GPPZ and the Holographic Triforce against Scalars

Jean-François Vaduret

Marjorie Schillo, Project supervisor
Lisa Freyhult, Examiner

E-mail: jean-francois.vaduret@physics.uu.se

ABSTRACT: We use gauge-invariant cosmological perturbation theory to compute one-point functions of active and inert scalar fields of the GPPZ RG-flow in AdS₅. Linearized Einstein equations are computed and made gauge-invariant for D-dimensional Euclidean domain-wall geometry. We briefly review the procedure of holographic renormalization for the GPPZ RG-flow in AdS₅ to get different one-point functions. The source-dependant vev of the operator dual to the Δ = 3 active scalar field in the GPPZ solution is computed and agrees with literature. We also find the source-dependant one-point function of the operator dual to the Δ = 3 inert scalar.
1 Introduction

Dualities relate two theories via a well-defined procedure or equation. They can be powerful tools, especially when it links the well-understood formalism of a theory to a more elusive one. That is the case of the AdS/CFT correspondence [1]. On one side the theory of gravity in AdS spaces that has been studied extensively but still holds mysteries. In the same way, conformal field theories (CFTs) have been greatly investigated along the years but still contains unknowns. But with the apparition of the gauge-gravity duality [2, 3], there is a way to understand CFTs by looking at the dual gravity theory.

Among the newly solvable puzzles, there are renormalization group flows which are tedious to understand in a pure-CFT framework. In fact they now have a dual description through gravity. It has been found that 4-dimensional superconformal field theories can be described by 5-dimensional domain-wall solutions of supergravity. These domain-wall solutions are described by a 5-dimensional space admitting symmetries from the Poincaré group on a 4-dimensional space slice. In particular, it is possible to describe a flow between two CFTs by traveling between two fixed points of the dual space — also called bulk. Each said fixed points in the bulk correspond to a CFT, these points are thus named CFT
fixpoints. In the case treated here, one of the CFT fixpoint is the long distance boundary of the radial dimension. A flow depicted this way is called a holographic RG-flow. The interesting feature of this discovery is that when the domain-wall geometry of a RG-flow is found, it is possible to travel to the other end of the flow, through the bulk. Using bulk data near one of the CFT fixpoints, one can compute fundamental quantities like vacuum expectation values (vevs) or even correlation functions [4] thanks to information about the other fixed point entirely contained in the geometry of the dual space.

Vacuum expectation values are found by studying the behaviour of space perturbations. One starts at the first CFT fixpoint and requires the space perturbations to be well behaved at this point. Then travels to the other CFT fixpoint and solves the equation of motion of the perturbations there. In the case of AdS/CFT, the duality links the generating functional of both theories. However, infinities appear in the supergravity action when approaching the boundary. This makes the gauge-gravity duality difficult to use, unless the action can be renormalized. A consistent procedure of renormalization in holography has been developed in [5]. Fields and metric in the bulk action all have a near-boundary expansion where the leading term is called boundary data or source, given that sources generate the vevs of the dual operators. When using these expansions in the classical gravity action, infinities appear explicitly. Holographic renormalization [5, 15] uses near-boundary expansions and Einstein equations to express the divergences in terms of bulk fields and curvature quantities. Then counterterms can be constructed on the boundary. Once the on-shell gravity action has been renormalized it is possible to take its functional derivative with respect to the fields sources. This is how one gets the vev of dual operators to the bulk theory.

The goal of this report is to review this procedure as it was done in [6] but by taking a different approach regarding bulk perturbations. Indeed, we apply the theory of cosmological perturbations [7] to study the gravity action in a consistently gauge-invariant way. This treatment of bulk perturbations is different from the one done in [6] essentially regarding the definition of gauge-invariant variables and thus leads to different equations of motions. Indeed, gauge invariance of the Einstein equations is required as we strive to find the solution to an equation of motion. The solution of the eom is then studied near a CFT fixpoint, and this analysis gives information on the CFT dual to the fixpoint. Thus gauge invariance guarantees generality of the results. This was argued in [6] when comparing to the results of [12] where a gauge choice was imposed early on. Using the procedure from [7] makes the Einstein equations ready to be used in the “domain-wall/cosmology correspondence” [8–10] which we will do in future work. To assess the differences due to the different methodology, we compute the vev of the active scalar present in the GPPZ flow [11] and compare it to the one found in [6]. We also take the time to compute the source-dependent vev of the inert scalar present in the GPPZ flow which has not been done in [6].

This report is organized as follow. In Sec 2 we apply the theory of cosmological perturbations [7] to the domain-wall geometry in an arbitrary number of dimensions. We do this by defining and computing gauge-invariant linearised Einstein equations. The section ends with the computation of the equation of motion for one gauge-invariant fluctuation
variable in $D = 5$ dimensions. Then Sec 3 is devoted to an introduction on the holographic framework. This contains a definition of the holographic principle, $RG$-flows and renormalization. We end the section by using holographic renormalization to find an expression for the vevs of operators generated by scalars in the GPPZ flow. Finally, Sec 4 is devoted to finding the source-dependent expression for the dual operators to the active and inert scalar fields in the GPPZ flow. We also briefly discuss similarities with [6] and differences with [12].

2 Cosmological perturbations theory

In this section we go through a quick review of the theory of cosmological perturbations as developed in [7]. We follow their work to derive the Einstein equations for domain-wall geometry in a fully gauge invariant way. We go through the calculations for domain-walls with Euclidean slice in $D$ dimensions. We will be particularly interested by theories containing one active scalar and $N$ inert scalars. And finally, we limit ourselves to a $D = 5$ domain-wall which will be of particular use in the next sections.

2.1 Gravitational background and perturbations

Perturbation theory relies on the fact that a model deviates from its nominal value, the background, by an amount small in comparison to the background. In this case the background under consideration is an asymptotically AdS spacetime metric relevant to the study of $RG$-flows. We will add a perturbation metric which will be defined differently whether we consider a Lorentzian or Euclidean slice. Unlike in [12], we will not impose a gauge choice early on, we will consider, in full generality, perturbations that will mix the radial and transverse coordinates. This is needed to define gauge-invariant variable as it was done in [7].

The background bulk space is defined by the following line element

$$ds^2 = e^{2A(r)} \gamma_{ij} dx^i dx^j + dr^2 = g_{\mu \nu} dx^\mu dx^\nu$$

where $\gamma_{ij}$ will be the Kronecker delta for a Euclidean slice and the regular Minkowski metric, with mostly plus signature, for a Lorentzian slice. We dub $r$ as radial coordinate and $x^i$, $i = 0, 1, \ldots, d$, make up the transverse space. For convenience we will sometime use the notation $a(r) = e^{A(r)}$. Additionally, for this line element to describe an asymptotically AdS space, we need $A(r) \sim r/L$ as $r \to +\infty$ with $L$ the AdS radius which shall be set to one unless stated explicitly.

The main goal of this work will be to treat linearized Einstein equations. To do so we will need to define the perturbed bulk metric. It is also important to determine which kind of perturbations will be treated here. As it is well-understood, there are three types of metric perturbations: scalar, vector and tensor perturbations. The vector and tensor types will not be treated in this report. So our focus is set on scalar perturbations and we leave the study of vector and tensor perturbations to other work. In the same manner as
we explicitly define the perturbation metric as

\[ h_{\mu\nu} = e^{2\Lambda(r)} \left( 2(\psi_{ij} - \partial_i \partial_j W) \frac{\partial_i \xi}{\partial_j \xi} - \frac{\partial_i \chi}{2\chi} - 2A(r) \right) \] (2.2)

where the scalar fields \( \psi, W, \xi \) and \( \chi \) all depend on the coordinates of the \( d \)-dimensional slice and \( r \). Note that we consider the most general perturbation metric. Whereas in [5, 6], the scalar \( \xi \) is not considered. It is important to mention that one has to be cautious as to how one defines the inverse of the metric after adding \( h_{\mu\nu} \). Indeed from linear algebra properties of matrices’ differentiation, one can show that if the metric is \( g_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu} \) then its inverse is \( g^{\mu\nu} = g^{\mu\nu} - g^{\mu\rho} h_{\rho\sigma} g^{\sigma\nu} \) provided that \( h_{\mu\nu} \) is small compared to \( g^{\mu\nu} \).

Let us explicitly state the definition of the Einstein equations for convention purposes

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa^2 T_{\mu\nu} \] (2.3)

where \( R_{\mu\nu} \) is the Ricci tensor, \( R = R_{\mu}^{\mu} \) is the Ricci scalar and \( T_{\mu\nu} \) is the energy-momentum tensor. In the same way that the metric is linearized, the Einstein equations will also be linearized and solved order by order. Namely

\[ \delta G_{\mu\nu} = \kappa^2 \delta T_{\mu\nu}, \]

\[ G^{(0)}_{\mu\nu} = \kappa^2 T^{(0)}_{\mu\nu} \] (2.4)

where the \( \delta \) prefix refers to the first order perturbation of the quantity and the superscript (0) refers to the background value. Let us start by giving the background expression of the Einstein tensor. We will work with \( \gamma_{ij} \) being the Kronecker delta, i.e. we will work in Euclidean signature. Although our calculations are inspired by [7], it is important to note that they are different because we will define and compute all equations for a Euclidean space. This is an important note as we have observed that \( D \)- or \( d \)-dependant factors in the different equations, can change depending on whether the space is Euclidean or Lorentzian.

\[ G^{(0)}_{ij} = a^2 (d - 1) \left( \frac{d}{2} H^2 + H' \right) \delta_{ij}, \]

\[ G^{(0)}_{rr} = \frac{d(d-1)}{2} H^2, \]

\[ G^{(0)}_{ri} = 0. \] (2.5)

We defined the quantity \( H(r) \), analogous to the Hubble parameter as it is defined as \( H = \dot{a}'/a = A' \) with the prime referring to differentiation with respect to \( r \). Now we compute the first order Einstein tensor by simply plugging (2.2) in (2.3) which gives

\[ \delta G_{ij} = \{ [dH^2 + 2H'] (d - 1)(\psi - \chi) - (d - 1)a a' \} \delta_{ij}, \]

\[ + d(d-1)aa' \psi' + (d - 1)a^2 \psi'' + p' \Lambda \} \delta_{ij} \]

\[ - \partial_i \partial_j \left\{ \Lambda + [dH^2 + 2H'] (d - 1)W \right\}, \]

\[ \delta G_{rr} = a^{-2} (d - 1) \left\{ \partial_r \left[ \psi - aa' (\xi + W') \right] + d \psi' \right\}, \]

\[ \delta G_{ri} = \partial_i \left\{ (d - 1)(H \chi - \psi') + \chi \left( \frac{d(d-1)}{2} H^2 + (d - 1)H' \right) \right\}. \] (2.6)
Where $\Lambda = \chi + (d-2)\psi - da'(\xi + W') - a^2(\xi + W')'$ which has no connection whatsoever to the cosmological constant.

Now concerning the energy-momentum tensor, we will consider the most general bulk action with one active scalar $\Phi_a$. The active scalars are defined as non-constant scalar fields, in particular we will work with active scalars with a background depending only on $r$. The relevant part of the Euclidean action is then

$$S = \int_M d^{d+1}x \sqrt{g} \left[ \frac{1}{4} R + \frac{1}{2} g^{\mu\nu} \partial_\mu \Phi_a \partial_\nu \Phi_a + V(\Phi_a, \Phi_i) \right] - \frac{1}{2} \int_{\partial M} d^d x \sqrt{\sigma} K \quad (2.7)$$

with $K$ being the trace of the second fundamental form and $\sigma$ is the metric on the boundary. Note the presence of inert scalars $\Phi_i$ which are constant scalar fields. We restrict the following study to one inert scalar but it will be shown that the same study can be conducted with an arbitrary number of them. We fixed $\kappa^2 = 2$. Since we are interested in the first order perturbation $\delta T_{\mu\nu}$, we need to define perturbations of the fields

$$\Phi_a = \phi_a(r) + \tilde{\phi}_a(r, x) \quad \text{and} \quad \Phi_i = \phi_i + \tilde{\phi}_i(r, x). \quad (2.8)$$

These definitions then give the following tensors

$$T_{ij}^{(0)} = -a^2 \left( \frac{1}{2} (\phi'_a)^2 + V \right); \quad T_{rr}^{(0)} = \frac{1}{2} (\phi'_a)^2 - V; \quad T_{ri}^{(0)} = 0,$$

$$\delta T_{ij} = a^2 \delta_{ij} \left[ \chi (\phi'_a)^2 - \tilde{\phi}'_a \phi'_a - \tilde{\phi}_a V_{\phi_a} - \tilde{\phi}_i V_{\phi_i} \right] - 2a^2 (\psi \delta_{ij} - \partial_i \partial_j W) \left[ \frac{1}{2} (\phi'_a)^2 + V \right],$$

$$\delta T_{rr} = -2\chi V + \tilde{\phi}'_a \phi'_a - \tilde{\phi}_a V_{\phi_a} - \tilde{\phi}_i V_{\phi_i},$$

$$\delta T_{ri} = \partial_i \xi a^2 \left( \frac{1}{2} (\phi'_a)^2 + V \right) + \partial_i \tilde{\phi}_a \phi'_a.$$

Now that we have every component of the Einstein equations (EE) we have to investigate their behaviour under an infinitesimal gauge transformation. Indeed, imposing boundary conditions directly on the EE leaves some of the boundary data undetermined. A way to solve this problem is simply to build gauge-invariant variables and solve their equation of motion.

### 2.2 Gauge-invariant variables and equations

We defined the metric with the line element (2.1) to which we added the first order perturbations (2.2). But one can see that there still is gauge freedom remaining. That can be fixed by considering the most general diffeomorphisms, such that

$$g_{\mu\nu} \to g_{\mu\nu} - \nabla_\mu \epsilon_\nu (x, r) - \nabla_\nu \epsilon_\mu (x, r), \quad (2.10)$$

where $\epsilon^\mu$ is the generator of an infinitesimal coordinate transformation on the metric. This generator is defined in a similar way to what is done in [7]. It is argued there that gauge transformation on the transverse space is parametrized by

$$\epsilon^i = \epsilon_i^r + \gamma^i_j \partial_j \epsilon^r, \quad (2.11)$$
where \( \epsilon \) satisfies
\[
\partial^j \partial_i \epsilon = \partial_j \epsilon^j.  \tag{2.12}
\]
And the vector \( \epsilon_{t r}^i \) is such that \( \partial_i \epsilon_{t r}^i = 0 \). Since this is an infinitesimal transformation, it can also be seen as metric perturbation which means that the background metric is left invariant under this class of diffeomorphisms. Meaning that we effectively have
\[
h_{\mu \nu} \rightarrow h_{\mu \nu} - \nabla_\mu \epsilon_\nu(x, r) - \nabla_\nu \epsilon_\mu(x, r).  \tag{2.13}
\]
Obviously enough, since we want to keep our studies to first order perturbations, the covariant derivatives are performed on the background metric. This interpretation of the gauge transformation also means that \( \epsilon_{t r}^i \) only contributes to vector-like perturbation. Since we are only interested in scalar fluctuations, \( \epsilon_{t r}^i \) is left out and the gauge transformation of the transverse space is only parametrized by \( \epsilon(r, x) \). Now we can explicitly compute the transformation for each components of \( h_{\mu \nu} \) resulting in
\[
\begin{align*}
\psi & \rightarrow \psi - H \epsilon^r, \\
W & \rightarrow W + \epsilon, \\
\xi & \rightarrow \xi - \frac{1}{a^2} \epsilon^r - \partial_r \epsilon, \\
\chi & \rightarrow \chi - \partial_r \epsilon^r.
\end{align*}  \tag{2.14}
\]
By studying these rules, we can build gauge invariant variables. There is an infinite number of such variables as many combinations of the above variables can be invariant. We will choose the combinations we consider the simplest, i.e.
\[
\begin{align*}
R & = \chi - \left[ a^2 (\xi + W') \right]' \\
\Psi & = \psi - a \alpha'(\xi + W').
\end{align*}  \tag{2.15}
\]
Let us immediately note the difference in definition between our variable called \( R \) and the one from [6] with the same symbol which we will call \( R_S \). Indeed, these differ simply in their definition, while \( R_S \) is defined using both metric perturbation and scalar field perturbation, ours is defined using only metric perturbation. It is therefore difficult to see an easy physical interpretation to \( R_S \) whereas \( R \) can be seen as a physical space perturbation. Now, one could argue that it cannot be interpreted as such because of the existence of \( \Psi \). In this case, one argues that there should be only one function describing physical perturbations. But as we will show later, \( R = -(d-2)\Psi \) so \( R \) and \( \Psi \) as well — can be interpreted as physical space perturbation.

While we defined properly how the metric transforms under (2.10), the scalar fields will also be modified under these diffeomorphisms. As mentioned above, \( \epsilon^\mu \) is seen as a perturbation, which means that the background scalar fields are gauge invariant. On the other hand, the perturbation scalar field \( \tilde{\phi}_a \) will be modified under (2.10) such that
\[
\tilde{\phi}_a \rightarrow \tilde{\phi}_a - \phi_a^r \epsilon^r.  \tag{2.16}
\]
One can create a gauge invariant variable from $\tilde{\phi}_a$ in the same fashion as with $R$ and $\Psi$. Once again, out of an infinite number of configuration we choose the one we consider the simplest, i.e.

$$\tilde{\phi}^{\text{GI}}_a = \tilde{\phi}_a - a^2 \phi^i_a \left( \xi + W' \right). \tag{2.17}$$

The superscript GI denotes the gauge invariant version of a function. Now, recall that the inert scalar field is defined as being constant. This automatically means that $\tilde{\phi}_i$ is gauge invariant by itself. Every gauge invariant variables defined above have been built in the same way as done in [7]. And we will continue to follow the same method to now define the gauge invariant Einstein equations.

Indeed, under coordinate transformation, tensors transform in a specific way. Not only the energy-momentum tensor, in Euclidean space, it yields as for a Euclidean slice up to sign conventions in the energy-momentum tensor. Now for the Minkowski metric, and it turns out that the gauge-invariant EEs are exactly the same in the same way as done in [7]. And we will continue to follow the same method to now define the gauge invariant Einstein equations.

Now, recall that the energy-momentum tensor, in Euclidean space, it yields as for a Euclidean slice up to sign conventions in the energy-momentum tensor. Now for the Minkowski metric, and it turns out that the gauge-invariant EEs are exactly the same in the same way as done in [7]. And we will continue to follow the same method to now define the gauge invariant Einstein equations.

Indeed, under coordinate transformation, tensors transform in a specific way. Not only do we need to replace the different perturbation fields by the variables in (2.15) and (2.17), but we also have to consider the tensor transformation

$$\tilde{T}_{\mu\nu}(x) = \frac{\partial x^\sigma}{\partial x'_{\mu}} \frac{\partial x'^\rho}{\partial x^\nu} T_{\sigma\rho}(x). \tag{2.18}$$

Here the tilde only denotes a new coordinate system, it has no link to the fluctuations. This rule is valid for any tensor $T$. In our case, under (2.10), $\tilde{x}^\mu = x^\mu + \epsilon^\mu$. After first order expansion, the explicit calculation of the rule above, gives

$$\begin{align*}
\delta G_{ij} &= \delta G_{ij} - \partial_i \epsilon^m G^{(0)}_{mj} - \partial_j \epsilon^n G^{(0)}_{ni} - (G^{(0)}_{ij})' \epsilon^r, \\
\delta G_{rr} &= \delta G_{rr} - 2G^{(0)}_{rr} \partial_r \epsilon^r - (G^{(0)}_{rr})' \epsilon^r, \\
\delta G_{ri} &= \delta G_{ri} - G^{(0)}_{ri} \partial_i \epsilon - G^{(0)}_{rr} \partial_i \epsilon^r.
\end{align*} \tag{2.19}$$

As $T_{\mu\nu}$ and $G_{\mu\nu}$ have the same properties, the same transformations hold for the energy-momentum tensor. One can easily see how to build the GI tensors using these relations and (2.14). In the end, we get

$$\begin{align*}
\delta G_{ij}^{\text{GI}} &= \left\{ -(d-1)a \left[ R' - d \Psi' \right] - a^2 \left[ dH^2 + 2H' \right] (d-1)(R - \Psi) \\
&\quad + (d-1) \Psi' a^2 + \nabla^2 (R + (d-2)\Psi) \right\} \delta_{ij} - \partial_i \partial_j \left[ R + (d-2)\Psi \right]; \\
\delta G_{rr}^{\text{GI}} &= \frac{(d-1)}{a^2} \left[ da' a \Psi' + \nabla^2 \Psi \right]; \\
\delta G_{ri}^{\text{GI}} &= (d-1) \partial_i \left[ HR - \Psi \right].
\end{align*} \tag{2.20}$$

One can see that these expressions are indeed gauge invariant since only $R$ and $\Psi$ are appearing. We also carried out the computation of the Einstein equations with $\gamma_{ij}$ being the Minkowski metric, and it turns out that the gauge-invariant EEs are exactly the same as for a Euclidean slice up to sign conventions in the energy-momentum tensor. Now for the energy-momentum tensor, in Euclidean space, it yields

$$\begin{align*}
\delta T_{ij}^{\text{GI}} &= a^2 \delta_{ij} \left[ R(\phi'_a)^2 - (\tilde{\phi}^{\text{GI}}_a)^2 \right] - \tilde{\phi}_a^{\text{GI}} V_{\phi_a} - \tilde{\phi}_a V_{\phi_a} - 2\Psi \left( \frac{1}{2} (\phi'_a)^2 + V \right) \\
\delta T_{rr}^{\text{GI}} &= -2RV + (\tilde{\phi}^{\text{GI}}_a)^2 \phi'_a - \tilde{\phi}_a^{\text{GI}} V_{\phi_a} - \tilde{\phi}_a V_{\phi_a}, \\
\delta T_{ri}^{\text{GI}} &= \partial_i \tilde{\phi}_a^{\text{GI}} \phi'_a. 
\end{align*} \tag{2.21}$$
These expressions are also clearly gauge invariant as they depend only on $R$, $\Psi$, $\widetilde{\phi}^G$ and $\tilde{\phi}_i$. Note that we don’t add the GI superscript to $\tilde{\phi}_i$ because it would make the notation redundant.

We now have all the components needed to build the Einstein equations. The following step is to piece everything together and find an equation of motion (EOM) for either $R$ or $\Psi$. If such an EOM is solvable, we will then have enough equations to determine all of the boundary data.

### 2.3 Einstein equations and perturbative EOM

We found the gauge invariant Einstein tensor (2.20) and energy-momentum tensor (2.21). This allows us to search for an EOM for either $R$ or $\Psi$. One could think that we have a choice to make as to which variable we want to treat. However when looking at the $(ij)$ Einstein equation with $i \neq j$, we can see that $\partial_i \partial_j [R + (d - 2)\Psi] = 0$ which means we can set $R = -(d - 2)\Psi$. This is freeing us from the previously mentioned choice.

Now let us search for the EOM of $R$. It can be done by taking the combination of EEs: $(ij)-a^2(rr)$, $i = j$. Doing this and using $R = -(d - 2)\Psi$ we get

$$
(d-2)HR' + [dH^2 + 2H'] (d-1)R - \nabla^2 R + R'' = \frac{d-2}{d-1} \kappa^2 \left[ 3R \left( \frac{1}{2} (\phi'_a)^2 + V \right) - 2(\phi^G_a)' \phi'_a \right].
$$

(2.22)

Now, we use the $(ri)$ Einstein equation to express $(\tilde{\phi}^G_a)'$ in terms of $R$ and $H$. We also use the background EEs to express $\phi'_a$ and $V$ as ODEs of $H$. The EOM of $R$ is then given by

$$
R'' + \left( (d-2)H - \frac{H''}{H'} \right) R' + (d-2) \left[ 2H' - \frac{H''}{H'} - \frac{\alpha^{-2} p^2}{d-2} \right] R = 0.
$$

(2.23)

Note that we went from position space to momentum space thus introducing the transverse momentum $p$. It is possible to show that (2.23) agrees with equation (4.13) of [6]. To do so, one has to realize that $p^2 R = H' a^2 R_S$ and use the first order flow equations we define below. We now have the equation of motion of $R$ which will be useful in the next sections as, if it’s solvable, it will help us determine all of the boundary data.

To find an expression for $\tilde{\phi}_i$, one can use either the $(rr)$ or $(ij)$ Einstein equation — or a combination of both — and both the solutions for $R$ and $\tilde{\phi}^G_a$. However, it is possible that $V_{\tilde{\phi}_i}$ vanishes when the inert scalar background goes to zero. In this case, the only way to find a solution for $\tilde{\phi}_i$ is to derive its Klein-Gordon equation by expanding the action integrand to second order in $\tilde{\phi}_i$. The inert scalar field equation should be solvable — in a fairly simple background — since it does not depend on space perturbations because the inert scalar perturbation does not couple to space fluctuations at second perturbative order. Conversely, the active scalar field equation can be more difficult to solve since $\tilde{\phi}_a$ does couple to space fluctuations at second perturbative order. Hence, we recommend finding a solution for $\tilde{\phi}^G_a$ by using the $(ri)$ Einstein equation in which we plug the solution to (2.23) — provided that a solution exists.

Below, we introduce the holographic framework and make clear the link between the gravity theory we described above and a field theory living on its boundary.
3 Holographic Triforce

This section is dedicated to a brief introduction to holographic principle, RG-flows and renormalization. We begin by giving a definition of the former. Then we go on to introduce well-known results of holographic RG-flows that will be of later use. The end of the section will be devoted to a general demonstration of holographic renormalization as pioneered in [5] which we apply to the GPPZ flow in the next section.

3.1 The holographic principle

The holographic principle [3] states that a \( D \)-dimensional theory of gravity can be described by a \( (D-1) \)-dimensional field theory. These two theories are respectively called bulk and boundary theories. This principle predicts the existence of a dictionary between the two theories [2]. Although the introduction of holography uses basic realizations from both bulk and boundary theory, finding the correspondence dictionary is a complicated task. The AdS/CFT correspondence [1] is one correspondence predicted by holography for which the mathematical framework is well established.

AdS/CFT has two sides, as suggested by the name. The first one concerns the bulk string theory compactified — in our case — on AdS\(_5\) which at low energy becomes a gauged supergravity theory. This bulk theory may contain different fields, but in this report we only consider the metric and scalar fields. By virtue of the correspondence, the partition function of the bulk theory with imposed boundary conditions equates to the generating functional of the boundary CFT vevs. This in the end means that the boundary value of bulk fields directly correspond to sources for operators living in the boundary CFT. It is therefore possible to compute one- and subsequently n-point functions of boundary operators [4] from bulk field boundary data.

3.2 Holographic RG-flows

In the previous section, we defined the action (2.7)

\[
S = \int_M d^5x \sqrt{\mathcal{g}} \left[ \frac{1}{4} \mathcal{R} + \frac{1}{2} \epsilon^{\mu\nu} \partial_\mu \Phi_a \partial_\nu \Phi_a + V(\Phi_a, \Phi_i) \right] - \frac{1}{2} \int_{\partial M} d^4x \sqrt{\mathcal{g}} K. \tag{3.1}
\]

Domain-wall solutions of this action which take the form (2.1) can be interpreted as descriptions of RG-flows, characterized by the expression of the potential \( V(\Phi) \). In this report, the flow we study is supersymmetric. It is important however to note that the literature also contains fake-supersymmetric flows. By which we mean that the potential \( V(\Phi) \) can be written in a supersymmetric-like expression as a function of a fake supersymmetric potential \( W(\Phi) \)

\[
V(\Phi) = \frac{1}{2L^2} (\partial_\Phi W)^2 - \frac{4}{3L^2} W^2. \tag{3.2}
\]

One can relate metric and field quantities via the first order flow equations [13]

\[
\frac{dA(r)}{dr} = -\frac{2}{3L} W(\Phi), \quad \frac{d\Phi(r)}{dr} = \frac{1}{L} \partial_\Phi W(\Phi) \tag{3.3}
\]
computed from Killing spinor conditions in the bulk theory. Note that (3.2) and (3.3) are for fixed $D = 5$ as the coefficients are depending on the number of dimensions. One can see as well that we made the AdS radius explicit as there are subtleties to mention. As mentioned in [6], one has to be careful when setting $L = 1$, instead the AdS radius should be set to one in string units, i.e. $L^2 = \alpha'$. It will also be useful to make it appear to fix dimensional analysis in some equations to come.

The idea of holographic RG-flow resides in the correspondence between distances in the bulk and energy scale in the field theory. Indeed, the radial coordinate $r$ has a natural interpretation as energy scale. Flowing between two CFT fixpoints in the bulk is related to a flow between two CFTs. Near a such a point one can expand the scalar field in a modified power law. There is a more suitable radial coordinate $\rho$ for near-boundary expansion such that $\frac{d\rho}{d\rho} = -\frac{2\rho}{L}$. The general AdS bulk metric is then

$$ds^2 = \frac{d\rho^2}{4\rho^2} + \frac{1}{\rho} g_{ij}(x, \rho) dx^i dx^j. \quad (3.4)$$

Near the boundary $\rho = 0$, $g_{ij}(x, \rho)$ admits the following expansion

$$g = g_{(0)} + \rho g_{(2)} + \ldots + \rho^{d/2} \left[ g_{(d)} + \sum_{k \geq 1} h_{k(d)} \log^k \rho \right] + O \left( \rho^{d/2+1} \right). \quad (3.5)$$

The logarithmic terms only appear in the case of even dimension $d$. For $d = 4$ the expansion is then

$$g(x, \rho) = g_{(0)} + g_{(2)} \rho + \rho^2 \left[ g_{(4)} + h_{1(4)} \log \rho + h_{2(4)} \log^2 \rho \right] + \ldots. \quad (3.6)$$

As we will show in the next section, in the case of the GPPZ flow, the metric (2.1) does satisfy the near-boundary expansion (3.4).

In this report we are most interested in duals to scalar fields. To find the correlators of these operators, we will need to solve the EOM we found in Section 2 and consider solutions with fields satisfying the modified Dirichlet condition

$$\Phi(\rho, x) \to \rho^{\frac{d-\Delta}{2}} \phi_{(0)}(x) \quad \text{as} \quad \rho \to 0. \quad (3.7)$$

We use the usual notation $\Delta$ for the conformal weight of the dual operator $O_{\Phi}$ to $\Phi$. The scalar fields actually admit a full expansion as $\rho \to 0$ satisfying (3.7)

$$\Phi(x, \rho) = \rho^{\frac{d-\Delta}{2}} \left[ \phi_{(0)} + \rho \phi_{(2)} + \ldots + \rho^{\frac{2\Delta-d}{2}} \left( \phi_{(2\Delta-d)} + \psi_{(2\Delta-d)} \log \rho \right) \right] + O \left( \rho^{\frac{\Delta+2}{2}} \right). \quad (3.8)$$

Like in the expansion of $g_{ij}$, the logarithmic term only appears for $d$ even. The functions $g_{(0)ij}(x)$ and $\phi_{(0)}(x)$ are respectively sources for the stress tensor $T_{ij}$ and operator $O$ in the boundary CFT and the other coefficients in 3.8 are called boundary data. Bulk and boundary theories are connected by the simple equation

$$\left\langle \exp \left( -S_{QFT} \left[ g_{(0)} \right] - \int d^4x \sqrt{g_{(0)}} O(x) \phi_{(0)}(x) \right) \right\rangle = \exp \left( -S_{SG} \left[ g_{(0)}, \phi_{(0)} \right] \right). \quad (3.9)$$

Where the left side is the generating functional of the boundary theory minimally coupled to the source $g_{(0)ij}(x)$. The right side is the generating functional of the bulk theory where
$S_{SG} [g_{(0)}, \phi_{(0)}]$ is the action (2.7) evaluated on-shell with boundary data defined above. However, the on-shell action is not finite but actually divergent because of the behaviour of the near-boundary solution of $g_{ij}$ and $\Phi$. It is easy to see it with $\Phi$ because we require its solution to satisfy (3.7). Then, taking $\rho$-derivatives of $\Phi$ makes negative powers of $\rho$ arise which are divergent when $\rho \to 0$. Thus we need to regularize and renormalize the action. Thus, we describe holographic renormalization in the next subsection.

### 3.3 Holographic renormalization

This subsection will introduce the renormalization method for holography developed in [5]. Indeed, when considering holographic $RG$-flows there is a need for renormalization as the on-shell action suffers from infinities. The process consist of first regularizing the action by restricting the integral to $\rho \geq \epsilon$. Obviously enough we will then take the limit $\epsilon \to 0$ but before that we use the near-boundary expansions (3.8) and (3.6) in the on-shell action $S$. This way one can express the divergence arising in $S$ in terms of the sources. It is also possible to “reverse” the expansions (3.8) and (3.6) to express the sources as functions of $\Phi$, the metric and other curvature-related quantities. This is done by plugging the asymptotic expansions in the Einstein equations to get relations between the different coefficients in the expansion. So in the end, we can cancel the divergences by creating boundary counter-terms depending on the fields and curvature-related quantities. This is indeed possible thanks to the Einstein equations we gave above, as it allows us to relate coefficients of the asymptotic expansion to one another. We present here general results from [6] and then apply the found formulas to the case of the GPPZ flow.

#### 3.3.1 Correlators of the GPPZ flow

Let us first define what we call GPPZ flow. It is a $\mathcal{N} = 1$ supersymmetric kink solution of $5D$ gauged supergravity found in [11]. It describes a flow between $\mathcal{N} = 4$ SYM, deformed by a relevant operator, to pure $\mathcal{N} = 1$ SYM. By relevant operator we mean that the conformal weight $\Delta$ of the operator must be such that $\Delta \leq 4$ in 5 dimensions. Indeed, in our analysis both active and inert scalars have $\Delta = 3$ which are considered relevant operators. Unlike what has been done in [6], we consider both active and inert scalars, respectively dubbed $m$ and $\sigma$ in [11] which we renamed $\Phi_a$ and $\Phi_i$. The inert scalar field is naturally equal to zero but as we will linearize the equations, we need to make $\Phi_i$ appear explicitly.

The superpotential is

\[
W(\Phi_a, \Phi_i) = -\frac{3}{4} \left[ \cosh \left( \frac{2\Phi_a}{\sqrt{3}} \right) + \cosh(2\Phi_i) \right] \quad (3.10)
\]

and its expansion around $\Phi_a = \Phi_i = 0$ yields

\[
W \sim -\frac{3}{2} - \frac{1}{2} \Phi_a^2 - \frac{3}{2} \Phi_i^2 - \frac{1}{18} \Phi_a^4 - \frac{1}{2} \Phi_i^4 + \mathcal{O}(\Phi^5) \quad (3.11)
\]

And so with this superpotential, one finds a background domain-wall solution when

\[
\phi_a = \frac{\sqrt{3}}{2} \log \frac{1 + \sqrt{\rho}}{1 - \sqrt{\rho}}, \quad e^{2A} = \frac{1 - \rho}{\rho} \quad (3.12)
\]
According to (3.8), the bulk fields with $\Delta = 3$ will have an expansion of the form
\[
\Phi = \sqrt{\rho} \left[ \phi^{(0)} + \rho \left( \psi^{(2)} + \phi^{(2)} \log \rho \right) \right] + \mathcal{O} \left( \rho^{5/2} \right). \tag{3.13}
\]
The near-boundary expansion in the $\rho$-variable of the background is now explicit
\[
\phi_a = \rho^{1/2} \left[ \sqrt{3} + \rho \frac{1}{\sqrt{3}} + \mathcal{O}(\rho^2) \right], \quad e^{2A} = \frac{1}{\rho} (1 - \rho). \tag{3.14}
\]
As required, the expansion has the same form as (3.8). Since both active and inert scalars have the same near-boundary expansion, the calculations below are carried out for a generic field $\Phi$ of conformal weight $\Delta = 3$. $\Phi$ contains both background and perturbations. As it turns out, the only difference between the active and inert scalars in our treatment, is that one of them has vanishing background value, i.e. the inert scalar. So one can do this treatment, as it was done in [6], and apply the equations to the inert scalar just by setting its background value to zero in the end results. The same philosophy goes for linearizing the equations, we compute everything for the generic field $\Phi$ and linearize the end result.

Now we need to regularize the action. If one replaces $\gamma_{ij}$ by $\delta_{ij}$ in (2.1) then one finds that the regularized action is
\[
S_{\text{reg}} = \int_{\rho=\epsilon} d^4 x \sqrt{\sigma} W[\Phi_a, \Phi_i], \tag{3.15}
\]
where $\sigma_{ij} = e^{2A} \delta_{ij}$ is the induced metric at the cut-off surface. Knowing this and looking at (3.10), we quickly realize that the constant and quadratic terms in the expansion of the superpotential will lead to infinities. This is indeed the case as the induced metric carries negative powers of $\epsilon$ and so its determinant does as well. Thus the lowest order terms in the asymptotic expansion of the fields will eventually lead to divergences in the action.

However, the last two terms we made explicit in (3.10) will lead to finite terms. These should be dealt with by adding a finite counterterm. This makes the renormalized action $S_{\text{ren}}$ equal to zero. Through holography this is equivalent to a choice of scheme in field theory. Indeed, fixing $S_{\text{ren}} = 0$ means that we get a vacuum energy to be zero. This is a requirement of supersymmetry. So we must add this finite counterterm, so that the on-shell value of the action is zero for this RG-flow.

An important remark we should add here, as it was briefly noted in [6], is that it is primordial to consider the most general counterterms possible. What we mean by that, is to consider the most general boundary metric and not only fixing it to be $\delta_{ij}$ as we did above. In fact, this is because not all counterterms appear in what we did above. It turns out that by considering a metric similar to (3.4), the most general counterterm [6] for the GPPZ solution is
\[
S_{\text{ct}} = \int_{\rho=\epsilon} d^4 x \sqrt{\sigma} \left( \frac{3}{2} - \frac{1}{8} R + \frac{1}{2} \Phi^2 \right. \right.
\]
\[
- \log \epsilon \left[ \frac{1}{32} (R_{ij} R^{ij} - \frac{1}{3} R^2) + \frac{1}{4} (\Phi \Box_{\sigma} \Phi + \frac{1}{6} R \Phi^2) \right]. \tag{3.16}
\]
Here $\Box_{\sigma}$ is the Laplacian of the induced metric $\sigma$ at $\rho = \epsilon$. Note also that all curvature-related quantities in this equation are those of $\sigma$. It is now clear that indeed new terms
appear in the counterterm, terms that did not show up in the analysis based solely on the simplest domain wall solution. Fortunately, we also find the terms we expected to show up. The finite counterterm

$$S_{ct, fin} = \int d^4x \sqrt{\sigma} \frac{1}{18} \Phi^4$$

(3.17)

has to be added in order to fix the scheme, as discussed earlier.

There is one caveat however that needs to be noted. We are able use results from [6] and [5] since the inert scalar and its perturbation don’t appear in the EE. Indeed, by definition of the superpotential and the fact that $\phi_i = 0$, at first perturbative order the inert scalar doesn’t couple to anything but itself. One could think that we need to compute the general one-point function of the inert scalar but if we are only interested in the first order perturbation we can use the expression of the active scalar. One could argue that it’s not possible because inert scalars don’t couple to gravity but active scalars do. Fair enough but as we will see shortly, we need to expand (3.16) at the second perturbative order to get the one-point function of the first perturbation. In doing so one realizes that the fluctuation of the inert scalar behaves like an active scalar but the coupling to gravity drops out because $\phi_i = 0$. However, the expression for $\langle O \rangle$ we will get from the equations above cannot be used as is since the inert field is not normalized like the active one. This problem is easily remedied by rescaling the field.

Finally the renormalized action is

$$S_{ren} = \lim_{\epsilon \to 0} [S_{reg} + S_{ct} + S_{ct, fin}].$$

(3.18)

Obviously enough, both scalar fields are now included in the counterterms. The holographic one-point function of the operator dual to the active scalar is

$$\langle O_a \rangle = \frac{1}{\sqrt{g(0)}} \frac{\delta S_{ren}}{\delta \phi_a(0)} = \lim_{\epsilon \to 0} \left( \frac{1}{\epsilon^{3/2}} \frac{1}{\sqrt{7}} \frac{\delta S_{ren}}{\delta \Phi_a(x, \epsilon)} \right) = -2 \left( \phi_a(2) + \psi_a(2) \right) + \frac{2}{9} \phi_a^3(0).$$

(3.19)

And the holographic one-point function of the operator dual to the inert scalar is

$$\langle O_i \rangle = \frac{1}{\sqrt{g(0)}} \frac{\delta S_{ren}}{\delta \phi_i(0)} = \lim_{\epsilon \to 0} \left( \frac{1}{\epsilon^{3/2}} \frac{1}{\sqrt{7}} \frac{\delta S_{ren}}{\delta \Phi_i(x, \epsilon)} \right) = -2\sqrt{3} \left( \phi_i(2) + \psi_i(2) \right) + \frac{2\sqrt{3}}{3} \phi_i^3(0).$$

(3.20)

Now that we have the one-point functions expressed as functions of the asymptotic expansion’s coefficients, we need to find their expression. The Einstein equations were used to find the expressions of the metric’s coefficients which gave us the counterterms. It is also important to point out that not all coefficients are determined. For example, in our case with $d = 4$ and $\Delta = 3$, in the expansion of the scalar fields $\phi(2)$ is undetermined by simply plugging the expansions in the equations. However $\psi(2)$ is fully determined by the Klein-Gordon equation. For the metric the first undetermined coefficient is $g_4(0)$ but its trace and divergence can be found [6].

To find these unknowns, we need new data and this data is found by making the Einstein equations gauge invariant and solving the EOM of one of the gauge invariant quantities. This means that to have all the information possible on the boundary theory, we need information about the deep interior of the bulk. This is exactly the purpose of the next section in which we find this extra bulk information needed.
4 The GPPZ solution

As it was mentioned at the end of the previous section, we miss data to express the one-point functions only in terms of the sources. This will be resolved in this section by writing and solving the equation of motion of \( R \), defined in subsection 2.3, for the GPPZ solution. We also investigate the Klein-Gordon equations of the perturbation fields. Indeed, some necessary expression can only be found by using these equations. And finally, we end this section by giving the one-point functions expressed only in terms of the sources, just as we set out to do from the beginning of this report.

Let us first define the asymptotic expansion of some fields in (2.2) in order to fix the notation

\[
\psi = \frac{1}{1 - \rho} \left[ h_{(0)}(0) + \rho h_{(2)} + \rho^2 h_{(4)} + \rho h_{(6)}(\log \rho) + h_{(8)}(\log^2 \rho) + \ldots \right],
\]

\[
W = \frac{1}{1 - \rho} \left[ H_{(0)}(0) + \rho H_{(2)} + \rho^2 H_{(4)} + \rho hH_{(6)}(\log \rho) + \ldots \right].
\] (4.1)

A keen eye can notice the factor \( 1/(1 - \rho) \) in the expansion. Indeed it was introduced to have a direct match between the coefficients in (4.1) and (3.6). Note also that the expansions of \( \xi \) and \( \chi \) don’t appear as we impose the radial gauge. This means that \( \chi = 0 \) and we take \( \xi = 0 \) to match the literature on holographic RG-flows. In order to make the equations lighter, we drop the tilde notation over the perturbations. This means that all equations are now of first order perturbations and the tilde will be reinstated in case ambiguity arises.

Recall the domain-wall geometry of the GPPZ solution given above in equation (3.14). The solution can be plugged in (2.23) to give in the \( \rho \)-variable

\[(1 - \rho)R'' + \frac{1}{\rho} \left[ -(1 + 2\rho)R' + \left( \frac{1}{\rho} - \frac{L^2 \rho^2}{4} \right) R \right] = 0.\] (4.2)

The prime refers to differentiation with respect to \( \rho \). This equation resembles Euler’s hypergeometric differential equation and it admits the following solution

\[R(p^2, \rho) = \rho F \left( \frac{3}{2} + \sqrt{1 - p^2 L^2}, \frac{3}{2} - \sqrt{1 - p^2 L^2}; 3, 1 - \rho \right).\] (4.3)

Equation (4.2) actually admits two different solution but one of them is not regular in the bulk hence the presence of only one of the solutions. We consider a solution to be regular when it doesn’t admit any infinities in the bulk or the boundary.

The hypergeometric function \( F(a, b; c; z) \) can be expressed as a series defined everywhere in the bulk [16]

\[
F(a, b; a + b + m; z) = \frac{\Gamma(m)}{\Gamma(a + m)\Gamma(b + m)} \sum_{n=0}^{m-1} \frac{(a)_n(b)_n}{(1 - m)_n n!} (1 - z)^n
\]
\[+ \frac{(1 - z)^m(-1)^m}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{(a + m)_n(b + m)_n}{(n + m)_n n!} [\alpha_n - \log(1 - z)] (1 - z)^n.\] (4.4)
Where the $\alpha_n$ coefficients are defined as
\[ \alpha_n = \psi(n + 1) + \psi(n + m + 1) - \psi(a + n + m) - \psi(b + n + m). \] (4.5)

The $\psi$ function is the logarithmic derivative of the usual $\Gamma$-function. Considering equation (4.3), we have to take $m = 0$ which leads us to interpret the first sum of (4.4) as being null. Now we can easily see that to apply the expansion to the solution of $R$, we have $a = (3 + \sqrt{1 - p^2 L^2})/2$ and $b = (3 - \sqrt{1 - p^2 L^2})/2$. It is important to mention, that $p^2$ can take any value. It is the case only because if $L^2 p^2 > 1$ then $a = b$ and this makes (4.3) real for any real $p^2$. Now, for obvious clarity purposes we define
\[ \gamma = \Gamma \left( \frac{3}{2} - \frac{1}{2} \sqrt{1 - L^2 p^2} \right) \Gamma \left( \frac{3}{2} + \frac{1}{2} \sqrt{1 - L^2 p^2} \right). \] (4.6)

The solution of $R$ (4.3) and its expansion (4.4) give us the missing information on the bulk that we needed to determine all asymptotic expansion coefficients of the fields and metric. In the end, the few equations above are the key to expressing one-point functions in terms of the sources. We complete this task in the next two subsections.

### 4.1 Active scalar

Let us recall that the only unknown we need to compute the one-point function (3.19) is $\phi_{a(2)}$. The expression for $\psi_{a(2)}$ is found in [6] by using the Klein-Gordon equation of $\tilde{\phi}_a$. One can find the field equation by expanding (2.7) to second order in $\tilde{\phi}_a$, calculation that we do not need to carry out in this report.

Before reusing the Einstein equations, we can get new relations using the fact that $R = -2\Psi$. We utilize the different asymptotic expansions defined up until this point and the definition (2.15) of $\Psi$ to find
\[ h(2) = \frac{1}{\gamma}; \quad H(2) = -\frac{L^2}{2} h(0) - H(0); \quad hH_{2(4)} = 0; \quad hH_{1(4)} = \frac{L^2}{4\gamma}. \] (4.7)

Previously, the Einstein equations were used to find the asymptotic solution of the metric which allowed us to find the most general counter-term for this flow. However, we now have new information coming from the solution of $R$. So instead of using raw Einstein equations, we use the (ri) equation and apply the same method as above. This yields multiple expressions such as
\[ \phi_{a(2)} = -\frac{\sqrt{3}}{4\gamma} \left[ 8 + L^2 p^2 + 8\alpha_0 - (8 + L^2 p^2)\alpha_1 \right] - \frac{\sqrt{3}}{\gamma} \phi_{a(0)} - 2\sqrt{3}h(0), \]
\[ \psi_{a(2)} = -\frac{\sqrt{3}}{4\gamma} L^2 p^2, \]
\[ \phi_{a(0)} = -\frac{\sqrt{3}}{\gamma} \left( \frac{1}{\gamma} + h(0) \right). \] (4.8)

We used (4.7) in these equations. To link together all the expressions we have found above, we need a relation between the $\alpha_n$. Relation which can be found by plugging (4.4) in (4.2), giving
\[ \alpha_1 \left( 8 + L^2 p^2 \right) - \alpha_0 \left( 8 + L^2 p^2 \right) - 2 \left( 2 + L^2 p^2 \right) = 0. \] (4.9)
In the end, this constraint and the relations above are combined to get
\[ \phi_{a(2)} + \psi_{a(2)} = \phi_{a(0)} - \psi_{a(2)} \alpha_0. \] (4.10)

This is exactly what is found in [6]. We now need the expression of \( \psi_{a(2)} \). The main difference between our derivation of the asymptotic solutions and the one done in [6], is that \( \psi_{a(2)} \) can be found without using the field equation. From the last two expressions of (4.8), we can clearly see that
\[ \psi_{a(2)} = \frac{L^2 p^2}{4} \left( \phi_{a(0)} + \sqrt{3} h_{(0)} \right). \] (4.11)

Now we can plug this expression in the equation above and finally find
\[ \phi_{a(2)} + \psi_{a(2)} = \phi_{a(0)} - \frac{L^2 p^2}{4} \left( \phi_{a(0)} + \sqrt{3} h_{(0)} \right) \alpha_0. \] (4.12)

And this result indeed agrees with the work done in [6, 14]. So we now find the same expression for the one-point function of the active scalar perturbation
\[ \langle O_a \rangle = -2 \left( \tilde{\phi}_{a(2)} + \tilde{\psi}_{a(2)} \right) + \frac{2}{3} \phi_{a(0)}^2 \tilde{\phi}_{a(0)} = \frac{L^2 p^2}{2} \left( \phi_{a(0)} + \sqrt{3} h_{(0)} \right) \alpha_0. \] (4.13)

As expected, the vev of the operator depends on a metric quantity. Indeed, it was anticipated as the active scalar is coupled to gravity in the bulk theory. On the other hand, the inert scalar is not coupled to gravity and so we trust that the expectation value of its dual operator will not depend on metric sources.

4.2 Inert scalar

As in the case of the active scalar, we need to find expressions for \( \phi_{i(2)} \) and \( \psi_{i(2)} \) that we will plug in (3.20). Both of these can be determined with the field equation of \( \tilde{\phi}_i \) which we find by expanding the action to second perturbative order. However, only \( \psi_{i(2)} \) can be computed with the Klein-Gordon equation. To find \( \tilde{\phi}_{i(2)} \) we will need to solve the equation for \( \tilde{\phi}_i \). So in the end, we do not use the Einstein equation at any point for the inert scalar. That is simply because the first derivative of the potential with respect to \( \phi_i \) vanishes when \( \phi_i = 0 \). And this makes the Einstein equations independant of the inert scalar perturbation.

Now, the field equation for the inert scalar \( \tilde{\phi}_i \) is given by
\[ \tilde{\phi}_i'' + \frac{1}{\rho(1 - \rho)} \left[ \left( \frac{3(1 + 3\rho)\rho}{4\rho} - \frac{L^2 p^2}{4} \right) \tilde{\phi}_i - (1 + \rho) \tilde{\phi}_i' \right] = 0. \] (4.14)

This equation admits two hypergeometric functions as solutions but only one is regular in the bulk. The solution is then
\[ \tilde{\phi}_i(p^2, \rho) = \rho^{3/2} F \left( \frac{3}{2}, \frac{1}{2} \sqrt{9 - L^2 p^2}; \frac{3}{2}, \frac{1}{2} \sqrt{9 - L^2 p^2}; 2; 1 - \rho \right). \] (4.15)

One can note that \( p^2 \) can take any value in this solution as well, for the same reason we gave earlier. As (4.3) and (4.15) are quite different, they admit different analytic continuations.
The one for $\tilde{\phi}_i$ is given by

$$\frac{F(a, b; a + b - m; z)}{\Gamma(a + b - m)} = \frac{\Gamma(m)(1 - z)^{-m} \sum_{n=0}^{m-1} (a - m)_n (b - m)_n}{\Gamma(a) \Gamma(b)} (1 - m)_n n! (1 - z)^n$$

$$+ \frac{(-1)^m}{\Gamma(a - m) \Gamma(b - m)} \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(n + m)! n!} \left[ \bar{\sigma}_n - \log(1 - z) \right] (1 - z)^n.$$  

(4.16)

Where, this time, the $\bar{\sigma}_n$ coefficients are defined as

$$\bar{\sigma}_n = \psi(1 + n) + \psi(1 + n + m) - \psi(a + n) + \psi(b + n).$$  

(4.17)

In this case we have $m = 1$, $a = (3 + \sqrt{9 - L^2 p^2})/2$ and $b = (3 - \sqrt{9 - L^2 p^2})/2$. To make the equations below more compact, we define

$$\gamma_i(n) = \Gamma(a - n) \Gamma(b - n).$$  

(4.18)

As before we drop the tilde on the perturbations and will reinstate it when needed.

Now, to find the asymptotic solution of $\psi_i(2)$, we plug the expansion of the inert scalar field (3.13) in the field equation (4.14). This yields

$$\psi_i(2) = \left( \frac{L^2 p^2}{4} - 2 \right) \phi_i(0).$$  

(4.19)

As expected, $\phi_i(2)$ in undetermined but we can find it by matching the coefficients in (3.13) with the ones in the analytic continuation (4.16). We get

$$\phi_i(0) = \frac{1}{\gamma_i(0)}; \quad \phi_i(2) = -\frac{\bar{\sigma}_0}{\gamma_i(1)}; \quad \psi_i(2) = \frac{1}{\gamma_i(1)}.$$  

(4.20)

From $\Gamma$-function properties, it is straightforward to check these results against (4.19). In the end, combining these solutions, gives

$$\phi_i(2) + \psi_i(2) = \phi_i(0) \left( \frac{L^2 p^2}{4} - 2 \right) (1 - \bar{\sigma}_0).$$  

(4.21)

The expectation value for the operator dual to inert scalars is given by

$$\langle O_i \rangle = -2\sqrt{3} \left( \tilde{\phi}_i(2) + \bar{\psi}_i(2) \right) + 2\sqrt{3}\phi^2_i(0) \tilde{\phi}_i(0) = \tilde{\phi}_i(0) \sqrt{3} \left( 4 - \frac{L^2 p^2}{2} \right) (1 - \bar{\sigma}_0).$$  

(4.22)

As we mentioned at the end of the previous subsection, the vev of duals to inert scalars does not depend on metric perturbation. One can also see that the whole treatment of the inert scalar is the same for an arbitrary number of such scalars. That is because they do not couple to gravity and so are not present in the Einstein equation.

The result we got in (4.22) differ from the one found in [12] by the simple fact that we made the dependence on the source $\tilde{\phi}_i(0)$ explicit. Whereas the authors of [12] kept $\langle O_i \rangle$ expressed as a pure function of the transverse momentum.
5 Conclusions

In this report we applied the theory of cosmological perturbations from [7] to the domain-wall solution of the GPPZ flow and computed the vevs of operators dual to scalar fields. We took a different approach than [6] by considering different gauge invariant variables. A brief review of holographic renormalization was given in order to compute vevs of operators in the boundary theory. The expectation value for the operator dual to the active scalar was found identical to the results presented in [6]. We also computed the source dependent vev of the operator dual to the inert scalar which was not considered in [6] and the source dependence was not made explicit in [12]. However more can be done, as we did not investigate the behaviour of the inert scalar correlation function. We also hope to investigate — in future work — the implications of the “domain-wall/cosmology correspondence” from [8–10] on holographic RG-flows like the GPPZ flow.

6 Acknowledgments

I am grateful to my supervisor M. Schillo for her boundless patience and invaluable help throughout the realization of this project. I would also like to thank G. Dibitetto for being available when I had all kinds of questions during this project.

References


