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Exploring strategies in Monopoly using Markov chains and simulation

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1 Introduction

Monopoly is a classic that almost everyone has played or at least heard of. It is a game which bring out joy and disputes in as well as family as friends. There seems to be a universal idea that the more expensive the better and that buying much will generate great result and it is easy to see why. If you make a big investment it is intuitive to expect to be rewarded. However is this in fact the case? In this project Monopoly was examined and an attempt to find the best possible strategy was made. It was investigated if all streets are equally probable, how the expected return change when buying houses, which colour will give you the most bang for the buck and which are the best colours to obtain.

The first part of this paper the focus lie in finding the probabilities of landing at each tile. That is, one player will be imagined walking around the board without making any decision or transactions. The second part will be more of an experimental part as to the first being more theoretical. This part includes simulating players buying different colours and see which works the best. One could say that this kind of test the conclusions of the first part.

1.1 Introduction to the game

Monopoly is a game about buying streets, collecting money, ruining your opponents and is won by being the last to have money left. The board game consist of forty tiles and each player traverse these by rolling two dice and move accordingly. If a player rolls a double, namely rolls two of the same number, she plays her turn as usual but after making decisions and paying eventual fees she plays another turn. If a player rolls three doubles in a row the player goes to jail. While in jail you have three options to get out, either you use a get-out-of jail card, pay 1000 kr or roll a double. Only one option can be chosen each turn meaning that you cannot try rolling the dice and then choose another option in the same turn. A player can spend a maximum of 3 turns in jail before being released. If a player has been in jail 3 rounds she is transferred to visit-jail and can on next turn play as usual.

- In preparation to start one player is chosen as a banker whose purpose is the distribution of assets between the bank and the players. Each player start at the tile GO with 30 000 kr.

- On a players turn the player must roll her dice. If the player wishes to trade, improve buildings or alike it must be done before she roll.
- If a player lands on a property not yet owned, the player is asked to buy the property. If she accept the player pays the bank and obtains the property. Otherwise the property goes to auction where the highest bidding player obtains the property and pays the given amount to the bank.
- If a player lands on a street that her self own nothing happens. However if landing on another players street the player must pay rent **if** the owner notice. If however one lands on someones utility the player must pay 80 times what they rolled. If all utilities are owned by same owner it is instead 200 times.
- If a player lands on a community chest or chance tile the player must draw a card, do what the cards says and then put the card at the bottom of the pile.
- If a player lands on the tile luxury tax they must pay 2000 kr if instead landing on income tax the player must pay 4000 kr.
- Landing on the go-to-jail tile means that the player transfers directly to jail without passing GO. However landing on the jail tile means just visiting and no penalties apply.
- If passing GO the player is rewarded with 4000 kr from the bank.
- If a player owns every street in one colour they may buy up to 4 houses or 1 hotel (hotels can only be bought after owning 4 houses on that street) for each property belonging to that colour. It does not have to be the players turn nor does the player have to be on the property that she choose to place a house/hotel on. However the house/hotels must be distributed evenly in a colour meaning that a house can only be placed on the street with fewest amount of house on. A house can be sold back to the bank for half of its initial cost.
- A property may be mortgaged for half of its price and cannot be developed nor can it collect rent while in this state. To end the mortgaged

the player must pay the mortgage price with 10% interest. If a mortgaged property is traded between players the new player must pay 10 % interest directly and then the same rules applies as above.

1.2 Delimitations

To make Monopoly more comprehensible to model some limitations of the game rules were made:

- Mortgaging properties or selling houses/properties is not allowed.
- Only 2-player games will be considered.
- Get-out-of jail cards and paying to get out will not be considered.
- Utilities will charge 7 times 80/200 instead of the sum of the dice times 80/200
- Trade will not be considered.
- No auction of the property if a player decide not to buy.
- The cards are put back randomly.

The limitations is purely motivated by that they reduced the complexity of the game and made it easier to simulate. For utilities instead of charging the dice roll times 80/200 the sum of the dice roll were replaced with 7 since it is the mean of the sum of 2 dice. Trading was not considered since in the second part of the paper where it was relevant, only 2-player games were considered and then trade does not make sense. If a player want to make an efficient trade then in some sense, she wants to reduce the probability of getting ruined. But intuitively in doing so the player have to increase another players probability of losing. Then of course if there are only two players, someone will make a bad trade and should not trade. If a player does not buy a property it will not be auctioned out. The reason for this is that it simply does not make sense to hold an auction for 1 person. Lastly the reason for putting the cards back randomly instead of at the bottom of the pile is to simplify the distribution of the cards and also to obtain time-homogeneity of the Markov chain.

1.3 Purpose and problem formulation

The main purpose of this essay is to try to investigate if there exist a way to play Monopoly good or if it is simply a game of chance. If it is more than a game of luck then how should a player behave? This leads to the main question of this essay: **Does an optimal strategy of Monopoly exist?** However since this is a difficult and rather broad question it will be divided in to sub-questions. It should be noted that these questions does not necessary capture the larger question as a whole but is important for drawing conclusion towards the goal.

- Do the probabilities of the tiles differ and which colour is most probable to land on?
- What is the optimal amount of houses to own for each street?
- Which colour generates most money?
- Which are the best colours to obtain?

2 Theory

The main purpose of this chapter is to build up basic theory about Markov chains and stating requirements for existence of stationary distribution for discrete Markov chains. The reason for this is that as will be seen later it is possible to model Monopoly using a Markov chain with discrete time and discrete state space. By finding the stationary distribution it is possible to find the probability of landing at each tile independent of where you currently stand.

2.1 Markov Chains

In order to define a Markov chain the concept of stochastic processes must first be introduced.

Definition 2.1. Given a probability space $(\Omega, \mathcal{F}, P)$, a stochastic process is a function $X : [0, \infty] \times \Omega \to \mathbb{R}$. We denote the function as $X_t = X(t, \omega)$ [1].

Definition 2.2. A Markov Chain is a stochastic process satisfying

$$P(X_n = j|X_{n-1} = i_{n-1}, ..., X_0 = i_0) = P(X_n = j|X_{n-1} = i_{n-1})$$

This is called the Markov property [2].

This paper will only consider Markov chains with discrete state space and discrete time. Furthermore the focus will lie solely in time-homogeneous Markov chains. This means that the probability of going from state i to j does not depend on n, that is

$$P(X_n = j|X_{n-1} = i_{n-1}) = P(X_1 = j|X_0 = i_{n-1})$$

.

Note that

$$p_{ij} = P(X_n = j|X_{n-1} = i)$$

is called the transition probability from state i to j. Now suppose $\{X_t\}_{t\in T}$ is a Markov chain. Then matrix P where the entry (i,j) equals p_{ij} such that $i, j \in S$ is called the transition matrix of the Markov chain. This matrix has the property that

$$\sum_{j\in S} p_{ij} = 1 \ \forall i \in S$$

and $p_{ij} \geq 0 \ \forall i, j \in S$

making P a stochastic matrix [2].

2.1.1 Stationary distribution for discrete Markov chains

The stationary distribution can be intuitively thought of as the distribution in the long run that is, the probabilities of reaching the different states independent of where you start. More formally, a stationary distribution π is the distribution satisfying the equation

$$\pi = \pi P$$

[2]. To reach the theorem that tells us when a stationary distribution exist it is necessary to introduce some properties and classifications of Markov chains.

Definition 2.3. A Markov chain is said to be irreducible if $p_{ij} > 0$ for all i, j in the chain.

Definition 2.4. Let T_n be the first time of return to state n. Then the state n is called **recurrent** if

$$P(T_n < \infty) = 1$$

Moreover it is called **positive recurrent** if

$$E(T_n) < \infty$$

[2] This means that a recurrent state will with probability 1 be revisited after a finite number of step. Now it is possible to state the following useful theorem:

Theorem 2.1. Suppose $\{X_n\}$ is an irreducible Markov chain then

1. If one state is positive recurrent then all states are positive recurrent.
2. All states are positive recurrent $\iff$ A stationary distribution exist.

Apart from this we also have the following useful theorem.

Theorem 2.2. Suppose $\{X_n\}$ is a Markov chain defined on a finite state space S. Then at least one j $\in S$ is recurrent and every recurrent state is positive recurrent. [3]

3 Modelling monopoly as a Markov chain

3.1 State space and description of model

To get a better understanding of the game a reasonable first step of analysis was to solely look at the board and finding the probabilities of landing on each tile. Therefore the first model of the game excluded money as well as any decisions and instead focused on capturing how one player move around on the board. To motivate why this might be interesting, a player wins the game by ruin its opponents. This is essentially done by establishing streets where the player can charge the opponents for rent. But in order to claim rent a player must stand on your tile and the players will of course tend to stand more often on the most probable tiles.

To get a grip of how the probabilities of the tiles arise it is key to realize how a player can move on the board. There are two primary ways, rolling the two dice and drawing cards. An additional feature of the dice is rolling doubles. Now in order to model this as a Markov chain its tempting to just say that there are 40 states in the state space, one for each tile. But by giving it a quick ponder one realize fast that it is not sufficient. The reason for this is that this fails to incorporate how many doubles a player has thrown and thus fails to satisfy the Markov property. For example, a player who has thrown two doubles has in its state remarkably higher probability to go to jail than a player that has not yet rolled a double standing in the same position. It is similar when a player is in jail. Then instead a player who has stood there one turn can only get out by rolling a double while a player who has stood there for three turns will get out with a probability of 1 and is in the same state as a player just visiting jail. With that said to get a correct model the state space must keep track of how many doubles the player has thrown for each non-jail tile and how many doubles the player has not thrown if in jail. This gives that the state space will have $3 \cdot 39 + 3 = 120$ states. Note that a player has 0 probability of ending a turn on the go-to-jail tile since the player will be directly sent to jail. However that position (30) will be used to keep track of people in jail and the jail tile will only keep track of visitings. Each state will be represented as $(doubles, position, timeJail)$. The variable $position$ will be between 0 and 39 and $doubles$ and $timeJail$ will be between 0 and 3. $Doubles$ and $timeJail$ can not be non-zero at the same time since if a player is in jail and throw a double the player will be released and timeJail will be set to 0. Now it is possible to presented all the states as following:

doubles	position	timeJail
0	k	0
1	k	0
2	k	0
0	30	0
0	30	1
0	30	2

Table 1: All possible states. $k \in \{0, ...39\} \setminus 30$

3.2 Transition probabilities

Now that it is clear which states the model has it remains to find the transition probabilities between these states. It should be noted that there are $120 \cdot 120 = 14400$ possible transitions so Matlab was used to create the matrix and find the stationary distribution. Since movement depends on the two dice and the cards, the probability distribution of these must be found.

3.2.1 Dice

Each time it is a players turn she throws two dice. The sum of these are the amount of steps the player walk. The sum can vary between 2 and 12 and the probability can easily be calculated. Although since a players transitions depends on if she rolls a double or not its important to separate them. Throwing a specific double is always done with a probability of 1/36 since there are exactly one unique pair where the eyes of the dice are the same for every number on a die. Furthermore the sum of a double can only be even. The distribution looks as following:

Sum	Probability (non-double)	Probability (double)
2	0	1/36
3	1/18	0
4	1/18	1/36
5	1/9	0
6	1/9	1/36
7	1/6	0
8	1/9	1/36
9	1/9	0
10	1/18	1/36
11	1/18	0
12	0	1/36

Table 2: Probability distribution of the sum of two dice.

From each non-jail state there are three transitions that can happen; A player does not roll a double and transition to the new position with 0 doubles, a player rolls a double and goes to the new position with one more double or a player rolls its third double in a row and goes to jail. If instead in jail a player can either roll a double and get out, stay another turn or get out because she has been there for 3 rounds. Note that the matrix becomes 120×120 since there are 120 states.

Current state	New State	Transition prob
(d,pos,0)	(0,pos+3 mod 40,0)	1/18
(d,pos,0)	(0,pos+4 mod 40,0)	1/18
(d,pos,0)	(0,pos+5 mod 40,0)	1/9
(d,pos,0)	(0,pos+6 mod 40,0)	1/9
(d,pos,0)	(0,pos+7 mod 40,0)	1/6
(d,pos,0)	(0,pos+8 mod 40,0)	1/9
(d,pos,0)	(0,pos+9 mod 40,0)	1/9
(d,pos,0)	(0,pos+10 mod 40,0)	1/18
(d,pos,0)	(0,pos+11 mod 40,0)	1/18
(d,pos,0)	(d+1,pos+doubleValue mod 40,0)	1/36
(2,pos,0)	(0,30,0)	1/6
(0,30,t)	(0,10+doubleValue,0)	1/36
(0,30,t)	(0,30,t+1)	5/6
(0,30,3)	(0,10,0)	1

Table 3: Possible transitions by throwing two dice. Note that in this table $d \in \{0, 1, 2\}$, $t \in \{0, 1\}$ and $doubleValue \in \{2, 4, 6, 8, 10, 12\}$

3.2.2 Cards

There are two types of cards, community chest cards and chance cards. Most of the cards in both involve some sort of money transaction but since money is excluded from the model these will be transition back to the same state.

Current state	New State	Transition prob
(d,chancePos,0)	(d,chancePos,0)	1/2
(d,chancePos,0)	(d,11,0)	1/16
(d,chancePos,0)	(d,0,0)	1/16
(d,chancePos,0)	(d,24,0)	1/16
(d,chancePos,0)	(d,1,0)	1/16
(d,chancePos,0)	(0,30,0)	1/16
(d,chancePos,0)	(d,15,0)	1/16
(d,chancePos,0)	(d,39,0)	1/16
(d,chancePos,0)	(d,chancePos-3 ,0)	1/16
(d,chestPos,0)	(d,chestPos,0)	14/16
(d,chestPos,0)	(d,0,0)	1/16
(d,chestPos,0)	(d,30,0)	1/16
(d,pos,j)	(d,pos,k)	1

Table 4: Transitions by drawing cards. $chancePos \in \{7, 22, 36\}$, $chestPos \in \{2, 17, 36\}$ and $pos \in \{1, ..., 39\} \setminus cardPos$, where $cardPos = \{2, 7, 17, 22, 33, 36\}$

Note that the last entry of the table is the probability of staying in the same position when not standing on a card tile. This is clearly 1 since you cannot draw a card there. Moreover the transition matrix here will be of size 120×120 with ones on the diagonal for the non-card tiles and for the card tiles it follows the distribution shown in table 4.

3.2.3 Transition matrix

In order to obtain the transition matrix for the whole model the transition matrix of the cards must be combined with the transition matrix of the two dice. To do this its important to realize that a player first throws the dice then draw a card and that what card a player draws is independent of what the sum of the dice was. This means that the probability of going from state i to j will be the transition probability of rolling the dice times the transition probability of the cards. Moreover if not standing on a card tile the transition probability will follow the dice distribution. Since in this essay probability vectors will be row vectors the transition matrix becomes $P = DC$ where D is the transition matrix for the dice and C for the cards. Note that we in fact can multiply the matrices together since both are of size 120×120.

3.3 Stationary distribution and result

The chain is irreducible. This can be seen by noting that given every position it is possible to walk around the board until every other position is reached. Since it is irreducible and finite we have by the first part of **Theorem 2.1** together **Theorem 2.2** that the chain only has positive recurrent states and therefore by part 2 of **Theorem 2.1** there exist a unique stationary distribution.

To find the stationary distribution the equation $\pi = \pi P$ has to be solved for a π with only positive components that sum to one. Since $\pi = \pi P$ is the equivalent to that π is the eigenvector of P corresponding to the eigenvalue 1 it suffices to find that eigenvector.

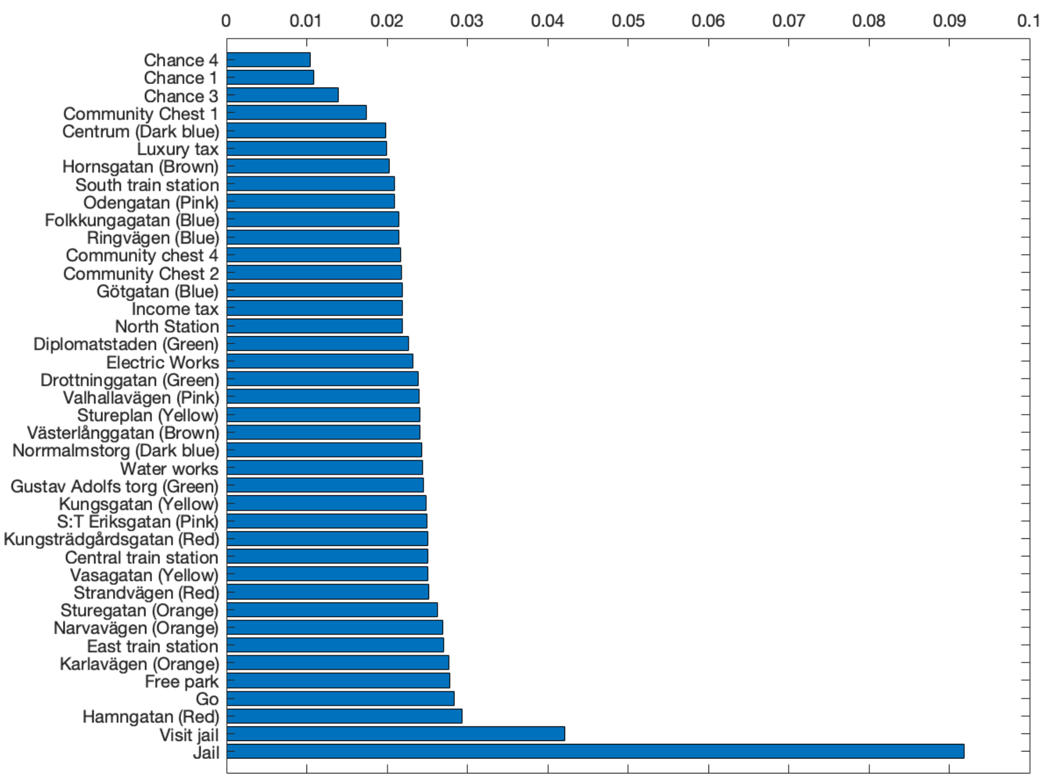

Figure 1: Probability of landing on each tile.

It should be noted that Chance 1 refers to the chance tile on the first row after GO and so on. Furthermore looking at the probabilities one can quickly say some things about the game. The first most obvious is the fact that all streets are not equally probable. This is key since this reinforces the suspicion that Monopoly is a game where it is possible to find ways to make good decisions. However it should be noted that the differences is small. Hamngatan and Centrum which have the highest and lowest probability of the buyable tiles only differs with 0.95 %. This means that for a game with 200 rounds you could expect about 2 more visits to Hamngatan than Centrum. Another fact that catch the eye is the high probability of the going to jail. This make sense since the game rules basically revolves around going to jail. If a player rolls 3 doubles in a row, draws a go-to-jail card or land on the go-to-jail tile she goes to jail. It is also notable that all of the orange streets have high probability and could partly be a consequence of being able to transition to them directly from jail. However since you cannot buy houses until you own all streets of the same colour its interesting to consider the probability of the colours as a whole.

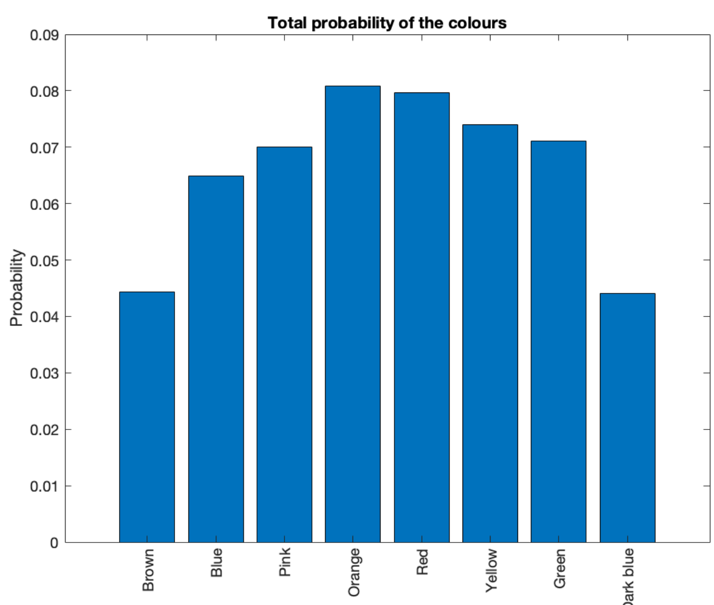

Figure 2: Total probability of landing on each colour

As seen in the picture orange and red are the most probable and brown and dark blue are the least probable. It should be noted that brown and dark blue are the only colours with 2 streets but they also contain some of the streets with lowest probabilities with Centrum (Dark blue) being the buyable-tile with the lowest probability. Now unfortunately it is not enough to just look at the probabilities. One also have to consider the potential return of investing in different colours. A player surely want the most bang for the buck. This leads to consideration of houses. How does the expected return change when a player invest in houses? Return here refers to the expected amount of money it will generate on a turn divided by cost. Immediately by

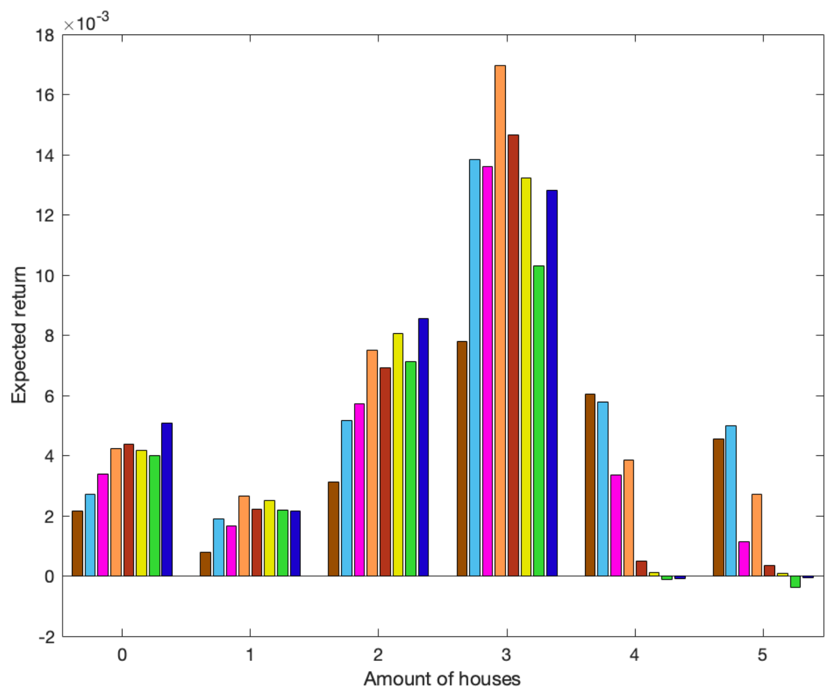

Figure 3: How the expected return change when developing houses. For 0 houses it is only the expected return.

looking at figure 3 one can notice some interesting things. Firstly there seems to be a peak in the change of expected return at 3 houses. This means that going from 2 to 3 houses has the largest increase of expected return. Orange has especially large expected increase with as much as 1.7 %. Otherwise it seems to be the fact that the expected return increase as you buy more houses. There are however two expectations, dark blue and green. When increasing from 3 and 4 houses per street a player actually get less for what she pays for. In addition to see how the return changes it is also interesting to consider how many opponent turns until profit is expected to be made.

colour	no house	1 house	2 houses	3 houses	4 houses	5 houses
brown	469	341	165	73	51	41
blue	370	218	103	43	35	30
pink	295	198	93	41	37	35
orange	237	146	70	32	29	27
red	215	147	73	35	35	34
yellow	240	150	68	36	36	36
green	251	162	76	43	43	44
dark blue	197	139	64	35	36	36

Table 5: Number of opponent turns to profit.

This corresponds as expected well with figure 3. For example for green the expected turns to profit increase when going from 4 houses to 5 which is reasonable since we saw in figure 3 that the expected return actually decreased. For the other colours the amount of turns either remained the same or decrease. The reason why dark blue does not have an increase in turns is due to that the decrease in expected return is so small. Green also has the most amount of turns to profit for 5 houses. Orange has the lowest expected rounds to profit for 3 houses or more. But apart from when a colour makes profit it is also of interest to considered how much profit each colour is expected to generate over time. Time in this case simply means how many turns the opponents have. The expected return will be restricted to 5 houses since it was the investment with greatest expected return for the majority of the colours.

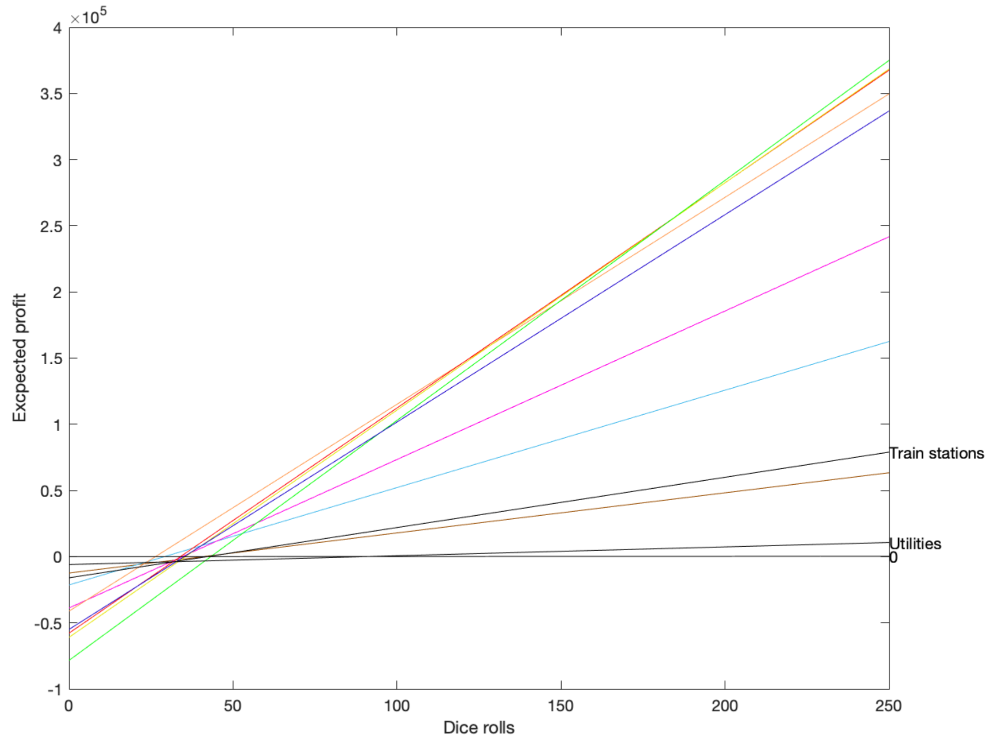

Figure 4: How the expected profit evolves over opponents turns.

From figure 4 we can see that orange is one of the streets that is expected to generate most profit with 5 houses. If comparing orange to dark blue it can be seen that orange is cheaper to invest in and have more expected profit and thus making orange strictly better than dark blue with 5 houses. Red and yellow seems to be about the same and green generates slightly better profit however it cost 17 400 kr more than the second most expensive (yellow) and it is a small difference between them in profit after 250 dice rolls. Brown, train stations and the utilities generate poor profit and might be bad investments. But we noticed previously that yellow and red almost had no increase in expected return after 3 houses and dark blue and green actually had an decrease, hence it is interesting to see what happens when we stop with 3 houses for them.

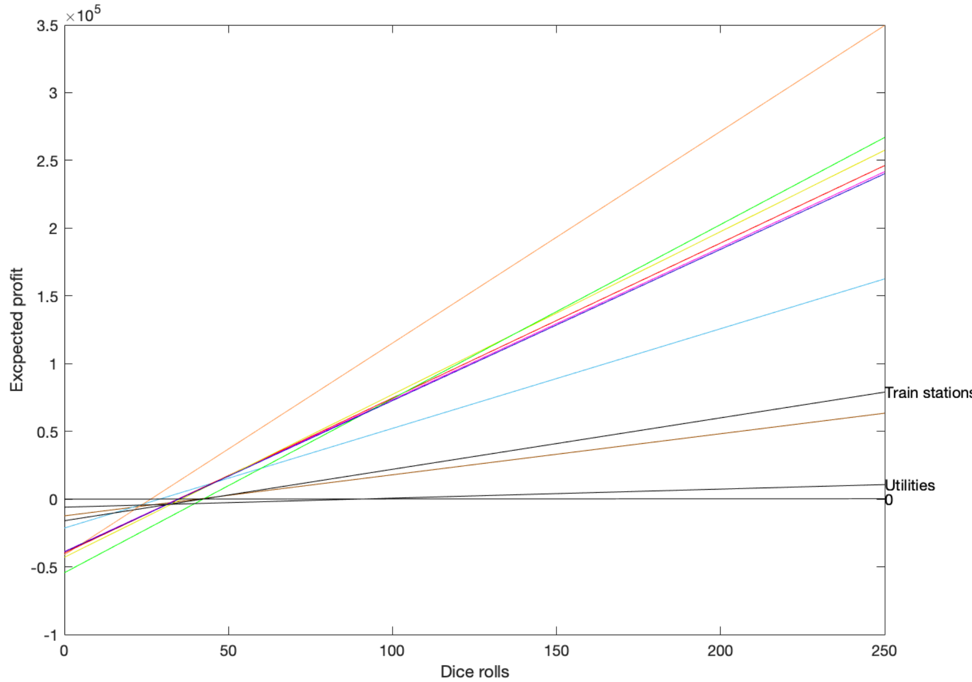

Figure 5: How the expected profit evolves over opponents turns were yellow,green, dark blue and red have 3 houses and the rest have 5.

Here we have that dark blue, red, pink and yellow are very similar in both price and how much money they are expected to generate over time. From figure 5 we see that red, dark blue and yellow are probably to prefer over pink since they perform as well with 3 houses as pink does with 5, making them more flexible and better in the long run since they can be developed further. Orange is very interesting here. It clearly has the best performance and yet cost about as much pink, dark blue, yellow and red and is strictly better than green with lower cost and greater expected profit.

4 Extended model of Monopoly

Even though it was possible to obtain interesting result from previous chapter the model does not involve an important feature of the game, the money. Furthermore it does not include picking up the streets and develop them as the turns go. In this chapter the goal will be to try to incorporate money and test previous results. However when including money the states space become infinite since the game rules does not have an upper limit on how much money there can be in a game. Even if the money was limited to be finite there is still all possible money for all players in the game times 120 times all possible owned streets and houses. This is the motivation to why this part will be more of a simulation.

4.1 The structure of the simulation

The first to consider was how a player would be represented. The way this was done was to represent it as a 3-tuple as before but extend it with money and way to keep track of what properties and houses the player has. The representation looked as follows: $(doubles, pos, timeJail, cash, streets)$ where $streets$ keeps track of which streets the player have with what amount of houses. A game of two players could now be represented as $(player1, player2, turn)$ where $turn$ was simply which players turn it was. So for example if $player1$ starts then its $(player1, player2, player1)$. Then after $player1$ plays her turn it became $(player1, player2, player2)$ and so on. From this it was possible to simulate 2-player games where different players took different decisions. It should be noted that in the simulations the cards will not be put back randomly anymore put instead in bottom of the pile as the rules state.

4.2 Simulations

The first simulation was to compare the colours against each other and see how they performed. It was of interest to look how the colours fared against each other with 3 houses since they had the most increase in return. The simulation was done as follows. Every player started at GO with 30 000 kr and moved according to the rules. A player did only develop its chosen colour and did only develop to max 3 houses on each street she owned. Houses could only be bought if 3 times the cost of it was less than the money the player owned. The reason for this was that so the player would not buy her self ruined/close to ruined. It should however be noted that the 3 is arbitrary. Every colour was simulated against every other 100 000 games.

colour	win	tie	loss	win prob
brown	782	239324	459894	0.001
blue	9582	347619	342799	0.014
pink	100290	366928	232782	0.143
orange	250880	287291	161829	0.358
red	330622	193946	175432	0.472
yellow	359704	156950	183346	0.514
green	446849	59978	193173	0.638
dark blue	408447	133652	157901	0.583

Table 6: Colours with 3 houses simulated against each other.

Some key features to be taken away from the simulation is that firstly, brown, blue and pink performed poorly especially brown. This results agree with the previous chapter. Moreover there seems to be a trend that the more expensive to better. However looking at orange losses it is actually the second lowest after dark blue. This indicates that orange might be a safe bet.

In similar manners a simulation was done for every pair of colours. Every pair met 15 different setups and each setup met each other 100 000 times.

colour	win rate
dark blue and orange	67 %
orange and red	63 %
red and dark blue	62 %
orange and yellow	61 %
pink and dark blue	60 %
yellow and dark blue/orange and green/ pink and orange	59 %
green and dark blue	56 %
red and yellow/pink and red	55 %

Table 7: The top 11 performers.

In table 7 we can see that orange is in the best performing pair together with dark blue and that orange is in 5 of 11 of the top performing pairs. Moreover there seems to be a trend that middle priced colours together with more expensive colours work well. Lastly blue and brown does not appear in the top 11.

5 Summary and discussion

To recap, let us recall the problem formulation of this paper. The main question that was asked was: **Does an optimal strategy of Monopoly exist?** However since this was a rather difficult question to answer a heuristic were made containing the following sub-questions :

- Do the probabilities of the tiles differ and which colour is most probable to land on?
- What is the optimal amount of houses to own for each street?
- Which colour generate most money?
- Which is the best colours to obtain?

There were differences in the probabilities of the tiles and the most probable colour to land on was orange. For every colour except dark blue and green the expected return of the houses increased for every bought house up to 5 houses but the greatest increase in expected return was when going from 2 to 3 houses for every colour. When it comes to which colour generates the most money it depends. Orange is the colour that is the quickest to make profit but eventually green will generate most. However when having 5 houses orange cost a lot less and will generate almost as much money as green. There is a even smaller difference between green, yellow and red but both yellow and red are cheaper, but not as cheap as orange. Moreover orange seems to be the best colour. It is relative cheap, is the quickest to gain profit and has one of the best the expected profit. From the simulations it had the second least losses against all other colours and when looking at pairs it was in 5 out of the 11 best performing pairs including the best performing pair, dark blue and orange. Otherwise yellow, dark blue and red seems good to buy. Green is probably too expensive and have to poor expected return to prioritize. Moreover blue seems to be reasonable to buy early since its cheap and have decent return.

It is still hard to give an answer if it exist an optimal strategy but some conclusions can be made. There are interesting differences in the colours that a player probably can use to gain advantage. For example based on the simulation of pairs of colours a middle priced colour like orange or pink seems to perform well combined with a more expensive street like dark blue. If looking at the gathered evidence one should probably skip brown, train stations and utilities or have them as last resort. Since red and yellow barley increased in expected return and green and dark blue decreased in expected return when increasing houses from 3 a player should probably stop investing in them when she reach 3 houses and instead focus on developing other colours further. Since these had very similar traits with pink with 5 houses they are probably better to buy than pink since they can be develop further making them more dynamic. However to be sure one would have to do a more in-depth analysis and the question still remain when to buy and in which order. More simulations should perhaps have been done that allowed 5 houses and also test letting red, yellow, green and dark blue remain with 3 houses until the other colour of the pair were fully developed. But then the question arise of how long do you want to wait and if you have money and are not in risk of becoming ruined then its probably best to invest the money anyway. Also it should be mentioned that just buying 1 or 2 colours are not realistic in a 2-player game but the simulations nevertheless gave some intuition of which colours seems to be good.

6 Further exploration

Although some results were made there is still features to explore towards finding good strategies. I think that a first thing to do would be to test more of the results and trying out decision-policies based on what was discovered in chapter 3. For example testing a strategy where if the player buys green, yellow, red or dark blue it only develop to 3 houses until finished developed other streets. Then test it without brown, utilities and train station and maybe even exclude green due to it being very expensive. Then start to look into blocks. With blocks I mean that even if a player are not interested in a colour the player can block it from an opponent by buying one of the streets so the opponent cannot develop houses there. However I think these decision rules might be to simple and miss a lot of important features like how much money does your opponent has, where is your opponent currently on the board, what is her expected return, how does your decision change your probability of getting ruined and so on. In order to find more advance and better strategies I think Markov decision processes (MDP) are an interesting next step. Intuitively a Markov decision process is a Markov process were the transitions happens partly due to decisions, partly to chance. If possible to model the game as a MDP there exist ways to find the optimal decision-policy, namely the optimal strategy for the game. Another interesting extension is to increase the amount of players. Monopoly is not a game that is often played with only 2 players but instead with maybe 4 or 5. By increasing the amount of players one could from there make some further natural extensions by allowing mortgaging properties, selling houses, trades, auctions and decisions of how to get out of jail. These would make the game have more decisions which I think will increase the effect of making good decisions and making the game less about chance. For example it is easier to get the good streets if they get auctioned out. From that you could for example also investigate how much a player should be willing to pay for the street given that the player is in a certain state.

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