The CHY Representation
Application on Tree Level YM+$\phi^3$ Amplitudes
# List of Figures

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.1</td>
<td>Relations between kinematic factors of different theories in CHY representation. The figure was taken from [CHY14].</td>
</tr>
<tr>
<td>3.2</td>
<td>The output of the Mathematica implementation for $\Pi(1, 2, 3, 4)$.</td>
</tr>
<tr>
<td>3.3</td>
<td>The output of the Mathematica implementation for $\text{Pf}^r \Pi(1, 2, 3, 4)$.</td>
</tr>
<tr>
<td>3.4</td>
<td>The output of the Mathematica implementation for the Half-Integrand $C_2(3, 4) Pf^r \Pi(1, 2, 3, 4)$.</td>
</tr>
<tr>
<td>3.5</td>
<td>Resulting Amplitude</td>
</tr>
</tbody>
</table>
List of Abbreviations

**EYM**  Einstein-Yang-Mills

**YM**  Yang-Mills

**CHY**  Cachazo-He-Yuan

**CK**  Color-kinematic

**BCJ**  Bern-Carrasco-Johansson

**KLT**  Kawai-Lewellen-Tye

**DDM**  Del Duca-Dixon-Maltoni

**KK**  Kleiss-Kuijf
In many gravitational theories the scattering amplitudes exhibit a double copy structure. They can be constructed from gauge-theory building blocks. It has been observed that such a construction exists for Einstein-Yang Mills (EYM) theory amplitudes. The gauge theory constituents of the double-copy constructions of these amplitudes are pure Yang-Mills (YM) theory and Yang-Mills theory coupled to a biadjoint scalar (YM+ϕ³) \cite{Chi+14}.

EYM = YM ⊗ YM + ϕ³

The aim of the Masters thesis ensuing this report will be to do new one-loop computations of EYM with the Cachazo-He-Yuan (CHY) representation. In the CHY representation scattering amplitudes are expressed as integrals over the moduli space of an \(n\) punctured Riemann sphere. It has remarkable connections to the double-copy construction.

Constituting the first part of the Master-thesis project, the aim of the research project in Uppsala was to calculate some of the YM+ϕ³ tree-level amplitudes, first from Feynman diagrams and then using the CHY representation.

This report summarises the theory behind these methods specifically the CHY representation and shows a simple result as an example.

The first chapter contains a very short review about the color-kinematics (CK) duality and the double-copy construction.

In the second chapter the CHY integral for tree-level amplitudes is explained. First, the scattering equations that are a central part of the CHY measure are introduced and identities showing a connection to the double-copy construction are presented. Then, CHY integrands for different theories are discussed. In the end of the chapter a simple example for the application of CHY for a YM+ϕ³ tree-level amplitude is given.

In the Conclusion an outlook on the next steps of the Master thesis project brings the report to an end.
CHAPTER 2

The double-copy construction - a very short review

The rather short and conceptual review in the following chapter is mainly based on the review [Ber+19] and partly on the report [Wei16].

Basic Concepts

The CK duality is satisfied for a theory if there is a representation such that for diagrams with at least cubic vertices in the perturbative expansion of the amplitudes a one-to-one map between the Lie Algebra identities of the color factors and identities for the kinematic factors exist [BCJ08]. Originally CK-duality was found to be satisfied by YM theory tree-level amplitudes. Since then it has been generalised to many gauge theories and others like NLSM and even string theories.

If the duality is manifest in a gauge theory amplitude, its constituents can be used as building blocks for the double copy construction. With this construction a gravity theory amplitude can be built from two gauge theory amplitudes.

\[
\text{gravity theory} = \text{gauge theory} \otimes \text{gauge theory}
\]

More concretely this means: An \( m \) point \( L \)-loop scattering amplitude in a \( D \) dimensional non-abelian gauge theory that satisfies CK duality can be organised as a sum over cubic-vertex graphs

\[
\mathcal{A}_m^L = i^{L-1}g^{m-2+2L} \sum_i \int \frac{d^{LD} l}{(2\pi)^{LD}} \frac{1}{S_i} \frac{c_i n_i}{D_i} 
\]

where \( S_i \) are symmetry factors, \( D_i \) are products of the Feynman propagators of the internal lines and \( l \) are loop momenta. The color-factors \( c_i \) satisfy identities derived form the Jacobi identity and due to the CK duality the kinematic factors \( n_i \) can conjecturally be made to satisfy similar relations. From two such amplitudes \( A_m^L \) and \( \tilde{A}_m^L \) with the same color-factors we can build a gravity amplitude with the double-copy construction by replacing the color factor \( c_i \) of \( A_m^L \) with the kinematic factor \( \tilde{n}_i \) of \( \tilde{A}_m^L \). The result
is the double-copy amplitude $\mathcal{M}_m$:

$$\mathcal{M}_m^L = A^L_m |_{c_i \to \tilde{n}_i, g \to \tilde{n}} = i^{L-1} \left( \frac{\kappa}{2} \right)^{m-2+2L} \sum_i \int \frac{d^{L+1}1}{(2\pi)^L} \frac{1}{S_i} \frac{n_i \tilde{n}_i}{D_i}. \quad (2.2)$$

It has been shown that many gravity theories can be constructed with this method. A simple example is Einstein Gravity coupled to a B-field and a dilaton, which can be constructed from two YM theories. Another example is Einstein Yang-Mills Theory. EYM amplitudes can be constructed from a pure Yang-Mills (YM) theory and a Yang-Mills theory minimally coupled to a biadjoint scalar (YM+φ³).

In the following paragraphs, two important sets of identities in the context of the double copy construction are introduced: the fundamental Bern-Carrasco-Johansson (BCJ) and the Kawai-Lewellen-Tye (KLT) relations.

### BCJ Relations

Tree level amplitudes in an adjoint-only theory satisfying the CK duality, like YM theory, can be written in the DDM color decomposition [DDM99]:

$$A^\text{tree}_m = g^{m-2} \sum_{\sigma \in S_{m-2}} A^\text{tree}_m (1, \sigma(2), \ldots, \sigma(m-1), m) \left( f^{\alpha(2)} f^{\alpha(3)} \ldots f^{\alpha(m-1)} \right)_{a_1 a_m}.$$

where the color-factor is a product of the gauge group structure constants. The kinematic factors are the objects $A^\text{tree}_m (1, \sigma(2), \ldots, \sigma(m-1), m)$, called the partial amplitudes. We denote by $p_i$ the lightlike external momenta. The partial amplitudes satisfy the fundamental BCJ relations

$$\sum_{i=2}^{m-1} p_1 \cdot (p_2 + \ldots + p_i) A^\text{tree}_m (2, \ldots, i, 1, i + 1, \ldots, m) = 0. \quad (2.3)$$

The additional $(m-2)! - (m-3)!$ constraints imposed by these relations reduce the number of independent partial amplitudes to $(m-3)!$.

### KLT Relations

The KLT relations were derived from string theory [KLT86]. They give an explicit expression for the double-copy tree-level amplitudes in terms of the color-ordered partial amplitudes of two gauge theories that satisfy CK duality and the KLT kernel. The $m$-particle KLT relation is [Ber+98, KLT86]:

$$\mathcal{M}^\text{tree}_m = -i \sum_{\sigma, \rho \in S_{m-3}(2, \ldots, m-2)} A^\text{tree}_m (1, \sigma, m-1, m) S[\sigma | \rho] \tilde{A}^\text{tree}_m (1, \rho, m, m-1). \quad (2.4)$$
The KLT kernel $S[\sigma|\rho]$ is a function of the momenta defined as
\[
S[\sigma|\rho] = \prod_{i=2}^{m-2} \left[ 2p_1 \cdot p_{\sigma_i} + \sum_{j=2}^{i} 2p_{\sigma_i} \cdot p_{\sigma_j} \theta(\sigma_j, \sigma_i)_\rho \right]. \tag{2.5}
\]

The function $\theta(\sigma_j, \sigma_i)_\rho$ is defined, such that $\theta(\sigma_j, \sigma_i)_\rho = 1$ if $\sigma_j$ is before $\sigma_i$ in the permutation $\rho$ and it is zero otherwise. These identities have a close relation with the KLT orthogonality of Park-Taylor factors in the context of the CHY representation presented in the following chapter.
CHAPTER 3

The CHY representation for tree-level amplitudes

The CHY representation was developed by Freddy Cachazo, Song He and Ellis Ye Yuan. A general n-point tree level amplitude of massless particles in CHY representation takes the form of an integral over the moduli space of n-punctured Riemann spheres\cite{Yua15, CHY14}:

\[
M_n = \int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta \left( \sum_{b=1, b \neq a}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}} \right) I_n(\{k, \varepsilon, \sigma\}) =: \int d\mu I_n(\{k, \varepsilon, \sigma\}). \tag{3.1}
\]

The locations of the punctures on the sphere are denoted by \(\sigma_i\) and \(\sigma_{ij} := \sigma_i - \sigma_j\). The products denoted by \(\prod_{a=1}^{n} \) are defined as \(\prod_{a=1}^{n} := (\sigma_{ij}\sigma_{jk}\sigma_{ki}) \prod_{a=1, a \neq i, j, k}^{n}\). The indices \(i, j, k\) that are left out of the product can be chosen freely among all of the indices. The shorthand \(\int d\mu_n\) was introduced for \(\int \prod_{a=1}^{n} d\sigma_a \prod_{a=1}^{n} \delta \left( \sum_{b=1, b \neq a}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}} \right)\). The different parts of this formula are discussed in more detail in the following sections, starting with the scattering equations that are part of the CHY measure and the integrands for different theories leading up to an explicit example in YM+\(\phi^3\) theory.

3.1. The Scattering Equations

The delta functions

\[
\delta \left( \sum_{b=1, b \neq a}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}} \right) \tag{3.2}
\]

in the CHY measure impose the scattering equations

\[
\sum_{b=1, b \neq a}^{n} \frac{k_a \cdot k_b}{\sigma_{ab}} = 0 \text{ for } a \in \{1, ..., n\}. \tag{3.3}
\]

In the context of field theory scattering amplitudes they were first studied by Cachazo\cite{Cac12}. They can be obtained by constructing a correspondence between the \(n\) punctures on a Riemann sphere and the \(n\) external momenta of a scattering amplitude and
imposing specific constraints \cite{CHY13a,Yua15}. Only \((n-3)\) of the scattering equations are linearly independent. Three of the puncture locations can be fixed using the \(SL(2, \mathbb{C})\) invariance of the equations. From a solution of the scattering equations different Park Taylor factors

\[
C_n(\omega(1), \ldots, \omega(n)) = \frac{1}{(\sigma_\omega(1) - \sigma_\omega(2)) \cdots (\sigma_\omega(n-1) - \sigma_\omega(n)) (\sigma_\omega(n) - \sigma_\omega(1))}
\]

can be built with different permutations \(\omega \in S_n\). Here we denote the Park-Taylor factor including \(n\) punctures as \(C_n\). Later in the report, the index \(n\) will often be left out, when the number of \(\sigma_i\) is obvious or does not need to be stressed. The Park-Taylor factors are cyclic. Furthermore they satisfy the Kleiss-Kuijf (KK) relations \cite{CHY13b}:

\[
C_n(1, A, n, B) = (-1)^{|B|} \sum_{\sigma \in A \shuffle B^t} C_n(1, \sigma, n).
\]

\(B^t\) is the transposed word \(B\). The shuffle symbol \(\shuffle\) is defined inductively by

\[
u \shuffle E = E \shuffle \nu = \nu
\]

\[
u a \shuffle \nu b = (\nu \shuffle \nu b) a + (\nu a \shuffle \nu) b
\]

where \(E\) symbolises the empty word, \(a\) and \(b\) are words of length 1 and \(v\) and \(u\) are arbitrary words. The KK relations allow for a basis of \((n-2)!\) Parke-Taylor factors \(C_n(1, \ldots, n)\) with positions 1 and \(n\) fixed. Additionally, the Parke-Taylor factors on the support of the scattering equations satisfy the fundamental BCJ relations:

\[
\sum_{i=2}^{n-1} p_1 \cdot (p_2 + \ldots + p_i) C_n(2, \ldots, i, 1, i+1, \ldots, n) = 0.
\]

Due to the \((n-2)\) additional constraints contributed by these relations, the Parke-Taylor factors can be expressed in a basis of \((n-3)!\) factors with positions 1, \(n-1\) and \(n\) fixed.

An important property of the Parke-Taylor factors built from the solutions of the scattering equations is KLT orthogonality \cite{CHY13a}:

\[
\sum_{\sigma, \rho \in S_{n-3}} C_n^{(i)}(1, \sigma, n-1, n) \kappa[\sigma|\rho] C_n^{(j)}(1, \rho, n, n-1) = \delta^{ij} (\text{Det}' \phi_n))^{(i)}
\]

with the KLT kernel on the LHS of the relation defined as in \cite{2.5} and the Jacobian on the RHS defined as

\[
\text{Det}' \phi_n := \frac{\text{Det}[\phi_n]_{ijk}}{\sigma_{ab} \sigma_{bc} \sigma_{ca} \sigma_{ij} \sigma_{jk} \sigma_{kl}} \quad \text{with} \quad (\phi_n)_{a,b} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}} & a \neq b \\ -\sum_{c \neq a} \frac{k_a \cdot k_c}{\sigma_{ac}} & a = b \end{cases}
\]

The notation \(C_n^{(i)}\) signifies that the \(i\)th solution of the scattering equations is used.

Let \(A_n^\alpha\) and \(A^\beta_n\) be two partial \(n\)-point tree-level amplitudes in a non-abelian gauge theory that satisfy the CK duality. In CHY representation their integrands consist of
two half-integrands, a Parke-Taylor factor $C_n(\{\sigma_i\})$ and the kinematic half-integrand $E_n(\{\sigma_i, k_i, \varepsilon_i\})$ that contains the kinematic data, the external momenta and the transverse polarizations $\varepsilon_i$, from the scattering process.

\[ I^A_n = C_n(\{\sigma_i\}) E_n^a(\{\sigma_i, k_i, \varepsilon_i\}) \quad \text{and} \quad I^B_n = C_n(\{\sigma_i\}) E_n^b(\{\sigma_i, k_i, \varepsilon_i\}). \] (3.6)

The n-point KLT relation [2.4] with the above partial amplitudes as building blocks reads:

\[ \sum_{\alpha\beta \in S_{n-3}} A_n^a[1, \alpha, n-1, n] S[\alpha|\beta] A_n^b[1, \beta, n, n-1]. \] (3.7)

Using the integrands in (3.6) as well as the CHY integral expressed as a sum over the solutions of the scattering equations

\[ \sum_{i=1}^{(n-3)!} \frac{\tau^{(i)}_n}{(\text{Det}^\prime \phi_n)^{n}} \] we put the CHY representation of the partial amplitudes into 3.8 and obtain:

\[ \sum_{\alpha\beta} \sum_{i=1}^{(n-3)!} C_n^{(i)}[1, \alpha, n-1, n] E_n^a(\{\sigma_i, k_i, \varepsilon_i\}) \frac{(n-3)!}{(\text{Det}^\prime \phi_n)^{(i)}} S[\alpha|\beta] \sum_{j=1}^{(n-3)!} C_n^{(j)}[1, \beta, n, n-1] E_n^b(\{k, \varepsilon\}). \]

Applying the KLT orthogonality (3.5) reduces this expression to

\[ \int d\mu_n \ E_n^a E_n^b = \mathcal{M}_n = \sum_{\alpha\beta \in S_{n-3}} A_n^a[1, \alpha, n-1, n] S[\alpha|\beta] A_n^b[1, \beta, n, n-1]. \] (3.8)

3.2. The CHY-Integrand

We have already established that for a color-decomposed partial amplitude in a non-abelian gauge theory the CHY integrand is a product of a Parke-Taylor factor and a kinematic half-integrand that contains the momenta and polarizations of the scattering process.

The YM-theory CHY-integrand is [Yua15] [CHY14]

\[ I_{YM}^n(1, 2, ..., n) := C(1, 2, ..., n) E_n(\{\sigma_i, k_i, \varepsilon_i\}) = \frac{E_n(\{\sigma_i, k_i, \varepsilon_i\})}{\sigma_1 \sigma_2 \sigma_3 \cdots} \] (3.9)

where the kinematic half-integrand is defined as $E_n(\{\sigma_i, k_i, \varepsilon_i\}) := \text{Pf}^\prime \psi_n(\{\sigma_i, k_i, \varepsilon_i\})$ with the matrix $\psi_n$ being

\[ \psi_n := \begin{pmatrix} A & -C^T \\ C & B \end{pmatrix} \] (3.10)
where
\[ A_{ab} = \begin{cases} \frac{k_a \cdot k_b}{\sigma_{ab}}, & a \neq b \\ 0, & a = b \end{cases} \quad B_{ab} = \begin{cases} 0, & a \neq b \\ \sum_{c=1}^{n} \frac{z_{ac} k_c}{\sigma_{ab}} - \frac{z_{ab} k_b}{\sigma_{ab}}, & a = b \end{cases} \quad C_{ab} = \begin{cases} 0, & a \neq b \\ \frac{\epsilon_a \cdot \epsilon_b}{\sigma_{ab}}, & a = b \end{cases} \]

The primed Pfaffian Pf' \psi is defined as:
\[
Pf' \psi = 2 \frac{(-1)^{i+j}}{\sigma_{ij}} Pf([[\psi]_{ij}]^2).
\]

In the last definition, the notation [[\psi]_{ij}] is used to express the matrix that results from the matrix \psi, when its \(i^{th}\) and \(j^{th}\) rows and columns are deleted.

In addition to the YM theory integrand we can already construct integrands for two more theories just from simple combinations of Parke-Taylor factors and the kinematic factor of YM theory. Combining two Parke-Taylor factors results in the integrand for a scalar theory with cubic vertices:
\[
T_{n}^{\text{Scalar}} := C_n(1, 2, \ldots, n) C_n(1, 2, \ldots, n).
\]

The CHY integrals resulting from this expression are the doubly partial amplitudes. These are useful for the calculation of other CHY integrals. The combination of two kinematic half-integrands generates the integrand for Einstein gravity:
\[
T_{n}^{\text{Einstein}} := E_n(\{\sigma_i, k_i, \varepsilon_i\}) E_n(\{\sigma_i, k_i, \varepsilon_i\}).
\]

Beyond these, many (gravity) theories can be constructed in the CHY representation. Some examples of different theories that can be constructed in the CHY representation and their integrands are presented in the following table [Yua15]:

<table>
<thead>
<tr>
<th>Theory</th>
<th>Integrand</th>
</tr>
</thead>
<tbody>
<tr>
<td>Einstein gravity</td>
<td>Pf' \psi_n Pf' \psi_n</td>
</tr>
<tr>
<td>Yang-Mills</td>
<td>C_n Pf' \psi_n</td>
</tr>
<tr>
<td>(\phi^3) flavored in (U(N) \times U(\tilde{N}))</td>
<td>C_n C_n</td>
</tr>
<tr>
<td>Einstein-Maxwell</td>
<td>Pf[\chi_n]<em>\gamma Pf' [\psi_n]</em>\gamma Pf' \psi_n</td>
</tr>
<tr>
<td>Einstein-Yang-Mills</td>
<td>C_{tr_i} C_{tr_i} Pf' \Pi(h, tr_1, \ldots, tr_i) Pf' \psi_n</td>
</tr>
<tr>
<td>Yang-Mills-Scalar</td>
<td>C_n Pf[\chi_n] Pf' [\psi_n]_s</td>
</tr>
<tr>
<td>Generalized Yang-Mills-Scalar</td>
<td>C_n C_{tr_i} C_{tr_i} Pf' (g, tr_1, \ldots, tr_i)</td>
</tr>
<tr>
<td>Born-Infeld</td>
<td>Pf' \psi_n (Pf' A_n)^2</td>
</tr>
<tr>
<td>Dirac-Born-Infeld</td>
<td>Pf[\chi_n]_s Pf'[\psi_n]_s (Pf' A_n)^2</td>
</tr>
<tr>
<td>extended Dirac-Born-Infeld</td>
<td>C_{tr_i} \ldots C_{tr_i} Pf' \Pi(\gamma, tr_1, \ldots, tr_i) (Pf' A_n)^2</td>
</tr>
<tr>
<td>U(N) non-linear sigma model</td>
<td>C_n (Pf' A_n)^2</td>
</tr>
<tr>
<td>special Galileon</td>
<td>(Pf' A_n)^4</td>
</tr>
</tbody>
</table>

The crucial step in the construction of the tree-level amplitudes in these theories is the construction of the kinematic half-integrands, like \(\Pi\). Many of those are the modified Pfaffians of matrices that can be obtained from the matrix \psi and each other with certain modifications (squeezing, compactifying, generalizing; more details in [CHY14]). The following figure represents how certain theories are related to each other by such modifications of their kinematic factors is presented in the following figure:
3.3. Example: YM+$\phi^3$

YM+$\phi^3$ theory is defined by the Lagrangian \[ \text{Chi+14} \]:

\[
\mathcal{L}^{YM} := -\frac{1}{4} F_{\mu\nu}{}^a F^{a\mu\nu} \\

\mathcal{L}^{YM+\phi^3} := \mathcal{L}^{YM} + \frac{1}{2} (D_\mu \phi^A)^a (D^\mu \phi^A)^a \\
- \frac{g^2}{4} f^{abc} f^{cde} \phi^A \phi^B \phi^C \phi^D + \frac{\lambda g}{3!} F_{ABC} f^{abc} \phi^A \phi^B \phi^C
\]

with independent coupling constants $g$ and $\lambda$ and

\[
(D_\mu \phi^A)^a := \partial_\mu \phi^A_a + g f^{abc} A_{\mu}^b \phi^A_c \\
F_{\mu\nu}^c = \partial_\mu A_{\nu}^c - \partial_\nu A_{\mu}^c + g f^{abc} A_{\mu}^b A_{\nu}^c.
\]
It contains two kinds of particles, gluons and scalars. The indices of two gauge groups are distinguished by upper and lower case letters. The gauge group with the lower case letters attached to both kinds of particles and structure constants \( f^{abc} \) is the SU(N).

The CHY integrand for a single-trace tree-level YM+\(\phi^3\) amplitude of \(n\) particles, with gluons labeled with indices from \(g := \{1, \ldots, m\}\) and scalars labeled with indices from \(s := \{m + 1, \ldots, n\}\) is:

\[
\mathcal{I}_{\text{YM}+\phi^3} := \mathcal{C}_n(1, 2, \ldots, n)\mathcal{C}_{m-n}(m+1, \ldots, n)\text{Pf}'(1, \ldots, n)
\]  

(3.11)

The first Parke-Taylor factor \(\mathcal{C}_n(1, \ldots, n)\) is related to the color both kinds of particles carry and the second one, \(\mathcal{C}_{m-n}(m+1, \ldots, n)\) is related to the color, only the scalars carry. The kinematic half-integrand contains the kinematic data from both kinds of particles in the form of the Pfaffian of the matrix \(\Pi\), that can be constructed from the matrix \(\psi\).

We begin with the \((2m) \times (2m)\) matrix \(\psi\) for only the gluons only defined as in (3.10) by:

\[
\psi_m := \left( \begin{array}{ccc}
\begin{array}{c}
A_{ab} \\
\sum_{c \in s} k_c \cdot k_b 
\end{array} & -C_{ba} \\
C_{ab} & \sum_{c \in s} k_c \cdot k_b
\end{array} \right).
\]

The kinematic data from the additional scalars in the scattering process is inserted by adding two rows and two columns (More details in [CHY14].)

\[
\Pi := \left( \begin{array}{cccc}
A_{ab} & \sum_{c \in s} k_c \cdot k_b & -C_{ba} & \sum_{c \in s} k_c \cdot k_b \\
\sum_{c \in s} k_c \cdot k_b & 0 & \sum_{c \in s} k_c \cdot k_b & \sum_{c,d \in s} \epsilon_{a,b} \cdot k_c \cdot k_d \\
C_{ab} & \sum_{d \in s} \epsilon_{a,b} \cdot k_d & B_{ab} & \sum_{d \in s} \epsilon_{a,b} \cdot k_d \cdot k_b \\
\sum_{c,d \in s} \epsilon_{a,b} \cdot k_c \cdot k_d & \sum_{c \in s} k_c \cdot k_b & 0 & 1
\end{array} \right) \quad (3.12)
\]

The result is an antisymmetric \(2(m+1) \times 2(m+1)\) matrix containing the kinematic data of the scattering process.

The modified Pfaffian of the matrix \(\Pi\) can be calculated in several equivalent ways:

\[
\text{Pf}' \Pi := \text{Pf} \Pi(i^a) = \left( \frac{(-1)^a}{\sigma_a} \right) \text{Pf} \Pi(j^a) = \left( \frac{(-1)^a}{\sigma_a} \right) \text{Pf} \Pi(i^a) = \left( \frac{(-1)^{a+b}}{\sigma[a,b]} \right) \text{Pf} \Pi^{(ab)}.
\]

(3.13)

After the discussion of all relevant building blocks of CHY-representation YM+\(\phi^3\) integrands a simple example is calculated in the following paragraphs: the amplitude of a scattering with two external gluons and two external scalars. The CHY integrand for this amplitude, with the gluons labeled by 1, 2 and the scalars by 3, 4 is:

\[
\mathcal{C}_2(1, 2, 3, 4)\mathcal{C}_2(3, 4)\text{Pf}'(1, 2, 3, 4).
\]

(3.14)

The object \(\text{Pf}'(1, 2, 3, 4)\) was calculated using Mathematica. Relevant parts of implementation used to obtain the results given here are attached in appendix A.1. The result for the matrix \(\Pi(1, 2, 3, 4)\) is:
There are several methods to calculate CHY amplitudes, once the integrand has been obtained. One method, also used in [Nan+16], is the following procedure:

First, the integrand is organised in terms of products of two Parke-Taylor factors \( C(\sigma(1), ..., \sigma(n))C(\omega(1), ..., \omega(n)) \). For amplitudes of only a few particles this is often trivial. For more complicated cases with many particles identities derived from relations presented in the preceding sections, like the scattering equations and the KK-relations, can be used to obtain that structure. Then, each of the integrals has the form of a doubly partial amplitude of bi-adjoint scalars

\[
\int d\mu C(\sigma(1), ..., \sigma(n))C(\omega(1), ..., \omega(n)).
\]

These can be calculated using a Berends-Giele recursion as described in [Mat16] or from a polygon decomposition [CHY13c]. The Mathematica implementation of the Berends-Giele recursion that was used for the following results can be found in appendix A.2. The amplitude in the current example 3.14 has a common Parke-Taylor factor \( C(1, 2, 3, 4) \) in each term. So, we organise the half-integrand \( C_2(3, 4)Pf'II(1, 2, 3, 4) \), which is

in terms of different tensor structures and then organise functions of \( \sigma_i \) attached to each

Figure 3.2.: The output of the Mathematica implementation for \( II(1, 2, 3, 4) \).

Figure 3.3.: The output of the Mathematica implementation for \( Pf'II(1, 2, 3, 4) \).

Figure 3.4.: The output of the Mathematica implementation for the Half-Integrand \( C_2(3, 4)Pf'II(1, 2, 3, 4) \).
of these tensor structures as Parke-Taylor factors. For example: The terms proportional to $(\varepsilon_1 \cdot k_3)(\varepsilon_2 \cdot k_1)$ are

$\varepsilon_1 \cdot (c_1 - c_2) (c_1 - c_3) (c_2 - c_4) (c_3 - c_4)^2 \cdot (c_1 - c_2) (c_1 - c_4) (c_2 - c_4) (c_3 - c_4)^2$.

This sum is equivalent to $C(1, 2, 4, 3)$. So, the amplitude contains a term

$\frac{(\varepsilon_1 \cdot k_3)(\varepsilon_2 \cdot k_1)}{k_1 \cdot k_2} \int d\mu_n C(1, 2, 3, 4)C(1, 2, 4, 3) = -\frac{(\varepsilon_1 \cdot k_3)(\varepsilon_2 \cdot k_1)}{k_1 \cdot k_2}$

Applying this method to all tensor structures of the half-integrand results in the amplitude

Figure 3.5.: Resulting Amplitude

The expression $s[a, b, ...]$ denotes $(k_a + k_b + ...)^2$. For the conventions used in this report this means $s[a, b] = 2k_a \cdot k_b$. A Berends-Giele recursion for the YM+$\phi^3$ theory was performed and the results agreed up to normalisation. Some relevant aspects of the code for this recursion can be found in appendix A.2.
In chapter two of this report, the double-copy construction was reviewed. The main point of this construction is, that gravity theory amplitudes can be built from two copies of gauge theory amplitudes. This principle is also reflected in the construction of amplitudes in the CHY representation. In CHY representation an \( n \)-point tree-level scattering amplitude is calculated as an integral over the moduli space of \( n \) punctures spheres. A discussion of this method was presented in the third chapter of the report.

The last section of this chapter about the application of the CHY representation on YM+\( \phi^3 \) theory tree level amplitudes shows the groundwork for the Masters thesis. We have started calculating several YM+\( \phi^3 \) tree level amplitudes for up to 6 particles and checking the results with a Berends Giele recursion.

The next steps will be the calculation of one-loop YM+\( \phi^3 \) amplitudes using these results and the methods presented in [HS17] [Gey+15] [HSZ17] and then to obtain EYM amplitudes from a double copy.
Mathematica Implementation

A.1. YM+$\phi^3$ amplitude in CHY representation

The Pfaffian was implemented recursively as

\[
\text{amp}[\text{mat}_-, i_-, j_-] := \text{Transpose} \left[ \text{Delete} \left[ \text{Delete} \left[ \text{Transpose} \left[ \text{Delete} \left[ \text{Delete} \left[ \text{mat}, \text{Max}[i, j]], \text{Min}[i, j]]], \text{Max}[i, j]], \text{Min}[i, j]] \right] \right] \right] \text{Max}[i, j], \text{Min}[i, j]]
\]

\[
\text{pf}[1, \{\{a_1_, a_2\}, \{a_3_, a_4\}\}] := a_2
\]

\[
\text{pf}[n_, \text{mat}_-] := \sum (-1)^i \text{mat}[[1]][[i]] \text{pf}[n-1, \text{amp}[\text{mat}, 1, i]], \{i, 2, 2n\}
\]

and used in the implementation of the modified Pfaffian from 3.13 as

\[
\text{pfpi4}[\text{mat}_-, \text{anzahlh}_-, \text{anzahltraces}_-, \text{a}_-, \text{b}_-, \text{gluonsinab}_\text{List}] := ((-1)^{(a + b)} / \text{\Sigma} [\text{gluonsinab}[[a]], \text{gluonsinab}[[b]]])
\text{pf}[\text{anzahlh} + \text{anzahltraces} - 1, \text{amp}[\text{mat}, \text{a}, \text{b}]].
\]

The Parke-Taylor factors for a list of particles were put in as

\[
\text{per}[\text{listtparticles}_-, j_] := \text{Permutations}[\text{Delete}[\text{listtparticles}, 1]][[j]]
\]

\[
\text{parktaylor}[\text{listtparticles}_-] := 1 / \left( \text{Product} \left[ \left[ \text{\Sigma} \right] [\text{listtparticles}[[i]], \text{listtparticles}[[i + 1]]], \{i, 1, \text{Length} [\text{listtparticles}] - 1\} \right] \right) / \left[ \text{\Sigma} \right] [\text{listtparticles}[[\text{Length} [\text{listtparticles}]]], \text{listtparticles}[[1]]] / . \text{ordnung}
\]

\[
\text{parktaylortrace}[i_] := \text{parktaylor}[\text{elementstr}[i]]
\]

The components of the matrix $\psi$ from the kinematic factor of the YM-theory were implented as

\[
\text{Amatrixelement}[n_-, \text{alle}_-, \text{a}_-, \text{b}_-] := \text{If} [\text{a} \neq \text{b}], \left( \text{Dot} [\text{Subscript} [k, \text{a}], \text{Subscript} [k, \text{b}]] / \text{Subscript} [\left[ \text{\Sigma} \right] [\text{a}], \text{a}]ight) -
\]
Subscript \((\Sigma, b))\), 0 / . ordnung

amatrix\(_n\) := Table[amatrixelement\(_n, n, i, j\), \{i, 1, n\}, \{j, 1, n\}]

B

bmatrixelement\(_n, \text{alle}, a, b\) := If[a != b, (Dot[Subscript[\(\epsilon\), a], Subscript[\(\epsilon\), b]] / (Subscript[\(\Sigma\), a] - Subscript[\(\Sigma\), b])), 0] / . ordnung

bmatrix\(_n\) := Table[bmatrixelement\(_n, n, i, j\), \{i, 1, n\}, \{j, 1, n\}]

C
cmatrixelement\(_n, \text{alle}, a, b\) := If[a != b, 
(Dot[Subscript[\(\epsilon\), b], Subscript[\(\epsilon\), a]] / (Subscript[\(\Sigma\), a] - Subscript[\(\Sigma\), b])), 
- Sum[If[c == a, 0, 
(Dot[Subscript[\(\epsilon\), b], Subscript[k, c]] / (Subscript[\(\Sigma\), a] - Subscript[\(\Sigma\), c])), \{c, 1, \text{alle}\}]]

cmatrix\(_n, \text{alle}\) := Table[cmatrixelement\(_n, n, \text{alle}, i, j\), \{i, 1, n\}, \{j, 1, n\}]

The command / . ordnung

used several times in this procedure orders products of commuting variables such that the element with the smallest index is on the left side of the product. This makes it easier to find coefficients of specific tensor structures later.

In a similar way, the matrix \(\Pi\) was implemented but the explicit expression is left out here due to its length. It was constructed such that a specific number of gluons, traces and scalars can be chosen freely. For the example presented in section 2.3 it is evaluated as

\(\text{pigluonscalarmatrix}\([\{1, 2\}, 1]\) / . ordnung // MatrixForm

The list that constitutes the first variable, in this case 1, 2, gives the indices of the gluons and the second variable is the number of traces, in this case one. It was implemented, such that it can be evaluated for more traces, but those cases are not yet relevant in the context of this report. The output of this is presented in figure 3.2. The evaluation of the Pfaffian,

\(\text{pfpi4}\left[\text{pigluonscalarmatrix}\([\{1, 2\}, 1]\), 2, 1, 1, 2, \{1, 2\}\right] / . \text{ordnung}\)
gives the output presented in figure 3.3

elementstr[1] = \{3, 4\} \(\text{pfpi4}\left[\text{pigluonscalarmatrix}\([\{1, 2\}, 1]\), 2, 1, 1, 2, \{1, 2\}\right]\) parktaylortrace[1] / . ordnung // Expand

generates the result for \(C_2(3,4) Pf'\Pi(1,2,3,4)\) in 3.4. The result was then separated in terms of different tensor structures by hand. The doubly partial amplitudes from the Park Taylor factors attached to each of these terms were calculated from the Berends-Giele recursion presented in A.2. The resulting amplitude is the one in figure 3.5.
A.2. Berends-Giele recursion

Doubly partial amplitudes

The Berends-Giele recursion can be used to calculate scattering amplitudes recursively from the equation of motions. The implementation is based on the discussion of the recursion in [Ma16].

In[9]:= \( \text{phi2}[\{x_\}, \{y_\}] := \text{If}[x == y, 1, 0] \)

In[10]:= \( \text{recphi2} = \{ \text{phi2}[\text{listA}_\text{List}, \text{listB}_\text{List}] \mapsto 1/s[\text{listA}] \}
\)

In[13]:= \( \text{amp}[\text{listA}_\_, \text{listB}_\_] := s[\text{listA}] \text{phi2}[\text{listA}, \text{listB}] \)

In[14]:= \( \text{m}[n_, \text{listA}_\_, \text{listB}_\_] :=
\quad \text{If}[\text{Length}[\text{listA}] == n, 
\quad \text{If}[\text{Length}[\text{listA}] == \text{Length}[\text{listB}], 
\quad \text{amp}[\text{listA}, n], \text{Delete}[\text{listA}, n]]], 
\quad \text{Delete}[\text{listB}, n]], 
\quad \text{PROBLEM}], 
\quad \text{PROBLEM2}] \)

YM+\(3\) tree-level amplitudes

The Berends-Giele recursion for YM+\(3\) tree-level amplitudes was also based on the explanations in [Ma16]. Some relevant parts of it are presented here, but various conventions and definitions are left out, as well as the calculations that were done on paper work out the relevant recursions.

The recursion for the scalars was implemented as

\[
\text{recphi} = \{ \phi[\text{listA}_\text{List}, a_] \mapsto 1/s[\text{listA}] \}
\]

\[
\left( \sum 2 g \left( \text{Subscript}[\text{J}_\text{listA}[[i + 1 ;; \text{Length}[\text{listA}]]], a], \backslash[\text{Mu}] \right) \right) 
\text{Superscript}[\text{k}_\text{listA}[[1 ;; i]], \backslash[\text{Mu}]] \phi[\text{listA}[[1 ;; i]], a] 
\]

\[
\left( \text{Subscript}[\text{J}_\text{listA}[[1 ;; i]], a], \backslash[\text{Mu}] \right) \text{Superscript}[\text{k}_\text{listA}[[i + 1 ;; \text{Length}[\text{listA}]]], \backslash[\text{Mu}]] \phi[\text{listA}[[i + 1 ;; \text{Length}[\text{listA}]], a]] 
\]

\]
$$F[a,\, \text{alphap1}[a],\, \text{alphap2}[a]] \equiv \frac{\Lambda}{3} \left( \phi[\text{listA}[[i + 1 ;; \text{Length}[\text{listA}]],\, \text{alphap1}[a]] - \phi[\text{listA}[[1 ;; i]],\, \text{alphap2}[a]] \right) \phi[\text{listA}[[i + 1 ;; \text{Length}[\text{listA}]],\, \text{alphap2}[a]]), \{i, 1, \text{Length}[\text{listA}] - 1\}$$

$$+ \left( \sum_{i} \left( \text{g}^2 \left( 2 \text{Subscript}[\text{J}[[\text{listA}[[1 ;; i]], a], \text{\textmu}], \text{\textmu}] \right) \text{Subscript}[\text{J}[[\text{listA}[[i + 1 ;; \text{Length}[\text{listA}]], a], \text{\textmu}], \text{\textmu}] \phi[\text{listA}[[i + 1 ;; j]], a] \right) - \text{Subscript}[\text{J}[[\text{listA}[[1 ;; i]], a], \text{\textmu}], \text{\textmu}] \text{Subscript}[\text{J}[[\text{listA}[[i + 1 ;; j]], a], \text{\textmu}], \text{\textmu}] \phi[\text{listA}[[j + 1 ;; \text{Length}[\text{listA}]], a] \right)$$

$$+ \text{g}^2 \left( \phi[\text{listA}[[1 ;; i]], a] \phi[\text{listA}[[i + 1 ;; j]], a] \phi[\text{listA}[[1 ;; i]], \text{alphap1}[a]] \phi[\text{listA}[[j + 1 ;; \text{Length}[\text{listA}]], a] \phi[\text{listA}[[1 ;; i]], \text{alphap1}[a]] \right)$$

$$+ \phi[\text{listA}[[j + 1 ;; \text{Length}[\text{listA}]], a] \phi[\text{listA}[[1 ;; i]], \text{alphap1}[a]] \phi[\text{listA}[[i + 1 ;; j]], \text{alphap1}[a]]$$

$$- 2 \phi[\text{listA}[[i + 1 ;; j]], a] \phi[\text{listA}[[j + 1 ;; \text{Length}[\text{listA}]], \text{alphap1}[a]] \phi[\text{listA}[[1 ;; i]], \text{alphap1}[a]]) \right)$$

$$\{i, 1, \text{Length}[\text{listA}] - 2\}, \{j, i + 1, \text{Length}[\text{listA}] - 1\}$$

// Expand)

with

$$\phi[{x_-,} , a_-] := \Delta[c, x]$$

for the one-particle case. The one-particle case for the gluons is

$$\text{Subscript}[\text{J}[[x_-], a_-], j_-] :=$$

$$\text{Subscript}[\text{\textmu}, x]$$

$$\text{Subscript}[\text{\textmu}, x]$$

$$\text{Subscript}[\text{\textmu}, x]$$

and their recursion is implemented as

$$\text{recIndex} = \{\text{Superscript}[\text{J}[[\text{listA}_, a_-], \text{\textmu}], \text{\textmu}] \Rightarrow 1/s[\text{listA}] \}$$

$$\text{Sum}[\text{g} (\text{Superscript}[\text{J}[[x_-], a_-], \text{\textmu}]$$

21
\[ J[\text{listA}[[i + 1 ;; \text{Length}[\text{listA}]]], \]
\[ a], \backslash\{\text{Nu}\} \] \[) \] \[\text{Superscript} \]
\[ J[\text{listA}[[1 ;; i]], \backslash\{\text{Nu}\}] \] \[) \] \[\text{Superscript} \]
\[ k[\text{listA}[[ i + 1 ;; \text{Length}[\text{listA}]], \backslash\{\text{Nu}\}] \] \[) \] \[\text{Superscript} \]
\[ J[\text{listA}[[i + 1 ;; \text{Length}[\text{listA}]]], \]
\[ a], \backslash\{\text{Mu}\}] \[) \] \[\text{Superscript} \]
\[ J[\text{listA}[[1 ;; i]], a], \backslash\{\text{Nu}\}] \] \[) \] \[\text{Superscript} \]
\[ k[\text{listA}[[ i + 1 ;; \text{Length}[\text{listA}]], \backslash\{\text{Mu}\}] \] \[) \] \[\text{Superscript} \]
\[ k[\text{listA}[[ i + 1 ;; \text{Length}[\text{listA}]], \backslash\{\text{Nu}\}] \] \[) \] \[\text{Superscript} \]
\[ J[\text{listA}[[1 ;; i]], a], \backslash\{\text{Nu}\}] \] \[) \] \[\text{Superscript} \]
\[ k[\text{listA}[[1 ;; i]], a], \backslash\{\text{Nu}\}] \] \[) \] \[\text{Superscript} \]
\[ k[\text{listA}[[ i + 1 ;; \text{Length}[\text{listA}]], \backslash\{\text{Nu}\}] \] \[) \] \[\text{Superscript} \]
\[ J[\text{listA}[[1 ;; i]], a], \backslash\{\text{Nu}\}] \] \[) \] \[\text{Superscript} \]
\[ \phi[\text{listA}[[i + 1 ;; \text{Length}[\text{listA}]]], a] \] \[) \] \[\text{Superscript} \]
\[ k[\text{listA}[[1 ;; i]], \backslash\{\text{Mu}\}] \] \[\phi[\text{listA}[[1 ;; i]], a]] \[) \] \[\text{Superscript} \]
\[ \phi[\text{listA}[[1 ;; i]], a] \] \[*\text{Normierung beachten}* \]
\[ g^2 ((2 \phi[\text{listA}[[1 ;; i]], a]) \] \[\text{Superscript} \]
\[ J[\text{listA}[[1 ;; j]], a], \backslash\{\text{Mu}\}] \phi[\text{listA}[[1 ;; j]], a] \]
\[ \phi[\text{listA}[[1 ;; j]], a] \] \[\text{Superscript} \]
\[ \phi[\text{listA}[[1 ;; j]], a] \]
\[ J[\text{listA}[[j + 1 ;; \text{Length}[\text{listA}]]], a], \backslash\{\text{Mu}\}] \phi[\text{listA}[[j + 1 ;; j]], a] \]
\[ \phi[\text{listA}[[j + 1 ;; j]], a] \]
\[ J[\text{listA}[[j + 1 ;; \text{Length}[\text{listA}]]], a], \backslash\{\text{Nu}\}] \] \[) \] \[\text{Superscript} \]
\[ J[\text{listA}[[i + 1 ;; j]], a], \backslash\{\text{Mu}\}] \] \[) \] \[\text{Superscript} \]
The amplitudes can then be obtained from the functions

\[
\text{AmplitudeJ}[	ext{listD}, a] := \\
\left( s[\text{listD}[[1 ;; \text{Last[listD]} - 1]]] \\
\left(\text{Superscript} [\text{J}[	ext{listD}[[1 ;; \text{Length[listD]} - 1]], a], \text{Nu}] \\
\text{Subscript}[\text{J}[[\text{Length[listD]}], a], \text{Mu}]] \right) \right) /\text{Expand}
\]

\[
\text{Amplitudephi}[	ext{listD}, a] := ( s[\text{listD}[[1 ;; \text{Last[listD]} - 1]]] \\
\left(\text{phi}[	ext{listD}[[1 ;; \text{Length[listD]} - 1]], a] \\
\\text{\textcopyright}[[\text{Delta}]] \right) \text{Subscript}[\text{c}, \text{Last[listD]}], a) \right) \right) /\text{Expand}
\]

\[
\text{A}[	ext{listD}, a] := ( \text{AmplitudeJ}[	ext{listD}, a] + \text{Amplitudephi}[	ext{listD}, a])
\]

This implementation is used to check the tree-level CHY results.
### Bibliography

<table>
<thead>
<tr>
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<th>Year</th>
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