r}, 1)'$. Assuming $(F), (G1),$ and $(G2),$ the vector $v_1$ is the unique right eigenvector of $A$ associated with $\lambda_1$ for which $a \cdot v_1 = 1$.

**Proof.** We verify that $v_1$ is a right eigenvector of $A$ associated with $\lambda_1$. We can look instead at $A'_{\lambda_1}$ which is introduced in the previous proof. Since only row operations were used to get from $A - \lambda I$ to $A'_{\lambda_1}$, then $(A - \lambda I) v_1 = 0$ if and only if $A'_{\lambda_1} v_1 = 0$. We therefore need only to verify that $A'_{\lambda_1} v_1 = 0$ (where all instances of $\lambda$ are replaced with $\lambda_1$).

Along the $(r + 1)$-th row of $A'_{\lambda_1}$ for any $j = 1, \ldots, r$ the $j$-th entry is given by (26), but with $\lambda$ replaced by $\lambda_1$, which is exactly (29), and so is equal to 0 by the calculations performed above. From (27), the $(r + 1)$-th entry in the $(r + 1)$-th row is simply $\lambda_1 - \lambda_1 = 0$. Therefore, the last row of $A'_{\lambda_1}$ is all zeros and the $(r + 1)$-th entry of the vector $A'_{\lambda_1} v_1$ is 0.

The top $r \times (r + 1)$ submatrix of $A'_{\lambda_1}$ is the same as the top $r \times (r + 1)$ submatrix of $A - \lambda_1 I$. After rearranging the equality (30) as

$$f(k_i) + \sum_{j=1}^{i-1} w_{k_j} g(k_i - k_j) v_{1,j} = v_{1,i} (\lambda_1 + w_{k_i}(1 - g(0))) \quad (33)$$

and recalling the entries $a_{ij}$ of $A$ from (25), we see that for $i = 1, \ldots, r$, the $i$-th entry of the vector $A'_{\lambda_1} v_1$ is

$$\sum_{j=1}^{i-1} a_{ij} v_{1,j} + (a_{ii} - \lambda_1) v_{1,i} + \sum_{j=i+1}^{r} a_{ij} v_{1,j} + a_{i,i+1} v_{1,i+1}$$

$$= f(k_i) \left( \sum_{j=1}^{r} w_{k_j} v_{1,j} + v_{1,r+1} \right) + \sum_{j=1}^{i-1} w_{k_j} g(k_i - k_j) v_{1,j} + v_{1,i} (w_{k_i}(g(0) - 1) - \lambda_1)$$

$$= f(k_i) + \sum_{j=1}^{i-1} w_{k_j} g(k_i - k_j) v_{1,j} + v_{1,i} (w_{k_i}(g(0) - 1) - \lambda_1) \quad \text{(by (31))}$$

$$= v_{1,i} (\lambda_1 + w_{k_i}(1 - g(0))) + v_{1,i} (w_{k_i}(g(0) - 1) - \lambda_1) = 0. \quad \text{(by (33))}$$

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Therefore \( A_{\lambda_1} v_1 = 0 \). Furthermore,
\[
a \cdot v_1 = \sum_{j=1}^r w_{k_j} v_{1,j} + \left( 1 - \sum_{j=1}^r w_{k_j} v_{1,j} \right) = 1.
\]

Since \( \lambda_1 \) has algebraic (and geometric) multiplicity 1, then \( v_1 \) is the unique vector satisfying the statement of the lemma.

### 3.3 Proofs of main results

Recall the definitions of \( f(k) \) from (1) and (11), and \( g(k) \) from (2) and (12) for a set of blocks \( C \). Recall also that \( w_k = \chi k + \rho \). Let \( k_1 < \cdots < k_r \) be the first \( r \) admissible (out)degrees for hooking networks or bipolar networks grown from \( C \).

We start by proving that the assumptions (F), (G1), and (G2) of the previous Section 3.2 hold for our networks:

**Lemma 3.3.** For \( f(k) \) defined in (1) and (11), and \( g(k) \) defined in (2) and (12), the assumptions (F), (G1), and (G2) hold.

**Proof.** In the interest of space, the lemma is proved for both hooking networks and bipolar networks simultaneously. The notation \((\text{out})\text{degree}\) is used, and is interpreted as degree for hooking networks and outdegree for bipolar networks.

If \( f(k) \neq 0 \), then there is a positive probability that at any step in the growth of the network, a new vertex (that is not the master hook or the master source) appears with \((\text{out})\text{degree}\) \( k \). By Definition 1.1 and by Proposition 1.2, \( k \) is an admissible \((\text{out})\text{degree}\) in this case, and so if \( k \leq k_r \), then \( k \in \{k_1, \ldots, k_r\} \), proving that (F) holds. The assumption (G1) holds since \( \sum_{k \geq 0} g(k) = p_1 + \cdots + p_m = 1 \), where \( p_i \) is the weight of the block \( G_i \) or \( B_i \). As for the assumption (G2), assume that \( g(k - k_j) \neq 0 \) for some admissible \((\text{out})\text{degree}\) \( k_j \leq k_r \). Since \( k_j \) is an admissible \((\text{out})\text{degree}\), there is a positive probability that some vertex \( v \) (that is not the master hook or the master source) has \((\text{out})\text{degree}\) \( k_j \). By definition, there is a probability of \( g(k - k_j) \) that the \((\text{out})\text{degree}\) of \( v \) is increased to \( k \) if a hook is fused to \( v \). Therefore, there is a positive probability that there is a vertex with \((\text{out})\text{degree}\) \( k \), and so \( k \) is an admissible \((\text{out})\text{degree}\), again by Definition 1.1 and Proposition 1.2. If \( k \leq k_r \), then \( k \in \{k_1, \ldots, k_r\} \), and so (G2) holds.

We now prove Theorem 1.3; the multivariate normal limit law for the degrees of hooking networks. Our main results for bipolar networks can be proved in a very similar manner, and we only outline the differences in the proofs.

**Proof of Theorem 1.3.** For a positive integer \( r \), let \( k_1 < \cdots < k_r \) be the smallest \( r \) admissible degrees. We look at two cases: when a block is attached to a latch that is not the master hook of the network with degree less than or equal to \( k_r \), and when a block is attached to a latch of degree greater than \( k_r \) or to the master hook of the network. Recall that the master hook of the network is represented by
balls of special type in the urn.

**Case I:** Let \( k_j \leq k_r \) be an admissible degree and suppose that at some step in the growth of the network a vertex \( v \) is chosen as a latch where \( \deg(v) = k_j \) and \( v \) is not the master hook of the network. Suppose a block is attached to \( v \). This corresponds to choosing a ball of type \( k_j \). Let \( k_i \leq k_r \) be an admissible degree. Other than the latch, the expected number of new vertices of degree \( k_i \) added to the network is equal to \( f(k_i) \). If \( k_i > k_j \), the probability that the degree of \( v \) is increased to \( k_i \) is equal to the probability of choosing a block whose hook has degree \( k_i > k_j \), which is exactly \( g(k_i - k_j) \). For \( k_i, k_j \leq k_r \) and with \( \mathbb{E}(\xi_{k_j,k_i}) \) being the expected change in the number of balls of type \( k_i \) in the networks when a ball of type \( k_j \) is chosen, then the arguments above show that

\[
\mathbb{E}(\xi_{k_j,k_i}) = \begin{cases} 
  f(k_i) & i < j \\
  f(k_i) - 1 & i = j \\
  f(k_i) + g(k_i - k_j) & i > j.
\end{cases}
\]

For every \( k \) that is an admissible degree greater than \( k_r \), balls of special type are added instead of balls of type \( k \). By a similar argument as above, the expected number of new balls of special type added corresponding to vertices of degree \( k \) when a latch of degree \( k_j \) is chosen is \( w_k(f(k) + g(k - k_j)) \). Summing over all admissible degrees greater than \( k_r \), then the expected number of balls of special type added when a ball of type \( k_j \) is chosen is

\[
\mathbb{E}(\xi_{k_j,*}) = \sum_{k > k_r} w_k(f(k) + g(k - k_j)).
\]

**Case II:** Now suppose at some step the latch \( v \) is either the master hook of the network or that \( \deg(v) > k_r \). In either case this corresponds to choosing a ball of special type in our urn; recall that the master hook is represented by balls of special type. Suppose that a block is attached to \( v \). For an arbitrary admissible degree \( k_i \leq k_r \), the expected number of new vertices added with degree \( k_i \) is \( f(k_i) \). Therefore with \( \mathbb{E}(\xi_{*,k_i}) \) being the expected number of balls of type \( k_i \) added when a ball of special type is chosen,

\[
\mathbb{E}(\xi_{*,k_i}) = f(k_i).
\]

For any \( k \geq 1 \), the probability that the degree of \( v \) is increased by \( k \) is \( g(k) \). In this case, the ball of special type is placed back in the urn along with \( \chi k \) new balls of special type. For any \( k > k_r \), the expected number of new vertices with degree \( k \) is once again \( f(k) \). Therefore, summing over all values of \( k \), the expected change in the number of balls of special type in the urn is

\[
\mathbb{E}(\xi_{*,*}) = \sum_{k > k_r} w_k f(k) + \sum_{k \geq 1} \chi k g(k).
\]

Let \( \mathbb{E}(\xi_{k_i}) := (\mathbb{E}(\xi_{k_i,k_1}), \ldots, \mathbb{E}(\xi_{k_i,k_r}), \mathbb{E}(\xi_{k_i,\ast})) \) for \( j = 1, \ldots, r \) and for the special type \( \ast \) let \( \mathbb{E}(\xi_{\ast}) := (\mathbb{E}(\xi_{*,k_1}), \ldots, \mathbb{E}(\xi_{*,k_r}), \mathbb{E}(\xi_{*,\ast})) \). The activity of each ball of type
Since assumptions (F), (G1), and (G2) hold by Lemma 3.3, then by Lemma 3.1, the intensity matrix $A$ has largest real eigenvalue $\lambda_1 = \sum_{k \geq 1} (w_k f(k) + \chi(k g(k))) > 0$, and the other eigenvalues $-w_{k1}, -w_{k2}, \ldots, -w_{kr}$ are all negative (and so less than $1/2$).

The vector $v_1$ defined in (32) with $g(0) = 0$ and restricted to the first $r$ entries is exactly the vector $\nu$ defined in (5). Theorem 1.3 now follows immediately from Lemma 3.2, and Theorem 2.1.

Proof of Corollary 1.4. Every time a new block $G_i$ with hook $h_i$ is attached to the hooking network by fusing $h_i$ with the latch $v$, any new vertex $u$ of $G_i$ added to the network is represented either by a ball of type $\deg(u)$ (with activity $\deg(u) + \chi$) or by $\deg(u) + \chi$ balls of special type (with activity 1). As for the latch $v$, one of the following cases applies:

- a ball of activity $\deg(v) + \rho$ is removed and replaced with a ball of activity $\chi(\deg(v) + \deg(h_i)) + \rho$,
- a ball of activity $\deg(v) + \rho$ is removed and replaced with $\chi(\deg(v) + \deg(h_i)) + \rho$ balls of special type (with activity 1), or
- an additional $\deg(h_i)$ balls of special type are added.

In any case the change in the total activity of the urn is

$$s_i = \chi \deg(h_i) + \sum_{u \in V(G_i) \setminus \{h_i\}} (\chi \deg(u) + \rho) = 2 \chi |E(G_i)| + \rho(|V(G_i)| - 1),$$

where the last equality holds thanks to the handshaking lemma (the sum of the degrees in a graph is twice the number of edges). Suppose that all $s_i$ are equal. Then the change in total activity is equal at every step, independent of which block is attached. Therefore, the corresponding urn is balanced. By [7, Remark 1.9], the urn satisfies the conditions of [7, Theorem 1.1], and so by Remark 2.4, Corollary 1.4 holds.

Theorem 1.6 and Corollary 1.7 are proved in a similar manner to the two proofs above. We therefore omit the details, and only specify where the proofs differ.

Proof of Theorem 1.6: The probability that the degree of a latch $v$ is increased by $k$ is now the probability of choosing a block whose north pole had outdegree $k + 1$ (since an arc is removed from $v$ when a block is attached). This probability is exactly defined to be $g(k)$. If a north pole has outdegree 1, then the outdegree of $v$ is not changed, and so the probability that the outdegree of $v$ is unchanged is $g(0)$. With similar arguments as in the proof of Theorem 1.3, we can calculate
the intensity matrix. The only differences between the intensity matrix for bipolar networks and that for hooking networks are the first \( r \) diagonal entries, which are

\[
    w_{ki}^E(\xi_{ki}, k_i) = w_{ki}^E(f(k_i) - g(0) - 1)
\]

for \( i = 1, \ldots, r \) in the case of bipolar networks. The value \( E(\xi_s, s) \) is the same as before since \( \chi kg(k) = 0 \) when \( k = 0 \).

From Lemma 3.1, we get that the largest real eigenvalue of the intensity matrix is \( \lambda_1 = \sum_{k \geq 1} (w_{ki}f(k) + \chi kg(k)) > 0 \) and the other eigenvalues are

\[
    w_{k1}(g(0) - 1), w_{k2}(g(0) - 1), \ldots, w_{kr}(g(0) - 1).
\]

Since \( g(0) \leq 1 \), then each eigenvalue \( \lambda \neq \lambda_1 \) is non-positive, and so is less than \( \lambda_1/2 \). The vector \( v_1 \) defined in (32) restricted to the first \( r \) entries is exactly the vector \( \psi \) defined in (15), and the result now follows just as in the proof of Theorem 1.3.

**Proof of Corollary 1.7.** Since an arc is removed at each step, the total change in activity when block \( B_i \) is attached is (by similar argument to the proof of Corollary 1.4)

\[
    s_i = \chi \deg^+(N_i) - \chi + \sum_{u \in V(B_i) \setminus \{N_i, S_i\}} (\chi \deg^+(u) + \rho) = \chi(|E(B_i) - 1| + \rho(|V(B_i) - 1|).
\]

If all the \( s_i \)'s are equal for every block, then once again the urn is balanced and Corollary 1.7 holds by [7, Theorem 1.1] and Remark 2.4.

**Remark 3.4.** From Remark 2.2 we know that the initial configuration of our urn does not effect the limiting distribution. This means that we may let the original block used to make \( G_0 \) or \( B_0 \) to be chosen at random, or to be deterministic. It also means that if we wanted to change the probability of choosing the master hook of a hooking network or the master source of a bipolar network, we can simply change the number of balls of special type at the beginning of the urn.

**Remark 3.5.** We can say something more about the covariance matrices \( \Sigma \) of Theorems 1.3 and 1.6. With the activity vector \( a = (w_{k1}, \ldots, w_{kr}, 1) \), then

\[
    a \cdot \Sigma(\xi_s) = \sum_{k \geq 1} (w_{ki}f(k) + k\xi g(k)) = \lambda_1,
\]

and for \( j = 1, \ldots, r \),

\[
    a \cdot \Sigma(\xi_{kj}) = \sum_{k \geq 1} w_{ki}f(k) + \sum_{i \geq j} w_{ki}g(k_i - k_j) - w_{kj}
\]

\[
    = \sum_{k \geq 1} w_{ki}f(k) + \sum_{i \geq j} (w_{kj} + \chi(k_i - k_j))g(k_i - k_j) - w_{kj}
\]

\[
    = \sum_{k \geq 1} w_{ki}f(k) + w_{kj} \sum_{k \geq 0} g(k) + \sum_{k \geq 0} \chi kg(k) - w_{kj}
\]

\[
    = \sum_{k \geq 1} w_{ki}f(k) + \sum_{k \geq 0} \chi kg(k) = \lambda_1.
\]
with the last line following from the fact that $\sum_{k \geq 0} g(k) = 1$. Thus, we see that Theorem 2.1 (ii) applies with $c = \lambda_1$ and $\Sigma = \lambda_1 \Sigma_1$, where $\Sigma_1$ is defined in (21).

**Remark 3.6.** Furthermore, if $\chi > 0$, then the values $w_k = \chi k + \rho$ are all different, and so from Lemma 3.1, all of the eigenvalues of $A$ are different. In this case, the matrix $A$ is then diagonalizable, and so Theorem 2.1 (iii) applies and $\Sigma$ can be calculated from (23). The diagonalizability of $A$ does not hold in general, see for example the matrix $A$ of (24).

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**References**


Distances in hooking networks

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Abstract
A hooking network is built by stringing together components randomly chosen from a set of building blocks (graphs with hooks). The nodes are endowed with "affinities" which dictate the attachment mechanism. We study the distance from the master hook to a node in the network chosen according to its affinity after many steps of growth. Such a distance is commonly called the depth of the chosen node.

We present an exact average result and a rather general central limit theorem for the depth. The affinity model covers a wide range of attachment mechanisms, such as uniform attachment and preferential attachment, among others. Naturally, the limiting normal distribution is parametrized by the structure of the building blocks and their probabilities.

We also take the point of view of a visitor uninformed about the affinity mechanism by which the network is built. To explore the network, such a visitor chooses the nodes uniformly at random. We show that the distance distribution under such a uniform choice is similar to the one under random choice according to affinities.

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1 Introduction

Several types of networks grow by repeatedly attaching components chosen from a given set of building blocks. For instance, a social network may start out from a clique of friends, and later additional cliques adjoin themselves to the network. Many species of random trees grow this way, where the blocks are simply paths of length 1. Recursive trees built from blocks that are themselves trees are investigated in [4].

In the context of networks, a building block is a connected graph $G = (V,E)$ with a set of vertices $V$ (also called nodes) and a set of edges $E$ containing at least one edge. A particular vertex in $V$ is distinguished as the hook, the node designated for connecting to larger graphs. A set of building blocks is used as components strung together into a bigger network in discrete time steps. A network grown from one building block is investigated in [7], where the initial building block is called a seed. Because all the components in a collection of size one are the same, the author of [7] calls such a network a self-similar hooking network. The focus in [7] is on local and global degree profiles.

As we consider here a set of possibly more than one building block, we think it is more appropriate to call such networks simply hooking networks. Desmarais and Holmgren [3] considers the global degree profile for a hooking network grown from a set of building blocks, as a generalization of [7]. In the present manuscript, we consider distances in hooking networks.

2 The setup of hooking networks

We have a finite set of random building blocks (each equipped with a hook) from which we grow the hooking network. By random, we mean there is a probability distribution on the set of blocks. More precisely, we build a sequence $G_0, G_1, G_2, \ldots$ of hooking networks from a collection of blocks $C = \{G_1, G_2, \ldots, G_m\}$, where each $G_i$, for $i = 1, \ldots, m$, has a labeled vertex $h_i$ called the hook and a selection probability $p_i \in [0,1]$, and of course $\sum_{i=1}^{m} p_i = 1$.

We denote the degree of vertex $v$ in a graph by $\deg(v)$. To cover a wide variety of growth models, we endow the network with two real parameters: $\chi$ and $\rho$. The affinity, or “attractive power” of a vertex $v$ in the hooking
network $G_{n-1}$ is the value
\[ \chi \deg(v) + \rho, \]
where $\deg(v)$ is the degree of $v$ in $G_{n-1}$. The sum of these quantities over all vertices in $G_{n-1}$, denoted by $\tau_{n-1}$, is the total affinity of $G_{n-1}$. The network $G_n$ is grown from $G_{n-1}$ by “randomly” choosing a latch among the vertices of $G_{n-1}$, where the probability of choosing $v$ as a latch is
\[ \frac{\chi \deg(v) + \rho}{\tau_{n-1}}. \]
So, when $\chi = 0$, the choice is made uniformly at random among all the vertices of $G_{n-1}$, and when $\rho = 0$, the choice is made proportionally to the degree of $v$ in $G_{n-1}$. Once a latch is identified, we choose a block $G_i \in C$ (with probability $p_i$), and attach a copy isomorphic to it to $G_{n-1}$ by joining the hook $h_i$ to the latch. The initial graph $G_0$ is chosen from the collection $C$; the block $G_i$ is chosen with probability $p_i$.

Let $B_n$ denote the $n$th block attached to the hooking network. The vertex corresponding to the hook of the initial block, $B_0$, used to launch the network with $G_0$, is called the master hook of the network and is denoted by $H$.

3 Illustrative example

Let $G_1, G_2, G_3$ be the blocks in Figure 1, with probabilities $p_1 = 1/5$, $p_2 = 1/5$, $p_3 = 3/5$. The hook of $G_i$, for $i = 1, 2, 3$, is labeled with $h_i$. Figure 2 shows the step-by-step evolution of one possible realization of the hooking networks $G_0, G_1, G_2, G_3$ grown from $C = \{G_1, G_2, G_3\}$.

![Figure 1: A collection of random building blocks. The probability of a block appears above the block.](image-url)
Under uniform attachment ($\chi = 0$), the probability of $G_3$ is
\[
\frac{1}{5} \times \left( \frac{1}{4} \times \frac{1}{5} \right) \times \left( \frac{1}{7} \times \frac{1}{5} \right) \times \left( \frac{1}{10} \times \frac{3}{5} \right) = \frac{3}{175000},
\]
and under preferential attachment ($\rho = 0$), the probability of $G_3$ is
\[
\frac{1}{5} \times \left( \frac{1}{4} \times \frac{1}{5} \right) \times \left( \frac{1}{16} \times \frac{1}{5} \right) \times \left( \frac{5}{24} \times \frac{3}{5} \right) = \frac{1}{64000}.
\]

4 Scope

Define $D_n$ to be the shortest distance from a random node $v$ in the network $G_n$ to the master hook $H$. Here, random means according to the node affinity in the network. We call this distance the *depth* of node $v$ in the network at time (age) $n$. For certain choices of $C$, we find the exact average of $D_n$, and prove a normal limit law. This result holds in the case of uniform attachment when all the blocks have the same number of vertices, and in the case of preferential attachment when all the blocks have the same number of vertices and the same number of edges. The theorem holds in more generality, which is explained below.

We also consider an uninformed view of “random” depth, $\bar{D}_n$, according to which the network explorer is not aware of the underlying affinity. The explorer then resorts to a uniform choice of a random node. The result for $\bar{D}_n$ is similar to that for $D_n$.

Let us introduce some notation:
• Denote the cardinality of a set \( E \) by \(|E|\).

• As we have many graphs, we need to qualify which graph is the subject of attention when we speak of vertices and edges of a graph. For a graph \( G \), we use \( V(G) \) for the set of its vertices and \( E(G) \) for the set of its edges.

• Let \( v_i = |V(G_i)| \). We call this the size of \( G_i \).

• Let \( e_i = |E(G_i)| \).

• Let

\[
s_i = 2\chi e_i + (v_i - 1)\rho.
\]

This quantity is the affinity of a copy \( B_n \) of the block \( G_i \) when it is hooked in the network. Note that after hooking, \( h_i \) is absorbed in a node of a previous network \( G_n \), so only \( v_i - 1 \) vertices of \( B_n \) participate in future hooking, while all the edges are intact and participate fully in all the future stages of hooking.

The collections of building blocks we consider for the presentation of Gaussian laws have blocks with the same affinity, which we call \( s \). This requirement can be attained in the following cases:

(a) the blocks have the same number of vertices, when \( \chi = 0 \), in which case \( v := v_1 = v_2 = \cdots = v_m \), and all \( s_i \)’s are the same; we simply set \( s := (v - 1)\rho = s_1 = \cdots = s_m \).

(b) the blocks have the same number of edges, when \( \rho = 0 \), in which case \( e = e_1 = e_2 = \cdots = e_m \), and all \( s_i \)’s are the same, and we simply set \( s := 2\chi e = s_1 = \cdots = s_m \).

(c) the blocks have the same number of vertices and the same number of edges, when both \( \chi \) and \( \rho \) are positive. In this case, we have \( v := v_1 = v_2 = \cdots = v_m \), and \( e := e_1 = e_2 = \cdots = e_m \). All \( s_i \)’s are the same, and we simply set \( s := 2\chi e + (v - 1)\rho = s_1 = \cdots = s_m \).

(d) There are numerous other cases where the blocks have the same affinity, only requiring that the equations \( 2\chi e_i + (v_i - 1)\rho = s \) to have positive integer solutions for \( v_i \) and \( e_i \). For example, if \( \chi = \rho = 1 \), a path of length 3 and a graph of two vertices connected via 4 multiple edges have the same affinity \( s = 9 \).
• Let $\delta_i$ be a random variable that measures the depth of a vertex chosen according to the affinity in (1) within a block (against the total affinity of the block) in the network that is a copy of $G_i$. That is, the distance of a vertex $v$ in the block from the hook $h_i$, where a vertex $v \neq h_i$ is chosen with probability $(\chi \deg(v) + \rho)/s_i$, while the hook is chosen with probability $(\chi \deg(h_i))/s_i$.

In our running example, we have $v := v_1 = v_2 = v_3 = 4, e := e_1 = e_2 = e_3 = 4$, and $s := s_1 = s_2 = s_3 = 8\chi + 3\rho$. It follows that

$$P(\delta_1 = 0) = \frac{2\chi}{s}, \quad P(\delta_1 = 1) = \frac{2\chi + \rho}{s}, \quad P(\delta_1 = 2) = \frac{2\chi + \rho}{s};$$

the multiple 2 in $P(\delta_1 = 1)$ accounts for two nodes at distance 1 from the hook of $G_1$.

Let $\mathbb{I}_A$ be the indicator of event $A$. Finally, define $\Delta_n$ to be the depth within the $n$th block added to the network of a vertex $v$ chosen according to the probabilities in (1). This depth has the representation

$$\Delta_n = \sum_{i=1}^{m} \mathbb{I}_{(B_n = G_i)} \delta_i,$$

where $P(B_n = G_i) = p_i$. Note that $\Delta_n$ has an “average distribution” over the $\delta_i$’s. In other words, $\Delta_n$ is a mixture of the distributions of $\delta_i$, where the mixing is taken according to the probabilities $p_i$.

For our running example, we have

$$P(\Delta_n = 0) = p_1 P(\delta_1 = 0) + p_2 P(\delta_2 = 0) + p_3 P(\delta_3 = 0)$$
$$= \frac{1}{5} \times \frac{2\chi}{s} + \frac{1}{5} \times \frac{3\chi}{s} + \frac{3}{5} \times \frac{\chi}{s}$$
$$= \frac{8\chi}{5s}.$$

Likewise, we have

$$P(\Delta_n = 1) = \frac{21\chi + 8\rho}{5s}, \quad P(\Delta_n = 2) = \frac{11\chi + 7\rho}{5s}.$$

In the general case, each $\Delta_n$ is identically and independently distributed as a generic random variable $\Delta$. In the running example, the mixed distribution is

$$\Delta = \begin{cases} 
0, & \text{with probability } 8\chi/(5s); \\
1, & \text{with probability } (21\chi + 8\rho)/(5s); \\
2, & \text{with probability } (11\chi + 7\rho)/(5s).
\end{cases} $$

6
The average of this distribution is

\[ \mathbb{E}[\Delta] = \frac{43\chi + 22\rho}{40\chi + 15\rho}, \]

and the variance is

\[ \mathbb{V}ar[\Delta] = \frac{751\chi^2 + 523\chi\rho + 56\rho^2}{25(8\chi + 3\rho)^2}. \]

To understand why we defined \( s_i, \delta_i \) and \( \Delta_n \) the way we did, consider what happens when we choose a latch. Figure 3 depicts \( G_3 \) in our running example, with the blocks \( B_0, B_1, B_2, \) and \( B_3 \) indicated.

Figure 3: The network of the running example at age 3, with the building blocks in the various steps identified.

A latch \( v \) is chosen from \( G_{n-1} \) with probability proportional to \( \chi \text{deg}(v) + \rho \). For example, the probability of choosing \( v_1 \) to latch in \( B_4 \) is

\[ \frac{4\chi + \rho}{32\chi + 13\rho}. \]

Note that \( 32\chi + 13\rho = 4s + \rho \).

For the general case, the networks we consider have constant \( s \). Each block added increases the total affinity by \( s \) (as its hook is absorbed) to a network starting with affinity \( \tau_0 = s + \rho \). Note that the initial block retains
its \( \rho \) as the master hook never gets absorbed. The total affinity in the entire network is then

\[ \tau_n = ns + \tau_0 = (n + 1)s + \rho. \]

The depth of \( v_1 \) in \( G_3 \) is 3, as are the depths of \( v_2, v_3 \) and \( v_4 \). The probability of choosing a vertex of depth 3 to latch in \( B_4 \) is then

\[ \frac{4\chi + \rho}{32\chi + 13\rho} + \frac{3\chi + \rho}{32\chi + 13\rho} + \frac{\chi + \rho}{32\chi + 13\rho} + \frac{3\chi + \rho}{32\chi + 13\rho} = \frac{11\chi + 4\rho}{32\chi + 13\rho}. \] (2)

Another way to see the depth of \( v_1 \) is to notice that the hook of \( B_3 \) (which is at \( u \) in \( G_3 \)) is at depth 2, while \( v_1 \) is at depth 1 within \( B_3 \). The hook of \( B_1 \) (which is also \( u \) in \( G_3 \)) is at depth 2, and all of \( v_2, v_3 \) and \( v_4 \) are at depth 1 within \( B_1 \). The hook of \( B_2 \) (which is at \( v_2 \) in \( G_3 \)) is at depth 3.

When choosing a vertex \( v \) to be a latch, we can think of first choosing one of the blocks \( B_0, B_1, B_2 \) or \( B_3 \), and then choosing a vertex within that block. The blocks \( B_1, B_2 \) and \( B_3 \) all have an affinity of \( s \), while \( B_0 \) has an affinity of \( s + \rho \) (since the master hook has not been absorbed). Since \( B_1 \) is a copy of \( G_2 \), \( B_2 \) is a copy of \( G_1 \), and \( B_3 \) is a copy of \( G_3 \), the probability calculated in (2) is alternatively attained with the following calculation:

\[ \frac{s}{4s + \rho} \mathbb{P}(\delta_2 = 1) + \frac{s}{4s + \rho} \mathbb{P}(\delta_1 = 0) + \frac{s}{4s + \rho} \mathbb{P}(\delta_3 = 1) = \frac{s}{4s + \rho} \left( \frac{5\chi + 3\rho}{s} + \frac{2\chi}{s} + \frac{4\chi + \rho}{s} \right). \]

Notice that \( v_2 \) contributes \( \chi + \rho \) to the affinity of \( B_1 \), while also contributing \( 2\chi \) to the affinity of \( B_2 \).

### 5 Main results

We are now poised to state the main theorems. They come in two flavors: exact and asymptotic. Let \( X_n \) be the depth of the latch \( v \) chosen in \( G_{n-1} \) to construct \( G_n \), and let \( D_n \) be the depth of a vertex chosen at random according to its affinity in this latter graph.

We use \( \psi_W(u) := \mathbb{E}[e^{wu}] \) for the moment generating function of any generic random variable \( W \). An exact moment generating function for \( X_n \) is key to the extraction of its exact moments and serves as a starting point for its asymptotic distribution.
When choosing the \( n \)th latch, we can think of first choosing one of the blocks that were previously added, and then choosing a vertex within that block. The depth of the hook of the \( i \)th block added is \( X_i \). Because of the absorption of the hook in a node of a previous network, a block contributes only according to the amount of affinity in its nonhook nodes and the edges connected to the hook (but not the hook itself), the only exception being the initial block as its master hook remains a viable candidate as a latch throughout all the stages. Therefore, the contribution of \( B_i \), for \( i \geq 1 \), is the depth \( X_i \), while that of \( B_0 \) is an adjusted depth \( \Delta_0 \).

To find the relation between \( \psi_\Delta(u) \) and \( \psi_\Delta(u) \), recall the definition of \( \delta_i \).

This random variable has the moment generating function

\[
\psi_\delta_i(u) = \sum_{v \in V_i - \{h_i\}} \frac{1}{s} \left( \chi \text{deg}(v) + \rho \right) e^{d(v)u} + \frac{\chi}{s} \text{deg}(h_i),
\]

with \( d(v) \) being the distance of \( v \) from \( h_i \). Going further, we express this as

\[
\psi_\delta_i(u) = \sum_{v \in V_i} \frac{1}{s} \left( \chi \text{deg}(v) + \rho \right) e^{d(v)u} - \frac{\rho e^{d(h_i)u}}{s} = \sum_{v \in V_i} \frac{1}{s} \left( \chi \text{deg}(v) + \rho \right) e^{d(v)u} - \frac{\rho}{s}.
\]

The distribution of \( \Delta_n \) is obtained by mixing the distributions in the blocks according to their probabilities. More precisely, \( \Delta_n \) has the moment generating function

\[
\psi_\Delta(u) = \sum_{i=1}^{m} p_i \psi_{\delta_i}(u) = \frac{s + \rho}{s} \psi_{\Delta_0}(u) - \frac{\rho}{s}.
\]

**Remark 5.1.** We alert the reader that in the distribution of \( \Delta_0 \), the probability that a node in \( B_0 \) is used as a latch is its affinity relative to the total affinity in \( B_0 \). The particular block \( B_0 \) has an adjusted total affinity equal to \( 2\chi e + \nu \rho \).

We now have the distribution

\[
X_n = \begin{cases} 
X_0 + \Delta_0, & \text{with probability } (s + \rho)/\tau_{n-1}; \\
X_i + \Delta_i, & \text{with probability } s/\tau_{n-1}, \text{for } i = 1, \ldots, n-1,
\end{cases}
\]

and \( X_0 = 0 \).

We next develop \( \psi_{X_n}(u) \) in exact form.
Lemma 5.2.

\[ \psi_{X_n}(u) = \psi_{\Delta_0}(u) \prod_{i=1}^{n-1} \frac{\rho + si + s\psi_{\Delta}(u)}{\rho + s(i + 1)}. \]

**Proof.** In the following derivation, we use the fact that, for \( i \geq 1 \), the variables \( X_i \) and \( \Delta_i \) are independent, and that all \( \Delta_i \) are identically distributed to the random variable \( \Delta \). We obtain

\[
\psi_{X_n}(u) = \frac{s + \rho}{sn + \rho} E[e^{(X_0 + \Delta_0)u}] + \frac{s}{sn + \rho} \sum_{i=1}^{n-1} E[e^{(X_i + \Delta_i)u}]
\]

\[ = \frac{s + \rho}{sn + \rho} \psi_{\Delta_0}(u) + \frac{s\psi_{\Delta}(u)}{sn + \rho} \sum_{i=1}^{n-1} \psi_{X_i}(u). \tag{3} \]

Differenting for \( n \geq 1 \), we get

\[(\rho + sn)\psi_{X_n}(u) - (\rho + s(n-1))\psi_{X_{n-1}}(u) = s\psi_{\Delta}(u)\psi_{X_{n-1}}(u),\]

which we arrange in the iterable form

\[ \psi_{X_n}(u) = \frac{\rho + s((n-1) + \psi_{\Delta}(u))}{\rho + sn} \psi_{X_{n-1}}(u). \]

Unwinding all the way back to \( \psi_{X_1}(u) = \psi_{\Delta_0}(u) \) gives us

\[ \psi_{X_n}(u) = \psi_{\Delta_0}(u) \prod_{i=1}^{n-1} \frac{\rho + si + s\psi_{\Delta}(u)}{\rho + s(i + 1)}, \]

for \( n \geq 2 \). \( \square \)

The exact average depth of the latch can be found mechanically from \( \psi_{X_n}(u) \) by taking derivatives. It appears in a form involving the generalized rth harmonic number

\[ H_r(y) = \frac{1}{y+1} + \cdots + \frac{1}{y+r}. \]

**Theorem 5.3.** Let \( X_n \) be the depth of the latch \( v \) chosen to construct \( G_n \) and suppose \( s = s_1 = s_2 = \cdots = s_m \). For \( n \geq 1 \), the exact expected value of \( X_n \) is given by

\[ \mathbb{E}[X_n] = \left( H_{n-1} \left( \frac{\rho}{s} + 1 \right) + \frac{s}{s + \rho} \right) \mathbb{E}[\Delta]. \]
Proof. Take the derivative of the moment generating function in Lemma 5.2, then evaluate at $u = 0$, to get

$$
\mathbb{E}[X_n] = \psi'_{\Delta_0}(0) \prod_{i=1}^{n-1} \frac{\rho + si + s\psi_{\Delta}(0)}{\rho + s(i + 1)}
$$

$$
+ \psi_{\Delta_0}(0) \prod_{i=1}^{n-1} \frac{1}{\rho + s(i + 1)} \prod_{i=1}^{n-1} (\rho + si + s\psi_{\Delta}(0))
$$

$$
\times \sum_{i=1}^{n-1} \frac{s\psi'_{\Delta}(0)}{\rho + si + s\psi_{\Delta}(0)}
$$

$$
= \mathbb{E}[\Delta_0] + \sum_{i=1}^{n-1} \frac{\mathbb{E}[\Delta]}{\rho/s + i + 1}
$$

$$
= \left( \mathcal{H}_{n-1} \left( \frac{\rho}{s} + 1 \right) + \frac{s}{\rho + s} \right) \mathbb{E}[\Delta].
$$

$\square$

From the approximate value of the generalized harmonic number, we immediately get the following result.

**Corollary 5.4.** As $n \to \infty$, we have

$$
\mathbb{E}[X_n] \sim \mathbb{E}[\Delta] \ln n.
$$

**Remark 5.5.** The formula in Theorem 5.3 is exact for any $v$, any $e$, and any $n$. It does not require that $v$ be a constant. For example, we can decide on a large $n$ and choose to build the network under uniform attachment from paths of length $4n$ edges as building blocks. In this case, as $n \to \infty$, we have

$$
\mathbb{E}[X_n] \sim 2n \ln n.
$$

**Theorem 5.6.** Let $X_n$ be the depth of the latch $v$ chosen to construct $G_n$. If $s_1 = s_2 = \cdots = s_m$, then

$$
\frac{X_n - \mathbb{E}[\Delta] \ln n}{\sqrt{\ln n}} \overset{d}{\to} \mathcal{N} \left( 0, \text{Var}[\Delta] + \mathbb{E}^2[\Delta] \right).
$$
Proof. We use Lemma 5.2 to develop the moment generating function of \((X_n - E[\Delta] \ln n)/\sqrt{\ln n}\). By Stirling’s approximation to the ratio of Gamma functions, as \(n \to \infty\), we have

\[
\psi_{X_n} \left( \frac{u}{\sqrt{\ln n}} \right) = \psi_{\Delta_0} \left( \frac{u}{\sqrt{\ln n}} \right) \frac{\Gamma(\rho/s + \psi_\Delta((u/\sqrt{\ln n}) + n)) \Gamma(\rho/s + 2)}{\Gamma(\rho/s + 1 + n) \Gamma(\rho/s + 1 + \psi_\Delta((u/\sqrt{\ln n}))}
\]

\[
\sim \frac{\Gamma(\rho/s + \psi_\Delta((u/\sqrt{\ln n}) + n))}{\Gamma(\rho/s + 1 + n)}
\]

\[
\sim n^{\psi_\Delta((u/\sqrt{\ln n}) - 1)}
\]

\[
= \exp \left( \left( \psi_\Delta \left( \frac{u}{\sqrt{\ln n}} \right) - 1 \right) \ln n \right).
\]

We use the asymptotic relation

\[
\psi_\Delta \left( \frac{u}{\sqrt{\ln n}} \right) = 1 + \frac{u E[\Delta]}{\sqrt{\ln n}} + \frac{u^2 (\text{Var}[\Delta] + E^2[\Delta])}{2 \ln n} + O \left( \frac{1}{\ln^3 n} \right),
\]

to get

\[
\psi_{X_n} \left( \frac{u}{\sqrt{\ln n}} \right) e^{-\frac{\psi_\Delta((u/\sqrt{\ln n}) \ln n)}{\sqrt{\ln n} u}} \sim \exp \left( \left( \frac{u^2 (\text{Var}[\Delta] + E^2[\Delta])}{2 \ln n} \right) \ln n \right)
\]

\[
\rightarrow e^{(\text{Var}[\Delta] + E^2[\Delta]) u^2/2}.
\]

The last expression is the moment generating function of a centered normal distribution with variance \(\text{Var}[\Delta] + E^2[\Delta]\). According to Lévy’s continuity theorem (see [5, Theorem 9.1], for example), convergence of moment generating functions implies the convergence in distribution stated in the theorem.

Theorem 5.6 has an immediate translation in terms of the depth of a randomly chosen node. Here, the random choice of a node \(v\) to examine its depth means that the node is chosen with probability proportional to its affinity in \(G_n\), that is, with probability \((\chi \deg(v) + \rho)/\tau_n\).1 When nodes are taken according to their affinity, the depth \(D_n\) has the distribution of \(X_{n+1}\). Consequently, preserving the conditions of the theorem about the epithets of the building blocks, we have

\[
\frac{D_n - E[\Delta] \ln n}{\sqrt{\ln n}} \to \mathcal{N} \left( 0, \text{Var}[\Delta] + E^2[\Delta] \right).
\]

1In the next section, we consider a different interpretation of random depth.
6 The uninformed visit

A visitor to the network may be uninformed about the affinity mechanism that constructed it. In the absence of such prior information, the visitor may decide to explore the network according to a uniform choice of a node, in which all the nodes in the network are equally likely.

When choosing a node uniformly at random, we can once again think of first choosing a block and then a node within that block. If the node is in $B_i$, its depth in the network is $X_i$ plus the depth of a uniformly random node in $B_i$. The latter is an $O(1)$ random variable. Let $B$ be the random block chosen and suppose that all blocks have the same number of vertices, that is, $v = v_1 = v_2 = \cdots = v_m$. From the vantage point of uniform randomness, the probability of choosing the node from block $B_i$, with $i \geq 1$, is given by

$$P(B = B_i) = \frac{v - 1}{(v - 1)n + v} \sim \frac{1}{n}.$$ 

The probability of $B_0$ is special:

$$P(B = B_0) = \frac{v}{(v - 1)n + v} = O\left(\frac{1}{n}\right).$$

**Theorem 6.1.** Let $\widetilde{D}_n$ be the the depth of a vertex chosen uniformly at random in the graph $G_n$. If $s_1 = s_2 = \cdots = s_m$ and $v = v_1 = v_2 = \cdots = v_m$, then we have

$$\frac{\widetilde{D}_n - \mathbb{E}[\Delta] \ln n}{\sqrt{\ln n}} \xrightarrow{d} \mathcal{N}\left(0, \mathbb{V}ar[\Delta] + \mathbb{E}^2[\Delta]\right).$$

**Proof.** Let $B$ be the block in which the chosen node falls. We compute

$$\psi_{\widetilde{D}_n}(u) = \sum_{i=0}^{n} \mathbb{E}\left[e^{\widetilde{D}_n u} \mid B = B_i\right] P(B = B_i)$$

$$= O\left(\frac{1}{n}\right) + \frac{v - 1}{(v - 1)n + v} \sum_{i=1}^{n} \mathbb{E}\left[e^{(X_i + O(1))u}\right].$$

The $O(1)$ in the exponent is the distance of a node chosen uniformly at random from the collection of building blocks. This random distance is in-
dependent of \( X_i \). Upon changing the scale of \( u \), we find
\[
\psi_{\tilde{D}_n} \left( \frac{u}{\sqrt{\ln n}} \right) \sim \frac{1}{n} \sum_{i=1}^{n} \mathbb{E} \left[ e^{X_i + O(1)} \frac{u}{\sqrt{\ln n}} \right] \\
\sim \frac{\mathbb{E} \left[ e^{O(1)} \frac{u}{\sqrt{\ln n}} \right]}{n} \sum_{i=1}^{n} \mathbb{E} \left[ e^{X_i} \frac{u}{\sqrt{\ln n}} \right] \quad \text{(by independence)}
\]
\[
\sim \frac{1}{n} \sum_{i=1}^{n} \psi_{X_i} \left( \frac{u}{\sqrt{\ln n}} \right).
\]
From (3), the moment generating function of \( X_{n+1}/\sqrt{\ln n} \) can be written as
\[
\psi_{X_{n+1}} \left( \frac{u}{\sqrt{\ln n}} \right) = \frac{s + \rho}{sn + \rho} \psi_{\Delta_0} \left( \frac{u}{\sqrt{\ln n}} \right) + \frac{s\psi_{\Delta} \left( \frac{u}{\sqrt{\ln n}} \right)}{sn + \rho} \sum_{i=1}^{n} \psi_{X_i} \left( \frac{u}{\sqrt{\ln n}} \right)
\]
\[
\sim \frac{1}{n} \sum_{i=1}^{n} \psi_{X_i} \left( \frac{u}{\sqrt{\ln n}} \right).
\]
Therefore, the moment generating functions of \( \tilde{D}_n/\sqrt{\ln n} \) and \( X_{n+1}/\sqrt{\ln n} \) have the same asymptotic behavior, and the convergence stated in the theorem follows from Theorem 5.6.

7 Concluding remarks

To get a feel for what is involved in the central limit theorem and see the various nuances of the network parameters, we give some concrete forms relating to the running example. Under pure uniform attachment (\( \chi = 0 \)), we have
\[
\frac{D_n - \frac{22}{15} \ln n}{\sqrt{\ln n}} \overset{d}{\to} \mathcal{N} \left( 0, \frac{12}{5} \right),
\]
whereas for preferential attachment (\( \rho = 0 \)), we have
\[
\frac{D_n - \frac{43}{36} \ln n}{\sqrt{\ln n}} \overset{d}{\to} \mathcal{N} \left( 0, \frac{13}{8} \right).
\]
A case of mixed uniform and preferential attachments with \( \chi = 2.4 \) and \( \rho = 5 \) gives rise to the central limit theorem
\[
\frac{D_n - \frac{1066}{855} \ln n}{\sqrt{\ln n}} \overset{d}{\to} \mathcal{N} \left( 0, \frac{112}{57} \right).
\]
Observe that the asymptotic average depth in the case of uniform attachment is higher than that of preferential attachment, because attaching preferentially “favors” older nodes in the network, and thus has the effect of “lifting” the joining blocks closer to the master hook. Of course, the asymptotic average depth in the mixed case is somewhere in between uniform and preferential attachments.

Similar interpretations are manifest under a uniformly random choice of a node to visit. The intuition is that, though the choice at that stage is uniformly random, it is an averaging over the $X_i$’s, for $i = 1, \ldots, n$, which were driven by the affinities in the first place, plus uniform perturbations (different from the affinity perturbations, but equally negligible asymptotically). So, the uniform choice in the depth $D_n$ inherits the properties of the affinity view of the depth $D_n$.

The class of random graphs we study in the article lives in the “small world,” a phrase commonly used nowadays to refer to networks with average inter-node distances of order $\log n$. The author of [6] reports on a variety of networks arising in nature and technology that exhibit small world behavior. An important application appears in [1]. For a wider scope on small-world network, see [2, 8].

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**References**


Degree distributions of generalized hooking networks

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Abstract
A hooking network is grown from a set of graphs called blocks, each block with a labelled vertex called a hook. At each step in the growth of the network, a vertex called a latch is chosen from the hooking network, and a block is attached by joining the hook of the block with the latch. These graphs generalize trees, which are hooking networks grown from a single edge as the only block. Using Pólya urns, we show multivariate normal limit laws for the degree distributions of hooking networks. We extend previous results by allowing for more than one block in the growth of the network and by studying arbitrarily large degrees.

1 Introduction
Let $\mathcal{S} = \{G_1, G_2, \ldots, G_m\}$ be a finite set of connected graphs with at least 2 vertices, each with a labelled vertex $h_i$. We call $G_i$ a block and $h_i$ its hook. A sequence of hooking networks $G_0, G_1, G_2, \ldots$ is constructed recursively as follows: choose one of the blocks $G_i$ in $\mathcal{S}$ and let $G_0$ be a copy of $G_i$. The graph $G_n$ is constructed from $G_{n-1}$ by choosing a vertex $v$ of $G_{n-1}$, called a latch, choosing a block $G_i$, and attaching a copy of $G_i$ to $G_{n-1}$ by joining the hook $h_i$ and the latch $v$. The vertex corresponding to the hook of the initial block used to make $G_0$ is called the root of the network.

As an example, consider the set of blocks in Figure 1.

![Figure 1: A set of graphs as blocks.](image)

The graphs $G_0, G_1, G_2,$ and $G_3$ in Figure 2 are examples of hooking networks constructed by choosing a copy of $G_3$ as $G_0$ and attaching a copy of $G_4$, then a copy of $G_2$, and finally a copy of $G_1$. The root of the network is labelled $r$ and at each step, the vertex chosen to be the latch is denoted by $\star$.

![Figure 2: A sequence of hooking networks constructed from $G_1, G_2, G_3$ and $G_4$ in Figure 1.](image)

Several well-known graphs can be considered as hooking networks. Any hooking network constructed from a single block consisting of an edge $K_2$ is simply a tree; the process of attaching $K_2$ to a latch $v$ is equivalent to adding a child to $v$. The definition of hooking networks above generalizes self-similar hooking networks introduced by Mahmoud [11], which are hooking networks grown from a single block. A block graph is a hooking network whose blocks are complete graphs, and a cactus graph is a hooking network whose blocks are cycles and that may include an edge $K_2$ in the set of blocks. The study of these graphs has been very active recently [1, 2, 7], with applications in genome comparison [13] as well as in telecommunication networks and material handling networks (see [4]).

In this extended abstract we only consider hooking networks where at every step in the growth of the network the latch is chosen uniformly at random among all of the vertices of the network, and the block to be attached is chosen uniformly at random among all of...
the blocks. We prove that for any positive integer \( k \),
the number of vertices of degree \( k \) in such a hooking network converges to a normal distribution. In fact, we prove convergence to multivariate normal distributions for vectors of random variables counting the number of vertices with fixed degrees. We make use of the theory of generalized Pólya urns developed in [8] to prove our results. Our methods also work for more general hooking networks (see Remark 1.1).

In the example of hooking networks of Figure 2, all of the hooks of the blocks have even degrees, and every other vertex has an odd degree. As a result, during the growth of the hooking networks, only the root of the network \( r \) will have an even degree, while every other vertex will have an odd degree. In this case, we say the odd numbers are admissible degrees. More generally, we define admissible degrees as follows:

**Definition 1.1.** Given a set of blocks \( S \), the positive integer \( k \) is called an admissible degree if there is a positive probability that for some \( n \), the hooking network \( G_n \) grown from the blocks \( S \) has at least two vertices with degree \( k \).

Note that the definition of admissible degrees used here differs slightly from that of [3] and [11]. It is easy to verify that the root of the network is the only vertex that may have a degree that is not admissible.

For a set \( S \) of blocks, let \( N_d \) be the set of admissible degrees less than or equal to \( d \), and let

\[
k_1 < k_2 < \cdots < k_r
\]

be the elements of \( N_d \).

**Theorem 1.1.** Let \( X_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,r}) \), where \( X_{n,i} \) is the number of vertices with degree \( k_i \) in \( G_n \), and \( G_n \) is a hooking network constructed from the set of blocks \( S \) grown by choosing latches and blocks uniformly at random for \( n \) iterations. Let

\[
\mu_n := \mathbb{E}X_n = (\mathbb{E}X_{n,1}, \ldots, \mathbb{E}X_{n,r}).
\]

Then

\[
n^{-1/2}(X_n - \mu_n) \xrightarrow{d} \mathcal{N}(0, \Sigma)
\]

for some covariance matrix \( \Sigma \), where \( \mathcal{N}(0, \Sigma) \) denotes a multivariate normal distribution.

**Remark 1.1.** In the journal version of this paper (to appear), we prove similar results when the latch is chosen preferentially according to its degree, and when the blocks are chosen proportional to a fixed weight assigned to each block. Our methods also show similar results for blocks trees (see [5] for a definition) and bipolar networks (see [3] for a definition).

The above theorem generalizes several previous results. A random recursive tree is equivalent to a hooking network grown from a single edge \( K_2 \). Theorem 1.1 therefore includes previously known results for the asymptotic degree distribution for random recursive trees [12, 9]. Blocks trees can be thought of as hooking networks with trees for blocks. Gopaladesikan, Mahmoud, and Ward [5] proved asymptotic results for the number of leaves in randomly grown blocks trees. Since leaves are vertices of degree 1, Theorem 1.1 extends their work to arbitrary degrees. Mahmoud [11] studied the distribution of vertices with the two smallest admissible degrees in self-similar hooking networks. Our theorem implies this result as well.

2 Pólya urns

A generalized Pólya urn process \((X_n)_{n=0}^{\infty}\) is defined as follows. There are \( q \) types (or colours) \( 1, 2, \ldots, q \) of balls and for each vector \( X_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,q}) \), the entry \( X_{n,i} \geq 0 \) is the number of balls of type \( i \) in the urn at time \( n \), starting with a given (random or not) vector \( X_0 \). Each type \( i \) is assigned an activity \( a_i \in \mathbb{R}_{\geq 0} \) and a random vector \( \xi = (\xi_1, \xi_2, \ldots, \xi_q) \) satisfying \( \xi_{i,j} \geq 0 \) for \( i \neq j \) and \( \xi_{i,i} \geq -1 \). At each time \( n \geq 1 \), a ball is drawn at random so that the probability of choosing a ball of type \( i \) is

\[
\frac{a_i X_{n-1,i}}{\sum_{j=1}^{q} a_j X_{n-1,j}}.
\]

If the drawn ball is of type \( i \), then it is replaced along with \( \Delta X_{n,j} \) balls of type \( j \) for each \( j = 1, \ldots, q \), where the vector \( \Delta X_{n} = (\Delta X_{n,1}, \Delta X_{n,2}, \ldots, \Delta X_{n,q}) \) has the same distribution as \( \xi \) and is independent of everything else that has happened so far. We allow for \( \Delta X_{n,i} = -1 \), in which case the drawn ball is not replaced.

The intensity matrix of the Pólya urn is the \( q \times q \) matrix

\[
A := (a_i \xi_{i,j})_{i,j=1}^{q}.
\]

By the choice of \( \xi_{i,j} \), the matrix \( I + A \) has non-negative entries and so by the standard Perron-Frobenius theory, \( A \) has a real eigenvalue \( \lambda_1 \) such that all other eigenvalues \( \lambda \neq \lambda_1 \) satisfy \( \text{Re}\lambda < \lambda_1 \).

We use basic assumptions on the Pólya urn, these are (A1)–(A7) gathered in [6] ((A1)–(A6) are also in [8]). A ball of type \( i \) is said to be dominating if with positive probability, every other ball of type \( j \) can be found at some time in an urn starting with a single ball of type \( i \). The urn (and its matrix \( A \)) is irreducible if every type is dominating. In our applications, all of the urns will be irreducible. Using the Perron-Frobenius theorem, it is easy to verify that (A1)–(A6) are satisfied when the intensity matrix is
irreducible. We will only use balls with positive activity and so (A7) is satisfied.

Denote column vectors as \( v \) with \( v' \) as its transpose. The transpose of a matrix \( A \) is also denoted as \( A' \). Let \( a = (a_1, \ldots, a_q)' \) denote the vector of activities, and let \( u_1' \) and \( v_1 \) be the left and right eigenvectors of \( A \) corresponding to the eigenvalue \( \lambda_1 \) normalized so that \( a \cdot v_1 = a' v_1 = v_1' a = 1 \) and \( u_1 \cdot v_1 = u_1' v_1 = v_1' u_1 = 1 \). Define \( P_{\lambda_1} = v_1 u_1' \) and \( P_1 = I_q - P_{\lambda_1} \). Define the matrices

\[
B_i := \mathbb{E}(\xi_i^2)
\]

for every \( i = 1, \ldots, q \), denote \( v_1 = (v_{1,1}, v_{1,2}, \ldots, v_{1,q})' \), and define the matrix

\[
B := \sum_{i=1}^q v_{1,i} a_i B_i.
\]

In the case where \( \text{Re} \lambda < \lambda_1/2 \) for every eigenvalue \( \lambda \neq \lambda_1 \), define

\[
\Sigma_1 := \int_0^\infty P_t e^{sA} B e^{sA'} P_t e^{-\lambda_1 s} ds,
\]

where \( e^{tA} = \sum_{j=0}^\infty t^j A^j / j! \).

The result we use from [8] guarantees that if (A1)-(A7) hold and \( \text{Re} \lambda < \lambda_1/2 \) for all eigenvalues \( \lambda \neq \lambda_1 \), then

\[
n^{-1/2}(X_n - n\mu) \overset{d}{\to} \mathcal{N}(0, \Sigma)
\]

for some \( \mu = (\mu_1, \ldots, \mu_q) \) and \( \Sigma = (\sigma_{i,j})_{i,j=1}^q \). We use lemmas stated in [8] and gathered in [6, Theorem 4.1] to calculate \( \mu \) and \( \Sigma \).

**Theorem 2.1. (\cite[Thm. 3.22 and Lem. 5.4]{8})**

Assume (A1)-(A7) and that the right and left eigenvectors corresponding to \( \lambda_1 \) are normalized as above. Assume that \( \text{Re} \lambda < \lambda_1/2 \) for each eigenvalue \( \lambda \neq \lambda_1 \).

(i) Then, as \( n \to \infty \),

\[
n^{-1/2}(X_n - n\mu) \overset{d}{\to} \mathcal{N}(0, \Sigma)
\]

with \( \mu = \lambda_1 v_1 \) and some covariance matrix \( \Sigma \).

(ii) Suppose further that, there exists \( c > 0 \) so that for every \( i = 1, \ldots, q \),

\[
a \cdot \mathbb{E}(\xi_i) = c.
\]

Then the covariance matrix is given by \( \Sigma = c \Sigma_1 \), with \( \Sigma_1 \) as defined in (2.1).

**Remark 2.1.** We will use Theorem 2.1 to show that the degree vectors of hooking networks converge in distribution to a multivariate normal distribution. A recent result by Janson and Pouyanne [10] guarantees convergence in moments for certain generalized Pólya urns. By [10, Remark 1.9], our urns satisfy the conditions needed for convergence in moments as well. In particular, from [10, Theorem 1.1], we get that

\[
\mathbb{E}X_n/n \to \mu.
\]

### 3 Proof of main result

We start by describing the degrees of the vertices in the growth of a hooking network as balls in the evolution of a Pólya urn, and then show that the intensity matrix of our urn satisfies the conditions of Theorem 2.1.

#### 3.1 Degrees in hooking networks as balls in Pólya urns

We start by first looking at an urn with infinitely many types. We assign a type to each degree in the network so that a ball of type \( k \) represents a vertex of degree \( k \). Each network starts as a copy of a graph from the list of blocks. This corresponds to starting a Pólya urn with a ball of the matching type for the degree of each vertex in the block. In the evolution of the network, the hook of a block being attached to a latch \( v \) corresponds to choosing a ball in the urn of type corresponding to the degree of \( v \), and replacing it with a ball representing the new degree of \( v \) along with balls representing the degrees of the rest of the vertices of the block. Since a latch is chosen uniformly at random, each ball simply has activity 1.

The Pólya urn described above has infinitely many types, and so Theorem 2.1 does not apply. We would like to instead use an urn with finitely many types in a similar manner as was done in [6] and [9]. The urn is replaced with the following Pólya urn: let \( d \) be a positive integer corresponding to the largest degree we wish to study in this instance of the model. A new ball of special type \( * \) with activity \( a_* = 1 \) is introduced, and for every \( k > d \), each ball of type \( k \) is replaced with a ball of special type \( * \). In this way, the probability of choosing a ball of special type in the new urn is equal to the probability of choosing a ball of type greater than \( d \) in the old urn. If a vertex \( v \) of degree \( k \leq d \) is chosen as a latch, and a hook is attached so that \( v \) now has degree \( k + j > d \), then the ball of type \( k \) is removed and a ball of special type is added. If a latch of degree \( k > d \) is chosen and a hook is attached, then the ball of special type that was chosen is simply placed back in the urn. At each step we again also place balls representing the degrees of the vertices of the block that are not the hook.

Instead of adding a ball of type corresponding to the degree of the root of the network, a ball of special type is added. This guarantees that all types of balls
in the urn that are not special types correspond only to degrees that are admissible; recall from Definition 1.1 that a degree is admissible if there is a positive probability that at some point in the growth of the network at least two vertices will have that degree. For a positive integer \( d \), the possible types of balls present in the urn are exactly the elements of \( N_d \), the set of admissible degrees less than or equal to \( d \), together with a ball of special type \( \ast \). In our intensity matrix, we can then omit the rows and columns corresponding to types that will never be present in the urn. By restricting to admissible degrees, it can be verified that now every ball in the urn is of dominating type, and so the urn (and its intensity matrix) is irreducible. As discussed in Section 2, it is easy to verify that the assumptions (A1)-(A7) are satisfied for irreducible urns. To avoid confusion, we will label the type of a ball with the degree of the vertex it represents.

To show how to calculate the intensity matrix for an urn corresponding to the growth of a hooking network, consider the example given in Figure 2 of the introduction. Suppose we look at all admissible degrees less than or equal to 9, so \( N_9 = \{1, 3, 5, 7, 9\} \) (recall from the introduction that the admissible degrees for this example are all odd degrees). The images in Figure 3 illustrate the possibilities for replacing a ball of type \( k \), corresponding to attaching a block to a latch \( v \) in the hooking network. Since the blocks are chosen uniformly at random, each has a probability of 1/4 of being attached to a latch.

The intensity matrix for this urn will have 6 rows and columns: one corresponding to the degrees 1,3,5,7,9, and the last one for balls of special type \( \ast \). Let’s consider what happens when a block is attached to a vertex with degree 3. This corresponds to choosing a ball of type 3. For example, with probability 1/4, the graph \( G_1 \) is attached to \( v \). The hook has degree 2 and there are 2 vertices of degree 1. Then the ball of type 3 is removed and replaced with a ball of type 5 along with 2 balls of type 1. This explains the first column vector in the sum below. Using Figure 3 for the other blocks, we can calculate \( E_{k3} \) to be the following sum:

\[
\frac{1}{4} \begin{pmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 4 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 \\ 3 \\ 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}.
\]

Remember that rows and columns for even types are removed, and so the first row represents balls of type 1, the second row for balls of type 3, etc, and the final row for balls of special type.

\[
\begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ -1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 4 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.
\]

Now consider what happens when a ball of type 7 is chosen. If a hook of degree 4 is attached to a vertex of degree 7, the resulting vertex will have degree 11. Recall that instead of adding a ball of type 11, we add a ball of special type. Then \( E_{k7} \) is the following sum:

\[
\frac{1}{4} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 2 \\ 0 \\ 1 \\ 0 \\ 1 \\ 1 \end{pmatrix} + \frac{1}{4} \begin{pmatrix} 0 \\ 4 \\ 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}.
\]

Finally consider choosing a ball of special type; that is either a block was attached to a vertex of degree greater than 9, or a block was attached to the root of the network \( r \). This time the ball of special type is simply replaced back in the urn. Thus, for the special type,
$\Xi_n$ is the following sum:

$$
\begin{bmatrix}
\frac{2}{4} \\
0
\end{bmatrix}
+ \frac{1}{4}
\begin{bmatrix}
4 \\
0
\end{bmatrix}
+ \frac{1}{4}
\begin{bmatrix}
2 \\
0
\end{bmatrix}
+ \frac{1}{4}
\begin{bmatrix}
0 \\
0
\end{bmatrix}.
$$

By finishing the calculations for the remaining degrees, we get the following intensity matrix:

$$
A = \frac{1}{4}
\begin{bmatrix}
4 & 8 & 8 & 8 & 8 \\
8 & 2 & 6 & 6 & 6 \\
2 & 2 & -4 & 0 & 0 \\
0 & 2 & 2 & 0 & 0 \\
0 & 0 & 2 & 4 & 0
\end{bmatrix}.
$$

The matrix $A$ has eigenvalues

$$7/2, -1, -1, -1, -1$$

and so Theorem 2.1 applies. The right eigenvector of $A$ corresponding to $\lambda_1$, normalized as is described in Section 2, is calculated (using MATHEMATICA) to be

$$v_1 = \left(\begin{array}{c}
4 \\
31 \\
67 \\
346 \\
949 \\
716
\end{array}\right).$$

Let $X_{n,i}$ be the number of vertices of degree $2i - 1$ for $i = 1, 2, 3, 4, 5$, and let $X_{n,6}$ be the number of vertices of degree greater than 9. Let $X_n = (X_{n,1}, \ldots, X_{n,6})$. Then by Theorem 2.1,

$$n^{-1/2}(X_n - n\mu) \overset{d}{\to} \mathcal{N}(0, \Sigma),$$

where $\mu = \frac{7}{2}v_1$.

For $\Sigma$, every ball in the urn has activity $a_i = 1$, and since all columns sum to $\frac{7}{2}$, then (2.3) holds, and we can calculate the covariance matrix $\Sigma$ by following the steps laid out in Section 2. The right eigenvector of $A$ associated with $\lambda_1$ normalized as is described in Section 2 is simply the vector of all 1’s. Therefore $P_{\lambda_1}$ is the matrix with copies of $v_1$ making up every column. We can use Figure 3 to calculate $B_1, B_3, B_5, B_7, B_9$ and $B_{11}$. For example,

$$B_3 = \frac{1}{4}b_1b'_1 + \frac{1}{4}b_2b'_2 + \frac{1}{4}b_3b'_3 + \frac{1}{4}b_4b'_4,$$

where $b_1, b_2, b_3,$ and $b_4$ are given below:

$$b_1 = \begin{bmatrix}
2 \\
1 \\
0 \\
0
\end{bmatrix},
\quad b_2 = \begin{bmatrix}
4 \\
-1 \\
0 \\
0
\end{bmatrix},
\quad b_3 = \begin{bmatrix}
2 \\
1 \\
0 \\
0
\end{bmatrix},
\quad b_4 = \begin{bmatrix}
0 \\
3 \\
1 \\
0
\end{bmatrix}.$$

The matrix $\Sigma$ was calculated according to (2.1) using MATHEMATICA, giving the covariance matrix

$$\Sigma = \lambda_1 \Sigma_I = \frac{7}{2} \Sigma_I,$$

which is added in Appendix A.

### 3.2 Proof of Theorem 1.1

We start by introducing some useful notation. Let $\mathcal{S}$ be a set of blocks and let each block $G_i \in \mathcal{S}$ have vertex set $V(G_i)$ and hook $h_i$. For every positive integer $j$, define

$$f(j) = \sum_{G_i \in \mathcal{S}} |\{v \in V(G_i) \setminus \{h_i\} : \text{deg}(v) = j\}|$$

and

$$g(j) = |\{h_i : \text{deg}(h_i) = j, G_i \in \mathcal{S}\}|.$$

The number $f(j)$ is the number of vertices that are not hooks of degree $j$ among all the blocks, and $g(j)$ is the number of hooks of degree $j$. Define $m = |\mathcal{S}|$ to be the number of blocks. Note also that $m = \sum_{j \geq 1} g(j)$. For the set $\mathcal{S}$ of blocks in Figure 1 as an example, $f(1) = 8$, $f(3) = 6$, $g(2) = 2$, $g(4) = 2$, and $m = 4$.

It is useful to note for the proof below that if $j$ is an admissible degree (recall from Definition 1.1) and $k$ is not, then there are no hooks of degree $k - j$. Otherwise, there is a positive probability of attaching a hook of degree $k - j$ to a latch of degree $j$ and increase the degree of the latch to $k$. As a result, $g(k - j) = 0$ whenever $j$ is admissible and $k$ is not admissible. Also, if $k$ is not an admissible degree, then $f(k) = 0$.

**Proof.** We start by calculating the intensity matrix for our Polya urn defined in Section 3.1. Suppose that we want to look at vertices of degree at most $d$. Recall that we let $N_d$ be the set of admissible degrees less than or equal to $d$, and that $k_1 < \cdots < k_r$ are the elements of $N_d$.

We will look at two cases: when a block is attached to a vertex that is not the root of the network with degree less than or equal to $d$, and when a block is attached to the root of the network or to a vertex of degree greater than $d$. Recall that the root of the network is the vertex corresponding to the hook of the initial block used to start the network, and is represented by a ball of special type $*$ in the urn.

**Case I:** Let $j \in N_d$ and suppose that at some step in the growth of the network a vertex $v$ is chosen as a latch where $\text{deg}(v) = j$ and $v$ is not the root of the network, and a block is attached to $v$. This corresponds to choosing a ball of type $j$. Let $i \in N_d$ be an arbitrary admissible degree less than or equal to $d$. The probability that the degree of $v$ is increased to $i$
is equal to the number of hooks of degree \(i - j\) divided by the total number of blocks, that is, \(g(i - j)/m\). Other than the latch, the expected number of new vertices of degree \(i\) added to the network is equal to the number of vertices of degree \(i\) that are not a hook divided by the total number of blocks, that is, equal to \(f(i)/m\). For \(i, j \in N_d\) and with \(\mathbb{E}(\xi_{i,j})\) being the expected change in the number of balls of type \(i\) added when a ball of type \(j\) is chosen, then the arguments above show that

\[
\mathbb{E}(\xi_{i,j}) = \begin{cases} 
  f(i)/m & i < j, \\
  f(i)/m - 1 & i = j, \\
  f(i)/m + g(i - j)/m & i > j.
\end{cases}
\]

For every \(i\) that is an admissible degree greater than \(d\), balls of special type are added instead of balls of type \(i\). So by similar arguments as above and summing over all admissible degrees greater than \(d\), the number of balls of special type added if a ball of type \(j\) is chosen is

\[
\mathbb{E}(\xi_{*,j}) = \sum_{k > d} f(k)/m + g(k - j)/m.
\]

Case II: Look at what happens when a ball of special type is chosen. This corresponds to choosing a vertex \(v\) of degree greater than \(d\) or the root of the network as a latch. Suppose that a block is attached to \(v\). Again, for an arbitrary admissible degree \(i \in N_d\), the expected number of new vertices of degree \(i\) added is \(f(i)/m\). Therefore for \(i \in N_d\),

\[
\mathbb{E}(\xi_{*,i}) = f(i)/m.
\]

The ball of special type that was chosen is simply placed back in the urn. We also add balls of special type for every new vertex other than the latch that is of degree greater than \(d\). The expected number of balls of special type added is then

\[
\mathbb{E}(\xi_{*,*}) = \sum_{k > d} f(k)/m.
\]

The activity of every ball is simply \(a_i = 1\), and so after removing rows and columns corresponding to degrees that are not admissible, we then get the following intensity matrix \(A\):

\[
\begin{pmatrix}
\frac{f(k_1)}{m} - 1 & \cdots & \frac{f(k_1)}{m} & \frac{f(k_1)}{m} \\
\frac{f(k_2) + g(k_2 - k_1)}{m} & \cdots & \frac{f(k_2)}{m} & \frac{f(k_2)}{m} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{f(k_r) + g(k_r - k_1)}{m} & \cdots & \frac{f(k_r - 1)}{m} & \frac{f(k_r)}{m} \\
\sum_{k > d} \frac{f(k) + g(k - k_1)}{m} & \cdots & \sum_{k > d} \frac{f(k) + g(k - k_2)}{m} & \sum_{k > d} \frac{f(k)}{m}
\end{pmatrix}
\]

The first \(r\) rows and columns of \(A\) correspond to \(k_1, k_2, \ldots, k_r\) and the last row and column corresponds to the ball of special type \(\ast\). From here, the eigenvalues of \(A\) can be calculated directly. Look at \(A - \lambda I\). Subtract column \(r + 1\) from all other columns. Then add rows 1, 2, \ldots, \(r\) to row \(r + 1\). Using the fact that \(\sum_{j \geq 1} g(j) = m\), we get the matrix

\[
A' = \begin{pmatrix}
-1 - \lambda & \cdots & 0 & \frac{f(k_1)}{m} \\
\frac{f(k_2 - k_1)}{m} & \cdots & 0 & \frac{f(k_2)}{m} \\
\vdots & \ddots & \vdots & \vdots \\
\frac{f(k_r - k_1)}{m} & \cdots & -1 - \lambda & \frac{f(k_r)}{m} \\
0 & \cdots & 0 & \sum_j \frac{f(j)}{m} - \lambda
\end{pmatrix}
\]

Since the determinant of a matrix is unchanged by adding one row to another or by subtracting one column from another, both \(A - \lambda I\) and \(A'\) have the same determinant. We can calculate the determinant of \(A'\) by expanding along the bottom row and see that \(A\) has characteristic polynomial

\[
\pm \left( \lambda - \frac{1}{m} \sum_{j \geq 1} f(j) \right) (\lambda + 1)^r.
\]

Therefore \(A\) has largest eigenvalue \(\lambda_1 = \frac{1}{m} \sum_{j \geq 1} f(j)\), and all other eigenvalues are \(\lambda = -1\). Theorem 2.1 applies, proving Theorem 1.1.

Remark 3.1. By studying the matrix \(A\) in the proof above, we see that the entries of the last column sum to \(\lambda_1 = \frac{1}{m} \sum_{j \geq 1} f(j)\). Furthermore, since \(\sum_{j \geq 1} g(j) = m\), then the entries of any other column sum to

\[
\frac{1}{m} \sum_{j \geq 1} f(j) + \frac{1}{m} \sum_{j \geq 1} g(j) - 1 = \frac{1}{m} \sum_{j \geq 1} f(j) = \lambda_1.
\]

Therefore, (2.3) is satisfied with \(c = \lambda_1\), and the covariance matrix \(\Sigma\) in Theorem 1.1 is \(\Sigma = \lambda_1 \Sigma_I\), where \(\Sigma_I\) is given by (2.1).

Recall the example of Figure 2 in the Introduction. We calculated at the end of Section 3.1 the intensity matrix \(A\) and its eigenvalues when we look at vertices with degrees less than or equal to 9. With \(f(1) = 8\), \(f(3) = 6\), \(g(2) = 2\), \(g(4) = 2\), and \(m = 4\), then the proof above shows that \(A\) has eigenvalues

\[
7/2, -1, -1, -1, -1, -1,
\]

which is what was calculated at the end of Section 3.1.
4 More examples: cactus graphs

Recall from the introduction that a cactus graph is a hooking network grown from a set of cycles that may or may not include a single edge $K_2$. For a positive integer $m$, consider a hooking network whose blocks are the $m$ smallest cycles: $C_3, C_4, \ldots, C_{m+2}$. The set of admissible degrees in such a hooking network consists of all even positive integers. For a positive integer $r$, suppose we want to count the vertices of degree $2, 4, \ldots, 2r$. We can do a similar analysis as was done in Section 3.1 to calculate the intensity matrix, or we can simply refer to the proof of Theorem 1.1 in Section 3.2. We calculate that $g(2) = m$ and that

$$f(2) = 2 + 3 + \ldots + m + 1 = \frac{m(m + 3)}{2}.$$ 

Then the intensity matrix is the $(r+1) \times (r+1)$ matrix

$$A = \begin{pmatrix} \frac{m+3}{2} & 1 & 1 & \cdots & 1 & 0 \\ 1 & \frac{m+3}{2} & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \frac{m+3}{2} & 1 \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix},$$

where for $i = 1, 2, \ldots, r$, the $i$-th row and column represent balls of type $2i$, and row and column $r+1$ represent balls of special type. By using the proof of Theorem 1.1 (or by direct calculation), we know that $A$ has a largest real eigenvalue $\lambda_1 = \frac{m+3}{2}$. It is then elementary to verify that the right eigenvector $v_1$ associated with $\lambda_1$ whose entries sum to 1 is the vector

$$(m+3) \left( 1, \frac{m+3}{2}, \cdots, \frac{2r-1}{m+5)^r}, \cdots, \frac{2r-1}{m+5)^r}, \cdots, \frac{2r-1}{m+5)^r} \right)'$$

For $i = 1, 2, \ldots, r$ let $X_{n,i}$ be the number of vertices of degree $2i$ in a hooking network at time $n$ grown from the blocks $C_3, C_4, \ldots, C_{m+2}$, and let $X_{n,r+1}$ be the number of vertices of degree greater than $2r$ in the same hooking network at time $n$. Let $X_n = (X_{n,1}, X_{n,2}, \ldots, X_{n,r}, X_{n,r+1})$. Then by Theorem 1.1,

$$n^{-1/2} (X_n - \mathbb{E}X_n) \overset{d}{\to} \mathcal{N}(0, \Sigma)$$

where

$$\mathbb{E}X_n/n \to \frac{m+3}{2} v_1$$

by Remark 2.1 and $\Sigma = \frac{m+3}{2} \Sigma_I$, where $\Sigma_I$ can be calculated by (2.1). For example, the matrix $\Sigma$ when $m = 3$ and $r = 5$ is included in Appendix B.

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References


A The covariance matrix calculated for the example in Figure 2

\[
\begin{pmatrix}
1358 & -125300 & -226030 & -76650812 & 525120302 & 408156658 \\
891 & 88209 & 8732691 & 864536409 & 85589104491 & 85589104491 \\
-125300 & 12903359 & -86970905 & 5297948383 & 1858133830 & 208893864151 \\
88209 & 8732691 & 864536409 & 171178208982 & 8473321344609 & 16946642689218 \\
-226030 & -86970905 & 40236500857 & -11135904377813 & -18077457766130 & -838588813116291 \\
8732691 & 864536409 & 171178208982 & 8473321344609 & 16946642689218 & 171178208982 \\
-76650812 & 5297948383 & 12903359 & -11135904377813 & -2559128002394561 & -388588813116291 \\
864536409 & 8732691 & 864536409 & 171178208982 & 8473321344609 & 16946642689218 \\
525120302 & 1858133830 & -41135904377813 & -1493303871893519 & -2559128002394561 & 84570373610799818 \\
85589104491 & 8473321344609 & 171178208982 & 8473321344609 & 16946642689218 & 171178208982 \\
408156658 & 208893864151 & -18077457766130 & -2559128002394561 & -84570373610799818 & 247872194436271685 \\
85589104491 & 16946642689218 & 838588813116291 & 166094044997025618 & 821655227352768891 & 8221655227352768891 \\
\end{pmatrix}
\]

B The covariance matrix calculated for a cactus graph example

\[
\begin{pmatrix}
\frac{1}{8} & -\frac{61}{352} & \frac{337}{15488} & \frac{11771}{681472} & \frac{193993}{29984768} & \frac{83779}{29984768} \\
\frac{61}{352} & \frac{15488}{681472} & \frac{15488}{681472} & \frac{15488}{681472} & \frac{15488}{681472} & \frac{15488}{681472} \\
\frac{337}{15488} & \frac{11771}{681472} & \frac{11771}{681472} & \frac{11771}{681472} & \frac{11771}{681472} & \frac{11771}{681472} \\
\frac{11771}{681472} & \frac{193993}{29984768} & \frac{193993}{29984768} & \frac{193993}{29984768} & \frac{193993}{29984768} & \frac{193993}{29984768} \\
\frac{681472}{29984768} & \frac{1193297992}{88055010848} & \frac{1193297992}{88055010848} & \frac{1193297992}{88055010848} & \frac{1193297992}{88055010848} & \frac{1193297992}{88055010848} \\
\frac{193993}{29984768} & \frac{1193297992}{88055010848} & \frac{1193297992}{88055010848} & \frac{1193297992}{88055010848} & \frac{1193297992}{88055010848} & \frac{1193297992}{88055010848} \\
\frac{83779}{29984768} & \frac{29984768}{88055010848} & \frac{29984768}{88055010848} & \frac{29984768}{88055010848} & \frac{29984768}{88055010848} & \frac{29984768}{88055010848} \\
\end{pmatrix}
\]
Errata

In the extended abstract proceeding this errata, it is stated (in Remark 2.1) that the urns studied in the proof satisfy conditions needed for convergence in moments. These conditions are in fact not met in general, and while the authors still believe convergence in moments to hold in all the cases studied, we have not yet found a proof. In the statement of Theorem 1, the convergence in distribution

\[ n^{-1/2}(X_n - \mu_n) \xrightarrow{d} \mathcal{N}(0, \Sigma), \]

where \( \mu_n := \mathbb{E}X_n \), would have followed by convergence in moments. Though again, we still believe this to be true, the convergence that we do in fact prove is

\[ n^{-1/2}(X_n - n\mu) \xrightarrow{d} \mathcal{N}(0, \Sigma) \]

for some vector \( \mu \). The final part of Remark 2.1, that \( \mathbb{E}X_n/n \to \mu \), still holds by dominated convergence (see Remark 2.3 from paper 1).

We apologize for the mistakes.

There are some cases where convergence in moments can in fact be proved, and we refer the reader to the results of paper 1 (specifically Corollary 1.4) for these cases.