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Introduction to early algebra in Estonia, Finland and Sweden – some distinctive features identified in textbooks for Grades 1-3

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This cross-cultural study focuses on specific features concerning the introduction to early algebra in three neighbouring countries – Estonia, Finland and Sweden – that, despite having relatively similar school systems, have different student results in international evaluations. The data consists of commonly used mathematics textbooks for Grades 1-3, and the focus of analysis is particularly on the treatment of expressions, equations, inequalities, and the meaning of the equal sign. The results show differences not only between the countries but also between different language groups within the same country, and similarities between the same language groups in different countries. We exemplify the most significant characteristics identified in different contexts in relation to various approaches to early algebra described in research literature, and discuss the implications of the findings for our research field.

Keywords: early algebra, comparative analysis, mathematics curriculum, primary education

Background

Early introduction to algebra has been in focus in recent research, and several researchers point out that it is beneficial to start working with important ideas within algebra already in primary school (e.g. Blanton, Stephens, Knuth, Murphy Gardiner, Isler & Kim, 2015). For example, an intervention study by Blanton et al. (2015) shows that students can improve their understanding of so-called big ideas of algebra with an appropriate teaching approach and relevant material. International comparisons show that there are great differences between approaches to the teaching and learning of algebra in different countries (e.g. Cai, Lew, Morris, Moyer, Ng & Schmittau, 2005; Leung, Park, Holton & Clarke, 2014). These comparisons have mainly concerned countries with significantly different cultures and school systems. In contrast, the present study investigates the early introduction to algebra in the three neighbouring countries of Estonia, Finland and Sweden, where the school systems are quite similar. For example, all three have nine years of compulsory school with no tracking. Moreover, national steering in the three countries is relatively weak, with no approval process for curriculum materials that are commercially produced. We deem that the similarity of the school systems facilitates the identification of specific features concerning the views on early progression in algebra in these countries. In international evaluations, there are differences between the countries in students’ learning outcomes, making it interesting to discuss different approaches to early algebra in relation to this variation. While Estonian and Finnish students’ results in international comparisons have been relatively good, algebra has not been the strongest area in Finland (Yang Hansen et al., 2011). In Sweden, students’ results in algebra have
been low since the 1960s regardless of the variation in other mathematical topics (Hemmi et al., 2018).

Curriculum materials are produced within certain educational traditions, and may therefore be shaped by national perspectives on the specific school subjects (cf. Andrews, 2007). In all three countries, mathematics textbooks are an important part of classroom work. Therefore, in this paper, we exemplify and discuss typical characteristics of the approaches to early algebra in Estonia, Finland and Sweden as identified in our textbook analysis concerning Grades 1-3 (ages 7-9 years). We have also included textbooks produced by Finnish Swedish authors in the analysis, as there are differences between the two language groups in Finland when it comes to students’ learning outcomes, as well as teachers’ relation to mathematics textbooks (e.g. Pehkonen, Hemmi, Krzywacki & Laine, 2018). We acknowledge that the textbooks can be used by teachers in various ways when designing and enacting mathematics lessons (e.g. Lepik et al., 2015; Röj-Lindberg, Partanen & Hemmi, 2017; Kilhamn, 2014). However, we deem these kinds of studies important as we learn how textbook authors interpret the introduction of early algebra in different cultural-educational contexts. The following research question guides the study: What are the specific characteristics of early algebra in textbooks series produced in four different contexts?

Relevant research and theories

Few comparative studies address the progression of algebra in different countries’ curricula. Cai, Lew, Morris, Moyer, Ng and Schmittau (2005) study how algebraic concepts are developed and represented in the curricula of the US (Investigations curriculum), China, Singapore, South Korea and Russia (Davydov curriculum). The study by Cai et al. (2005) shows that the three Asian countries build informal equation solving on the use of inverse operations and the related doing-undoing. In the Russian Davydov curriculum for Grades 1-3, children develop algebraic thinking by exploring and comparing quantities before the study of arithmetic and, in contrast to the US Investigations curriculum, letters are used from the very beginning (Cai et al., 2005). Both the Russian and the US curricula address real-life problems, but in fundamentally different manners. In the former, they are carefully planned and sequenced to help students develop a theoretical understanding of mathematical concepts and an ability to analyse problem situations (Cai et al., 2005). Rather than proceeding inductively from the concrete to the abstract as Investigations does, the Davydov curriculum develops students’ ability to see the same abstract relationships across different concrete contexts. Investigations engages students in mathematical problems embedded in authentic contexts and these applied problem-solving activities require US students to explore problems in depth, construct their own strategies and approaches utilizing a variety of tools (e.g., manipulatives, computers, calculators), and communicate their mathematical reasoning through drawing, writing, and talking (Cai et al., 2005).

Based on recent research on early algebra and the language of learning progression, Blanton et al. (2015) have identified five so-called big ideas connected to algebraic thinking. These big ideas

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1 About 5% of the Finnish population has Swedish as their mother tongue, and have the right to receive education in Swedish.
consist of: 1) equivalence, expressions, equations and inequalities; 2) generalized arithmetic; 3) functional thinking; 4) proportional reasoning; and 5) variables. In a recent study, we used the big ideas as the basis for an analytical tool in order to analyse the algebraic content in the current Swedish curriculum for mathematics as well as two Swedish textbook series for Grades 1-6 (Bråting, Madej & Hemmi, 2019). In the study reported in this paper, we look more in-depth at Grades 1-3 and the first two big ideas. The first – equivalence, expressions, equations and inequalities – includes relational understanding of the equal sign; representing and reasoning with expressions, equations, and inequalities; as well as relationships between and among generalized quantities. The second big idea – generalized arithmetic – involves reasoning about structures of arithmetic expressions (rather than their computational value) as well as generalizations of arithmetical relationships, which include fundamental properties of numbers and operations (e.g., the commutative property of addition) and inverse properties of operations (Blanton et al., 2015).

**Method**

We chose the textbooks to be analysed based on our knowledge of their popularity in each region. In order to examine the variation within different contexts, we chose two textbook series each in Estonia (Matemaatika by Koolibri and Matemaatika by Avita), Finnish Finland (hereafter FinFin) (Kymppi and Tuhattaituri), and Sweden (Eldorado and Matte Safari). Only one series has been produced in the Finnish Swedish context (hereafter FinSwe), Lyckotal. In the data analysis, we identified the sections connected to the big ideas of equivalence, expressions, equations, and inequalities as well as generalized arithmetic. The insider(s) in each region conducted the analysis according to our collective interpretation of the categories based on our thorough discussions, and the analyses were then discussed by the whole research team. The analysis was qualitative, identifying the character of the texts and their order/place of appearance throughout the three grades in order to understand the progression of this big idea within various topics. The only quantification that took place in the analysis was connected to the identification of possible dominating topics. For example, if a certain kind of task occupied a great number of pages in a textbook or occurred regularly throughout the grades, it was marked in the analysis protocol. While it is not possible to report the results of the entire study within the scope of this paper, in the following we will describe, exemplify, and discuss the most striking differences and similarities displayed in the textbooks from the four regions.

**The results with illustrative examples**

**Estonia**

Addition and subtraction are introduced simultaneously in Grade 1, with subtraction introduced as the inverse of addition. In Grade 2, multiplication and division using whole numbers are introduced in a similar manner. Students are to find missing values using the inverse property of the operations (Figure 1). They should learn the names of the components and results of the four arithmetic operations in order to be able to reason about informal equation solving using the inverse property of the operation at a more general level. In addition, the trial and error strategy is introduced for searching the value of an unknown. At this stage, various symbols (box, blank, letter) are widely used in the textbooks to represent a variable or an unknown. Children are expected to learn to use
letters to express the relationships between quantities and determine the correct order of operations in expressions (parentheses, multiplication/division and addition/subtraction) from Grade 2. The notions of equivalence and inequality of two expressions (incl. letter expressions) are introduced and practiced using the formal signs (=, <, >) from the beginning.

Figure 1. Division as the reverse of multiplication in Estonian Grade 2 textbook, *Matemaatika by Koolibri*

Rather than seeing equations as simply objects to manipulate, in Estonian primary textbooks from Grade 2 the equations describe relationships between varying quantities that arise from contextualized situations. Children are required to schematically represent the internal quantitative relationships in word problems and to write expressions of these relationships (Figure 2). This process is sequenced by the authors so that the same abstract relationship appears in different concrete contexts. Textbooks also systematically guide students in composing their own word problems for given expressions.

Figure 2. Modelling word problems in Estonian Grade 2 textbook *(Matemaatika by Koolibri)*

Finland

In the FinFin textbooks, equalities and inequalities are presented with formal signs (> , <, =) in parallel from the very beginning of Grade 1 and are used consistently throughout the three grades. First, only numbers are compared, and later during Grade 1, expressions are compared. This approach is different from the FinSwe style, in which the equal sign is introduced in Grade 1 while the signs for inequalities are only briefly introduced and practiced. Similar to the Estonian textbooks, one of the FinFin textbooks *(Tuhattaituri)* introduces the inverse property of addition and subtraction and guides students in using it in informal equation solving during Grade 1. The same is done with multiplication and division in Grade 2. This is not the case in the FinSwe textbooks, however, where equations start to appear at the beginning of Grade 1 in connection with the partition of specific numbers, for example with the number 5 in Figure 3.
The inverse property of addition and subtraction is introduced very briefly in the FinSwe textbooks during the first school year using so-called number families, whereby the student is tasked with writing out different expressions with the numbers 10, 3 and 7 in the shape of a triangle with + and – signs included. Hence, the students are expected to see how the members of this “family” are connected to each other via the operations. No explicit connections are made to guide students in using the inverse properties in equation solving. In the FinFin textbooks, the informal equation solving proceeds step-by-step by first focusing only on equations in which the same term of the operations is missing, while in the FinSwe books, diverse and sometimes open equations start appearing quite early in Grade 1. For example, students are asked to create two different expressions that are equal to each other during the first school term by filling out empty spaces (e.g. \( +_1 = +_2 \)), or with expressions in which the operations are missing (e.g. \( 4 - 3 = 5 - 2 \)).

![Figure 3](image)

**Figure 3. Partition of the number 5 in the FinSwe textbook (Lyckotal)**

Contrary to the FinFin textbooks, at no point during Grade 1 are students asked to check their solutions by using the inverse operation. Neither the FinFin nor the FinSwe textbooks use letters for variables or unknown quantities during Grades 1-3; instead, they use gaps or pictures such as flowers and hearts. Thus, in the textbooks from the two language contexts students are expected to write the expressions of operations based on pictures or real-world problems from the beginning. The FinSwe textbooks also guide students in finding real-world problems for expressions, which we do not find in the FinFin textbooks.

![Figure 4](image)

**Figure 4. Handling of expressions (a) and informal equation systems (b) in the FinFin mathematics textbooks**

In the FinFin context, the priority rules are presented in Grade 2 and students learn to systematically calculate the value of expressions with two operations, stepwise marking every new phase below the previous expression. In Grade 3, students are to handle expressions with three operations in a similar matter, as in Figure 4a, where the first line includes two divisions and one addition. For Grade 3 the FinSwe textbooks state that the multiplicative operation is done before the additive, but all calculations are done in a straight line, i.e., not each below the previous as in the Finnish textbook Kymppi, shown in Figure 4a. The commutative properties of operations are presented during Grade 1 (addition) and Grade 2 (multiplication) in the FinFin textbooks. The formal names
of the operations are used from the beginning, and in Grade 3 students are to choose the correct expressions for sentences, for instance “Subtract from the quotient of 18 and 2 the product of 2 and 3.” A special feature of the Finnish textbook is its repeated exercises from the very beginning in solving systems of equations in an informal manner (see Figure 4b).

**Sweden**

What characterizes the Swedish textbook series, especially *Matte Safari*, is the emphasis on open number sentences (see Figure 5). Inverse expressions occur in the context of number families, but the connection between addition and subtraction is barely visible in practice. As in the FinSwe textbooks, no explicit connections are used to guide students in applying the inverse properties in equation solving. We found only one example in a single textbook (*Eldorado*) in which students were to check their solutions using the inverse property.

![Figure 5. Practicing of open number sequences (*Matte Safari*)](image)

Both textbook series present the commutative property of addition and state that subtraction is not commutative. Multiplication is introduced in Grade 2, and the commutative property of multiplication is shown in both textbook series. The inverse relation between multiplication and division is only presented in *Eldorado*. Expressions are addressed in both textbook series, but more often in *Eldorado*. In both series, the expressions go “both directions”, i.e. students are expected to write an expression from a real-world problem or a picture, but also to create their own text problem for a given expression. This is similar to the Estonian and FinSwe textbook series. An example of this is the task “Draw and write text problems” in Grade 1, in which students are to create text problems for the expressions 5+3+2 and 12-5.

![Figure 6. Examples of equations in the Swedish Grade 1 textbook (*Eldorado*)](image)

Priority rules are not considered in either of the two Swedish textbook series; and neither are inequalities, even though one of the textbooks guides students in using the symbols for “greater than” and “less than” to express which of two numbers is larger or smaller. However, this appears only one single time and in the other textbook these symbols are not used at all. The relational meaning of the equal sign is emphasized in both textbook series, especially *Matte Safari*. For example, students are asked to write = or ≠ between expressions like 9 - 2 and 4 + 4 or 7 + 3 and 2 + 8. There are also fully open number sentences, such as _ + _ = _ + _, in line with the FinSwe
textbooks. Neither of the textbook series introduces formal equation solving. Yet, one of the textbooks introduces “the symbol x” as a placeholder in Grade 1, using the sentence “The symbol x has a value which makes the equality true”, and lets students find the value of x through inspection (Figure 6). However, this is not followed up in Grade 2 or 3.

**Discussion and conclusion**

Our analysis reveals both differences and similarities between the four regions. Next, this will be discussed in light of relevant research.

The Estonian textbooks and one of the Finnish textbook series frequently use inverse properties of operations in connection with informal equation solving, which is similar to how informal equation solving is prescribed in the Singaporean and Chinese curriculum documents (Cai et al., 2005). Meanwhile, the Finnish Swedish and the Swedish textbook series use inverse properties of operations in the context of “number families” (Figure 3), but not explicitly in connection with informal equation solving. Instead, they both frequently use open number sentences to introduce students to equation solving. In Sweden, this approach might be connected to the emphasis on the meaning of the equal sign, which is explicitly pointed out in the current Swedish mathematics curriculum for Grades 1-3 (Bråting, Madej & Hemmi, 2019). In all textbook series from the four regions, the handling of expressions appears to varying degrees and students are expected to write expressions based on real-world problems. However, it is only in the Estonian textbooks that students are expected to create letter expressions and equations, which is in conformity with the Russian Davydov curriculum in which letters are used from the very beginning (Cai et al., 2005). The Swedish and the FinSwe series guide the students in finding real-world problems for expressions by means of writing and drawing. This approach is analogous to the US Investigations curriculum, in which students are required to explore problems in depth and construct their own strategies using a variety of tools (Cai et al., 2005).

We know that both Estonian and Finnish students have had outstanding results on PISA (Programme for International Student Assessment), in which concrete real-life problems are solved. Both Estonian and FinFin textbooks focus on the creation of quite complicated expressions in Grades 1-3, which might enhance students’ structural understanding of problems. This is something to delve into more deeply in further studies. Perhaps the most striking result is the difference we found between the Finnish language contexts. Although they follow the same national curriculum, the FinFin textbooks contain elements similar to the Estonian textbooks, while the FinSwe textbooks seem to have been influenced by Swedish traditions or vice versa. In fact, there seems to be a greater difference between the two Swedish textbook series than between one of the Swedish series and the Finnish Swedish textbook series. There is no government auditing of the textbooks in the three countries, and we are seeing an increased exchange of curriculum material between countries (e.g. Hemmi, Krzywacki & Liljekvist, 2018). In addition, some publishing companies have emerged across the countries. This might lead to greater variation between textbook series within single contexts, but also similarities across various contexts, making our understanding of the connections between curriculum materials and student outcome within a certain context more difficult.
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