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Cells and 2-representations of bimodules over Nakayama algebras

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Cell structure of bimodules over radical square zero Nakayama algebras

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ABSTRACT
In this paper, we describe the combinatorics of the cell structure of the tensor category of bimodules over a radical square zero Nakayama algebra. This accounts to an explicit description of left, right and two-sided cells.

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1. Introduction and description of the results

Let $\Bbbk$ be an algebraically closed field of characteristic zero. For a positive integer $n > 1$, let Q_n denote the quotient of the path algebra of the quiver

$$\begin{tikzcd}
 & & & 1 \arrow[dlll, "\alpha_1"'] & & \\
2 \arrow[r, "\alpha_2"'] & 3 \arrow[r, "\alpha_3"'] & 4 \arrow[r, "\alpha_4"'] & \dots \arrow[r, "\alpha_{n-2}"'] & n-1 \arrow[r, "\alpha_{n-1}"'] & n \arrow[ulll, "\alpha_n"']
\end{tikzcd}$$

of affine Coxeter type $\tilde{A}_{n-1}$ by the relations that all paths of length two in this quiver are equal to zero. We note that we compose paths from right to left.

We also denote by Q_1 the algebra $\Bbbk[x]/(x^2)$ of dual numbers over $\Bbbk$. This algebra is isomorphic to the quotient of the path algebra of the quiver

$$\begin{tikzcd}
0 \arrow[loop right, "\alpha_0"]
\end{tikzcd}$$

by the relation that the path of length two in this quiver is equal to zero. For each positive integer n, the algebra Q_n is a Nakayama algebra, see [15]. In what follows we fix n and set $A = Q_n$.

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Consider the tensor category A-mod-A of all finite dimensional A-A-bimodules and let $\mathcal{S}$ denote the set of isomorphism classes of indecomposable objects in A-mod-A. Note that $\mathcal{S}$ is an infinite set. For an A-A-bimodule X, we will denote by $[X]$ the class of X in $\mathcal{S}$. By [12, Section 3], the set $\mathcal{S}$ has the natural structure of a multisemigroup, cf. [9], defined as follows: for two indecomposable A-A-bimodules X and Y, we have

$$[X] \star [Y] := \{[Z] \in \mathcal{S} \ : \ Z \text{ is isomorphic to a direct summand of } X \otimes_A Y\}.$$

The basic combinatorial structure of a multisemigroup (or, as a special case, of a semigroup) is encoded into the so-called Green's relations, see [4, 5, 9]. For $\mathcal{S}$, these Green's relations are defined as follows.

For two indecomposable A-A-bimodules X and Y, we write $[X] \geq_L [Y]$ provided that there is an indecomposable A-A-bimodule Z such that $[X] \in [Z] \star [Y]$. The relation $\geq_L$ is a partial preorder on $\mathcal{S}$, called the *left preorder* and equivalence classes for $\geq_L$ are called *left cells*. Similarly, one defines the *right preorder* $\geq_R$ and *right cells* using $[Y] \star [Z]$, and also the *two-sided preorder* $\geq_J$ and *two-sided cells* using $[Z] \star [Y] \star [Z']$. We will abuse the language and often speak about cells of bimodules (and not of isomorphism classes of bimodules).

The main aim of the present paper is an explicit description of left, right, and two-sided cells in $\mathcal{S}$. As the algebra $A \otimes_{\mathbb{k}} A^{\mathrm{op}}$ is special biserial, cf. [1, 17], all indecomposable A-A-bimodules split into three types, see Section 2 for details:

- string bimodules;
- band bimodules;
- non-uniserial projective-injective bimodules.

Following [8, 14], for a string bimodule X we consider a certain invariant $\mathbf{v}(X)$ called the *number of valleys* in the graph of X. The structure of two-sided cells is described by the following:

Theorem 1.

a. *All band bimodules form a two-sided cell denoted $\mathcal{J}_{\text{band}}$.*
b. *For each positive integer k, all string bimodules with $\mathbf{v}(X) = k$ form a two-sided cell denoted $\mathcal{J}_k$.*
c. *All $\mathbb{k}$-split bimodules in the sense of [13] form a two-sided cell denoted $\mathcal{J}_{\text{split}}$.*
d. *All string bimodules with $\mathbf{v}(X) = 0$ which are not $\mathbb{k}$-split form a two-sided cell denoted $\mathcal{J}_0$.*
e. *All two-sided cells are linearly ordered as follows:*

$$\mathcal{J}_{\text{split}} \geq_J \mathcal{J}_0 \geq_J \mathcal{J}_1 \geq_J \mathcal{J}_2 \geq_J \dots \geq_J \mathcal{J}_{\text{band}}.$$

Also following [8, 14], all non-$\mathbb{k}$-split string bimodules can be divided into four different types, M, W, N, or S, depending on the action graph. For each non-$\mathbb{k}$-split string bimodule, in Section 2.4 we introduce three invariants: the *initial vertex*, *width*, and *height*. The structure of left and right cells is then given by the following.

Theorem 2.

a. *The two-sided cell $\mathcal{J}_{\text{band}}$ is also a left and a right cell.*
b. *Left cells in $\mathcal{J}_{\text{split}}$ are indexed by indecomposable right A-modules. For an indecomposable right A-module N, the left cell of N consists of all $M \otimes_{\mathbb{k}} N$, where M is an indecomposable left A-module.*
c. *Right cells in $\mathcal{J}_{\text{split}}$ are indexed by indecomposable left A-modules. For an indecomposable left A-module M, the right cell of M consists of all $M \otimes_{\mathbb{k}} N$, where N is an indecomposable right A-module.*
d. *For a non-negative integer k, a left cell in $\mathcal{J}_k$ consists of all bimodules in $\mathcal{J}_k$ which have the same second coordinate of the initial vertex and the same width. Two bimodules in the same left cell are necessarily either of the same type or of type M or N alternatively of type W or S.*

e. *For a non-negative integer k, a right cell in $\mathcal{J}_k$ consists of all bimodules in $\mathcal{J}_k$ which have the same first coordinate of the initial vertex and the same height. Two bimodules in the same right same are necessarily either of the same type of or type M or S alternatively of type W or N.*

f. *All two-sided cells, with the exception of $\mathcal{J}_{\text{band}}$, are strongly regular in the sense of [11, Section 4.8].*

The paper is organized as follows: In Section 2, we recall the classification of indecomposable A-A-bimodules and collect all necessary preliminaries. After some preliminary computations of tensor products in Section 3, we prove Theorem 1 in Section 4 and Theorem 2 in Section 5.

For the special case of A being the algebra of dual numbers, some of the results of this paper were obtained in author's Master Thesis [8]. In fact, in the case of dual numbers, Ref. [8] provides detailed (very technical) explicit formulae for decomposition of tensor product of indecomposable bimodules which we decided not to include in the present paper. The results of this paper can also be seen as a generalization and an extension of the results of [14] which describes the cell combinatorics of the tensor category of bimodules over the radical square zero quotient of a uniformly oriented Dynkin quiver of type A_n. The results of this paper can also be compared with the results of [3, 6, 7].

2. Indecomposable *A*-*A*-bimodules

2.1. Quiver and relations for A-A-bimodules

The category $A\text{-mod-}A$ is equivalent to the category $A \otimes A^{\text{op}}$-mod of finitely generated left $A \otimes A^{\text{op}}$-modules. The latter is equivalent to the category of modules over the path algebra of the discrete torus given by the following diagram, where we identify the first row with the last row and the first column with the last column, modulo the relations that

- the composition of any two vertical arrows or horizontal arrows is 0,
- all squares commute.

(1)
$$
\begin{tikzcd}
1|1 \arrow[d] & 1|2 \arrow[l] \arrow[d] & \dots \arrow[l] & 1|n \arrow[l] \arrow[d] & 1|1 \arrow[l] \arrow[d] \\
2|1 \arrow[d] & 2|2 \arrow[l] \arrow[d] & \dots \arrow[l] & 2|n \arrow[l] \arrow[d] & 2|1 \arrow[l] \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
n|1 \arrow[d] & n|2 \arrow[l] \arrow[d] & \dots \arrow[l] & n|n \arrow[l] \arrow[d] & n|1 \arrow[l] \arrow[d] \\
1|1 & 1|2 \arrow[l] & \dots \arrow[l] & 1|n \arrow[l] & 1|1 \arrow[l]
\end{tikzcd}
$$

Here indices of nodes are identified modulo n (i.e. are in $\mathbb{Z}_n$), vertical arrows correspond to the left component A and horizontal arrows correspond to the right component A^{op}.

We denote by ε_i the primitive idempotent of A corresponding to the vertex i.

If an A-A-bimodule X is considered as a representation of (1), we denote by $X_{i|j}$ the value of X at the vertex $i|j$.

2.2. $\Bbbk$-split A-A-bimodules

A $\Bbbk$-split A-A-bimodule, cf. [13], is a bimodule of the form $M \otimes_{\Bbbk} N$, where M is a left A-module and N is a right A-module. The bimodule $M \otimes_{\Bbbk} N$ is indecomposable if and only if both M and N are indecomposable. The additive closure in A-mod-A of all $\Bbbk$-split A-A-bimodules is the unique minimal tensor ideal. Therefore, all indecomposable $\Bbbk$-split A-A-bimodules belong to the same two-sided cell, denoted $\mathcal{J}_{\text{split}}$. This proves Theorem 1(c).

Note that

$$M' \otimes_{\Bbbk} A \otimes_A M \otimes_{\Bbbk} N \cong (M' \otimes_{\Bbbk} N)^{\oplus \dim_{\Bbbk} M}.$$

This implies that left cells in $\mathcal{J}_{\text{split}}$ are of the form

$$\{[M \otimes_{\Bbbk} N] \ : \ M \text{ is indecomposable}\}$$

and N is fixed. This implies Theorem 2(b) and Theorem 2(c) is obtained similarly.

As a special case of $\Bbbk$-split A-A-bimodules, we have projective-injective bimodules (they all are non-uniserial) which correspond to the commuting squares in the diagram (1). For $n > 1$ and fixed i and j, the non-zero part of the corresponding projective-injective bimodule $P(i|j) \cong I(i+1|j-1)$ realized as a representation of the quiver (1) looks as follows:

$$\begin{tikzcd}
\Bbbk_{i|j-1} \arrow[d, "\mathrm{id}"'] & \Bbbk_{i|j} \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\
\Bbbk_{i+1|j-1} & \Bbbk_{i+1|j} \arrow[l, "\mathrm{id}"]
\end{tikzcd}$$

Here $\Bbbk_{s|t}$ denotes a copy of $\Bbbk$ at the vertex $s|t$. For $n = 1$, we have the following picture:

$$\begin{tikzcd}
\Bbbk^4_{1|1} \arrow[d, "\varphi"'] & \Bbbk^4_{1|2} \arrow[l, "\psi"'] \arrow[d, "\varphi"] \\
\Bbbk^4_{2|1} & \Bbbk^4_{2|2} \arrow[l, "\psi"]
\end{tikzcd}$$

where φ and ψ are given by the respective matrices

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Another special case of $\Bbbk$-split A-A-bimodules are simple A-A-bimodules $L(i|j)$, for fixed i and j. The bimodule $L(i|j)$ is one-dimensional and the non-zero part of $L(i|j)$ realized as a representation of the quiver (1) looks as $\Bbbk_{i|j}$.

Finally, we also have the bimodules $S^{(0)}_{i|j}$ and $N^{(0)}_{i|j}$ given by their respective non-zero parts of realizations as a representation of the quiver (1):

$$\Bbbk_{i|j-1} \xleftarrow{\mathrm{id}} \Bbbk_{i|j} \qquad \text{and} \qquad \begin{tikzcd} \Bbbk_{i|j} \arrow[d, "\mathrm{id}"] \\ \Bbbk_{i+1|j}, \end{tikzcd}$$

for $n > 1$ and

$$
\begin{tikzcd}
\Bbbk^2_{1|1} \arrow[d, "0"'] & \Bbbk^2_{1|2} \arrow[l, "\varphi"'] \arrow[d, "0"] \\
\Bbbk^2_{2|1} & \Bbbk^2_{2|2} \arrow[l, "\varphi"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Bbbk^2_{1|1} \arrow[d, "\varphi"'] & \Bbbk^2_{1|2} \arrow[l, "0"'] \arrow[d, "\varphi"] \\
\Bbbk^2_{2|1} & \Bbbk^2_{2|2} \arrow[l, "0"]
\end{tikzcd}
$$

where φ denotes the matrix

$$
\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \tag{2}
$$

for $n=1$.

2.3. Band A-A-bimodules

Band A-A-bimodules, as classified in [1, 17], are A-A-bimodules $B(k,m,\lambda)$, where $k \in \mathbb{Z}_n$, m is a positive integer and λ is a non-zero number. We now recall their construction. The foundational band A-A-bimodule is the regular A-A-bimodule ${}_AA_A = B(1,1,1)$. For $n=1$ and $n=2$, here are the respective realizations of this bimodule as a representation of (1) (here φ is given by (2)):

$$
\begin{tikzcd}
\Bbbk^2_{1|1} \arrow[d, "\varphi"'] & \Bbbk^2_{1|2} \arrow[l, "\varphi"'] \arrow[d, "\varphi"] \\
\Bbbk^2_{2|1} & \Bbbk^2_{2|2} \arrow[l, "\varphi"]
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\Bbbk_{1|1} \arrow[d, "\mathrm{id}"'] & \Bbbk_{1|2} \arrow[l, "0"'] \arrow[d, "0"] & \Bbbk_{1|3} \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\
\Bbbk_{2|1} \arrow[d, "0"'] & \Bbbk_{2|2} \arrow[l, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \Bbbk_{2|3} \arrow[l, "0"] \arrow[d, "0"'] \\
\Bbbk_{3|1} & \Bbbk_{3|2} \arrow[l, "0"] & \Bbbk_{3|3} \arrow[l, "\mathrm{id}"]
\end{tikzcd}
$$

For a non-zero $\lambda \in \Bbbk$, denote by η_λ the automorphism of A given by multiplying α_1 with λ and keeping all other basis elements of A untouched. For an A-A-bimodule X, we can consider the A-A-bimodule X^{η_λ} in which the right action of A is twisted by η_λ. Then we have

$$
B(1,1,\lambda) = B(1,1,1)^{\eta_\lambda}.
$$

For $n=2$, the realization of $B(1,1,\lambda)$ as a representation of (1) is as follows:

$$
\begin{tikzcd}
\Bbbk_{1|1} \arrow[d, "\mathrm{id}"'] & \Bbbk_{1|2} \arrow[l, "0"'] \arrow[d, "0"] & \Bbbk_{1|3} \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\
\Bbbk_{2|1} \arrow[d, "0"'] & \Bbbk_{2|2} \arrow[l, "\lambda\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & \Bbbk_{2|3} \arrow[l, "0"] \arrow[d, "0"'] \\
\Bbbk_{3|1} & \Bbbk_{3|2} \arrow[l, "0"] & \Bbbk_{3|3} \arrow[l, "\mathrm{id}"]
\end{tikzcd}
$$

The bimodule $B(1,m,\lambda)$ fits into a short exact sequence

$$
0 \to B(1,1,\lambda) \to B(1,m,\lambda) \to B(1,m-1,\lambda) \to 0.
$$

The definition of $B(1,m,\lambda)$ is best explained by the following example. For $n=2$, the realization of $B(1,m,\lambda)$ as a representation of (1) is as follows:

$$\begin{tikzcd}
\mathbb{k}^m_{1|1} \arrow[d, "\mathrm{id}"'] & \mathbb{k}^m_{1|2} \arrow[l, "0"'] \arrow[d, "0"] & \mathbb{k}^m_{1|3} \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] \\
\mathbb{k}^m_{2|1} \arrow[d, "0"'] & \mathbb{k}^m_{2|2} \arrow[l, "J_m(\lambda)"] \arrow[d, "\mathrm{id}"'] & \mathbb{k}^m_{2|3} \arrow[l, "0"] \arrow[d, "0"'] \\
\mathbb{k}^m_{3|1} & \mathbb{k}^m_{3|2} \arrow[l, "0"] & \mathbb{k}^m_{3|3} \arrow[l, "\mathrm{id}"]
\end{tikzcd}$$

where $J_m(\lambda)$ denotes the Jordan $m \times m$-cell with eigenvalue λ. For $n = 1$, the bimodule $B(1, m, \lambda)$ is given by

$$\begin{tikzcd}
\mathbb{k}^{2m}_{1|1} \arrow[d, "\psi"'] & \mathbb{k}^{2m}_{1|2} \arrow[l, "\varphi"'] \arrow[d, "\psi"] \\
\mathbb{k}^{2m}_{2|1} & \mathbb{k}^{2m}_{2|2} \arrow[l, "\varphi"]
\end{tikzcd}$$

where ψ and φ are respectively given by the matrices

$$\begin{pmatrix} 0 & 0 \\ E & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ J_m(\lambda) & 0 \end{pmatrix}$$

(here each matrix is a 2×2 block matrix with $m \times m$ blocks and E is the identity matrix). Note that $B(1, m, \lambda) \cong B(1, m, 1)^{\eta_\lambda}$ for any m and λ.

Finally, let θ denote the automorphism of A given by the elementary rotation of the quiver which sends each ε_i to ε_{i+1} and each α_i to α_{i+1}. Clearly, $\theta^n = \mathrm{Id}$. For an A-A-bimodule X and any $k \in \mathbb{Z}_n$, we can consider the A-A-bimodule X^{θ^k} in which the right action of A is twisted by θ^k. Then, for all m and λ, we have

$$B(k, m, \lambda) \cong B(1, m, \lambda)^{\theta^{k-1}}.$$

For $n = 2$, the realization of $B(2, 1, 1)$ as a representation of (1) is as follows:

$$\begin{tikzcd}
\mathbb{k}_{1|1} \arrow[d, "0"'] & \mathbb{k}_{1|2} \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"] & \mathbb{k}_{1|3} \arrow[l, "0"'] \arrow[d, "0"] \\
\mathbb{k}_{2|1} \arrow[d, "\mathrm{id}"'] & \mathbb{k}_{2|2} \arrow[l, "0"] \arrow[d, "0"'] & \mathbb{k}_{2|3} \arrow[l, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \\
\mathbb{k}_{3|1} & \mathbb{k}_{3|2} \arrow[l, "\mathrm{id}"] & \mathbb{k}_{3|3} \arrow[l, "0"]
\end{tikzcd}$$

Hence, any band A-A-bimodule can be constructed from $B(1, 1, 1)$ using extensions and twists by θ^k and η_λ.

2.4. String A-A-bimodules

String bimodules are best understood using covering techniques, see e.g. [16]. Consider the infinite quiver

(3)

$$\begin{tikzcd}
\cdots & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \\
\cdots & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] & \bullet \arrow[l] \arrow[d] & \cdots \arrow[l] \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{tikzcd}$$

with the vertex set $\mathbb{Z}^2$ and the same relations as in (1), that is all squares commute and the composition of any two horizontal or any two vertical arrows is zero. The group $\mathbb{Z}^2$ acts on this quiver such that the standard generators act by horizontal and vertical shifts by n, respectively. In this way, the above quiver is a covering of (1) in the sense of [16]. We denote by Θ the usual functor from the category of finite dimensional modules over (3) to the category of finite dimensional modules over (1). All indecomposable string A-A-bimodules are obtained from indecomposable finite dimensional representations of (3) using Θ.

The relevant representations of (3) are denoted $V(x, c, l)$, where the parameter $v = i|j$ is a vertex of (3) such that $1 \leq i, j \leq n$, the parameter c (the *course*) takes values in $\{r, d\}$ where r is a shorthand for "right" and d is a shorthand for "down," and, finally, l, the *length*, is a non-negative integer (if $l = 0$, then the parameter c has no value). The representation $V(x, c, l)$ has total dimension $l + 1$ and is constructed as follows:

- start at the vertex v;
- choose the initial course (or direction) given by c;
- make an alternating right/down walk from v of length l;
- put a one-dimensional vector space at each vertex of this walk;
- put the identity operator at each arrow of this walk;
- set all the remaining vertices and arrows to zero.

For example, the non-zero part of the module $V(1|2, r, 3)$ is as follows:

$$\begin{tikzcd}
\Bbbk_{1|2} & \Bbbk_{1|3} \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] & \\
& \Bbbk_{2|3} & \Bbbk_{2|4} \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

Note that $\Theta(V(v, c, l))$ is $\Bbbk$-split if $l \leq 1$. Therefore from now one we assume that $l \geq 2$. We will say that

- $V(v, c, l)$ if *of type* M if $c = r$ and l is even;
- $V(v, c, l)$ if *of type* N if $c = r$ and l is odd;
- $V(v, c, l)$ if *of type* W if $c = d$ and l is even;
- $V(v, c, l)$ if *of type* S if $c = d$ and l is odd.

Here are examples of modules of respective types M and N:

$$\begin{tikzcd}
\Bbbk_{1|1} & \Bbbk_{1|2} \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] & \\
& \Bbbk_{2|2} & \Bbbk_{2|3} \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] \\
& & \Bbbk_{3|3}
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk_{1|1} & \Bbbk_{1|2} \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] & \\
& \Bbbk_{2|2} & \Bbbk_{2|3} \arrow[l, "\mathrm{id}"']
\end{tikzcd}$$

Here are examples of modules of respective types W and S:

$$\begin{tikzcd}
\Bbbk_{1|1} \arrow[d, "\mathrm{id}"'] & & \\
\Bbbk_{2|1} & \Bbbk_{2|2} \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] & \\
& \Bbbk_{3|2} & \Bbbk_{3|3} \arrow[l, "\mathrm{id}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\Bbbk_{1|1} \arrow[d, "\mathrm{id}"'] & \\
\Bbbk_{2|1} & \Bbbk_{2|2} \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] \\
& \Bbbk_{3|2}
\end{tikzcd}$$

The vertex v of $V(v,c,l)$ is called the *initial vertex*. The *width* of $V(v,c,l)$ is the number of non-zero columns. For modules of type M and N, the width is given by $\lceil\frac{l+2}{2}\rceil$. For modules of type W and S, the width is given by $\lceil\frac{l+2}{2}\rceil$.

The *height* of $V(v,c,l)$ is the number of non-zero rows. For modules of type M and N, the height is given by $\lfloor\frac{l+2}{2}\rfloor$. For modules of type W and S, the height is given by $\lceil\frac{l+2}{2}\rceil$.

A *valley* of a module is a vertex of the form

$$\begin{tikzcd}
\cdots \arrow[d] & \\
\bullet & \cdots \arrow[l]
\end{tikzcd}$$

where both incoming arrows are non-zero. The number of valleys in V is denoted $\mathbf{v}(V)$. We have

$$\mathbf{v}(V(v,c,l)) = \begin{cases} \lfloor\dfrac{l-1}{2}\rfloor, & c = r; \\ \lfloor\dfrac{l}{2}\rfloor, & c = d. \end{cases}$$

Note that the module $V(v,c,l)$ is uniquely determined by its type, v and the number of valleys. For any type $X \in \{M,N,W,S\}$, we denote by $X_{i|j}^{(k)}$ the A-A-bimodule obtained using Θ from the corresponding representation of (3) of type X, the initial vertex $i|j$ and having k valleys.

From [1, 17] it follows that any indecomposable A-A-bimodule is isomorphic to one of the bimodules defined in this section (i.e. is $\Bbbk$-split, a band or a string bimodule).

3. Preliminary computations of tensor products

3.1. Tensoring band bimodules

The aim of this subsection is to prove the following explicit result.

Proposition 3. *For all possible values of parameters, we have*

$$B(k_1, m_1, \lambda_1) \otimes_A B(k_2, m_2, \lambda_2) \cong \bigoplus_s B(k_1 + k_2 - 1, s, \lambda_1\lambda_2),$$

where s runs through the set

$$\{|m_1 - m_2| + 1, |m_1 - m_2| + 3, \dots, m_1 + m_2 - 1\}. \tag{4}$$

To prove Proposition 3, we need some preparation. For any A-A-bimodule X and any automorphism φ of A, we denote by ${}^{\varphi}X$ and X^{φ} the bimodule obtained from X by twisting the left (resp. right) action of A by φ. Note that, for indecomposable X, both ${}^{\varphi}X$ and X^{φ} are indecomposable.

Lemma 4. *For any A-A-bimodules X and Y, we have* $X^{\varphi} \otimes_A Y \cong X \otimes_A {}^{\varphi^{-1}}Y$.

Proof. The correspondence $x \otimes y \mapsto x \otimes y$ from $X^{\varphi} \otimes_A Y$ to $X \otimes_A {}^{\varphi^{-1}}Y$ induced a well-defined map due to the fact that $x\varphi(a) \otimes y = x \otimes ay = x \otimes \varphi^{-1}(\varphi(a))y$, for all $x \in X, y \in Y$ and $a \in A$. Being well defined, this map is, obviously, an isomorphism of bimodules. □

Lemma 5. *For all possible values of parameters, we have*

$${}^{\theta}B(k, m, \lambda) \cong B(k-1, m, \lambda) \quad \text{and} \quad {}^{\eta_\mu}B(k, m, \lambda) \cong B(k, m, \lambda\mu^{-1}).$$

Proof. An indecomposable band A-A-bimodule X is uniquely defined by its *dimension vector* (as a representation of (1)) together with the trace of the linear operator

$$R := (\cdot\alpha_{j-1}^{-1})(\alpha_n\cdot)(\cdot\alpha_{j-2}^{-1})\dots(\alpha_2\cdot)(\cdot\alpha_j^{-1})(\alpha_1\cdot)$$

on the vector space $X_{1|j}$, where j denotes the unique element in $\mathbb{Z}_n$ for which we have $\alpha_1 \cdot X_{1|j} \neq 0$.

Now, the isomorphism ${}^{\theta}B(k, m, \lambda) \cong B(k-1, m, \lambda)$ follows by comparing the dimension vectors of $B(k, m, \lambda)$ and $B(k-1, m, \lambda)$ using the definition of θ and also noting that the twist by θ does not affect the trace of R.

Similarly, the isomorphism ${}^{\eta_\mu}B(k, m, \lambda) \cong B(k, m, \lambda\mu^{-1})$ follows by noting that the twist by η_μ does not affect the dimension vector but it does multiply the eigenvalue of R by μ^{-1}. The claim follows. □

Proof of Proposition 3. Using Lemmata 4 and 5, in the product

$$B(k_1, m_1, \lambda_1) \otimes_A B(k_2, m_2, \lambda_2)$$

we can move all involved twists by automorphism to the right. Hence the claim of Proposition 3 reduces to the case

$$B(1, m_1, 1) \otimes_A B(1, m_2, 1) \cong \bigoplus_s B(1, s, 1), \tag{5}$$

where s runs through (4). The latter formula should be compared to the classical formula for the tensor product of Jordan cells, see [2, Theorem 2],

$$J_{m_1}(1) \otimes_{\Bbbk} J_{m_2}(1) \cong \bigoplus_s J_s(1)$$

where s runs through (4); and also to the formula for tensoring simple finite dimensional $\mathfrak{sl}_2$-modules, see e.g. [10, Theorem 1.39].

This comparison implies that, by induction, (5) reduces to the case

$$B(1,2,1)\otimes_A B(1,m,1)\cong\begin{cases}B(1,2,1), & m=1;\\ B(1,m-1,1)\oplus B(1,m+1,1), & m>1;\end{cases}$$

and the symmetric case $B(1,m,1)\otimes_A B(1,2,1)$. The latter is similar to the former and left to the reader. The case $m=1$ is obvious as $B(1,1,1)$ is the regular A-A-bimodule. We now consider the case $m>1$. We also assume $n>1$, the case $n=1$ is similar but (as we have already seen several time) requires a change of notation as, for example, the dimension vector of the regular bimodule does not fit into the $n>1$ pattern.

Let $a^{(1)}_{i|i}, a^{(2)}_{i|i}, a^{(1)}_{i+1|i}, a^{(2)}_{i+1|i}$, where $i\in\mathbb{Z}_n$, be the elements of the standard basis of $B(1,2,1)$ at the vertices $i|i$ and $i+1|i$, respectively (the value of the bimodule $B(1,2,1)$ at all other verities is zero). Similarly, let $b^{(s)}_{i|i}$ and $b^{(s)}_{i+1|i}$, for $s=1,2,\ldots,m$, be elements of the standard basis of $B(1,m,1)$. Then a basis of $B(1,2,1)\otimes_A B(1,m,1)$, which we will call *standard*, is given by all elements of the form

$$a^{(1)}_{i|i}\otimes b^{(s)}_{i|i},\quad a^{(2)}_{i|i}\otimes b^{(s)}_{i|i},\quad a^{(1)}_{i+1|i}\otimes b^{(s)}_{i|i},\quad a^{(2)}_{i+1|i}\otimes b^{(s)}_{i|i}. \tag{6}$$

That the left action of every α_i in this basis is given by the identity operator follows from the corresponding property for $B(1,2,1)$. A similar, but slightly more involved observation is that the right action of every α_i, where $i\neq 1$, is also given by the identity operator. This follows from the corresponding property for $B(1,m,1)$ and also the fact that the left such α_i on any side of both $B(1,2,1)$ and $B(1,m,1)$ is given by the identity operator.

It remains to compute the right action of α_1. It is given by $J_2(1)$ on $B(1,2,1)$ and by $J_m(1)$ on $B(1,m,1)$. To compute the right action on an element of the form $a^{(t)}_{2|2}\otimes b^{(s)}_{2|2}$, we first apply the right action on $B(1,m,1)$ which acts on the $b^{(s)}_{2|2}$-component via $J_m(1)$. The outcome is a linear combination of elements of the form $a^{(t)}_{2|2}\otimes b^{(s)}_{2|1}$ which now have to be rewritten in the basis (6). For this we write each $b^{(s)}_{2|1}$ as $\alpha_1\cdot b^{(s)}_{1|1}$ and move α_1 through the tensor product sign. The element α_1 acts on $a^{(t)}_{2|2}$ from the right via $J_2(1)$. Altogether, we get exactly the Kronecker product of Jordan cells $J_2(1)\otimes_{\Bbbk} J_m(1)$ written in the basis (6). Using the classical decomposition of the latter, see [2, Theorem 2], the necessary result follows. □

3.2. Tensoring string bimodules with band bimodules

The aim of this section is to prove the following result.

Proposition 6. *Let X be a string A-A-bimodule and Y a band A-A-bimodule.*

a. *The bimodule $X\otimes_A Y$ decomposes as a direct sum of string A-A-bimodules, moreover, both the type, height and width of any summand coincides with that of X.*
b. *The bimodule $Y\otimes_A X$ decomposes as a direct sum of string A-A-bimodules, moreover, both the type, height and width of any summand coincides with that of X.*

Proof. We prove the first claim, the proof of the second claim is similar. Let us start be noting that both X^θ and X^{η_λ}, for any $\lambda\in\Bbbk\setminus\{0\}$, are obviously string bimodules of the same height as X. Therefore, from Proposition 3 it follows that it is enough to consider the case $Y=B(1,2,1)$. We use the same notation for the standard basis in $B(1,2,1)$ as in the proof of Proposition 3.

Let $\{v_{i|j}\}$ be the standard basis of X, where $v_{i|j}$ is a basis of $\Bbbk_{i|j}$ whenever the space in the vertex $i|j$ of (3) is non-zero. Then a basis in $X\otimes_A Y$ is given by all elements of the form

$$v_{i|j} \otimes a^{(1)}_{j|j} \qquad \text{and} \qquad v_{i|j} \otimes a^{(2)}_{j|j}.$$

Similarly to the proof of Proposition 3, all non-zero left actions of all α_i in this basis are given by the appropriate identity operators. Similarly for the right actions of all α_i, where $i \neq 1$. The action of α_1 is given by a direct sum of copies of $J_2(1)$. The picture, written as a representation of (3), looks as follows:

$$\begin{tikzcd}
\Bbbk^2 \arrow[d, "\mathrm{id}"'] & & & & \\
\Bbbk^2 & \Bbbk^2 \arrow[l, "J_2(1)"'] \arrow[d, "\mathrm{id}"'] & & & \\
& \Bbbk^2 & \Bbbk^2 \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] & & \\
& & \Bbbk^2 & \Bbbk^2 \arrow[l, "\mathrm{id}"'] \arrow[d, "\mathrm{id}"'] & \\
& & & \Bbbk^2 & \Bbbk^2 \arrow[l, "J_2(1)"']
\end{tikzcd}$$

It is clear that we can choose new bases in all spaces such that in these new bases all actions are given by the identity operators. The claim follows. $\square$

3.3. Tensoring string bimodules

The aim of this section is to prove the following result.

Proposition 7. *Let X and Y be string A-A-bimodules. Then the bimodule $X \otimes_A Y$ decomposes as a direct sum of string and $\Bbbk$-split A-A-bimodules, moreover, both the number of valleys and the width of each string summand does not exceed that of Y, and both the number of valleys and the height of each string summand does not exceed that of X.*

Proof. We view both X and Y as representations of (3). Let $x_{i|j}$ be the standard basis of X and $y_{i|j}$ be the standard basis of Y. Then $x_{i|j} \otimes y_{s|t}$ is a basis of $X \otimes_{\Bbbk} Y$.

Consider a directed graph Γ whose vertices are all $x_{i|j} \otimes y_{s|t}$ and arrows are defined as follows:

- There is a (vertical) arrow from $x_{i|j} \otimes y_{s|t}$ to $x_{i+1|j} \otimes y_{s|t}$ if $x_{i+1|j} = \alpha_i x_{i|j}$ in X.
- There is a (horizontal) arrow from $x_{i|j} \otimes y_{s|t}$ to $x_{i|j} \otimes y_{s|t-1}$ if $y_{s|t-1} = y_{s|t}\alpha_{t-1}$ in Y.

The graph Γ represents the action of A (on the left via vertical arrows and on the right via horizontal arrows) on the $\Bbbk$-split A-A-bimodule $X \otimes_{\Bbbk} Y$. Therefore each connected component of Γ looks as follows:

$$\bullet\,, \qquad \begin{tikzcd} \bullet & \bullet \arrow[l] \end{tikzcd}, \qquad \begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd}, \qquad \begin{tikzcd} \bullet \arrow[d] & \bullet \arrow[l] \arrow[d] \\ \bullet & \bullet \arrow[l] \end{tikzcd}.$$

We would like to understand what happens with the graph Γ under the projection $X\otimes_{\Bbbk}Y\twoheadrightarrow X\otimes_A Y$, that is under factoring out the relations $xa\otimes y=x\otimes ay$, for $x\in X, y\in Y$ and $a\in A$.

First we note that, if $j\not\equiv s \pmod n$, then $x_{i|j}\otimes y_{s|t}=0$ in $X\otimes_A Y$. As j and s are the same for all vertices in a connected component of Γ, we can just throw away all connected components for which $j\not\equiv s \pmod n$.

Next consider the valleys in Γ. These exist only for rectangular connected components (the last one in the list above). Let $x_{i|j}\otimes y_{s|t}$ be such a valley. Then one of the following two possible cases occurs.

Case 1. Neither $x_{i|j}$ nor $y_{s|t}$ is a valley of the graphs of X and Y, respectively. This happens exactly when X is of type M or S and $x_{i|j}$ is the "last," that is south-east vertex in X, and Y is of type M or N and $x_{i|j}$ is the "first," that is north-west vertex in Y. In this case, it is easy to see that the whole connected component survives the projection onto $X\otimes_A Y$ and gives there a copy of a ($\Bbbk$-split) projective-injective bimodule.

Case 2. At least one of $x_{i|j}$ or $y_{s|t}$ is a valley in the respective graphs of X and Y. Let us consider the case when $x_{i|j}$ is a valley (the other case is similar). Then $x_{i|j}=x_{i|j+1}\alpha_j$. At the same time, $\alpha_j y_{s|t}=0$ as $y_{s|t}$ is in the image of the right action of A (and the left action annihilates the image of the right action on string bimodules). Therefore such a valley of Γ is mapped to zero in $X\otimes_A Y$.

A similar argument show that, apart from some vertices of Γ being sent to zero via the projection $X\otimes_{\Bbbk}Y\twoheadrightarrow X\otimes_A Y$, one can also have identifications in which the left vertex of one of the components of the form

$$\bullet\,, \qquad \bullet \longleftarrow \bullet\,, \qquad \begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[d] \\ & \bullet \end{tikzcd}$$

(here the last component is obtained from a rectangular component of Γ the valley of which gets killed) is identified with the bottom vertex of one of the components of the form

$$\bullet\,, \qquad \begin{tikzcd} \bullet \arrow[d] \\ \bullet \end{tikzcd}, \qquad \begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[d] \\ & \bullet \end{tikzcd}.$$

This implies that the action graph of A on any non $\Bbbk$-split direct summand of $X\otimes_A Y$ is a tree and hence this direct summand must be a string bimodule.

It remains to prove the claims about the width and the height. We prove the first one, the second one is similar. Note that the width of any connected component of Γ is at most one. To get a summand of greater width, we need the identifications as described in the previous paragraph to happen between components of the form

$$\bullet \longleftarrow \bullet\,, \qquad \begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[d] \\ & \bullet \end{tikzcd} \tag{7}$$

where the first one can appear at most ones, on the far right as it appearance stops farther possible identifications on the right. As follows from the above, potential identifications correspond to valleys of Y. Indeed, we need a fragment in Y of the form $\bullet \longleftarrow$ to have one of the components in (7) and we need a fragment in Y of the form

$$\begin{array}{c} \Big\downarrow \\ \bullet \end{array}$$

for the same vertex in order to be able to do identification. Putting all this together, we see that both the width and the number of valleys cannot increase. The claim follows. □

3.4. Some explicit computations

Lemma 8. *For any $k > 0$, the bimodule $M^{(k-1)}_{1|2}$ is a direct summand of the bimodule $M^{(k)}_{1|1} \otimes_A M^{(k)}_{1|1}$.*

Proof. Let $\{x_{i|j}\}$ be the standard basis of $M^{(k)}_{1|1}$ considered as a representation of (3). Following the arguments in the proof of Proposition 7, we see that $M^{(k)}_{1|1} \otimes_A M^{(k)}_{1|1}$ contains a direct summand which is the span of the following elements:

$$x_{1|1} \otimes x_{1|2} = x_{1|2} \otimes x_{2|2}, \quad x_{1|2} \otimes x_{2|3}, \quad x_{2|2} \otimes x_{2|3} = x_{2|3} \otimes x_{3|3}, \dots,$$
$$x_{k+1|k+1} \otimes x_{k+1|k+2} = x_{k+1|k+2} \otimes x_{k+2|k+2}.$$

This direct summand is isomorphic to $M^{(k-1)}_{1|2}$, the claim follows. □

Lemma 9. *For any $k > 0$, the bimodule $N^{(k)}_{1|1}$ is a direct summand of the bimodule $W^{(k)}_{1|1} \otimes_A M^{(k)}_{1|1}$.*

Proof. Let $\{x_{i|j}\}$ be the standard basis of $W^{(k)}_{1|1}$ and $\{y_{i|j}\}$ be the standard basis of $M^{(k)}_{1|1}$, both considered as a representation of (3). Following the arguments in the proof of Proposition 7, we see that $W^{(k)}_{1|1} \otimes_A M^{(k)}_{1|1}$ contains a direct summand which is the span of the following elements:

$$x_{1|1} \otimes y_{1|1}, \quad x_{1|1} \otimes y_{1|2}, \quad x_{2|1} \otimes y_{1|2} = x_{2|2} \otimes y_{2|2},$$
$$x_{2|2} \otimes y_{2|3}, \quad \dots, \quad x_{k+1|k+1} \otimes y_{k+1|k+2}.$$

This direct summand is isomorphic to $N^{(k)}_{1|1}$, the claim follows. □

Lemma 10. *For any $k > 0$, the bimodule $M^{(k)}_{1|1}$ is a direct summand of the bimodule $S^{(k)}_{1|1} \otimes_A N^{(k)}_{1|1}$.*

Proof. Let $\{x_{i|j}\}$ be the standard basis of $S^{(k)}_{1|1}$ and $\{y_{i|j}\}$ be the standard basis of $N^{(k)}_{1|1}$, both considered as a representation of (3). Following the arguments in the proof of Proposition 7, we see that $S^{(k)}_{1|1} \otimes_A N^{(k)}_{1|1}$ contains a direct summand which is the span of the following elements:

$$x_{1|1} \otimes y_{1|1}, \quad x_{1|1} \otimes y_{1|2}, \quad x_{2|1} \otimes y_{1|2} = x_{2|2} \otimes y_{2|2},$$
$$x_{2|2} \otimes y_{2|3}, \quad \dots, \quad x_{k+1|k+1} \otimes y_{k+1|k+2}, \quad x_{k+2|k+1} \otimes y_{k+1|k+2}.$$

This direct summand is isomorphic to $M^{(k)}_{1|1}$, the claim follows. □

Lemma 11. *For any $k > 0$, the bimodule $W^{(k)}_{1|1}$ is a direct summand of the bimodule $N^{(k)}_{1|1} \otimes_A S^{(k)}_{2|1}$.*

Proof. Let $\{x_{i|j}\}$ be the standard basis of $N^{(k)}_{1|1}$ and $\{y_{i|j}\}$ be the standard basis of $S^{(k)}_{1|1}$, both considered as a representation of (3). Following the arguments in the proof of Proposition 7, we see that $N^{(k)}_{1|1} \otimes_A S^{(k)}_{2|1}$ contains a direct summand which is the span of the following elements:

$$x_{1|2} \otimes y_{2|1}, \quad x_{2|2} \otimes y_{2|1} = x_{2|3} \otimes y_{3|1}, \quad x_{2|3} \otimes y_{3|2}, \quad \dots, x_{k+1|k+2} \otimes y_{k+2|k+1}.$$

This direct summand is isomorphic to $W^{(k)}_{1|1}$, the claim follows. □

Lemma 12. *For any $k>0$, the bimodule $S_{1|2}^{(k)}$ is a direct summand of the bimodule $M_{1|1}^{(k)} \otimes_A W_{2|2}^{(k)}$.*

Proof. Let $\{x_{i|j}\}$ be the standard basis of $M_{1|1}^{(k)}$ and $\{y_{i|j}\}$ be the standard basis of $W_{2|2}^{(k)}$, both considered as a representation of (3). Following the arguments in the proof of Proposition 7, we see that $M_{1|1}^{(k)} \otimes_A W_{2|2}^{(k)}$ contains a direct summand which is the span of the following elements:

$$x_{1|2} \otimes y_{2|2}, \quad x_{2|2} \otimes y_{2|2} = x_{2|3} \otimes y_{3|2}, \quad x_{2|3} \otimes y_{3|3}, \quad \dots, x_{k+2|k+2} \otimes y_{k+2|k+2}.$$

This direct summand is isomorphic to $S_{1|2}^{(k)}$, the claim follows. □

4. Proof of Theorem 1

4.1. Proof of Theorem 1(a)

This follows immediately from Propositions 3, 6, and 7.

4.2. Proof of Theorem 1(b)

After Propositions 7, we only need to argue that all string bimodules with the same number of valleys belong to the same two-sided cell. If we fix a type, then, looking at the action graph of the bimodule, it is clear that all bimodules with fixed number of valleys of this particular type can be transformed into each other by twisting by some powers of θ on both sides. Twisting by θ is the same as tensoring with A^{θ} or ${}^{\theta}A$. Therefore the fact that all bimodules of the same type having the same number of valleys belong to the same two-sided cell follows from Proposition 7.

Lemmata 9 and 10 imply that the bimodules of type M and N with the same number of valleys belong to the same two-sided cell. Lemmata 9 and 11 imply that the bimodules of type W and N with the same number of valleys belong to the same two-sided cell. Finally, Lemmata 10 and 12 imply that the bimodules of type M and S with the same number of valleys belong to the same two-sided cell.

4.3. Proof of Theorem 1(c)

This is proved in Section 2.2.

4.4. Proof of Theorem 1(d)

All non-$\mathbb{k}$-split string bimodules with zero valleys are of type M. Therefore the fact that they all form a two-sided cell follows from Propositions 3, 6 and 7 and the observation in the first paragraph of Section 4.2.

4.5. Proof of Theorem 1(e)

It is clear that $\mathcal{J}_{\text{band}}$ is the minimum and $\mathcal{J}_{\text{split}}$ is the maximum two-sided cells. Hence, we just need to prove that $\mathcal{J}_{k-1} \geq_J \mathcal{J}_k$. However, this follows from Lemma 8.

5. Proof of Theorem 2

5.1. Proof of Theorem 2(a)

This follows from Proposition 3.

5.2. Proof of Theorem 2(b)

This is proved in Section 2.2.

5.3. Proof of Theorem 2(c)

This is proved in Section 2.2.

5.4. Proof of Theorem 2(d)

Let X be a string representation of (3). The *right support* of X is the set of all $j \in \mathbb{Z}$ for which there is $i \in \mathbb{Z}$ such that the value of X at the vertex $i|j$ is non-zero. The *left support* of X is the set of all $i \in \mathbb{Z}$ for which there is $j \in \mathbb{Z}$ such that the value of X at the vertex $i|j$ is non-zero. Both the left and the right support of a string module is an integer interval. The width of a module is simply the cardinality of the right support of this module plus one. The height of a module is simply the cardinality of the left support of this module plus one. Therefore the initial vertex and the width uniquely determine the right support. The initial vertex and the height uniquely determine the left support.

If Y is a string A-A-bimodule and X is any A-A-bimodule, then, by Propositions 6 and 7, every direct summand of $X \otimes_A Y$ is a string A-A-bimodule. Moreover, directly from the definitions it follows that the right support of each such direct summand is a subset of the right support of Y. Therefore, to be in the same left cell, two string A-A-bimodules must have the same left support. By Theorem 1(d), the two bimodules should also have the same number of valleys. This implies that they either have the same type of type M or N alternatively of type W or S.

If two string A-A-bimodules X and Y are of the same type and have the same right support, then $X \cong^{\theta^k} Y$, for some power of k and hence they belong to the same right cell by the same argument as in the first paragraph of Section 4.2. From Lemmata 9 and 10 it thus follows that two bimodules of type M and N and with the same right support belong to the same left cell. From Lemmata 11 and 12 it follows that two bimodules of type W and S and with the same right support belong to the same left cell.

5.5. Proof of Theorem 2(e)

Mutatis mutandis the proof of Theorem 2(d).

5.6. Proof of Theorem 2(f)

Strong regularity of a two-sided cell $\mathcal{J}$ in the sense of [11, Subsection 4.8] means that the intersection of any left and any right cell inside $\mathcal{J}$ consists of one element. For $\mathcal{J}_{\text{split}}$, the claim follows directly from the description of left and right cells in Theorem 2(b) and Theorem 2(c). For $\mathcal{J}_k$, the claim follows from the description of left and right cells in Theorem 2(d) and Theorem 2(e).

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ON SIMPLE TRANSITIVE 2-REPRESENTATIONS OF BIMODULES OVER THE DUAL NUMBERS

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ABSTRACT. We study the problem of classification of simple transitive 2-representations for the (non-finitary) 2-category of bimodules over the dual numbers. We show that simple transitive 2-representations with finitary apex are necessarily of rank 1 or 2, and those of rank 2 are exactly the cell 2-representations. For 2-representations of rank 1, we show that they cannot be constructed using the approach of (co)algebra 1-morphisms. We also propose an alternative definition of (co-)Duflo 1-morphisms for finitary 2-categories and describe them in the case of bimodules over the dual numbers.

1. Motivation, introduction and description of the results

Classification problems form an important and intensively studied class of questions in modern representation theory. One of the natural examples of these kinds of problems is the problem of classification of all "simple" representations of a given mathematical object. During the last 20 years, a lot of attention was attracted to the study of representations of tensor categories and 2-categories, see [EGNO, Ma1] and references therein. In particular, there are by now a number of interesting tensor categories (and 2-categories) for which the structure of "simple" representations is well-understood. To the best of our knowledge, the first deep results of this kind can be found in [Os1, Os2], we refer to [EGNO] for more details.

Around 2010, Mazorchuk och Miemietz started a systematic study of representation theory of finitary 2-categories, see the original series [MM1, MM2, MM3, MM4, MM5, MM6] of papers by these authors. Finitary 2-categories can be considered as natural 2-analogues of finite dimensional algebras, in particular, they have various finiteness properties, analogous to those of the category of projective modules over a finite dimensional algebra. The paper [MM5] introduces the notion of *simple transitive* 2-representations of finitary 2-categories and provides convincing arguments, including an adaptation of the Jordan-Hölder theorem, on why these 2-representations are a natural 2-analogue for the notion of a simple module over an associative algebra. This motivated the natural problem of classification of simple transitive 2-representations for various classes of finitary 2-categories. This problem was considered and solved in a number of special cases, see [MM5, Zh2, Zi1, Zi2, MZ1, MZ2, MMZ1, MMZ2, MaMa, KMMZ, MT, MMMT, MMMZ, MMMTZ] and also [Ma2] for a slightly outdated overview on the status of that problem.

Arguably, one of the most natural examples of a finitary 2-category is the 2-category $\mathscr{C}_A$ of *projective bimodules* over a finite-dimensional associative algebra A, introduced in [MM1, Subsection 7.3]. Classification of simple transitive 2-representation of $\mathscr{C}_A$ is given in [MMZ2], with the special case of a self-injective A treated already in [MM5, MM6]. The reason to restrict to projective bimodules is the observation that, in the general case, the tensor category A-mod-A of *all* finite dimensional

A-A-bimodules is not finitary because it has infinitely many indecomposable objects. The only basic connected algebras A, for which A-mod-A is finitary, are the radical square zero quotients of the path algebras of uniformly oriented type A Dynkin quivers, see [MZ2]. Moreover, for almost all A, the category A-mod-A is wild, that is the associative algebra $A \otimes_{\Bbbk} A^{\mathrm{op}}$, whose module category is equivalent to A-mod-A, has wild representation type, and hence indecomposable objects of A-mod-A are not even known (and, perhaps, never will be known).

The smallest example of the algebra A for which the category A-mod-A is not finitary, but is, at least, tame, is the algebra $D := \Bbbk[x]/(x^2)$ of *dual numbers*. The combinatorics of tensor product of indecomposable objects in D-mod-D is described in [Jo1, Jo2]. In particular, although not being finitary itself, D-mod-D has a lot of finitary subcategories and subquotients. The main motivation for the present paper is to understand simple transitive 2-representations of D-mod-D which correspond to simple transitive 2-representations of its finitary subquotients.

Our main result is Theorem 1 which can be found in Subsection 3.2. It asserts that simple transitive 2-representations of D-mod-D with finitary apex are necessarily of rank 1 or 2 and, in the latter case, each such 2-representation is necessarily equivalent to a so-called *cell 2-representation*, which is an especially nice class of 2-representations. Unfortunately, at this stage we are not able to classify (or, for that matter, even to construct, with one exception) rank 1 simple transitive 2-representations. One possible reason for that is given in Theorem 21 in Subsection 8.4 which asserts that potential simple transitive 2-representations of rank 1 cannot be constructed using the approach of (co)algebra 1-morphisms, developed in [MMMT] for the so-called *fiat 2-categories*, that is finitary 2-categories with a weak involution and adjunction morphism. Needless to say, neither D-mod-D nor any of its finitary subquotients is fiat.

Section 7 and 8 summarize, in some sense, the outcome of our failed attempt to adjust the approach of [MMMT] at least for construction of simple transitive 2-representations of D-mod-D. Due to the fact that D-mod-D is not fiat, several classical notions for fiat 2-categories require non-trivial adaptation to the more general setup of D-mod-D. One of these, discussed in detail in Section 7, is that of a *Duflo 1-morphism*. Originally, it is defined in [MM1] in the fiat setup and slightly adjusted in [Zh1] to a more general finitary setting. Here we propose yet another alternative definition of Duflo 1-morphisms (and the dual notion of co-Duflo 1-morphisms) using certain universal properties, see Subsections 7.3 and 7.5. We show in Proposition 13 that our notion agrees with the notion of Duflo 1-morphisms from [MM1] in the fiat case. We show that some left cells in D-mod-D have a Duflo 1-morphism and that some other left cells have a co-Duflo 1-morphism, see Subsections 7.4 and 7.5. In Section 8, we further show that these Duflo and co-Duflo 1-morphisms admit the natural structure of coalgebra and algebra 1-morphisms, respectively.

All necessary preliminaries are collected in Section 2. Our main Theorem 1 has four statements. The first one is proved in Subsection 3.3. The other three are proved in Sections 4, 5 and 6, respectively.

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2. Preliminaries

2.1. **2-categories.** A *2-category* $\mathscr{C}$ consists of

- objects $\mathtt{i}, \mathtt{j}, \mathtt{k}$,...;
- for each pair of objects $\mathtt{i}, \mathtt{j}$, a small category $\mathscr{C}(\mathtt{i}, \mathtt{j})$ of *morphisms* from $\mathtt{i}$ to $\mathtt{j}$, objects of $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called 1-morphisms F, G, H,..., and morphisms of $\mathscr{C}(\mathtt{i}, \mathtt{j})$ are called 2-morphisms α, β,...;
- for each object $\mathtt{i}$, an identity 1-morphism $\mathbb{1}_{\mathtt{i}}$;
- bifunctorial composition $\circ : \mathscr{C}(\mathtt{j}, \mathtt{k}) \times \mathscr{C}(\mathtt{i}, \mathtt{j}) \to \mathscr{C}(\mathtt{i}, \mathtt{k})$.

This datum is supposed to satisfy the obvious set of strict axioms. The internal composition of 2-morphisms in $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is called *vertical* and denoted by $\circ_v$. The composition of 2-morphisms induced by $\circ$ is called *horizontal* and denoted $\circ_h$.

Let $\Bbbk$ be a field. Important examples of 2-categories are

- **Cat**, the 2-category whose objects are small categories, 1-morphisms are functors, and 2-morphisms are natural transformations of functors;
- $\mathfrak{A}_{\Bbbk}^{f}$, the 2-category whose objects are finitary $\Bbbk$-linear categories, 1-morphisms are additive $\Bbbk$-linear functors, and 2-morphisms are natural transformations of functors;
- $\mathfrak{R}_{\Bbbk}$, the 2-category of finitary $\Bbbk$-linear abelian categories, whose objects are small categories equivalent to module categories of finite-dimensional associative $\Bbbk$-algebras, 1-morphisms are right exact additive $\Bbbk$-linear functors, and 2-morphisms are natural transformations of functors.

2.2. **2-representations.** Let $\mathscr{C}$ be a 2-category. A *2-representation* of $\mathscr{C}$ is a strict 2-functor $\mathbf{M} : \mathscr{C} \to \mathbf{Cat}$.

For example, given an object $\mathtt{i}$ in $\mathscr{C}$, we can define the *principal representation* $\mathbf{P}_{\mathtt{i}} = \mathscr{C}(\mathtt{i}, -)$. A *finitary 2-representation* of $\mathscr{C}$ is a strict 2-functor $\mathbf{M} : \mathscr{C} \to \mathfrak{A}_{\Bbbk}^{f}$.

A finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$ is called *transitive* if, for any indecomposable object $X \in \mathbf{M}(\mathtt{i})$ and $Y \in \mathbf{M}(\mathtt{j})$, there is a 1-morphism U in $\mathscr{D}$ such that Y is isomorphic to a direct summand of $\mathbf{M}(U)X$. We, further, say that $\mathbf{M}$ *simple* if it has no proper nonzero $\mathscr{C}$-stable ideals. While simplicity implies transitivity, we follow [MM5] and speak of simple transitive 2-representations to emphasize the two levels (objects and morphisms) of the involved structure.

All 2-representations of $\mathscr{C}$ form a 2-category, see [MM3, Subsection 2.3] for details. In particular, two 2-representations $\mathbf{M}$ and $\mathbf{N}$ of $\mathscr{C}$ are *equivalent* if there is a 2-natural transformation $\Phi : \mathbf{M} \to \mathbf{N}$ which restricts to an equivalence $\mathbf{M}(\mathtt{i}) \to \mathbf{N}(\mathtt{i})$ for every object $\mathtt{i} \in \mathscr{C}$.

If $\mathscr{C}$ has only one object $\mathtt{i}$, we say that a finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$ has *rank* r if the category $\mathbf{M}(\mathtt{i})$ has exactly r isomorphism classes of indecomposable objects.

2.3. **Abelianization.** For every finitary 2-representation $\mathbf{M}$ of $\mathscr{C}$, we can consider its *(projective) abelianization* $\overline{\mathbf{M}}$ as defined in [MMMT, Section 3]. Then $\overline{\mathbf{M}}$ is a 2-functor from $\mathscr{C}$ to $\mathfrak{R}_{\Bbbk}$ and, up to equivalence, $\mathbf{M}$ is recovered by restricting $\overline{\mathbf{M}}$ to the subcategories of projective objects in the underlying abelian categories of the *abelian* 2-representation $\overline{\mathbf{M}}$.

There is also the dual notion of *(injective) abelianization* $\underline{\mathbf{M}}$.

2.4. **Cells and cell 2-representations.** One the set of isomorphism classes of indecomposable 1-morphisms in $\mathscr{C}$, define the *left preorder* $\leq_L$ by $F \leq_L G$ if there is some H such that G is a direct summand of $H \circ F$. The induced equivalence relation $\sim_L$ is called *left equivalence*, and the equivalence classes *left cells*. Similarly, we can define the right preorder $\leq_R$ by composing with H from the right, and two-sided preorder $\leq_J$ by composing with H_1 and H_2 from both sides. Right and two-sided equivalence and cells are also defined analogously.

For any transitive 2-representation $\mathbf{M}$ of $\mathscr{C}$, there is, by [CM], a unique two-sided cell, maximal with respect to the two-sided preorder, which is not annihilated by $\mathbf{M}$. This two-sided cell is called the *apex* of $\mathbf{M}$.

A two-sided cell $\mathcal{J}$ is called *idempotent* if it contains F, G and H such that H is isomorphic to a direct summand of $F \circ G$. The apex of a 2-representation is necessarily idempotent.

Let $\mathcal{L}$ be a left cell in $\mathscr{C}$ and let $\mathtt{i} = \mathtt{i}_{\mathcal{L}}$ be the object such that all 1-morphisms in $\mathcal{L}$ start at $\mathtt{i}$. Then the principal representation $\mathbf{P}_{\mathtt{i}}$ has a subrepresentation given by the additive closure of all 1-morphisms F such that $F \geq_L \mathcal{L}$. This, in turn, has a unique simple transitive quotient which we call the *cell 2-representation* associated to $\mathcal{L}$ and denote by $\mathbf{C}_{\mathcal{L}}$. We refer to [MM1, MM2] for more details.

2.5. **Action matrices.** Let $\mathbf{M}$ be a finitary 2-representation of $\mathscr{C}$ and F a 1-morphism in $\mathscr{C}(\mathtt{i},\mathtt{j})$. Let $X_1, X_2, \ldots, X_k$ be a complete list of representatives of isomorphism classes of indecomposable objects in $\mathbf{M}(\mathtt{i})$ and $Y_1, Y_2, \ldots, Y_m$ be a complete list of representatives of isomorphism classes of indecomposable objects in $\mathbf{M}(\mathtt{j})$. Then we can define the *action matrix* $[F]$ of F as the integral $m \times k$-matrix $(r_{ij})_{i=1,\ldots,m}^{j=1,\ldots,k}$, where r_{ij} is the multiplicity of Y_i as a direct summand of $\mathbf{M}(F)X_j$. Clearly, we have $[FG] = [F][G]$.

If $\mathscr{C}$ has only one object, then $\mathbf{M}$ is transitive if and only if all coefficients of $[F]$ are positive, where F is such that it contains, as direct summands, all indecomposable 1-morphisms in the apex of $\mathbf{M}$.

If $\overline{\mathbf{M}}(F)$ is exact, then we can also consider the matrix $[[F]]$ which bookkeeps the composition multiplicities of the values of $\overline{\mathbf{M}}(F)$ on simple objects in $\overline{\mathbf{M}}(\mathtt{i})$.

3. Bimodules over the dual numbers and the main result

3.1. **The 2-category of bimodules over the dual numbers.** In the remainder of the paper, we work over an algebraically closed field $\Bbbk$ of characteristic 0. Denote by $D = \Bbbk[x]/(x^2)$ the dual numbers. Fix a small category $\mathcal{C}$ equivalent to D-mod. Let $\mathscr{D}$ be the 2-category which has

- one object $\mathtt{i}$ (which we identify with $\mathcal{C}$),
- as 1-morphisms, all endofunctors of $\mathcal{C}$ isomorphic to tensoring with finite dimensional D-D-bimodules,
- as 2-morphisms, all natural transformations of functors (these are given by homomorphisms of the corresponding D-D-bimodules).

Indecomposable D-D-bimodules can be classified, up to isomorphism, following [BR], [WW]. Using the notation from [Jo2], they are the following.

- The (unique) projective-injective bimodule $D \otimes_{\Bbbk} D$.
- The band bimodules $B_k(\lambda)$, indexed by $k \in \mathbb{Z}_{>0}$ and $\lambda \in \Bbbk \setminus \{0\}$. The bimodule $B_k(\lambda)$ can be depicted as follows:

$$\begin{tikzcd} \Bbbk^k \arrow[d, bend right, "x\cdot\_=\mathrm{Id}"'] \arrow[d, bend left, "\_\cdot x=Q_k(\lambda)"] \\ \Bbbk^k \end{tikzcd}$$

  where $Q_k(\lambda)$ is the $k \times k$ Jordan cell with eigenvalue λ. In particular, the regular bimodule ${}_D D_D$ is isomorphic to the band bimodule $B_1(1)$.
- String bimodules of four shapes W, S, N and M indexed by $k \in \mathbb{Z}_{\geq 0}$. For a string bimodule U, the integer k is the number of *valleys* in the graph representing this bimodule, alternatively, $k = \dim(DU \cap UD)$. The graphs representing the bimodules W_1, S_1, N_1 and M_1 look, respectively, as follows (here vertices $\bullet$ and $\circ$ represent a fixed basis with $\circ$ depicting the valley, the non-zero right action of $x \in D$ is depicted by horizontal arrows, the non-zero left action of $x \in D$ is depicted by vertical arrows and all non-zero coefficients of both actions are equal to 1):

$$\begin{tikzcd} \bullet \arrow[d] & \\ \circ & \bullet \arrow[l] \end{tikzcd}, \quad \begin{tikzcd} \bullet \arrow[d] & \\ \circ & \bullet \arrow[l] \arrow[d] \\ & \bullet \end{tikzcd}, \quad \begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[d] & \\ & \circ & \bullet \arrow[l] \end{tikzcd}, \quad \begin{tikzcd} \bullet & \bullet \arrow[l] \arrow[d] & \\ & \circ & \bullet \arrow[l] \arrow[d] \\ & & \bullet \end{tikzcd}.$$

An indecomposable bimodule is called $\Bbbk$-split if it is of the form $U \otimes_{\Bbbk} V$ for indecomposables $U \in D$-mod and $V \in$ mod-D. The $\Bbbk$-split bimodules $D \otimes D$, W_0, S_0 and N_0 form the unique maximal two-sided cell $\mathcal{J}_{\Bbbk\text{-split}}$, with left cells inside it indexed by indecomposable right D-modules and right cells inside it indexed by indecomposable left D-modules, cf. [MMZ1].

As was shown in [Jo2], band bimodules form one cell (both left, right and two-sided), which we denote $\mathcal{J}_{\text{band}}$. Moreover, for each positive integer k, the four string bimodules with k valleys form a two sided cell $\mathcal{J}_k$, see Section 3.4 for more details. The string bimodule M_0 forms its own two-sided cell $\mathcal{J}_0$. The two-sided cells are linearly ordered as follows:

$$\mathcal{J}_{\Bbbk\text{-split}} >_J \mathcal{J}_{M_0} >_J \mathcal{J}_1 >_J \mathcal{J}_2 >_J \dots >_J \mathcal{J}_{\text{band}}.$$

All two-sided cells except $\mathcal{J}_{M_0}$ are idempotent. Note also that all two-sided cells except the minimal cell $\mathcal{J}_{\text{band}}$ are finite.

3.2. **The main result.** The following theorem is the main result of this paper.

Theorem 1. (*i*) *Any simple transitive 2-representation of $\mathscr{D}$ with apex $\mathcal{J}_{\Bbbk\text{-split}}$ is equivalent to a cell 2-representation.*

(*ii*) *Any simple transitive 2-representation of $\mathscr{D}$ with apex $\mathcal{J}_k$, where $k \geq 1$, has rank 1 or rank 2.*

(*iii*) *Any simple transitive 2-representation of $\mathscr{D}$ with apex $\mathcal{J}_k$, where $k \geq 1$, of rank 2 is equivalent to the cell 2-representation $\mathbf{C}_{\mathcal{L}}$, where $\mathcal{L} = \{M_k, N_k\}$ (or, equivalently, $\mathcal{L} = \{W_k, S_k\}$).*

(*iv*) *There exists a simple transitive 2-representation of $\mathscr{D}$ with apex $\mathcal{J}_1$ which has rank 1.*

Taking Theorem 1 into account, the following conjecture seems very natural.

Conjecture 2. *For each $k \geq 1$, there exists a unique, up to equivalence, simple transitive 2-representation of $\mathscr{D}$ of rank 1 with apex $\mathcal{J}_k$.*

3.3. **Proof of Theorem 1(i).** For an arbitrary indecomposable $\Bbbk$-split D-D-bimodule $U \otimes_{\Bbbk} V$, using adjunction and projectivity of both V and $\mathrm{End}_{D\text{-}}(U)$ as $\Bbbk$-modules, we have

$$\begin{aligned}
\mathrm{End}_{D\text{-}D}(U \otimes_{\Bbbk} V) &\cong \mathrm{Hom}_{D\text{-}D}(U \otimes_{\Bbbk} V, U \otimes_{\Bbbk} V)\\
&\cong \mathrm{Hom}_{\text{-}D}(V, \mathrm{Hom}_{D\text{-}}(U, U \otimes_{\Bbbk} V))\\
&\cong \mathrm{Hom}_{\text{-}D}(V, \mathrm{Hom}_{D\text{-}}(U, U) \otimes_{\Bbbk} V)\\
&\cong \mathrm{Hom}_{\Bbbk}(\Bbbk, \mathrm{Hom}_{\text{-}D}(V, \mathrm{Hom}_{D\text{-}}(U, U) \otimes_{\Bbbk} V))\\
&\cong \mathrm{Hom}_{\Bbbk}(\Bbbk, \mathrm{Hom}_{D\text{-}}(U, U) \otimes_{\Bbbk} \mathrm{Hom}_{\text{-}D}(V, V))\\
&\cong \mathrm{End}_{D\text{-}}(U) \otimes_{\Bbbk} \mathrm{End}_{\text{-}D}(V).
\end{aligned} \tag{1}$$

Consider the finite dimensional algebra $A = \mathrm{End}_{D-}({}_D D \oplus {}_D\Bbbk)$ (note that it can be described as the path algebra of the quiver

$$\begin{tikzcd}
1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, "\beta"]
\end{tikzcd}$$

modulo the relation $\alpha\beta = 0$). Then we have the 2-category $\mathscr{C}_A$ of projective A-A-bimodules. By [MMZ2, Theorem 12], any simple transitive 2-representation of $\mathscr{C}_A$ is equivalent to a cell 2-representation.

Denote by $\mathscr{A}$ the 2-full 2-subcategory of $\mathscr{D}$ given by the additive closure inside $\mathscr{D}$ of the regular D-D-bimodule and all $\Bbbk$-split D-D-bimodules. The computation in (1) implies that the 2-categories $\mathscr{C}_A$ and $\mathscr{A}$ are biequivalent. Consequently, any simple transitive 2-representation of $\mathscr{A}$ is equivalent to a cell 2-representation.

Let $\mathbf{M}$ be a simple transitive 2-representation of $\mathscr{D}$ with apex $\mathcal{J}_{\Bbbk\text{-split}}$. Then the restriction of $\mathbf{M}$ to $\mathscr{A}$ is also simple transitive and hence this restriction is equivalent to a cell 2-representation of $\mathscr{A}$ by the previous paragraph. Now, the arguments similar to the ones in [MM5, Theorem 18] imply that $\mathbf{M}$ is equivalent to a cell 2-representation of $\mathscr{D}$. This proves Theorem 1(i).

3.4. **The two-sided cell $\mathcal{J}_k$, where $k \geq 1$.** Fix a positive integer k. Recall from [Jo2] that the two-sided cell $\mathcal{J}_k$ has the following egg-box diagram in which columns are left cells and rows are right cells.

W_k	N_k
S_k	M_k

Modulo the two-sided cells that are strictly larger with respect to the two sided order, the multiplication table of $\mathcal{J}_k$ is as follows.

(2)

$\otimes_D$	W_k	S_k	N_k	M_k
W_k	W_k	W_k	N_k	N_k
S_k	S_k	S_k	M_k	M_k
N_k	W_k	W_k	N_k	N_k
M_k	S_k	S_k	M_k	M_k

Lemma 3. *For any $k \geq 0$, the pair $(S_k \otimes_D -, N_k \otimes_D -)$ is an adjoint pair of endofunctors of D-mod.*

Proof. By [MZ2, Lemma 13], it is enough to show that S_k is projective as a left D-module, and that $\mathrm{Hom}_{D-}(S_k, D) \simeq N_k$ as D-D-bimodules. As a left module, S_k is a direct sum of $k+1$ copies of the left regular module ${}_D D$. This also implies that $\mathrm{Hom}_{D-}(S_k, D)$ is projective as a right module. Moreover

$$\dim \mathrm{Hom}_{D-}(S_k, D) = \dim \mathrm{Hom}_{D-}(D^{\oplus k+1}, D) = (k+1) \dim \mathrm{End}_{D-}(D) = 2(k+1).$$

Note that D is a symmetric algebra and thus ${}_D D_D \cong {}_D D^*_D$. Hence, by adjunction, we get

$$\mathrm{Hom}_{D-}(S_k, D) \simeq \mathrm{Hom}_{D-}(S_k, \mathrm{Hom}_{\Bbbk}(D, \Bbbk)) \simeq \mathrm{Hom}_{\Bbbk}(D \otimes_D S_k, \Bbbk) \simeq \mathrm{Hom}_{\Bbbk}(S_k, \Bbbk)$$

so that $\mathrm{Hom}_{D-}(S_k, D)$ and S_k^* are isomorphic as D-D-bimodules. Since S_k is indecomposable as a D-D-bimodule, so is $\mathrm{Hom}_{D-}(S_k, D)$.

The indecomposable, right projective, $2(k+1)$-dimensional D-D-bimodules are:

- N_k,
- $B_{k+1}(\lambda)$,
- $D \otimes_{\Bbbk} D$ (in the case $k = 1$).

To show that $\mathrm{Hom}_D(S_k, D) \simeq N_k$, note first that

$$\mathrm{Hom}_{D-}(S_0, D) = \mathrm{Hom}_{D-}(D \otimes_{\Bbbk} \Bbbk, D) \simeq \mathrm{Hom}_{\Bbbk}\big(\Bbbk, \mathrm{Hom}_D(D, D)\big) \simeq \mathrm{Hom}_{\Bbbk}(\Bbbk, D),$$

so it is clear that $\mathrm{Hom}_{D-}(S_0, D) \simeq N_0 = \Bbbk \otimes_{\Bbbk} D$, as D-D-bimodules. Now, for any $k \geq 1$, there is a short exact sequence of D-D-bimodules

$$0 \to S_{k-1} \to S_k \to S_0 \to 0.$$

Apply the functor $\mathrm{Hom}_{D-}(-, D)$ to this sequence. As the regular D-D-bimodule is injective as a left module, this functor is exact. Therefore we get a short exact sequence of D-D-bimodules

$$0 \to \mathrm{Hom}_{D-}(S_0, D) \to \mathrm{Hom}_{D-}(S_k, D) \to \mathrm{Hom}_{D-}(S_{k-1}, D) \to 0. \tag{3}$$

Hence $\mathrm{Hom}_{D-}(S_0, D) \simeq N_0$ is a submodule of any $\mathrm{Hom}_{D-}(S_k, D)$, implying that $\mathrm{Hom}_{D-}(S_k, D)$ is not a band bimodule. This proves the statement for $k \neq 1$. Moreover, by setting $k = 2$ in (3), we see that $\mathrm{Hom}_{D-}(S_1, D)$ is the quotient of $\mathrm{Hom}_{D-}(S_2, D) \simeq N_2$ by $\mathrm{Hom}_{D-}(S_0, D) \simeq N_0$, that is

$$\mathrm{Hom}_{D-}(S_1, D) \simeq N_2/N_0 \simeq N_1.$$

This concludes the proof. □

The following statement is an adjustment of [Zi2, Theorem 3.1] to a slightly more general setting, into which simple transitive 2-representations of $\mathscr{D}$ with apex $\mathcal{J}_k$, where $k \geq 1$, fit.

Theorem 4. *Let $\mathscr{C}$ be a 2-category with finitely many objects and such that each $\mathscr{C}(\mathtt{i}, \mathtt{j})$ is $\Bbbk$-linear, idempotent split and has finite dimensional spaces of 2-morphisms. Let $\mathbf{M}$ be a finitary simple transitive 2-representation of $\mathscr{C}$ such that the apex $\mathcal{J}$ of $\mathbf{M}$ is finite. Assume that $F \in \mathcal{J}$. Then the following holds.*

(*i*) *For every object X in any $\overline{\mathbf{M}}(\mathtt{i})$, the object $\overline{\mathbf{M}}(F)X$ is projective.*

(*ii*) *If $\overline{\mathbf{M}}(F)$ is left exact, then $\overline{\mathbf{M}}(F)$ is a projective functor.*

Proof. We can restrict to the finitary 2-subcategory of $\mathscr{C}$ given by the identities and the apex and then apply [Zi2, Theorem 3.1]. $\square$

Corollary 5. *Let* $\mathbf{M}$ *be a simple transitive 2-representation of* $\mathscr{D}$ *with apex* $\mathcal{J}_k$*, where* $k \geq 1$*. Then the functor* $\overline{\mathbf{M}}(N_k)$ *is a projective functor (in the sense that it is given by tensoring with a projective bimodule over the underlying algebra of the 2-representation).*

Proof. From Lemma 3 it follows that $\overline{\mathbf{M}}(N_k)$ is left exact. Therefore we may apply Theorem 4 and the claim follows. $\square$

4. Combinatorial results

Fix a simple transitive 2-representation $\mathbf{M}$ of $\mathscr{D}$ with apex $\mathcal{J}_k$, where $k \geq 1$. Let B be a basic associative $\Bbbk$-algebra for which $\mathbf{M}(\mathtt{i})$ is equivalent to B-proj. Let $1 = \varepsilon_1 + \ldots + \varepsilon_r$ be a decomposition of the identity in B into a sum of pairwise orthogonal primitive idempotents. Denote by P_i the i'th indecomposable projective left B-module $B\varepsilon_i$, and denote by L_i its simple top.

The aim of this section is to prove the following.

Proposition 6. *Let* $\mathbf{M}$ *be a simple transitive 2-representation of* $\mathscr{D}$ *with apex* $\mathcal{J}_k$*, where* $k \geq 1$*. Then the action matrices of indecomposable 1-morphisms in* $\mathcal{J}_k$ *are, up to renumbering of projective objects in* $\mathbf{M}(\mathtt{i})$*, either all equal to* $[1]$ *or*

$$[N_k] = [W_k] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad [M_k] = [S_k] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

In particular, Proposition 6 implies Theorem 1(ii). The remainder of this section is devoted to the proof of Proposition 6.

Lemma 7. *(i) If the matrix* $[N_k]$ *has a zero column, then the corresponding row in* $[S_k]$ *must be zero.*

(ii) If the matrix $[S_k]$ *has a zero column, then the corresponding row in* $[N_k]$ *must be zero.*

Proof. By Lemma 3, the functor N_k is exact. By [MM5, Lemma 10], we have $[[N_k]] = [S_k]^{\mathrm{tr}}$. If column i in the matrix $[N_k]$ is zero, then $N_kP_i = 0$. As L_i is the top of P_i, the object N_kL_i must be zero as well. This proves (i). On the other hand, if column i of $[S_k]$ is zero, then row i of $[S_k]^{\mathrm{tr}} = [[N_k]]$ is zero. This means that nothing in the image of N_k can have L_i as a simple subquotient. In particular, P_i cannot occur in the image of N_k, and so row i of $[N_k]$ must be zero. This proves (ii). $\square$

Note that W_k, S_k, N_k and M_k are all idempotent modulo strictly greater two-sided cells. Setting $F = W_k + S_k + N_k + M_k$ yields $F \otimes F = F^{\oplus 4}$. Hence the action matrix of F must be an irreducible positive integer matrix satisfying $[F]^2 = 4[F]$. The set of such matrices are classified in [TZ]. They are, up to permutations of rows and columns, the following.

$$[4], \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}, \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Since the functors $\mathbf{M}(W_k)$, $\mathbf{M}(S_k)$, $\mathbf{M}(N_k)$ and $\mathbf{M}(M_k)$ are all idempotent, their action matrices are idempotent as well. The rank of an idempotent matrix equals its trace. The trace of $[F]$ is 4, so the action matrices $[W_k], [S_k], [N_k], [M_k]$ must all have trace and rank 1. The action matrices also inherit left, right and two-sided preorders and equivalences, so we speak of these notions for 1-morphisms and action matrices interchangeably. Directly from the multiplication table we can also conclude the following about the action matrices.

- If $A \sim_R B$, then $AB = B$ and $BA = A$. This also implies
$$\begin{aligned} \operatorname{im}(B) &= \operatorname{im}(AB) \subseteq \operatorname{im}(A) \\ \operatorname{im}(A) &= \operatorname{im}(BA) \subseteq \operatorname{im}(B), \end{aligned}$$
so that $\operatorname{im} A = \operatorname{im} B$. For matrices of rank 1 this means that all nonzero columns of A and B are linearly dependent.
- If $A \sim_L B$, then $AB = A$ and $BA = B$. This also implies
$$\begin{aligned} \ker(A) &= \ker(AB) \supseteq \ker(B) \\ \ker(B) &= \ker(BA) \supseteq \ker(A), \end{aligned}$$
so that $\ker A = \ker B$. Hence A and B have the same zero columns.

Lemma 8. *(i)* $[W_k] = [S_k]$ *if and only if* $[N_k] = [M_k]$.

(ii) $[W_k] = [N_k]$ *if and only if* $[S_k] = [M_k]$.

(iii) If $[W_k] = [M_k]$ *or* $[S_k] = [N_k]$, *then* $[W_k] = [S_k] = [N_k] = [M_k]$.

Proof. If $[W_k] = [S_k]$, then
$$[N_k] = [W_k][N_k] = [S_k][N_k] = [M_k].$$
On the other hand, if $[N_k] = [M_k]$, then
$$[W_k] = [N_k][W_k] = [M_k][W_k] = [S_k].$$
This proves claim (i); claim (ii) is similar. Finally, if $[W_k] = [M_k]$, then
$$[W_k] = [W_k][W_k] = [W_k][M_k] = [N_k]$$
and
$$[W_k] = [W_k][W_k] = [M_k][W_k] = [S_k].$$
This proves one of the implications in (iii), the other is similar. □

In particular, Lemma 8 implies that, if the matrix $[F]$ has 1 as an entry, then the matrices $[W_k]$, $[S_k]$, $[N_k]$ and $[M_k]$ are all different.

We now do a case-by-case analysis depending on the rank of the 2-representation (i.e. the size of action matrices).

4.1. **Rank 1.** If $F = [4]$, then $[W_k] = [S_k] = [N_k] = [M_k] = [1]$.

4.2. **Rank 2.** Consider first the case $F \in \left\{ \begin{bmatrix} 3 & 3 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \right\}$. Since F has entries equal to 1, the action matrices of W_k, S_k, N_k and M_k must all be different. They all have trace 1 and their sum has diagonal $(3, 1)$, so we must have four different matrices with non-negative integer entries:

$$A = \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & c \\ d & 0 \end{bmatrix}, \ C = \begin{bmatrix} 1 & e \\ f & 0 \end{bmatrix}, \ G = \begin{bmatrix} 0 & g \\ h & 1 \end{bmatrix}.$$

Two of those with diagonal $(1, 0)$, say A and B, must belong to the same left cell. Then $AB = A$, i.e.

$$A = \begin{bmatrix} 1 & a \\ b & 0 \end{bmatrix} = AB = \begin{bmatrix} 1 + ad & c \\ d & bc \end{bmatrix}$$

which implies $a = c$ and $b = d$, so that $A = B$, a contradiction.

Assume now $F \in \left\{ \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \right\}$. Then W_k, S_k, N_k and M_k will be given by the following matrices:

$$A = \begin{bmatrix} 1 & * \\ * & 0 \end{bmatrix}, \ B = \begin{bmatrix} 1 & * \\ * & 0 \end{bmatrix}, \ C = \begin{bmatrix} 0 & * \\ * & 1 \end{bmatrix}, \ G = \begin{bmatrix} 0 & * \\ * & 1 \end{bmatrix}.$$

We see that $AB, BA \in \{A, B\}$. This implies that either $A \sim_L B$ or $A \sim_R B$.

If $A \sim_L B$, then $A \not\sim_R B$, so we can assume $A \sim_R C$ and $B \sim_R G$. By comparing images, and using that all ranks are 1, we get

$$A = B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \ C = G = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

$A \sim_L B$ and $C \sim_L G$ tells us that left equivalent functors are represented by the same matrix. By symmetry we can set

$$[N_k] = [M_k] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \quad \text{and} \quad [S_k] = [W_k] = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}.$$

However, now the second column of $[N_k]$ is zero, but the second row of $[S_k]$ is nonzero. This contradicts Lemma 7(i), so we discard this case.

If, instead, $A \sim_R B$ and $C \sim_R G$, we can assume $A \sim_L C$ and $B \sim_L G$. Since the first column of A is nonzero, so is the first column of C. At the same time, the second column of C is nonzero, so the second column of A is as well. Together with ranks being 1 and right equivalences, this yields

$$A = B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } C = G = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

By symmetry we can set

$$[N_k] = [W_k] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } [M_k] = [S_k] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

4.3. **Rank 3.** $F \in \left\{ \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \right\}$. Either choice of the matrix of F has 1 as entry, so all of $[W_k]$, $[S_k]$, $[N_k]$ and $[M_k]$ have to be different. As the diagonal of F is $(2,1,1)$, they must be represented by idempotent matrices A,B,C,G, all of rank 1, as follows.

$$A=\begin{bmatrix}1&&*\\&0&\\*&&0\end{bmatrix},\ B=\begin{bmatrix}1&&*\\&0&\\*&&0\end{bmatrix},\ C=\begin{bmatrix}0&&*\\&1&\\*&&0\end{bmatrix},\ G=\begin{bmatrix}0&&*\\&0&\\*&&1\end{bmatrix}.$$

Note that $AB, BA \in \{A,B\}$, so A and B are either left or right equivalent. We consider these two cases separately.

Assume first $A \sim_L B$, $C \sim_L G$, so that C and G have the same kernel. Hence the third column of C and the second of G are nonzero. Since the ranks are 1 we get

$$C=\begin{bmatrix}0&&*\\&1&1\\*&0&0\end{bmatrix},\ G=\begin{bmatrix}0&&*\\&0&0\\*&1&1\end{bmatrix}.$$

Taking into account that the lower right submatrix of F has all entries 1, this implies

$$A=\begin{bmatrix}1&&*\\&0&0\\*&0&0\end{bmatrix},\ B=\begin{bmatrix}1&&*\\&0&0\\*&0&0\end{bmatrix}.$$

Since $A \sim_L B$, we have $A \not\sim_R B$. We can thus assume $A \sim_R C$ and $B \sim_R G$. Then A and C have the same image, and B and G have the same image, so that

$$A=\begin{bmatrix}1&0&0\\1&0&0\\0&0&0\end{bmatrix},\ B=\begin{bmatrix}1&0&0\\0&0&0\\1&0&0\end{bmatrix},\ C=\begin{bmatrix}0&1&1\\0&1&1\\0&0&0\end{bmatrix},\ G=\begin{bmatrix}0&1&1\\0&0&0\\0&1&1\end{bmatrix}.$$

Then $\{S,N\}$ is either $\{A,G\}$ or $\{B,C\}$. Any such choice contradicts Lemma 7(ii).

Assume now $A \sim_R B$ and $C \sim_R G$, so that C and G have the same image. Then

$$C=\begin{bmatrix}0&&*\\&1&0\\*&1&0\end{bmatrix},\ G=\begin{bmatrix}0&&*\\&0&1\\*&0&1\end{bmatrix}.$$

By considering the lower right 2×2-submatrix of F, we conclude

$$A=\begin{bmatrix}1&&*\\&0&0\\*&0&0\end{bmatrix},\ B=\begin{bmatrix}1&&*\\&0&0\\*&0&0\end{bmatrix}.$$

$A \sim_R B$ implies $A \not\sim_L B$, so we can assume $A \sim_L C$ and $B \sim_L G$. Using now that left equivalence means common kernel, together with all ranks being 1, we get

$$A=\begin{bmatrix}1&1&0\\0&0&0\\0&0&0\end{bmatrix},\ B=\begin{bmatrix}1&0&1\\0&0&0\\0&0&0\end{bmatrix},\ C=\begin{bmatrix}0&0&0\\1&1&0\\1&1&0\end{bmatrix},\ G=\begin{bmatrix}0&0&0\\1&0&1\\1&0&1\end{bmatrix}.$$

Then $\{S,N\}$ is either $\{A,G\}$ or $\{B,C\}$. Any such choice contradicts Lemma 7(i).

4.4. **Rank 4.** Assume that W_k, S_k, N_k and M_k are given by

$$A=\begin{bmatrix}1&&&*\\&0&&\\&&0&\\*&&&0\end{bmatrix},\ B=\begin{bmatrix}0&&&*\\&1&&\\&&0&\\*&&&0\end{bmatrix},\ C=\begin{bmatrix}0&&&*\\&0&&\\&&1&\\*&&&0\end{bmatrix},\ G=\begin{bmatrix}0&&&*\\&0&&\\&&0&\\*&&&1\end{bmatrix}.$$

As all entries of F are 1, we have that, for each position (i,j), one of A,B,C,G has entry 1 at this position, while the others have entry 0 at this position.

We can, without loss of generality, assume that $A \sim_R B$, $C \sim_R G$, $A \sim_L C$ and $B \sim_L G$. This gives us immediately

$$A = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \; C = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}, \; G = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

Then $\{S_k, N_k\}$ is either $\{A, G\}$ or $\{B, C\}$. Any such choice contradicts Lemma 7. This completes the proof of Proposition 6.

5. Each simple transitive 2-representation of rank 2 is cell

Fix a simple transitive 2-representation $\mathbf{M}$ of $\mathscr{D}$ with apex $\mathcal{J}_k$, where $k \geq 1$.

Let $\mathcal{L}$ be the left cell $\{N_k, M_k\}$. As seen in Proposition 6, the action matrices of $\mathbf{M}(U_k)$, where $U_k \in \mathcal{J}_k$, are as follows:

$$[N_k] = [W_k] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \qquad [M_k] = [S_k] = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

Let us see what this says about the basic algebra B underlying $\overline{\mathbf{M}}(\mathtt{i})$. The rank is two, so we have a decomposition $1 = \varepsilon_1 + \varepsilon_2$ of the identity in B into primitive orthogonal idempotents. Denote by $P_1 = B\varepsilon_1$ and $P_2 = B\varepsilon_e$ the indecomposable projective left B-modules, and by L_1, L_2 their respective simple tops. Then, for $i = 1, 2$, we have

$$\mathbf{M}(N_k)P_i \simeq \mathbf{M}(W_k)P_i \simeq P_1 \qquad \text{and} \qquad \mathbf{M}(S_k)P_i \simeq \mathbf{M}(M_k)P_i \simeq P_2.$$

Moreover, since

$$[[N_k]] = [S_k]^t = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix},$$

we have $\overline{\mathbf{M}}(N_k)L_1 = 0$ and $\overline{\mathbf{M}}(N_k)L_2$ has simple subquotients L_1, L_2. By Theorem 4, it follows that $\overline{\mathbf{M}}(N_k)L_1$ must be isomorphic to a number of copies of P_1. Therefore we see that $\overline{\mathbf{M}}(N_k)L_1 \simeq P_1$, and P_1 has length 2 with socle L_2. In the underlying quiver of B this means that we have exactly one arrow α from 1 to 2, and no loops at 1. If there is an arrow β from 2 to 1 then $\beta\alpha = 0$. Moreover, if there is a loop γ at 2 then $\gamma\alpha = 0$:

(4)
$$\begin{tikzcd} 1 \arrow[r, bend left, "\alpha"] & 2 \arrow[l, bend left, dotted, "\beta"] \arrow[loop right, dotted, "\gamma"] \end{tikzcd}$$

This also yields

$$\dim(\varepsilon_1 B \varepsilon_1) = \dim \mathrm{Hom}_B(P_1, P_1) = 1$$
$$\dim(\varepsilon_2 B \varepsilon_1) = \dim \mathrm{Hom}_B(P_2, P_1) = 1.$$

Since $\overline{\mathbf{M}}(N_k)$ is exact and only has P_1 in its image, it must be of the form

$$\overline{\mathbf{M}}(N_k) \simeq B\varepsilon_1 \otimes \varepsilon_1 B^{\oplus a} \oplus B\varepsilon_1 \otimes \varepsilon_2 B^{\oplus b},$$

for some nonnegative integers a and b. Since

$$\overline{\mathbf{M}}(N_k)L_1 = 0,$$

we must have $a = 0$. Then

$$\overline{\mathbf{M}}(N_k)L_2 \simeq P_1$$

implies that $b = 1$, so that

$$\mathbf{M}(N_k) \simeq B\varepsilon_1 \otimes \varepsilon_2 B \otimes_B -.$$

As seen in Lemma 3, (S_k, N_k) is an adjoint pair, so this also gives

$$\mathbf{M}(S_k) \simeq B\varepsilon_2 \otimes (B\varepsilon_1)^* \otimes_B -,$$

cf. [MM1, Subsection 7.3]. Again, using that (S_k, N_k) is an adjoint pair, yields

$$\begin{aligned}\dim(\varepsilon_2 B\varepsilon_2) &= \dim \mathrm{Hom}_B(P_2, P_2) = \\ &= \dim \mathrm{Hom}_B(\mathbf{M}(S_k)P_1, P_2) = \\ &= \dim \mathrm{Hom}_B(P_1, \mathbf{M}(N_k)P_2) = \\ &= \dim \mathrm{Hom}_B(P_1, P_1) = \\ &= 1.\end{aligned}$$

In the quiver (4), this rules out loops at 2. Moreover, it implies

$$\mathbf{M}(W_k) \simeq \mathbf{M}(N_k)\mathbf{M}(S_k) \simeq B\varepsilon_1 \otimes (B\varepsilon_1)^* \otimes_B -.$$

Because W_k is idempotent, $\dim\big((B\varepsilon_1)^* \otimes_B B\varepsilon_1\big) = 1$. Hence, it follows that

$$\mathbf{M}(M_k) = \mathbf{M}(S_k)\mathbf{M}(N_k) = B\varepsilon_2 \otimes \varepsilon_2 B \otimes_B -.$$

Consider now $(B\varepsilon_1)^*$. As seen above, $P_1 = B\varepsilon_1$ has Jordan-Hölder series L_1, L_2, so $(B\varepsilon_1)^*$ has top L_2^* and socle L_1^* (these are simple right B-modules). This implies that $(B\varepsilon_1)^*$ is exactly the projective right module $\varepsilon_2 B$. Hence we conclude

$$\begin{aligned}\mathbf{M}(N_k) \simeq \mathbf{M}(W_k) \simeq B\varepsilon_1 \otimes \varepsilon_2 B \otimes_B - \\ \mathbf{M}(S_k) \simeq \mathbf{M}(M_k) \simeq B\varepsilon_2 \otimes \varepsilon_2 B \otimes_B -.\end{aligned}$$

We have that the Cartan matrix of $\mathbf{M}$ is

$$\begin{bmatrix} 1 & c \\ 1 & 1 \end{bmatrix}$$

where $c = \dim \mathrm{Hom}_B(P_1, P_2)$ remains unknown.

Since $\dim \mathrm{Hom}_B(P_2, P_2) = 1$, and P_1 has Jordan Hölder series L_1, L_2, we must have a short exact sequence

$$L_1^{\oplus c} \xrightarrow{g} P_2 \to L_2.$$

In the quiver (4), this corresponds to the fact that we have exactly c arrows $\beta_1, \dots, \beta_c : 2 \to 1$ and the relations

$$\alpha\beta_i = 0 = \beta_i\alpha.$$

Let us sum up what we know so far:

- P_1 has basis $\{\varepsilon_1, \alpha\}$,
- P_2 has basis $\{\varepsilon_2, \beta_1, \dots, \beta_c\}$,
- $\mathrm{Hom}_B(P_1, P_2)$ has a basis $\{f_1, \dots, f_c\}$ where $f_i(\alpha) = 0$ and $f_i(\varepsilon_1) = \beta_i$.

However, all functors above are of the form

$$\overline{\mathbf{M}}(U) = B\varepsilon_i \otimes \varepsilon_2 B \otimes_B -.$$

The module $\varepsilon_2 B$ has basis $\{\varepsilon_2, \alpha\}$, and, as seen above, we have

$$\alpha\beta_i = 0 = \varepsilon_2\beta_i.$$

Thus, for $U \in \mathcal{J}_k$, we have

$$\overline{\mathbf{M}}(U)(f_i)(\varepsilon_1) = 0,$$

so that $\overline{\mathbf{M}}(U)(f_i) = 0$. But then the f_i's generate a proper $\mathscr{D}$-invariant ideal in $\mathbf{M}(\mathtt{i})$. By simplicity of $\mathbf{M}$, this ideal is $\{0\}$. Thus $c = 0$ and the Cartan matrix is

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}.$$

The rest of the proof now goes as in e.g. [MM5, Proposition 9] or [MaMa, Subsection 4.9]. Consider the principal 2-representation $\mathbf{P}_{\mathtt{i}}$ and the subrepresentation $\mathbf{N}$ with $\mathbf{N}(\mathtt{i}) = \mathrm{add}\{F \mid F \geq_L \mathcal{L}\}$. Recall that there is a unique maximal ideal $\mathbf{I}$ in $\mathbf{N}$ such that $\mathbf{N}/\mathbf{I} \simeq \mathbf{C}_{\mathcal{L}}$. The map

$$\Phi : \mathbf{P}_{\mathtt{i}} \to \overline{\mathbf{M}}$$
$$\mathbb{1}_{\mathtt{i}} \mapsto L_2$$

extends to a 2-natural transformation by the Yoneda Lemma, [MM2, Lemma 9]. Since

$$\overline{\mathbf{M}}(N_k)L_2 = P_1 \quad \text{and} \quad \overline{\mathbf{M}}(M_k)L_2 = P_2,$$

Φ induces a 2-natural transformation $\Psi : \mathbf{N} \to \overline{\mathbf{M}}_{\mathrm{proj}}$. Note that $\overline{\mathbf{M}}_{\mathrm{proj}}$ is equivalent to $\mathbf{M}$. By uniqueness of the maximal ideal $\mathbf{I}$ the kernel of Ψ is contained in $\mathbf{I}$, so Ψ factors through $\mathbf{C}_{\mathcal{L}}$. On the other hand, the Cartan matrices of $\mathbf{M}$ and $\mathbf{C}_{\mathcal{L}}$ coincide. Consequently, Ψ induces an equivalence of 2-representations between $\mathbf{C}_{\mathcal{L}}$ and $\mathbf{M}$.

This proves Theorem 1(iii).

6. A simple transitive 2-representation of rank 1 with apex $\mathcal{J}_1$

Recall that we have the two-sided cell $\mathcal{J}_0$ containing only the 1-morphism M_0. We have $\mathcal{J}_{\Bbbk\text{-split}} \geq_J \mathcal{J}_0 \geq_J \mathcal{J}_1$. The cell $\mathcal{J}_0$ is not idempotent, since

$$M_0 \otimes_D M_0 \simeq D \otimes D \oplus \Bbbk.$$

However, for all $U \in \mathcal{J}_1$, we have

$$U \otimes_D M_0 \simeq M_0 \oplus V,$$

where all indecomposable direct summands of V are $\Bbbk$-split. Since $\mathcal{J}_0$ contains only one element, it is also a left cell. Therefore the cell 2-representation $\mathbf{C}_{\mathcal{J}_0}$ is a simple transitive 2-representation of $\mathscr{D}$ with apex $\mathcal{J}_1$. Note that the matrix describing the action of each 1-morphism in $\mathcal{J}_1$ is [1], agreeing with Proposition 6.

This proves Theorem 1(iv) and thus completes the proof of Theorem 1.

7. (Co-) Duflo 1-morphisms

7.1. 2-morphisms to and from $\mathbb{1}_{\mathtt{i}}$.

7.1.1. *String bimodules.* In what follows we will need a more detailed description of string bimodules. We will index the basis elements of M_k and W_k as follows:

$$
\begin{tikzcd}
m_1 & m_2 \arrow[l] \arrow[d] & & \\
& m_3 & \ddots \arrow[l] \arrow[d] & \\
& & m_{2k+1} & m_{2k+2} \arrow[l] \arrow[d] \\
& & & m_{2k+3}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
w_1 \arrow[d] & & & \\
w_2 & w_3 \arrow[l] \arrow[d] & & \\
& \ddots & w_{2k-1} \arrow[l] \arrow[d] & \\
& & w_{2k} & w_{2k+1} \arrow[l]
\end{tikzcd}
$$

With this convention, we have $N_k \simeq M_k/\mathrm{span}\{m_{2k+3}\}$, $S_k \simeq M_k/\mathrm{span}\{m_1\}$ and $W_k \simeq M_k/\mathrm{span}\{m_1, m_{2k+3}\}$.

Lemma 9. *Let k be a positive integer.*

(*i*) *The only element of $\mathcal{J}_k$ admitting a D-D-bimodule morphism to $\mathbb{1}_{\mathtt{i}}$ which does not factor through the simple bimodule is M_k.*

(*ii*) *The only element of $\mathcal{J}_k$ admitting a D-D-bimodule morphism $\mathbb{1}_{\mathtt{i}} \to U_k$ which does not factor through the simple bimodule is W_k.*

Proof. The regular bimodule $\mathbb{1}_{\mathtt{i}} \simeq {}_D D_D$ has standard basis $\{1, x\}$.

There is a D-D-bimodule morphism $\varphi_k : M_k \to \mathbb{1}_{\mathtt{i}}$ given by

$$\varphi_k(m_j) = \begin{cases} 1, & j \text{ even} \\ x, & j \text{ odd} \end{cases}.$$

That is, φ_k maps standard basis elements from $\mathrm{rad}(M_k)$ to $x \in {}_D D_D$, and the rest of the standard basis elements to 1. We prove that any D-D-bimodule morphism $\varphi : W_k \to \mathbb{1}_{\mathtt{i}}$ factors through the simple bimodule, and similar arguments for S_k and N_k complete the proof of part (i). Assume that $\varphi : W_k \to \mathbb{1}_{\mathtt{i}}$ is a D-D-bimodule morphism. Consider the standard basis vector w_1. Since $w_1 x = 0$ we must have $\varphi(w_1) \in \mathrm{span}\{x\}$. Thus

$$\varphi(w_2) = \varphi(x w_1) = x\varphi(w_1) = 0.$$

As $w_2 = w_3 x$ this, in turn, implies $\varphi(w_3) \in \mathrm{span}\{x\}$ and so on. We will have $\varphi(w_{2j}) = 0$ for all j, i.e. φ annihilates $\mathrm{rad}(W_k)$. Thus φ factors through the simple bimodule.

For part (ii), it is straightforward to check that $\psi_k : \mathbb{1}_{\mathtt{i}} \to W_k$ given by

$$\begin{aligned} \psi_k(1) &= w_1 + w_3 + \ldots + w_{2k+1}, \\ \psi_k(x) &= w_2 + w_4 + \ldots + w_{2k}, \end{aligned}$$

is a homomorphism of D-D-bimodules. If $\eta : \mathbb{1}_{\mathtt{i}} \to M_k$ is a bimodule morphism, then

$$\eta(1) = \sum_{j=1}^{2k+3} \lambda_j m_j,$$

for some $\lambda_j \in \Bbbk$. Then

$$\eta(x) = \eta(1)x = \sum_{j=1}^{k+1} \lambda_{2j} m_{2j-1} = x\eta(1) = \sum_{j=1}^{k+1} \lambda_{2j} m_{2j+1}.$$

Comparing the coefficients, we conclude that $\lambda_{2j} = 0$, for $j = 1, \dots, k+1$, so that $\eta(1) \in \mathrm{rad}(M_k)$. Thus φ factors through the simple bimodule. For S_k and N_k, the proof is similar. □

Note that $\varphi_0 : M_0 \to \mathbb{1}_{\mathtt{i}}$ is also defined. If we fix integers $l \le k$, then φ_l factors through φ_k, and ψ_l factors through ψ_k. Indeed, M_k has a submodule isomorphic to M_l spanned by $\{m_j \mid j = 1, \dots, 2l+3\}$. Letting $\iota_{l,k} : M_l \to M_k$ be the inclusion of M_l into M_k, it is clear that $\varphi_l = \varphi_k \circ \iota_{l,k}$. Similarly, denote by $\pi_{k,l} : W_k \to W_l$ the projection whose kernel is spanned by $\{w_j \mid j \ge 2l+2\}$. Then $\psi_l = \pi_{k,l} \circ \psi_k$.

Let us now address the problem of uniqueness of φ_k and ψ_k. For a non-negative integer k, denote by V_k the subspace of $\mathrm{Hom}_{D\text{-}D}(M_k, D)$ consisting of all homomorphisms which factor through the simple D-D-bimodule. For a positive integer k, denote by $\hat{V}_k$ the subspace of $\mathrm{Hom}_{D\text{-}D}(D, W_k)$ consisting of all homomorphisms which factor through the simple D-D-bimodule.

Corollary 10.

(i) *For any non-negative integer k, we have* $\dim \mathrm{Hom}_{D\text{-}D}(M_k, D)/V_k = 1$.

(ii) *For any positive integer k, we have* $\dim \mathrm{Hom}_{D\text{-}D}(D, W_k)/\hat{V}_k = 1$.

Proof. Assume that $\varphi \in \mathrm{Hom}_{D\text{-}D}(M_k, D) \setminus V_k$. Then $\varphi(m_2) \in D \setminus \Bbbk\langle x\rangle$, in particular, $x\varphi(m_2) = \varphi(xm_2) = \varphi(m_3) \neq 0$. Using the right action of x, we have $\varphi(m_3) = \varphi(m_4 x) = \varphi(m_4)x$, which uniquely determines the image of $\varphi(m_4)$ in $D/\Bbbk\langle x\rangle$. Similarly, the image of each $\varphi(m_i)$, where i is even, in $D/\Bbbk\langle x\rangle$ is uniquely determined. As $\Bbbk\langle x\rangle \subset D$ is a simple D-D-bimodule, claim (i) follows. Claim (ii) is proved similarly. □

7.1.2. *Band bimodules.* From the definition of band bimodules, it follows directly that, for all $n \ge 2$, there are short exact sequences of D-D-bimodules

$$0 \to B_1(1) \xrightarrow{\alpha_n} B_n(1) \to B_{n-1}(1) \to 0$$

and

$$0 \to B_{n-1}(1) \to B_n(1) \xrightarrow{\beta_n} B_1(1) \to 0.$$

It is a technical but not difficult exercise to verify that, for any n and k, the morphism φ_k factors through β_n, and the morphism α_n factors through ψ_k.

7.2. **Duflo 1-morphisms in fiat 2-categories.** Following [MM1], recall that a finitary 2-category $\mathscr{C}$ is called *fiat* if it has a weak involution $\star$ such that each pair $(F, F^\star)$ of 1-morphisms is an adjoint pair via some choice of adjunctions morphisms between the compositions $FF^\star$, $F^\star F$ and the relevant identities.

Let $\mathscr{C}$ be a fiat 2-category and $\mathcal{L}$ a left cell in $\mathscr{C}$. Let $\mathtt{i} = \mathtt{i}_{\mathcal{L}}$ be the object such that all 1-morphisms in $\mathcal{L}$ start in $\mathtt{i}$. A 1-morphism $G \in \mathcal{L}$ is called a *Duflo 1-morphism* for $\mathcal{L}$, cf. [MM1, Subsection 4.5], if the indecomposable projective module $P_{\mathbb{1}_{\mathtt{i}}}$ in $\overline{\mathbb{P}}_{\mathtt{i}}(\mathtt{i})$ has a submodule K such that

(1) $P_{\mathbb{1}_{\mathtt{i}}}/K$ is annihilated by all $F \in \mathcal{L}$,

(2) there is a surjective morphism $P_G \to K$.

By [MM1, Proposition 17], any left cell in a fiat 2-category $\mathscr{C}$ has a unique Duflo 1-morphism. These Duflo 1-morphisms play a major role in the construction of cell 2-representations, cf. [MM1].

7.3. **Duflo 1-morphisms for finitary 2-categories.** The paper [Zh1] gives a different definition of the notion of Duflo 1-morphisms which is also applicable for general finitary 2-categories. One significant difference with [MM1] is that, in the general case, Duflo 1-morphisms in the sense of [Zh1] do not have to exist, and if they exist, they do not have to belong to the left cell they are associated to. Below we propose yet another alternative.

Let $\mathscr{C}$ be a finitary 2-category, $\mathcal{L}$ a left cell in $\mathscr{C}$ and $\mathtt{i} = \mathtt{i}_{\mathcal{L}}$ the object such that all 1-morphisms in $\mathcal{L}$ start at it.

Definition 11.

(i) A 1-morphism G in $\mathscr{C}$ is *good* for $\mathcal{L}$ if there is a 2-morphism $\varphi : G \to \mathbb{1}_{\mathtt{i}}$ such that $F\varphi : FG \to F$ is right split, for any $F \in \mathcal{L}$ (i.e. there is $\xi : F \to FG$ such that $F\varphi \circ_v \xi = \mathrm{id}_F$).

(ii) A 1-morphism G in $\mathscr{C}$ is *great* for $\mathcal{L}$ if it is good for $\mathcal{L}$, and, for any G' with $\varphi' : G' \to \mathbb{1}_{\mathtt{i}}$ which is also good for $\mathcal{L}$, there is a 2-morphism $\beta : G \to G'$ such that $\varphi = \varphi' \circ \beta$.

Remark 12. That $\mathscr{C}$ is finitary is not necessary for to state Definition 11.

For fiat 2-categories, the following proposition relates the latter notion to that of Duflo 1-morphisms.

Proposition 13. *Let $\mathscr{C}$ be a fiat 2-category and $\mathcal{L}$ a left cell in $\mathscr{C}$. Then $G \in \mathcal{L}$ is great for $\mathcal{L}$ if and only if G is the Duflo 1-morphism of $\mathcal{L}$.*

Proof. The proof goes as follows: we first prove that the Duflo 1-morphism of $\mathcal{L}$ is good for $\mathcal{L}$. Then we prove that if G is great for $\mathcal{L}$, then G is the Duflo 1-morphism for $\mathcal{L}$. Finally, we prove that the Duflo 1-morphism is great for $\mathcal{L}$.

Assume first that G is the Duflo 1-morphism of $\mathcal{L}$. Let $K \subseteq P_{\mathbb{1}_{\mathtt{i}}}$ be the submodule from the definition and $\alpha : P_G \to K$ a surjective morphism. Let $f : P_G \to P_{\mathbb{1}_{\mathtt{i}}}$ be the composition $P_G \xrightarrow{\alpha} K \xhookrightarrow{\iota} P_{\mathbb{1}_{\mathtt{i}}}$. The morphism f is given by a morphism $\varphi : G \to \mathbb{1}_{\mathtt{i}}$ as represented on the commutative diagram

$$\begin{tikzcd} P_G \arrow[d, "f"] & = & 0 \arrow[r] \arrow[d] & G \arrow[d, "\varphi"] \\ P_{\mathbb{1}_{\mathtt{i}}} & = & 0 \arrow[r] & \mathbb{1}_{\mathtt{i}}. \end{tikzcd} \tag{5}$$

Consider short exact sequences

$$\mathrm{ker} \hookrightarrow P_G \xrightarrow{\alpha} \!\!\!\!\rightarrow K$$

$$K \xhookrightarrow{\iota} P_{\mathbb{1}_{\mathtt{i}}} \twoheadrightarrow P_{\mathbb{1}_{\mathtt{i}}}/K.$$

As $\mathscr{C}$ is fiat, each 1-morphism of $\mathscr{C}$ acts as an exact functor on each abelian 2-representation of $\mathscr{C}$. Therefore applying $F \in \mathcal{L}$ yields short exact sequences

$$F\,\mathrm{ker} \hookrightarrow FP_G \xrightarrow{F\alpha} \!\!\!\!\rightarrow FK$$

$$FK \xhookrightarrow{F\iota} FP_{\mathbb{1}_{\mathtt{i}}} \twoheadrightarrow F\big(P_{\mathbb{1}_{\mathtt{i}}}/K\big).$$

By assumption $F\big(P_{\mathbb{1}_\mathtt{i}}/K\big)=0$, so $F\iota : FK \to FP_{\mathbb{1}_\mathtt{i}}$ is an isomorphism, in particular, it is surjective. Thus $Ff = F\iota \circ F\alpha : FP_G \to F\mathbb{1}_\mathtt{i}$ is also surjective, implying that it is right split.

By considering the right column of the diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & FG \arrow[d, "F\varphi"] \\ 0 \arrow[r] & F, \end{tikzcd}$$

we see that $F\varphi$ is right split. Therefore G is good for $\mathcal{L}$. This completes the first step of our proof.

To prove the second step, assume that G is great for $\mathcal{L}$. Let $\varphi : G \to \mathbb{1}_\mathtt{i}$ be the corresponding 2-morphisms from the definition. This extends to a morphism $P_G \to P_{\mathbb{1}_i}$ in $\overline{\mathbf{P}}_\mathtt{i}$ as in (5) and the submodule K of $P_{\mathbb{1}_i}$ is the image of this morphism. We now have a short exact sequence

$$0 \to K \xrightarrow{\overline{f}} P_{\mathbb{1}_\mathtt{i}} \xrightarrow{g} P_{\mathbb{1}_\mathtt{i}}/K \to 0.$$

Applying exact $F \in \mathcal{L}$, we get a short exact sequence

$$0 \to FK \xrightarrow{F\overline{f}} P_F \xrightarrow{Fg} F\big(P_{\mathbb{1}_\mathtt{i}}/K\big) \to 0.$$

Note that, since $F\varphi$ is right split, the induced morphism $K \to P_F$ in $\overline{\mathbf{P}}_\mathtt{i}$ is also right split and therefore surjective. Hence $F\overline{f} : FK \to P_F$ is an isomorphism. By exactness, we obtain $F\big(P_{\mathbb{1}_\mathtt{i}}/K\big)=0$.

To conclude that G is the Duflo 1-morphism in $\mathcal{L}$, it remains to show that $G \in \mathcal{L}$. Assume that H is the Duflo 1-morphism of $\mathcal{L}$, and that $K_H \subseteq P_{\mathbb{1}_\mathtt{i}}$ is the submodule from the definition. We shall prove that $G = H$. By the above, H is good for $\mathcal{L}$, with the corresponding morphism $H \to \mathbb{1}_\mathtt{i}$, so there is a morphism $\alpha : G \to H$ making the following diagram commutative:

$$\begin{tikzcd} H \arrow[r] & \mathbb{1}_\mathtt{i} \\ G \arrow[u, "\alpha"] \arrow[ur, "\varphi"'] & \end{tikzcd}$$

Therefore $K \subseteq K_H \subseteq P_{\mathbb{1}_\mathtt{i}}$. Note that K_H has simple top L_H. By [MM1, Proposition 17(b)], for all $F \in \mathcal{L}$, the object FL_H has simple top L_F, in particular $FL_H \neq 0$. Since $F(P_{\mathbb{1}_\mathtt{i}}/K)=0$, for all $F \in \mathcal{L}$, we conclude that $K_H \subseteq K$. Thus $K_H = K$. But K_H has simple top L_H and K has simple top L_G, so $H = G$ is the Duflo 1-morphism of $\mathcal{L}$. This completes the second step of our proof.

Finally, let G be the Duflo 1-morphism of $\mathcal{L}$. We have already seen that G is good for $\mathcal{L}$, it remains to prove that it is great. Assume that H is also good for $\mathcal{L}$, with $\psi : H \to \mathbb{1}_\mathtt{i}$ being the morphism such that $F\psi$ is right split, for all $F \in \mathcal{L}$.

As above, $\operatorname{im}\varphi$ and $\operatorname{im}\psi$ give submodules K_G and K_H of $P_{\mathbb{1}_\mathtt{i}}$ with $F(P_{\mathbb{1}_\mathtt{i}}/K_G)=0$ and $F(P_{\mathbb{1}_\mathtt{i}}/K_H)=0$, for all $F \in \mathcal{L}$. Since the top of K_G is L_G and L_G is not annihilated by $F \in \mathcal{L}$, there is a nonzero morphism $K_G \to K_H$ such that the diagram

$$\begin{tikzcd} P_G \arrow[r, two heads] & K_G \arrow[r] \arrow[d] & P_{\mathbb{1}_{\mathtt{i}}} \arrow[d, equal] \\ P_H \arrow[r, two heads] & K_H \arrow[r] & P_{\mathbb{1}_{\mathtt{i}}} \end{tikzcd}$$

commutes. Since P_G is projective, there is a morphism $\alpha : P_G \to P_H$ making the left square commute. Thus the whole diagram commutes and we obtain a factorization

$$\begin{tikzcd} P_G \arrow[r] \arrow[d, "\alpha"] & P_{\mathbb{1}_{\mathtt{i}}} \\ P_H \arrow[ur] & \end{tikzcd}$$

implying that G is great for $\mathcal{L}$. $\square$

7.4. **Duflo 1-morphisms in $\mathscr{D}$.** For a positive integer m, we denote by $\mathscr{D}^{(m)}$ the 2-full 2-subcategory of $\mathscr{D}$ given by the additive closure of all 1-morphisms in all two-sided cells $\mathcal{J}$ such that $\mathcal{J} \geq_J \mathcal{J}_m$, together with $\mathbb{1}_{\mathtt{i}}$. Note that $\mathscr{D}^{(m)}$ is a finitary 2-category.

The following proposition suggests that M_k is a very good candidate for being called a *Duflo* 1-*morphism* in its left cell in $\mathscr{D}^{(k)}$.

Proposition 14. *For any $m \geq k \geq 1$, the* 1*-morphism M_k of the finitary* 2*-category $\mathscr{D}^{(m)}$ is great for $\mathcal{L} = \{N_k, M_k\}$.*

Proof. Let us first establish that M_k is good for $\mathcal{L}$. It is easy to check, by a direct computation (see Subsection 8.2), that the composition $M_k \otimes M_k$ has a direct summand isomorphic to M_k spanned by

$$\{m_2 \otimes m_1,\ m_j \otimes m_j,\ m_{j+1} \otimes m_j \mid j = 2, 4, \ldots, 2k+2\},$$

and that the projection onto this summand is a right inverse to $M_k\varphi_k$. Using $N_k \simeq M_k/\mathrm{span}\{m_{2k+3}\}$, gives also that $N_k\varphi_k$ is right split. Therefore M_k is good for $\mathcal{L}$ with respect to the morphism $\varphi_k : M_k \to \mathbb{1}_{\mathtt{i}}$. Note also that, by Corollary 10, the choice of φ_k is unique up to a non-zero scalar and up to homomorphisms which factor through the simple D-D-bimodule.

Let now F be a 1-morphism in $\mathscr{D}^{(k)}$ which is good for $\mathcal{L}$ via the map $\alpha : F \to D$. To start with, we argue that α does not factor through the simple D-D-bimodule. Indeed, if α does factor through the simple D-D-bimodule, it is not surjective as a map of D-D-bimodules. Applying the right exact functor $M_k \otimes_D {}_-$ to the exact sequence

$$F \xrightarrow{\alpha} D \longrightarrow \mathrm{Coker} \to 0,$$

we get the exact sequence

$$M_k \otimes_D F \xrightarrow{M_k \otimes_D \alpha} M_k \longrightarrow M_k \otimes_D \mathrm{Coker} \to 0.$$

Note that Coker is the simple D-D-bimodule and that $M_k \otimes_D \mathrm{Coker} \neq 0$. Therefore $M_k \otimes_D \alpha$ is not right split. This implies that α is surjective as a map of D-D-bimodules.

Now we show that if F has an indecomposable direct summand $G \in \mathcal{J}_l$, $k < l \leq m$, such that the restriction of α to G does not factor through the simple D-D-bimodule, then φ_k factors through α. Indeed, by Lemma 9 the only such possibility is $G \simeq M_l$, and by Corollary 10 the restriction of α to this summand is a scalar multiple of φ_l. As noted in Section 7.1.1, φ_k factors via φ_l for $k \leq l$, so this provides a factorization of φ_k through α.

As the next step, we show that if the condition of the previous paragraph is not satisfied, then F contains a summand isomorphic to either D or M_k such that the restriction of α to this summand does not factor through the simple D-D-bimodule. Indeed, assume that this is not the case. Then, by Lemma 9, the only possible indecomposable summands G of F for which the restriction of α does not factor through the simple D-D-bimodule come from two-sided cells $\mathcal{J}$ such that $\mathcal{J} >_J \mathcal{J}_k$. However, for such G, the composition M_kG cannot have any summands in $\mathcal{J}_k$ since $\mathcal{J} >_J \mathcal{J}_k$. Since M_k is indecomposable, it follows that any morphism $M_kG \to M_k$ is a radical morphism. That $M_kG \to M_k$ is a radical morphism, for any summand G isomorphic to D or M_k, follows from our assumption by the arguments in the previous paragraph. Therefore $M_k \otimes_D \alpha$ is a radical morphism and hence not right split, as M_k is indecomposable, a contradiction.

Because of the previous paragraph, there is a direct summand G of F isomorphic to either M_k or D such that the restriction of α to G does not factor through the simple D-D-bimodule. If $G \cong D$, then the restriction of α to it is an isomorphism. We can pull back φ_k via this isomorphism and define the map from M_k to all other summands of F as zero. This provides the necessary factorization of φ_k via α.

If $G \cong M_k$, we can pull back φ_k using first Corollary 10 and then correction via morphisms from M_k to the socle of G (such morphisms factor through the simple D-D-bimodule). In any case, the constructed factorization implies that M_k is great for $\mathcal{L}$ and completes the proof of our proposition. □

7.5. **Co-Duflo 1-morphisms in $\mathscr{D}$.** We can dualize Definition 11. Given a 2-category $\mathscr{C}$ and a left cell $\mathcal{L}$ in $\mathscr{C}$ with $\mathtt{i} = \mathtt{i}_{\mathcal{L}}$, we say that a 1-morphism H in $\mathscr{C}$ is *co-good* for $\mathcal{L}$ if there is a 2-morphism $\psi : \mathbb{1}_{\mathtt{i}} \to H$ such that $F\psi$ is left split, for all $F \in \mathcal{L}$. Moreover, we say that H is *co-great* for $\mathcal{L}$ if H is co-good for $\mathcal{L}$ and, for any H' which is co-good for $\mathcal{L}$ with $\psi' : \mathbb{1}_{\mathtt{i}} \to H'$, there is a 2-morphism $\gamma : H' \to H$ such that $\psi = \gamma \circ \psi'$.

The following proposition suggests that W_k is a very good candidate for being called a *co-Duflo* 1*-morphism* in its left cell in $\mathscr{D}^{(k)}$.

Proposition 15. *For any $m \geq k \geq 1$, the* 1*-morphism W_k of the finitary* 2*-category $\mathscr{D}^{(m)}$ is co-great for the left cell $\mathcal{L} = \{W_k, S_k\}$.*

Proof. Consider the 2-morphism $\psi_k : \mathbb{1}_{\mathtt{i}} \to W_k$. By a direct calculation, it is easy to check that $W_k \otimes W_k$ has a unique direct summand isomorphic to W_k and that $S_k \otimes W_k$ has a unique direct summand isomorphic to S_k. The projections onto these summands provide left inverses for $W_k\psi_k$ and $S_k\psi_k$, respectively. This implies that W_k is co-good for $\mathcal{L}$ via ψ_k.

Assume now that F is co-good for $\mathcal{L}$ via some $\alpha : \mathbb{1}_{\mathtt{i}} \to F$. We need to construct a factorization $F \to W_k$. Since multiplication with x is a nilpotent endomorphism of D, the endomorphism $W_k \otimes_A x$ is a nilpotent endomorphism of W_k. In particular, this endomorphism is a radical map. By a direct computation, one can check that, for any $\beta : D \to W_k$ which factors through the simple D-D-bimodule, the endomorphism $W_k \otimes_A \beta$ is not injective, in particular, it is a radical map.

Now, using arguments similar to the ones in the proof of Proposition 14, one shows that there must exist a summand G of F, the restriction of α to which does not factor through the simple D-D-bimodule and that this summand must be isomorphic to either D or W_l for some $l \geq k$. In the former case, the restriction of α to G is an isomorphism and the necessary factorization $F \to W_k$ is constructed via $G \to W_k$ using this isomorphism. In the latter case, the necessary factorization is constructed via $G \to W_k$ using Corollary 10 and the observation that φ_l factors via φ_k for $k < l$, and then correction via morphisms from G to W_k which factor through the simple D-D-bimodule. □

8. Some algebra and coalgebra 1-morphisms in $\mathscr{D}$

8.1. **Algebra and coalgebra 1-morphisms.** Let $\mathscr{C}$ be a 2-category. Recall that an *algebra structure* on a 1-morphism $A \in \mathscr{C}(\mathtt{i},\mathtt{i})$ is a pair (μ,η) of morphisms $\mu : AA \to A$ and $\eta : \mathbb{1}_{\mathtt{i}} \to A$ which satisfy the usual associativity and unitality axioms

$$\mu \circ_v (\mu \circ_h \mathrm{id}) = \mu \circ_v (\mathrm{id} \circ_h \mu), \quad \mathrm{id} = \mu \circ_v (\mathrm{id} \circ_h \eta), \quad \mathrm{id} = \mu \circ_v (\eta \circ_h \mathrm{id}).$$

Similarly, a *coalgebra structure* on a 1-morphism $C \in \mathscr{C}(\mathtt{i},\mathtt{i})$ is a pair (δ,ε) of morphisms $\delta : C \to CC$ and $\varepsilon : C \to \mathbb{1}_{\mathtt{i}}$ which satisfy the usual coassociativity and counitality axioms

$$(\delta \circ_h \mathrm{id}) \circ_v \delta = (\mathrm{id} \circ_h \delta) \circ_v \delta, \quad \mathrm{id} = (\mathrm{id} \circ_h \varepsilon) \circ_v \delta, \quad \mathrm{id} = (\varepsilon \circ_h \mathrm{id}) \circ_v \delta.$$

In the case of fiat 2-categories, it is observed in [MMMT, Section 6] that a Duflo 1-morphism often has the structure of a coalgebra 1-morphism (as suggested by the existence of a map from the identity to a Duflo 1-morphism) This is particularly interesting as it is shown in [MMMT] that any simple transitive 2-representation of a fiat 2-category can be constructed using categories of certain comodules over coalgebra 1-morphisms.

Let (A,μ,η) be an algebra 1-morphism in $\mathscr{C}$. A *right module* over A is a pair (M,ρ), where M is a 1-morphism in $\mathscr{C}$ and $\rho : MA \to M$ is such that the usual associativity and unitality axioms are satisfied:

$$\rho \circ_v (\rho \circ_h \mathrm{id}) = \rho \circ_v (\mathrm{id} \circ_h \mu), \quad \mathrm{id} = \rho \circ_v (\mathrm{id} \circ_h \eta).$$

Dually, one defines the notion of a comodule over a coalgebra. Morphisms between (co)modules are defined in the obvious way. We denote by $\mathrm{mod}_{\mathscr{C}}(A)$ the category of all right A-modules in $\mathscr{C}$, and by $\mathrm{comod}_{\mathscr{C}}(C)$ the category of all right C-comodules in $\mathscr{C}$.

8.2. **Coalgebra structure on Duflo 1-morphisms.** Given the results from the previous section, it is natural to ask whether M_k is a coalgebra 1-morphism in $\mathscr{D}$.

Proposition 16. *For a positive integer k, the 1-morphism M_k has the structure of a coalgebra 1-morphism in $\mathscr{D}$. Moreover, the 1-morphism N_k has the structure of a right M_k-module.*

Proof. Recall the standard basis of the bimodule M_k from Section 7.1. The tensor product $M_k \otimes M_k$ has a unique direct summand isomorphic to M_k with a basis given by

$$\begin{tikzcd}[column sep=small]
m_2 \otimes m_1 & m_2 \otimes m_2 \arrow[l] \arrow[d] & & & \\
& m_3 \otimes m_2 & \ddots \arrow[l] \arrow[d] & & \\
& & m_{2k+1} \otimes m_{2k} & m_{2k+2} \otimes m_{2k+2} \arrow[l] \arrow[d] \\
& & & m_{2k+3} \otimes m_{2k+2}.
\end{tikzcd}$$

Moreover, we have $m_{2j+1} \otimes m_{2j} = m_{2j+2} \otimes m_{2j+1}$, for $j = 1, \dots, k$. We define the comultiplication $\delta : M_k \to M_k \otimes M_k$ explicitly as follows:

$$\begin{cases} \delta(m_{2j}) &= & m_{2j} \otimes m_{2j}, & 1 \le j \le k+1 \\ \delta(m_{2j+1}) &= & m_{2j+1} \otimes m_{2j} = m_{2j+2} \otimes m_{2j+1}, & 1 \le j \le k+1 \\ \delta(m_1) &= & m_2 \otimes m_1 & \\ \delta(m_{2k+3}) &= & m_{2k+3} \otimes m_{2k+2} & \end{cases}$$

As a counit, we take the morphism φ_k from Section 7.1. The counitality and comultiplication axioms are now checked by a lengthy but straightforward computation.

To prove that N_k is a right M_k-comodule, we recall that $N_k \simeq M_k/\mathrm{span}\{m_{2k+3}\}$. Let $\pi : M_k \to N_k$ be the canonical projection. Then $\rho = \pi \circ_h \mu$ makes N_k a right M_k-comodule.

Indeed, all necessary properties for ρ follow directly from the corresponding properties for μ. $\square$

Corollary 17. *The 2-representation $\mathscr{C}M_k \subset \mathrm{comod}_{\mathscr{C}}(C)$ of $\mathscr{C}$ has a unique simple transitive quotient, moreover, this quotient is equivalent to the cell 2-representation $\mathbf{C}_{\mathcal{L}}$, where $\mathcal{L} = \{M_k, N_k\}$.*

Proof. As M_k is indecomposable, the unique simple transitive quotient $\mathbf{M}$ of $\mathscr{C}M_k$ is the quotient of $\mathscr{C}M_k$ by the sum of all $\mathscr{C}$-stable ideals in $\mathscr{C}M_k$ which do not contain id_{M_k}. Clearly, M_k does not annihilate M_k. At the same time, for any $F >_J M_k$, we have that $\mathscr{C}FM_k$ does not contain id_{M_k}. Therefore any such F is killed by $\mathbf{M}$. This means that $\mathbf{M}$ has apex $\mathcal{J}_k$.

Further, N_kM_k does not have any copy of M_k as a direct summand. Therefore the rank of $\mathbf{M}$ is at least 2. Now the claim of our corollary follows from Theorem 1(iii). $\square$

8.3. **Algebra structure on co-Duflo algebra 1-morphisms.** Similarly to the previous section, it is natural to ask whether W_k is an algebra 1-morphism in $\mathscr{D}$.

Proposition 18. *For a positive integer k, the 1-morphism W_k has the structure of an algebra 1-morphism in $\mathscr{D}$. Moreover, the 1-morphism S_k has the structure of a right W_k-module.*

Proof. The tensor product $W_k \otimes W_k$ has a unique direct summand isomorphic to W_k, namely, the direct summand with the basis

$$\begin{tikzcd}
w_1 \otimes w_1 \arrow[d] & & & \\
w_2 \otimes w_1 & w_3 \otimes w_3 \arrow[l] \arrow[d] & & \\
& \ddots & w_{2k-1} \otimes w_{2k-1} \arrow[l] \arrow[d] & \\
& & w_{2k} \otimes w_{2k-1} & w_{2k+1} \otimes w_{2k+1} \arrow[l]
\end{tikzcd}$$

moreover, $w_{2j} \otimes w_{2j-1} = w_{2j+1} \otimes w_{2j}$, for $1 \leq j \leq k$. This allows us to define multiplication μ as the projection onto this direct summand. As the unit morphism, we take ψ_k from Section 7.1. All necessary axioms are checked by a straightforward computation.

The projection onto the unique summand of $S_k \otimes W_k$ isomorphic to S_k provides S_k with the structure of a right W_k-module. Note that letting $\theta : M_k \to S_k$ and $\zeta : M_k \to W_k$ be the canonical projections (see Section 7.1.1), and $\pi_{M_k} : M_k \otimes M_k \to M_k$ the projection as in the proof of Proposition 14, the projection $S_k \otimes W_k \to W_k$ makes the following diagram commute.

$$\begin{tikzcd}
M_k \otimes M_k \arrow[r, "\pi_{M_k}"] \arrow[d, "\theta\otimes\zeta"'] & M_k \arrow[d, "\theta"] \\
S_k \otimes W_k \arrow[r] & S_k
\end{tikzcd}$$

Verifying that this gives S_k the structure of a right W_k-module is done by straightforward computation. □

Corollary 19. *The* 2*-representation* $\mathscr{C}W_k \subset \mathrm{comod}_{\mathscr{C}}(C)$ *of* $\mathscr{C}$ *has a unique simple transitive quotient, moreover, this quotient is equivalent to the cell* 2*-representation* $\mathbf{C}_{\mathcal{L}}$*, where* $\mathcal{L} = \{W_k, S_k\}$*.*

Proof. Mutatis mutandis Corollary 17. □

8.4. **Rank 1 representations are non-constructible.** In this last subsection we would like to emphasize one major difference between the 2-representation theory of $\mathscr{D}$ and that of fiat 2-categories.

Definition 20. Let $\mathscr{C}$ be a (finitary) 2-category and let $\mathscr{B} \in \{\mathscr{C}, \underline{\mathscr{C}}, \overline{\mathscr{C}}\}$. A 2-representation $\mathbf{M}$ of $\mathscr{C}$ is called *$\mathscr{B}$-constructible* if there is a (co)algebra 1-morphism C in $\mathscr{B}$, a $\mathscr{C}$-stable subcategory $\mathcal{X}$ of the category of right C-(co)modules, and a $\mathscr{C}$-stable ideal $\mathcal{I}$ in $\mathcal{X}$ such that $\mathbf{M}$ is equivalent to $\mathcal{X}/\mathcal{I}$.

If $\mathscr{C}$ is fiat, then any simple transitive 2-representation of $\mathscr{C}$ is both $\underline{\mathscr{C}}$- and $\overline{\mathscr{C}}$-constructible by [MMMT]. From [MMMTZ, Section 3] it follows that faithful simple transitive 2-representation of $\mathcal{J}$-simple fiat 2-categories are even $\mathscr{C}$-constructible.

Corollary 17 implies that, for each $k \geq 1$, the cell 2-representation $\mathbf{C}_{\mathcal{L}}$ of $\mathscr{D}^{(k)}$, where $\mathcal{L} = \{M_k, N_k\}$, is $\mathscr{D}^{(k)}$-constructible.

The following statement, in some sense, explains why the statement of Theorem 1(iv) is as it is.

Theorem 21. *Let k and m be positive integers such that $2 \leq k \leq m$. Let $\mathbf{M}$ be a rank 1 simple transitive 2-representation of $\mathscr{D}^{(m)}$ with apex $\mathcal{J}_k$. Then $\mathbf{M}$ is not $\mathscr{D}^{(m)}$-constructible.*

Proof. Assume towards contradiction that $\mathbf{M}$ is $\mathscr{D}$-constructible. Let $\mathcal{X}$ be as in Definition 20 and consider some object $X \in \mathcal{X}$ which is nonzero in the quotient by $\mathcal{I}$. Then, for each $U \in \mathcal{J}_k$, we must have $UX \simeq X + \mathcal{I}$.

If $X \in \mathrm{add}\{\mathcal{J} \,|\, \mathcal{J} >_J \mathcal{J}_k\}$, then the action of $\mathcal{J}_{k-1}$ is nonzero on X, implying that $\mathcal{J}_k$ is not the apex of the representation (note that $k > 1$). This means that all indecomposable summands of X which matter for the computations in $\mathcal{X}/\mathcal{I}$ are in $\mathcal{J}_k$.

From (2), we obtain that, modulo higher two-sided cells, $N_k X_m \in \mathrm{add}\{N_k \oplus W_k\}$ while $M_k X_m \in \mathrm{add}\{M_k \oplus S_k\}$. Since $\mathcal{J}_k$ is the apex of $\mathbf{M}$, both $N_k X_m$ and $M_k X_m$ are non-zero. This contradicts the assumption that $\mathbf{M}$ has rank 1. □

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