Sequential Testing for Diffusion Processes

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Abstract

In this project we are looking at a one-dimensional stochastic process $S_t$ where the drift depends on a Bernoulli distributed random variable $\mu$ giving the process two possible drifts. For this process we are interested in determining the value of $\mu$ by minimizing the penalty of making the wrong decision and the time of waiting for the decision, which is an optimal stopping problem. In general this will not depend only on $S_t$ but also on a $\pi_t$ process that is the probability of $\mu$ having a specific value given $S_t$ leading to a two-dimensional problem. For some processes $S_t$ we can write the $\pi_t$ process as a function of time and the current value of $S_t$ which enable us to reduce the dimension of the optimal stopping problem, which is the main result of this project.

Here we look at the conditions on the stochastic processes needed for one to be able to write the $\pi_t$ process as a function of time and the current value of $S_t$. We also see two examples of when this is possible and the optimal stopping problem for those two processes.
1 Introduction

Stochastic processes are used to model various things, where two examples are the price of stocks or the diffusion of molecules, and are therefore interesting to study. Here we consider statistical processes where the form of the drift is unknown, and where we like to determine what form the drift has. As an example we can look at a stochastic process $S_t$ that is a simple Brownian motion with drift

$$dS_t = \mu dt + dW_t$$

$$S_0 = s_0$$

We can assume that $\mu$ actually is a random variable that can take the value 0, giving just a Brownian motion, or 1, giving a Brownian motion with drift. More generalized the process will be

$$dS_t = A(S_t, \mu)dt + B(S_t)dW_t$$

$$S_0 = s_0$$

where we again assume that $\mu$ is a random variable with a Bernoulli distribution giving two different possible drifts. For this process we do not know the value of $\mu$, but we can make a guess for the probability of the different drifts and update the guess as new values of $S_t$ become available. We denote the conditional probability at time $t$ that $\mu$ is 1 as $\pi_t = P(\mu = 1 | F_t^\mathcal{Y})$ given the probability that $\mu$ is 0 as $1 - \pi_t$.

Now, we would like to observe the process until we can make a good guess at the value of $\mu$. We will then have a stopping time $\tau$ when we make a decision $d \in \{0, 1\}$ of the value of the unknown variable $\mu$. If we have penalties for making the wrong choice ($a$ and $b$) and a penalty for waiting ($c$), we would like to minimize the penalty, which is to calculate

$$V = \inf_{\tau, d} (aP(d = 0, \mu = 1) + bP(d = 1, \mu = 0) + cE_{\pi}[\tau])$$

This is a stochastic optimization problem where the solution in general will depend on both $s$ (the current position of $S_t$) and $\pi$ (current position of $\pi_t$) and might not be simple to solve. Here we will look at the specific case when we can express $\pi_t$ as a function, $H$, of time $t$ and the current value of the stochastic process $S_t$. Then we can reduce the problem to depend only on $S_t$ and $t$ by substituting $\pi_t$ with $H(t, S_t)$, or, by substituting $S_t$ with $H^{-1}(t, \pi_t)$, the problem will depend only on $\pi_t$ and $t$.

In the example above with the process being either a Brownian motion or a Brownian motion with a drift we can find an equation for $\pi_t$ that is

$$\pi_t = \frac{\pi_0}{(1 - \pi_0) \exp(\frac{1}{2}t) \exp(s_0 - S_t) + \pi_0}$$
and as we will see it is possible to do so in other cases too.
2 Background material

Before exploring the main part we need to define some basic building blocks, and we start with a stochastic process [Oks13].

**Definition 1** A stochastic process is a collection of random variables \( \{S_t\}_{t \in T} \) defined on a probability space \((\Omega, \mathcal{F}, P)\) and with range in \(\mathbb{R}^n\). The collection is parametrized by a set \(T\).

Here the set \(T\) is commonly taken to be \(\mathbb{R}_+\). An important example of a stochastic process is the Wiener process (also known as a Brownian Motion) that will be used later as the noise in our processes and is defined by [RS 77a].

**Definition 2** A stochastic process \(\{W_t\}_{t \in T}\) on the probability space \((\Omega, \mathcal{F}, P)\) is a Wiener process if

1. \(W_0 = 0\) a.s.
2. \(W_t\) has stationary independent increments
3. \(W_t\) has increments that have a normal distribution s.t.
   \[ E[W_{t_2} - W_{t_1}] = 0, \quad \text{Var}[W_{t_2} - W_{t_1}] = \sigma^2|t_2 - t_1| \forall t_1, t_2 \in T \text{ such that } t_1 < t_2 \]
4. \(t \mapsto W_t(\omega)\) is continuous for almost all \(\omega \in \Omega\)

We will use the Wiener process to define stochastic integrals and Itô processes, for that we first need the following definitions for \(\mathcal{F}_t\)-adapted [Oks13] and for the set of functions we use when defining the Itô process [Oks13].

**Definition 3** Given a increasing family of \(\sigma\)-algebras of subsets of \(\Omega\), \(\{\mathcal{F}_t\}_{t \geq 0}\), a process \(g(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}^n\) is called \(\mathcal{F}_t\)-adapted if for each \(t \geq 0\)
\(\omega \to g(t, \omega)\) is \(\mathcal{F}_t\)-measurable.

**Definition 4** \(\mathcal{V}(U, T)\) is a class of functions \(f(t, \omega) : [0, \infty) \times \Omega \to \mathbb{R}\) that satisfy

1. \((t, \omega) \to f(t, \omega)\) is measurable
2. \(f(t, \omega)\) is \(\mathcal{F}_t\)-adapted
3. \(E\left[\int_U f(t, \omega)^2 dt\right] < \infty\)

From this we can set \(\mathcal{V} = \bigcap_{T>0} \mathcal{V}(0, T)\) which is used to define an Itô process as [Oks13]

**Definition 5** Given a Wiener process \(W_t\) on \((\Omega, \mathcal{F}, P)\) an Itô process is a stochastic
process $S_t$ on $(\Omega, \mathcal{F}, P)$ given as

$$S_t = \int_0^t A(u, \omega) du + \int_0^t B(u, \omega) dW_u$$

where $A$ is $\mathcal{F}_t$-adapted, $P[\int_0^t |A(u, \omega)| du < \infty \forall t \geq 0] = 1$ and $B \in \mathcal{V}$

This is often given in differential form as

$$dS_t = Adt + BdW_t$$

An Itô process for which $A$ and $B$ do not depend on anything except current value ($S_t$) and time is an Itô diffusion. A stochastic process can also be described by its infinitesimal generator $A$ which is defined for a time-homogeneous Itô diffusion, $dS_t = A(S_t)dt + B(S_t)dW_t$, and the $C^2_0$ function $g(s)$ as [Oks13]

$$\mathcal{A}g(s) = A(s)\frac{\partial g}{\partial s}(s) + \frac{1}{2}B(s)^2\frac{\partial^2 g}{\partial s^2}(s)$$

A concept of importance for our stochastic optimisation problem is the stopping time [Oks13] that will be used as the time when we make a decision of which drift we assume that the process have. It is defined as

**Definition 6** Given an increasing family of σ-algebras on $\Omega$, $\{\mathcal{N}_t\}$, a stopping time with respect to $\{\mathcal{N}_t\}$ is a function $\tau : \Omega \to [0, \infty)$ such that

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{N}_t, \forall t \geq 0$$

An example of a stopping time is the first exit time of $S_t$ from a Borel set $U$ defined as

$$\tau_U = \inf\{t > 0; S_t \notin U\}$$

2.1 Bessel process

A process we will use is the Bessel process which can be constructed from a $\delta$-dimensional Wiener process where we assume that $\delta \geq 2$ and $\delta \in \mathbb{N})$. The process has the components $W^{(i)}_t \ i \in \{1, \ldots, \delta\}$ that all are assumed to start in zero. Now if we measure the distance from the origin to the current position of the Wiener process with

$$\sqrt{(W^{(1)}_t)^2 + \cdots + (W^{(\delta)}_t)^2}$$

we get a new process which is the Bessel process [And02].
can derive the SDE for a Bessel process by setting 
\[ S_t = f(W) = \sqrt{\sum_{i=1}^{\delta} (\bar{W}_t^{(i)})^2} \]
then we get the derivatives (where we put in \( S_t \) when possible)

\[
\begin{align*}
    f_t &= 0 \\
    f_i &= \frac{\bar{W}_t^{(i)}}{S_t} \\
    f_{ij} &= \frac{\delta_{ij}}{S_t} - \frac{W_t^{(i)}W_t^{(j)}}{S_t^2}
\end{align*}
\]

Using these derivatives with Itô's formula we get

\[
\begin{align*}
    dS_t &= f_t dt + \sum_{i=1}^{\delta} f_i d\bar{W}_t^{(i)} + \sum_{i=1}^{\delta} \sum_{j=1}^{\delta} f_{ij} \bar{W}_t^{(i)}d\bar{W}_t^{(j)} \\
    &= \sum_{i=1}^{\delta} \frac{\bar{W}_t^{(i)}}{S_t} d\bar{W}_t^{(i)} + \frac{\delta}{2} \sum_{i=1}^{\delta} \sum_{j=1}^{\delta} \left( \frac{\delta_{ij}}{S_t} - \frac{W_t^{(i)}W_t^{(j)}}{S_t^2} \right) d\bar{W}_t^{(i)}d\bar{W}_t^{(j)} \\
    &= \sum_{i=1}^{\delta} \frac{\bar{W}_t^{(i)}}{S_t} d\bar{W}_t^{(i)} + \left( \frac{\delta}{2S_t} - \frac{1}{2S_t^2} \sum_{i=1}^{\delta} (\bar{W}_t^{(i)})^2 \right) dt \\
    &= \sum_{i=1}^{\delta} \frac{\bar{W}_t^{(i)}}{S_t} d\bar{W}_t^{(i)} + \left( \frac{\delta}{2S_t} - \frac{S_t^2}{2S_t^2} \right) dt \\
    &= \frac{\delta-1}{2S_t} dt + \sum_{i=1}^{\delta} \frac{\bar{W}_t^{(i)}}{S_t} d\bar{W}_t^{(i)}
\end{align*}
\]

Here \( \int \sum_{i=1}^{\delta} \frac{W_t^{(i)}}{S_t} d\bar{W}_t^{(i)} \) is a Brownian motion and setting it to \( W_t \) gives the Bessel process

\[
    dS_t = \frac{\delta - 1}{2S_t} dt + dW_t
\]

We see from the SDE that close to the origen the drift will be large so that the process is pushed away from zero while far away it will behave as a Brownian motion.

2.2 Filtering theory

If we have some process that we measure (or some process derived from it) we have an interest in estimating the parameters of that process. We assume that we have some stochastic process \( S_t \) driven by a Wiener process \( W_t \) which we cannot observe and instead we observe the stochastic process \( U_t \) driven by a Wiener process \( V_t \) (independent of \( W_t \) and \( S_0 \)) that depends on \( S_t \) such as

\[
    dU_t = C(t, S_t) dt + D(t, S_t) dV_t
\]
Now for observations of $U$ from $U_0$ up to $U_t$ we can try to find the best estimate of $S_t$ and it can be found to be $\hat{S}_t = E[S_t|\mathcal{F}_t^U]$ [Oks13].

Now we have an Itô process but in this case we assume that the drift of the process depends on some random variable $\mu$ (assumed to take values in a discrete set) so the drift is unknown. For this process we can be interested in the probability, $\pi_i(t)$, that the random variable $\mu$ takes a specific value $\mu_i$. From an initial guess, $\pi_i(0)$, of this probability we will update the probability given how the process changes. From this we will get a stochastic differential equation for the probability

**Theorem 1** Let $\mu$ be a discrete distributed random variable with values in the set $E$, $S_t$ a random process given by the stochastic differential $dS_t = A_t(\mu, S_t)dt + B_t(S_t)dW_t$ where $W_t$ is a Wiener process and $\pi_i(t) = P(\mu = \mu_i|\mathcal{F}_t^S)$. Then $\pi_i(t)$ satisfy

$$\pi_i(t) = \pi_i(0) + \int_0^t \pi_i(u) \frac{A_u(\mu_i, S_u) - \bar{A}_u(S_u)}{B_u(S_u)} d\bar{W}_u$$

(10)

where

$$\bar{A}_u(S_u) = \sum_{j \in E} A_u(\mu_j, S_u) \pi_j(u)$$

(11)

and $\bar{W}_t$ a Wiener process defined by

$$\bar{W}_t = \int_0^t \frac{dS_u - \bar{A}_u(S_u)du}{B_u(S_u)}$$

(12)

A proof can be found in [RS 77b].
3 Dimension reduction with two drifts

In this section we look at when it is possible to write \( \pi_t \) as a function of time and the current value of \( S_t \) and what constraints are needed for that to be possible. We start by assuming that the process we have has a drift that depends on a parameter \( \mu \) and has the form

\[
\begin{align*}
    dS_t &= A(t, S_t, \mu)dt + B(t, S_t)dW_t \\
    S_0 &= s_0
\end{align*}
\]

Here we assume that \( \mu \) only takes two values (here using 1 and 2) such that we have two possibilities for the drifts, here called \( F_1 \) and \( F_2 \). For this process we introduce \( \pi_t = P(\mu = |F_t^S) \) and using Theorem (1) we get

\[
\begin{align*}
    d\pi_t &= F_2(t, S_t)\pi_t(1 - \pi_t)dt + B(t, S_t)dW_t \\
    dS_t &= (F_1(t, S_t)(1 - \pi_t) + F_2(t, S_t)\pi_t)dt + B(t, S_t)dW_t
\end{align*}
\]

where \( W_t \) is a Wiener process. Now we assume that \( \pi_t \) can be written as a function of time and the current value of \( S_t \) so that we can use \( \pi_t = H(t, S_t) \) with Ito’s formula to get (where subscripts on \( H \) denote differentiation)

\[
\begin{align*}
    d\pi_t &= (H_t + H_s(F_1(t, S_t)(1 - \pi_t) + F_2(t, S_t)\pi_t) + \frac{1}{2}H_{ss}B(t, S_t)^2)dt + H_sB(t, S_t)dW_t
\end{align*}
\]

We know that both forms of \( d\pi_t \) given above must be the same so if we set them to be equal to each other the result is the following theorem.

**Theorem 2** Given a process

\[
\begin{align*}
    dS_t &= A(t, S_t, \mu)dt + B(t, S_t)dW_t \\
    S_0 &= s_0
\end{align*}
\]

where \( \mu \) can take the value either 1 and 2 giving the possible drifts \( F_1(t, S_t) \) and \( F_2(t, S_t) \). For this process we can calculate

\[
D(t, s) = -\int_{s_0}^{s} \frac{F_2(t, u) - F_1(t, u)}{B(t, u)^2}du
\]

and

\[
f = \frac{B(t, s)^2}{2} \left( \frac{\partial}{\partial s} D(t, s) \right)^2 + \frac{B(t, s)^2}{2} \frac{\partial^2}{\partial s^2} D(t, s) + \frac{\partial}{\partial t} D(t, s) + F_2(t, s) \frac{\partial}{\partial s} D(t, s)
\]
Given this we have that the conditional probability of $\mu$ being 2, $\pi_t = P(\mu = 2 | F^S_t)$, can be written as a function of $t$ and $S_t$ when $f$ only depends on $t$ (or is constant). Then $\pi_t$ is written

$$\pi_t(S_t) = \frac{1}{\exp(-\int f(t)dt) \exp(D(t, S_t)) + 1}$$  \hspace{1cm} (19)$$

Calculations proving the theorem can be found in Appendix (A).

3.1 Brownian motion vs. time independent drift process

We like to find possible processes when we can write $\pi_t$ as a function of time and the current value of the process. Here we will look at the case of a Brownian motion vs. a process with time independent drift. This we get if we assume that $F_1(t, s) = 0$ and $B(t, s) = \sigma$ to get the Brownian motion and assume $F_2(s) = g(s)$ for the process with time independent drift. This added to Theorem (2) will result in

$$D(s) = -\int^{s}_{S_0} \frac{F_2(u) - F_1(u)}{B(t, u)^2} du = -\int^{s}_{S_0} \frac{g(u)}{\sigma^2} du = -\frac{G(s) - G(S_0)}{\sigma^2}$$  \hspace{1cm} (20)$$

from which we get the derivatives

$$D_t(s) = 0$$
$$D_s(s) = -\frac{g(s)}{\sigma^2}$$
$$D_{ss}(s) = -\frac{g_s(s)}{\sigma^2}$$  \hspace{1cm} (21)$$

Inserting these derivatives into the equation for $f$ we get

$$f = \frac{B(t, s)^2}{2} \left( \frac{\partial}{\partial t} D(t, s) \right)^2 + \frac{B(t, s)^2}{2} \frac{\partial^2}{\partial s^2} D(t, s) + \frac{\partial}{\partial t} D(t, s) + F_2(t, s) \frac{\partial}{\partial s} D(t, s)$$
$$= \frac{\sigma^2}{2} \left( \frac{g(s)}{\sigma^2} \right)^2 - \frac{\sigma^2}{2} \frac{g_s(s)}{\sigma^2} - g(s) \frac{g(s)}{\sigma^2}$$
$$= -\frac{1}{2\sigma^2} \left( g(s)^2 + \sigma^2 g_s(s) \right)$$  \hspace{1cm} (22)$$

This must be constant for us to be able to write $\pi_t$ as a function of time and the current value of the process and this gives the ODE $g(s)^2 + \sigma^2 g_s(s) = d$, $d \in \mathbb{R}$ which we can solve in three parts. We can have that, $g(s)^2 + \sigma^2 g_s(s) < 0$, $g(s)^2 + \sigma^2 g_s(s) = 0$ or $g(s)^2 + \sigma^2 g_s(s) > 0$. 

3.1.1 The case \( g(s)^2 + \sigma^2 g_s(s) < 0 \)

If we have \( g(s)^2 + \sigma^2 g_s(s) < 0 \) we can assume that \( g(s)^2 + \sigma^2 g_s(s) = -a \) for some \( a > 0 \). This is an ODE that has the solutions \( g(s) = \sqrt{-a} \) and

\[
g(s) = \sqrt{-a} \tan(k_a \sqrt{-a} - s \sqrt{-a}) + c
\]

for some constant \( k_a \).

3.1.2 The case \( g(s)^2 + \sigma^2 g_s(s) = 0 \)

Here the ODE has the solutions \( g(s) = 0 \) and

\[
g(s) = \frac{\sigma^2}{s - \sigma^2 k_0}
\]

for some constant \( k_0 \). If we set \( k_0 = 0 \) and \( \sigma = 1 \) we get a Bessel process as the alternative process which will be looked at later.

3.1.3 The case \( g(s)^2 + \sigma^2 g_s(s) > 0 \)

Here we have the ODE \( g(s)^2 + \sigma^2 g_s(s) = b \) for some \( b > 0 \) that has the solution \( g(s) = \sqrt{b} \) and

\[
g(s) = \sqrt{b} \tanh(k_b \sqrt{b} + s \sqrt{b}) + c
\]

for some constant \( k_b \). When we have \( g(s) = \sqrt{b} \) the alternative process is a Brownian motion with drift, which we will look closer at below.

3.1.4 Brownian motion vs. Brownian motion with drift

The first time independent drift we want to look at is when we can have either a Brownian motion or a Brownian motion with drift. In this case we have \( g(s) = c \) and the primitive function is

\[
G(s) = cs
\]

Using Theorem (2) we get that the form of \( \pi_t \) is

\[
\pi_t(S_t) = \frac{\pi_0}{(1 - \pi_0) \exp\left(\frac{c_1}{\pi_0} t\right) \exp\left(c(S_0 - S_t)\right)} + \pi_0
\]

That this formula behaves as expected can be seen from the example \( \pi_t \) trajectory for a specific \( S_t \) trajectory (both for \( \mu = 1 \) and \( \mu = 2 \) when \( c = 1 \), \( \pi_0 = 0.4 \) and \( S_0 = 1 \) in

11
Figure (1) where the same random numbers are used for both processes. We see that for the $\mu = 1$ process the probability that $\mu = 2$ goes down over time while for the $\mu = 2$ process the probability that $\mu = 2$ stays high, as one would expect.

Figure 1: In the top row (a and b) the simulated trajectory of the process $dS_t = (\mu - 1)dt + dW_t$ for $\mu = 1$ (a) and $\mu = 2$ (b) with the same random numbers used in both simulation. In the bottom graphs (c and d) $\pi_t$ is calculated from $S_t$ for both $\mu = 1$ (c) and $\mu = 2$ (d). Here the initial values of the processes are $S_0 = 1$ and $\pi_0 = 0.5$.

3.1.5 Brownian motion vs. Bessel

The second time independent drift case is that of a Brownian motion vs. a Bessel process where we have the ODE

$$g(s)^2 + g_s(s) = 0$$

with the solution

$$g(s) = \frac{1}{s - k_0}$$

for some constant $k_0$ that we here will set to 0 to get the Bessel process with the primitive function

$$G(s) = \log(s)$$

Using Theorem (2) with this $g(s)$ gives us the function for $\pi_t$
\[ \pi_t(S_t) = \frac{\pi_0}{(1 - \pi_0) \frac{S_t}{S_0} + \pi_0} \]  

(31)

when \( S_t > 0 \) and 0 otherwise, as if \( S_t \) goes below zero the process must be a Brownian motion.

The \( \pi_t \) trajectory for a specific \( S_t \) trajectory (both for \( \mu = 1 \) and \( \mu = 2 \)) when \( \pi_t = 0.5 \) and \( S_t = 1 \) can be seen in Figure (2) where the same random numbers are used in both simulations. Here, again we see that as the \( \mu = 1 \) process moves closer to zero the probability of it being a \( \mu = 2 \) process goes down and becomes zero when the process takes negative values, while the \( \mu = 2 \) process’ probability generally is a bit higher than the initial guess. But in this example the probabilities stay closer to the initial guess than the previous example of Brownian motion v.s Brownian motion with drift.

![Figure 2](image)

Figure 2: In the top row (a and b) the simulated trajectory of the process \( dS_t = \mu \frac{S_t}{S_0} dt + dW_t \) with the same random numbers used in both simulation. In the bottom graphs (c and d) \( \pi_t \) is calculated from \( S_t \) for both \( \mu = 1 \) (c) and \( \mu = 2 \) (d). Here the initial values of the processes are \( S_0 = 1 \) and \( \pi_0 = 0.5 \).

3.2 Bessel vs Bessel

Above we had a process that either was a Brownian motion or a Bessel process but we can also look at the case when the process could be two different Bessel processes, so that we have \( F_1 = \frac{\delta_0 - 1}{2S_0} \), \( F_2 = \frac{\delta_1 - 1}{2S_0} \) and \( B = 1 \) where \( \delta_0 < \delta_1 \). Now using Theorem (2) we get
\[ D(t, s) = \frac{\delta_1 - \delta_0}{2} \ln \left( \frac{s_0}{s} \right) \]  

(32)

and

\[ f = \frac{1}{2} \left( \frac{\delta_1 - \delta_0}{2s} \right)^2 + \frac{1}{2} \frac{\delta_1 - \delta_0}{2s^2} - \frac{\delta_1 - \delta_0}{2s} \frac{\delta_1 - 1}{2s} \]

(33)

As \( f \) cannot depend on \( s \) we must have that one of the factors must be zero, so we have either \( \frac{\delta_1 - \delta_0}{2} = 0 \) which does not satisfy \( \delta_0 < \delta_1 \) or \( 2 - \frac{\delta_1 + \delta_0}{2} = 0 \) giving that \( \delta_1 + \delta_0 = 4 \). From this we get the equation of \( \pi_t \) to be

\[ \pi_t(S_t) = \frac{\pi_0}{1 - \pi_0} \left( \frac{s_0}{S_t} \right)^{\frac{\delta_1 - \delta_0}{2}} + \pi_0 \]

(34)

This reduces to the equation of \( \pi_t \) for Brownian motion vs. Bessel when \( \delta_1 = 3 \) and \( \delta_0 = 1 \), where we must think of a Brownian motion as a one-dimensional Bessel process even though the Brownian motion can take negative values which a Bessel process cannot.

**Remark.** There are also other ways to reduce the dimensionality of the problem than the one given here. In the case [Pet18] when the drifts are different Bessel process as before but now \( \delta_0 \) and \( \delta_1 \) can take any values as long as \( 2 \leq \delta_0 < \delta_1 \) is satisfied, a time change can be used by defining

\[ A_t = \int_0^t \frac{du}{\pi_u} \]

(35)

\[ T_t = A_t^{-1} \]

from which a new process can be define by

\[ (\hat{\pi}_t, \hat{S}_t) = (\pi_{T_t}, S_{T_t}) \]

(36)

Here the problem can be reduced to an one-dimensional problem, \( \hat{\pi}_t = \hat{\pi}_t(\hat{S}_t) \), for details see [Pet18].

### 3.3 Brownian motion vs. Ornstein–Uhlenbeck process

It is not always possible to find an equation for \( \pi_t \) that depends on the current value of \( S_t \) and \( t \). Here we will apply Theorem (2) to a process that either can be an Ornstein–Uhlenbeck process or a Brownian motion. An Ornstein–Uhlenbeck process is described by the equation

\[ dS_t = -\mu S_t dt + \sigma dW_t \]

(37)
with both $\mu$ and $\sigma$ being larger than zero meaning that the Ornstein–Uhlenbeck process’ drift is towards zero. We can reason that this process (vs. a Brownian motion) cannot have a function for $\pi_t$ that depends only on the current value of $S_t$ and $t$. This by assuming that we have a process that moves far from zero and then moves back close to zero. We would guess that this process must be a Brownian motion since the Ornstein–Uhlenbeck process always has a drift that keeps it close to zero and it is unlikely that it would go far from zero. But at the same time if we only think of the current value and ignore the previous values we are more likely to guess that it is an Ornstein–Uhlenbeck process as it is close to zero where we expect it to be. This means that we would not assume that it is possible to find a function for $\pi_t$ of only current value of $S_t$ and $t$.

This we can also show by applying Theorem (2) to the Ornstein–Uhlenbeck process equation above there we assume that $\mu$ can be either 1 (Ornstein–Uhlenbeck process) or 0 (Brownian motion) and to simplify we assume that $\sigma = 1$. We then get that

$$D(t, s) = \int_{S_0}^{s} \mu u du = \frac{\mu}{2} (s^2 - S_0^2)$$

with the derivatives

$$D_t = 0$$
$$D_s = \mu s$$
$$D_{ss} = \mu$$

This we plug into the equation for $f$ in Theorem (2) to get

$$f = \frac{B(t, s)^2}{2} \left( \frac{\partial}{\partial s} D(t, s) \right)^2 + \frac{B(t, s)^2}{2} \frac{\partial^2}{\partial s^2} D(t, s) + \frac{\partial}{\partial t} D(t, s) + F_2(t, s) \frac{\partial}{\partial s} D(t, s)$$

$$= \frac{1}{2} (\mu s)^2 + \frac{1}{2} \mu + \mu s - \mu s \mu s$$

$$= \mu (s - \frac{1}{2} \mu s^2) + \frac{1}{2} \mu$$

From this we see that $f$ can only be constant if $\mu = 0$ which would make the process being one of two identical Brownian motions. So we can conclude that it is not possible to find a function for $\pi_t$ depending only on the current value of $S_t$ and $t$. 

15
4 Stochastic optimization

If we have a model for a phenomenon we might want to optimize something that is dependent on the phenomenon. In our case the model of the phenomenon would be a stochastic process $S_t$ and we want to optimize the expectation of a function of $S_t$. A simple example of this would be to model a stock as a stochastic process and then try to maximize the profit gained from the stock.

4.1 Optimal stopping

A specific stochastic optimization problem is that of optimal stopping [Gor06a] [Gor06b] where we will observe the process until we stop and make a decision. This is to optimize the expectation of a function of the stochastic process for a family of stopping times. We can assume that we have a gain function $G$ that we are to maximize the expectation of, then given a starting point $s$ of the process we can define the value function $V$ that depends on the starting point $s$.

$$V(s) = \sup_{\tau} E_s[G(S_\tau)]$$ (41)

For the value function we have that $V(s) \geq G(s)$ from which we can define two disjoint sets.

**Definition 7** Given $V(s)$ as above we have

$$C = \{ s \in E : V(s) > G(s) \}$$

$$D = \{ s \in E : V(s) = G(s) \}$$ (42)

Here the first is called the continuation set and the second the stopping set. With these sets we introduce the first exit time from the set $C$ (or equivalently, first entry time into $D$) as

$$\tau_C = \inf\{ t \geq 0 : X_t \notin C \}$$ (43)

This is the stopping time that we are after when trying to find the maximum of the expectation of the gain function, we stop when we exit the continuation set. For this gain function we have the value function that have a specific property defined as

**Definition 8** $F : X \rightarrow Y$ a measurable function is called superharmonic with respect to $S_t$ if

$$E_s[F(S_\tau)] \leq F(s) \forall s \in X$$ (44)

and all stopping times $\tau$

proved by the following theorem [Gor06c]
Theorem 3 Given an optimal stopping time (if existing) \( \tau^* \) so that

\[
V(s) = E_s[G(S_{\tau^*})] \forall s \in E
\]  

(45)

Then \( V \) is the smallest superharmonic function that dominates \( G \) on \( E \).

The object is optimizing the gain function to obtain a value function, in our case the gain function will have a specific form which is seen in the theorem [Gor06b] below. Here we assume that the time does not have an upper limit (meaning that we are, in theory, prepared to wait forever to make a decision)

Theorem 4 Given the value function defined by

\[
V(s) = \sup_{\tau} E_s[M(S_{\tau}) + \int_0^{\tau} L(S_t)dt]
\]  

(46)

then \( V(s) \) can be found by solving (for \( C \) and \( D \) defined as in Definition (7)) the following equations

\[
\begin{align*}
A_s V(s) &= -L(s) \text{ in } C \\
V(s) &\leq M(s) \text{ at } C \\
V(s) &= M(s) \text{ at } D
\end{align*}
\]  

(47)

To this equations we can add smooth fit condition stating that \( \frac{\partial V}{\partial s}(s) = \frac{\partial M(s)}{\partial s}(s) \) at \( \partial C \).

4.2 Process with two possible drifts

Now coming back to the kind of process we are interested in, we have an SDE with two different possibilities for the drift and we would like to determine which alternative is the correct one where we have penalties for mistakes. If we assume that the drift is \( F_1 (\mu = 1) \) but it is actually \( F_2 (\mu = 2) \) the penalty will be \( a \) and for the opposite the penalty will be \( b \), where \( a \) and \( b \) are positive constants. In addition to this we will have a penalty that accounts for the time elapsed before making the decision. Here the constant is 1 as we can move it outside the infimum by adjusting the other constants. Assuming that the decision taken is \( d \) we have a gain function by summing the penalties and this is the function we like to minimize to get the value function (here the process is in general two dimensional so the value function has two parameters)

\[
V(\pi, s) = \inf_{\tau, d} \{aP(d = 1, \mu = 2) + bP(d = 2, \mu = 1) + E_\pi[\tau]\}
\]  

(48)

which can be rewritten [Gor06d], in the form used in Theorem (4), as
\[
V(\pi, s) = \inf_{\tau} E_{\pi}[\min(a\pi_\tau, b(1 - \pi_\tau)) + \int_0^\tau 1dt]
\qquad (49)
\]

Now we want to find calculate \( V \) to get the best guess of the drift, which is to find the continue set \( C \) and the stopping set \( D \). For the current problem that is to find the function \( V \in C^1([0, 1] \times \mathbb{R}_+) \cap C^2([0, 1] \times \mathbb{R}_+ \setminus B) \), where \( B \) is the set of the boundary of \( C \) (or \( D \)), that solves the equation

\[
\min\{A_\pi V(\pi, s) + 1, \min(a\pi, b(1 - \pi)) - V(\pi, s)\}
\qquad (50)
\]

### 4.2.1 Brownian motion vs. Brownian motion with drift

An example of determine the drift of a stochastic process with unknown drift is when the drift is either zero or some positive constant \( k \) (see [Gor06e] for a slightly different equation than the one calculated here). This give the SDE for \( \pi_t \) as

\[
d\pi_t = F_2(t, S_t) - F_1(t, S_t)B_t \pi_t (1 - \pi_t) d\bar{W}_t = k \sigma \pi_t (1 - \pi_t) d\bar{W}_t
\qquad (51)
\]

Since the equation above does not depend on \( S_t \) we have that \( V \) only depend on \( \pi \) and the infinitesimal generator will have the form

\[
A_\pi = \frac{1}{2} \frac{k^2}{\sigma^2} \pi^2 (1 - \pi)^2 \frac{\partial^2}{\partial \pi^2}
\qquad (52)
\]

This result in the ODE for the value function in the set \( C \) to be

\[
\frac{1}{2} \frac{k^2}{\sigma^2} \pi^2 (1 - \pi)^2 \frac{\partial^2}{\partial \pi^2} V(\pi) = -1
\qquad (53)
\]

Here we can observe that the ODE only depends on \( \pi \) and therefore the problem is one-dimensional and the value function will only depend on \( \pi \) and has the solution

\[
V(\pi) = \frac{2\sigma^2}{k^2} (1 - 2\pi) \log(\frac{\pi}{1 - \pi}) + K_1 \pi + K_2
\qquad (54)
\]

with the derivative

\[
\frac{\partial V}{\partial \pi}(\pi) = K_1 - \frac{4\sigma^2}{k^2} \log(\frac{\pi}{1 - \pi}) + \frac{2\sigma^2}{k^2} \frac{1 - 2\pi}{\pi(1 - \pi)}
\qquad (55)
\]
Since the problem is reduced to a one-dimensional problem we have that the border between the $C$ and $D$ set is two points, $A < B$, giving the smooth fit

\[
V(A, S_t) = aA \\
\frac{\partial V}{\partial \pi}(A, S_t) = a \\
V(B, S_t) = b(1 - B) \\
\frac{\partial V}{\partial \pi}(B, S_t) = -b
\]  

Choosing the right value $B$ to calculate the function and plugging in $\frac{\partial V}{\partial \pi}(B) = -b$ gives

\[
K_1 = \frac{4\sigma^2}{k^2} \log\left(\frac{B}{1 - B}\right) - \frac{2\sigma^2}{k^2} \frac{1 - 2B}{B(1 - B)} - b
\]  

which can be plugged back to give

\[
V(\pi) = \frac{2\sigma^2}{k^2} \left((1 - 2\pi) \log\left(\frac{\pi}{1 - \pi}\right) + 2\pi \log\left(\frac{B}{1 - B}\right) - \pi \frac{1 - 2B}{B(1 - B)}\right) - b\pi + K_2
\]

Now using $V(B) = (1 - B)b$ we get

\[
K_2 = \frac{2\sigma^2}{k^2} \left(\frac{1 - 2B}{1 - B} - \log\left(\frac{B}{1 - B}\right)\right) + b
\]

and plugging this back in gives

\[
V(\pi) = \frac{2\sigma^2}{k^2} \left((1 - 2\pi) \left(\log\left(\frac{\pi}{1 - \pi}\right) - \log\left(\frac{B}{1 - B}\right)\right) + (1 - \pi) \frac{1 - 2B}{B(1 - B)}\right) + b(1 - \pi)
\]

from which $A$ and $B$ can be solved by the equations (where the remaining smooth fit conditions are used)

\[
V(A) = aA \\
\frac{\partial V}{\partial \pi}(A) = a
\]

Doing this for three different choices of $a$ and $b$ can be seen in Figure (3) where the $k = 1$. 

19
Figure 3: Graph of $V(\pi)$ used to determine if the drift is either zero or one. The dotted lines is the bounding maximal misclassification error function. Here for three different choices of $a$ and $b$. In all three cases we have that $k = 1$.

When one penalty is larger, we require more extreme values of $\pi_t$ before choosing that drift. In the left graph of Figure (3) we have that $b$ is larger than $a$ meaning that the penalty of assuming that the process has a drift when it is just a Brownian motion is larger that assuming the opposite and we require a larger value of $\pi_t$ before assuming the process has a non-zero drift. The stopping set $(D_t, \text{in this case } [0, A] \cup [B, 1])$ we found here is for the probability of non-zero drift, while in any reality we will just observe the process. We can transform $A$ and $B$ in to the space of the process by inverting Equation (27), in this case the bounds will depend on the time as seen in Figure (4).

Figure 4: Stopping bound for $S_t$ for three different values of $a$ and $b$. In all three cases we have that $k = 1$.

Now we can see that larger $b$ cause the stopping bound for when we assume that the process has a non-zero drift to be higher.
4.3 Brownian motion vs. Bessel

As we see in the last example, when the infinitesimal generator does not contain \( S_t \) dependency we can write Equation (56), together with smooth fit criteria, as

\[
\begin{align*}
\mathcal{A}_\pi V(\pi) &= -1 & \pi \in (A, B) \\
V(A) &= aA \\
\frac{\partial V}{\partial \pi}(A) &= a \\
V(B) &= b(1 - B) \\
\frac{\partial V}{\partial \pi}(B) &= -b \\
V(\pi) &< \min(a\pi, b(1 - \pi)) & \pi \in (A, B) \\
V(\pi) &= \min(a\pi, b(1 - \pi)) & \pi \notin (A, B)
\end{align*}
\]  

(62)

But when that is not the case we cannot directly do that. In the case of Bessel vs. Brownian motion \((F_1(s) = 0, F_2(s) = 1/s \text{ and } B(s) = 1)\) we have

\[
d\pi_t = \frac{F_2(S_t) - F_1(S_t)}{B(S_t)} \pi_t (1 - \pi_t) d\bar{W}_t \\
= \frac{1}{S_t} \pi_t (1 - \pi_t) d\bar{W}_t
\]  

(63)

This will give a infinitesimal generator that does not only depend on \( \pi_t \) so the problem will be two-dimensional. But, in this case we have an equation relating \( \pi_t \) and \( S_t \) given by Theorem (2) and previously computed as

\[
\pi_t(S_t) = \frac{\pi_0}{(1 - \pi_0) \frac{S_0}{S_t} + \pi_0}
\]  

(64)

From the equation above we see that the \( \pi_t \) depends on \( \pi_0 \) but we are also minimizing the value function with respect to \( \pi \) which is the initial value of \( \pi_t \). These two \( (\pi \text{ and } \pi_0) \) represent the same thing but are separate and therefore do not cause any problem in the calculations. The equation can be inverted to gives

\[
\frac{1}{S_t} = \frac{\pi_0}{1 - \pi_0} \frac{1 - \pi_t}{\pi_t} \frac{1}{S_0}
\]  

(65)

which inserted into the equation of \( d\pi_t \) above gives

\[
d\pi_t = \sigma \frac{\pi_0}{1 - \pi_0} \frac{1 - \pi_t}{\pi_t} \frac{1}{S_0} \pi_t (1 - \pi_t) d\bar{W}_t \\
= K_0 (1 - \pi_t)^2 d\bar{W}_t
\]  

(66)

with the constant \( K_0 = \sigma \frac{\pi_0}{1 - \pi_0} \frac{1}{S_0} \) which give the infinitesimal generator

\[
\mathcal{A}_\pi = \frac{1}{2} K_0^2 (1 - \pi_t)^4 \frac{\partial^2}{\partial \pi^2}
\]  

(67)

and the ODE
\frac{1}{2} K_0^2 (1 - \pi)^4 \frac{\partial^2}{\partial \pi^2} V(\pi) = -1 \tag{68}

which has the solution

\[ V(\pi) = K_1 \pi + K_2 - \frac{1}{3 K_0^2 (1 - \pi)^2} \tag{69} \]

for some constants \( K_1 \), and \( K_2 \). We also have its derivative

\[ \frac{\partial V}{\partial \pi}(\pi) = K_1 - \frac{2}{3 K_0^2 (1 - \pi)^3} \tag{70} \]

where we can plug in \( \frac{\partial V}{\partial \pi}(B) = -b \) to get

\[ K_1 = \frac{2}{3 K_0^2 (1 - B)^3} - b \tag{71} \]

Plugging \( K_1 \) back gives

\[ V(\pi) = K_2 + \frac{1}{3 K_0^2} \left( \frac{2 \pi}{(1 - B)^3} - \frac{1}{(1 - \pi)^2} \right) - b \pi \tag{72} \]

Now with \( V(B) = b(1 - B) \) we get \( K_2 \) that we plug back to get

\[ V(\pi) = \frac{1}{3 K_0^2} \left( \frac{2 \pi}{(1 - B)^3} + \frac{1 - 3 B}{(1 - B)^3} - \frac{1}{(1 - \pi)^2} \right) + b(1 - \pi) \tag{73} \]

which has the derivative

\[ \frac{\partial V}{\partial \pi}(\pi) = \frac{2}{3 K_0^2} \left( \frac{1}{(1 - B)^3} - \frac{1}{(1 - \pi)^3} \right) - b \pi \tag{74} \]

into which we can plug

\[ V(A) = aA \]
\[ \frac{\partial V}{\partial \pi}(A) = a \tag{75} \]

to solve for \( A \) and \( B \). We have that for values of \( a \) larger than some constant that \( A \) must be equal to zero and that \( \frac{\partial V}{\partial \pi}(A) < a \). Here again we can solve for three different values of \( a \) and \( b \) as shown in Figure (5) where \( K_0 = 1 \).
Figure 5: Graph of $V(\pi)$ used to determine if the process is a Bessel process or a Brownian Motion. The dotted lines is the bounding maximal misclassification error function. Here for three different choices of $a$ and $b$. In all three graphs we have $K_0 = 1$.

Here we see that for two of the graphs we have that $A$ is zero. We also see that when the constant $a$ increases we will get that $A$ is zero since we tolerate less mistake in the determination and therefore need to wait until it falls below zero to be sure that the process is not a Bessel process. It is also interesting to think how the starting probability $\pi_0$ affects the bounds $A$ and $B$ (in $\pi$ space) as we have the constant $K_0$ (that depends on $\pi_0$ proportionally to $\frac{\pi_0}{1-\pi_0}$) that affects the form of $V(\pi)$. In the above figure we have $K_0 = 1$ giving that $\pi_0 = 0.5$ if $S_0 = 1$ but if we vary the value of $\pi_0$ we get different bounds as seen in Figure (6).

Figure 6: The stopping bounds $A$ and $B$ (black lines where $A$ is left of $B$) for different initial probabilities $\pi_0$, the blue line is the diagonal $\pi = \pi_0$ plotted for comparison. Here for three different choices of $a$ and $b$.

If comparing where the blue diagonal is in relation to the bounds we see that if we have a low initial probability $\pi_0$ we are in the stopping set of a Brownian motion. This is not that unexpected as we do not believe that the process is a Bessel process and are likely
to make a decision given any reason. In the same way a large initial probability of a Bessel process is close to the stopping set for a Bessel process (for some values of $a$ and $b$, inside it). We could also vary the starting position of the process as the constant $K_0$ is equal to $\frac{1}{S_0}$ when $\pi_0 = 0.5$. This we can see for different values of $a$ and $b$ in Figure (7).

![Figure 7: The stopping bounds $A$ and $B$ (black lines where $A$ is left of $B$) for different starting point $S_0$ of the process. Here for three different choices of $a$ and $b$.](image)

Here we see that as $S_0$ increases the stopping bounds come closer together, this due to that if we are starting farther from zero the drift will be small and a Bessel process will behave similarly to a Brownian motion. Then the cost of waiting will be large if one has to wait until it reaches zero (or close to it) to be able to assume that the process is a Brownian motion, then it might be better to make a quick decision to avoid the cost of waiting for a decision. We also see that when $a$ and $b$ are larger the stopping bounds are less close to each other as one can afford to wait to make a better guess.
5 Discussion

We have derived conditions on the stochastic process $S_t$ in Theorem (2) which enable us to see if, for a process with specific options of the drifts and the diffusion, the $\pi_t$ process can be given as a function of time and the current value of $S_t$. Here we looked at two different processes where both has constant diffusion and one possible drift was zero giving a Brownian motion. For the other drift possibility for one process it was constant, giving a Brownian motion with drift, and for the other process the drift was inversely proportional to $S_t$, giving a Bessel process.

To be able to write $\pi_t$ as a function of time and the current value of $S_t$ is helpful when, as done here, trying to find which drift the process has. This can have applications where one’s action differs depending on the drift. If the process is a stock we like to buy if we can assume that the drift is large to be able to sell at a profit later. If we are able to make an educated guess at the drift this can be use when taking the decision whether to buy the stock. Here we used optimal stopping to find the bounds where we could assume the form of the drift. In general this problem will depend on both the $S_t$ process and the $\pi_t$ process but for processes where $\pi_t$ can be written as a function of current value of $S_t$ and $t$, we can reduce the problem to depend on only $S_t$ or $\pi_t$. 

25
References


A

Calculation of \( \pi_t(S_t) \) for two states

We will here present the calculations behind the result in Theorem (2). We start by assuming that we have a process with unknown drift, given by the SDE

\[
dS_t = A(t, \mu, S_t)dt + B(t, S_t)dW_t
\]

\[S_0 = s_0\]  \hspace{1cm} (76)

where \( \mu \) is a random variable making the drift unknown. We now assume that \( \mu \) can take the values 1 or 2, where we assume that \( A(t, 1, S_t) = F_1(t, S_t) \) and \( A(t, 2, S_t) = F_2(t, S_t) \). Let \( \pi \) be \( P(\mu = 2) \) and the estimation of \( \pi \) be \( \pi_t = P(\mu = 2|F_t^S) \). Then using Theorem (1) we have

\[
\bar{A}(t, S_t) = A(t, 1, S_t)(1 - \pi_t) + A(t, 2, S_t)\pi_t = F_1(t, S_t)(1 - \pi_t) + F_2(t, S_t)\pi_t
\]

\[
dW_t = \frac{dS_t - \bar{A}(t, S_t)dt}{B(t, S_t)} = \frac{dS_t - (F_1(t, S_t)(1 - \pi_t) + F_2(t, S_t)\pi_t)dt}{B(t, S_t)}\]  \hspace{1cm} (77)

This gives

\[
d\pi_t = \pi_t \frac{F_2(t, S_t) - \bar{A}(t, S_t)}{B(t, S_t)} d\bar{W}_t
\]

\[
= \pi_t \frac{F_2(t, S_t) - (F_1(t, S_t)(1 - \pi_t) + F_2(t, S_t)\pi_t)}{B(t, S_t)} d\bar{W}_t
\]

\[
= \frac{F_2(t, S_t) - F_1(t, S_t)}{B(t, S_t)} \pi_t (1 - \pi_t) d\bar{W}_t
\]

\[
(78)
\]

and

\[
dS_t = (F_1(t, S_t)(1 - \pi_t) + F_2(t, S_t)\pi_t)dt + B(t, S_t)dW_t
\]

\[
(79)
\]

I we assume that \( \pi_t = H(t, S_t) \) we can by Ito’s formula

\[
d\pi_t = H_t dt + H_s dS_t + \frac{1}{2} H_{ss}(dS_t)^2
\]

\[
= H_t dt + H_s ((F_1(t, S_t)(1 - \pi_t) + F_2(t, S_t)\pi_t)dt + B(t, S_t)d\bar{W}_t) + \frac{1}{2} H_{ss} B(t, S_t)^2 dt
\]

\[
= (H_t + H_s (F_1(t, S_t)(1 - \pi_t) + F_2(t, S_t)\pi_t) + \frac{1}{2} H_{ss} B(t, S_t)^2) dt + H_s B(t, S_t) d\bar{W}_t
\]

\[
(80)
\]

where we can compare with the formula for \( d\pi_t \) in equation (78) to get the equations

\[
0 = H_t + H_s (F_1(t, S_t)(1 - \pi_t) + F_2(t, S_t)\pi_t) + \frac{1}{2} H_{ss} B(t, S_t)^2
\]

\[
H_s B(t, S_t) = \frac{F_2(t, S_t) - F_1(t, S_t)}{B(t, S_t)} \pi_t (1 - \pi_t)
\]

\[
(81)
\]
Here the last equation can be worked on to give

\begin{align*}
H_s B(t, s) &= \frac{F_2(t, s) - F_1(t, s)}{B(t, s)} \pi_t (1 - \pi_t) \\
H_s B(t, s) &= \frac{F_2(t, s) - F_1(t, s)}{B(t, s)} H(1 - H) \\
\frac{\partial H}{\partial s} B(t, s) &= \frac{F_2(t, s) - F_1(t, s)}{B(t, s)} H(1 - H) \\
\frac{\partial H}{\partial s} &= \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} H(1 - H) \\
\frac{\partial H}{H(1-H)} \partial H &= \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s \\
\int_{\pi_0}^{\pi_t} \frac{\partial H}{H(1-H)} \partial H + \int_{\pi_0}^{\pi_t} \frac{\partial H}{1-H} &= \frac{S_t}{S_0} \int_{S_0}^{S_t} \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s \\
\ln(\frac{\pi_t}{1-\pi_t}) + C(t) &= \frac{S_t}{S_0} \int_{S_0}^{S_t} \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s
\end{align*}

(82)

We can rearrange this to get

\begin{align*}
\exp(\ln(\frac{\pi_t}{1-\pi_t})) &= \exp\left(\int_{S_0}^{S_t} \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s - C(t)\right) \\
\frac{\pi_t}{1-\pi_t} &= \exp(-C(t)) \exp\left(\int_{S_0}^{S_t} \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s\right) \\
\pi_t &= (1 - \pi_t) \exp(-C(t)) \exp\left(\int_{S_0}^{S_t} \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s\right) \\
\pi_t (1 + \exp(-C(t)) \exp\left(\int_{S_0}^{S_t} \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s\right)) &= \exp(-C(t)) \exp\left(\int_{S_0}^{S_t} \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s\right) \\
\pi_t &= \frac{\exp(-C(t)) \exp\left(\int_{S_0}^{S_t} \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s\right)}{1 + \exp(-C(t)) \exp\left(\int_{S_0}^{S_t} \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s\right)} \\
\pi_t &= \frac{\exp(C(t)) \exp\left(\int_{S_0}^{S_t} \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s\right)}{1 + \exp(-C(t)) \exp\left(\int_{S_0}^{S_t} \frac{F_2(t, s) - F_1(t, s)}{B(t, s)^2} \partial s\right) + 1}
\end{align*}

(83)

Now identify

\begin{align*}
K(t) &= \exp(C(t)) \\
D(t, s) &= -\int_{S_0}^{s} \frac{F_2(t, u) - F_1(t, u)}{B(t, u)^2} \partial u
\end{align*}

(84)

we get

\begin{align*}
\pi_t(S_t) &= \frac{1}{K(t) \exp(D(t, S_t)) + 1}
\end{align*}

(85)

First the differentials of \( H \) are calculated

\begin{align*}
H_t &= -\frac{K \exp(D) + D_t K \exp(D)}{(K \exp(D) + 1)^2} \\
H_s &= -\frac{D_s K \exp(D)}{(K \exp(D) + 1)^2} \\
H_{ss} &= -\frac{D_{ss} K \exp(D) + D_s^2 K \exp(D) + 2 D_s K^2 \exp(2D)}{(K \exp(D) + 1)^3} + 2 D_s^2 K^2 \exp(2D)
\end{align*}

(86)
Now this can be used for the second formula of equation (81)

\[
0 = H_t + H_s(F_1(1 - H) + F_2 H) + \frac{1}{2} H_s B^2 \\
0 = -\frac{K_e \exp(D) + D_s K \exp(D)}{(K \exp(D) + 1)^2} + \frac{D_s K \exp(D)}{(K \exp(D) + 1)^2} (F_1 K \exp(D) + F_2 K \exp(D) + 1) + \\
\frac{1}{2} (\frac{D_s K \exp(D) + D_s^2 K \exp(D)}{(K \exp(D) + 1)^2} + 2 \frac{D_s K^2 \exp(2D)}{(K \exp(D) + 1)^2}) B^2 \\
0 = -\frac{B^2 D_s^2 K \exp(D) - B^2 D_s \exp(D) - K_t - D_t K + B^2 D_s^2 K \exp(D) - D_s K \exp(D) F_1 - D_s K F_2}{K \exp(D) + 1} \\
0 = -\frac{B^2 D_s^2 K - B^2 D_s \exp(D) - K_t - D_t K + D_s K F_2}{F_2} \\
0 = -\frac{B^2 D_s^2 K - B^2 D_s \exp(D) - K_t - D_t K - D_s K F_2}{K \exp(D) + 1}
\]

Now looking at \(\frac{B^2 D_s^2 K - D_s K F_1}{D_s K F_2}\), we can use that \(D_s = \frac{F_1 - F_2}{B^2}\) to do

\[
\frac{B^2 D_s^2 K - D_s K F_1}{D_s K F_2} = \frac{B^2 F_1 - F_2 - F_1}{F_1 - F_2 - F_1} - 1
\]

Now returning to the previous equation

\[
0 = -\frac{B^2 D_s^2 K - B^2 D_s \exp(D) - K_t - D_t K + D_s K F_2}{K \exp(D) + 1} \\
0 = -\frac{B^2 D_s^2 K - B^2 D_s \exp(D) - K_t - D_t K + D_s K F_2}{K \exp(D) + 1} \\
0 = -\frac{B^2 D_s^2 K - B^2 D_s \exp(D) - K_t - D_t K - D_s K F_2}{K \exp(D) + 1} \\
K_t = (\frac{-B^2 D_s^2}{B^2 D_s} - \frac{B^2 D_s}{B^2 D_s} - D_t - D_s F_2) K \\
K_t = -\frac{B^2 D_s^2}{B^2 D_s} + \frac{B^2 D_s}{B^2 D_s} + D_t + D_s F_2)
\]

Now this gives a ODE where \(\frac{B^2 D_s^2}{B^2 D_s} + \frac{B^2 D_s^2}{B^2 D_s} + D_t + D_s F_2\) must be equal to a constant of function depending only on \(t\) and this puts a constraint on the possible forms of \(F_1\), \(F_2\) and \(B\). We also see that the form of \(K(t)\) must be so that \(K(0) = \frac{1 - \pi_0}{\pi_0}\).