Sudden Death of Entanglement
for Non-locality and Concurrence

A review of entanglement sudden death behaviour of non-locality and concurrence in commonly used entangled state classes under influence of decay and dephasing noise dynamics

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Abstract

When entanglement dies it can be zero in a finite time (i.e. exhibit sudden death of entanglement) or tend to zero in an infinite time (i.e. exhibit entanglement convergence). In this thesis entanglement sudden death behaviour is investigated for non-locality and concurrence for two-qubit states including Bell states, maximally entangled states (MES), non-maximally entangled states (Non-MES), Werner states, and maximally entangled mixed states (MEMS). The dynamics chosen are local operations acting on the one-qubit subsystems constituting the entangled two-qubit state, with decay and dephasing as noise sources. For decay, non-locality sudden death (NESD) is certain. For dephasing, pure parity is required for a state to show non-local entanglement convergence. For dephasing all states with rank < 4 show concurrence entanglement convergence, and for decay an initial state where the excited state is unpopulated show concurrence entanglement convergence.

1 Sammanfattning

Sammanflätning, initialt ett av kvantmekanikens stora mysterier, är numera ett väl undersökt fenomen. Sammanflätning kan beskrivas som en kommunikation mellan två olika partiklar. Låt oss säga att vi har två olika partiklar A och B där partikel A har 50% chans att vara antingen röd eller grön när man mäter den, och partikel B har 50% chans att vara antingen svart eller vit. Vi kan sammaflätta de här partiklarna så att när A till exempel är röd, så är B vit, och när A är grön så är B svart. En mätning på partikel A skulle då kunna avgöra färgen på B. Sammanflätning är ett centralt koncept i kvantcryptoeringprotokoll och i kvantdatorer, och förståelsen för sammanflätning kommer vara viktig i framtiden då forskningen om kvantdatorer har framskridit.

För att mäta och kvantifiera sammanflätning har man behövt införa relevanta kvantiteter. Två av dessa kvantiteter är icke-lokalitet och concurrence [1]. Icke-lokalitet är ett mått på hur mycket ett tillstånd bryter Bells olikhet [2], medans concurrence kvantifierar måttet "entanglement of formation" [3].

Om två sammanflätade system skulle bli utsatta för störningar så kommer även sammanflätningen bli påverkad. Om störningarna är tillräckligt starka så kommer sammanflätningen att tillslut bli noll, men det är viktigt att se att sammanflätningen när noll på olika sätt. Ibland kan processen liknas vid en halveringstid där sammanflätningen går mot noll då tiden går mot oändligheten, och ibland där sammanflätningen tvärt vilket har getts namnet "sammanflätning plötslig död", eller ESD (entanglement sudden death) [4]. I den här uppsatsen undersöks sammanflätning plötslig död för de vanligaste sammanflätade systemet, där måttet icke-lokalitet och concurrence används som ett mått på sammanflätningen.

2 Introduction

Coherence is an important concept in physics. But coherence is also a forced state since few systems behave coherently in nature, and it is necessary to shield the coherence in an experimental setting to avoid information loss. The loss of information, called decoherence, could be seen as amplitude or phase errors as a result of energy losses or interaction with distant electrical charges. It is therefore of interest to investigate and gain intuition about decoherence processes. These processes are to some extent well understood. As an example, energy losses occur in a particle as a transition from excited to ground state. This could be a simulated process, but could also be due to spontaneous emission which is a decoherence process where the system will decay with a half-life and tend to zero as $t \to \infty$. However, in some cases we might have two entangled subsystems where the entanglement is intended as a resource in some quantum information processing scheme. The two subsystems are subjected to the same type of "local" decoherence, but here the matter of interest is how the entanglement resource is affected by local losses on the individual subsystems. This is non-trivial since entanglement introduces a type of "non-local coherence". It was shown by Eberly and Yu [4, 3] that the disentanglement of two two-level atoms where each one of them is coupled to an environment is found to have a finite decay time in some cases. The finite decay time of entanglement is called entanglement sudden death or ESD [4].

How to measure entanglement is not a straight forward question. There are multiple choices of entanglement measures to choose from depending on the context and properties of the system of interest. As an example, concurrence [1] quantifies the entanglement of formation, i.e. how many copies of a given bipartite state can be formed from copies of a maximally entangled state by using LOCC (local operations and classical communication), and Bell’s inequality
violation [2] disproves local hidden variables of states and introduces the concept of non-locality. In the analysis of Eberly and Yu [4, 5], concurrence has been chosen as an entanglement measure.

In this project the noise evolution settings for common quantum states are determined, extending the analysis made by Eberly and Yu to include non-locality as a quantum measure as well as concurrence. In analogy to the case of concurrence, some states show finite time non-local entanglement sudden death (NESD) [6] while some states show death of non-locality only as \( t \to \infty \), a phenomenon that will be called entanglement convergence. An evaluation of non-locality decoherence is essential in fundamental research, and would provide understanding and intuition of NESD behaviour. In addition, the results can be verified since the art of measuring non-locality has been known for years.

The outline is the following. In section 3 a historical introduction of entanglement is presented, from the EPR paper to concurrence measure. In section 4 the necessary prerequisites for this thesis to be self-contained are presented. The result section is divided into two parts. In the “Simulation Results” section 7 non-locality and concurrence for decay and dephasing noise is calculated numerically and analytically for seven types of states. In the “Theoretical Results” section 8 some calculations are made to verify the general trend of ESD and NESD behaviour, if any. The results are summarized in the “Discussion” section 9. The plots belonging to the “Simulation Results” sections are collected in section 10.

3 Historical introduction

3.1 EPR paradox

The well established Heisenberg uncertainty principle has contributed greatly to the understanding of quantum mechanics and is considered essential in the quantum theory. The Heisenberg uncertainty principle delimits the precision by which a pair of quantum mechanical variables can be known. These variables are associated with physical properties of quantum systems such as position and momentum, mathematically formulated as operators \( \mathcal{O} \) acting on wavefunction eigenvectors like

\[
\mathcal{O}\psi = a\psi, \tag{1}
\]

where the measurement outcome \( a \) is the eigenvalue of the operator \( \mathcal{O} \) acting on the state given by \( \psi \). Operators, like position and momentum, represent measurements connecting quantum theory to human experience and may be mathematically formulated as non-commuting variables. The non-commutativity introduces hardships in how we define such operators since it introduces a basis dependency of the wavefunction - meaning that a measurement on a wavefunction will have different outcomes depending on which observable you choose to measure. Position \( Q \) and momentum \( P \) operators are hence defined as

\[
\text{Position representation:} \quad P = -i\hbar \frac{\partial}{\partial x}, \quad Q = x, \\
\text{Momentum representation:} \quad P = p, \quad Q = i\hbar \frac{\partial}{\partial p}. \tag{2}
\]

As an example, let a quantum mechanical state have the form of a plane wave in the position basis

\[
\psi(x) = e^{i\frac{px_0}{\hbar}}, \tag{3}
\]

where \( p_0, \hbar \) are constants, and \( x \) is a variable. The momentum of this system is found with the momentum operator

\[
P = -i\hbar \frac{\partial}{\partial x}, \tag{4}
\]

which gives

\[
P\psi(x) = -i\hbar \frac{\partial}{\partial x} \left( e^{i\frac{px_0}{\hbar}} \right) = p_0\psi(x). \tag{5}
\]

The momentum for the state \( \psi(x) \) takes the sharp value \( p_0 \). The position on the other hand is a multiplication by a variable,

\[
Q\psi(x) = x\psi(x) \tag{6}
\]
and hence not an eigensystem. Instead, we have a relative probability

\[ P(x_1, x_2) = \int_{x_1}^{x_2} |\psi|^2 dx = x_2 - x_1 = \Delta x, \]  

(7)

implying that all values within \( x_1 \) and \( x_2 \) are equally probable and cannot be known with full certainty. A measurement could show the particle position, but would at the same time disturb the state and alter it. The conclusion is the today well known claim which has nearly reached the level of a quantum mechanical axiom; if the momentum of a particle is known, its position cannot have a physical reality (and vice versa). This quantum mechanical claim clearly does not have a counterpart in the macroscopic “real” world, and impedes the intuition of quantum behaviour. When early quantum physicists were faced with these claims, some attempts were made to test the theory. This includes Einstein’s, Podolsky’s, and Rosen’s famous 1935 paper Can Quantum-Mechanical Description of Physical Reality Be Considered Complete? [7]. The paper, today referred to as the “EPR paper”, was intended to point out a contraction of quantum theory but ended up being accepted as a strange quantum mechanical mystery. Below, a brief variation of the argument from the original EPR paper as discussed by Bohm and Aharonov [8] follows: A molecule with zero total spin consists of two atoms with opposite spin. The wavefunction of the molecule is

\[ \psi = \frac{1}{\sqrt{2}} (\psi_+ (1) \psi_- (2) - \psi_- (1) \psi_+ (2)), \]  

(8)

where \( \psi_+ (1) \) refers to the wavefunction of the atom 1 having spin \( +\hbar/2 \), etc. Now assume that the two atoms are separated and any desired component of the spin of atom 1 is measured. Since the two atoms constituting the molecule have opposite spin, the spin of atom 2 will immediately be known from the measurement of atom 1 without disturbing or measuring atom 2 at all. This implies a hidden correlation between the two atoms. Hence, by measuring a subsystem of two combined systems we can, without disturbing the second system, gain information about the position or momentum of the second system. How could this be? One possibility could be that the information was there at all times, that the particles are deterministic and that their variables could be predicted with probability 1 (this is called hidden variables in quantum systems). But if the variables would be deterministic, position and momentum could simultaneously be asserted for a system and this would in turn violate the Heisenberg uncertainty relation. Another possibility for this strange behaviour could be that the two subsystems have faster than light communication. This is questionable, since relativity have had great success in physics. Despite the questionable nature of the EPR paradox, the quantum theory turned out to accurately describe reality when experimental verification years later disproved local hidden variables.

### 3.2 Bell’s inequality

Any serious consideration of a physical theory must take into account the distinction between the objective reality, which is independent of any theory, and the physical concepts with which the theory operates. These concepts are intended to correspond with the objective reality, and by means of these concepts we picture this reality to ourselves.

A. Einstein, B. Podolsky, and N. Rosen [7]

The EPR paper sparked controversy and researchers turned to the objective reality to settle this mystery. Almost 30 years later, John Stewart Bell published a paper titled “On the Einstein Podolsky Rosen paradox” [2], where he presented his inequality, known as Bell’s inequality, intended as a tool to test hidden variable theories. The Bell’s inequalities are fulfilled for theories which are inherently local and realistic. In this chapter Bell’s inequality is presented. Let us start with a system of two opposite spins which we measure in the direction of three unit vectors. Zero net angular momentum ensures that they are opposite spins, meaning that if \( (\hat{a}+, \hat{b}+, \hat{c}+) \) are measured for particle 1, then \( (\hat{a}−, \hat{b}−, \hat{c}−) \) are measured for particle 2. There are 8 possibilities given this configuration,
from the resulting state vector where the rotational direction is set to \( \hat{\mathbf{n}} \). We hence relate

Assume that observer 1 measures \( \hat{\mathbf{S}} \) and obtains +, and observer 2 measures \( \hat{\mathbf{b}} \) and obtains +. We then know that \( \hat{c} \) might be either + or − and that the system might belong to either C3 or C4 and the probability

\[
P(\hat{a}+, \hat{b}+) = \frac{C3 + C4}{\sum_i C_i}
\]

Let us say that the observers 1 and 2 choose to measure other vectors with the respective probabilities

\[
P(\hat{a}+, \hat{c}+) = \frac{C2 + C4}{\sum_i C_i}, \quad P(\hat{c}+, \hat{b}+) = \frac{C3 + C7}{\sum_i C_i}
\]

The derivation follows,

\[
(C2 + C4) + (C3 + C7) - (C3 + C4) = C2 + C7 \geq 0,
\]

\[
\Rightarrow (C3 + C4) \leq (C2 + C4) + (C3 + C7),
\]

\[
\Rightarrow P(\hat{a}+, \hat{b}+) \leq P(\hat{a}+, \hat{c}+) + P(\hat{c}+, \hat{b}+),
\]

where eq. (11) is Bell’s inequality, [9].

How does this hold when applied to a quantum mechanical system? Let us calculate the probability \( P(\hat{a}+, \hat{b}+) \) for two entangled particles allowed to produce the states listed in Table (1). The state \( |a\rangle_{12} \) is in superposition between

\[
|a\rangle_{12} = \frac{1}{\sqrt{2}} (|+, -\rangle_{12} + |-, +\rangle_{12}),
\]

with 50%/50% probability to obtain either + or - for particle 1 with particle 2 always yielding inverse result. When measuring \( |b\rangle_{12} \) we slowly change the rotation angle of system 2. For entangled polarized photons spatially separated into paths 1 and 2 the rotation could be accomplished by rotating the angle of a wave-plate connected to path 2, and in the case of two entangled spin-\( \frac{1}{2} \) electrons the rotation would correspond to the spin precession of electron 2. We can hence relate \( |b\rangle_{12} \) with \( |a\rangle_{12} \) through a rotation in a chosen direction \( \hat{n} \),

\[
|b\rangle_{12} = \mathbb{I} \otimes \exp \left( -\frac{i \hat{S}_2 \cdot \hat{n} \theta}{\hbar} \right) |a\rangle_{12}.
\]

In matrix form, the state of \( |b\rangle_{12} \) after obtaining \( |a\rangle_{12} = |+, -\rangle \) is

\[
|b\rangle_{12} = \begin{bmatrix}
\cos \left( \frac{\theta}{2} \right) & -\sin \left( \frac{\theta}{2} \right) \\
\sin \left( \frac{\theta}{2} \right) & \cos \left( \frac{\theta}{2} \right)
\end{bmatrix}
\begin{bmatrix}
|1\rangle_1 \\
|0\rangle_1
\end{bmatrix}
\otimes
\begin{bmatrix}
|0\rangle_2 \\
|1\rangle_2
\end{bmatrix}
\]

where the rotational direction is set to \( \hat{n} = (0, 1, 0) \). The probability to obtain \( \hat{b}+ \) for particle two can be acquired from the resulting state vector

\[
|b\rangle_{12} = \begin{bmatrix}
|0\rangle_1 \\
|1\rangle_1
\end{bmatrix}
\otimes
\begin{bmatrix}
|\sin \left( \frac{\theta}{2} \right) \\
|\cos \left( \frac{\theta}{2} \right)
\end{bmatrix}
\]

Table 1: Possible configurations of two particles with opposite spin using three spatial axes.

<table>
<thead>
<tr>
<th>Config.</th>
<th>Particle 1</th>
<th>Particle 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>C1</td>
<td>(( \hat{a}+, \hat{b}+, \hat{c}+ ))</td>
<td>(( \hat{a}−, \hat{b}−, \hat{c}− ))</td>
</tr>
<tr>
<td>C2</td>
<td>(( \hat{a}+, \hat{b}+, \hat{c}− ))</td>
<td>(( \hat{a}−, \hat{b}−, \hat{c}+ ))</td>
</tr>
<tr>
<td>C3</td>
<td>(( \hat{a}+, \hat{b}−, \hat{c}+ ))</td>
<td>(( \hat{a}−, \hat{b}+, \hat{c}− ))</td>
</tr>
<tr>
<td>C4</td>
<td>(( \hat{a}+, \hat{b}−, \hat{c}− ))</td>
<td>(( \hat{a}−, \hat{b}+, \hat{c}+ ))</td>
</tr>
<tr>
<td>C5</td>
<td>(( \hat{a}−, \hat{b}+, \hat{c}+ ))</td>
<td>(( \hat{a}+, \hat{b}−, \hat{c}− ))</td>
</tr>
<tr>
<td>C6</td>
<td>(( \hat{a}−, \hat{b}+, \hat{c}− ))</td>
<td>(( \hat{a}+, \hat{b}−, \hat{c}+ ))</td>
</tr>
<tr>
<td>C7</td>
<td>(( \hat{a}−, \hat{b}−, \hat{c}+ ))</td>
<td>(( \hat{a}+, \hat{b}−, \hat{c}− ))</td>
</tr>
<tr>
<td>C8</td>
<td>(( \hat{a}−, \hat{b}−, \hat{c}− ))</td>
<td>(( \hat{a}+, \hat{b}−, \hat{c}+ ))</td>
</tr>
</tbody>
</table>
\[ |b\rangle_{12} = |+\rangle_1 \otimes \left( \cos \left( \frac{\theta}{2} \right) |+\rangle_2 - \sin \left( \frac{\theta}{2} \right) |\rangle_2 \right) \]  

and is given by

\[ P(\hat{a}+, \hat{b}+) = \frac{1}{2} \sin^2 \left( \frac{\theta}{2} \right). \]  

The factor \( \frac{1}{2} \) is added from the 50% probability to obtain \( \hat{a}+ \) for particle 1. Bell’s inequality can then be written as

\[ \sin^2 \left( \frac{\theta_{ab}}{2} \right) \leq \sin^2 \left( \frac{\theta_{ac}}{2} \right) + \sin^2 \left( \frac{\theta_{cb}}{2} \right). \]  

A common choice for the angles are \( \theta_{ab} = 2\theta \), \( \theta_{ac} = \theta_{cb} = \theta \), for which (17) is violated between \( 0 < \theta < \frac{\pi}{2} \) [9].

### 3.3 The CHSH Inequality

Bell’s inequality comes in a number of variations for detecting non-locality in various quantum systems. One of the most widely used is the CHSH inequality. CHSH stands for Clauser, Horne, Shimony, and Holt, who presented the inequality in their 1969 paper [10]. It can be derived starting from an entangled state of horizontal and vertical polarization,

\[ |\Psi\rangle = \frac{1}{\sqrt{2}} (|V, H\rangle_{12} - |H, V\rangle_{12}), \]  

where we have chosen the basis \( |V\rangle \doteq \begin{pmatrix} 1 \\ 0 \end{pmatrix} \) and \( |H\rangle \doteq \begin{pmatrix} 0 \\ 1 \end{pmatrix} \) to represent vertical and horizontal polarization. Qubit 1 can be measured and manipulated by observer 1, and qubit 2 equally by observer 2. Observer 1 performs her measurement in the basis

\[ |V_\alpha\rangle = \cos \alpha |V\rangle - \sin \alpha |H\rangle, \]
\[ |H_\alpha\rangle = \sin \alpha |V\rangle + \cos \alpha |H\rangle, \]  

and equally observer 2 obtains similar results when measuring in the basis

\[ |V_\beta\rangle = \cos \beta |V\rangle - \sin \beta |H\rangle, \]
\[ |H_\beta\rangle = \sin \beta |V\rangle + \cos \beta |H\rangle. \]  

The probability that observer 1 measures \( |V_\alpha\rangle \) and observer 2 measures \( |V_\beta\rangle \) is

\[ P_{VV}(\alpha, \beta) = |\langle V_\alpha, V_\alpha |\Psi\rangle|^2 = \frac{1}{2} |\sin \alpha \cos \beta - \sin \beta \cos \alpha|^2 = \frac{1}{2} \sin^2(\alpha - \beta) \]  

and in the same way

\[ P_{HH}(\alpha, \beta) = \frac{1}{2} \sin^2(\alpha - \beta), \]
\[ P_{VH}(\alpha, \beta) = \frac{1}{2} \cos^2(\alpha - \beta), \]
\[ P_{HV}(\alpha, \beta) = \frac{1}{2} \cos^2(\alpha - \beta). \]  

Note that if \( \alpha = \beta \), then \( P_{VV}(\alpha, \beta) = P_{HH}(\alpha, \beta) = 0 \) and \( P_{VH}(\alpha, \beta) = P_{HV}(\alpha, \beta) = \frac{1}{2} \), and if \( \alpha - \beta = \frac{\pi}{2} \) then \( P_{VV}(\alpha, \beta) = P_{HH}(\alpha, \beta) = \frac{1}{2} \) and \( P_{VH}(\alpha, \beta) = P_{HV}(\alpha, \beta) = 0 \). With the help of these probabilities the correlation measure \( E(\alpha, \beta) \) is introduced,
\[ E(\alpha, \beta) = P_{VV}(\alpha, \beta) + P_{HH}(\alpha, \beta) - P_{VH}(\alpha, \beta) - P_{HV}(\alpha, \beta) \]
\[ = \sin^2(\alpha - \beta) - \cos^2(\alpha - \beta) = -\cos(2(\alpha - \beta)), \quad (23) \]

which has maxima and minima

\[ \alpha - \beta = \frac{\pi}{2} \quad \Rightarrow \quad E(\alpha, \alpha - \pi/2) = 1, \]
\[ \alpha = \beta \quad \Rightarrow \quad E(\alpha, \alpha) = -1. \quad (24) \]

Finally a measure \( S \) is constructed from 4 different polarization angles,

\[ S = E(\alpha, \beta) - E(\alpha, \beta') + E(\alpha', \beta) + E(\alpha', \beta'). \quad (25) \]

We have the option to redefine the angles as

\[ 2(\alpha - \beta) = \theta_{\alpha\beta}, \quad 2(\alpha' - \beta) = \theta_{\alpha'\beta}, \ldots \quad (26) \]

where \( \theta_{\alpha\beta} \) is spanned by the unit vectors \( \hat{\alpha} \) and \( \hat{\beta} \), \( \theta_{\alpha'\beta} \) is spanned by \( \hat{\alpha}' \) and \( \hat{\beta}' \),... where \( \hat{\alpha}, \hat{\alpha}', \hat{\beta}, \hat{\beta}' \) are all unit vectors in \( \mathbb{R}^3 \). Then \( S \) can be redefined since

\[ E(\alpha, \beta) = -\cos(\theta_{\alpha\beta}) = -\hat{\alpha} \cdot \hat{\beta} \quad (27) \]

which justifies the CHSH operator

\[ S \mapsto \hat{\mathcal{B}} = \hat{\alpha} \cdot (\hat{\beta} - \hat{\beta}') + \hat{\alpha}' \cdot (\hat{\beta} + \hat{\beta}') \quad (28) \]

which can be written in the Pauli matrix-basis as

\[ \hat{\mathcal{B}} = \hat{\alpha} \cdot \sigma \otimes (\hat{\beta} - \hat{\beta}') \cdot \sigma + \hat{\alpha}' \cdot \sigma \otimes (\hat{\beta} + \hat{\beta}') \cdot \sigma. \quad (29) \]

As we can see from eq. (24), \( S \) is bounded by \(-2 \leq S \leq 2\), and the CHSH inequality can be reformulated as either

\[ |S| \leq 2, \quad |\langle \hat{\mathcal{B}} \rangle| \leq 2. \quad (30) \]

If \( |S| \) or \( |\langle \hat{\mathcal{B}} \rangle| \) is larger than 2, the tested theory cannot be reproduced by local hidden variables. An example of this violation could be the choice of angles

\[ \alpha = \frac{\pi}{4}, \quad \alpha' = 0, \quad \beta = \frac{\pi}{8}, \quad \beta' = -\frac{\pi}{8}. \quad (31) \]

For these angles, \( |S| \) from eq. (30) is

\[ |E(\frac{\pi}{4}, \frac{\pi}{8}) - E(\frac{\pi}{4}, -\frac{\pi}{8}) + E(0, \frac{\pi}{8}) + E(0, -\frac{\pi}{8})| \]
\[ = | -\cos \left( \frac{\pi}{4} \right) + \cos \left( \frac{3\pi}{4} \right) - \cos \left( \frac{\pi}{4} \right) - \cos \left( \frac{\pi}{4} \right) | = 2\sqrt{2}, \quad (32) \]

which is larger than 2. The angles in eq. (31) hence violate Bell’s inequality. The CHSH inequality was used in early Bell test experiments disproving local hidden variables. An early example of experimental research that shows good agreement with quantum mechanical predictions is Aspect et al. [11] who is considered a pioneer in the field. His 1982 experimental setup is built upon the idea of counting coincidence rates of photon pairs passing through polarizers in different orientations.
3.4 Teleportation and quantum measures as resources

Entanglement is a highly useful resource used to share and communicate information between distant observers. One of the most noted communication schemes is quantum teleportation, which was first theoretically proposed in 1993 by Bennett et al. [12] and experimentally realized four years later by Bouwmeester et al. [13]. Teleportation, as known from science fiction, means to make a person or object disappear and later reappear somewhere else. In the 1993 quantum scheme, teleportation involves Alice dividing the information encoded in a qubit $|\phi\rangle$ into a classical and non-classical part which is then transferred to Bob. This makes it possible for Bob to construct an accurate replica of $|\phi\rangle$, destroying the qubit in the process (no-cloning theorem). It is done in the following way: Consider a simple case of two spin-$\frac{1}{2}$ particles prepared in a Bell state

$$|\Psi_{23}^{(-)}\rangle = \frac{1}{\sqrt{2}} (|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle),$$

(33)

where particle 2 is given to Alice and particle 3 is given to Bob. In addition to particle 2, Alice also has a particle prepared in an unknown state

$$|\phi\rangle_1 = a |\uparrow\rangle + b |\downarrow\rangle,$$

(34)

with $|a|^2 + |b|^2 = 1$. Then a combined state $|\Psi_{123}\rangle$ can be expressed as

$$|\Psi_{123}\rangle = |\phi\rangle_1 \otimes |\Psi_{23}^{(-)}\rangle = \frac{1}{\sqrt{2}} (a |\uparrow\downarrow\rangle - a |\uparrow\uparrow\rangle + b |\downarrow\uparrow\rangle - b |\downarrow\downarrow\rangle),$$

(35)

where no measurement on $|\Psi_{23}^{(-)}\rangle$ can give any information about $|\phi\rangle$. $|\Psi_{123}\rangle$ can be expressed in the well known Bell basis

$$|\Psi^{(\pm)}\rangle = \frac{1}{\sqrt{2}} ((|\uparrow\downarrow\rangle \pm |\downarrow\uparrow\rangle),$$

$$|\Phi^{(\pm)}\rangle = \frac{1}{\sqrt{2}} ((|\uparrow\uparrow\rangle \pm |\downarrow\downarrow\rangle),$$

(36)

like

$$|\Psi_{123}\rangle = \frac{1}{2} \left( |\Psi_{12}^{(-)}\rangle \left[ - a |\uparrow\rangle_3 - b |\downarrow\rangle_3 \right] + |\Psi_{12}^{(+)}\rangle \left[ - a |\uparrow\rangle_3 + b |\downarrow\rangle_3 \right] 
+ |\Phi_{12}^{(-)}\rangle \left[ a |\downarrow\rangle_3 + b |\uparrow\rangle_3 \right] + |\Phi_{12}^{(+)}\rangle \left[ a |\downarrow\rangle_3 - b |\uparrow\rangle_3 \right] \right).$$

(37)

Alice performs a measurement in the Bell basis on her two qubits where all outcomes are equally likely with probability $\frac{1}{4}$. Bob’s particle 3 is then projected into one of the pure states,

$$|\psi_1\rangle_3 = -a |\uparrow\rangle_3 - b |\downarrow\rangle_3, \quad |\psi_2\rangle_3 = -a |\uparrow\rangle_3 + b |\downarrow\rangle_3,$$

$$|\psi_3\rangle_3 = a |\downarrow\rangle_3 + b |\uparrow\rangle_3, \quad |\psi_4\rangle_3 = a |\downarrow\rangle_3 - b |\uparrow\rangle_3,$$

(38)

which are related to Alice’s original state $|\psi\rangle_1$ in different ways. For the first case $|\psi_1\rangle_3$, Bob’s and Alice’s states are the same up to an irrelevant phase factor. In the other three cases, Bob needs to perform a unitary operation corresponding to a rotation of $\pi$ around the $\hat{z}$, $\hat{x}$, or $\hat{y}$ direction, respectively, to get a state identical to Alice’s original state. This is where the classical communication plays a role. In order for Bob to perform the correct measurement, Alice needs to classically communicate (for example by a phone call) the result of her measurement, after which Bob applies the corresponding rotation. After all this is done Alice is left with the particles 1 and 2 in one of the Bell states, losing her original copy.


3.5 Experimental teleportation and concurrence

Following Bennett et al. [12], researchers started to investigate questions connected to experimental realization of teleportation scheme. One of these operational questions concerned impurities and noise in quantum channels. Bennett’s initial scheme could be considered an ideal case where Alice has two qubits and Bob has one, but a more realistic scenario is a system coupled to an environment where decoherence is introduced. The appropriate question to ask is therefore if pure entangled states which can be used in teleportation can be extracted from these “impure” (i.e. mixed) states. As this proved to be a rather difficult, the opposite question ergo “How many pure states are required to create a given mixed state?” was investigated by Bennett et al. [3]. The process is summarized as follows: Consider the entanglement of a pure state $\Psi_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$, where Alice has access to $\mathcal{H}_A$ and Bob to $\mathcal{H}_B$, given by its entropy

$$E(\Psi_{AB}) = S(\rho_A) = S(\rho_B),$$

where $S(\rho) = -\text{Tr}[\rho \log \rho]$ is the von Neumann entropy, and $\rho_A(\rho_B)$ is the reduced density matrix of Alice (Bob). The construction of quantum mixed states from pure states are done in the following way: Assume that the pure state $\Psi_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ is entangled, and then separated into two parts where Alice and Bob act on their respective parts using noise processes $N_A$ and $N_B$. Under influence of these noise processes the pure entangled state will evolve into a mixed state $M$. To quantitatively measure this process the measure Entanglement of formation $E(M)$ came to describe the cost of entanglement needed to construct $M$. In other words, $E(M)$ is the amount of entanglement of the ensemble of pure states $\{\Psi\}$ that Alice and Bob require to create the mixed state $M$. The entanglement of formation for one pure state is the von Neumann entropy of the reduced density matrix defined in eq. (39). For a set of pure states $\mathcal{P} = \{p_i \Psi_i\}$, the entanglement of formation is just the ensemble average

$$E(\mathcal{P}) = \sum_i p_i E(\Psi_i),$$

and the entanglement of formation for a mixed state $M$ is just the minimum of this ensemble average realizing $M$

$$E(M) = \min \{E(\mathcal{P})\},$$

which was the main result of Bennett et al. It was not until two years later that Wootters [1] found a method of how to calculate eq. (41) for two qubits. This is done with the help of the spin flip transformation

$$|\tilde{\Psi}\rangle = \sigma_2 |\Psi^*\rangle,$$

where $|\Psi^*\rangle$ is the complex conjugation of $|\Psi\rangle$. For two qubits, the spin flip can be generalized as

$$\tilde{\rho} = (\sigma_2 \otimes \sigma_2) \rho^* (\sigma_2 \otimes \sigma_2).$$

Next, the concurrence $C$ is introduced. For a pure and mixed two-qubit state respectively,

$$C(\Psi) = |\langle \Psi | \tilde{\Psi} \rangle|,$$

$$C(\rho) = \max \{0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4}\},$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ are eigenvalues of the non-Hermitian matrix $\rho \tilde{\rho}$. With this at hand, the Entanglement of formation can be written as

$$E(\mathcal{X}) = h \left( \frac{1 + \sqrt{1 - C(\mathcal{X})^2}}{2} \right),$$

$$h(x) = -x \log_2 x - (1 - x) \log_2 (1 - x),$$

where $\mathcal{X}$ can be a pure state $\Psi$ or a two-qubit mixed state $\rho$. Even if the concurrence was initially intended as a tool to define Entanglement of formation, it ended up being used as a measure of quantum entanglement by itself. This could be motivated by the fact that $E(C)$ is monotonically increasing with $C$ and conveniently $C = 1$ for maximally entangled states and $C = 0$ for separable states.
4 Prerequisites

4.1 State classes

Below common two-qubit classes are presented.

4.1.1 Bell states

The four Bell states

\[ |\Phi^+\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle), \]
\[ |\Phi^-\rangle = \frac{1}{\sqrt{2}} (|00\rangle - |11\rangle), \]
\[ |\Psi^+\rangle = \frac{1}{\sqrt{2}} (|01\rangle + |10\rangle), \]
\[ |\Psi^-\rangle = \frac{1}{\sqrt{2}} (|01\rangle - |10\rangle), \]

(48)

can form the maximally entangled two-qubit states

\[ \hat{\rho}_{\Phi^+} = |\Phi^+\rangle\langle \Phi^+|, \quad \hat{\rho}_{\Phi^-} = |\Phi^-\rangle\langle \Phi^-|, \]
\[ \hat{\rho}_{\Psi^+} = |\Psi^+\rangle\langle \Psi^+|, \quad \hat{\rho}_{\Psi^-} = |\Psi^-\rangle\langle \Psi^-|, \]

(49)

which have the matrix representations

\[ \rho_{\Phi^{(+,-)}} = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \pm 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\pm 1 & 0 & 0 & 1 \end{pmatrix}, \quad \rho_{\Psi^{(+,-)}} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\
0 & 1 & \pm 1 & 0 \\
0 & \pm 1 & 1 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}. \]

(50)

4.1.2 MES - maximally entangled states

MES are short for maximally entangled states and are written as

\[ |MES_+\rangle = \frac{1}{\sqrt{2}} (\alpha |00\rangle + \beta |01\rangle - \beta^* |10\rangle + \alpha^* |11\rangle), \]
\[ |MES_-\rangle = \frac{1}{\sqrt{2}} (\alpha |00\rangle + \beta |01\rangle + \beta^* |10\rangle - \alpha^* |11\rangle). \]

(51)

The parameters \( \alpha \) and \( \beta \) rotate the space vector in Hilbert space. A usual ansatz is

\[ \alpha = e^{i\phi} \cos(\theta), \quad \beta = e^{i\phi} \sin(\theta). \]

(52)

4.1.3 Non-MES

Non-MES are pure states constructed from the state vectors

\[ |\Theta+\rangle = \cos \theta |00\rangle + \sin \theta |11\rangle, \]
\[ |\Theta-\rangle = \cos \theta |01\rangle + \sin \theta |10\rangle, \]

(53)

where the + and − are referring to the parity of the state. The parameter \( \theta \) controls the ratios between ground and excited state. The density matrices are
\[ \rho_{\Theta^+} = \begin{pmatrix} \cos^2 \theta & 0 & 0 & \cos \theta \sin \theta \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \cos \theta \sin \theta & 0 & 0 & \sin^2 \theta \end{pmatrix}, \] (55)

\[ \rho_{\Theta^-} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \cos^2 \theta & \cos \theta \sin \theta & 0 \\ 0 & \cos \theta \sin \theta & \sin^2 \theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (56)

The Bell states are obtained by \( \theta = \pi/4 \). The positive partial transpose (PPT) criterion guarantees entanglement for all values of \( \theta \) except \( \theta = \frac{(2n-1)\pi}{2}, \, n = 1, 2, 3... \)

### 4.1.4 Werner states

Werner states are mixed states defined as [14]

\[ \hat{\rho}_w = \frac{(1-p)}{4} \mathbb{1} + p |\Psi^-\rangle \langle \Psi^-|, \]

\[ \rho_w = \frac{1}{4} \begin{pmatrix} 1-p & 0 & 0 & 0 \\ 0 & 1+p & -2p & 0 \\ 0 & -2p & 1+p & 0 \\ 0 & 0 & 0 & 1-p \end{pmatrix}. \] (57)

where \( p \) ranges from \( p = [0,1] \). By the PPT criterion, \( \rho_w \) is entangled for \( p > \frac{1}{3} \).

### 4.1.5 MEMS - maximally entangled mixed states

MEMS are states that give the maximal concurrence values for a given degree of mixing (linear entropy), and is given by [15]

\[ \rho_{MEMS} = \begin{pmatrix} g(p) & 0 & 0 & p/2 \\ 0 & 1-2g(p) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ p/2 & 0 & 0 & g(p) \end{pmatrix}, \] (58)

where the function \( g(p) \) is divided into two regions,

\[ g(p) = \begin{cases} \frac{p}{2}, & p \geq 2/3, \\ \frac{1}{3}, & p < 2/3. \end{cases} \] (59)

where \( p \) ranges from \( p = [0,1] \).

### 4.1.6 X-states

The X-states, named after the distinct form given by the matrix elements of the density matrix, are a class of states of the form [16]

\[ \rho_X = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23} & \rho_{33} & 0 \\ \rho_{14} & 0 & 0 & \rho_{44} \end{pmatrix}, \] (60)

normalized as \( \rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} = 1 \) and having eigenvalues
\[
\lambda_{1,2} = \frac{1}{2}(\rho_{11} + \rho_{44} \pm \sqrt{(\rho_{11} - \rho_{44})^2 + 4|\rho_{14}|^2}),
\]
\[
\lambda_{3,4} = \frac{1}{2}(\rho_{22} + \rho_{33} \pm \sqrt{(\rho_{22} - \rho_{33})^2 + 4|\rho_{23}|^2}).
\]

(61)

Since \( \{\lambda_i\} \geq 0 \), the matrix entries must fulfil

\[
\rho_{11}\rho_{44} \geq |\rho_{14}|^2, \quad \rho_{22}\rho_{33} \geq |\rho_{23}|^2.
\]

(62)

The X-states include important subgroups like Bell states, Werner states, MEMS, and non-MES. X-states are useful in the sense that the X-structure is preserved under the evolution of certain physically relevant types of noise like dephasing and decay.

4.2 Entanglement measures for two-qubit density matrices.

4.2.1 Concurrence

The concurrence of a two-qubit state is first introduced in section 3.5, and is given by [1]

\[
C = \operatorname{Max} \left[ 0, \sqrt{\lambda_1} - \sqrt{\lambda_2} - \sqrt{\lambda_3} - \sqrt{\lambda_4} \right]
\]

(63)

where \( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4 \) is the eigenvalues of the matrix \( \hat{\rho} \), where

\[
\hat{\rho} = (\sigma_2 \otimes \sigma_2)^\dagger(\sigma_2 \otimes \sigma_2).
\]

(64)

4.2.2 Non-locality

In section 3.3 the CHSH inequality is presented as a possible method for determining whether a state admits local hidden variables or not. However, the CHSH inequality might not be suited for mixed states (i.e. states that cannot be written as state vectors in Hilbert space). A sufficient criterion where mixed states are included was presented by the Horodecki family [17] in 1995 and the main results are presented here. A general two-qubit matrix can be expressed in the standard Bloch representation as

\[
\rho = \frac{1}{4} \left( I \otimes I + \mathbf{x} \cdot \sigma \otimes I + I \otimes \mathbf{y} \cdot \sigma + \sum_{i,j=1}^{3} T_{ij} \sigma_i \otimes \sigma_j \right),
\]

(65)

where \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) and

\[
T_{ij} = \operatorname{Tr} [\rho(\sigma_i \otimes \sigma_j)],
\]

(66)

is the correlation matrix. The correlation matrix gives us direct access to the nonlocality measure \( M \), defined as

\[
M = \max_{j \neq i} |h_i + h_j| \leq 2,
\]

(67)

where \( h_m, m = 1, 2, 3 \), are the eigenvalues of the symmetric matrix \( \tau = T^T T \) [17]. Bell’s inequality is violated for \( M > 1 \). The measure used to quantify non-locality is given by the function \( B \)

\[
B = \sqrt{\operatorname{Max}[0, M - 1]},
\]

(68)

which has the property of being equal the measure for concurrence for two-qubit pure states [18]. For a local state, \( B \) is zero, while \( B = 1 \) for a state that maximally violates Bell’s inequality. The measure \( B \) is from now on referred to as the “non-locality of a state” or just the “non-locality”.

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4.2.3 Expression for X-state

The concurrence and non-locality for the X-states defined in eq. (60) can be handily expressed in terms of matrix elements $\rho_{ij}$, $i,j = 1, 2, 3$. The concurrence for an arbitrary X-state in terms of its matrix elements is given by [16]

$$C = 2\text{Max}[0, \kappa_1, \kappa_2], \quad \kappa_1 = |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}}, \quad \kappa_2 = |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}.$$  \hfill (69)

Equally, the non-locality measure $B$ from eq. (68) can be redefined in terms of the functions [16]

$$u_1 = 4 (|\rho_{14}| + |\rho_{23}|)^2, \quad u_2 = (\rho_{11} + \rho_{44} - \rho_{22} - \rho_{33})^2, \quad u_3 = 4 (|\rho_{14}| - |\rho_{23}|)^2,$$

$$B = \sqrt{\text{Max}(0, Ud - 1)}, \quad \text{Max}[U1, U2], \quad U1 = u_1 + u_2, \quad U2 = u_1 + u_3,$$

which are the eigenvalues of $\tau = TTT$, where $T$ is defined in eq. (65). Then eq. (68) is

$$B = \sqrt{\text{Max}(0, Ud - 1)},$$

where

$$Ud = \text{Max}[U1, U2], \quad U1 = u_1 + u_2, \quad U2 = u_1 + u_3,$$

since $U1 > U3$.

5 Decoherence

5.1 Decoherence and master equations

The density matrix of a pure single-qubit state can be written as

$$\rho = \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix},$$

(73)

with $|b| = \sqrt{1 - |a|^2}$. Pure states are rare, if not non-existent, in nature. An example of photons in a pure state could for example be lasers, which produce a continuous stream of photons with definite wavelength. If an experimental physicist is exposed to the environment called “outside-the-lab”, he or she would most likely only encounter photons in mixed states. The reason why pure states are rare in nature is because they couple to their surrounding environment via which noise is introduced. If a pure state like eq. (73) would be thrown out of the lab, it would soon lose its coherence and end up in

$$\rho = \begin{pmatrix} |a|^2 & 0 \\ 0 & 1 - |a|^2 \end{pmatrix}.$$

(74)

A system that is isolated is called a closed system, and a system that is coupled to an environment is called an open system. This process where an open quantum system loses its off-diagonal elements (also called the coherences) is called decoherence and is often modelled through master equations. The von Neumann equation for the evolution of a density matrix in a closed quantum system driven by a time independent Hamiltonian $H$ is

$$\frac{d\rho}{dt} = -i[H, \rho].$$

(75)

By solving eq. (75) the solution obtained is evolution through unitary transformation

$$\rho(t) = e^{-iHt} \rho(0) e^{iHt}.$$
A master equation on the other hand dictates the evolution of an open quantum system, and is written

$$\frac{d\rho}{dt} = -i \{H, \rho\} + f(\Gamma)\rho,$$

(77)

where $f(\Gamma)\rho$ is a so called superoperator, which is an operator acting on another operator, in this case $\rho$. The superoperator describes the type of noise, and $\Gamma$ is a set of parameters that dictates the loss rate. The two most common sources of noise are decay (amplitude damping) and dephasing (phase damping).

 Dephasing describes a loss of information without losing energy. A system subjected to dephasing picks up phases between the energy eigenstates and loses the relative phase between them. An example of this kind of noise could be a charged atom interacting with other distant electrical point charges. Dephasing can be modelled using eq. (77) with

$$f(\Gamma)\rho = \frac{\Gamma}{2} (\sigma_x \rho \sigma_x - \rho).$$

(78)

Making the assumption that the evolution of the state is wide open i.e. $H = 0$ the evolution through dephasing on the state given by eq. (73) is

$$\rho(t) = \left( \begin{array}{cc} |a|^2 & ab^* e^{-\Gamma t} \\ a^* b e^{-\Gamma t} & |b|^2 \end{array} \right).$$

(79)

By letting $t \to \infty$, $\rho(t)$ will take the form of the mixed state in eq. (74).

Decay describes energy losses in quantum systems. This could for example be atoms emitting photons (spontaneous emission), or photons in cavities subjected to scattering. To model decay the master equation in eq. (77) is used with

$$f(\Gamma)\rho = \frac{\Gamma}{2} (2\sigma_- \rho \sigma_+ - \{\sigma_- \sigma_+, \rho\}).$$

(80)

Again assuming $H = 0$ the evolution through decay on the state given by eq. (73) is

$$\rho(t) = \left( \begin{array}{cc} e^{-\Gamma t/2} |a|^2 & ab^* e^{-\Gamma t/2} \\ a^* b e^{-\Gamma t/2} & 1 + e^{-\Gamma t} (|b|^2 - 1) \end{array} \right).$$

(81)

As $t \to \infty$, the state $\rho(t)$ converges to

$$\rho(t) = \left( \begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right),$$

(82)

which has the physical interpretation that decay “pushes” the state back into its ground state.

### 5.2 Kraus operators

As we have seen, the time evolution of an open quantum system cannot be described by unitary transformations. Instead the evolution is described in terms of operational elements or Kraus operators $\rho_k = \langle e_k | U | e_0 \rangle$, where $\{|e_k\}\}$ is an ON-basis for the environment

$$\rho(t) = \sum_k E_k \rho E_k^\dagger.$$

(83)

Equation (83) is known as the operator-sum representation of the evolution of $\rho$. The operational elements $E_k$ can be trace-preserving, $\sum_k E_k^\dagger E_k = I$, or non-trace-preserving, $\sum_k E_k^\dagger E_k < I$. A non trace-preserving operation describes quantum measurements or processes where information is gained by the observer [19].
5.2.1 Kraus operators for dephasing

The operational elements describing dephasing are given by [4]

\[
\text{Dephasing: } E_0 = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix},
\]

where the summation gives

\[
\sum_k E_k^\dagger E_k = \begin{pmatrix} 1 & 0 \\ 0 & \gamma^2 + \omega^2 \end{pmatrix},
\]

which is trace preserving if the parameters are normalized, \( \omega = \sqrt{1 - \gamma^2} \). The time evolution of a pure state as in eq. (73) is found by combining eqs. (84) and (83),

\[
\rho(t) = \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \gamma \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}
\]

\[
= \begin{pmatrix} |a|^2 & \gamma ab^* \\ \gamma a^*b & |b|^2 \end{pmatrix},
\]

which is exactly the same as in eq. (79) if the ansatz \( \gamma = e^{-\Gamma t/2} \) is made.

5.2.2 Kraus operators for decay

The operational elements describing the process of energy losses are [4]

\[
\text{Decay: } E_0 = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & 0 \\ \omega & 0 \end{pmatrix}
\]

The sum of the operational elements is

\[
\sum_k E_k^\dagger E_k = \begin{pmatrix} \gamma^2 + \omega^2 & 0 \\ 0 & 1 \end{pmatrix},
\]

which is trace-preserving if normalized. The time evolution of a pure state subjected to decay is found by combining eq. (87), (83), and (73), which result in

\[
\rho(t) = \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix} \begin{pmatrix} |a|^2 & ab^* \\ a^*b & |b|^2 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \omega \end{pmatrix}
\]

\[
= \begin{pmatrix} \gamma^2 |a|^2 & ab^* \gamma \\ \gamma a^*b \gamma & 1 + \gamma^2 (|b|^2 - 1) \end{pmatrix}.
\]

This is exactly the same as in eq. (81) if the ansatz is put to \( \gamma = e^{-\Gamma t/2} \).

5.3 Kraus operators for two qubits

In this thesis, different types of entangled two-qubit systems are investigated. An entangled two-qubit system can be described by two entangled two-level atoms each placed in a separate cavity as depicted in fig. 1. Similarly to eq. (83), the time evolution of a two-qubit system is given by

\[
\rho(t) = \sum_{k=0}^3 K_k \rho \tilde{K}_k^\dagger.
\]

where the four Kraus operators \( K_i = E_i^A \otimes E_i^B \) are constructed from the operational elements
Figure 1: Two two-level atoms are placed in separate cavities, sharing entanglement in-between each other without direct interaction. The atoms are subjected to local environment noise processes described by Kraus operators $E_i^A$ and $E_i^B$.

\begin{align}
\text{Decay:} &\quad E_0^A = \begin{pmatrix} \gamma_A & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1^A = \begin{pmatrix} 0 & 0 \\ \omega_A & 0 \end{pmatrix}, \\
&\quad E_0^B = \begin{pmatrix} \gamma_B & 0 \\ 0 & 1 \end{pmatrix}, \quad E_1^B = \begin{pmatrix} 0 & 0 \\ \omega_B & 0 \end{pmatrix}, \\
\text{Dephasing:} &\quad E_0^A = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_A \end{pmatrix}, \quad E_1^A = \begin{pmatrix} 0 & 0 \\ 0 & \omega_A \end{pmatrix}, \\
&\quad E_0^B = \begin{pmatrix} 1 & 0 \\ 0 & \gamma_B \end{pmatrix}, \quad E_1^B = \begin{pmatrix} 0 & 0 \\ 0 & \omega_B \end{pmatrix},
\end{align}

where $E_i^A$ ($E_i^B$) are the operational elements describing the noise processes in system $A$ ($B$). To simplify, the assumption is made that both subsystems $A$ and $B$ follow the same evolution, i.e. $\omega_A = \omega_B = \omega$ and $\gamma_A = \gamma_B = \gamma$. The Kraus operators $K_i$ for two-qubit systems are

\begin{align}
\text{Decay:} &\quad K_0 = E_0^A \otimes E_0^B = \begin{pmatrix} \gamma^2 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \\
&\quad K_1 = E_0^A \otimes E_1^B = \begin{pmatrix} \gamma \omega & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \omega \end{pmatrix}, \\
&\quad K_2 = E_1^A \otimes E_0^B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \gamma \omega & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \end{pmatrix}, \\
&\quad K_3 = E_1^A \otimes E_1^B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \omega^2 & 0 & 0 & 0 \end{pmatrix},
\end{align}
and

\[
K_0 = E_0^A \otimes E_0^B = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \gamma & 0 & 0 \\
0 & 0 & \gamma & 0 \\
0 & 0 & 0 & \gamma^2
\end{pmatrix},
\]

\[
K_1 = E_0^A \otimes E_1^B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \omega & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \gamma\omega
\end{pmatrix},
\]

\[
K_2 = E_1^A \otimes E_0^B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & \omega & 0 & 0 \\
0 & 0 & 0 & \gamma\omega
\end{pmatrix},
\]

\[
K_3 = E_1^A \otimes E_1^B = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \omega^2
\end{pmatrix}.
\]

The state change on a density matrix which is real in the computational basis

\[
\rho = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\
\rho_{12} & \rho_{22} & \rho_{23} & \rho_{24} \\
\rho_{13} & \rho_{23} & \rho_{33} & \rho_{34} \\
\rho_{14} & \rho_{24} & \rho_{34} & \rho_{44}
\end{pmatrix},
\]

(95)

using eq. (90) is

\[
\rho \mapsto \rho^{(A)}(\gamma) = \begin{pmatrix}
\gamma^4\rho_{11} & \gamma^3\rho_{12} & \gamma^3\rho_{13} & \gamma^2\rho_{14} \\
\gamma^3\rho_{12} & \gamma^2\omega^2\rho_{11} + \gamma^2\rho_{22} & \gamma^2\rho_{23} & \gamma\omega\rho_{21} + \gamma\rho_{24} \\
\gamma^3\rho_{13} & \gamma^2\rho_{23} & \gamma^2\omega^2\rho_{11} + \gamma^2\rho_{33} & \gamma\omega^2\rho_{21} + \gamma\rho_{34} \\
\gamma^2\rho_{14} & \gamma\omega^2\rho_{13} + \gamma\rho_{24} & \gamma\omega^2\rho_{12} + \gamma\rho_{34} & \rho_{44} + \omega^2\rho_{22} + \omega^2\rho_{33} + \omega^4\rho_{11}
\end{pmatrix}
\]

(96)

for decay, and

\[
\rho \mapsto \rho^{(p)}(\gamma) = \begin{pmatrix}
\rho_{11} & \gamma\rho_{12} & \gamma\rho_{13} & \gamma^2\rho_{14} \\
\gamma\rho_{12} & (\gamma^2 + \omega^2)\rho_{22} & \gamma\rho_{23} & \gamma\rho_{24} \\
\gamma\rho_{13} & \gamma\rho_{23} & (\gamma^2 + \omega^2)\rho_{33} & \gamma\rho_{34} \\
\gamma^2\rho_{14} & \gamma\rho_{24} & \gamma\rho_{34} & (\gamma^2 + \omega^2)^2\rho_{44}
\end{pmatrix}
\]

(97)

for dephasing, where the indexes \((A)\) (for amplitude noise) and \((p)\) (for phase noise) denote the respective dynamics.

### 6 Entanglement sudden death

#### 6.1 ESD according to Eberly and Yu [4]

Classical correlations, as well as quantum correlations, decay if subjected to noise processes. However, quantum entanglement does not always obey the half life rule. In fact, entanglement can reach zero in a finite time for specific states in specific systems. The term used for the finite time disentanglement is ESD, entanglement sudden death, also referred to as early-stage disentanglement [4, 20]. The state examined in the 2004 paper by Eberly and Yu [4] is a single parameter mixed state,

\[
\rho = \frac{1}{3} \begin{pmatrix}
a & 0 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 - a
\end{pmatrix}.
\]

(98)
Assuming decay noise given by the Kraus operator in eq. (93), normalized as $\gamma^2 + \omega^2 = 1$, the state change is

$$\rho^{(A)} = \frac{1}{3} \begin{pmatrix} \gamma^4 a & 0 & 0 & 0 \\ 0 & \gamma^2 - \gamma^2 a(1 - \gamma^2) & \gamma^2 - \gamma^2 a(1 - \gamma^2) & 0 \\ 0 & \gamma^2 & 0 & 3 - 2a\gamma^2 + a\gamma^4 \\ 0 & 0 & 0 & 3 - 2a\gamma^2 + a\gamma^4 \end{pmatrix}, \quad (99)$$

with concurrence given by

$$C = \frac{2}{3} \max \left[0, \gamma^2 - a\gamma^2(1 - a + 2(1 - \gamma^2)^2 + a(1 - \gamma^2)^4)\right]. \quad (100)$$

The concurrence reaches zero if

$$a \left(1 - a + 2(1 - \gamma^2)^2 + a(1 - \gamma^2)^4\right) \geq 1. \quad (101)$$

Choosing the ansatz $\gamma = \exp(-\Gamma t/2)$ the finite disentanglement time is calculated for $a = 1$ and is

$$t(a = 1) = \frac{1}{\Gamma} \ln \left(\frac{2 + \sqrt{2}}{2}\right). \quad (102)$$

The lower limit of $a$ for which the state no longer exhibits ESD is at $a \leq 1/3$. Below this point, the state converges to a point of entanglement and is said to exhibit entanglement convergence. The results are shown in fig. 2.

![Figure 2: Entanglement using concurrence as an entanglement measure subjected to dynamics of spontaneous emission, i.e. decay noise. For $a \leq 1/3$ the state dynamics show entanglement convergence, and for $a > 1/3$ ESD is shown. The blue line represents the boundary where ESD occurs, also called the concurrence zero line which is denoted $a_{kd}(\gamma)$.](image)
6.2 The geometry of two-qubit ESD

We classify the geometry of ESD by introducing categories for the possible scenarios in two-qubit ESD, as described by Terra Cunha [21].

The first distinction to be made is whether the system converges to one or more asymptotic points. For the latter case we denote the set of asymptotic points with \( R \). For the case of one convergent point, the asymptotic point might be within the region of separable, entangled, or boundary states as shown in fig. 3. If the system converges to one entangled point it means that no ESD can be observed (shown in fig. 3, red line). The noise processes for such a system would need to involve some kind of pumping for the entanglement to be stable, and this case is therefore not relevant for this thesis where we only investigate local noise sources. The one point case which converges to a separable point show ESD (fig. 3, blue line), and lastly the limiting point (fig. 3, green line) holds some interesting dynamics since the limit points contain both separable and entangled states. In this category, some initial states might exhibit ESD while other might not.

If the system tends to more than one asymptotic point \( R \), the points might be a union of separable, entangled, or boundary states in analogy with the categories of one convergent point, as shown in fig. 4. In addition, one scenario which involves the set of asymptotic points \( R \) containing both separable and entangled states (and hence also the boundary state) is presented in fig. 5. In this situation, all outcomes of ESD are possible, as well as the revival of entanglement after ESD.

![Figure 3: Three different classes of one-point convergence is shown. The set \( S \) is the set of separable states and \( S \not\subset E \), where \( E \) is the set of entangled states. The points i., ii., and iii. represent different types of convergence categories where i. tends to the group of one asymptotic point which is entangled, ii. tends to a boundary point group where the state can have a certain amount of entanglement and iii. is the separable group where ESD is observed. The case i. requires some additional coupling and is not relevant for this thesis.](image)

Figure 4: The picture shows three different classes of more than one point convergence, where \( R \) denotes the set of asymptotic points. The scenarios are analogous to that of the one-point convergence. Again, case ii. would require a coupling and is therefore not considered in this thesis.

Figure 5: The case where the set \( R \) of asymptotic states is both in \( E \) and in \( S \). In this situation, we might have revival of entanglement after ESD.

7 Simulation results

The concurrence and non-locality as described in section 4.2 are investigated with specific emphasis on sudden death behaviour. For states which are subsets of the X-states, the simplified measures presented in section 4.2.3 are used. The time evolution for the initial states presented in section 4.1 is calculated using the Kraus operators given by eq. (90) together with eqs. (93) or (94), using the assumptions that \( \gamma \) and \( \omega \) are normalized \( \omega^2 + \gamma^2 = 1 \), and \( \gamma \) evolves according to \( \gamma = e^{-\Gamma t/2} \), where \( \Gamma \) is associated with the half-life of the system, and \( \Gamma = 1 \) in plots. A mathematical formulation is required to detect sudden death behaviour. NESD(ESD) is characterized by vanishing non-locality(concurrence) for a finite time \( \gamma \in (0,1] \). The sudden death conditions are hence

\[
\text{NESD}: \quad B = \sqrt{\text{Max}[0,Ud-1]} = 0, \quad \Rightarrow \text{Max}(U1,U2) \leq 1,
\]

\[
\text{ESD}: \quad C = 2\text{Max}[0,\kappa_1,\kappa_2] = 0, \quad \Rightarrow \text{Max}(\kappa_1,\kappa_2) \leq 0.
\] (103)

We call \( \text{Max}[U1,U2] = Ud \) and \( \text{Max}[\kappa_1,\kappa_2] = \kappa_i \) the dominant functions. Hence, to ensure NESD, a non-locality zero point must exist for a finite value of \( \gamma \) which is given by the dominant \( Ud = 1 \). Analogously, to ensure ESD a
concurrence zero point must exist for a finite value of $\gamma$, given by the dominant $\kappa_i = 0$. Note that the non-negativity of $B$ and $C$ must be assumed separately when examining $\kappa_i$ and $Ud$. In addition to the parameter $\gamma$ which dictates the time evolution, the system might be dependent on another parameter $x \in [a, b]$, in which case there might be a set of concurrence zero-points and non-locality zero-points which trace out a zero line in $(x, \gamma)$. We define the non-locality zero line as $x_{Ud}(\gamma)$ and concurrence zero line as $x_{\kappa d}(\gamma)$ as the set of points which fulfil eq. (103). Such a line is plotted in blue in fig. 2. Below, we present the functions $U1$, $U2$, $\kappa_1$ and $\kappa_2$ for all the state classes presented in section 4.1, and define the non-locality and concurrence zero lines for the dominant function(s). Analytical results are presented within the limits of the problem’s complexity.

7.1 Bell state $\rho_{\Phi(+,-)}$

The time evolutions for the Bell state $\rho_{\Phi(+,-)}$ are

$$\rho_{\Phi(+,-)}^{(A)}(\gamma) = \frac{1}{2} \begin{pmatrix} \gamma^4 & 0 & 0 & \pm\gamma^2 \\ 0 & \gamma^2(1-\gamma^2) & 0 & 0 \\ 0 & 0 & \gamma^2(1-\gamma^2) & 0 \\ \pm\gamma^2 & 0 & 0 & 2 - 2\gamma^2 + \gamma^4 \end{pmatrix},$$

(104)

$$\rho_{\Phi(+,-)}^{(p)}(\gamma) = \frac{1}{2} \begin{pmatrix} 1 & 0 & 0 & \pm\gamma^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \pm\gamma^2 & 0 & 0 & 1 \end{pmatrix},$$

(105)

7.1.1 $\rho_{\Phi(+,-)}$: Decay - non-locality

$$U1 = \gamma^4 + (1 - 2\gamma^2 + 2\gamma^4)^2, \quad U2 = 2\gamma^4.$$

(106)

$Ud = \text{Max}[U1, U2]$ is calculated numerically and is found to be $Ud = U2$ within the boundary $Ud \geq 1$. The non-locality zero point is found by solving $U2 = 1$ which gives

$$\gamma = \frac{1}{\sqrt{2}},$$

(107)

which is finite, giving decay NESD.

7.1.2 $\rho_{\Phi(+,-)}$: Dephasing - non-locality

$$U1 = \gamma^4 + 1, \quad U2 = 2\gamma^4.$$

(108)

$U1$ is always larger than $U2$, which can be seen by

$$U1 - U2 \geq 0 \quad \implies \quad 1 \geq \gamma \geq 0.$$

(109)

The non-locality zero point is found by

$$U1 = 1, \quad \implies \quad \gamma^4 = 0,$$

(110)

resulting in entanglement convergence for the state subjected to dephasing.

7.1.3 $\rho_{\Phi(+,-)}$: Decay - concurrence

$$\kappa_1 = \frac{\gamma^4}{2}, \quad \kappa_2 = -\frac{\gamma^2}{2} \sqrt{2 - 2\gamma^2 + \gamma^4}.$$

(111)

$\kappa_2$ is always negative, and the solution to $\kappa_1 = 0$ clearly shows entanglement convergence.
7.1.4 $\rho_{\Phi(\pm,-)}$: Dephasing - concurrence

$$\kappa_1 = \gamma^2, \quad \kappa_2 = -\frac{1}{2}.$$  \hspace{1cm} (112)

The solution to $\kappa_1 = 0$ clearly shows entanglement convergence.

7.2 Bell state $\rho_{\Psi(\pm,-)}$

The time evolutions for the Bell states are

Decay:
$$\rho_{\Psi(\pm,-)}^{(A)}(\gamma) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \gamma^2 & \pm\gamma^2 & 0 \\ 0 & \pm\gamma^2 & \gamma^2 & 0 \\ 0 & 0 & 0 & 2 - 2\gamma^2 \end{pmatrix},$$ \hspace{1cm} (113)

Dephasing:
$$\rho_{\Psi(\pm,-)}^{(p)}(\gamma) = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & \pm\gamma^2 & 0 \\ 0 & \pm\gamma^2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$ \hspace{1cm} (114)

7.2.1 $\rho_{\Psi(\pm,-)}$: Decay - non-locality

The functions are

$$U_1 = \gamma^4 + (1 - 2\gamma^2)^2, \quad U_2 = 2\gamma^4,$$ \hspace{1cm} (115)

where $U_2 - U_1 \geq 0$ for $\gamma \geq \frac{1}{3}$. In this limit, $U_2 = 1$ gives a point of NESD for

$$\gamma = \frac{1}{\sqrt{2}}.$$ \hspace{1cm} (116)

7.2.2 $\rho_{\Psi(\pm,-)}$: Dephasing - non-locality

The functions

$$U_1 = \gamma^4 + 1, \quad U_2 = 2\gamma^4,$$ \hspace{1cm} (117)

are identical to those of section 7.1.2. Hence, $\rho_{\Psi(\pm,-)}$ exhibit non-local entanglement convergence for dephasing.

7.2.3 $\rho_{\Psi(\pm,-)}$: Decay - concurrence

The functions

$$\kappa_1 = -\left|\frac{\gamma^2}{2}\right|, \quad \kappa_2 = \frac{\gamma^2}{2},$$ \hspace{1cm} (118)

clearly exhibit concurrence entanglement convergence for the state subjected to decay.

7.2.4 $\rho_{\Psi(\pm,-)}$: Dephasing - concurrence

The functions

$$\kappa_1 = \frac{\gamma^2}{2}, \quad \kappa_2 = -\frac{1}{2},$$ \hspace{1cm} (119)

clearly exhibit entanglement convergence for the state subjected to dephasing.
7.3 MES

For MES, the ansatz

\[ \alpha = \cos(\theta), \quad \beta = \sin(\theta), \quad (120) \]

is used. The results are calculated numerically and presented in figs. 16 - 19. For dephasing, NESD and ESD are observed except for the states which coincide with the Bell states \( \rho_{MES}(\theta = 0) = \rho_{\Phi}^- \) and \( \rho_{MES}(\theta = \pi/2) = \rho_{\Psi^+} \). For decay, NESD is observed for all values of \( \theta \) while concurrence shows entanglement convergence.

7.4 Werner states

The time evolutions for Werner states are

Decay:

\[ \rho_w^{(A)}(\gamma) = \frac{1}{4} \begin{pmatrix} \gamma^4(1-p) & 0 & 0 & 0 \\ 0 & 2\gamma^2 - \gamma^4 + \gamma^4p & -\gamma^22p & 0 \\ 0 & -\gamma^22p & 2\gamma^2 - \gamma^4 + \gamma^4p & 0 \\ 0 & 0 & 0 & (4 - 4\gamma^2 + \gamma^4 - \gamma^4p) \end{pmatrix}, \quad (121) \]

Dephasing:

\[ \rho_w^{(p)}(\gamma) = \frac{1}{4} \begin{pmatrix} 1 - p & 0 & 0 & 0 \\ 0 & 1 + p & -\gamma^22p & 0 \\ 0 & -\gamma^22p & 1 + p & 0 \\ 0 & 0 & 0 & 1 - p \end{pmatrix}, \quad (122) \]

7.4.1 Werner: Decay - non-locality

The functions \( U1 \) and \( U2 \) are

\[ U1 = \gamma^4p^2 + (\gamma^4 - \gamma^4p - 2\gamma^2 + 1)^2, \]
\[ U2 = 2\gamma^4p^2. \quad (123) \]

\( Ud = \text{Max}[U1, U2] \) is calculated numerically and is found to be \( Ud = U2 \) within the boundary \( Ud \geq 1 \). \( U2 = 1 \) gives the solution

\[ p(\gamma) = \frac{1}{\gamma^2\sqrt{2}}, \quad (124) \]

which is finite for all values of \( \gamma \) within \( p \in [1/3, 1] \). The Werner states hence exhibit NESD if subjected to decay.

7.4.2 Werner: Dephasing - non-locality

The functions \( U1 \) and \( U2 \) are

\[ U1 = p^2(1 + \gamma^4), \quad U2 = 2\gamma^4p^2, \quad (125) \]

where \( U1 \) is larger than \( U2 \), which can be seen by subtracting \( U1 - U2 = p^2(1 - \gamma^4) > 0 \). Hence \( U2 \) is discarded for the analysis and the point of NESD is

\[ p(\gamma) = \frac{1}{\sqrt{\gamma^4 + 1}}. \quad (126) \]

If \( \gamma = 1 \) we get the boundary point

\[ p(\gamma) = \frac{1}{\sqrt{2}} \quad (127) \]

and a point of entanglement convergence is found as

\[ \lim_{\gamma \to 0} p(\gamma) \to 1. \quad (128) \]

For \( p = 1 \) the Werner state takes on the form of a separable Bell state, which exhibits entanglement convergence.
7.4.3 Werner: Decay - Concurrence

The functions $\kappa_1$ and $\kappa_2$ are

\[
\kappa_1 = -\frac{1}{4}2\gamma^2 - \gamma^4 + \gamma^4 p, \\
\kappa_2 = |\gamma^2| \left[ p - \frac{1}{4} \sqrt{(4 - 4\gamma^2 + \gamma^4 - \gamma^4 p)(1 - p)} \right],
\]

where $\text{Max}(\kappa_1, \kappa_2) = \kappa_2$. The concurrence zero line is given by $\kappa_2 = 0$. Rearranging the expression we get

\[
\kappa_2 = 0 \implies p^2 + p\frac{2\gamma^4 + 4 - 4\gamma^2}{4 - \gamma^4} - \frac{(2 - \gamma^2)^2}{4 - \gamma^4} = 0,
\]

which has the two solutions

\[
p_{1,2} = -\frac{2\gamma^4 + 4 - 4\gamma^2}{2(4 - \gamma^4)} \pm 2\sqrt{\frac{5 - 6\gamma^2 + 2\gamma^4}{(\gamma^4 - 4)^2}}.
\]

Clearly, $p_2$ is negative for all values of $\gamma$. Since $p \in [0, 1]$, we can discard $p_2$ and is left with the concurrence zero line,

\[
p(\gamma) = -\frac{2\gamma^4 + 4 - 4\gamma^2}{2(4 - \gamma^4)} + 2\sqrt{\frac{5 - 6\gamma^2 + 2\gamma^4}{(\gamma^4 - 4)^2}}.
\]

One point of entanglement convergence on the concurrence zero line is found,

\[
\lim_{\gamma \to 0} p(\kappa_2) \to \frac{\sqrt{5} - 1}{2}
\]

recognized as the golden ratio. Hence, ESD is obtained for Werner states if $p \in \left[\frac{1}{3}, \frac{\sqrt{5} - 1}{2}\right)$, and entanglement convergence is observed for $p \in \left[\frac{\sqrt{5} - 1}{2}, 1\right]$.

7.4.4 Werner: Phase - Concurrence

The functions $\kappa_1$ and $\kappa_2$ are

\[
\kappa_1 = -\frac{1}{4}(1 + p), \\
\kappa_2 = \frac{\gamma^2 p}{2} - \frac{1}{4}\sqrt{(1 - p)^2},
\]

where $\kappa_1 < 0$ and can be discarded. A concurrence zero line is then

\[
\kappa_2 = 0 \implies p(\gamma) = \frac{1}{2\gamma^2 + 1},
\]

which will have a point of entanglement convergence for $p = 1$. 

23
7.5 MEMS

The MEMS have the output states

\[ \rho_{\text{MEMS}}^{(A)}(\gamma) = \begin{pmatrix}
0 & \gamma^2 - f_2(\gamma) - f_1(\gamma) & 0 & \frac{p\gamma^2}{2} \\
0 & 0 & f_2(\gamma) - f_1(\gamma) & 0 \\
\frac{p\gamma^2}{2} & 0 & 0 & 1 - \gamma^2 + f_1(\gamma)
\end{pmatrix}, \tag{136} \]

\[ \rho_{\text{MEMS}}^{(p)}(\gamma) = \begin{pmatrix}
h & 0 & 0 & \frac{p\gamma^2}{2} \\
0 & 1 - 2h & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{p\gamma^2}{2} & 0 & 0 & h
\end{pmatrix}, \tag{137} \]

where

\[ f_1(\gamma) = \begin{cases} 
p\gamma^4/2, & p \geq 2/3 \\
\gamma^4/3, & p < 2/3
\end{cases}, \quad f_2(\gamma) = \begin{cases} 
p\gamma^2/2, & p \geq 2/3 \\
\gamma^2/3, & p < 2/3
\end{cases}, \tag{138} \]

and

\[ h = \begin{cases} 
p/2, & p \geq 2/3 \\
1/3, & p < 2/3
\end{cases}. \tag{139} \]

7.5.1 MEMS: Decay - non-locality

The functions are

\[ U_1 = p^2\gamma^4 + (1 + 4f_1(\gamma) - 2\gamma^2)^2, \quad U_2 = 2p^2\gamma^4. \tag{140} \]

Max\(U_2,U_1) = U_2\) for Max\(U_2,U_1) \geq 1\) is solved numerically. The point of decay NESD is found to be

\[ p_{U2}(\gamma) = \frac{1}{\gamma^2\sqrt{2}}, \tag{141} \]

which is the same as for Werner states in section 7.4.1.

7.5.2 MEMS: Dephasing - non-locality

The functions are

\[ U_1 = \begin{cases} 
p^2\gamma^4 + (2p - 1)^2, & p \geq 2/3 \\
p^2\gamma^4 + 1/9, & p < 2/3
\end{cases}, \quad U_2 = 2p^2\gamma^4. \tag{142} \]

In the case \(p < 2/3, \) Max\(U_2,U_1) = U_2.\) In the case \(p \geq 2/3, \)

\[ \text{Max}(U_2,U_1) = \begin{cases} 
U_1, & \gamma \geq \sqrt{2(\sqrt{2} - 1)} \\
U_2, & \gamma \leq \sqrt{2(\sqrt{2} - 1)}
\end{cases}, \tag{143} \]

and the solution for the non-locality line is

\[ p < 2/3: \quad p_{U2}(\gamma) = \frac{1}{\gamma^2\sqrt{2}}, \]

\[ p \geq 2/3: \quad p_{U2}(\gamma) = \frac{1}{\gamma^2\sqrt{2}}, \quad \gamma \geq \sqrt{2(\sqrt{2} - 1)}, \]

\[ p_{U2}(\gamma) = \frac{1}{\gamma^2\sqrt{2}}, \quad \gamma \leq \sqrt{2(\sqrt{2} - 1)}. \tag{144} \]
7.5.3 MEMS: Decay - concurrence

The functions are

\[
\begin{align*}
\kappa_1 &= \begin{cases} 
\frac{\gamma^2}{2} \left( p - \sqrt{2p(1 - \gamma^2) + p^2(\gamma^4 - 1)} \right), & p \geq 2/3 \\
\frac{\gamma^2}{2} \left( p - \frac{3}{4} \sqrt{3 - 3\gamma^2 + \gamma^4} \right), & p < 2/3.
\end{cases} \\
\kappa_2 &= -\sqrt{f_1(\gamma)(1 - \gamma^2 + f_1(\gamma))}.
\end{align*}
\]

(145)

(146)

Obviously, \( \max(\kappa_1, \kappa_2) = \kappa_1 \) and the concurrence zero line is

\[
p_{\kappa_1}(\gamma) = \begin{cases} 
\frac{2(\gamma^2 - 1)}{\gamma^2 - 2}, & p \geq 2/3 \\
\frac{2}{3} \sqrt{3 - 3\gamma^2 + \gamma^4}, & p < 2/3.
\end{cases}
\]

(147)

7.5.4 MEMS: Dephasing - concurrence

The functions are

\[
\begin{align*}
\kappa_1 &= \frac{p\gamma^2}{2}, \\
\kappa_2 &= -|h|.
\end{align*}
\]

(148)

Clearly, \( \max(\kappa_1, \kappa_2) = \kappa_1 \) and \( \kappa_1 = 0 \) results in entanglement convergence.

7.6 Non-MES \( |\Theta^+\rangle \)

Decay and dephasing noise produces the output states

\[
\rho^{(A)}_{\Theta^+}(\gamma) = \begin{pmatrix} 
\gamma^4 \cos^2 \theta & 0 & 0 & \frac{\gamma^2 \cos \theta \sin \theta}{\gamma^2 - 2} \\
0 & \gamma^2(1 - \gamma^2) \cos^2 \theta & 0 & 0 \\
\gamma^2 \cos \theta \sin \theta & 0 & \gamma^2(1 - \gamma^2) \cos^2 \theta & 0 \\
0 & 0 & 0 & 1 + \gamma^2(\gamma^2 - 2) \cos^2 \theta
\end{pmatrix},
\]

(149)

\[
\rho^{(B)}_{\Theta^+}(\gamma) = \begin{pmatrix} 
\frac{\cos^2 \theta}{\gamma^2 \cos \theta \sin \theta} & 0 & \frac{\gamma^2 \cos \theta \sin \theta}{\gamma^2 - 2} \\
0 & 0 & 0 & 0 \\
\gamma^2 \cos \theta \sin \theta & 0 & 0 & \sin^2 \theta
\end{pmatrix}.
\]

(150)

7.6.1 \( |\Theta^+\rangle \): Decay - non-locality

The functions are

\[
U_1 = 4\gamma^4 |\cos^2 \theta \sin^2 \theta| + (1 + 4\gamma^2 (\gamma^2 - 1) \cos^2 \theta)^2, \quad U_2 = 8\gamma^4 |\cos^2 \theta \sin^2 \theta|.
\]

(151)

The non-locality \( B \) is examined numerically, resulting in NESD as shown in fig. 25.

7.6.2 \( |\Theta^+\rangle \): Dephasing - non-locality

The functions are

\[
U_1 = 4\gamma^4 |\cos^2 \theta \sin^2 \theta| + 1, \quad U_2 = 8\gamma^4 |\cos^2 \theta \sin^2 \theta|.
\]

(152)

\[
\max(U_1, U_2) = U_1 \text{ can be found by subtracting } U_2 \text{ from } U_1,
\]

\[
U_1 - U_2 = -4\gamma^4 |\cos^2 \theta \sin^2 \theta| + 1 \geq 0, \\
\implies 1 \geq \gamma^2 |\sin(2\theta)|.
\]

(153)
The non-locality line is found by solving

$$\gamma^2 |\sin(2\theta)| = 0,$$

(154)

which has no non-trivial solution and exhibits entanglement convergence for all $\theta$.

7.6.3 $|\Theta^+\rangle$: Decay - concurrence

The functions are

$$\kappa_1 = \gamma^2 |\cos \theta \sin \theta| - \gamma^2 (1 - \gamma^2) |\cos^2 \theta|,$$
$$\kappa_2 = -\gamma^2 |\cos \theta| \sqrt{1 + \gamma^2 (\gamma^2 - 2) \cos^2 \theta},$$

(155)

where $\kappa_2$ is negative and can be discarded. The concurrence zero line is found by solving

$$\kappa_1 = 0 \implies |\gamma^2 \cos \theta \sin \theta| - \gamma^2 (1 - \gamma^2) |\cos^2 \theta| = 0,$$

(156)

which gives

$$\theta_{\kappa_1}(\gamma) = \tan^{-1}(1 - \gamma^2).$$

(157)

Entanglement convergence point if found by taking the limit $\gamma \to 0$ for which

$$\lim_{\gamma \to 0} \theta_{\kappa_1}(\gamma) = \tan^{-1}(1) = \frac{\pi}{4}.$$  

(158)

Hence, entanglement convergence exists within the limits $\theta \in [0, \frac{\pi}{4})$ and ESD if $\theta \in (\frac{\pi}{4}, 1]$ for non-MES under decay.

7.6.4 $|\Theta^+\rangle$: Dephasing - concurrence

The functions are

$$\kappa_1 = \gamma^2 |\cos \theta \sin \theta|,$$
$$\kappa_2 = -|\cos \theta \sin \theta|.$$  

(159)

$\kappa_2$ is always negative, and the concurrence zero line is found by solving

$$\kappa_1 = 0, \implies \gamma^2 |\cos \theta \sin \theta| = 0,$$

(160)

which has no non-trivial solution and exhibits entanglement convergence for all $\theta$.

7.7 Non-MES $|\Theta^\pm\rangle$

Decay and dephasing noise produces the output states

Decay:

$$\rho^{(A)}_{\Theta^\pm}(\gamma) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \gamma^2 \cos^2 \theta & \gamma^2 \cos \theta \sin \theta & 0 \\
0 & \gamma^2 \cos \theta \sin \theta & \gamma^2 \sin^2 \theta & 0 \\
0 & 0 & 0 & (1 - \gamma^2)
\end{pmatrix},$$

(161)

Dephasing:

$$\rho^{(D)}_{\Theta^\pm}(\gamma) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \cos^2 \theta & \gamma^2 \cos \theta \sin \theta & 0 \\
0 & \gamma^2 \cos \theta \sin \theta & \sin^2 \theta & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.$$  

(162)
7.7.1 \(|\Theta^−\rangle\): Decay - non-locality

\[ U_1 = 4\gamma^4|\cos^2 \theta \sin^2 \theta| + (1 - 2\gamma^2)^2, \quad U_2 = 8\gamma^4|\cos^2 \theta \sin^2 \theta|. \]  (163)

The non-locality \(B\) is examined numerically, resulting in NESD as shown in fig. 29.

7.7.2 \(|\Theta^−\rangle\): Dephasing - non-locality

\[ U_1 = 4\gamma^4|\cos^2 \theta \sin^2 \theta| + 1, \quad U_2 = 8\gamma^4|\cos^2 \theta \sin^2 \theta|. \]  (164)

Similarly to chapter (7.6.2), this will result in entanglement convergence for the non-locality of \(|\Theta^−\rangle\) subjected to dephasing.

7.7.3 \(|\Theta^−\rangle\): Decay - concurrence

\[ \kappa_1 = -\gamma^2|\cos \theta \sin \theta|, \quad \kappa_2 = \gamma^2|\cos \theta \sin \theta|. \]  (165)

Similarly to chapter (7.6.4), this will result in entanglement convergence for the concurrence of \(|\Theta^−\rangle\) subjected to decay.

7.7.4 \(|\Theta^−\rangle\): Dephasing - concurrence

\[ \kappa_1 = -|\cos \theta \sin \theta|, \quad \kappa_2 = \gamma^2|\cos \theta \sin \theta|. \]  (166)

Similarly to chapter (7.6.4), this will result in entanglement convergence for the concurrence of \(|\Theta^−\rangle\) subjected to dephasing.

7.8 Summary

The results are summarized using the following notation: \(\dagger\) is used for states that show sudden death behaviour, \(\rightarrow\) is used for the states that show entanglement convergence. If the states show both, \((\dagger, \rightarrow)\) is used. The Bell states are subgroups of all the state classes investigated in this thesis, and in 75% of all the \((\dagger, \rightarrow)\) cases, the entanglement convergence is seen only for the points within the group which are Bell states. This scenario it is denoted in the table below with an index \(b\) as \((\dagger, \rightarrow)_b\).

<table>
<thead>
<tr>
<th>State</th>
<th>Non-locality</th>
<th>Concurrence</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Decay</td>
<td>Dephasing</td>
</tr>
<tr>
<td>Bell</td>
<td>(\dagger)</td>
<td>(\rightarrow)</td>
</tr>
<tr>
<td>MES</td>
<td>(\dagger)</td>
<td>((\dagger, \rightarrow)_b)</td>
</tr>
<tr>
<td>MEMS</td>
<td>(\dagger)</td>
<td>((\dagger, \rightarrow)_b)</td>
</tr>
<tr>
<td>Werner</td>
<td>(\dagger)</td>
<td>((\dagger, \rightarrow)_b)</td>
</tr>
<tr>
<td>(\Theta^+)</td>
<td>(\dagger)</td>
<td>(\rightarrow)</td>
</tr>
<tr>
<td>(\Theta^−)</td>
<td>(\dagger)</td>
<td>(\rightarrow)</td>
</tr>
</tbody>
</table>

8 Theoretical results

The X-states preserve their shape under decay and dephasing dynamics, and are therefore simpler to analyse. A general state (see eq. (95)) is considered a non-X-state if at least one of its elements \(\rho_{12}, \rho_{13}, \rho_{24}, \rho_{34}\) is non-zero. We call these elements undefined-parity elements since they are not invariant under the parity operator. Non-pure non-X-states have rarely been studied in the literature since they yield cumbersome expressions when subjected to quantum operations. To efficiently analyse the non-locality and concurrence, some understanding is required regarding the appearance of non-X-states. In section 8.1 we demonstrate that the undefined-parity elements of a general density matrix \(\rho\) are maximized when the density matrix is separable. In section 8.2 we show that a density matrix of pure parity cannot have additional undefined-parity elements which are non-zero.
8.1 Non-X-states and separability

In this section, we will prove that the undefined-parity elements \(\rho_{12}, \rho_{13}, \rho_{24}, \) and \(\rho_{34}\) of a density matrix \(\rho\) defined in (95) are maximized when \(\rho\) is pure. Firstly we note that given a Hermitian positive semi-definite matrix \(\rho\), the product \(M\rho M^\dagger\), where \(M\) is any matrix, will also be positive semi-definite. We define six matrices

\[
M_{12} = \begin{pmatrix} 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{13} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \\
M_{24} = \begin{pmatrix} 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{32} = \begin{pmatrix} 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \\
M_{14} = \begin{pmatrix} 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{34} = \begin{pmatrix} 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}.
\]

(167)

These result in the products

\[
M_{12}\rho M_{12}^\dagger = \begin{pmatrix} \rho_{22} & \rho_{12} & 0 & 0 \\
\rho_{12} & \rho_{11} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{13}\rho M_{13}^\dagger = \begin{pmatrix} \rho_{11} & \rho_{13} & 0 & 0 \\
\rho_{13} & \rho_{33} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \\
M_{24}\rho M_{24}^\dagger = \begin{pmatrix} \rho_{44} & \rho_{24} & 0 & 0 \\
\rho_{24} & \rho_{22} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{34}\rho M_{34}^\dagger = \begin{pmatrix} \rho_{33} & \rho_{34} & 0 & 0 \\
\rho_{34} & \rho_{44} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \\
M_{14}\rho M_{14}^\dagger = \begin{pmatrix} \rho_{11} & \rho_{14} & 0 & 0 \\
\rho_{14} & \rho_{44} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}, \quad M_{23}\rho M_{23}^\dagger = \begin{pmatrix} \rho_{33} & \rho_{23} & 0 & 0 \\
\rho_{23} & \rho_{22} & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix}.
\]

(168)

which need to be positive semi-definite. Then, by Sylvester’s criterion [22], all principal minors of \(M\rho M^\dagger\) must be non-negative. The upper 2x2 principal minors provide us with the inequalities

\[
\rho_{22}\rho_{11} \geq \rho_{12}^2, \quad \rho_{11}\rho_{33} \geq \rho_{13}^2, \quad \rho_{44}\rho_{22} \geq \rho_{24}^2, \quad \rho_{33}\rho_{44} \geq \rho_{34}^2, \\
\rho_{44}\rho_{11} \geq \rho_{14}^2, \quad \rho_{22}\rho_{33} \geq \rho_{23}^2,
\]

(169)

which need to be fulfilled for a positive semi-definite \(\rho\). The largest allowed values for the undefined-parity elements are obtained if

\[
\rho_{22}\rho_{11} = \rho_{12}^2, \quad \rho_{11}\rho_{33} = \rho_{13}^2, \quad \rho_{44}\rho_{22} = \rho_{24}^2, \quad \rho_{33}\rho_{44} = \rho_{34}^2, \\
\rho_{44}\rho_{11} = \rho_{14}^2, \quad \rho_{22}\rho_{33} = \rho_{23}^2.
\]

(170)

Put \(\rho_{11} = a^2, \rho_{22} = b^2, \rho_{33} = c^2, \rho_{44} = d^2\), then

\[
ab = \rho_{12}, \quad ac = \rho_{13}, \quad bd = \rho_{24}, \quad cd = \rho_{34}, \\
ad = \rho_{14}, \quad bc = \rho_{23}.
\]

(171)

Making sure that the signs of \(a, b, c\) and \(d\) match the density matrix definitions, the above ansatz gives
\[ \rho_p = \begin{pmatrix} a^2 & ab & ac & ad \\ ab & b^2 & bc & bd \\ ac & bc & c^2 & cd \\ ad & bd & cd & d^2 \end{pmatrix}, \] (172)

corresponding to the pure state
\[ |\Psi\rangle = a|00\rangle + b|10\rangle + c|01\rangle + d|11\rangle. \] (173)

### 8.2 Undefined-parity cannot be combined with pure parity

States that have either negative or positive parity include the Bell states and Non-MES, and states with mixed parity include Werner states, MES, and MEMS. The parity operator is defined as
\[ P_z = \prod_{i=1}^{N} \sigma_z^{(i)}. \] (174)

A state with positive parity \( P_z \rho_+ = \rho_+ \) has the form
\[ \rho_+ = \begin{pmatrix} \rho_{11} & 0 & 0 & \rho_{14} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \rho_{14} & 0 & 0 & \rho_{44} \end{pmatrix}. \] (175)

A state with negative parity \( P_z \rho_- = -\rho_- \) has the form
\[ \rho_- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \rho_{22} & \rho_{23} & 0 \\ 0 & \rho_{23} & \rho_{33} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \] (176)

Finally, a matrix with neither positive or negative parity (undefined-parity) has the form
\[ \rho_n = \begin{pmatrix} 0 & \rho_{12} & \rho_{13} & 0 \\ \rho_{12} & 0 & 0 & \rho_{24} \\ \rho_{13} & 0 & 0 & \rho_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & \rho_{44} \end{pmatrix}. \] (177)

From this follows that a density matrix cannot have undefined-parity only. Using the matrices defined in eq. (168) it can be shown that the states
\[ \rho_{+n} = \begin{pmatrix} \rho_{11} & \rho_{12} & \rho_{13} & \rho_{14} \\ \rho_{12} & 0 & 0 & \rho_{24} \\ \rho_{13} & 0 & 0 & \rho_{34} \\ \rho_{14} & \rho_{24} & \rho_{34} & \rho_{44} \end{pmatrix}, \quad \rho_{-n} = \begin{pmatrix} 0 & \rho_{12} & \rho_{13} & 0 \\ \rho_{12} & 0 & \rho_{22} & \rho_{23} \\ \rho_{13} & \rho_{23} & 0 & \rho_{33} \\ 0 & \rho_{24} & \rho_{34} & 0 \end{pmatrix}, \] (178)
cannot exist. The inequalities in eq. (169) for \( \rho_{+n} \) and \( \rho_{-n} \) are

\[ \rho_{+n} : \quad 0 \geq \rho_{12}^2, \quad 0 \geq \rho_{13}^2, \quad 0 \geq \rho_{24}^2, \quad 0 \geq \rho_{34}^2, \quad \rho_{44} \rho_{11} \geq \rho_{14}^2, \quad 0 \geq \rho_{23}^2, \]

\[ \rho_{-n} : \quad 0 \geq \rho_{12}^2, \quad 0 \geq \rho_{13}^2, \quad 0 \geq \rho_{24}^2, \quad 0 \geq \rho_{34}^2, \quad 0 \geq \rho_{14}^2, \quad \rho_{22} \rho_{33} \geq \rho_{23}^2, \] (179)

which conclude that states with combinations positive and undefined-parity, or negative and undefined-parity, are not allowed. Thus a state with non-zero undefined-parity elements must have both non-zero negative and positive parity elements.
8.3 Non-locality decay analysis for $O(\gamma^4)$

If an entangled state is subjected to decay noise, the evolution of the state in the limit $\gamma \to 0$ will be

$$
\lim_{\gamma \to 0} \rho^{(A)}(\gamma) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
$$

(180)

which is separable. NESD for decay could be avoided if a local state can be found such that the neighbouring vicinity is not local. Assume a general state given by eq. (96). The corresponding symmetric matrix $\tau = T^T T$ given by the correlation matrix defined in eq. (66) is calculated analytically and have eigenvalues

$$
\text{Eigenvalues}[\tau] = \left( 0, 0, 1 - 4\gamma^2(\rho_{11} + 1) + 4\gamma^2(\rho_{44} + (\rho_{13} + \rho_{24})^2 + (\rho_{12} + \rho_{34})^2) \right) + O(\gamma^4),
$$

(181)

which results in a non-locality measure

$$
B = \sqrt{\text{Max} \left[ 0, 4\gamma^2 \left( (\rho_{13} + \rho_{24})^2 + (\rho_{12} + \rho_{34})^2 - (1 + \rho_{11} - \rho_{44}) \right) \right]}. 
$$

(182)

The criterion for NESD to occur is hence

$$
(\rho_{13} + \rho_{24})^2 + (\rho_{12} + \rho_{34})^2 < 1 + \rho_{11} - \rho_{44},
$$

(183)

meaning that if eq. (183) is fulfilled, the state shows NESD. For X-states, eq. (183) becomes

$$
\rho_{44} - \rho_{11} < 1.
$$

(184)

If we can find a state such that eq. (184) is violated, the state will exhibit non-local entanglement convergence for decay. This is the case for a state where

$$
\rho_{44} = \rho_{11} + 1 + \epsilon,
$$

(185)

where $\epsilon > 0$ is a small real number. Then $\text{Tr} \rho = 2\rho_{11} + \rho_{22} + \rho_{33} + 1 + \epsilon = 1$ implies

$$
\rho_{33} = -2\rho_{11} - \rho_{22} - |\epsilon|,
$$

(186)

is negative, which would result in a negative definite matrix. Therefore X-states always exhibit NESD under decay. When it comes to non-X-states, we have shown in section 8.1 that the off-diagonal entries $\rho_{13}$, $\rho_{24}$, $\rho_{12}$, and $\rho_{34}$ are maximized when $\rho$ is pure. The most likely state to violate eq. (183) will therefore be a pure state. Assume a two-qubit pure state defined in equation (173), normalized as $\langle \Psi | \Psi \rangle = 1$. For simplicity $a$, $b$, $c$, and $d$ are taken to be real valued. For such a state, eq. (183) will be

$$
(a^2 + d^2)(b^2 + c^2) + 4abcd \leq 1 + a^2 - d^2.
$$

(187)

Reducing the number of variables by using the normalization $b = \sqrt{1 - a^2 - d^2 - c^2}$ results in the inequality

$$
F(a, c, d) \leq 1,
$$

(188)

where

$$
F(a, c, d) = 4adc\sqrt{1 - a^2 - d^2 - c^2} - a^4 - 2a^2d^2 + 2d^2 - d^4.
$$

(189)
Maximizing $F(a, c, d)$ yields

$$\frac{\partial}{\partial a} F(a, c, d) = 4ac\sqrt{1 - a^2 - d^2 - c^2} - \frac{4acd^2}{\sqrt{1 - a^2 - d^2 - c^2}} + 4d - 4a^2d - 4d^3,$$

$$\frac{\partial}{\partial c} F(a, c, d) = 4ad\sqrt{1 - a^2 - d^2 - c^2} - \frac{4ac^2d}{\sqrt{1 - a^2 - d^2 - c^2}},$$

$$\frac{\partial}{\partial d} F(a, c, d) = 4cd\sqrt{1 - a^2 - d^2 - c^2} - \frac{4a^2cd}{\sqrt{1 - a^2 - d^2 - c^2}} - 4a^3 - 4ad^2$$

which give a system of equations

$$\begin{cases}
\frac{\partial}{\partial a} F(a, c, d) = 0, \\
\frac{\partial}{\partial c} F(a, c, d) = 0, \\
\frac{\partial}{\partial d} F(a, c, d) = 0,
\end{cases}$$

with possible (real) solutions,

$$c_{\text{max}} = \sqrt{1 - a_{\text{max}}^2 - d^2},$$

$$a_{\text{max}} = \frac{d^2}{2^{3/2}(9d^3 + 2\sqrt{3} \sqrt[6]{7}d^6 - 27d^3 + 27d^2 - 18d)^{1/3}} - \frac{d}{2}.$$  

These give a function dependent on a single variable, $F(a_{\text{max}}, c_{\text{max}}, d) = F(d)$. This function is evaluated as a plot in fig. 6, with the result $F(d) \leq 1$. Hence, we can conclude that for a non-X-state with real entries will always show NESD if influenced by decay noise.

![Figure 6](image-url)

Figure 6: Plot of $F(d)$. Since $F(d) \leq 1$, a separable state cannot violate the criteria (183).
8.4 Non-locality dephasing analysis for $O(\gamma^4)$

Assume a general state given by eq. (97). The corresponding symmetric matrix $\tau = T^T T$ given by the correlation matrix defined in eq. (66) is calculated analytically and has eigenvalues

$$\text{Eigenvalues}[\tau] = \left(0, 0, 4 \gamma^2 ((\rho_{13} - \rho_{24})^2 + (\rho_{12} - \rho_{34})^2) + (\rho_{11} + \rho_{44} - \rho_{22} - \rho_{33})^2\right) + O(\gamma^4),$$

which result in a non-locality measure

$$B = \sqrt{\text{Max} \left|0, 4 \gamma^2 ((\rho_{13} - \rho_{24})^2 + (\rho_{12} - \rho_{34})^2) - 4 (\rho_{11} + \rho_{44} - 1) (\rho_{22} + \rho_{33} - 1)\right|}.$$

For NESD to occur, the criterion

$$\gamma^2 ((\rho_{13} - \rho_{24})^2 + (\rho_{12} - \rho_{34})^2) \leq (\rho_{11} + \rho_{44} - 1) (\rho_{22} + \rho_{33} - 1) \tag{195}$$

needs to be fulfilled. Could this be violated for any state? There are two possible cases. First case is when the RHS of eq. (195) is zero, and the second case is when the RHS is negative. The first case corresponds to a system in which the density matrix has pure parity, but we have shown in section 8.2 that a state of pure parity has off-diagonal elements $\rho_{13} = \rho_{24} = \rho_{12} = \rho_{34} = 0$, which is a trivial case for eq. (195). The pure parity states require an extended analysis addressed in section 8.5. The second case of violation is if the RHS of eq. (195) is negative. Let us investigate this case. As $\gamma \to 0$, eq. (195) converges to

$$\epsilon + (\rho_{11} + \rho_{44})^2 \leq \rho_{11} + \rho_{44}, \tag{196}$$

where $\epsilon \propto \gamma^2$ is a small number. This inequality is violated for a small enough $\epsilon$ if $\rho_{11} + \rho_{44} \geq 1$. This would violate the definition of positive-definiteness since $\rho_{11} + \rho_{44} \geq 1$ introduces a negative number in the diagonal elements due to the constraint $\text{Tr} \rho = 1$. Hence, eq. (195) is always fulfilled and NESD is given for a state with mixed parity.

8.5 Non-locality dephasing analysis X-STATES

For X-states an inequality for NESD can be manageable for $O(\gamma^6)$. Assume a general state given by eq. (97). The corresponding symmetric matrix $\tau = T^T T$ given by the correlation matrix defined in eq. (66) is calculated analytically and have eigenvalues

$$\text{Eigenvalues}[\tau] = \left(4 \gamma^4 (\rho_{14} - \rho_{23})^2, 4 \gamma^4 (\rho_{14} + \rho_{23})^2, (\rho_{22} + \rho_{33} - \rho_{11} - \rho_{44})^2\right) + O(\gamma^6) \tag{197}$$

such that the non-locality measure $M$ can be either

$$M_1 = 8 \gamma^4 (\rho_{14}^2 + \rho_{23}^2),$$
$$M_2 = 4 \gamma^4 (\rho_{14} + \rho_{23})^2 + (\rho_{22} + \rho_{33} - \rho_{11} - \rho_{44})^2. \tag{198}$$

The non-locality $B$ is given by the largest of the two functions

$$B_1 = \sqrt{\text{Max} \left|0, 8 \gamma^4 (\rho_{14}^2 + \rho_{23}^2) - 1\right|},$$
$$B_2 = \sqrt{\text{Max} \left|0, 4 \gamma^4 (\rho_{14} + \rho_{23})^2 + (\rho_{22} + \rho_{33} - \rho_{11} - \rho_{44})^2 - 1\right|}, \tag{199}$$

and a condition for NESD would be either one of

$$1 \geq 8 \gamma^4 (\rho_{14}^2 + \rho_{23}^2),$$
$$(\rho_{11} + \rho_{44} - 1)(\rho_{22} + \rho_{33} - 1) \geq \gamma^4 (\rho_{14} + \rho_{23})^2. \tag{200}$$
Let us turn to the topic of parity. In section 8.4 dephasing NESD has been proven for non-X-states. We will now extend the proof to X-states, and show that non-locality for pure parity states exhibits entanglement convergence, while the states with mixed parity all show NESD. Assume a general X-state

\[ \rho_X = \mu \rho_+ + \nu \rho_- \]  

normalized as \( \mu(\rho_{11} + \rho_{44}) + \nu(\rho_{22} + \rho_{33}) = 1 \). Then eq. (200) will be

\[
\begin{align*}
1 & \geq 8\gamma^4(\mu^2 \rho_{14}^2 + \nu^2 \rho_{23}^2), \\
\chi & \geq \gamma^4(\mu \rho_{14} + \nu \rho_{23})^2,
\end{align*}
\]

where \( \chi = (\nu(\rho_{22} + \rho_{33}) - 1)(\mu(\rho_{11} + \rho_{44}) - 1) \) is the parity mixture parameter which is \( \chi = 0 \) for a state with given parity and \( \chi = 1 \) for a state with maximally mixed parity. The analysis is straightforward, for a mixed parity state \( (\chi > 0) \) we have

\[
\begin{align*}
1 & \geq 8\gamma^4(\mu^2 \rho_{14}^2 + \nu^2 \rho_{23}^2), \\
\chi & \geq \gamma^4(\mu \rho_{14} + \nu \rho_{23})^2,
\end{align*}
\]

where NESD is the rule. On the other hand, for a pure parity state where \( \chi = 0 \) implies either

\[
\mu = 1 \implies \gamma^4 \rho_{14}^2 \leq 0, \quad \nu = 1 \implies \gamma^4 \rho_{23}^2 \leq 0
\]

to be fulfilled for NESD to occur. This cannot be since \( \gamma^4 \rho_{14} \) and \( \gamma^4 \rho_{23} \) approach zero exponentially.

### 8.6 Concurrence dephasing

From the Simulation Results section 7, a relationship between ESD and the rank of a density matrix might be suggested. Full rank density matrices (i.e. Werner states and MES) exhibit dephasing ESD, while all other states show entanglement convergence. In this section we verify this relationship. First the eigenvalues of the Hermitian matrix \( \rho \hat{\rho} \) are considered for a specific class of Rank(3) matrices. These density matrices belong to one of the following sets,

\[
\begin{align*}
\rho_{1-} = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \rho_{22} & \rho_{23} & \rho_{24} \\
0 & \rho_{23} & \rho_{33} & \rho_{34} \\
0 & \rho_{24} & \rho_{34} & \rho_{44}
\end{pmatrix}, & \quad \rho_{2-} = \begin{pmatrix}
\rho_{11} & \rho_{12} & \rho_{13} & 0 \\
\rho_{12} & \rho_{22} & \rho_{23} & 0 \\
\rho_{13} & \rho_{23} & \rho_{33} & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}, \\
\rho_{1+} = \begin{pmatrix}
\rho_{11} & 0 & \rho_{13} & \rho_{14} \\
0 & 0 & 0 & 0 \\
\rho_{13} & 0 & \rho_{33} & \rho_{34} \\
\rho_{14} & 0 & \rho_{34} & \rho_{44}
\end{pmatrix}, & \quad \rho_{2+} = \begin{pmatrix}
\rho_{11} & \rho_{12} & 0 & \rho_{14} \\
\rho_{12} & \rho_{22} & 0 & \rho_{24} \\
0 & 0 & 0 & 0 \\
\rho_{14} & \rho_{24} & 0 & \rho_{44}
\end{pmatrix},
\end{align*}
\]

where the eigenvalues of the Hermitian matrices are related as \( \text{Eig}(\rho_{1+}\rho_{1+}) = \text{Eig}(\rho_{2+}\rho_{2+}) = \text{Eig}(\rho_{+}\rho_{+}) \) and \( \text{Eig}(\rho_{1-}\rho_{1-}) = \text{Eig}(\rho_{2-}\rho_{2-}) = \text{Eig}(\rho_{-}\rho_{-}) \), where \( \rho_+ \) and \( \rho_- \) are Rank(2) pure parity density matrices defined in eqs. (173) and (176), where

\[
\begin{align*}
\text{Eig}(\rho_-) = \left(0, 0, (\rho_{23} + \sqrt{\rho_{23}^2}), (\rho_{23}^2 - \sqrt{\rho_{23}^2})^2\right), \\
\text{Eig}(\rho_+) = \left(0, 0, (\rho_{14} + \sqrt{\rho_{14}^2}), (\rho_{14}^2 - \sqrt{\rho_{14}^2})^2\right).
\end{align*}
\]

Under influence of dephasing noise, the eigenvalues of the Hermitian matrix \( \rho_+\rho_+ \) have the evolution
Eig(\(\rho_–\hat{\rho}_–\)) \(\leftrightarrow\) Eig(\(\rho_–(\gamma)\hat{\rho}_–(\gamma)\))
\[
= \left(0, 0, (\gamma^2 \rho_{23} + \sqrt{\rho_{22}\rho_{33}})^2, (\gamma^2 \rho_{23} - \sqrt{\rho_{22}\rho_{33}})^2\right),
\]
Eig(\(\rho_+\hat{\rho}_+\)) \(\leftrightarrow\) Eig(\(\rho_+(\gamma)\hat{\rho}_+(\gamma)\))
\[
= \left(0, 0, (\gamma^2 \rho_{14} + \sqrt{\rho_{11}\rho_{44}})^2, (\gamma^2 \rho_{14} - \sqrt{\rho_{11}\rho_{44}})^2\right),
\]
with corresponding concurrence
\[
C = \text{Max}\left[0, |\gamma^2 \rho_{23} + \sqrt{\rho_{22}\rho_{33}}| - |\gamma^2 \rho_{23} - \sqrt{\rho_{22}\rho_{33}}|\right],
\]
\[
C = \text{Max}\left[0, |\gamma^2 \rho_{14} + \sqrt{\rho_{11}\rho_{44}}| - |\gamma^2 \rho_{14} - \sqrt{\rho_{11}\rho_{44}}|\right].
\]

Equation (208) is applicable to the entangled states in eq. (205), as well as the Rank(2) pure parity states \(\rho_+\) and \(\rho_-\) defined in eqs. (175) and (176). For these states entanglement convergence is shown, reaching zero only as \(\gamma \to 0\) in an infinite time.

8.7 Concurrence decay

We have shown in section 8.6 that \(\text{Rank}(\rho) < 4\) is a sufficient condition for concurrence entanglement convergence of a state \(\rho\). If a state is subjected to decay noise the rank of the system is not preserved asymptotically as it displays a “cascade” from excited to ground state. As an example, in the pure parity dynamics
\[
\rho_- \mapsto \rho^{(A)}_–(\gamma) = \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \gamma^2 \rho_{22} & \gamma^2 \rho_{23} & 0 \\
0 & \gamma^2 \rho_{23} & \gamma^2 \rho_{33} & 0 \\
0 & 0 & 0 & \omega^2 (\rho_{22} + \rho_{33})
\end{pmatrix},
\]
\[
\rho_+ \mapsto \rho^{(A)}_+(\gamma) = \begin{pmatrix}
\gamma^4 \rho_{11} & 0 & 0 & \gamma^2 \rho_{14} \\
0 & \gamma^2 \omega^2 \rho_{11} & 0 & 0 \\
0 & 0 & \gamma^2 \omega^2 \rho_{14} & 0 \\
\gamma^2 \rho_{14} & 0 & 0 & \rho_{44} + \omega^4 \rho_{11}
\end{pmatrix},
\]
both systems exhibit the cascade from excited to ground state. The difference is that eq. (209) evolves such that \(\text{Rank}(\rho^{(A)}_–(\gamma)) < 4\), and (210) evolves such that \(\text{Rank}(\rho^{(A)}_+(\gamma)) \leq 4\) and could exhibit ESD or entanglement convergence depending on the relation between \(\rho_{11}, \rho_{44},\) and \(\rho_{14}\).

9 Conclusion

Assuming two qubits \(\rho_A\) and \(\rho_B\) with real entries are subsystems of an entangled state \(\rho_{AB}\) such that both subsystems are subjected to local noise processes. The noise processes are described by Kraus operators in eq. (91) and (92) where subsystem \(A\) follows the same noise evolution as subsystem \(B\). Then the following conclusion can be made about the entanglement \(\rho_{AB}\):

- For non-locality decay, NESD is the rule. The geometry of non-locality decay is illustrated in fig. 7.
- For non-locality dephasing, a state of pure parity shows exponential decay, while all other states show NESD.
- For concurrence dephasing, states with rank < 4 show entanglement convergence.
Figure 7: **Geometry of locality and separability.** $E$ denotes entanglement, $L$ locality and $S$ separability. As a state approaches region $S$ (for example, by being pushed towards its ground state) it needs to pass through $L$. NESD is hence the rule for a state subjected to decay noise.

10 Plots

**Bell state** $\rho_{\Psi^\pm}$

Each graph shows the evolution of decay (blue line) and dephasing (orange line).

![Graph](image1.png)  
![Graph](image2.png)

Figure 8: Concurrence - identical evolution of decay and dephasing.  
Figure 9: Non-locality
Bell state $\rho_{\Phi_{\pm}}$

Each graph shows the evolution of decay (blue line) and dephasing (orange line).

Figure 10: Concurrence

Figure 11: Non-locality

Werner states Non-locality

Figure 12: Non-locality dephasing

Figure 13: Non-locality decay
Werner states Concurrence

Figure 14: Concurrence dephasing

Figure 15: Concurrence decay

MES Non-locality

Figure 16: Non-locality dephasing

Figure 17: Non-locality decay
MES Concurrence

Figure 18: Concurrence dephasing

Figure 19: Concurrence decay

MEMS-states Non-locality

Figure 20: Non-locality dephasing

Figure 21: Non-locality decay
MEMS-states Concurrence

Figure 22: Concurrence dephasing

Figure 23: Concurrence decay

Non-MES $|\Theta+\rangle$ Non-locality

Figure 24: Non-locality dephasing

Figure 25: Non-locality decay
Non-MES $|\Theta+\rangle$ Concurrence

Figure 26: Concurrence dephasing

Figure 27: Concurrence decay

Non-MES $|\Theta-\rangle$ Non-locality

Figure 28: Non-locality dephasing

Figure 29: Non-locality decay
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References


