Regularity and Boundary Behavior of Solutions to Complex Monge–Ampère Equations
Regularity and boundary behavior of solutions to complex Monge–Ampère equations

Björn Ivarsson
Abstract


In the theory of holomorphic functions of one complex variable it is often useful to study subharmonic functions. The subharmonic functions can be described using the Laplace operator. When one studies holomorphic functions of several complex variables one should study the plurisubharmonic functions instead. Here the complex Monge–Ampère operator has a role similar to that of the Laplace operator in the theory of subharmonic functions. The complex Monge–Ampère operator is nonlinear and therefore it is not as well understood as the Laplace operator. We consider two types of boundary value problems for the complex Monge–Ampère equation in certain pseudoconvex domains. In this thesis the right-hand side in the Monge–Ampère equation will always be smooth, strictly positive and meet a monotonicity condition. The first type of boundary value problem we consider is a Dirichlet problem where we look for plurisubharmonic solutions which are zero on the boundary of the domain. We show that this problem has a unique smooth solution if the domain has a smooth bounded plurisubharmonic exhaustion function which is globally Lipschitz and has Monge–Ampère mass larger than one everywhere. We obtain some results on which domains have such a bounded exhaustion function. The second type of boundary value problem we consider is a boundary blow-up problem where we look for plurisubharmonic solutions which tend to infinity at the boundary of the domain. Here we also assume that the right-hand side in the Monge–Ampère equation satisfies a growth condition. We study this problem in strongly pseudoconvex domains with smooth boundary and show that it has solutions which are Hölder continuous with arbitrary Hölder exponent $\alpha$, $0 \leq \alpha < 1$. We also show a uniqueness result. A result on the growth of the solutions is also proved. This result is used to describe the boundary behavior of the Bergman kernel.

Keywords and phrases: Complex Monge–Ampère operator, interior regularity, plurisubharmonic function, blow-up rate of solutions, bounded plurisubharmonic exhaustion function, globally Lipschitz, strongly pseudoconvex domain, hyperconvex domain, Bergman kernel.

2000 Mathematics Subject Classification. Primary 32W20; Secondary 32A25, 32U10, 35B40, 35B65, 35J60.

Björn Ivarsson, Department of Mathematics, Uppsala University, Box 480, SE-751 06 Uppsala, Sweden; bjorni@math.uu.se

©Björn Ivarsson 2002
ISSN 1401-2049

Printed in Sweden by Reprocentralen, Ekonomikum, Uppsala 2002
Distributor: Department of Mathematics, Box 480, SE-751 06 Uppsala, Sweden
This thesis consists of a summary and the following 4 papers:


III. B. IVARSSON, Regularity and uniqueness of solutions to boundary blow-up problems for the complex Monge–Ampère operator, Manuscript, 14 pp.

IV. B. IVARSSON, On the behavior of strictly plurisubharmonic functions near real hypersurfaces, Manuscript, 9 pp.

Paper I is reprinted with permission from the publisher.
1. Background

We begin by pointing out some phenomena which occur in the theory of several complex variables but not in the theory of one complex variable. First we introduce...
some notation. Let \((z_1, \ldots, z_n) = (x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n\),
\[
\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right)
\]
and
\[
dz_j = dx_j + idy_j, \quad d\overline{z}_j = dx_j - idy_j.
\]

With this notation the Cauchy–Riemann equation in one variable becomes
\[
\frac{\partial f}{\partial \overline{z}} = 0.
\]

A holomorphic function is a \(C^1\)-function which satisfies the Cauchy–Riemann equation. In several variables it should satisfy the Cauchy–Riemann equation in each variable separately. We introduce more notation. A form
\[
\alpha_{\beta} dz_{\alpha} \wedge d\overline{z}_{\beta}
\]
is said to be a \((p, q)\)-form. Here we have used multiindex notation, that is \(|\alpha| = p\) if \(\alpha \in \mathbb{Z}^p\), \(dz_{\alpha} = dz_{\alpha_1} \wedge \cdots \wedge dz_{\alpha_p}\), and \(d\overline{z}_{\beta} = d\overline{z}_{\beta_1} \wedge \cdots \wedge d\overline{z}_{\beta_q}\). Let \(A^{p,q}(\Omega)\) denote the \((p, q)\)-forms with smooth coefficients on \(\Omega\). By smooth we mean \(C^\infty\)-smooth. We define \(\partial: A^{0,0}(\Omega) \rightarrow A^{1,0}(\Omega)\) and \(\overline{\partial}: A^{0,0}(\Omega) \rightarrow A^{0,1}(\Omega)\) by
\[
\partial f = \sum_{j=1}^{n} \frac{\partial f}{\partial z_j} dz_j \quad \text{and} \quad \overline{\partial} f = \sum_{j=1}^{n} \frac{\partial f}{\partial \overline{z}_j} d\overline{z}_j.
\]

We now proceed inductively via the formulas
\[
\partial (f \wedge dz_j) = \partial f \wedge dz_j, \quad \partial (f \wedge d\overline{z}_j) = \partial f \wedge d\overline{z}_j
\]
and use linearity to define \(\partial: A^{p,q}(\Omega) \rightarrow A^{p+1,q}(\Omega)\). The definition of \(\overline{\partial}: A^{p,q}(\Omega) \rightarrow A^{p,q+1}(\Omega)\) is similar. Hence holomorphicity of functions can be expressed as \(\overline{\partial} f = 0\). We note that for a \((p, q)\)-form, with \(q \geq 1\), \(\overline{\partial} f = 0\) does not imply that the coefficients are holomorphic functions.

Assume that \(f\) is holomorphic in \(\Omega\) and \(C^1\) on \(\overline{\Omega}\). Then
\[
f(z) = \frac{1}{2\pi i} \int_{\partial \Omega} \frac{f(\zeta)}{\zeta - z} d\zeta
\]
for \(z \in \Omega\). This is the Cauchy integral formula. This generalizes to product domains \(\Omega = \Omega_1 \times \cdots \times \Omega_n\) in \(\mathbb{C}^n\). We get
\[
f(z) = \left( \frac{1}{2\pi i} \right)^n \int_{\partial\Omega_1} \cdots \int_{\partial\Omega_n} \frac{f(\zeta)}{\prod_{j=1}^{n}(\zeta_j - z_j)} d\zeta_1 \cdots d\zeta_n.
\]

Note that we do not integrate over the whole boundary \(\partial \Omega\). We need only integrate over the distinguished boundary \(\partial_0 \Omega = \partial_1 \Omega \times \cdots \times \partial_n \Omega\). This formula lets us conclude that a holomorphic function \(f\) can be expanded in a power series \(f(z) = \sum_{\alpha \in \mathbb{Z}^n} a_\alpha z^\alpha\) which converges locally. Here we use multiorder notation. That is, if \(\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n\) then \(z^\alpha = \prod_{j=1}^{n} z_j^{\alpha_j}\). When one uses multiorder notation we have \(|\alpha| = \sum_{j=1}^{n} |\alpha_j|\).

One might wonder if there is an integral representation of holomorphic functions on a domain \(\Omega\), not necessarily a product domain. The proof of the Cauchy integral formula, basically Stokes’ Theorem applied to the form \(f(\zeta)(\zeta - z)^{-1} d\zeta\), can be
modified so that applies to the case of several variables. Let \( \omega(\zeta) = d\zeta_1 \wedge \cdots \wedge d\zeta_n \) and
\[
\eta(\zeta - \overline{z}) = \sum_{j=1}^{n} (-1)^{j+1}(\zeta_j - z_j)d\zeta_1 \wedge \cdots \wedge d\zeta_{j-1} \wedge d\zeta_{j+1} \wedge \cdots \wedge d\zeta_n.
\]
If \( \Omega \subseteq \mathbb{C}^n \) with \( C^1 \)-boundary and \( f \) is a holomorphic function in \( \Omega \) which is \( C^1 \) on \( \overline{\Omega} \) then
\[
f(z) = \frac{(n-1)!}{(-1)^n(n-1)/2(2\pi i)^n} \int_{\partial\Omega} \frac{f(\zeta)\eta(\zeta - \overline{z}) \wedge \omega(\zeta)}{|\zeta - z|^{2n}}
\]
for \( z \in \Omega \). This is the Bochner–Martinelli formula. Notice that the Bochner–Martinelli kernel
\[
\frac{\eta(\zeta - \overline{z}) \wedge \omega(\zeta)}{|\zeta - z|^{2n}}
\]
is the Cauchy kernel if \( n = 1 \). The Cauchy kernel is holomorphic in \( \zeta \) when \( \zeta \neq z \). This is not true for the Bochner–Martinelli kernel when \( n \geq 2 \). That is, while it is true that
\[
\overline{\partial} \left( \frac{\eta(\zeta - \overline{z}) \wedge \omega(\zeta)}{|\zeta - z|^{2n}} \right) = 0
\]
it is not true that the coefficient functions are holomorphic functions in \( \zeta \). One important way of constructing holomorphic functions in the one variable case is to form convolutions of measures with the Cauchy kernel. If we form convolutions of measures with the Bochner–Martinelli kernel, in general we do not get a holomorphic function. There exists kernels which produces holomorphic functions when one forms a convolution of a measure with the kernel, but these kernels are often quite non-explicit: one example is the Bergman kernel. This difficulty makes the study of the \( \overline{\partial} \)-equation \( \overline{\partial} u = f \) more important in the multidimensional case than it is in the one dimensional case.

Let \( f \) be a \((p,q)\)-form, \( q \geq 1 \). Since \( d = \partial + \overline{\partial} \) and \( d^2 = 0 \) we see that \( \partial^2 = 0, \partial \overline{\partial} + \overline{\partial} \partial = 0 \) and \( \overline{\partial}^2 = 0 \). Thus a necessary condition for solvability of \( \overline{\partial} u = f \) is \( \overline{\partial} f = 0 \). Now let \( f \in A^{0,1}(\mathbb{C}) \) have compact support. We see that \( \overline{\partial} f = 0 \) is automatic. A solution of \( \overline{\partial} u = f \) is given by
\[
u(z) = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f(\zeta)}{\zeta - z} \ d\zeta \wedge d\zeta.
\]
This function does not have compact support. We can add any holomorphic function to \( u \) to produce a new solution. One realizes that no nonzero solution of \( \overline{\partial} u = f \) can have compact support in the one dimensional case. The situation changes when we consider the same problem in the multidimensional case. Take \( f = \sum_j f_j \ d\zeta_j \in A^{0,1}(\mathbb{C}^n) \) which satisfies \( \overline{\partial} f = 0 \) and has compact support. Then
\[
u_j(z) = -\frac{1}{2\pi i} \int_{\mathbb{C}} \frac{f_j(z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n)}{\zeta - z_j} \ d\zeta \wedge d\zeta
\]
is a solution of \( \overline{\partial} u = f \). This solution has compact support. Let \( U \) be the unbounded component of \( \mathbb{C}^n \setminus \text{supp } f \). Then \( \text{supp } u \subseteq \mathbb{C}^n \setminus U \). One sees this by choosing \( z \) so that \( \{z_1, \ldots, z_{j-1}, \zeta, z_{j+1}, \ldots, z_n; \zeta \in \mathbb{C} \} \) is contained in \( U \). We see that \( u_j \) is zero in an open set contained in \( U \) and since \( u_j \) is holomorphic outside the support of \( f \) the identity theorem for holomorphic functions, which holds since holomorphic functions can be expanded in a power series, implies that \( u_j \) must vanish in \( U \). Since the difference between two solutions of \( \overline{\partial} u = f \) is holomorphic, two solutions with compact support must be identical. Hence \( u_j = u_{j'} \) for any \( j \).
and $j'$ such that $1 \leq j, j' \leq n$. We see that there exists a unique solution with compact support. This gives rise to new and interesting phenomena in the theory of several complex variables which are not present in classical function theory. We illustrate this by presenting the Hartogs Phenomenon which is the following. Given a bounded domain $\Omega \subseteq \mathbb{C}^n$, $n \geq 2$, a compact set $K \subseteq \Omega$ such that $\Omega \setminus K$ is connected, and a holomorphic function $f$ on $\Omega \setminus K$ we can find a holomorphic function $F$ on $\Omega$ such that $F|_{\Omega \setminus K} = f$. Let us now see how this can be proved using the fact that $\overline{\partial}u = f$ has a compactly supported solution if $f$ has compact support. Let $\chi$ be a smooth function with compact support in $\Omega$ which is 1 on an open neighborhood of $K$. Let $\tilde{f}(z) = \begin{cases} (1-\chi(z))f(z) \text{ if } z \in \Omega \setminus K \\ 0 \text{ if } z \in K \end{cases}$ and put $\psi = \overline{\partial}\tilde{f}$. We see that $\overline{\partial}\psi = 0$ and $\psi$ has compact support in $\Omega$. Therefore there is a function $u$ with compact support in $\Omega$ that satisfies $\overline{\partial}u = \psi$. We see that $F = \tilde{f} - u$ is holomorphic in $\Omega$ and is equal to $f$ near the boundary of $\Omega$. Hence $F|_{\Omega \setminus K} = f$. This is in contrast to the situation in one variable. In a domain in the complex plane one can construct a holomorphic function which cannot be continued across any boundary point of the domain. Therefore it is interesting to study domains which has holomorphic functions which cannot be continued to any larger domain. Such a domain is called a domain of holomorphy.

Another interesting consequence of the Hartogs Phenomenon is that the zero-set of a holomorphic function of several variables cannot be contained in a compact set. Again this is not true for holomorphic functions of one variable. Good references for the theory of several complex variables are Hörmander’s book [16] and Krantz’s book [22].

2. Plurisubharmonic functions

When one studies holomorphic functions of one complex variable a related class of functions are often useful. This is the class of subharmonic functions. A subharmonic function on $\Omega \subseteq \mathbb{C}$ is a function $u: \Omega \to [-\infty, \infty)$ which is upper semicontinuous and, for $z \in \Omega$, satisfies

$$u(z) \leq \frac{1}{2\pi} \int_{0}^{\pi} u(z + re^{\theta}) \, d\theta$$

for all $r > 0$ such that $B_r(z) \subseteq \Omega$. If $u \in C^2(\Omega)$ then $u$ is subharmonic if and only if

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4 \frac{\partial^2 u}{\partial z \partial \overline{z}} \geq 0.$$ 

For a general subharmonic function we have $\Delta u \geq 0$ in the sense of distributions and if $v \in L^1_{loc}(\Omega)$ satisfies $\Delta v \geq 0$ in the sense of distribution then $v$ is equal to a subharmonic function almost everywhere. If $f: \Omega \to \mathbb{C}$ is holomorphic then $\log |f|$ is subharmonic (if we include $u \equiv -\infty$ among the subharmonic functions.) This link between function theory and the theory of subharmonic function has proven fruitful. When one studies function theory in several complex variables the related class is not the subharmonic functions. This is because a holomorphic function $f: \Omega \to \mathbb{C}$, $\Omega \subseteq \mathbb{C}^n$, is holomorphic as a function of one variable on every complex line. Hence $\log |f|$ is subharmonic on every complex line. A upper semicontinuous function $u: \Omega \to [-\infty, \infty)$ that is subharmonic on every complex line is said to be plurisubharmonic. Hence the related class in function theory of several variables
is the plurisubharmonic functions. We denote this class by $\mathcal{PSH}$. If $u \in C^2(\Omega)$ and plurisubharmonic then for $z \in \Omega$

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)w_j \overline{w}_k \geq 0$$

for all $w \in \mathbb{C}^n$. If the inequality is strict then $u$ is said to be strictly plurisubharmonic. For a plurisubharmonic function $u$ which is not twice differentiable we have in the sense of distributions

$$\sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)w_j \overline{w}_k \geq 0$$

for all $w \in \mathbb{C}^n$. That is, for any non-negative test function $\varphi \in C_0^\infty(\Omega)$ we have

$$\int_{\Omega} u(z) \sum_{j,k=1}^{n} \frac{\partial^2 \varphi}{\partial z_j \partial \overline{z}_k}(z)w_j \overline{w}_k \ d\lambda(z) \geq 0$$

for all $w \in \mathbb{C}^n$. If $v \in L^1_{\text{loc}}(\Omega)$ satisfies

$$\sum_{j,k=1}^{n} \frac{\partial^2 v}{\partial z_j \partial \overline{z}_k}(z)w_j \overline{w}_k \geq 0$$

for all $w \in \mathbb{C}^n$ in the sense of distribution then $v$ is equal to a plurisubharmonic function almost everywhere. One important result for plurisubharmonic functions is the main approximation theorem for plurisubharmonic functions which states the following. For any plurisubharmonic function $u$ and any relatively compact $\Omega'$ of $\Omega$ we can find a decreasing sequence of smooth plurisubharmonic functions $(u_j)_{j=1}^\infty$ defined on $\Omega'$ so that $\lim_{j \to \infty} u_j(z) = u(z)$ for $z \in \Omega'$. The functions $u_j$ are smooth regularizations of $u$. Let $0 \leq \chi \in C_0^\infty(B_1(0))$ be a radial function which satisfies $\int_{B_1(0)} \chi(z) \ d\lambda(z) = 1$. Here $d\lambda$ denotes Lebesgue measure. Let $\chi_\varepsilon(z) = \varepsilon^{-n} \chi(z/\varepsilon)$ and put $u_j = u \ast \chi_1/\varepsilon$. These functions are smooth plurisubharmonic functions that has all the properties we need. The main approximation theorem for plurisubharmonic functions imply that properties that hold for smooth plurisubharmonic functions often hold, at least in a generalized sense, for non-smooth plurisubharmonic functions. One can also replace plurisubharmonic by subharmonic and get a main approximation theorem for subharmonic functions.

We shall now describe some ways of constructing new plurisubharmonic functions. If $u \in \mathcal{PSH}(\Omega)$ then $cu \in \mathcal{PSH}(\Omega)$ if $0 \leq c \in \mathbb{R}$. We have already mentioned that if $f$ is holomorphic then $\log |f|$ is plurisubharmonic. The sum of two plurisubharmonic functions are plurisubharmonic. If $(u_j)_{j=1}^\infty$ is a decreasing sequence of plurisubharmonic functions then $\lim_{j \to \infty} u_j(z) = u(z)$ is plurisubharmonic. If $u_\alpha, \alpha \in A$, is a family of plurisubharmonic functions then $u(z) = \sup(\{u_\alpha(z); \alpha \in A\})$ is plurisubharmonic if it is bounded and upper semicontinuous. Consider a function $\varphi : \mathbb{R} \to \mathbb{R}$ which is convex and increasing. Set $\varphi(-\infty) = \lim_{x \to -\infty} \varphi(x)$. Then $\varphi \circ u$ is plurisubharmonic if $u$ is. Finally let $\Omega \subseteq \mathbb{C}^n$, $\Omega' \subseteq \mathbb{C}^m$ and $f : \Omega \to \Omega'$ be an analytic map. If $u \in \mathcal{PSH}(\Omega')$ then $u \circ f \in \mathcal{PSH}(\Omega)$. This implies that we can define plurisubharmonic functions on complex manifolds.

We can now give another description of domains of holomorphy. Let $\Omega$ be an open subset of $\mathbb{C}^n$. The set $\Omega$ is said to be pseudoconvex if one can find a continuous plurisubharmonic function $u$ on $\Omega$ such that $\{z \in \Omega; u(z) < c\} \subseteq \Omega$ for every $c \in \mathbb{R}$. Let $d_\Omega(z)$ denote the distance of $z$ to the boundary $\partial \Omega$. One
can show that if $\Omega$ is pseudoconvex then $-\log d\Omega(z)$ is plurisubharmonic in $\Omega$. Conversely if $-\log d\Omega(z)$ is plurisubharmonic in $\Omega$ then $\Omega$ is pseudoconvex. Also $\Omega$ is pseudoconvex if and only if it is a domain of holomorphy. If $\Omega$ have $C^2$-boundary then one find a $C^1$-function $\rho$ defined in a neighborhood of $\Omega$ such that $\Omega = \{z \in \Omega; \rho(z) < 0\}$, $\lim_{z \to z_0} \rho(z) = 0$ for all $z_0 \in \partial\Omega$ and $dp \neq 0$ on $\partial\Omega$. Then $\Omega$ is pseudoconvex if $\rho$ satisfies

$$
\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \bar{z}_k}(z)w_j \bar{w}_k \geq 0
$$

for $z \in \partial\Omega$ and $w \in \mathbb{C}^n$ such that

$$
\sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z)w_j = 0.
$$

This condition is called the Levi condition. If the inequality in the Levi condition is strict then $\Omega$ is said to be strongly pseudoconvex.

3. The complex Monge–Ampère operator

When one studies subharmonic functions the equation $\Delta u = f$ is important. If one wants to study plurisubharmonic functions an analogue to the Laplace operator is desirable. Using $d = \partial + \bar{\partial}$ and introducing the notation $d^c = i(\bar{\partial} - \partial)$ we have

$$
dd^c u = 2i \sum_{j,k=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} dz_j \wedge d\bar{z}_k.
$$

If $u \in C^2(\Omega)$ then one can form $(dd^c u)^2 = dd^c u \wedge dd^c u, \ldots,$

$$(dd^c u)^n = (2i)^n \det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) \right) dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_n \wedge d\bar{z}_n =$$

$$= 4^n n! \det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) \right) dV$$

where $dV = dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$ is the volume form. The counterpart of the Laplace operator when one studies plurisubharmonic functions is the nonlinear operator $(dd^c u)^n$ which is called the complex Monge–Ampère operator. So far we have only described the definition of the complex Monge–Ampère operator when applied to functions which are twice differentiable. One would like to extend this definition so that it can be applied to more general plurisubharmonic function.

In the following two subsections we shall describe two extensions. In both these extensions $(dd^c u)^n$ is a positive Borel measure. Before we describe these extensions we point out that is known that a definition of $(dd^c u)^n$ as a positive Borel measure for all plurisubharmonic functions $u$ is impossible, see for example Kiselman’s paper [19]. We also recall some basic material on currents and positivity of forms, that is elements of the exterior algebra of $\mathbb{C}^n$, differential forms and currents. Let $\mathcal{D}^{p,q}(\Omega)$ denote the differential forms of bidegree $(p, q)$ that have smooth coefficients with compact support in $\Omega$. We call these differential forms test forms of bidegree $(p, q)$. We give $\mathcal{D}^{p,q}(\Omega)$ a topology in the usual way. Let $\Omega_j$, $j \in \mathbb{N}$, be a sequence of relatively compact open subsets of $\Omega$ such that $\Omega_j \Subset \Omega_{j+1}$ and $\Omega = \bigcup_j \Omega_j$. Give $\mathcal{D}^{p,q}(\Omega_j)$ the topology that is defined by the countable set of seminorms

$$
\|u\|_{j,k} = \sup \left( \left| \frac{\partial^{\alpha+\beta} u_{JK}}{\partial x^\alpha \partial y^\beta}(z) \right| ; z \in \Omega_j, |J| = p, |K| = q; \alpha, \beta \in \mathbb{N}^n, |\alpha| + |\beta| \leq k \right)
$$
where \( u = \sum_{|J|=p,|K|=q} u_{JK} dz^J \wedge d\bar{z}^K \) and we have used multiform notation again.

With this topology \( \mathcal{D}^{p,q}(\Omega) \) is a Fréchet space. We equip \( \mathcal{D}^{p,q}(\Omega) \) with a topology such that \( u^j \to u \) in \( \mathcal{D}^{p,q}(\Omega) \) if and only if the following holds: (i) there is a compact subset \( K \subset \Omega \) such that \( \operatorname{supp} u_{JK}^j \subset K \) for all \( J, K \); (ii) \( u_{JK}^j \) and all its derivatives converge uniformly to \( u_{JK} \) and the corresponding derivatives on \( K \) for all \( J, K \).

The dual space \( \mathcal{D}^{n-p,n-q}(\Omega)' \) is the space of currents of bidegree \((p,q)\). We equip this space with the weak*-topology. In this topology a sequence \( T_j \) of currents of bidegree \((p,q)\) converges to \( T \) if and only if \( \lim_{j \to \infty} T_j(\varphi) = T(\varphi) \) for any \( \varphi \in \mathcal{D}^{p,q}(\Omega) \). A differential form \( u \) of bidegree \((p,q)\) defines a current \( T_u \) of bidegree \((p,q)\) via

\[
T_u(\varphi) = \int_{\Omega} u \wedge \varphi
\]

for \( \varphi \in \mathcal{D}^{n-p,n-q}(\Omega) \). Informally one can think of currents of bidegree \((p,q)\) as differential forms of bidegree \((p,q)\) with coefficients that are distributions.

We now describe the notion of positivity that we shall use, first for forms, then for differential forms and finally for currents. A 2n-form \( \omega \) is positive if \( \omega = \tau \, dV \) where \( dV \) is the volume form and \( \tau \geq 0 \). We say that a \((p,p)\)-form \( \omega \) is elementarily strongly positive if there are \( p \) linearly independent \( \mathbb{C} \)-linear mappings \( \eta_j : \mathbb{C}^n \to \mathbb{C} \), \( j = 1, \ldots, p \) such that

\[
\omega = \frac{i}{2} \eta_1 \wedge \bar{\eta}_1 \wedge \cdots \wedge \frac{i}{2} \eta_p \wedge \bar{\eta}_p.
\]

Forms \( \omega \) that can be written as

\[
\omega = \sum_{j=1}^n \lambda_j \omega_j
\]

for some \( m \), some \( \lambda_j > 0 \) and some elementarily strongly positive forms \( \omega_j \) are said to be strongly positive. A \((p,p)\)-form \( \omega \) is positive if for any strongly positive \((n-p, n-p)\)-form \( \eta \) the form \( \omega \wedge \eta \) is positive. All these concepts extend to differential forms by requiring that they should hold at all points in \( \Omega \). A \((p,p)\)-current \( T \) is positive if for any strongly positive test form \( \varphi \) of bidegree \((n-p, n-p)\) we have \( T(\varphi) \geq 0 \). For more information and details on plurisubharmonic functions and currents see Klimek’s book [20].

3.1. **Bedford’s and Taylor’s definition of the complex Monge–Ampère operator.** We now describe an extension of the complex Monge–Ampère operator so that it can be applied to plurisubharmonic functions \( u \in L^\infty_{loc}(\Omega) \). This extension was done by Bedford and Taylor in [3]. This extension is accomplished using the main approximation theorem for plurisubharmonic functions, an integration-by-parts formula and the Chern–Levine–Nirenberg estimate. The Chern–Levine–Nirenberg estimate is the following. Let \( u_1, \ldots, u_n \in \mathcal{PSh}(\Omega) \cap C^2(\Omega) \) and \( K \) be a compact subset of \( \Omega \). Then there exists a constant \( C \) and a compact set \( L \subseteq \Omega \setminus K \) such that

\[
\int_K dd^c u_1 \wedge \cdots \wedge dd^c u_n \leq C \prod_{j=1}^n \sup(|u_j(z)|; z \in L).
\]

Integration by parts yields

\[
\int_{\Omega} dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge \chi = \int_{\Omega} u_k \, dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c \chi
\]
where \( \chi \in \mathcal{D}^{n-k,n-k}(\Omega) \). Using this we can inductively define \( dd^c u_1 \wedge \cdots \wedge dd^c u_k \) as a positive \((k, k)\)-current for \( 1 \leq k \leq n \) when \( u_1, \ldots, u_n \in \mathcal{P}SH(\Omega) \cap L^\infty_{loc}(\Omega) \). Note that since \( u_j \) is plurisubharmonic then \( dd^c u_j \) is a positive \((1, 1)\)-current. Assume that \( dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \) has been defined as a positive \((k - 1, k - 1)\)-current. A positive current have coefficients which are positive Borel measures. Therefore the \( u_k \wedge dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \) measure coefficients and we use the integration-by-parts formula to define \( dd^c u_1 \wedge \cdots \wedge dd^c u_k \) as a \((k, k)\)-current. What remains to be shown is that the current we have just defined is positive. Let \( \chi \) be a strongly positive test form of bidegree \((n - k, n - k)\) and \( \Omega' \) be a relatively compact subset of \( \Omega \) which contains the support of \( \chi \). By the main approximation theorem for plurisubharmonic functions there exists a decreasing sequence \( \{u_{k,j}\}_{j \in \mathbb{N}} \) of smooth plurisubharmonic functions that converges pointwise to \( u_k \) in \( \Omega' \). For each \( j \) the test form \( dd^c u_{k,j} \wedge \chi \) is positive and therefore

\[
\int_{\Omega} dd^c u_1 \wedge dd^c u_{k-1} \wedge (dd^c u_{k,j} \wedge \chi) \geq 0.
\]

Because of the Chern–Levine–Nirenberg estimate the dominated convergence theorem can be applied and we get

\[
\int_{\Omega} dd^c u_1 \wedge \cdots \wedge dd^c u_k \wedge \chi = \int_{\Omega} u_k dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge \chi = \lim_{j \to \infty} \int_{\Omega} u_{k,j} dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c \chi = \lim_{j \to \infty} \int_{\Omega} dd^c u_1 \wedge \cdots \wedge dd^c u_{k-1} \wedge dd^c u_{k,j} \wedge \chi \geq 0.
\]

Hence \( dd^c u_1 \wedge \cdots \wedge dd^c u_k \) is a positive \((k, k)\)-current and therefore its coefficients are positive Borel measures. This gives us a definition of \((dd^c u)^n\) for \( u \in \mathcal{P}SH(\Omega) \cap L^\infty_{loc}(\Omega) \) as a positive Borel measure.

3.2. **Cegrell’s definition of the complex Monge–Ampère operator.** Often one wants to study plurisubharmonic functions which are not bounded in \( \Omega \). For such plurisubharmonic functions Bedford’s and Taylor’s definition of the complex Monge–Ampère operator is not applicable. We shall describe Cegrell’s definition of the complex Monge–Ampère operator. If we want the operator to have certain properties Cegrell’s definition has the largest domain of definition possible. We shall state these properties after we have given the definition. The justification of the definition consists of the same steps as Bedford’s and Taylor’s definition; an approximation theorem, integration by parts and a convergence proof. For details on how to perform these steps see Cegrell’s paper [10]. We begin by describing the approximation theorem. Let us denote the class of negative plurisubharmonic functions on \( \Omega \) by \( \mathcal{P}SH^-(\Omega) \). A bounded domain is said to be hyperconvex if we can find a continuous negative plurisubharmonic function \( \rho \) such that \( \{z \in \Omega : \rho(z) < c\} \subseteq \Omega \) for all \( c < 0 \). Such a function \( \rho \) is called a bounded exhaustion function for \( \Omega \).

**Theorem 3.1.** Suppose that \( \Omega \) is a hyperconvex domain and assume that \( u \in \mathcal{P}SH^-(\Omega) \). Then there is a decreasing sequence of functions \( u_j \in \mathcal{P}SH^-(\Omega) \cap C(\Omega) \) with \( u_j|_{\partial \Omega} = 0 \), for \( j = 1, 2, 3, \ldots \), \( \lim_{j \to \infty} u_j(z) = u(z) \) for all \( z \in \Omega \) and

\[
\int_{\Omega} (dd^c u_j)^n < \infty.
\]
The proof makes use of a continuous negative plurisubharmonic function \( v \) which satisfies \( \lim_{z \to z_0} v(z) = 0 \) for all \( z_0 \in \partial \Omega \) and
\[
\int_\Omega (dd^c v)^n < \infty.
\]

Let \((r_j)_{j=1}^{\infty}\) be a decreasing sequence such that
\[
0 < r_j < \inf \left( d_\Omega(z); z \in \Omega, v(z) < -\frac{1}{2j^2} \right)
\]
where \( d_\Omega(z) \) is the distance of \( z \) to the boundary \( \partial \Omega \). Let \( u_{r_j} = u \ast \chi_{r_j} \) be a smoothing of \( u \) which was described in Section 2. The smooth regularizations \( u_{r_j} \) are defined on \( \{ z \in \Omega; d_\Omega(z) > r_j \} \). Put
\[
u_j = \max_{j \leq m} \left( u_{r_m} - \frac{1}{m}, mv \right).
\]
We see that
\[
\max \left( u_{r_m} - \frac{1}{m}, mv \right)
\]
is plurisubharmonic on \( \Omega \), continuous on \( \overline{\Omega} \) and equal to zero on \( \partial \Omega \) for each \( m \geq j \).

One shows that the same is true for \( u_j \). One also checks that \( \lim_{j \to \infty} u_j(z) = u(z) \) and
\[
\int_\Omega (dd^c u_j)^n < j^n \int_\Omega (dd^c v)^n < \infty.
\]

Now one turns to integration by parts. For this we introduce the class \( \mathcal{E}_0(\Omega) \).
This is the convex cone of bounded plurisubharmonic functions \( u \) which satisfies \( \lim_{z \to z_0} u(z) = 0 \) for all \( z_0 \in \partial \Omega \) and \( \int_\Omega (dd^c u)^n < \infty \). Cegrell proves the following result.

**Theorem 3.2.** Suppose that \( \Omega \) is a hyperconvex domain, \( u, v \in \mathcal{P}SH^{-}(\Omega) \), \( u \) is not identically 0, \( \lim_{z \to z_0} u(z) = 0 \) for all \( z_0 \in \partial \Omega \). Assume that \( T \) is a positive and closed current of bidegree \((1, 1)\). Then \( dd^c u \wedge T \) is a well-defined positive measure on \( \Omega \). Furthermore, if
\[
\int_\Omega v \ dm \wedge T > -\infty
\]
then \( dd^c v \wedge T \) is also a well-defined positive measure on \( \Omega \) and
\[
\int_\Omega v \ dm \wedge T \leq \int_\Omega u \ dm \wedge T.
\]

As a corollary we get the following integration-by-parts formula.

**Corollary 3.3.** Suppose that \( u, v \in \mathcal{P}SH(\Omega) \) such that \( \lim_{z \to z_0} u(z) = 0 \) and \( \lim z \to z_0 v(z) = 0 \) for all \( z_0 \in \partial \Omega \) and that \( T \) is a positive closed current of bidegree \((1, 1)\). If
\[
\int_\Omega v \ dm \wedge T > -\infty
\]
then
\[
\int_\Omega u \ dm \wedge T > -\infty
\]
and
\[
\int_\Omega v \ dm \wedge T = \int_\Omega u \ dm \wedge T.
\]
Definition 3.5. For $u \in \mathcal{E}(\Omega)$. We say that $u \in \mathcal{E}(\Omega)$ if $u \in PSH^-(\Omega)$ and for every $z_0 \in \Omega$ there is a neighborhood $\omega$ of $z_0$ in $\Omega$ and a decreasing sequence $h_j \in \mathcal{E}_0(\Omega)$ such that $\lim_{j \to \infty} h_j(z) = u(z)$ for all $z \in \omega$ and
\[ \sup \left( \int_{\Omega} (dd^c h_j)^n ; j = 1, 2, 3, \ldots \right) < \infty. \]
We have

Theorem 3.4. Suppose that $u \in \mathcal{E}(\Omega)$. If $g_j \in \mathcal{E}_0(\Omega)$ decreases to $u$, $j \to \infty$, then $(dd^c g_j)^n$ is weak*-convergent and the limit measure is independent of the sequence $g_j$.

We can now define the complex Monge–Ampère operator on plurisubharmonic function $u \in \mathcal{E}(\Omega)$.

Definition 3.5. For $u \in \mathcal{E}(\Omega)$ define $(dd^c u)^n$ as the limit measure obtained in Theorem 3.4.

Let $\mathcal{K}(\Omega)$ be a class of negative plurisubharmonic functions which satisfies: (i) If $u \in \mathcal{K}$ and $v \in PSH^-(\Omega)$ then $\max(u, v) \in \mathcal{K}(\Omega)$. (ii) If $u \in \mathcal{K}(\Omega)$ and $(g_j)_{j=1}^\infty$ is a decreasing sequence such that $g_j \in PSH^-(\Omega) \cap L^\infty_{loc}(\Omega)$ and $\lim_{j \to \infty} g_j(z) = u(z)$, then $((dd^c g_j)^n)_{j=1}^\infty$ is weak*-convergent. It is possible to show that $\mathcal{K}(\Omega) \subseteq \mathcal{E}(\Omega)$ for any class $\mathcal{K}(\Omega)$. Also $\mathcal{E}(\Omega)$ meets (i) and (ii) and in this sense Cegrell’s definition of the complex Monge–Ampère operator is optimal. The area of mathematics which is the study of this operator is called pluripotential theory.

4. The Dirichlet problem for the complex Monge–Ampère operator

Assume that $\mu$ is a positive Borel measure on a domain $\Omega$ and $\varphi$ some function on the boundary of $\Omega$. Central to pluripotential theory is the study of the Dirichlet problem
\[ \begin{cases} (dd^c u)^n = \mu & \text{in } \Omega \\ u = \varphi & \text{on } \partial \Omega. \end{cases} \]
In this thesis we shall always have $\mu = f \, dV$ where $f$ is a function. We shall be considering the question of how regularity of $f$ implies regularity of $u$.

In potential theory for the Laplace operator the harmonic functions are solutions to the equation $\Delta u = 0$. It is tempting to believe that the pluriharmonic functions are solutions to the equation $(dd^c u)^n = 0$. However, while it is true that pluriharmonic functions solve this equation, there are functions which solve $(dd^c u)^n = 0$ which are not pluriharmonic. One realizes this by thinking about a function defined on $\mathbb{C}^n$ which depends on $n - 1$ variables only. One also realizes that solutions to $(dd^c u)^n = 0$ can be very irregular in spite of the regularity of the right-hand side. The plurisubharmonic solutions of $(dd^c u)^n = 0$ can be shown to be the maximally plurisubharmonic functions. The maximally plurisubharmonic functions are the plurisubharmonic functions $u$ on $\Omega \subseteq \mathbb{C}^n$ such that if $G \subseteq \Omega$ and $v \in PSH(G)$ satisfies $\liminf_{z \to z_0 \in \partial G} (u(z) - v(z)) \geq 0$ then $v \leq u$ in $G$. Even though in general solutions to $(dd^c u)^n = 0$ can be irregular if one demands that the boundary data be continuous then it can be proved in certain domains, as it was done by Walsh in [26], that the solution is continuous. Put
\[ PB_\varphi(z) = \sup \left( v(z); v \in PSH(\Omega) \text{ and } \limsup_{z \to z_0} v(z) \leq \varphi(z_0) \text{ for all } z_0 \in \partial \Omega \right). \]
It had been observed by Bremermann in [8] that if the problem
\[
\begin{aligned}
(ddc u)^n &= 0 \text{ in } \Omega \\
u &= \varphi \text{ on } \partial \Omega
\end{aligned}
\]
is solvable, the solution is the Perron–Bremermann envelope
\[(PB_\varphi)^*(z) = \limsup_{\zeta \to z} PB_\varphi(\zeta).
\]
The result Walsh obtained is the following.

**Theorem 4.1.** Suppose that \( \Omega \) is a bounded domain in \( \mathbb{C}^n \) and \( \varphi \in C(\partial \Omega) \). Assume that
\[
\liminf_{z \to z_0} PB_\varphi(z) = \limsup_{z \to z_0} PB_\varphi(z) = \varphi(z_0)
\]
for all \( z_0 \in \partial \Omega \). Then \( PB_\varphi \in C(\overline{\Omega}) \).

High order regularity is harder for the equation \((ddc u)^n = 0\) because of the example
\[
u(z, w) = (\max\{|z|^2 - 1/2, |w|^2 - 1/2, 0\})^2.
\]
This function is not smooth in \( B_1(0) \), satisfies \((ddc u)^n = 0\) and \( u|_{\partial B_1(0)} \in C^\infty(\partial B_1(0)) \). See also Bedford’s and Fornæss’ paper [2]. They showed that the problem
\[
\begin{aligned}
(ddc u)^n &= f(z, u(z)) \text{ in } \Omega \\
u &= 0 \text{ on } \partial \Omega
\end{aligned}
\]
does not always have a solution which satisfies \( u \in C^\infty(\overline{\Omega}) \). Here \( f^{1/j} \in C^\infty(\overline{\Omega}) \) for \( j = 1, 2, 3, \ldots \), and \( f \geq 0 \). Yet another example was given by Cegrell and Sadullaev in [11]. They showed that there is a strongly pseudoconvex domain \( \Omega \) with real analytic boundary and a real analytic \( \varphi \) on \( \partial \Omega \) such that
\[
\begin{aligned}
(ddc u)^n &= 0 \text{ in } \Omega \\
u &= \varphi \text{ on } \partial \Omega
\end{aligned}
\]
has a solution which is not real analytic. The first result on high order regularity was obtained in 1985 by Caffarelli, Kohn, Nirenberg and Spruck in [9]. Note that positivity of \( f \) is crucial in view of Bedford’s and Fornæss’ example.

**Theorem 4.2.** Suppose that \( \Omega \) is a bounded, strongly pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary. Let \( f \in C^\infty(\overline{\Omega} \times \mathbb{R}) \) be a strictly positive function which is increasing in the second variable. Suppose that \( \varphi \in C^\infty(\partial \Omega) \). Then the problem
\[
\begin{aligned}
\det\left(\frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z)\right) &= f(z, u(z)) \text{ in } \Omega \\
u &= \varphi \text{ on } \partial \Omega \\
u &\in \mathcal{PSH}(\Omega) \cap C^2(\Omega) \cap C(\overline{\Omega})
\end{aligned}
\]
has a unique solution. Moreover \( u \in C^\infty(\overline{\Omega}) \).

**Remark 4.3.** When we say that a function \( g: \mathbb{R} \to \mathbb{R} \) is increasing we mean that \( x \leq x' \) implies that \( g(x) \leq g(x') \). If \( x < x' \) implies that \( g(x) < g(x') \) we say that \( g \) is strictly increasing.

**Remark 4.4.** Actually Caffarelli, Kohn, Nirenberg and Spruck proved a more general result than that stated in Theorem 4.2. One can in fact allow the Monge-Ampère mass of \( u \) to depend on the gradient of \( u \) in a certain way. For details on this see [9].
Put $v^*(z) = \limsup_{w \to z} v(w)$. We say that a domain $\Omega$ in $\mathbb{C}^n$ is $B$-regular if it admits a strong plurisubharmonic barrier at every boundary point, that is, for every $z_0 \in \partial \Omega$ there exists $v \in \mathcal{P}SH(\Omega)$ such that $\lim_{z \to z_0} v(z) = 0$ and $v^*|_{\mathbb{C}^n \setminus \{z_0\}} < 0$. These domains were given a nice characterization by Sibony in [25]. He proved the following.

**Theorem 4.5.** For a bounded domain $\Omega$ in $\mathbb{C}^n$ the following are equivalent:

i) $\Omega$ is $B$-regular.

ii) In $\Omega$ there exists a smooth bounded exhaustion function $\rho$, that is, a strictly negative function such that $\lim_{z \to z_0} \rho(z) = 0$, which in addition satisfies

$$\sum_{j,k=1}^n \frac{\partial^2 \rho}{\partial z_j \partial z_k} \alpha_j \alpha_k \geq |\alpha|^2, \quad \alpha \in \mathbb{C}^n.$$

iii) For every $f \in C(\partial \Omega)$ there exists $v \in \mathcal{P}SH(\Omega) \cap C(\overline{\Omega})$ such that $v|_{\partial \Omega} = f$.

In Section 3.2 we described bounded hyperconvex domains. A domain $\Omega$ in $\mathbb{C}^n$ is hyperconvex if it admits a weak plurisubharmonic barrier at every boundary point, that is, for every $z_0 \in \partial \Omega$ there exists $v \in \mathcal{P}SH(\Omega)$ such that $v < 0$ and $\lim_{z \to z_0} v(z) = 0$. Kerzman and Rosay showed in [18] that it is equivalent to say that there exists a smooth bounded strictly plurisubharmonic exhaustion function $\rho$ in $\Omega$. This was improved upon by B wasim in [5] so that we can choose a smooth plurisubharmonic $\rho$ satisfying $\lim_{z \to z_0} \rho(z) = 0$ for all $z_0 \in \partial \Omega$ and

$$\det \left( \frac{\partial^2 \rho}{\partial z_j \partial z_k} (z) \right) \geq 1.$$

If we do not demand that the solutions should be smooth we can get the following, which was proved by B wasim in [4].

**Theorem 4.6.** Let $\Omega$ be a bounded, hyperconvex domain in $\mathbb{C}^n$. Assume that $f$ is nonnegative, continuous and bounded in $\Omega$. Suppose that $\varphi$ is continuous on $\partial \Omega$ and that it can be continuously extended to a plurisubharmonic function on $\Omega$. Then there exists a unique solution to the following problem

$$\begin{cases}
(ddc u)^n = f(z) & \text{in } \Omega \\
u = \varphi & \text{on } \partial \Omega \\
u \in \mathcal{P}SH(\Omega) \cap C(\overline{\Omega}).
\end{cases}$$

B wasim has also given a sufficient condition for smooth solution in convex domains in [6]. This result has also been announced in [7].

**Theorem 4.7.** Let $\Omega$ be a bounded, convex domain in $\mathbb{C}^n$. Assume that $f$ is a strictly positive, smooth function in $\Omega$ such that

$$\sup \left( \left| \frac{\partial f^{1/n}}{\partial x_1} (z) \right| : z \in \Omega \right) < \infty.$$

Then there exists a unique solution to the following problem

$$\begin{cases}
\det \left( \frac{\partial^2 u}{\partial z_j \partial z_k} (z) \right) = f(z) & \text{in } \Omega \\
\lim_{z \to z_0} u(z) = 0 & \text{for all } z_0 \in \partial \Omega \\
u \in \mathcal{P}SH(\Omega) \cap C^\infty(\Omega).
\end{cases}$$

In a convex domain we can find a convex function $v$ such that $v < 0$ and $\lim_{z \to z_0} v(z) = 0$ for all boundary points $z_0$. Hence a convex domain is hyperconvex since convex functions are plurisubharmonic. Also a hyperconvex domain is pseudoconvex since $\bar{\rho}(z) = -\log(-\rho(z))$ is plurisubharmonic and $\lim_{z \to z_0} \bar{\rho}(z) = \infty$. 
4.1. Boundary blow-up problems for the complex Monge–Ampère operator. In [12], Cheng and Yau studied the problem

\[
\begin{align*}
\det \left( \frac{\partial^2 u}{\partial z_i \partial z_j}(z) \right) &= f(z)e^{Ku(z)}, \ z \in \Omega \\
\lim_{z \to z_0} u(z) &= \infty \text{ for all } z_0 \in \partial \Omega,
\end{align*}
\]

where \( \Omega \) is a bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary, \( f \) is a smooth strictly positive function and \( K > 0 \). They showed that there is a unique smooth plurisubharmonic solution to this problem. We describe their motivation in Section 5.2. In Paper II and III we study a similar problem, which we call a boundary blow-up problem, namely

\[
\begin{align*}
\det \left( \frac{\partial^2 u}{\partial z_i \partial z_j}(z) \right) &= f(z, u(z)), \ z \in \Omega \\
\lim_{z \to z_0} u(z) &= \infty \text{ for all } z_0 \in \partial \Omega,
\end{align*}
\]

where the right-hand side is a function \( f \in C^\infty(\Omega \times \mathbb{R}) \) which is strictly positive, increasing in the second variable and satisfies the following three conditions:

A: There exist functions \( h \in C^\infty(\overline{\Omega}) \) and \( f_1 \in C^\infty(\mathbb{R}) \) and two strictly positive constants \( c_1 \) and \( c_2 \) such that

\[
\lim_{t \to \infty} \frac{f(z, t)}{f_1(t)} = h(z)
\]

uniformly in \( \Omega \) and \( c_1 f_1(t) \leq f(z, t) \leq c_2 f_1(t) \) for all \( (z, t) \in \Omega \times \mathbb{R} \).

B: The function \( f_1 \) is strictly positive and increasing.

C: The function

\[
\Psi_n(a) = \int_a^\infty ((n + 1)F(y))^{-\frac{1}{n+1}} \, dy
\]

exists for \( a > 0 \), where \( F'(s) = f_1(s) \) and \( F(0) = 0 \).

4.2. Comparison principles. We shall need the following two comparison principles, the first of which was proved by Bedford and Taylor in [3].

**Lemma 4.8.** Suppose that \( \Omega \) is a bounded domain in \( \mathbb{C}^n \) and \( v, w \in C(\overline{\Omega}) \cap \mathcal{PSH}(\Omega) \). Assume that \( (dd^c v)^n \geq (dd^c w)^n \). Then

\[
\min (w(z) - v(z); z \in \overline{\Omega}) = \min (w(z) - v(z); z \in \partial \Omega).
\]

The following lemma is sometimes useful.

**Lemma 4.9.** Let \( \Omega \) be a bounded domain in \( \mathbb{C}^n \). Assume that \( f \in C(\Omega \times \mathbb{R}) \) is a nonnegative function which is increasing in the second variable. Suppose that \( v, w \in C(\overline{\Omega}) \cap \mathcal{PSH}(\Omega) \). Then

\[
(dd^c w)^n \leq f(z, w), \ f(z, v) \leq (dd^c v)^n
\]

and \( v \leq w \) on \( \partial \Omega \) implies that \( v \leq w \) in \( \Omega \).

**Proof.** Put \( V(z) = v(z) - w(z) \). We want to show that \( V \leq 0 \) and do this by contradiction. Therefore assume that there exists \( z_0 \in \Omega \) such that \( V(z_0) > 0 \). Define \( \tilde{\Omega} = \{ z \in \Omega; V(z) > 0 \} \). By assumption \( \tilde{\Omega} \) is nonempty. Let \( \Omega_0 \) be the component of \( \tilde{\Omega} \) that contains \( z_0 \). In \( \Omega_0 \) we have

\[
(dd^c w)^n \leq f(z, w) \leq f(z, v) \leq (dd^c v)^n,
\]

since \( f \) is increasing in the second variable. We have \( v = w \) on the boundary of \( \Omega_0 \) and an application of Lemma 4.8 tells us that \( v = w \) in \( \Omega_0 \), which contradicts our assumption. \( \square \)
When one studies boundary blow-up problems the following is useful.

**Corollary 4.10.** Let $\Omega$ be a bounded, strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary and assume that $\varphi \in C^\infty(\partial\Omega)$. Assume that $f \in C^\infty(\overline{\Omega} \times \mathbb{R})$ is a strictly positive function which is increasing in the second variable. Let $v$ be a plurisubharmonic solution to

\[
\begin{cases}
\det \left( \frac{\partial^2 v}{\partial \bar{z}_j \partial z_k} (z) \right) = f(z, v(z)) & \text{in } \Omega \\
\lim_{z \to z_0} u(z) = \varphi(z_0) & \text{for all } z_0 \in \partial \Omega.
\end{cases}
\]

that is smooth on $\overline{\Omega}$ and $w$ be a smooth plurisubharmonic solution to

\[
\begin{cases}
\det \left( \frac{\partial^2 w}{\partial \bar{z}_j \partial z_k} (z) \right) = f(z, w(z)) & \text{in } \Omega \\
\lim_{z \to z_0} w(z) = \infty & \text{for all } z_0 \in \partial \Omega.
\end{cases}
\]

Then $w \leq u$ in $\Omega$.

4.3. **Regularity theory.** When one wants to study the regularity of the solution of

\[
\begin{cases}
(dd^c u)^n = f & \text{in } \Omega \\
u = \varphi & \text{on } \partial \Omega
\end{cases}
\]

or

\[
\begin{cases}
(dd^c u)^n = f & \text{in } \Omega \\
\lim_{z \to z_0} u(z) = \infty & \text{for all } z_0 \in \partial \Omega
\end{cases}
\]

one often proceed in the following way. One begins by constructing a sequence of plurisubharmonic functions $(u_l)_{l=1}^\infty$ which is monotone, increasing or decreasing. A candidate for a solution will be $u(z) = \lim_{l \to \infty} u_l(z)$. One then investigates the regularity of $u$. One does this by the establishing of a priori estimates for the functions $u_l$. The a priori estimates should imply that the sequence $(u_l)_{l=1}^\infty$ is bounded in the function space $C^k(\Omega')$ or the space $C^{k,\alpha}(\Omega')$ for compact $\Omega' \subseteq \Omega$, $k \in \mathbb{N}$ and $\alpha \in (0, 1)$. The space $C^k(\Omega')$ is a Banach space with the norm

\[
\|u\|_k = \sum_{m=0}^k \sum_{|\beta|+|\gamma|=m} \sup_{z \in \Omega'} \left| \frac{\partial^{3+\gamma} u}{\partial x^\beta \partial y^\gamma} (z) \right|
\]

and $C^{k,\alpha}(\Omega')$ a Banach space with the norm

\[
\|u\|_{k,\alpha} = \|u\|_k + \sum_{|\beta|+|\gamma|=k} \sup_{z \in \Omega'} \left| \frac{\partial^{3+\gamma} u}{\partial x^\beta \partial y^\gamma} (z) - \frac{\partial^{3+\gamma} u}{\partial x^\beta \partial y^\gamma} (w) \right| |z - w|^{-\alpha} ; z, w \in \Omega', z \neq w
\]

If the sequence is bounded in $C^k(\Omega')$ then the sequence is equicontinuous in $C^{k-1,\alpha'}(\Omega')$ for $\alpha' \in (0, 1)$. If we apply the Arzela-Ascoli Theorem we see that $u \in C^{k-1,\alpha'}(\Omega')$. If the sequence is bounded in $C^{k,\alpha}(\Omega')$ then it is equicontinuous in $C^{k,\alpha'}(\Omega')$, for $\alpha' \in (0, \alpha)$ and $u \in C^{k,\alpha'}(\Omega')$. Also note that if such estimates exists for all $\Omega' \subseteq \Omega$ we can conclude that $u \in C^k(\Omega)$ or $u \in C^{k,\alpha}(\Omega)$. Hence for proving regularity we establish a priori estimates. This is often quite technical. However it turns out that it is only necessary to prove that $(u_l)_{l=1}^\infty$ is bounded in $C^{2,\alpha}(\Omega')$ in order to prove that $u \in C^\infty(\Omega')$. This is a consequence of Schauder theory which we shall describe in the next section.
4.4. Schauder theory. Assume that \( u \in C^{2,\alpha}(\Omega) \) is a solution of
\[
\det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) \right) = f(z, u(z)) > 0
\]
where \( f \) is smooth. Then \( u \in C^\infty(\Omega) \) and \( \|u\|_k < C, k \geq 3 \) where \( C \) only depend on \( \|u\|_{2,\alpha} \). In this section we shall outline how this is proved. First differentiate
\[
\det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) \right) = f(z, u(z))
\]
to get
\[
\sum_{j,k=1}^{n} M_{jk}(z) \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \left( \frac{\partial u}{\partial x_l}(z) \right) = \frac{\partial f}{\partial x_l}(z, u(z)) + \frac{\partial u}{\partial x_l}(z) \frac{\partial f}{\partial u}(z, u(z))
\]
where \( M_{jk}(z) \) is the cofactor we get by deleting the \( j \)th column and \( k \)th row from
\[
\det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) \right).
\]
We see that \( \partial u/\partial x_l \) satisfies a second order linear equation with coefficients that are Hölder continuous functions with Hölder exponent \( \alpha \). One then uses the theory for such equations that says that solutions belongs to \( C^{2,\alpha} \). Hence \( u \in C^{3,\alpha} \).

Differentiating again yields a new linear equation of second order. We get that \( u \in C^{4,\alpha} \) and continuing in this way we see that \( u \in C^\infty \). For a more thorough discussion of Schauder theory see Beals’, Fefferman’s and Grossman’s paper [1].

5. Applications of pluripotential theory to the theory of several complex variables

Here we shall give some applications of pluripotential theory. In one complex variable there is the Riemann Mapping Theorem. The Riemann Mapping Theorem says that any two simply connected domains \( \Omega \) and \( \Omega' \) in \( \mathbb{C} \) is biholomorphically equivalent. That is, one can find a holomorphic function \( f : \Omega \to \Omega' \) which has holomorphic inverse. In the theory of several complex variables the Riemann Mapping Theorem is no longer true. In fact the unit ball in \( \mathbb{C}^n \) and most ellipsoids are not biholomorphically equivalent. One sees that the Riemann Mapping Theorem cannot be true in \( \mathbb{C}^n, n \geq 2 \), in the following way. Assume that \( \Omega_1 \) and \( \Omega_2 \) are strongly pseudoconvex domains with smooth boundary. Assume that \( f : \Omega_1 \to \Omega_2 \) is a biholomorphic mapping. In [13] Fefferman proved that \( f \in C^\infty(\Omega_1) \). Let \( p \in \partial \Omega_1 \), which we assume is the origin after a translation. After a rotation we can find a function \( \rho \) so that \( \partial \Omega_1 = \{ z \in \mathbb{C}^n ; x_1 = \rho(y_1, z_2, \ldots, z_n) \} \).

Let \( \rho_N \) be the \( N \)th order Taylor polynomial. This is a polynomial of degree \( N \)th in \( 2n - 1 \) real variables. Let \( V_N \) be the vector space of all such polynomials. Now we shall introduce an equivalence relation in \( V_N \). We say that \( \rho_N \sim \tilde{\rho}_N \) if there is a biholomorphic change of coordinates \( w = \Phi(z) \) near the origin so that the surface \( \{ x_1 = \rho_N(y_1, z_2, \ldots, z_n) \} \) is mapped into a surface \( \{ \text{Re } w_1 = \tilde{\rho}_N(\text{Im } w_1, w_2, \ldots, w_n) + O(|w|^{N+1}) \} \). We see that is enough to use the \( N \)th order Taylor expansion of \( \Phi \) to check whether \( \rho_N \) and \( \tilde{\rho}_N \) are equivalent. The \( N \)th order Taylor expansion of \( \Phi \) is given by \( n \) polynomials of degree \( N \) in \( n \) complex variables. Put \( W_N = V_N/\sim \). Since the dimension of \( V_N \) is larger, if \( n \geq 2 \) and \( N \) large, than the dimension of the space of \( N \)th order Taylor expansions of holomorphic mappings we see that \( \dim W_N > 0 \). Because of Fefferman’s result these equivalence classes are biholomorphic invariants. Informally one can
say that there is no Riemann Mapping Theorem in \( \mathbb{C}^n \), \( n \geq 2 \), since there are many more domains than there are holomorphic maps.

There is a proof of the Riemann Mapping Theorem that uses potential theory for subharmonic functions, see Ransford’s book [23]. Therefore one could expect that pluripotential theory should give some information on the biholomorphism problem, that is which domains are biholomorphically equivalent, in \( \mathbb{C}^n \). Kerzman, Kohn and Nirenberg showed in [17] how that problem is connected to the problem

\[
\begin{cases}
(dd^c u)^n = f(z) \text{ in } \Omega \\
u = 0 \text{ on } \partial \Omega \\
u \in \mathcal{P}SH(\Omega) \cap C^\infty(\overline{\Omega})
\end{cases}
\]

where \( f \in C^\infty(\overline{\Omega}) \) satisfies \( f^{1/j} \in C^\infty(\overline{\Omega}) \), \( j = 1, 2, 3, \ldots \), and \( f \geq 0 \). Assume that \( \Omega_1 \) and \( \Omega_2 \) are two strongly pseudoconvex domains with smooth boundary. If the Dirichlet problem above has a solution in \( \Omega_1 \) for some \( f \) then proper holomorphic mappings between \( \Omega_1 \) and \( \Omega_2 \) extends smoothly to the boundary. A mapping \( f \) is said to be proper if \( f^{-1}(K) \) is compact for any compact \( K \) in \( \Omega_2 \). Note that Theorem 4.2 then implies that any proper holomorphic mapping between two strongly pseudoconvex domains with smooth boundary extends smoothly to the boundary.

5.1. Boundary behavior of the Bergman kernel. The boundary behavior of the Bergman kernel is also important for the biholomorphism problem. We begin by recalling the definition of the Bergman kernel and some basic properties of it. Let \( \Omega \) be a domain in \( \mathbb{C}^n \) and \( O L^2(\Omega) = \mathcal{O}(\Omega) \cap L^2(\Omega) \). One can show that given a compact \( K \subseteq \Omega \) and \( k \in \mathbb{N} \) there exist a constant \( C_k \) such that

\[
\|f\|_{C^k(K)} \leq C_k \|f\|_{L^2(\Omega)}.
\]

Therefore \( O L^2(\Omega) \) is a closed subspace of \( L^2(\Omega) \) and hence a Hilbert space. Let \( \Phi_z: O L^2(\Omega) \to \mathbb{C} \) be the linear functional \( \Phi_z(f) = f(z) \). If we let \( K = \{z\} \) in the estimate above we see that \( \Phi_z \) is a bounded linear functional. The Riesz Representation Theorem gives that there is a function \( k_z \in O L^2(\Omega) \) such that

\[
f(z) = \Phi_z(f) = \int_{\Omega} f(\zeta \overline{k_z(\zeta)}) \, d\zeta.
\]

The function \( K_{\Omega}(z, \zeta) = k_z(\overline{\zeta}) \) is called the Bergman kernel and is holomorphic in the \( z \)-variable and anti-holomorphic in the \( \zeta \)-variable. One can show that \( K_{\Omega}(z, \zeta) = \overline{K_{\Omega}(\zeta, z)} \), in other words the Bergman kernel is conjugate symmetric. One can also show that the Bergman kernel is uniquely determined by the properties that it is an element of \( O L^2(\Omega) \) in \( z \), is conjugate symmetric, and reproduces \( O L^2(\Omega) \), that is

\[
f(z) = \int_{\Omega} K_{\Omega}(z, \zeta)f(\zeta) \, d\zeta
\]

for all \( f \in O L^2(\Omega) \). One interesting result is that

\[
K_{\Omega}(z, z) = \sup(\|f(z)\|^2; f \in O L^2(\Omega), \|f\|_{L^2(\Omega)} = 1)
\]

for \( \Omega \subseteq \mathbb{C}^n \). For the biholomorphism problem the following is important. Let \( \Omega_1 \) and \( \Omega_2 \) be domains in \( \mathbb{C}^n \) and \( f: \Omega_1 \to \Omega_2 \). Put

\[
J_{\zeta}f(\zeta) = \frac{\partial(f_1, \ldots, f_n)}{\partial(\zeta_1, \ldots, \zeta_n)}(\zeta).
\]
It is a fact that $\text{det} J_C f(\zeta) \neq 0$ if $f$ is biholomorphic. We have

$$\text{det} J_C f(z) K_{\Omega_2}(f(z), f(\zeta)) \text{det} J_C f(\zeta) = K_{\Omega_1}(z, \zeta)$$

if $f$ is biholomorphic. For a domain $\Omega$ let

$$h_{jk}(z) = \frac{\partial^2}{\partial z_j \partial \bar{z}_k} \log K_{\Omega}(z, z)$$

and put $ds^2 = \sum_{j,k=1}^n h_{jk}(z) \, dz_j \otimes d\bar{z}_k$. This Hermitian metric is called the Bergman metric. We can use how the Bergman kernel transforms under biholomorphic mappings to show that biholomorphisms are isometries if we equip $\Omega_1$ and $\Omega_2$ with the Bergman metric. Because of these properties both the Bergman kernel and the Bergman metric can provide valuable insights into the biholomorphism problem. However the Bergman kernel is very hard to calculate explicitly. In 1974 Fefferman [13] described the boundary behavior of the Bergman kernel in strongly pseudoconvex domains with smooth boundary. This was done by studying the behavior of geodesics for the Bergman metric near the boundary. One could also observe, as Fefferman did in [14], that solutions of the problem

$$\left\{ \begin{array}{l}
J(v)(z) = (-1)^n \text{det} \left( \frac{\partial v}{\partial \bar{z}_j} \frac{\partial v}{\partial z_k} \right) = 1 \text{ in } \Omega \\
\lim_{z \to z_0} v(z) = 0 \text{ for all } z_0 \in \partial \Omega
\end{array} \right.$$  

transforms like a negative power of $K_{\Omega}(z, z)$. Let $f: \Omega_1 \to \Omega_2$ be a biholomorphic mapping and $v_2: \Omega_2 \to \mathbb{R}$ satisfy $J(v_2) = 1$ in $\Omega_2$ and $\lim_{\zeta \to \zeta_0} v_2(\zeta) = 0$ for all $\zeta_0 \in \partial \Omega_2$. Define $v_1(z) = | \text{det} J_C f(z) |^{-2/(n+1)} v_2(f(z))$. Then $J(v_1) = 1$ in $\Omega_1$ and $\lim_{z \to z_0} v_1(z) = 0$ for all $z_0 \in \partial \Omega_1$. Hence solutions of this problem transform like $K_{\Omega}(z, z)^{-1/(n+1)}$. One might therefore suspect that they are identical. This is not true but it can be shown that the boundary behavior of these functions are the same. Fefferman developed a method to construct approximate solutions of

$$\left\{ \begin{array}{l}
J(v) = 1 \text{ in } \Omega \\
\lim_{z \to z_0} v(z) = 0 \text{ for all } z_0 \in \partial \Omega
\end{array} \right.$$  

in [14]. One can also put $u = - \log v$ and see that $u$ then satisfies

$$\left\{ \begin{array}{l}
\text{det} \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} (z) \right) = e^{(n+1)u(z)} \text{ in } \Omega \\
\lim_{z \to z_0} u(z) = \infty \text{ for all } z_0 \in \partial \Omega.
\end{array} \right.$$  

If we can solve this problem we see that the boundary behavior of $e^{(n+1)u(z)}$ is the same as $K_{\Omega}(z, z)$ since $e^{(n+1)u} = e^{- (n+1) \log v} \approx e^{-(n+1) \log (K_{\Omega}(z, z))^{-1/(n+1)}} = K_{\Omega}(z, z)$.

5.2. **Kähler–Einstein metrics.** Let $\Omega$ be a bounded strongly pseudoconvex domain in $\mathbb{C}^n$ with smooth boundary. In [12] Cheng and Yau studied

$$\left\{ \begin{array}{l}
\text{det} \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} (z) \right) = k(z) e^{K u(z)} \text{ in } \Omega \\
\lim_{z \to z_0} u(z) = \infty \text{ for all } z_0 \in \partial \Omega.
\end{array} \right.$$  

where $K > 0$ is a constant and $k(z)$ a strictly positive smooth function on $\Omega$. They showed that this type of problem has a unique smooth plurisubharmonic solution. Their motivation for solving this problem was to construct Kähler-Einstein metrics. We shall briefly outline how a solution of such a Monge-Ampère equation implies the existence of a Kähler-Einstein metric. Remember that a Hermitian metric $ds^2 = \sum_{j,k=1}^n h_{jk}(z) \, dz_j \otimes d\bar{z}_k$ has an associated form $\omega = (i/2) \sum_{j,k=1}^n h_{jk}(z) \, dz_j \wedge d\bar{z}_k$. The metric $ds^2$ is said to be a Kähler metric if
\[ d\omega = 0 \] and \( \omega \) is said to be a Kähler form. A plurisubharmonic function \( u \) gives rise to a Hermitian metric with \( \omega = \partial\overline{\partial}u \). In fact, this is a Kähler metric since \( d\omega = (\partial + \overline{\partial})\partial\overline{\partial}u = \partial\overline{\partial}\partial\overline{\partial}u = -\partial\overline{\partial}\partial\overline{\partial}u = 0 \). A metric is said to be an Einstein metric if its Ricci curvature tensor is a constant multiple of the metric tensor.

Curvature tensors are really defined in terms of connections and are in some sense independent of the metric. However, given a metric there is a choice of connection so that the connection is said to be compatible with the metric. In the case of a complex manifold there is also the concept of a connection being compatible with the complex structure. It is known that on a complex manifold with Hermitian metric there is a unique connection which is compatible with both the metric and the complex structure. With this choice of connection the Ricci curvature tensor is given by

\[
-\sum_{j,k=1}^{n} \frac{\partial^2}{\partial z_j \partial \overline{z}_k} \log \left( \det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) \right) \right) \, dz_j \otimes d\overline{z}_k.
\]

A good reference for this is Kobayashi’s book [21]. We see that a plurisubharmonic solution of

\[
\left\{ \begin{array}{l}
\det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) \right) = e^{Ku(z)} \text{ in } \Omega \\
\lim_{z \to z_0} u(z) = \infty \text{ for all } z_0 \in \partial \Omega
\end{array} \right.
\]

satisfies

\[
\log \left( \det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) \right) \right) = Ku(z)
\]

and hence gives rise to a metric which is both Kähler and Einstein, a Kähler-Einstein metric. Also since \( u(z) \) tends to infinity as \( z \) tends to boundary points the metric is complete.

6. THE RESULTS IN THE THESIS

We now summarize the results in Paper I, II, III and IV. The papers are intended to be self-contained and therefore there is some overlap in the papers.

6.1. Paper I: Interior regularity of solutions to a complex Monge–Ampère equation. In this paper the author studies the problem

\[
\left\{ \begin{array}{l}
\det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k}(z) \right) = f(z, u(z)) \text{ in } \Omega \\
\lim_{z \to z_0} u(z) = 0 \text{ for all } z_0 \in \partial \Omega
\end{array} \right.
\]

in a bounded hyperconvex domain \( \Omega \) with no smoothness assumption on the boundary. Here \( f \in C^\infty(\overline{\Omega} \times \mathbb{R}) \) is a strictly positive function which is increasing in the second variable. We construct a solution in the following way. Let \( (\Omega_m)_{m=1}^\infty \) be a sequence of strongly pseudoconvex domains which have smooth boundary such that \( \Omega_m \Subset \Omega_{m+1} \) and \( \Omega = \bigcup_{m=1}^\infty \Omega_m \). By Caffarelli’s, Kohn’s, Nirenberg’s and Spruck’s result, Theorem 4.2 (Theorem 1.2 in Paper I), we know that

\[
\left\{ \begin{array}{l}
\det \left( \frac{\partial^2 u_m}{\partial z_j \partial \overline{z}_k}(z) \right) = f(z, u_m(z)) \text{ in } \Omega_m \\
\lim_{z \to z_0} u(z) = 0 \text{ for all } z_0 \in \partial \Omega_m
\end{array} \right.
\]

has a solution \( u_m \in C^\infty(\overline{\Omega}_m) \). The natural candidate for a solution is \( u(z) = \lim_{m \to \infty} u_m(z) \) and we now study its regularity. This is done by a priori estimates. It is not hard to establish that \( u_{m+1}(z) \leq u_m(z) \) and that \( (u_m)_{m=1}^\infty \) is bounded. Proving that \( (u_m)_{m=1}^\infty \) is bounded in \( C^1(K) \) for a compact subset \( K \subset \Omega \) is the next step. This step is the hardest. Once it is taken we can use results by Schulz [24] (Proposition 5.2 in Paper I) and Błocki [6] (Proposition 5.3 in Paper...
I) to show that \((u_m)_{m=1}^\infty\) is bounded in \(C^{2,\alpha}(K)\). One then uses Schauder theory to conclude that \(u \in C^{\infty}(\Omega)\). We therefore need to estimate the \(C^1\)-norm of solutions to the Dirichlet problem. First an extension of an estimate of Blocki is proved (Proposition 3.1 in Paper I). This estimate works in convex domains. We would like to remove the convexity assumption. When we do this the estimate changes in a significant way. Now we get a maximum principle for first derivatives for solutions of the Dirichlet problem (Proposition 4.1 in Paper I). In order to use this estimate we need to know that the first derivatives of solutions are bounded near the boundary. Therefore we introduce the following class of hyperconvex domains.

**Definition 6.1.** We say that a hyperconvex domain \(\Omega\) satisfies the **non-precipitousness condition**, or for short the NP-condition, if we can find a smooth plurisubharmonic function \(\rho\) satisfying
\[
\lim_{z \to z_0 \in \partial \Omega} \rho(z) = 0
\]
and the condition
\[
\sup_{\Omega} \left( \left| \frac{\partial \rho}{\partial x_j} (z) \right| ; z \in \Omega \text{ and } j = 1, \ldots, 2n \right) < \infty.
\]

In this class of domains we can prove the following result (Theorem 5.1 in Paper I).

**Theorem 6.2.** Let \(\Omega\) be a bounded hyperconvex domain in \(C^n\) and \(f \in C^\infty(\overline{\Omega} \times \mathbb{R})\) be a strictly positive function which is increasing in the second variable. If \(\Omega\) satisfies the NP-condition, see Definition 6.1, then the problem
\[
\begin{align*}
\det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} (z) \right) &= f(z, u(z)) \text{ in } \Omega \\
\lim_{z \to z_0} u(z) &= 0 \text{ for all } z_0 \in \partial \Omega
\end{align*}
\]
has a unique smooth strictly plurisubharmonic solution \(u\), which moreover satisfies
\[
\sup_{\Omega} \left( \left| \frac{\partial u}{\partial x_l} (z) \right| ; z \in \Omega, \ l = 1, \ldots, 2n \right) < \infty.
\]

Conversely, if there is a smooth strictly plurisubharmonic solution \(u\) to the problem
\[
\begin{align*}
\det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) &= f(z, u(z)) \text{ in } \Omega \\
\lim_{z \to z_0} u(z) &= 0 \text{ for all } z_0 \in \partial \Omega,
\end{align*}
\]
which satisfies
\[
\sup_{\Omega} \left( \left| \frac{\partial u}{\partial x_l} (z) \right| ; z \in \Omega, \ l = 1, \ldots, 2n \right) < \infty
\]
then \(\Omega\) satisfies the NP-condition.

It is not clear which domains satisfy the NP-condition. One can show that strongly pseudoconvex domains with smooth boundary meets the NP-condition (Proposition 6.2 in Paper I). Also if \(\Omega_1, \ldots, \Omega_N\) satisfy the NP-condition then so does \(\bigcap_{l=1}^N \Omega_l\) (Proposition 6.1 in Paper I). Then we show that the bidisk \(D^2 = \{ z \in \mathbb{C}^2 ; |z_j| < 1, j = 1, 2 \}\) does not satisfy the NP-condition (Example 6.3 in Paper I). This example indicates that if the boundary is to flat then it might be problematic to construct bounded plurisubharmonic exhaustion functions of the type required in Definition 6.1. In Paper IV we give more examples of this phenomenon.
6.2. Paper II: The blow-up rate of solutions to boundary blow-up problems for the complex Monge–Ampère operator (joint work with J. Matero). Here the author in collaboration with Jerk Matero studies the boundary behavior of solutions to boundary blow-up problems of the type

\[
\begin{cases}
\det \left( \frac{\partial^2 u}{\partial z \partial \bar{z}} \right) = f(z, u(z)) \text{ in } \Omega \\
\lim_{z \to z_0} u(z) = \infty \text{ for all } z_0 \in \partial \Omega.
\end{cases}
\]

Here \( \Omega \) is a strongly pseudoconvex domain with smooth boundary and \( f \in C^\infty(\overline{\Omega} \times \mathbb{R}) \) which is strictly positive, increasing in the second variable and satisfies the following three conditions:

A: There exist functions \( h \in C^\infty(\overline{\Omega}) \) and \( f_1 \in C^\infty(\mathbb{R}) \) and two strictly positive constants \( c_1 \) and \( c_2 \) such that

\[\lim_{t \to \infty} \frac{f(z, t)}{f_1(t)} = h(z)\]

uniformly in \( \Omega \) and \( c_1 f_1(t) \leq f(z, t) \leq c_2 f_1(t) \) for all \((z, t) \in \Omega \times \mathbb{R}\).

B: The function \( f_1 \) is strictly positive and increasing.

C: The function

\[\Psi_n(a) = \int_a^\infty ((n + 1)F(y))^{-\frac{1}{n+1}} dy\]

exists for \( a > 0 \), where \( F'(s) = f_1(s) \) and \( F(0) = 0 \).

In [12] Cheng and Yau studies this problem with \( f(z, u(z)) = k(z) \exp(Ku(z)) \) where \( K > 0 \) and \( k \) is a strictly positive smooth function. In Section 5.2 we described how a solution of this problem gives a Kähler-Einstein metric on \( \Omega \). They showed that the boundary blow-up problem has a unique smooth plurisubharmonic solution in this case. In Paper II we mainly study how solutions, if they exist, behave near the boundary. We give one existence and regularity result, see Proposition 6.3 below (Proposition 2.1 in Paper II). First we quickly describe how the weak solutions are constructed. Using Theorem 4.2 we know that there are smooth plurisubharmonic functions \( u_N \) so that

\[
\begin{cases}
\det \left( \frac{\partial^2 u_N}{\partial z \partial \bar{z}} \right) = f(z, u_N(z)) \text{ in } \Omega \\
\lim_{z \to z_0} u_N(z) = N \text{ for all } z_0 \in \partial \Omega.
\end{cases}
\]

The function \( u(z) = \lim_{N \to \infty} u_N(z) \) is a weak solution of the problem

\[
\begin{cases}
\det \left( \frac{\partial^2 u}{\partial z \partial \bar{z}} \right) = f(z, u(z)) \text{ in } \Omega \\
\lim_{z \to z_0} u(z) = \infty \text{ for all } z_0 \in \partial \Omega.
\end{cases}
\]

We must then study the regularity of \( u \). In balls in \( \mathbb{C}^n \) we show

**Proposition 6.3.** Let \( R, c_1 \) and \( c_2 \) be strictly positive real numbers such that \( c_1 \leq c_2 \). Assume that \( k: [0, R] \to [c_1, c_2] \) is a smooth function such that \( k^{(2l+1)}(0) = 0 \) for all \( l \in \mathbb{N} \). Suppose that \( f_1 \in C^\infty(\mathbb{R}) \) satisfies assumptions B and C. Then

\[
\begin{cases}
\det \left( \frac{\partial^2 v}{\partial z \partial \bar{z}} \right) = k(|z|)f_1(v(z)) \text{ in } B_R(0) \\
\lim_{|z| \to R} v(z) = \infty
\end{cases}
\]

has a smooth solution. Moreover the solution is radial.

**Remark 6.4.** We require that derivatives of odd order vanishes at 0 because we want the function \( k(|z|) \) to be smooth at the origin.
In proving this proposition we make use of the rotational symmetry of the ball, the Monge–Ampère operator and the right-hand side. In general the a priori estimate needed to study the regularity of \( u \) is hard to establish and is not pursued in Paper II. Some a priori estimates are given in Paper III. In Paper III we also study another issue that is neglected in Paper II, namely uniqueness. It is clear that \( u \) is the smallest solution but it is not clear whether or not there also could be larger solutions.

We then study the boundary behavior of the solutions. We give an estimate, which we call a blow-up estimate below, in terms of \( \Psi_n \), the distance of \( z \) to the boundary \( d_\Omega(z) \) and the Levi form of \( \partial \Omega \). We introduce some notation (Definition 3.7 in Paper II).

**Definition 6.5.** Assume that \( \Omega = \{ z \in \mathbb{C}^n ; \rho(z) < 0 \} \) where \( \rho \in C^\infty(\overline{\Omega}) \). For \( z_0 \in \partial \Omega \) suppose that \( |\nabla \rho(z_0)| = 1 \). Then \( \Pi(z_0) \) is the product of the eigenvalues of the form

\[
\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}(z_0) \, dz_j \wedge d\overline{z}_k
\]

restricted to the vector space \( \{ w \in \mathbb{C}^n ; \sum_{j=1}^{n} \frac{\partial \rho}{\partial z_j}(z_0) w_j = 0 \} \).

We can now state the main result in Paper II (Theorem 3.8 in Paper II).

**Theorem 6.6.** Let \( \Omega \) be a bounded, strongly pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary. Let \( f \in C^\infty(\overline{\Omega} \times \mathbb{R}) \) be a strictly positive function which is increasing in the second variable and satisfies assumptions A, B and C. For boundary points \( z_0 \in \partial \Omega \) let \( \Pi(z_0) \) be defined as in Definition 6.5. Then \( u \), any solution to the problem

\[
\begin{cases}
\text{det} \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \right) = f(z, u(z)) \quad \text{in } \Omega \\
\lim_{z \to z_0} u(z) = \infty \quad \text{for all } z_0 \in \partial \Omega
\end{cases}
\]

meets the estimate

\[
\lim_{z \to z_0} \frac{\Psi_n(u(z))}{d_\Omega(z)} = 4^{\frac{n-1}{n+1}} h(z_0)^{\frac{1}{n+1}} \Pi(z_0)^{-\frac{1}{n+1}},
\]

where \( z_0 \in \partial \Omega \).

This result is first established in balls. We then pass to general strongly pseudoconvex domains by approximation of the domain by balls from the inside and outside. After a holomorphic coordinate change we can approximate \( \partial \Omega \) locally by boundaries of two balls whose radii is almost the same. The comparison principles Lemma 4.8, Lemma 4.9 and Corollary 4.10 are global in nature and we cannot apply them to our local problem directly. This is handled by adding and subtracting affine functions. The affine function do not change the blow-up rate of the solutions and this lets us extend the blow-up estimate to general strongly pseudoconvex domains with smooth boundary.

Paper II ends with an application of Theorem 6.6 to describe the boundary behavior of the Bergman kernel (Theorem 4.1 in Paper II). This result is well-known and was proven by Hörmander in [15]. In fact Hörmander’s result also holds for weighted Bergman kernels.

**Theorem 6.7.** Let \( \Omega \) be a bounded strongly pseudoconvex domain with smooth boundary. Let \( K_\Omega(z, w) \) be the Bergman kernel of \( \Omega \). For boundary points \( z_0 \) let
for all $z_0 \in \partial \Omega$.

6.3. Paper III: Regularity and uniqueness of solutions to boundary blow-up problems for the complex Monge–Ampère operator. In this paper we study regularity of solutions to boundary blow-up problems in strongly pseudo-convex domains with smooth boundary of the type studied in Paper II. We also show a uniqueness result. The regularity result is obtained on the assumption that the right-hand side is independent of the $z$-variable, that is $f(z, u(z)) = f(u(z))$. We also require that $f$ satisfies the technical condition

$$\frac{n-1}{n+1} \leq \frac{F(t)f'(t)}{f(t)^2}$$

where $F'(t) = f(t)$ and $F(0) = 0$. In order for the argument to work we have modify the construction of the solution a little. We begin by constructing a plurisubharmonic function which is a subsolution and have the right boundary behavior. Let us call this function $\varphi$ and put $\Omega_N = \{z \in \Omega; \varphi(z) < N\}$. Let $u_N$ be the solution of

$$\begin{cases}
\det\left(\frac{\partial^2 u_N}{\partial z_j \partial \bar{z}_k}(z)\right) = f(u_N(z)) \text{ in } \Omega_N \\
\lim_{z \to z_0} u_N(z) = N \text{ for all } z_0 \in \partial \Omega_N.
\end{cases}$$

We have that $\varphi \leq u_N$ in $\Omega_N$ and can show that $u_N \leq u_{N+1}$ in $\Omega_N$. One can show that $(u_N)_{N=1}^\infty$ is bounded from above on compact subset of $\Omega$. Put $u(z) = \lim_{N \to \infty} u_N(z)$. In order to estimate the gradient of $u_N$ we study the function $v_N(z) = |\nabla u_N(z)|^2(g^{-1})(u_N(z))^2$. Here

$$g^{-1}(t) = -\Psi_N(t) = -\int_t^{\infty} ((n+1)F(y))^{-\frac{1}{n-1}}dy.$$

We see that $v_N = |\nabla u_N|^2(g^{-1})(u_N)^2 \leq |\nabla \varphi|^2(g^{-1})(\varphi)^2$ on $\partial \Omega_N$. One now checks that $|\nabla \varphi|^2(g^{-1})(\varphi)^2$ is bounded on $\Omega$. If we can show that $v_N$ does not have interior maximum points we get as a consequence that $|\nabla u_N| = C(g^{-1})(u_N)^{-1}$ on $\Omega_N$. Since $u_N$ is uniformly bounded on compact subsets of $\Omega$ we see that $|\nabla u_N|$ also is. In order to show that $v_N$ has no interior maximum we assume there is an interior maximum point. One can show that at a critical point we have

$$|\nabla u_N|^2 < \frac{(n+1)F(u_N)}{2f(u_N)} \sum_{j=1}^{n} \frac{\partial^2 u_N}{\partial z_j \partial \bar{z}_j}.$$

At a maximum the inequality

$$\sum_{j=1}^{n} \frac{\partial^2 u_N}{\partial z_j \partial \bar{z}_j} \leq |\nabla u_N|^2 \left(\frac{f(u_N)}{F(u_N)} - \frac{f'(u_N)}{f(u_N)}\right)$$

holds and hence

$$|\nabla u_N|^2 < \frac{n+1}{2} \frac{F(u_N)}{f(u_N)} \sum_{j=1}^{n} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_j} \leq \frac{n+1}{2} \left(1 - \frac{F(u_N)f'(u_N)}{f(u_N)^2}\right) |\nabla u_N|^2$$

which gives a contradiction if

$$\frac{n+1}{2} \left(1 - \frac{F(u_N)f'(u_N)}{f(u_N)^2}\right) \leq 1,$$
that is if
\[ \frac{n - 1}{n + 1} \leq \frac{F(u_N) f'(u_N)}{f(u_N)^2}. \]

Using the Arzela-Ascoli Theorem we get (Theorem 3.2 in Paper III)

**Theorem 6.8.** Let \( \Omega \) be a bounded strongly pseudoconvex domain in \( \mathbb{C}^n \) with smooth boundary. Suppose that \( f \) satisfies \( \text{B, C} \) and
\[ \frac{n - 1}{n + 1} \leq \frac{f'(t) F(t)}{f(t)^2}. \]

Then the problem
\[
\begin{cases}
(dd^c u)^n = f(u(z)), & z \in \Omega \\
\lim_{z \to z_0} u(z) = \infty \text{ for all } z_0 \in \partial \Omega
\end{cases}
\]
has a solution \( u \in C^\alpha(\Omega) \) for any \( \alpha \) which satisfies \( 0 \leq \alpha < 1 \).

Note that \( f(t) = e^{Kt}, \ K > 0, \) satisfies all the conditions in the theorem. Also note that \( f(t) = t^\gamma, \ \gamma \geq (n - 1)/2 \) meets
\[ \frac{n - 1}{n + 1} \leq \frac{f'(t) F(t)}{f(t)^2}. \]

If we modify this \( f \) for \( t < 1 \) so that it satisfies \( \text{B and C} \) we see that the theorem can be applied to right-hand sides of other growth than exponential growth.

Next we turn to the question of uniqueness. Here we have right-hand sides that may depend on the \( z \)-variable. That is, \( f \in C^\infty(\overline{\Omega} \times \mathbb{R}) \) is a smooth strictly positive function which is increasing in the second variable and meets \( \text{A, B and C} \). We prove (Proposition 4.1 in Paper III)

**Proposition 6.9.** Let \( \Omega \) be a bounded strongly pseudoconvex domain with smooth boundary and assume that \( f \in C^\infty(\overline{\Omega} \times \mathbb{R}) \) is a strictly positive function, increasing in the second variable which satisfies \( \text{A, B and C} \). Assume also that
\[ \frac{\Psi_n(t)}{\Psi'_n(t)} \]
is bounded for large \( t \). If \( u \) and \( v \) are plurisubharmonic solutions of
\[
\begin{cases}
\det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \right) = f(z, u(z)) \text{ in } \Omega \\
\lim_{z \to z_0} u(z) = \infty \text{ for all } z_0 \in \partial \Omega
\end{cases}
\]
then \( u \equiv v \).

The condition that \( \Psi_n(t)/\Psi'_n(t) \) is for large \( t \) is fulfilled when \( f_1 \) has exponential growth but not when the growth is only polynomial. We have to impose such a condition to conclude that the difference \( u(z) - v(z) \) tends to zero as \( z \) tends to the boundary. If \( u > v \) at some interior point the difference \( u(z) - v(z) \) has an interior maximum. At this point
\[ \det \left( \frac{\partial^2 u}{\partial z_j \partial \overline{z}_k} \right) \leq \det \left( \frac{\partial^2 v}{\partial z_j \partial \overline{z}_k} \right) \]
which is a contradiction since \( f \) is increasing in the second variable.
6.4. Paper IV: On the behavior of strictly plurisubharmonic functions near real hypersurfaces. In this paper we prove a result which provides us with plenty of examples of domains which do not satisfy the NP-condition. The main result in this paper is formulated for real hypersurfaces rather than domains and therefore we need to introduce a local version of the NP-condition (Definition 2.1 in Paper IV).

Definition 6.10. Let $M$ be a smooth real hypersurface, $p \in M$ and $\rho$ be a defining function for $M$. Assume that the Levi form is positive semidefinite on $T^C_q(M)$ for all $q \in M$. We say that $p \in M$ satisfies the local non-precipitousness condition, or for short the local NP-condition, if we can find an open neighborhood $U$ of $p$ and a smooth plurisubharmonic function $u$ defined on $M^c = \{z \in U; \rho(z) < 0\}$ satisfying $\lim_{z \to z_0 \in M} u(z) = 0$ and

$$\det \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) \geq 1,$$

and the condition

$$\sup \left( \left| \frac{\partial u}{\partial x_j}(z) \right| ; z \in \Omega \text{ and } j = 1, \ldots, 2n \right) < \infty.$$

First we investigate the behavior of first derivatives of defining functions of ellipsoids. Let $a = (a_1, \ldots, a_n) \in \mathbb{R}^n$, $a_j > 0$, and put

$$\Omega_a = \left\{ z \in \mathbb{C}^n; \sum_{j=1}^n |z_j|^2/a_j^2 < 1 \right\}.$$

A defining function for $\Omega_a$ is $\rho_a = (\sum_{j=1}^n |z_j|^2/a_j^2) - 1$. The normalization $\tilde{\rho}_a = \left( \prod_{j=1}^n a_j^{2/n} \right) \rho_a$ meets

$$\det \left( \frac{\partial^2 \tilde{\rho}_a}{\partial z_j \partial \bar{z}_k} \right) = 1$$

and

$$\frac{\partial \tilde{\rho}_a}{\partial z_j} = \prod_{l=1}^n a_l^{2/n} a_j \left( \frac{z_j}{a_j} \right) \quad \text{and} \quad \frac{\partial \tilde{\rho}_a}{\partial \bar{z}_j} = \prod_{l=1}^n a_l^{2/n} a_j \left( \frac{\bar{z}_j}{a_j} \right).$$

In particular we see that

$$\frac{\partial \tilde{\rho}_a}{\partial z_1}(a_1, 0, \ldots, 0) = \frac{\partial \tilde{\rho}_a}{\partial \bar{z}_1}(a_1, 0, \ldots, 0) = a_1^{(2/n)-1} \prod_{j=2}^n a_j^{2/n}.$$

Thus we see that the normal derivative at this boundary point depends on the lengths of the semi-axes. This is then used to show that the polydisk $D^n$ in $\mathbb{C}^n$, $n \geq 3$, does not satisfy the NP-condition. We then use the interplay between $a_1$ and $a_2$, dots, $a_n$ to describe the behavior of strictly plurisubharmonic functions near some real hypersurfaces which do not necessarily contain complex lines. A complex curve is a holomorphic mapping $\gamma$ from an open neighborhood of $0 \in \mathbb{C}$ to $\mathbb{C}^n$ such that $\partial \gamma \neq 0$. The order of contact between $M$ and $\gamma$ at $p \in M$ is $l$ if $d_M(q) \leq C d(p, q)^l$ near $p$ and $l$ is the largest such number. Here $d_M(q)$ is the distance of $q$ to $M$ and $d(p, q)$ is the distance between $p$ and $q$. We then prove the following theorem (Theorem 2.2 in Paper IV).
Theorem 6.11. Let $M$ be a hypersurface in $\mathbb{C}^n$ which is pseudoconvex at $p \in M$. Assume that there are complex curves $\gamma_2, \ldots, \gamma_n$ whose order of contact with $M$ is $2l_2, \ldots, 2l_n$ respectively and that $\gamma'_2, \ldots, \gamma'_n$ are linearly independent. Suppose that

$$\frac{1}{n-1} \sum_{j=2}^{n} \frac{1}{l_j} < \frac{n-2}{n-1}$$

Then $p \in M$ does not satisfy the local NP-condition, see Definition 6.10.

In order to use this theorem it is nice to note that it is not necessary to find complex curves which have optimal order of contact. One should also note that the inequality

$$\frac{1}{n-1} \sum_{j=2}^{n} \frac{1}{l_j} < \frac{n-2}{n-1}$$

is sufficient but not necessary for the local NP-condition to fail. This is because the bidisk $D^2$ does not satisfy the NP-condition and in this case the inequality is not met. In fact the inequality is not met for any hypersurface in $\mathbb{C}^2$.

6.5. Discussion. Here we discuss some problems that might be interesting to consider. First we discuss the blow-up problem treated in Paper II and III. The blow-up estimates in Paper II gives a description of the boundary behavior which is good. However if one wants to study the boundary behavior of the Bergman metric one would need to know the asymptotic behavior of second derivatives of solutions to the blow-up problem. The result on the regularity of solutions in Paper III is probably not optimal. However the author has been unable to prove a priori estimates for second derivatives. On the other hand no counterexample is known to the author so high order regularity seems open. Establishing the estimates needed for high order regularity is probably, if at all possible, very technical.

Next let us consider the problem dealt with in Paper I and IV. It would be nice to have a characterization of the NP-condition. For hyperconvex domains with smooth boundary the author believes that a domain which satisfies the NP-condition is strongly pseudoconvex. The author feels that in order to understand the NP-condition better one needs to gain insight into the situation in $\mathbb{C}^2$. One might then use this insight to understand what happens when the inequality

$$\frac{1}{n-1} \sum_{j=2}^{n} \frac{1}{l_j} < \frac{n-2}{n-1}$$

fails.

Acknowledgements

I am greatly indebted to my advisor Professor Christer Kiselman for all the help and support he has provided during my work on this thesis. I would also like to thank him for the opportunities to travel that he has given me.

Next I would like thank Jerk Matero for introducing me to boundary blow-up problems. It has been a very good experience to have him as a co-worker. Also Tobias Ekholm has been a great help for me, especially when I was writing the last paper in this thesis.

I am grateful to the whole staff at the Department of Mathematics at Uppsala University. You have all contributed to making my time at the department a
pleasant and educational experience. Let me mention Sara Maad and Olov Viirmann who have been studying alongside me during my time at the department. I thank you both for all your support and encouragement.

Finally I would like to thank my family and friends outside of mathematics. You have been very important for this thesis by helping me relax and making my spare time a lot of fun.

References


DEPARTMENT OF MATHEMATICS, UPPSALA UNIVERSITY, P. O. BOX 480, SE–75106 UPPSALA, SWEDEN

E-mail address: bjorni@math.uu.se