UPPSALA
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UPTEC F 20007
Examensarbete 30 hp
Mars 2020

When to expect decreasing implied volatilities

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Abstract

When to expect decreasing implied volatilities

Tobias Nordahl

In this report the goal is to investigate general properties of implied volatility such as the relationship between implied volatility and local volatility as well as the relationship between implied volatility and the sign of the jumps in jump-diffusion models. More specific the question to investigate is whether the implied volatility is decreasing as the strike price increases under the assumption that the market is pricing under a monotonically decreasing local volatility model. To investigate the previously mentioned problem some derivations will have to be done. The core of the derivations are Dupire's equation and Black-Scholes formula. Not only will the result from the derivation show what is tried to be proven, the derived results will also be implemented to visualise the results. This means that the result both is backed by both theory and derivations but also computations which is plotted in the report.

An additional goal of this report is to shed light of the relationship between the consepts of implied volatility and local volatility, as well as on the computational aspects on how to determine the implied volatility.

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Examinator: Tomas Nyberg
ISSN: 1401-5757, UPTEC F 20007

1 populärvetenskaplig sammanfattning

En europeisk köpoption är ett finansiellt kontrakt som ger ägaren av kontraktet rätten men inte skyldigheten att köpa den underliggande tillgången som kontraktet är skrivet på för ett förbestämt pris på ett förbestämt datum. Den som säljer ett sådant kontrakt tar en risk eftersom priset som personen kan få köpa den underliggande tillgången är förbestämt från början. När man pratar om risk i aktiesammanhang så menar man ofta volatilitet. Volatilitet är ett mått på hur mycket något fluktuerar. Inom finans är volatilitet ett mått på hur mycket värdet på tillexempel en aktie fluktuerar. Om man vet priset på en europeisk köpoption och dess parametrar som optionen är skriven på så man med hjälp av Black Scholes formel räkna fram volatiliteten. Denna typ av volatilitet kallas för implicit volatilitet. Det finns olika tumregler för när man räknar fram den implicita volatiliteten beroende på vilken underliggande modell som används för att beskriva aktien som köpoptionen skrivs på. För den enklaste typen av underliggande modell, det vill säga den lokala volatilitetensmodellen där volatiliteten är en funktion av aktiepriset säger man att om den lokala volatiliteten avtar så kan man förvänta sig att den implicita volatiliteten avtar.

I den här rapporten används två olika typer av underliggande modeller och sedan utifrån dessa modeller ska den implicita volatiliteten räknas ut för att se om en avtagande implicit kan förväntas. Den första underliggande modellen som testas är en lokal volatilitetsmodell där en avtagande lokal volatilitet antags. Den andra underliggande modellen som används är en hoppmodell där volatiliteten är konstant och istället introduceras negativa hopp. Det som sedan kommer testas är om man kan förvänta sig en avtagande implicit om en lokal volatilitetsmodell används med avtagande lokal volatilitet eller om en hoppmodell används med negativa hopp och en konstant volatilitet. För att tag fram köpoptions priset när den lokala volatilitetsmodellen används så tas Dupires ekvation fram och sedan löses. När köpoptions priserna är framtagna från Dupires ekvation så används sedan Black Scholes formel för att få fram den implicita volatiliteten. I den här rapporten härleds Dupires ekvation för att få fram köpoptions priser och sedan härleds två metoder för att få fram den implicita volatiliteten. För att tag reda på om man kan förvänta sig en avtagande implicit volatilitet kommer två typer av resultat presenteras, ett visuellt och ett mer teoretisk.

Contents

2 Introduction

Volatility is a measure of how much the value of an asset fluctuates. The volatility can be computed in different ways and is therefore named differently depending on what method was used to compute the volatility. One way is to compute the historic volatility by trying to fit the parameters in the underlying model to the historical values. The other type of volatility is the implied volatility which is computed by using the market value of for example an call option and invert the Black-Scholes formula to get the volatility. Depending on what underlying model is used to describe the underlying asset there exists some rules of thumb for the implied volatility. Throughout this report the notation for local volatility is σ and the notation for implied volatility is φ. This project focuses on the implied volatility which the following rules of thumb exists.

- If the market is pricing using a local volatility model that decreases as the strike price increases the implied volatility also decreases as the strike price is increasing.
- If the market is pricing using a jump model with negative jumps the implied volatility will decrease as the strike price increases [4].

These two rules are often mentioned is literature as rules of thumb. However it is not clear how strict the rules are.

The procedure to investigate the points above is to first write down the underlying model and derive the Dupire's equation. After the model is written down and Dupire's formula is derived it is time to first assume a local volatility for example $\sigma(S) = \frac{1}{\sqrt{S}}$. Then depending on which of the two methods that will be used ether solve Dupire's equation and from that compute the implied volatility by inverting Black-Scholes formula or skip the middle step of solving Dupire's and solve a PDE for the implied volatility. The two methods will be explained later in the report, this is just to demonstrate the procedure for how the questions above will be tackled.

3 Theory

The option type that will be used in this project is the European call option.

- A **European call option** gives the holder of the contract the option but not the obligation to buy the underlying asset S at the price K at time of maturity T.

Another option which is very similar to the call option which uses the Black-Scholes equation as well is the put option. In this project only call options will be used, however one could similarly use put options in this project though the derivations would look a bit different.

- A **European put option** gives the holder of the contract the option but not the obligation to sell the underlying asset S at the price K at time of maturity T.

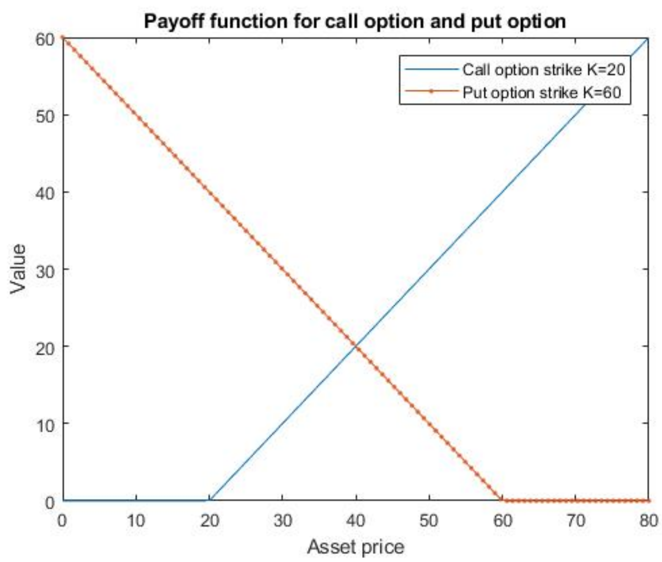

Figure 1: The payoff function for an European call potion(blue line) with strike price $K = 20$ and a put option(orange dotted line) with strike price $K = 60$.

Black Scholes formula will be mentioned from time to time during this project. Therefore it is important to formulate it so one understands where it comes from. If the underlying asset is based on the model,

$$dS_t = \mu(t, S_t)dt + \sigma(t, S_t)dW, \tag{1}$$

then the call-option price is V(t,S(t)), where V(t,S) solves the Black-Scholes equation,

$$
\begin{aligned}
V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0 \\
V(T,S) = (S-K)^+.
\end{aligned}
\tag{2}
$$

Here σ is the local volatility r is the drift and K is the strike price. In the case of constant volatility i.e $\sigma = constant$, the Black-Scholes equation can be solved explicitly using the Black-Scholes formula.

$$
V(t,s) = SN(d_1) - Ke^{-r(T-t)}N(d_2), \tag{3}
$$

where N is a normal distribution and,

$$
d_1 = \frac{ln\frac{S}{K} + (r + \frac{\varphi^2}{2})(T-t)}{\varphi\sqrt{T-t}}, d_2 = \frac{ln\frac{S}{K} + (r - \frac{\varphi^2}{2})(T-t)}{\varphi\sqrt{T-t}}. \tag{4}
$$

To simplify derivations and computations and since in this report the focus is on the volatiltiy, the interest rate is set to $r = 0$. Black-Scholes equation and Black-Scholes formula as well as Dupires's equation and formula are the base for all the derivations in this report. In the sections Method 1 and Method 2 Dupire's equation and Black-Scholes formula is used for example. It is the implied volatility from 3 and 4 which is being investigated in this report and will be plotted for different strike prices.

Intuitively one might expect the price of a call option to decrease if the volatility increases since the seller of the option is now carrying more risk. In fact it is the opposite, when using a higher volatility the option price will also be higher. This can be shown by differentiating Black-Scholes formula with respect to the volatility. Doing the chain rule one gets,

$$
V_\varphi = Sn(d_1)d_{1\varphi} - Ke^{-r(T-t)}n(d_2)d_{2\varphi}. \tag{5}
$$

Which can be simplified to,

$$
V_\varphi = Sn(d_1)\sqrt{T-t}. \tag{6}
$$

Equation 6 is named Vega ν and is one of the measurements when discussing the Greeks in finance. Plotting ν for different volatility φ and strike price K one gets.

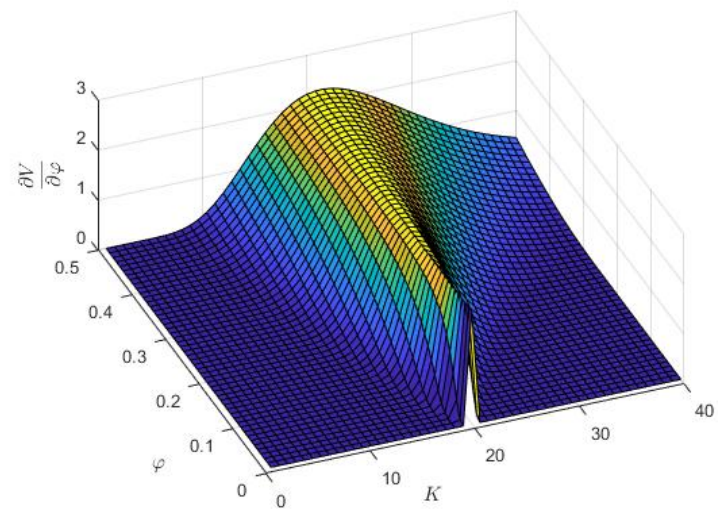

Figure 2: V_φ visualised for different volatility $\varphi \in [0, 0.5]$ and strike price $K \in [0, 2 * S]$.

In figure 2 above one can see that $V_\varphi(\varphi) \geq 0$ which means that the price of a call option increases as the volatility increases. This is visualised below in figure 3 by piloting the option price for different volatilities and strike price.

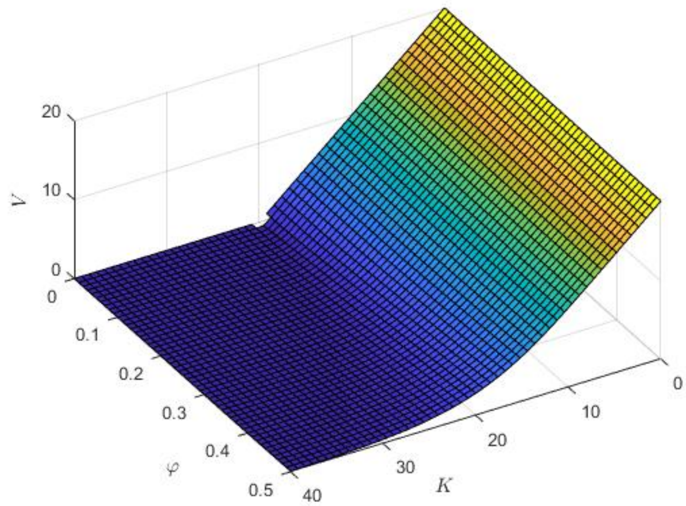

Figure 3: V visualised for different volatility $\varphi \in [0, 0.5]$ and strike price $K \in [0, 2 * S]$.

Figure 3 is the call option price plotted for different φ and K and it is clear if one looks around the strike price that the option price does increase when the volatility increases.

3.1 Dupire's equation

The local volatility is what the name say it is. It is a local volatility i.e. instead of treating the volatility as a constant one can treat it as a function $\sigma(t, S)$. If one solves equation 17 named the Dupire's equation after Bruno Dupire in 1994, one can get the local volatility at every strike K and time T. A key difference here apart from Dupire's equation containing derivatives with respect to the strike price K and Black-Scholes contain derivatives with respect to the stock price S is that in Dupire's equation where the local volatility is used or computed from the time derivative is with respect to the time of maturity T. That means that when solving the equation one is actually getting solutions for option-prices for different strike prices and different time of maturities. Another use for this equation is to assume a local volatility for example $\sigma(K) = \frac{1}{\sqrt{K}}$ and solve Dupire's equation to get the market option prices. To understand more how one can use the local volatility to compute the implied volatility a good start is to understand how Dupire's equation is derived. Below Dupire's equation is derived from Fokker Planck equation and knowledge about the underlying asset and call options [3]. The change of an assets value can be described as in equation 7 where $\mu = r - d$ where r is the interest rate, d is the dividend yield and σ is the drift and volatility respectively.

$$dS_t = \mu_t Stdt + \sigma(S_t, t)S_t dW_t \tag{7}$$

To describe the change of value for a European call option we need to know the Fokker Planck equation where $f(S, t)$ is the probability density function,

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial S}[\mu S f(S,t)] + \frac{1}{2}\frac{\partial^2}{\partial S^2}[\sigma^2 S^2 f(S,t)] \tag{8}$$

and the European call option.

$$C = e^{-r(T-t)}E^Q[(S_T - K)^+] = e^{-r(T-t)}\int_K^\infty (S-K)f(S,T)dS. \tag{9}$$

The goal is to evaluate the partial derivatives $\frac{\partial C}{\partial T}$, $\frac{\partial C}{\partial K}$ and $\frac{\partial^2 C}{\partial K^2}$ and plug it into Fokker Planck.

$$\frac{\partial C}{\partial K} = -e^{r(T-t)}\int_K^\infty f(S,T)dS. \tag{10}$$

Assuming the probability distribution function $f(S, T)$ goes to zero as $S \to \infty$. There for one can say $\lim_{S\to\infty} f(S,T) = 0$ the second derivative is

$$\frac{\partial^2 C}{\partial K^2} = e^{r(T-t)}f(K,T), \tag{11}$$

and the time derivative is

$$\frac{\partial C}{\partial T} = -rC + e^{-r(T-t)}\int_K^\infty (S-K)\frac{\partial f(S,T)}{\partial K}dS. \tag{12}$$

Now it is time to plug the time derivative at t=T from Fokker Planck into the time derivative from the call option,

$$\frac{\partial C}{\partial T} = -rC + \int_K^{\infty} (S-K)(-\frac{\partial}{\partial S}[\mu S f(S,T)] + \frac{1}{2}\frac{\partial^2}{\partial S^2}[\sigma^2 S^2 f(S,T)]), \quad (13)$$

which is rewritten as,

$$\begin{aligned} \frac{\partial C}{\partial T} = -rC - \mu \int_K^{\infty} (S-K)\frac{\partial}{\partial S}[S f(S,T)]dS \\ +\frac{1}{2}\int_K^{\infty} (S-K)\frac{\partial^2}{\partial S^2}[\sigma^2 S^2 f(S,T)]dS. \end{aligned} \quad (14)$$

The next step is to evaluate the two integrals in equation 14 above. The first one is called I_1 and the second I_2. Here the assumption that as S goes to infinity the probability density function goes to zero faster than S goes to infinity which will result in $\lim_{S\to\infty}[(S-K)Sf(S,T)] = 0$. Evaluating I_1 gives,

$$\begin{aligned} I_1 &= \int_K^{\infty} (S-K)\frac{\partial}{\partial S}[S f(S,T)]dS \\ &= [(S-K)Sf(S,T)]_{S=K}^{S=\infty} - \int_K^{\infty} Sf(S,T)dS \\ &= [0-0] - \int_K^{\infty} Sf(S,T)dS = -C + K\frac{\partial C}{\partial K} \end{aligned} \quad (15)$$

and evaluating I_2 gives,

$$\begin{aligned} I_2 &= \frac{1}{2}\int_K^{\infty} (S-K)\frac{\partial^2}{\partial S^2}[\sigma^2 S^2 f(S,T)]dS \\ &= [(S-K)\frac{\partial}{\partial S}[\sigma^2 S^2 f(S,T)]]_{S=K}^{S=\infty} - \int_K^{\infty} \frac{\partial}{\partial S}[\sigma^2 S^2 f(S,T)]dS \\ &= [0-0] + \sigma^2 K^2 f(K,T) = \sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}. \end{aligned} \quad (16)$$

Now insert the evaluated integrals and one gets,

$$\frac{\partial C}{\partial T} = -rC + \mu(C - K\frac{\partial C}{\partial K}) + \frac{1}{2}\sigma^2 K^2 \frac{\partial^2 C}{\partial K^2}. \quad (17)$$

Now when using the fact that $\mu = (r - q)$ and solve the equation for σ one gets the general Dupire's formula,

$$\sigma(K,T)^2 = \frac{\frac{\partial C}{\partial T} + qC - (r-q)K\frac{\partial C}{\partial K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}. \tag{18}$$

In this project both the interest rate r and the dividend yield q is assumed to be zero which means that the underlying model to describe the asset value S is,

$$dS_t = \sigma(S_t, t)S_t dW_t, \tag{19}$$

Thus Dupire's formula becomes,

$$\sigma(K,T)^2 = \frac{\frac{\partial C}{\partial T}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}. \tag{20}$$

Observe that Dupire's formula is solved for expiration dates i.e T compared to Black-Scholes formula which is solved in current time t. Computationally wise this is not a crucial observation since one could use $\tau = T - t$ which makes it easier with the computations.

3.2 Jump-diffusion model

A jump-model is a way to describe how the value of an asset changes as time goes. The underlying model that will be used in this project (equation 22) looks very similar to the model used in the previous sections except in this model a third term has been added. This term is the term which gives the jump in asset value.

The stock modeled by a jump-diffusion process is described as,

$$dS = (\mu - \lambda J)Sdt + \sigma SdW + JSdN. \tag{21}$$

Here J the negative proportional jump size and dN is the increment of a Poisson process with intensity λ. The jump size is actually the proportional jump size defined such that $\Delta S(t) = S(t) - S(t_-) = JS(t_-)$ which means that $\frac{\Delta S(t)}{S(t_-)} = J$ hence $J \in [0, -1]$. In this project the goal is to investigate the implied volatility when having a jump-model with negative jumps. Again μ is assumed to be 0 and define instead the proportional jump size as γ where $J = -\gamma$ and $\gamma \in [0, 1]$. Therefore dS wan be written as,

$$dS = \lambda\gamma S dt + \sigma S dW - \gamma S dN. \tag{22}$$

Where the drift is $\lambda\gamma$ since $\mu = 0$. If one assumes constant volatility one can compute the option price for a call option C_{Jump} in a similar way as one would do when using a model without jumps. However now one has to take the jumps into consideration as well. Below one can see the formula for computing the call option price. By conditioning on the number of jumps of the Poisson process a formula for the option price can be obtained as follows.

$$C(t, S)_{Jump} = E[(S_t - K)^+] = \sum_{k=0}^{\infty} E[(S_t - K)^+ | N_t = k] P(N_t = k). \tag{23}$$

The expectation can be treated as a call option where dividend is introduced hence the initial $S_0 = S(1-\gamma)^k$ where k in this case is treated as the number of dividends and since $\mu = 0$ the drift becomes $\lambda\gamma$. The probability that a Poisson random variable $N_t = k$ is $P(N_t = k) = e^{-\lambda(T-t)} \frac{(\lambda(T-t))^k}{k!}$. Hence the following formula for computing the option price is obtained, where C_{BS} is the Black-Scholes formula from equation 3,

$$C(t, S)_{Jump} = \sum_{k=0}^{\infty} C_{BS}(S(1-\gamma)^k, \sigma, K, \gamma\lambda, T, t) e^{-\lambda(T-t)} \frac{(\lambda(T-t))^k}{k!}. \tag{24}$$

In the formula above one treat $S(1-\gamma)^k$ as one treats the initial S_0 in the Black-Scholes formula when dividend is introduced. In [2] the pricing function with dividend $F_\delta(t, S)$ where δ is the dividend size and n are the number of dividends in the time $(t, T]$, can be written as the pricing function without dividend but with a new initial stock value $S(1-\delta)^n$. Hence, the pricing function can be written as $F(t, S(1-\delta)^n)$. The drift term can also be moved to the initial stock price so that the formula ends up with no drift term like in equation 25.

$$C(t,S)_{Jump} = \sum_{k=0}^{\infty} C_{BS}(S(1-\gamma)^k e^{\gamma\lambda(T-t)}, \sigma, K, 0, T, t) e^{-\lambda(T-t)} \frac{(\lambda(T-t))^k}{k!}. \tag{25}$$

Equation 25 is now similar to the Black-Scholes formula used earlier with the drift parameter set to 0.

4 Computing implied volatility

4.1 Local volatility model

When the option price is computed from Dupire's equation by using a known local volatility one can start computing the implied volatility. Here are two ways of computing the implied volatility. For the sake of making this friendly to read the two ways of computing the implied volatility is called method 1 and method 2 even though they are not two separate methods, they are related. In method 1 the implied volatility is computed by using the analytic solution to Black-Scholes equation and the option prices obtained from Dupire's equation. An observation to make is that the time derivative in Black-Scholes equation is not time of maturity T it is rather the current time t. In theory this means that when using Black-Scholes formula the time of maturity is fixed and the iteration is over the current time t compared to Dupire's formula where the time t is fixed and the iterations are done over maturities T. This however plays a minor role since below only time-independent local volatilities is used. In the paper [1] a single PDE is derived consisting of local volatility and derivatives of the implied volatility using the coordinate log-moneyness i.e. $x = log(\frac{S}{K})$ and time variable $\tau = T - t$. In this report method 2 derives the same PDE but with more intuitive coordinates strike price K and time of maturity T. Through lengthy derivations the implied volatility is computed without computing the option prices from Dupire's equation and instead use the local volatility and solve the PDE equation 53 for the implied volatility.

4.1.1 Method 1

The goal in the following section is to compute the implied volatility by solving Dupire's equation which is given below in equation 26. In equation 26 r and q are assumed to be 0 and the local volatility $\sigma(K)$ is a function with respect to the strike price K. Once the PDE is solved one can compute the implied volatility. To obtain the implied volatility one tries to minimize the equation 27 where the implied volatility φ is the only variable. Once the implied volatility is obtained both the local and implied volatility is plotted in the same plot against the used strike prices to check how similar they are.

$$\frac{\partial C}{\partial T} = \frac{1}{2}\sigma^2(K)K^2\frac{\partial^2 C}{\partial K^2} \tag{26}$$

$$0 = C_{Dupire}(T, K) - C_{BS-formula}(\varphi, T, K, S) \tag{27}$$

To solve equation 26 the domain is discretized in both time and space and the derivatives are replaced with finite differences. The discretized Dupire's equation below is the equation that is implemented. This method is also called the explicit Euler method.

$$\frac{u(t+1, K) - u(t, K)}{\Delta t} = \frac{1}{2}\sigma^2 K^2 \frac{u(t, K+1) - 2u(t, K) + u(t, K-1)}{\Delta K^2}, \tag{28}$$

which can be written as

$$u(t+1) = \mathrm{A}u(t), \tag{29}$$

where A is a tridiagonal matrix with 1-2q on the diagonal and q on the sides and

$$q = \frac{\sigma^2 K^2 \Delta t}{2\Delta K^2}. \tag{30}$$

The initial (IC) and boundary (BC) conditions to Dupire's equation are

$$IC : u(0, K) = max(S - K, 0) \tag{31}$$

$$BC : u(t, 0) = S \ , \ BC : u(t, K_{max}) = 0 \tag{32}$$

4.1.2 Method 2

For method 2 the goal is to derive a PDE for the implied volatility. When having only one equation that can compute the volatility one reduces the amount of total steps which can reduce the numerical error.

Theorem 1. *Assuming the local volatility σ is bounded between two constants $0 < \underline{\sigma} \leq \sigma(K) \leq \overline{\sigma} < \infty$ and the implied volatility φ is defined implicitly as $C(T, K) = u(T\varphi^2(T, K), K)$ where C is the call-option price at time T with strike K and u solves the Black-Sholes formula. Then the implied volatility φ satisfies*

$$\begin{aligned} \varphi^2 + 2\varphi\varphi_T T = \sigma^2(1 + Kln\frac{S}{K}\frac{\varphi_K}{\varphi})^2 + \\ \frac{\sigma^2}{2}(2TK^2(\varphi\varphi_K)_K + (2TK\varphi\varphi_K)^2(\frac{1}{2TK\varphi\varphi_K} - \frac{1}{2T\varphi^2} - \frac{1}{8})) \end{aligned} \tag{33}$$

Where $\varphi = \varphi(T, K)$ and $\sigma = \sigma(K)$. Moreover as $T \to 0$ one has that $\lim_{T\to 0}\varphi(T, K) = \psi(K)$ where $\psi(K)$ solves the ordinary differential equation (ODE) 34 below,

$$\psi^2\frac{\sigma}{Kln\frac{S}{K}} - \psi\frac{}{Kln\frac{S}{K}} = \psi_K \tag{34}$$

Proof. Start by defining Dupire's formula assuming no drift i.e. $r = 0$.

$$\begin{aligned} C_T = \frac{\sigma^2}{2}K^2C_{KK} \\ C(0, K) = (S - K)^+ \end{aligned} \tag{35}$$

Now define $u(\tau, K)$ which solves the equation below,

$$
\begin{aligned}
u_\tau &= \frac{K^2}{2} u_{KK} \\
u(0, K) &= (S - K)^+
\end{aligned}
\tag{36}
$$

Equation 36 can be solved analytically using the analytical solution to Black-Sholes equation i.e.

$$
u(\tau, K) = SN(d_1) - KN(d_2) \tag{37}
$$

where,

$$
\begin{aligned}
d_1 &= \frac{ln\frac{S}{K} + \frac{\tau}{2}}{\sqrt{\tau}}, \\
d_2 &= \frac{ln\frac{S}{K} - \frac{\tau}{2}}{\sqrt{\tau}}.
\end{aligned}
\tag{38}
$$

Now the implied volatility $\varphi(T, K)$ is defined implicitly by $C(T, K) = u(T\varphi^2(T, K), K)$. Now the goal is to work out the derivatives C_T and C_{KK} for u.

$$
C_T = u_\tau \varphi^2 + u_\tau 2\varphi\varphi_T T, \tag{39}
$$

$$
C_K = u_K + u_\tau 2\varphi\varphi_K T \tag{40}
$$

and

$$
C_{KK} = u_{KK} + 4T\varphi\varphi_K u_{\tau K} + 2T\varphi\varphi_{KK} u_\tau + 2T\varphi_K^2 u_\tau + (2T\varphi\varphi_K)^2 u_{\tau\tau} \tag{41}
$$

Now plug the derived derivatives into Dupire and for u_{KK} use $u_{KK} = \frac{2}{K^2} u_\tau$ from 36.

$$
\begin{aligned}
u_\tau(\varphi^2 + 2\varphi\varphi_T T) = u_\tau \frac{\sigma^2}{2}(2 + 4TK^2\varphi\varphi_K \frac{u_{\tau K}}{u_\tau} + 2TK^2\varphi\varphi_{KK} + \\
2TK^2\varphi_K^2 + (2TK\varphi\varphi_K)^2 \frac{u_{\tau\tau}}{u_\tau})
\end{aligned}
\tag{42}
$$

The next step is to work out the two second derivatives $u_{\tau K}$ and $u_{\tau\tau}$. To do that a few relations is needed. The relations can be seen below in the proposition,

Proposition 1.

1. $d_{1K} = d_{2K}$
2. $d_{1\tau} = -\frac{d_2}{2\tau}$
3. $Kn(d_2) = Sn(d_1)$

Where n is the normal probability density function $n(d_1) = \frac{1}{\sqrt{2\pi}} e^{-\frac{d_1^2}{2}}$.

Proof.

1.

$$d_{1K} = -\frac{1}{K} = d_{2K} \tag{43}$$

2.

$$\begin{aligned} d_{1\tau} = -\frac{1}{2}(ln\frac{S}{K} + \frac{\tau}{2})\tau^{-\frac{3}{2}} + \frac{1}{2}\tau^{-\frac{1}{2}} = \\ -\frac{1}{2}ln\frac{S}{K}\tau^{-\frac{3}{2}} - \frac{1}{2}\frac{\tau^{-\frac{1}{2}}}{2} + \frac{1}{2}\tau^{-\frac{1}{2}} = \\ -\frac{1}{2\tau}(ln\frac{S}{K} - \frac{\tau}{2})\tau^{-\frac{1}{2}} = -\frac{d_2}{2\tau} \end{aligned} \tag{44}$$

3.

$$\begin{aligned} Kn(d_2) = \frac{K}{\sqrt{2\pi}}e^{-\frac{d_2^2}{2}} = \frac{K}{\sqrt{2\pi}}e^{-\frac{(d_1-\sqrt{\tau})^2}{2}} = \frac{K}{\sqrt{2\pi}}e^{-\frac{(d_1^2-2\sqrt{\tau}d_1+\tau)}{2}} = \\ n(d_1)\frac{K}{\sqrt{2\pi}}e^{-\frac{2\sqrt{\tau}d_1+\tau}{2}} = n(d_1)Ke^{ln\frac{S}{K}} = Sn(d_1) \end{aligned} \tag{45}$$

□

Now when the three relations are proven it is time to work out u_τ $u_{\tau K}$ and $u_{\tau\tau}$.

$$u_\tau = Sn(d_1)d_{1\tau} - Kn(d_2)d_{2\tau} = [\text{relation 3}] = Sn(d_1)(d_{1\tau} - d_{2\tau}) = \\ [\text{relation 2}] = \frac{d_1 - d_2}{2\tau}Sn(d_1) = \frac{Sn(d_1)}{2\sqrt{\tau}} \tag{46}$$

$$u_{\tau\tau} = -\frac{Sd_1n(d_1)d_{1\tau}}{2\tau} - \frac{Sn(d_1)}{4\tau\sqrt{\tau}} = (-d_1d_{1\tau} - \frac{1}{2\tau})u_\tau = \\ [\text{Relation 2}] = (\frac{d_1d_2 - 1}{2\tau})u_\tau \tag{47}$$

$$u_{\tau K} = -\frac{Sd_1n(d_1)d_{1K}}{2\tau} = \frac{Sd_1n(d_1)}{2\tau}\frac{1}{K\sqrt{\tau}} = \frac{d_1}{K\sqrt{\tau}}u_\tau \tag{48}$$

Now plugging in the derived derivatives above into 42 and dividing both sides by u_τ one gets,

$$\varphi^2 + 2\varphi\varphi_T T = \frac{\sigma^2}{2}(2 + 4TK^2\varphi\varphi_K\frac{d_1}{K\sqrt{\tau}} + 2TK^2\varphi\varphi_{KK} + \\ 2TK^2\varphi_K^2 + (2TK\varphi\varphi_K)^2(\frac{d_1d_2 - 1}{2\tau})) \tag{49}$$

Plugging in d_1 and d_2 one gets,

$$\varphi^2 + 2\varphi\varphi_T T = \frac{\sigma^2}{2}(2 + 4TK^2\varphi\varphi_K\frac{\frac{ln\frac{S}{K} - \frac{\varphi^2T}{2}}{\varphi\sqrt{T}}}{K\varphi\sqrt{T}} + 2TK^2\varphi\varphi_{KK} + \\ 2TK^2\varphi_K^2 + (2TK\varphi\varphi_K)^2(\frac{\frac{(ln\frac{S}{K})^2}{\varphi^2T} - \frac{\varphi^2T}{4} - 1}{2\varphi^2T})), \tag{50}$$

which can be written as,

$$\varphi^2 + 2\varphi\varphi_T T = \frac{\sigma^2}{2}(2 + 4Kln\frac{S}{K}\frac{\varphi_K}{\varphi} + 2TK\varphi\varphi_K + \\ 2TK^2\varphi\varphi_{KK} + 2TK^2\varphi_K^2 + K^2(ln\frac{S}{K})^2\frac{\varphi_K^2}{\varphi^2} - \\ (2TK\varphi\varphi_K)^2\frac{1}{8} - (2TK\varphi\varphi_K)^2\frac{1}{2\varphi^2T}) \tag{51}$$

which can be simplified to,

$$\varphi^2 + 2\varphi\varphi_T T = \sigma^2(1 + Kln\frac{S}{K}\frac{\varphi_K}{\varphi})^2 + \frac{\sigma^2}{2}(2TK^2(\varphi\varphi_K)_K + (2TK\varphi\varphi_K)^2(\frac{1}{2TK\varphi\varphi_K} - \frac{1}{2T\varphi^2} - \frac{1}{8})) \tag{52}$$

which again can finally be simplified to,

$$\varphi_T = -\frac{\varphi}{2T} + \frac{\sigma^2}{2\varphi T}(1 + Kln\frac{S}{K}\frac{\varphi_K}{\varphi})^2 + \frac{\sigma^2 K^2 \varphi_{KK}}{2} + \frac{\sigma^2 K\varphi_K}{2} - \frac{\sigma^2 TK^2\varphi\varphi_K^2}{8} \tag{53}$$

□

To solve and compute the implied volatility one needs to first find the initial value of φ. To find $\varphi(0, K)$ one lets $T \to 0$ in the equation above and one then gets,

$$\varphi^2 = \sigma^2(1 + Kln\frac{S}{K}\frac{\varphi_K}{\varphi})^2. \tag{54}$$

The equation above can be by taking the square root on both sides rewritten as,

$$\varphi^2 \frac{1}{Kln\frac{S}{K}} - \varphi\frac{\sigma}{Kln\frac{S}{K}} = \varphi_K. \tag{55}$$

The equation above is a first order non-linear ODE and by solving it one gets the initial value to equation 53. It can be solved through regular ODE solving techniques starting by substituting $\varphi = v^m$ and $\varphi_K = mv^{m-1}v_K$ which gives,

$$v^{2m}p(K) - v^m f(K) = mv^{m-1}v_K, \quad p(K) = \frac{1}{\sigma Kln\frac{S}{K}} \text{ and } f(K) = \frac{1}{Kln\frac{S}{K}}. \tag{56}$$

Divide both side by v^{m-1} which gives

$$v^{m+1}p(K) - vf(K) = mv_K. \tag{57}$$

Now let $m = -1$ which means that $\varphi = \frac{1}{v}$ and $\varphi_K = -v_K$. The relation between φ and v will be used later but before that when using $m = -1$ one gets,

$$p(K) - vf(K) = -v_K \Rightarrow v_K - vf(K) = -p(K). \tag{58}$$

The equation above is a first order linear ODE which can be solved by first finding the integrating factor $g(K)$ and then plug it into the general solution,

$$v(K) = \frac{\int g(K)p(K)dK}{g(K)}. \tag{59}$$

The integrating factor is,

$$g(K) = e^{\int f(K)dK} = ln(\frac{S}{K}), \tag{60}$$

and by plugging it all into the general solution using $\sigma = \frac{1}{K^\gamma}$ one gets,

$$v(K) = \frac{\int ln\frac{S}{K}\frac{1}{K^{-\gamma}Kln\frac{S}{K}}dK}{ln\frac{S}{K}} = \frac{\int \frac{1}{K^{1-\gamma}}dK}{ln\frac{S}{K}} = -\frac{\frac{1}{\gamma}K^\gamma}{ln\frac{S}{K}}. \tag{61}$$

Now substituting back $\varphi(0,K) = \frac{1}{v(K)}$ one finally gets the initial value for $\varphi(0,K)$

$$\varphi(0,K) = \frac{1}{v(K)} = -\frac{ln\frac{S}{K}}{\frac{K^\gamma}{\gamma}} = -\gamma K^{-\gamma}ln\frac{S}{K} \tag{62}$$

The benefit of this approach is that the second method reduces the amount of steps needed to obtain the implied volatility. In method one to obtain the

implied volatility one needs to first solve Dupire's equation then use the solution to compute in implied volatility through optimization. In method two one only needs to solve the PDE seen in equation 53.
Now it is somewhat clear where equation 53 derives from. In the result section the solution is plotted to try visualize that if the local volatility is decreasing the implied volatility is decreasing as well if we have a decreasing initial condition. To strengthen the result a derivation is done as well to show with pen and paper that in theory we can expect that if the initial condition is decreasing in space then the implied volatility will not increase in space as the time goes.

4.2 Jump-diffusion model

The goal is to plot the implied volatility as a function of the strike price K and see if the implied volatility decreases when the underlying model uses negative jumps.

Procedure.

1. Define the underlying model and its parameters.
2. Compute the call-option prices for different strike prices, using the jump model as the underlying model.
3. Calibrate the implied volatility using the computed option prices and the Black-Scholes formula.
4. plot the implied volatility.

5 Result

5.1 Theoretical results

Informal Proof. Assume first that there is a point in time at (T_0, K_0) where φ_K goes from $\varphi_K < 0$ to $\varphi_K > 0$ i.e. at the point (T_0, K_0) $\varphi_K(T_0, K_0) = 0$. This means that at this point the derivative in K will never be positive and in fact along K at time T_0 this is a maximum point and we can write the different derivatives at this point as.

- $\varphi_K(T_0, K_0) = 0$

- $\varphi_{KK}(T_0, K_0) = 0$
- $\varphi_{KKK}(T_0, K_0) \leq 0$

Now this proof is called an informal proof and this is because to even start deriving the proof one has to make a few assumptions. First of all one has to assume that it even exists a point discussed above, also here it is assumed that the implied volatility is positive otherwise one can not be certain that $\partial_T \varphi_K \leq 0$ which will be shown below. What the derivation below shows is that $\partial_T \varphi_K \leq 0$ which means that the implied volatility will never increase in space as the time goes. Starting with what is trying to be shown one get.

$$\partial_T \varphi_K \leq 0. \tag{63}$$

Using what is known in equation 53 this can be written as,

$$\begin{aligned}\partial_T \varphi_K = \partial_K \varphi_T = \partial_K(-\frac{\varphi}{2T} + \frac{\sigma^2}{2\varphi T}(1 + Kln\frac{S}{K}\frac{\varphi_K}{\varphi})^2 + \\ \frac{\sigma^2 K^2 \varphi_{KK}}{2} + \frac{\sigma^2 K \varphi_K}{2} - \frac{\sigma^2 T K^2 \varphi \varphi_K^2}{8})\end{aligned} \tag{64}$$

now differentiation of the terms on the right hand sided with respect to K gives,

$$\begin{aligned}\partial_K \varphi_T = -\frac{\varphi_K}{2T} + (\frac{\sigma^2}{2T})_K \frac{1}{\varphi} - \frac{\sigma^2}{2T\varphi^2}\varphi_K + \partial_K(2Kln\frac{S}{K}\frac{\varphi_K}{\varphi}) + \\ \partial_K((Kln\frac{S}{K}\frac{\varphi_K}{\varphi})^2) + (\frac{\sigma^2}{2})_K K^2 \varphi_{KK} + \sigma^2 K \varphi_{KK} + \frac{\sigma^2 K^2}{2}\varphi_{KKK} + \\ \partial_K(\frac{\sigma^2 K \varphi_K}{2}) - \partial_K(\frac{\sigma^2 T K^2 \varphi \varphi_K^2}{8})\end{aligned} \tag{65}$$

Using $\varphi_K(T_0, K_0) = 0$ and $\varphi_{KK}(T_0, K_0) = 0$ one can now remove a lot of terms that will include either φ_K and φ_{KK}. This will result in the following expression.

$$\partial_T \varphi_K = \partial_K \varphi_T = \frac{\sigma^2 K^2}{2}\varphi_{KKK} + (\frac{\sigma^2}{2T})_K \frac{1}{\varphi}. \tag{66}$$

Knowing that something squared is always positive and that the third derivative is less or equal than zero the first term in equation 66 will be non-positive. The second term in equation 66 above consists of $(\frac{\sigma^2}{2T})_K$ which is non-positive if the function $\sigma(K)$ is monotonically decreasing in K. Thus the second term in equation 66 is non-positive as well and one can therefore say,

$$\partial_T \varphi_K = \partial_K \varphi_T = \frac{\sigma^2 K^2}{2} \varphi_{KKK} + (\frac{\sigma^2}{2T})_K \frac{1}{\varphi} \leq 0. \tag{67}$$

This informal proof is not entirely mathematically correct but it shows the tendency that if one plugs in something that is decreasing at time t=0 the next time step will not increase. Since the initial condition is $\varphi(0, x) = x^{-\gamma}, \gamma > 0$ which decreases in space the next time step should not tend to increase. In this project φ is defined to be positive which makes the second term in equation 66 non-positive. Now what happens if one does not make the assumption on φ. If φ is negative then equation 66 can be positive. However if φ increases in time it will eventually become positive and then equation 66 will become negative and φ will start decreasing. One can in fact write down two conditions to satisfy $\partial_T \varphi_K \leq 0$. This result bases on the assumption that $\varphi_K(t, K)$ never passes $\varphi_K(0, K) = 0$ and becomes positive so the largest value $\varphi_K(t, K)$ can have is 0. There fore one ends up with $\varphi_K(t, K) \leq 0$.

Conditions for the inequality in equation 67 to hold.

- The initial value of $\varphi(0, K) \geq 0$,$\forall$ $K \in [0, K_{max}]$.
- The initial value of $\varphi_K(0, K) \leq 0$,$\forall$ $K \in [0, K_{max}]$.
- The local volatility needs to be monotonically decreasing $\sigma_K \leq 0$,$\forall$ $K \in [0, K_{max}]$

Again since the proof that the implied volatility decreases in K above is sort of a weak proof, the conditions are also somewhat weak conditions. However if the conditions are held one can show theoretically that the implied volatility decreases in K.

5.2 Visualised results

5.2.1 Local volatility model

To compute the implied volatility it is important to check that the computed option prices is what one expects. The expected result is that the option price should follow the payoff function starting at C=S and go to zero value as K grows larger. This result is what can be seen in figure 4.

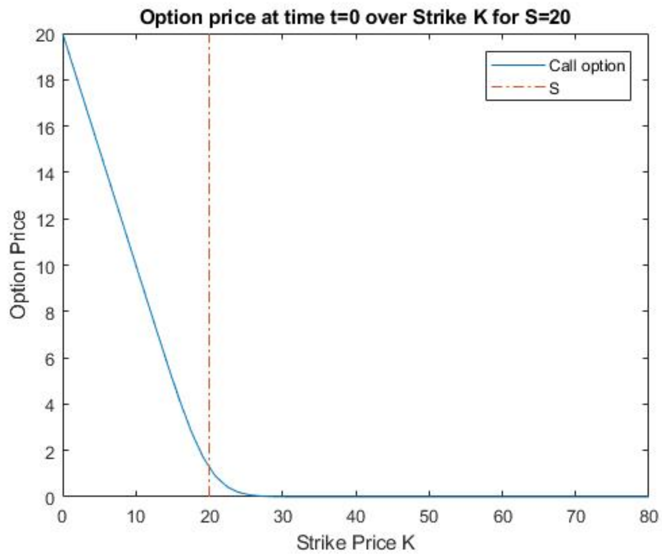

Figure 4: Option price for different strike prices at time t=0 and S=20.

Since the option prices is correct the implied volatility can now be computed by finding the volatility that minimizes the difference between computed option price and the analytical option price. The expected result is that the implied volatility follows the local volatility and more importantly that the implied volatility does not grow as K increases since the local volatility is decreasing with K.

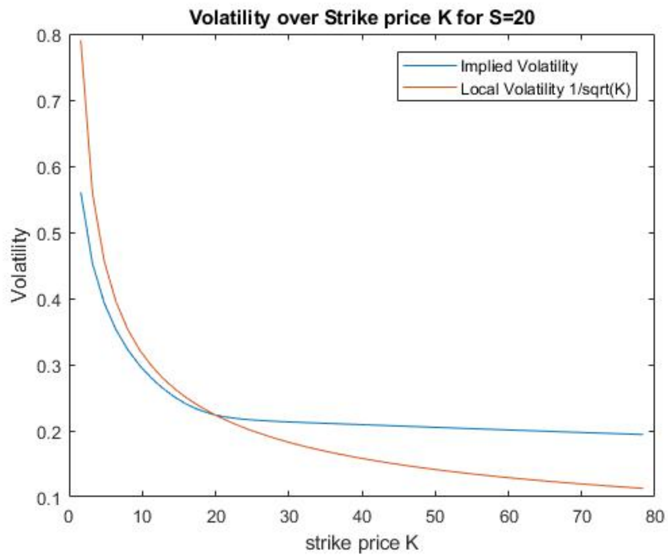

Figure 5: The initial local volatility (orange) used to compute the implied volatility (blue). Local volatility as a function of the strike price $\sigma = \frac{1}{\sqrt{K}}$

The aim is to test what was mentioned in the introduction that if the market is pricing using a decreasing local volatility model then the implied volatility also decreases as the strike price is increasing. In figure 6 different values of gamma have been tested and on the span $K = [0 : 4S]$ the implied volatility does not increase.

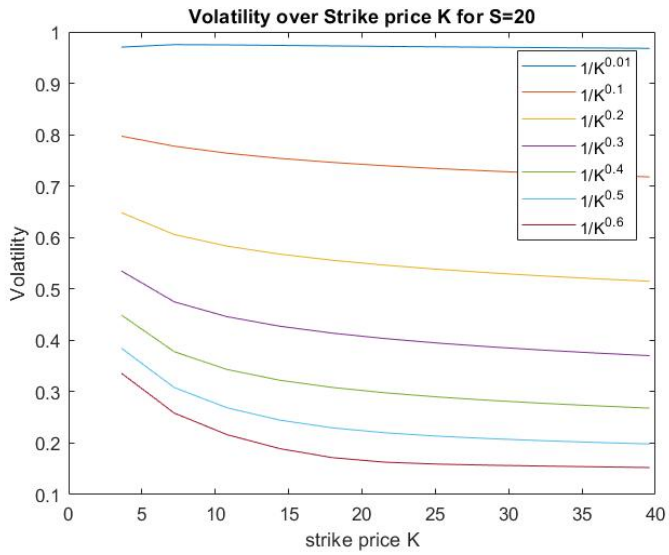

Figure 6: Here the implied volatility is plotted computed from the local volatility as a function of the strike price $sigma = K^{-\gamma}$ when varying γ.

In the four small figures below one can see what happens with different right side boundary condition (BC). The idea of this result is to show that it is important to choose a correct BC when solving Dupire's equation, or at least know what impact the different types of boundary conditions can have on the final solution. The BC for the four figures respectively is no BC, $u = 0$, $u_x = 0$ and $u_{xx} = 0$. No BC means that the boundaries when using finite differences are computed using forward and backward approximations.

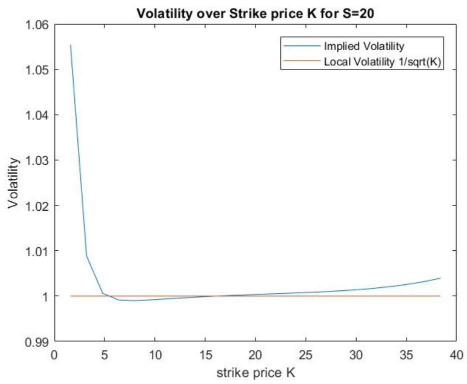

Figure 7: No right side boundary condition.

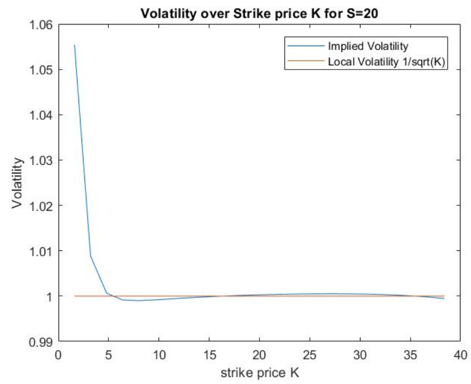

Figure 8: Right side boundary condition $u(\tau, K_{max}) = 0$.

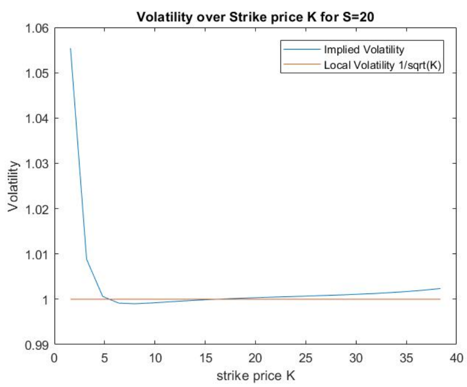

Figure 9: Right side boundary condition $u_x(\tau, K_{max}) = 0$.

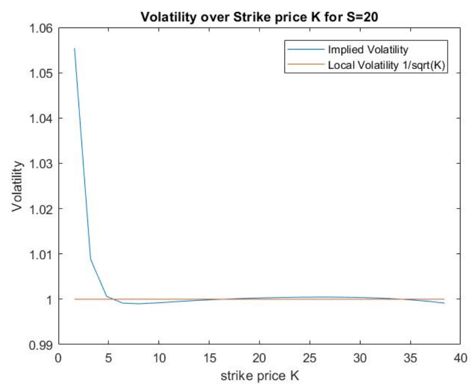

Figure 10: Right side boundary condition $u_{xx}(\tau, K_{max}) = 0$.

Figure 11 is the result from implementing equation 53 and the implied volatility is plotted together with the local volatility. From figure 11 one can see that the implied volatility decreases as the strike price increases similarly to the local volatility.

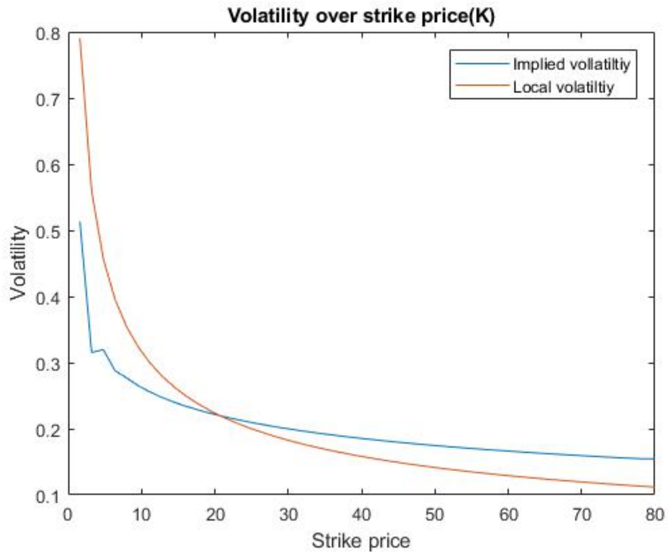

Figure 11: The solution to the PDE in equation 53 at time T=2 starting time $t_0 = 0.04$ with local volatility $\sigma(K)_{loc} = \frac{1}{\sqrt{K}}$

The two methods to compute the implied volatility both decreases in K and are plotted together with the local volatility in figure 12.

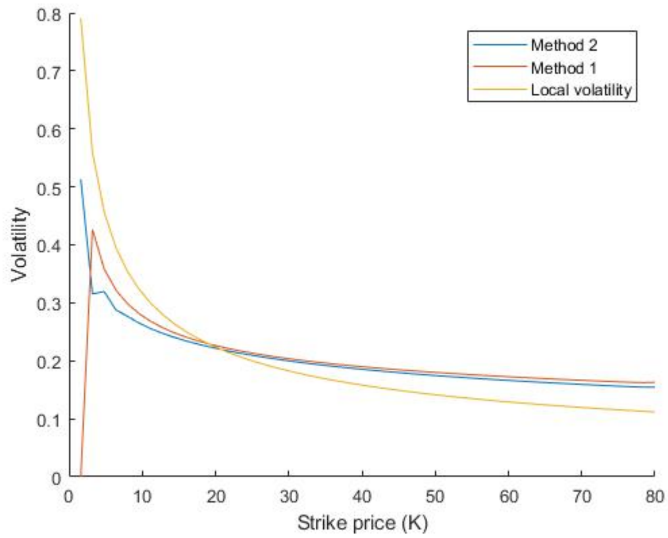

Figure 12: Implied volatility computed from method 2 and 1 and the local volatility with $t = 0.04, T = 2$ and $\sigma(K) = \frac{1}{\sqrt{K}}$.

The difference between the two methods of computing implied volatilities is smallest around the strike price as can be seen in figure 13. After the strike price the two methods slowly converges to different volatilities however method 2 ends up closer to the local volatility at larger strike price. For smaller strike price or close to the strike price method 1 is closer to the local volatility.

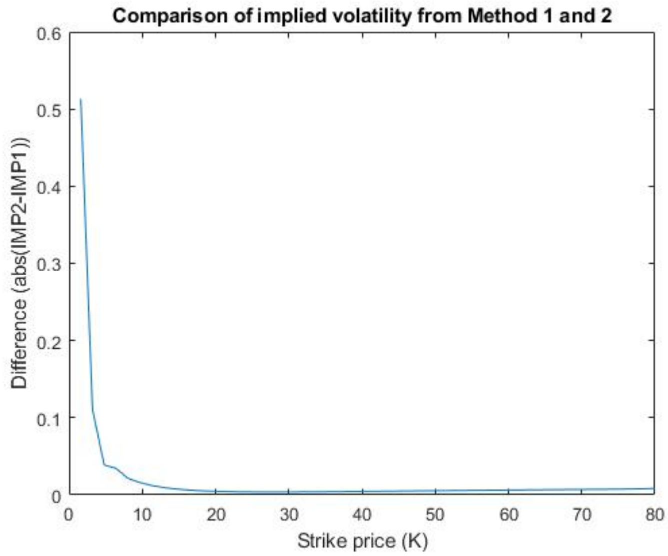

Figure 13: The absolute difference between the implied volatility from method 2 and the implied volatility from method 1. i.e. $\sqrt{\varphi_2^2 - \varphi_1^2}$.

5.2.2 Jump-diffusion model

In this section the results from the jump-diffusion model are visualized. First a comparison between the call option price using a local volatility model and a jump diffusion model is plotted so that one can see what effect a jump diffusion model can have compared to the standard local volatility model. After that the two parameters jump frequency and jump length are varied to visualise the effect the two parameters have on the option price and implied volatility.

In figure 14 the call option price is computed and plotted when using the blacksholes formula with no interest rate, no dividend yield, initial asset value is 20 and a volatility of 0.15 and the call option price when using a jump-diffusion process with the same parameters and a proportional jump size $\gamma = 0.2$ and jump frequency $\lambda = 1$.

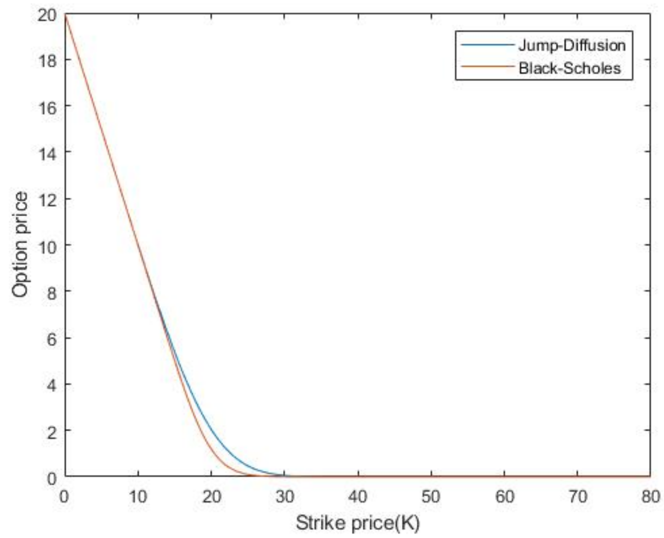

Figure 14: The price of a Call-option without jumps and the price for a call option based on an asset with jumps($\gamma = 0.2$, $\lambda = 1$, $\sigma = 0.15$).

In figure 15 the implied volatility is plotted. The implied volatility is computed from the two call-option prices above by minimizing the difference between the prices of the two call-options i.e. $0 = C_{Jump} - C_{BS}(\varphi)$.

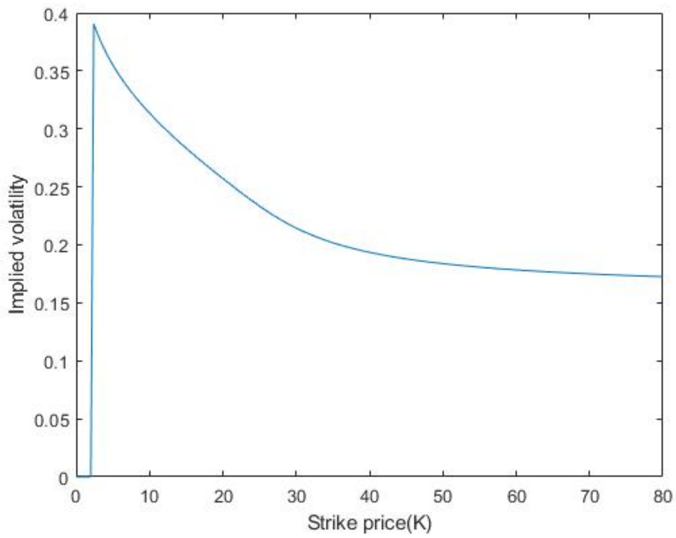

Figure 15: Implied volatility for a call option based on an asset with jumps($\gamma = 0.2$, $\lambda = 1$, $\sigma = 0.15$).

In figure 16 the jump intensity is fixed and the proportional jump size is varied. By adding negative jumps the call option price increases and by increasing the proportional jump size the call option price increases.

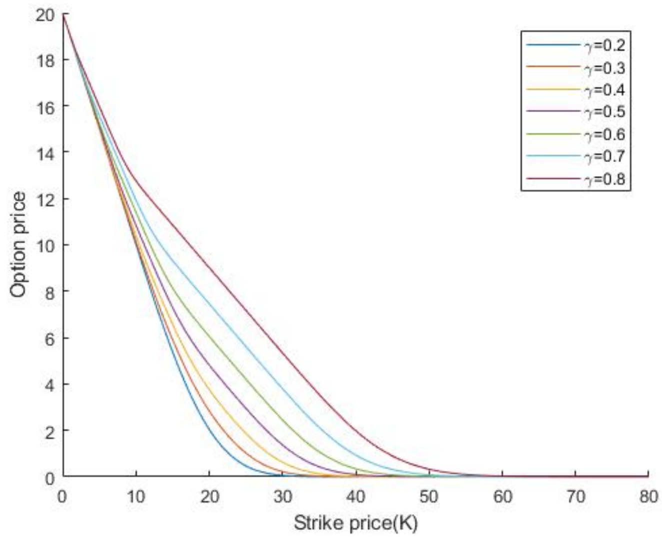

Figure 16: Call-option prices using $\lambda = 1$, $\sigma = 0.15$ $S = 20$ and varying gamma.

Figure 17 is the implied volatility computed from the result in figure 16. By increasing the proportional jump size the implied volatility is also increasing.

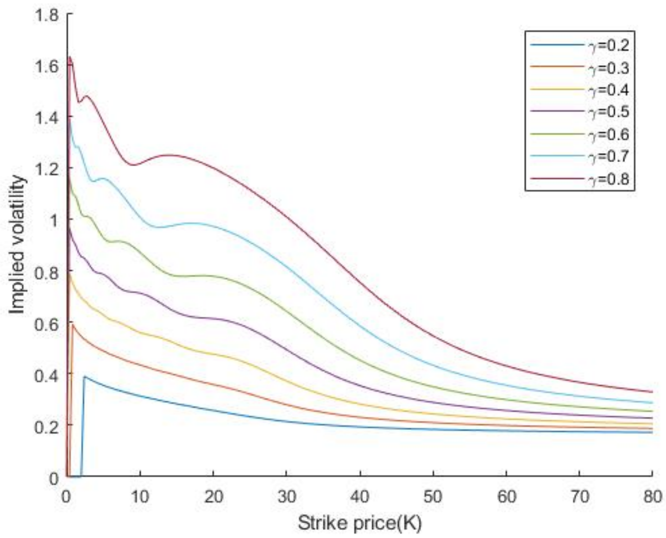

Figure 17: Implied volatility for call-option prices using $\lambda = 1$, $\sigma = 0.15$ $S = 20$ and varying gamma.

If one keep the proportional jump size γ fixed and instead varies the jump intensity λ one can see a similar effect on the option price. By increasing the jump intensity the option price increases as one can notice in figure 18.

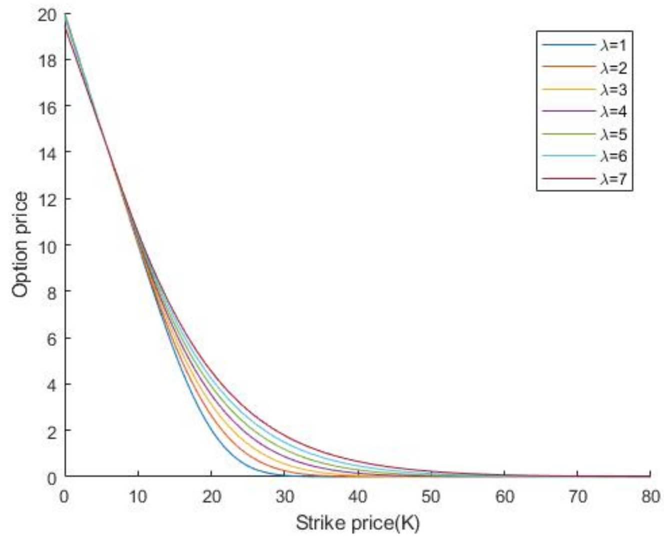

Figure 18: Call-option prices using $\gamma = 0.2$, $\sigma = 0.15$ $S = 20$ and varying lambda.

As in the previous plots the increase in option price also results in an increase in implied volatility where a higher jump intensity gives a higher implied volatility.

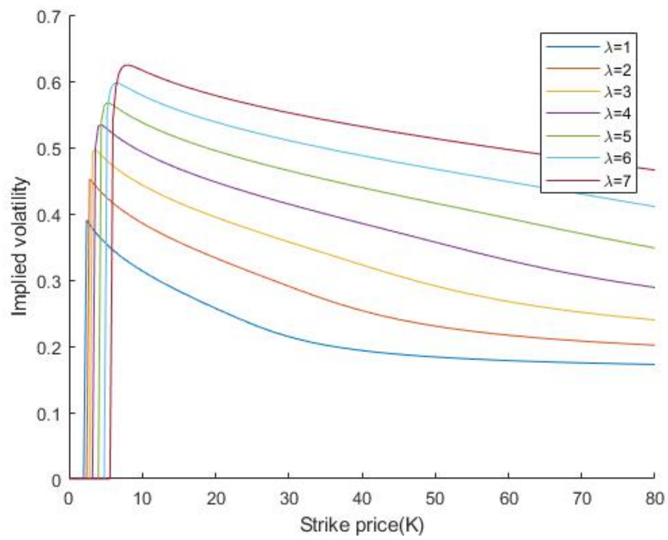

Figure 19: Implied volatility for call-option prices using $\gamma = 0.2$, $\sigma = 0.15$ $S = 20$ and varying lambda.

To summary the results from the jump-diffusion model one notices that by adding negative jumps one increases the option price and therefore also the implied volatility. By increasing the proportional jump size and/or the jump intensity the option price and implied volatility increases. What can also be seen is that negative jumps seem to give a implied volatility that decreases in K.

6 Discussion

6.1 Local voatility model

Regarding figure 7-10 where it is tried to visualise the impact of the right side boundary condition(BC) has on the solution and in the end on the implied volatility. To take a different view of this one can try to visualise the option price as a heating rod since Dupire's equation with r=0 is a diffusion process. The left hand side is fixed at a value here S which can be seen as the heated rod has a fixed temperature on the left side. As the time goes the heat diffuses along the rod and the process on the right side can be interpreted

differently depending on what type of BC at that point.
So no right-side BC would mean that there is no control on the right side which means that if the left side is kept at a constant temperature eventually the hole rod will have that temperature. If The right-side BC is equal to a constant for instance 0, that means that there is a fixed temperature on the right hand side as well and the temperature will decrease from S to 0 along the rod. If the derivative on the right-side is set to 0 this can be interpreted as the temperature is not fixed along the rod however the temperature difference along the rod will decrease in other word the heat flow at the right-side is constant and in this case 0. This problems comes into play when trying to solve Dupire's formula since the strike price range can not continue to infinity instead one has to choose where to stop computing and in this case $K_{max} = 4 * S$. Setting the solution at the right-side to zero would be to underestimate the solution at that point since $\lim_{K \to \infty} u(T, K) = 0$ and setting the first derivative to zero would therefor be an overestimate of the solution at $K = K_{max}$. Using the second derivative as a BC would mean that the heat-flow can be at most linear. Computations for method two did not start at initial time $t_0 = 0$ unfortunately instead they start at $t_0 = 0.04$ this is because as $t_0 \to 0$ the computations becomes unstable. This is due to the fact that for the first time steps t is small and when dividing by something small the result become large. This can potentially be solved by taking longer time steps so that one avoids the small times, however by doing so the time steps becomes larger and one needs to adjust the step length in K.

6.2 Jump-diffusion model

The results from the jump-diffusion model shows that when adding negative jumps one increases the option price. One reason for this comes from the fact that one has to keep the martingale when constructing the underlying model. What that means is that if the process Xt is a martingale then the expected value $E[X_{t_0} - X_{t_1}|F_0] = 0$ where F_0 denotes the information one knows up until $t = t_0$. When one adds negative one will not end up with with a martingale since $E[X_{t_0}|F_0] > E[X_{t_1}|F_0]$, therefore a positive drift is added to the model hence the option price increases as one can see in figure 14. Since negative jumps gave a positive drift, increasing the proportional jump size would increase the drift more hence the option price increases as can be seen in figure 16. By increasing the proportional jump size the implied volatility also increases. In figure 17 one can see for the larger jumps that the implied volatility drops a few times. The large but few drops for when using a large proportional jump size is due to when using large negative jumps the stock can only drop a few times before the option is worth nothing. When using smaller negative proportional jump sizes the stock can jump more times before the option becomes worthless, this is why one can see in figure 17 when γ is small the drops happens more frequent but they are smaller. In figure 19 the proportional jump size is set to $\gamma = 0.2$ thus the drops is barely nonexistent however the implied volatility is higher when increasing the jump intensity. This is due to the increase in the option price, more jumps under a time period means that the drift will become higher witch increases the option price.

7 Conclusion

In the section method 1 there is no derivation one can use to prove the asymptotics between local and implied volatility since that method takes an optimization approach where the optimization tool is a black box and the user only have to crate and set up the problem. However the results from the implementation shows that using a local volatility which decreases will result in the implied volatility having monotonicity as well. On the other hand for method 2 it is possible to show the mononticity by both derivation and implementation. In the end of the section method 2 it is shown what happens with φ_K as time goes and what was shown was that it is non-positive which means that if the local volatility is decreasing in K then φ should not increase in space as the time goes. Results from method 1 i.e. figure 5 and 6 and the derivation from method 2 which suggests that the monotonicity should be kept as the strike price increases together with figure 12 is a good argument that the monotonicity is kept for the implied volatility since two related but different methods show the same results. The initial value of the implied volatility is positive and the local volatility function is monotonically decreasing in K which means that the monotonicity is kept. Method 2 is gives a direct path between the local volatility and the implied volatility through the derived PDE. By using a decreasing local volatility one could from the PDE show that the implied volatility had the tendency to decrease in K as well. Theoretically method 2 gives PDE that can be derived with a initial condition that can be derived as well however computation wise method 2 needs further research. The derived PDE from method 2 is a non-linear PDE which has terms divided by the time which can cause problems when the time is small. For further research an interesting project is to implement the PDE and investigate it from a computational point of view. Pros for method 1 is that it is easily implemented and derived, however in method 1 one does not obtain a equation linking the implied volatility and local volatility as in method 2.

The goal of investigating a jump-diffusion model is to see if by using negative jumps the implied volatility decreases in K. The results from the jump-diffusion model showed that when adding negative jumps the option price increases and when one computes the implied volatility from those option prices one gets an implied volatility that deceases in K. The increase in option price comes from the effect getting a positive drift from adding negative jumps. One could also see that increasing the proportional jump size so that the negative jumps were larger resulted in a higher option price and higher implied volatility. A similar effect could be seen when increasing the jump intensity as well.

8 References

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[2] Tomas Björk Arbitrage theory in continuous time third edition chapter 16, p 235.

[3] Fabrice Douglas Rouah. *Derivation of Local Volatility*.

[4] Rama Cont and Peter Tankov Financial modeling with jump processes, p 359.