Interpolation of Hilbert spaces

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Abstract

(i) We prove that intermediate Banach spaces \( A, B \) with respect to arbitrary Hilbert couples \( H, K \) are exact interpolation iff they are exact \( K \)-monotonic, i.e. the condition \( f^0 \in A \) and the inequality \( K(t, g^0, K) \leq K(t, f^0, H), t > 0 \) imply \( g^0 \in B \) and \( \|g^0\|_B \leq \|f^0\|_A \) (\( K \) is Peetre’s \( K \)-functional). It is well-known that this property is implied by the following: for each \( \varrho > 1 \) there exists an operator \( T : H \rightarrow K \) such that \( Tf^0 = g^0 \), and \( K(t, T f; K) \leq \varrho K(t, f; H), f \in H_0 + H_1, t > 0. \) Verifying the latter property, it suffices to consider the “diagonal” case where \( H = K \) is finite-dimensional. In this case, we construct the relevant operators by a method which allows us to explicitly calculate them. In the strongest form of the theorem, it is shown that the statement remains valid when substituting \( \varrho = 1 \).

(ii) A new proof is given to a theorem of W. F. Donoghue which characterizes certain classes of functions whose domain of definition are finite sets, and which are subject to certain matrix inequalities. The result generalizes the classical L"owner theorem on monotone matrix functions, and also yields some information with respect to the finer study of monotone functions of finite order.

(iii) It is shown that with respect to a positive concave function \( \psi \) there exists a function \( h \), positive and regular on \( \mathbb{R}^+ \) and admitting of analytic continuation to the upper half-plane and having positive imaginary part there, such that \( h \leq \psi \leq 2h. \) This fact is closely related to a theorem of Foiaş, Ong and Rosenthal, which states that regardless of the choice of a concave function \( \psi \), and a weight \( \lambda \), the weighted \( \ell_2 \)-space \( \ell_2(\psi(\lambda)) \) is \( c \)-interpolation with respect to the couple \((\ell_2, \ell_2(\lambda))\), where we have \( c \leq \sqrt{2} \) for the best \( c. \) It turns out that \( c = \sqrt{2} \) is best possible in this theorem; a fact which is implicit in the work of G. Sparr.

(iv) We give a new proof and new interpretation (based on the work (ii) above) of Donoghue’s interpolation theorem; for an intermediate Hilbert space \( H_+ \) to be exact interpolation with respect to a regular Hilbert couple \( H \) it is necessary and sufficient that the norm in \( H_+ \) be representable in the form \( \|f\|_+ = (\int_{[0,\infty]} (1+t^{-1})K_2(t, f; H)^2 d\rho(t))^{1/2} \) with some positive Radon measure \( \rho \) on the compactified half-line \([0, \infty].\)

(v) The theorem of W. F. Donoghue (item (ii) above) is extended to interpolation of tensor products. Our result is related to A. Korányi’s work on monotone matrix functions of several variables.

Keywords and phrases: Interpolation, Hilbert space, \( K \)-functional, \( K \)-monotonicity, Calderón couple, Pick function, matrix monotone function, \( K_2 \)-functional, L"owner’s and Donoghue’s theorems.

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Overview of thesis

This thesis consists of six chapters, whereof the first chapter is introductory and the remaining five are individual papers,

A. *The Calderón problem for Hilbert couples*,
B. *On some classes of functions related to the interpolation of Hilbert spaces*,
C. *Note on a theorem of Sparr*,
D. *A new proof of Donoghue’s interpolation theorem*,
E. *Interpolation functions of several matrix variables*.

Here “A” is to be regarded as the central paper whereas the others contain natural extensions and applications of the theory developed therein, with the exception of “C”, which is independent of the other chapters. The papers have been submitted for publication.
CHAPTER I

Introduction

In this introductory chapter we provide a general background and summarize some of our results, sometimes giving additional details and slight variations which are not mentioned in the subsequent chapters. (We have chosen here to restrict our discussion to a special case; most of our theorems stated here shall be given more general formulations in the later chapters.)

The thesis can roughly be described as the study of two seemingly unrelated notions of monotonicity and their relations, namely (i) “$K$-monotonicity of spaces” and (ii) “matrix monotonicity of functions”.

We shall start by addressing (i) above, so let us proceed with some definitions from “finite-dimensional interpolation theory”. (A good introduction to interpolation theory is given in Bergh–Löfström [3].)

Let there be given two $n$-dimensional normed spaces over $\mathbb{R}$ (where $n \in \mathbb{N}$)

$$\mathcal{A}_i = (\mathcal{V}, \| \cdot \|_i), \quad i = 0, 1,$$

and denote by $\mathcal{A} = (\mathcal{A}_0, \mathcal{A}_1)$ the corresponding normed couple. A third normed space

$$\mathcal{A} = (\mathcal{V}, \| \cdot \|_A)$$

is by definition exact interpolation with respect to $\mathcal{A}$ iff

$$\|T\|_{\mathcal{L}(\mathcal{A})} \leq \max(\|T\|_{\mathcal{L}(\mathcal{A}_0)}, \|T\|_{\mathcal{L}(\mathcal{A}_1)}), \quad T \in \mathcal{L}(\mathcal{A}_0),$$

where we have used the following convention (operator norms)

$$\|T\|_{\mathcal{L}(\mathcal{X})} = \sup \|Tf\|_{\mathcal{X}} / \|f\|_{\mathcal{X}}.$$

Our first and foremost topic is the following: to characterize all exact interpolation spaces with respect to a couple $\mathcal{H}$ of euclidean spaces.

An important notion of interpolation theory is that of Peetre’s $K$-functional which is defined as follows (relative to the couple $\mathcal{A}$ and an element $f \in \mathcal{V}$)

$$K(t) = K(t, f) = K(t, f; \mathcal{A}_0, \mathcal{A}_1) = \inf_{f = f_0 + f_1} (\|f_0\|_0 + t\|f_1\|_1), \quad t > 0.$$

It is easy to see that $K(t)$ is increasing and $K(t) \leq \max(1, t/s)K(s)$. We have the following definition.

**Definition 1.** The space $\mathcal{A}$ is called exact $K$-monotonic relative to $\mathcal{A}$ iff for all $f, g \in \mathcal{V}$

$$K(t; g; \mathcal{A}) \leq K(t, f; \mathcal{A}), \quad t > 0$$

implies

$$\|g\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}}.$$
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We have now a well-known, basic lemma.

**Lemma 1.** Exact $K$-monotonicity is stronger than exact interpolation.

**Proof.** Let $\mathcal{A}$ be exact $K$-monotonic, $T \in \mathcal{L}(\mathcal{A}_0)$ and $f \in \mathcal{V}$. Then by the definition of the $K$-functional

$$K(t, Tf) = \inf_{f_0 + f_1} (\|Tf_0\|_0 + t\|Tf_1\|_1) \leq \inf_{f_0 + f_1} (\|T\|_{\mathcal{L}(\mathcal{A}_0)}\|f_0\|_1 + t\|T\|_{\mathcal{L}(\mathcal{A}_1)}\|f_1\|_1) \leq \max(\|T\|_{\mathcal{L}(\mathcal{A}_0)}, \|T\|_{\mathcal{L}(\mathcal{A}_1)}) K(t, f), \quad t > 0.$$

Hence by (3) and homogeneity of the norms

$$\|Tf\|_{\mathcal{A}} \leq \max(\|T\|_{\mathcal{L}(\mathcal{A}_0)}, \|T\|_{\mathcal{L}(\mathcal{A}_1)}) \|f\|_{\mathcal{A}}, \quad f \in \mathcal{V},$$

viz.

$$\|T\|_{\mathcal{L}(\mathcal{A})} \leq \max(\|T\|_{\mathcal{L}(\mathcal{A}_0)}, \|T\|_{\mathcal{L}(\mathcal{A}_1)}) \text{ i.e. } \mathcal{A} \text{ is exact interpolation.}$$

The two notions “exact interpolation” and “exact $K$-monotonicity” are in general different (cf. [3], Exercise 5.7.14). We have the following definition.

**Definition 2.** The couple $\overline{\mathcal{A}}$ is called

(i) **Calderón** iff exact interpolation is equivalent to exact $K$-monotonicity relative to $\overline{\mathcal{A}}$,

(ii) **$C$-monotonic** iff the property that $\mathcal{A}$ is exact interpolation relative to $\overline{\mathcal{A}}$ together with the condition (2) implies $\|g\|_{\mathcal{A}} \leq C\|f\|_{\mathcal{A}}$ ($C$ here is independent of $f$ and $g$.)

Let us review on some well-known theorems about couples of $\ell^n_p$’s and weighted $\ell^n_p(\lambda)$’s. (Here $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is some increasing sequence of positive numbers and

$$\|f\|_{\ell^n_p} = \left(\sum_{i=1}^{n} |f_i|^p\right)^{1/p}, \quad \|f\|_{\ell^n_p(\lambda)} = \left(\sum_{i=1}^{n} \lambda_i |f_i|^p\right)^{1/p}, \quad f \in \mathcal{V},$$

if $p < \infty$. The usual conventions apply for $p = \infty$.)

- The first instance of a Calderón couple, the couple $(\ell^n_1, \ell^n_\infty)$ was discovered by A. P. Calderón [5] in 1966 and (independently) by B. S. Mityagin [21] in 1965. Hence our use of the term “Calderón couple” may seem unfair, and indeed some authors prefer the term Calderón–Mityagin couple.
- Another early case of Calderón couples is the couple $(\ell^n_\infty(\lambda_0), \ell^n_\infty(\lambda_1))$ (where $\lambda_i$ are arbitrary weights, $i = 0, 1$). This result of folk-lore character was well-known in the mid 1960’s.
- An important case of Calderón couples was found by A. A. Sedaev and E. M. Semenov in 1971 [26], namely the pair $(\ell^n_p(\lambda_0), \ell^n_p(\lambda_1))$. For a new proof of this theorem along with a discussion of its more recent applications, see [8].
- In 1973 Sedaev [25] proved that the pair $(\ell^n_p(\lambda_0), \ell^n_p(\lambda_1))$ is $2^{1/p'}$-monotonic for all $1 \leq p \leq \infty$ ($\frac{1}{p} + \frac{1}{p'} = 1$). In particular, it yields that all euclidean couples $\overline{\mathcal{H}}$ are $\sqrt{2}$-monotonic.
A very general theorem was given by G. Sparr [27] in 1974: for any \( p_0, p_1 \) we have that \( (\ell_{p_0}^n(\lambda_0), \ell_{p_1}^n(\lambda_1)) \) is \( C(p_0, p_1) \)-monotonic, where the best \( C \) fulfills \( C \leq 2 \) for all \( p_0, p_1 \) and \( C(1, p) > 1 \) for \( 1 < p < \infty \). About the same results were obtained independently by M. Cwikel [7]. Alternative approaches have later been provided by Dmitriev [9] and Arazy–Cwikel [1].

Note that the last example by Sparr refutes the possibility of a general theorem stating that all couples of weighted \( \ell_p^p \)'s are Calderón. We have however the following positive result.

**Theorem 1.** Every euclidean couple \( \overline{\mathcal{H}} \) is Calderón.

We shall give a couple of different (well-known) interpretations of the property that a couple be Calderón. Our first version is formulated in terms of operators. More specifically, it shows that a Calderón couple is characterized by the property that the “unit ball” \( L_1(\overline{\mathcal{A}}) \) contains a very rich supply of elements, where the latter is defined as the following set:

\[
T \in L_1(\overline{\mathcal{A}}) \quad \text{iff} \quad T \in L(A_0), \quad \max(\|T\|_{L(A_0)}, \|T\|_{L(A_1)}) \leq 1.
\]

**Theorem 2.** For a couple \( \overline{\mathcal{A}} \) to be Calderón, it is necessary and sufficient that the following property holds: for all \( f, g \in \mathcal{V} \) the condition (2) implies the existence of an operator \( T \in L(A_0) \) such that

\[
T \in L_1(\overline{\mathcal{A}}) \quad \text{and} \quad Tf = g.
\]

**Proof.** The sufficiency is proved as in Lemma 1 wherefore we only prove the necessity. Let \( \overline{\mathcal{A}} \) be Calderón and pick \( f, g \) fulfilling (2). Let a normed space \( \mathcal{A} = (\mathcal{V}, \| \cdot \|_{\mathcal{A}}) \) be defined by

\[
\|h\|_{\mathcal{A}} = \inf_{Tf = h} \max(\|T\|_{L(A_0)}, \|T\|_{L(A_1)}), \quad h \in \mathcal{V}.
\]

(\( \mathcal{A} \) is an “orbit space” in the sense of Aronzjajn and Gagliardo, cf. [2].) It is easy to see that \( \mathcal{A} \) is exact interpolation with respect to \( \overline{\mathcal{A}} \). Since \( \overline{\mathcal{A}} \) is a Calderón couple, it yields that \( \mathcal{A} \) is exact \( K \)-monotonic, whence (2) yields \( \|g\|_{\mathcal{A}} \leq \|f\|_{\mathcal{A}} = 1 \). By definition of the space \( \mathcal{A} \) it yields that there exists for every \( \rho > 1 \) a matrix \( T_\rho \in L(A_0) \) such that

\[
T_\rho f = g \quad \text{and} \quad \max(\|T_\rho\|_{L(A_0)}, \|T_\rho\|_{L(A_1)}) < \rho.
\]

Applying this construction for each \( \rho > 1 \) it yields a net \( (T_\rho)_{\rho > 1} \) which has a cluster point \( T \) as \( \rho \downarrow 1 \) (use compactness of the unit ball of \( L(A_0) \)). It is clear that \( T \) satisfies (4) since every \( T_\rho \) does so.

Below is discussed a different, more geometric interpretation of the notion “Calderón couples”, involving the convex (Legendre-) duality relative to certain functionals. (Regarding these matters we have essentially followed Peetre–Sparr [24], sect. 3 and Sparr [27], pp. 236-237.)

The **Gagliardo indicator** \( \Gamma f \) of an element \( f \in \mathcal{V} \) with respect to the couple \( (A_0, A_1) \) is defined as the following plane convex set

\[
\Gamma f = \{(x_0, x_1) \in \mathbb{R}^2 : \exists f_0, f_1 \in \mathcal{V}; f = f_0 + f_1, x_i \geq \|f_i\|_i, i = 0, 1\}.
\]

It is easy to see that

\[
K(t, f) = \inf_{(x_0, x_1) \in \Gamma f} (x_0 + tx_1) = \inf_{(x_0, x_1) \in \partial \Gamma f} (x_0 + tx_1).
\]
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The following is immediate from Theorem 1.

**Theorem 3.** For a couple $\mathcal{A}$ to be Calderón it is necessary and sufficient that the following holds with respect to spaces $\mathcal{A}$: if $\mathcal{A}$ is exact interpolation, then for all $f, g \in \mathcal{V}$

$$
\Gamma g \subset \Gamma f \implies \|g\|_\mathcal{A} \leq \|f\|_\mathcal{A}.
$$

The $E$-functional (relative to the couple $\mathcal{A}$) is defined as follows

$$
E(t) = E(t, f) = E(t, f; \mathcal{A}_0, \mathcal{A}_1) = \inf_{\|f_0\|_{\mathcal{A}_0} \leq t} \|f - f_0\|_1.
$$

It is easy to see that the intersection of $\Gamma f$ with the line $x_0 = t$ is a halfline with end-point $(t, E(t))$, whence $E(t)$ may be regarded as the boundary curve of $\Gamma f$. In particular, the $E$-functional is convex, decreasing and

$$
K(t) = \inf_{s > 0} (s + tE(s)),
$$

which formula shows that $K$ is a kind of Legendre transform of $E$.

In the case when $\mathcal{A} = \mathcal{H}$ is euclidean it is particularly rewarding besides $K$, $E$ to study the functionals $K_2$, $E_2$ defined by

$$
K_2(t, f) = \inf_{f = f_0 + f_1} \left( \|f_0\|_0^2 + t\|f_1\|_1^2 \right)^{1/2}, \quad E_2(t, f) = \inf_{\|f_0\|_{\mathcal{A}_0} \leq t} \|f - f_0\|_1 = E(t^{1/2}, f).
$$

In terms of the Gagliardo indicator we have

$$
K_2(t)^2 = \inf_{(x_0, x_1) \in \Gamma f} (x_0^2 + tx_1^2) = \inf_{s > 0} (s + tE_2(s)^2),
$$

i.e. $K_2^2$ is the Legendre transform of $E_2^2$. The decreasingness and convexity of $E(t)$ clearly implies the convexity of $E_2(t)^2 = E(t^{1/2})^2$, whence making inverse Legendre transformations yields

$$
E(s) = \sup_{t > 0} \left( \frac{K(t)}{t} - \frac{s}{t} \right), \quad E_2(s)^2 = \sup_{t > 0} \left( \frac{K_2(t)^2}{t} - \frac{s}{t} \right).
$$
Under these circumstances we have the following equivalences

\[ K(t, g) \leq K(t, f), \ t > 0 \iff E(s, g) \leq E(s, f), \ s > 0 \]
\[ \iff E_2(s, g) \leq E_2(s, f), \ s > 0 \iff K_2(t, g) \leq K_2(t, f), \ t > 0. \]

As a consequence, we obtain the following.

**Theorem 4.** A couple \( \overline{A} \) is Calderón iff “exact interpolation” (1) is equivalent to the following property (“exact \( K_2 \)-monotonicity”)

\[ K_2(t, g; \overline{A}) \leq K_2(t, f; \overline{A}), \ t > 0 \implies \|g\|_A \leq \|f\|_A, \ f, g \in \mathcal{V}. \]

We note that our proof of Theorem 1 is based on \( K_2 \)-monotonicity rather than \( K \)-monotonicity, depending on the fact that \( K_2 \) admits of convenient calculation in the euclidean case, in a way which we describe below.

Given a euclidean couple \( \mathcal{H}_0 \) the function \( \| \cdot \|^2 \) defines a quadratic form on \( \mathcal{H}_0 \), which we represent as \( \|f\|^2_1 = (Af, f)_0 \) with some positive definite operator \( A \in \mathcal{L}(\mathcal{H}_0) \). Let \( \lambda_i \) be the eigenvalues of \( A \), ordered in the increasing order, and \( e_i \) the corresponding eigenvectors. Putting \( \lambda = (\lambda_i)^n_{i=1} \) and expanding a generic vector \( f \in \mathcal{H}_0 \) in the form \( f = \sum_{i=1}^n f_i e_i \), it yields that the couple \( \mathcal{H} \) is canonically isomorphic to the pair \((\ell^2_n, \ell^0_2(\lambda))\) defined by

\[ \|f\|^2 = \sum_{i=1}^n f_i^2, \ \ \|f\|_1 = \sum_{i=1}^n \lambda_i f_i^2, \ \ f = (f_i)^n_{i=1} \in \ell^2_n, \]
and the operator \( A \) becomes identified with the following matrix

\[ A = \text{diag}(\lambda_i). \]

It is easy to calculate the \( K_2 \)-functional with respect to \((\ell^0_2, \ell^0_2(\lambda))\) (minimalize for every \( t \))

\[ K_{2,\lambda}(t, f) = K_2(t, f; \ell^0_2, \ell^0_2(\lambda)) = \left( \sum_{i=1}^n \frac{t \lambda_i}{1 + t \lambda_i f_i^2} \right)^{1/2}, \]

which formula will turn out useful in the sequel.

Let us temporarily leave our discussion of \( K \)-monotonicity and instead turn to the second topic mentioned at the beginning of this chapter; matrix monotonicity of functions. We shall first need to define some classes of matrix functions.

In the sequel we shall identify \( \mathcal{L}(\ell^0_n) \) with the set \( M_n(\mathbb{R}) \) of real \( n \times n \) matrices. Given any self-adjoint matrix \( A \in M_n(\mathbb{R})_{sa} \) we use the spectral theorem to write \( A = \sum_{\lambda \in \sigma(A)} \lambda E_\lambda \) where \( E_\lambda \in M_n(\mathbb{R})_{sa} \) is the projection of \( \ell^0_n \) onto the eigenspace of \( A \) corresponding to the eigenvalue \( \lambda \). When \( h \) is a real function defined on \( \sigma(A) \) we define the functional calculus \( h(A) \in M_n(\mathbb{R})_{sa} \) by

\[ h(A) = \sum_{\lambda \in \sigma(A)} h(\lambda) E_\lambda. \]

On the set \( M_n(\mathbb{R})_{sa} \) is defined the partial order “\( \leq \)” by

\[ A \leq B \iff B - A \in M_n(\mathbb{R})_+, \]
where \( M_n(\mathbb{R})_+ \) denotes the cone of positive semi-definite matrices of order \( n \).
By a definition due to Karl Löwner [20] a positive function \( h \) defined on \( \mathbb{R}_+ \) is monotone of order \( n \) (written \( h \in P_n^\nu \)) iff for any positive definite matrices \( A, B \in M_n(\mathbb{R}) \) we have

\[
A \leq B \quad \text{implies} \quad h(A) \leq h(B).
\]

(An extensive study of matrix monotone functions is provided in Donoghue’s book [12]. Another nice introduction to the theory is Löwner’s lecture notes [19].)

Let \( \mathcal{H} = (\mathcal{H}_0, \mathcal{H}_1) \) be given, where the norms of \( \mathcal{H}_0, \mathcal{H}_1 \) are related by the equation \( \|f\|^2 = (Af, f)_0 \), \( A = \text{diag}(\lambda_i) \in \mathcal{L}(\ell^2_n) \). Let \( \| \cdot \|_* \) be a third euclidean norm on \( \mathcal{V} \); \( \|f\|^2_* = (Bf, f)_0 \) where \( B \in \mathcal{L}(\ell^2_n) \) is some other positive definite matrix. It will be convenient to use the following notations with respect to various matrix norms \( (T \in M_n(\mathbb{R}) = \mathcal{L}(\ell^2_n)) \)

\[
\|T\|^2 = \|T\|_{\mathcal{L}(\mathcal{H}_0)}^2 = \sup_{(f, f)_0 \leq 1} (T^* T f, f)_0, \quad \|T\|^2_A = \|T\|_{\mathcal{L}(\mathcal{H}_1)}^2 = \sup_{(Af, f)_0 \leq 1} (T^* A T f, f)_0.
\]

\[
\|T\|^2_B = \|T\|_{\mathcal{L}(\mathcal{H}_*)}^2 = \sup_{(Bf, f)_0 \leq 1} (T^* B T f, f)_0.
\]

It is immediate from the definitions that \( \mathcal{H}_* \) is exact interpolation with respect to \( \mathcal{H} \) iff

\[
\|T\|_B \leq \max(\|T\|_*, \|T\|_A), \quad T \in M_n(\mathbb{R})
\]

that is, iff

\[
T^* T \leq 1, \quad T^* A T \leq A \quad \text{implies} \quad T^* B T \leq B, \quad T \in M_n(\mathbb{R}),
\]

where all operations are being taken with respect to the inner product of \( \mathcal{H}_0 = \ell^2_n \).

We have the following elementary lemma (cf. [11], Lemma 1).

**Lemma 2.** Let \( \mathcal{H}_* \) is exact interpolation with respect to \( \mathcal{H} \). Then there exists a function \( h : \{\lambda_i\}_{i=1}^n \to \mathbb{R}_+ \) such that \( \|f\|^2 = (h(A)f, f)_0 = \sum_{i=1}^n h(\lambda_i)f_i^2, \ f \in \ell^2_n \).

**Proof.** It is well-known that for orthogonal projections \( E \), the inequality \( EAE \leq A \) is equivalent to that \( EA = AE \). Accordingly, it follows from (6) that every projection that commutes with \( A \) necessarily commutes with \( B \). By the double commutant theorem (\([6]\), ch. IX) it yields that \( BA = AB \) and that \( B = h(A) \) with a suitable function \( h \) defined on the eigenvalues \( \lambda_i \) of \( A \). \( \square \)

By Lemma 2 the set of exact euclidean interpolation norms with respect to \( \mathcal{H} \) is identified with the class of functions \( h \) defined on \( \sigma(A) \) satisfying (6). Let us introduce some terminology with respect to this and some related function classes:

**Definition 3.**
(a) We say that a positive function \( h \) defined on \( \mathbb{R}_+ \) belongs to the set \( C_n \) of interpolation functions of order \( n \) iff for all positive definite matrices \( A \in M_n(\mathbb{R}) \)

\[
\|T\|_{h(A)} \leq \max(\|T\|, \|T\|_A), \quad T \in M_n(\mathbb{R}).
\]

(b) Similarly, a positive function \( h \) defined on \( \sigma(A) \) is said to belong to the set \( C_A \) of interpolation functions with respect to \( A \) iff

\[
\|T\|_{h(A)} \leq \max(\|T\|, \|T\|_A), \quad T \in M_n(\mathbb{R}).
\]

We list some elementary facts relative to the classes \( P^\nu_n \) and \( C_n \):

- \( P^\nu_n \) and \( C_n \) are convex cones of functions, closed under pointwise convergence on \( \mathbb{R}_+ \),
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• $P'_n$ and $C_n$ decrease with $n$: $P'_n \supset P'_{n+1}$ and $C_n \supset C_{n+1}$,
• $P'_1$ is the set of positive, non-decreasing functions on $\mathbb{R}_+$ whereas $C_1$ is
  the set of all positive functions on $\mathbb{R}_+$,
• The function $h_1(\lambda) = \lambda^{1/2}$ belongs to all classes $P'_n$ but the function
  $h_2(\lambda) = \lambda^2$ does not belong to $P'_2$. (A closer study shows that $P'_2$
  is included in the cone of positive, concave, continuously differentiable functions
  on $\mathbb{R}_+$, cf. [19]).

We have seen above that an arbitrary euclidean couple is canonically isomorphic
to a pair of the form $(\ell^2_n, \ell^2_n(\lambda))$ and (Lemma 2) every exact euclidean interpolation
space $\mathcal{H}_*$ with respect to the latter couple can be represented in the form

$$\|f\|_*^2 = \sum_{i=1}^{n} h(\lambda_i) f_i^2, \quad f \in \ell^2_n$$

with some function $h$ defined on $\Lambda = \{\lambda_i\}_{i=1}^{n}$. 

Searching for a sufficient condition for (7) to define an exact interpolation norm
with respect to $(\ell^2_n, \ell^2_n(\lambda))$. Let $\rho$ be an arbitrary positive Radon measure on the
compactified halfline $[0, \infty]$ and assume that $h$ is of the special form

$$h(\lambda_i) = \int_{[0, \infty]} \frac{(1 + t)\lambda_i}{1 + t\lambda_i} \, d\rho(t), \quad i = 1, \ldots, n.$$ 

Then (5)

$$\|f\|_*^2 = \sum_{i=1}^{n} h(\lambda_i) f_i^2 = \int_{[0, \infty]} \left( \sum_{i=1}^{n} f_i^2 \frac{(1 + t)\lambda_i}{1 + t\lambda_i} \right) \, d\rho(t) = \int_{[0, \infty]} (1 + t^{-1}) K_{2,\lambda}(t, f)^2 \, d\rho(t).$$ 

The latter expression evidently defining an exact interpolation norm, it yields that
every function of the form (7) belongs to the cone $C_A$ of interpolation functions
with respect to $A = \text{diag}(\lambda_i)$. We shall see below that this condition is also
necessary for exact interpolation to hold.

**Definition 4.** The cone of functions on $\mathbb{R}_+$ representable in the form (8)
shall henceforth be denoted by the letter $P'$, cf. [11] (read: the set of positive Pick
functions). Given a subset $\Lambda \subset \mathbb{R}_+$ we denote by $P'|\Lambda$ the set of restrictions to $\Lambda$
of functions in the cone $P'$.

The following theorem is due to William Donoghue [11] who proved it using a
highly non-trivial extension [12] of Löwner’s theory of interpolation by rational
functions in the Pick class.

**Theorem 5.** $C_A = P'|\sigma(A)$.

In a later chapter we give a new proof of Theorem 5 depending on $K$-monotonicity
(Theorem 1). Here, we settle with noting the following Corollary of Theorem 5
and (9).

**Corollary 1.** For a euclidean space $\mathcal{H}_*$ to be exact interpolation with respect
to $\mathcal{F}$, it is necessary and sufficient that there exists a positive Radon measure $\rho$
on $[0, \infty]$ such that

$$\|f\|_*^2 = \int_{[0, \infty]} (1 + t^{-1}) K_{2}(t, f; \mathcal{F})^2 \, d\rho(t), \quad f \in \ell^2_n.$$
Let us now explain how Theorem 5 can be used to obtain information about the relations between the function classes $P'$, $P'_n$ and $C_n$. It is immediate from Theorem 5 that

$$P' = \bigcap_{n=1}^{\infty} C_n,$$

which identity is (essentially) due to C. Foiaş and J.-L. Lions [13].

Less obvious is the following fact ("Löwner’s theorem" [20])

$$P' = \bigcap_{n=1}^{\infty} P'_n.$$

In the latter identity, it is easy to verify the inclusion $P' \subset \bigcap_{n=1}^{\infty} P'_n$ (integrate with respect to $d\rho(t)$). Thus by (10) in order to prove (11) it suffices to show the inclusion

$$\bigcap_{n=1}^{\infty} P'_n \subset \bigcap_{n=1}^{\infty} C_n.$$

This is fairly simple, so let us show that $P'_n \subset C_n$, $n \in \mathbb{N}$. Let $h \in P'_2$ and $A \in M_n(\mathbb{R})$ a positive definite matrix. Then we have Hansen’s inequality (cf. [16] or [17])

$$T^* h(A) T \leq h(T^* A T), \quad T \in M_n(\mathbb{R}), \quad T^* T \leq 1.$$

So if $T^* A T \leq A$, then, using the monotonicity

$$T^* h(A) T \leq h(T^* A T) \leq h(A),$$

i.e. $h \in C_n$ and Löwner’s theorem (11) is proved. □

(A closer study shows that $P'_{n+1} \subset C_{2n} \subset P'_n$ for all $n$.)

The class $P'$ plays a prominent rôle in the interpolation theory of Hilbert spaces. Below are stated some elementary facts with respect to this class, which are of general interest.

It is easily verified that the elements of $P'$ are concave functions on $\mathbb{R}^+$, and that the set $P'$ is closed in the topology of pointwise convergence on $\mathbb{R}^+$ (Helly’s theorem). Less obvious is the fact that $P'$ coincides precisely with the set of functions, positive and regular on $\mathbb{R}^+$ which prolong to the upper half-plane and have positive imaginary parts there, cf. [10], sect. 2.

We have the following theorem.

**Theorem 6.** For each positive concave function $\psi$ on $\mathbb{R}^+$ there exists a function $h \in P'$ such that $h \leq \psi \leq 2h$.

We note that our proof of Theorem 6 is easy by comparing the extreme rays of the cone of positive concave functions with the extreme rays of $P'$.

It is not difficult to see that Theorem 5 and Theorem 6 leads to the following: for each positive, concave function $\psi$ on $\mathbb{R}^+$ we have

$$\|T\|_{\psi(A)} \leq \sqrt{2} \max(\|T\|, \|T\|_A), \quad T \in M_n(\mathbb{R}).$$

This inequality is due to C. Foiaş, S. C. Ong and P. Rosenthal [14] and can be regarded as a quantitative version of a theorem of Peetre [23] stating that a necessary and sufficient condition for the implication

$$\max(\|T\|, \|T\|_A) < \infty \quad \text{implies} \quad \|T\|_{\psi(A)} < \infty$$
to hold is that the function $\varphi$ be equivalent to a concave function, i.e. that there exists $c \geq 1$ and a concave function $\psi$ such that $(1/c)\psi \leq \varphi \leq c\psi$. In the language of interpolation theory, the Foiaș–Ong–Rosenthal inequality (12) says that the euclidean space $\mathcal{H}_1$ defined by $\|f\|_2^2 = (\psi(A)f, f)_0$ is $\sqrt{2}$-interpolation with respect to $\mathcal{H}$ regardless of the choice of a positive concave function $\psi$. The question was posed in [14] whether the constant $\sqrt{2}$ in (12) is smallest possible. It turns out that the answer to this question is implicit in the work of Gunnar Sparr [27], Lemma 5.1. We state it here as a theorem.

**Theorem 7.** The constant $\sqrt{2}$ in (12) is best possible.

We note also the following, purely function-theoretic theorem, which we obtained as a corollary of Theorem 7 (but can also be obtained by more direct methods).

**Corollary 2.** The constant 2 in Theorem 6 is best possible.

Finally let us mention a generalization of some of our results to interpolation of tensor products.

Let $A_i \in M_{n_i}(\mathbb{R})$ be some positive definite matrices, $i = 1, 2$. It is natural to call a function $h$ defined on $\sigma(A_1) \times \sigma(A_2)$ an interpolation function with respect to $A_1, A_2$ iff

$$
(T_1 \otimes T_2)^*(T_1 \otimes T_2) \leq 1, \quad (T_1 \otimes T_2)^*(A_1 \otimes A_2)(T_1 \otimes T_2) \leq (A_1 \otimes A_2)
$$

implies

$$
(T_1 \otimes T_2)^*h(A_1, A_2)(T_1 \otimes T_2) \leq h(A_1, A_2)
$$

(13)

where we have used the convention of A. Korányi [18]

$$
h(A_1, A_2) = \sum_{(\lambda_1, \lambda_2) \in \sigma(A_1) \times \sigma(A_2)} h(\lambda_1, \lambda_2)E_{\lambda_1} \otimes F_{\lambda_2}.
$$

(Here $E_{\lambda_1}$ is the projection of $\mathbb{R}^{n_1}$ onto the eigenspace of $A_1$ corresponding to the eigenvalue $\lambda_1$, etc.) We have the following theorem.

**Theorem 8.** A necessary and sufficient condition for $h$ to satisfy (13) is given by the existence of a positive Radon measure $\rho$ on $[0, \infty]^2$ such that

$$
h(\lambda_1, \lambda_2) = \int_{[0, \infty]^2} \frac{(1 + t_1)\lambda_1}{1 + t_1\lambda_1} \frac{(1 + t_2)\lambda_2}{1 + t_2\lambda_2} d\rho(t_1, t_2), \quad (\lambda_1, \lambda_2) \in \sigma(A_1) \times \sigma(A_2).
$$

(14)

This last result extends the analogy between interpolation theory and the theory of monotone matrix functions. Indeed Korányi [18] has shown that (14) is necessary and sufficient for a (sufficiently regular) function $h$ defined on $\mathbb{R}_+^2$ to be matrix monotonic of two variables, where the latter term is reserved for the class of functions $h(\lambda_1, \lambda_2)$ having the following property: $A_1, A_2$ positive definite and $A_1 \leq A_1', A_2 \leq A_2'$ imply

$$
h(A_1', A_2') - h(A_1', A_2) - h(A_1, A_2') + h(A_1, A_2) \geq 0.
$$
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References

CHAPTER II

The Calderón problem for Hilbert couples

ABSTRACT. We prove that intermediate Banach spaces \( A, B \) with respect to arbitrary Hilbert couples \( \overline{H}, \overline{K} \) are exact interpolation iff they are exact \( K \)-monotonic, i.e. the condition \( f^0 \in A \) and the inequality \( K(t, g^0; \overline{K}) \leq K(t, f^0; \overline{H}) \), \( t > 0 \) imply \( g^0 \in B \) and \( \| g^0 \|_B \leq \| f^0 \|_A \) (\( K \) is Peetre’s \( K \)-functional). It is well-known that this property is implied by the following: for each \( q > 1 \) there exists an operator \( T : \overline{H} \to \overline{K} \) such that \( Tf^0 = g^0 \), and \( K(t, Tf; \overline{K}) \leq \varrho K(t, f; \overline{H}) \), \( f \in \mathcal{H}_0 + \mathcal{H}_1 \), \( t > 0 \). Verifying the latter property, it suffices to consider the “diagonal case” where \( \overline{H} = \overline{K} \) is finite-dimensional, in which case we construct the relevant operators by a method which allows us to explicitly calculate them. In the strongest form of the theorem, it is shown that the statement remains valid when substituting \( q = 1 \).

1. Preliminaries

Before we formulate our basic problem let us fix some notation and review some notions from the theory of interpolation spaces (cf. [4], [5] or [6] for comprehensive accounts of that theory).

Let \( A, B \) be Banach spaces over the real or complex field. Following Gunnar Sparr [21] we denote by \( \mathcal{L}(A; B) \) the Banach space of bounded linear maps \( T : A \to B \) provided with the operator norm

\[
\| T \|_{\mathcal{L}(A; B)} = \sup \| Tf \|_B / \| f \|_A.
\]

Similarly for Banach couples \( \overline{A} = (A_0, A_1) \) and \( \overline{B} = (B_0, B_1) \) we define \( \mathcal{L}(\overline{A}; \overline{B}) \) as the set of linear operators \( T : A_0 + A_1 \to B_0 + B_1 \) such that the restriction of \( T \) to \( A_i \) belongs to \( \mathcal{L}(A_i; B_i), i = 0, 1 \). It is well-known that \( \mathcal{L}(\overline{A}; \overline{B}) \) is a Banach space under the norm

\[
\| T \|_{\mathcal{L}(\overline{A}; \overline{B})} = \max(\| T \|_{\mathcal{L}(A_0; B_0)}, \| T \|_{\mathcal{L}(A_1; B_1)}).
\]

For a given \( c \), denote \( \mathcal{L}_c(A; B) \) and \( \mathcal{L}_c(\overline{A}; \overline{B}) \) the balls of radius \( c \)

\[
T \in \mathcal{L}_c(A; B) \quad \text{iff} \quad T \in \mathcal{L}(A; B), \quad \| T \|_{\mathcal{L}(A; B)} \leq c,
\]

\[
T \in \mathcal{L}_c(\overline{A}; \overline{B}) \quad \text{iff} \quad T \in \mathcal{L}(\overline{A}; \overline{B}), \quad \| T \|_{\mathcal{L}(\overline{A}; \overline{B})} \leq c.
\]

In this notation, intermediate spaces \( A, B \) are interpolation with respect to \( \overline{A}, \overline{B} \) iff there exists \( c \) with the property that

(\( c \)-Int)

\[
\mathcal{L}_1(\overline{A}; \overline{B}) \subset \mathcal{L}_c(A; B),
\]

(where necessarily \( c \geq 1 \)). In the special case when \( c = 1 \),

(ExInt)

\[
\mathcal{L}_1(\overline{A}; \overline{B}) \subset \mathcal{L}_1(A; B),
\]

we speak about exact interpolation. Of particular interest is the diagonal case, \( A = B \) and \( \overline{A} = \overline{B} \), in which we simply say that \( A \) is exact interpolation with respect to \( \overline{A} \).

In the present study is considered the problem of characterizing all exact interpolation spaces with respect to arbitrary (possibly different) Hilbert couples. Our
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results sharpens the theorems of Sedaev ([17], Theorem 4) and Sparr ([21], Theorem 5.1) and implies the theorems of Donoghue [10] and Löwner [12] (see [3] for further details).

2. Main Results

It is well-known that many exact interpolation spaces can be described by Peetre’s $K$-functional,

$$ K(t, f) = K(t, f; X_0, X_1) = \inf_{f = f_0 + f_1} (\|f_0\|_0 + t\|f_1\|_1), \quad f \in X_0 + X_1, \ t > 0, $$

or more precisely by the quasi-order (relative to $\mathbf{A}$, $\mathbf{B}$) defined by

$$ g \leq f[K] \iff K(t; g; \mathbf{B}) \leq K(t; f; \mathbf{A}), \ t > 0. $$

We have the following basic lemma.

**Lemma 2.1.** (ExInt) is implied by the following property (“exact $K$-monotonicity”)

$$ f \in \mathbf{A} \text{ and } g \leq f[K] \implies g \in \mathbf{B} \text{ and } \|g\|_\mathbf{B} \leq \|f\|_\mathbf{A}. $$

**Proof.** See [21], Theorem 1.1, p. 232. \qed

In general the property (ExInt) is not the same as exact $K$-monotonicity, cf. [5] Exercise 5.7.14 where a three-dimensional counterexample is given (see sect. 7 for further remarks). However with respect to regular Hilbert couples is known the following weak form of equivalence between the two notions. (Recall that $(X_0, X_1)$ is regular if $X_0 \cap X_1$ is dense in $X_0$ and in $X_1$.)

**Theorem SS.** Let $\mathbf{A}$, $\mathbf{B}$ be exact interpolation relative to regular Hilbert couples $\mathcal{H}$, $\mathcal{K}$. Then they are “$\sqrt{2}$-monotonic” in the following sense:

$$ f \in \mathbf{A} \text{ and } g \leq f[K] \implies g \in \mathbf{B} \text{ and } \|g\|_\mathbf{B} \leq \sqrt{2}\|f\|_\mathbf{A}. $$

The theorem is due to Sedaev [17] in the diagonal case and to Sparr [21] in the general case. We have the following sharpening.

**Theorem 2.1.** With respect to regular Hilbert couples, (ExInt) is equivalent to exact $K$-monotonicity.

Couples having the property that (ExInt) coincides with exact $K$-monotonicity are (in this paper) called Calderón couples – after Alberto P. Calderón [7] who in 1966 found the non-trivial case $\mathcal{A} = (L_1(\mu), L_\infty(\mu))$ and $\mathcal{B} = (L_1(\nu), L_\infty(\nu))$, $\mu$, $\nu$ being arbitrary $\sigma$-finite measures. \footnote{An equivalent characterization of the exact interpolation spaces with respect to $(L_1, L_\infty)$ was independently discovered by B. S. Mityagin [13] in 1965, whence some authors prefer to speak about “Calderón–Mityagin couples”. Also the terms “$K$-adequate couple”, “$K$-monotone couple” and “$C$-couple” exist in the literature.} In this terminology, Theorem 2.1 states that regular Hilbert couples constitute a case of Calderón couples.

We have also the following, slightly stronger theorem.

**Theorem 2.2.** Let $\mathcal{H}$, $\mathcal{K}$ be regular Hilbert couples, and $f^0 \in \mathcal{H}_0 + \mathcal{H}_1$, $g^0 \in \mathcal{K}_0 + \mathcal{K}_1$ such that $g^0 \leq f^0[K]$. Then there exist an operator $T \in L_1(\mathcal{H}; \mathcal{K})$ such that $T f^0 = g^0$.

We note that our proof of the above results are fairly straightforward consequences of the following “key lemma”.

**Proof.** See [21], Theorem 1.1, p. 232. \qed
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Lemma 2.2. Let $\mathcal{H}$, $\mathcal{K}$ be regular Hilbert couples with elements $f^0 \in \mathcal{H}_0 + \mathcal{H}_1$, $g^0 \in \mathcal{K}_0 + \mathcal{K}_1$ fulfilling $K(t, g^0; \mathcal{K}) < \varrho^{-1} K(t, f^0; \mathcal{H})$, $t > 0$ with some number $\varrho > 1$. Then there exists an operator $T \in L_1(\mathcal{H}; \mathcal{K})$ such that $T f^0 = g^0$.

It is not hard to show that Theorem 2.1 is equivalent to the above lemma, and that Theorem 2.2 follows as a limiting case. The main part of this study is devoted to the proof of Lemma 2.2.

Before embarking on the details, it is fitting to compare our results with the work of William Donoghue [10] by which is known a complete description of the exact interpolation Hilbert spaces with respect to Hilbert couples. Indeed, Donoghue’s theorem is advantageous over our Theorem 2.1 in the respect that it yields an explicit representation formula for all possible norms in such spaces. However, with a modicum of effort, it is possible to incorporate that theorem as a natural part of our context, cf. [3].

Remark 2.1. Throughout this paper, we have been studious to avoid non-regular couples, although this restriction is strictly speaking not necessary. With minor modifications, our theorems and proofs extend to the non-regular case and to interpolation of quadratic semi-norms, cf. the remarks of Sparr [21], top of p. 235.

3. The functional $K_2$

With respect to $(X_0, X_1)$ we have the following functional (cf. [17])

$$K_2(t, f) = K_2(t, f; X_0, X_1) = \inf_{f = f_0 + f_1} (\|f_0\|^2_0 + t\|f_1\|^2_1)^{1/2}.$$  

Correspondingly, relative to $\mathfrak{A}$, $\mathfrak{B}$ is defined the following quasi-order

$$g \leq f[K_2] \iff K_2(t, g; \mathfrak{B}) \leq K_2(t, f; \mathfrak{A}), \quad t > 0.$$  

The following theorem is crucial for what follows.

Lemma 3.1. With respect to arbitrary Banach couples $\mathfrak{A}$, $\mathfrak{B}$, we have

$$g \leq f[K] \iff g \leq f[K_2].$$

Proof. This lemma is formulated in Sparr [21], Lemma 3.2, p. 236 for weighted $L_2$-couples. However, a short consideration of the proof shows that it holds equally well with arbitrary Banach couples.  

It is immediate from the definitions that $K(t, \cdot)$ and $K_2(t, \cdot)$ enjoy the property of being exact interpolation functors for all $t$, viz.

$$T f \leq ||T||_{L(\mathfrak{A}, \mathfrak{B})} f[K] \quad \text{and} \quad T f \leq ||T||_{L(\mathfrak{A}, \mathfrak{B})} f[K_2],$$

for any $\mathfrak{A}$, $\mathfrak{B}$, $T$, $f$. An advantage of using $K_2$ and not $K$ is that the former can be conveniently calculated in the regular Hilbert case in a way which we describe below.

Given a regular Hilbert couple $(\mathcal{H}_0, \mathcal{H}_1)$ the squared norm $\|f\|^2_1$ is an (in general unbounded, but densely defined) quadratic form on $\mathcal{H}_0$, which we represent in the form

$$\|f\|^2_1 = (Af, f)_0,$$

where $A$ is a positive, injective, densely defined linear operator in $\mathcal{H}_0$ henceforth referred to as the associated operator of $\mathcal{H}$. (Note that the domain of the positive
square-root $A^\frac{1}{2}$ is $\mathcal{H}_0 \cap \mathcal{H}_1$. As general sources to the spectral theory of self-adjoint operators we refer to [15] and [16].

Let us now fix $f \in \mathcal{H}_0 \cap \mathcal{H}_1$ and consider the optimal decomposition\(^2\)

$$f = f_0(t) + f_1(t), \quad K_2(t, f) = \|f_0(t)\|_0^2 + t\|f_1(t)\|_1^2.$$ 

Plainly $f_i(t) \in \mathcal{H}_0 \cap \mathcal{H}_1 = \text{domain}(A^\frac{1}{2})$, $i = 0, 1$ and moreover for all $\tilde{f}$ in this domain

$$\frac{d}{d\varepsilon} \{(f_0(t) + \varepsilon \tilde{f}, f_0(t) + \varepsilon \tilde{f})_0 + t(A(f_1(t) - \varepsilon \tilde{f}), f_1(t) - \varepsilon \tilde{f})_0\}|_{\varepsilon = 0} = 0,$$

so that

$$2R\{(f_0(t) - tAf_1(t), \tilde{f})_0\} = 0, \quad \tilde{f} \in \mathcal{H}_0 \cap \mathcal{H}_1.$$

By regularity, $f_0(t) = tAf_1(t)$ and $f = f_0(t) + f_1(t) = (1 + tA)f_1(t)$, which yields

$$f_0(t) = \frac{tA}{1 + tA}(f) \quad \text{and} \quad f_1(t) = \frac{1}{1 + tA}(f)$$

and

(3.1)

$$K_2(t, f)^2 = \|f_0(t)\|_0^2 + t\|f_1(t)\|_1^2 = \left(\frac{(tA)^2}{(1 + tA)^2}(f), f\right)_0 + t\left(\frac{A}{(1 + tA)^2}(f), f\right)_0 = (h_t(A)f, f)_0 \quad \text{where} \quad h_t(\lambda) = \frac{t\lambda}{1 + t\lambda}. $$

An important consequence of (3.1) it that $K_2(t, f)$ is a Hilbert space norm on $\mathcal{H}_0 + \mathcal{H}_1$ for every fixed $t > 0$. We shall denote by $\mathcal{H}_0 + t\mathcal{H}_1$ the space normed by $K_2(t, f)$; in particular $\mathcal{H}_0 + \mathcal{H}_1$ is considered as normed by $K_2(1, f)$.

Further consideration of (3.1) shows that

$$\|f\|_1^2 = \lim_{t \to \infty} K_2(t, f)^2, \quad \|f\|_0^2 = \lim_{t \to 0} t^{-1}K_2(t, f)^2,$$

which gives the following characterization of the unit ball $\mathcal{L}_1(\overline{\mathcal{H}}; \overline{\mathcal{K}})$ with respect to regular Hilbert couples $\overline{\mathcal{H}}, \overline{\mathcal{K}}$

(3.2) $T \in \mathcal{L}_1(\overline{\mathcal{H}}; \overline{\mathcal{K}})$ \iff $Tf \leq f[K_2], \quad f \in \mathcal{H}_0 + \mathcal{H}_1$,

which is to say that

(3.3) $\mathcal{L}_1(\overline{\mathcal{H}}; \overline{\mathcal{K}}) = \bigcap_{t>0} \mathcal{L}_1(\mathcal{H}_0 + t\mathcal{H}_1; \mathcal{K}_0 + t\mathcal{K}_1)).$

Below is shown that (3.3) implies a weak*-type compactness property of the set $\mathcal{L}_1(\overline{\mathcal{H}})$. (In the diagonal case we prefer to write $\mathcal{L}(\overline{\mathcal{H}})$ instead of $\mathcal{L}(\overline{\mathcal{H}}; \overline{\mathcal{H}})$, etc.)

**Lemma 3.2.** The subset $\mathcal{L}_1(\overline{\mathcal{H}}) \subset \mathcal{L}_1(\mathcal{H}_0 + \mathcal{H}_1)$ is compact relative to the weak operator topology\(^3\) on $\mathcal{L}_1(\mathcal{H}_0 + \mathcal{H}_1)$.

\(^2\)It is a simple exercise to verify that an optimal decomposition exists and is unique. Assume that $t = 1$ and use the convexity and closedness of the subset $\{(f_0, f_1) : f_i \in \mathcal{H}_i, i = 0, 1; f_0 + f_1 = f\}$ of the cartesian product space $\mathcal{H}_0 \times \mathcal{H}_1$ to obtain an element of minimal norm.

\(^3\)A net $T_i$ converges to $T$ in the weak operator topology on $\mathcal{L}(\mathcal{H})$ \iff $(T_i f, g)_\mathcal{H}$ converges to $(T f, g)_\mathcal{H}$ for all $f, g \in \mathcal{H}$. 
4. Further preparations

In this section we simplify and reduce our problems; below is shown that they boil down to the diagonal case (i.e. $\mathcal{H} = \mathcal{K}$) of Lemma 2.2.

**Lemma 4.1.**

(1) Lemma 2.2 is a consequence of its diagonal case.

(2) Theorem 2.2 is a consequence of its diagonal case.

(3) Theorem 2.2 is a consequence of Lemma 2.2.

(4) Theorem 2.1 is a consequence of Lemma 2.2.

**Proof.**

Proof of (1): Given $\mathcal{H}$, $\mathcal{K}$, we form the “direct sum” $\mathfrak{S} = (\mathcal{H}_0 \oplus K_0, \mathcal{H}_1 \oplus K_1)$. The splitting $S_0 + S_1 = (\mathcal{H}_0 + \mathcal{H}_1) \oplus (K_0 + K_1)$ easily yields the following expression for the $K_2$-functional with respect to a generic element $f \oplus g \in S_0 + S_1$,

$$K_2(t, f \oplus g; \mathfrak{S})^2 = K_2(t, f; \mathcal{H})^2 + K_2(t, g; \mathcal{K})^2.$$  

By Lemma 3.1, the assumptions of Lemma 2.2 translate to

$$K_2(t, 0 \oplus g^0; \mathfrak{S}) < g^{-1}K_2(t, f^0 \oplus 0; \mathfrak{S}), \quad t > 0.$$  

Applying the diagonal case of Lemma 2.2, it yields an operator $S \in \mathcal{L}(\mathfrak{S})$ such that $S(f^0 \oplus 0) = 0 \oplus g^0$ and $\|S\|_{\mathcal{L}(\mathfrak{S})} \leq 1$. Let $P$ denote the orthogonal projection

$$P : S_0 + S_1 \to K_0 + K_1,$$

evidently $\|P\|_{\mathcal{L}(\mathfrak{S}, \mathfrak{S})} = 1$. Hence putting

$$T : \mathcal{H}_0 + \mathcal{H}_1 \to K_0 + K_1 \quad : \quad f \mapsto PS(f \oplus 0)$$

yields $Tf^0 = g^0$ and $\|T\|_{\mathcal{L}(\mathcal{H}, \mathcal{K})} \leq 1$, as desired.

Proof of (2): This is very similar to (1), the simple modifications are left to the reader.

Proof of (3): By (1) and (2) it suffices to consider the diagonal case $\mathcal{H} = \mathcal{K}$. Let $g > 1$ be given together with any elements $g^0, f^0 \in \mathcal{H}_0 + \mathcal{H}_1$ such that $g^0 \leq f^0[K]$. The hypothesis that Lemma 2.2 holds true in the diagonal case then yields the existence for each $n \in \mathbb{N}$ of an operator $T_n \in \mathcal{L}_1(\mathcal{H})$ such that $T_n f^0 = \frac{n}{n+1}g^0$. By compactness (Lemma 3.2) the $T_n$’s cluster at a point $T \in \mathcal{L}_1(\mathcal{H})$, and it remains to check that $Tf^0 = g^0$.

To this end, it suffices to note that

$$(Tf^0, h)_{\mathcal{H}_0 + \mathcal{H}_1} = \lim(T_n f^0, h)_{\mathcal{H}_0 + \mathcal{H}_1} = (g^0, h)_{\mathcal{H}_0 + \mathcal{H}_1}, \quad h \in \mathcal{H}_0 + \mathcal{H}_1.$$  

Proof of (4): Recall (Lemma 2.1) that exact $K$-monotonicity implies (ExInt). Under the hypothesis that Lemma 2.2 holds true, we shall prove the reverse implication. Given exact interpolation spaces $\mathcal{A}$, $\mathcal{B}$ with respect to $\mathcal{H}$, $\mathcal{K}$ together with elements $g^0, f^0$ such that

$$g^0 \leq f^0[K] \quad \text{and} \quad f^0 \in \mathcal{A},$$
there then exists for each $\rho > 1$ an operator $T \in \mathcal{L}_1(\mathcal{H}; \mathcal{K})$ such that $Tg^0 = \rho^{-1}g^0$. Hence $T \in \mathcal{L}_1(\mathcal{A}; \mathcal{B})$ by (ExInt) whence
\[
\|g^0\|_B = \|\rho Tg^0\|_B \leq \rho\|f^0\|_A.
\]
Since $\rho > 1$ is arbitrary, it yields that $\mathcal{A}$, $\mathcal{B}$ are exact $K$-monotonic. $\square$

5. Solution of the problem

This section is devoted to the diagonal case of Lemma 2.2. By Lemma 4.1 this will yield our three main theorems stated in the introduction. Our proof is divided into two parts: (i) reduction to a finite-dimensional case and (ii) explicit solution of the problem in that case. We start with (i).

5.1. Reduction to finite dimension. To fix the problem, let $\mathcal{H}$ be given together with a number $\rho > 1$ and vectors $g^0$, $f^0 \in \mathcal{H}_0 + \mathcal{H}_1$ such that
\[
K_2(t, g^0; \mathcal{H})^2 < \rho^{-1}K_2(t, f^0; \mathcal{H})^2, \quad t > 0.
\]
We want to prove that there exists $T$ with the following properties
\[
T \in \mathcal{L}_1(\mathcal{H}) \quad \text{and} \quad Tf^0 = g^0.
\]
Let $A$ be the operator associated with $\mathcal{H}$, $E$ the spectral measure of $A$, and let a sequence of orthogonal projections in $\mathcal{H}_0$ be defined by
\[
P_n = E_{\sigma(A) \cap [1/n, n]}, \quad n \in \mathbb{N}.
\]

**Lemma 5.1.** To verify (5.2) we can besides (5.1) w.l.o.g. assume that $f^0$ and $g^0$ belong to the set $P_n(\mathcal{H}_0)$ with some $n \in \mathbb{N}$.

**Proof.** Since $P_n$ commutes with $A$ it is easily seen that $\|P_n\|_{\mathcal{L}(\mathcal{P})} = 1$ for all $n \in \mathbb{N}$. Moreover, as $n \to \infty$ the projections $P_n$ converge in the strong operator topology on $\mathcal{L}(\mathcal{H}_0 + \mathcal{H}_1)$ to the identity. Accordingly,
\[
K_2(t, P_ng^0)^2 \leq K_2(t, g^0)^2 < \rho^{-1}K_2(t, f^0)^2, \quad t > 0.
\]
Moreover, by the estimate $K_2(t, P_ng^0)^2 \leq C_n \min(1, t)$ and because the sequence of functions $K_2(t, P_mf^0)^2$ increase monotonically, converging uniformly on compact subsets of $\mathbb{R}_+$ to $K_2(t, f^0)^2$, it follows that, for each number $\rho_0$ such that $1 < \rho_0 < \rho$, we can choose $m = m(\rho_0, n) \geq n$ such that
\[
K_2(t, P_ng^0)^2 < \rho_0^{-1}K_2(t, P_mf^0), \quad t > 0.
\]
Thus under the hypothesis that the implication “(5.3)$\Rightarrow$(5.2)” holds true with respect to the vectors $P_mf^0$, $P_ng^0 \in P_m(\mathcal{H}_0)$, it yields an element $T_{nm} \in \mathcal{L}_1(\mathcal{H})$ such that $T_{nm}P_mf^0 = P_ng^0$. By Lemma 3.2 the $T_{nm}$’s cluster at a point $T \in \mathcal{L}_1(\mathcal{H})$, and one checks without difficulty that $Tf^0 = g^0$ whence (5.2) holds. $\square$

Define a subcouple $\mathcal{K} \subset \mathcal{H}$ by letting $\mathcal{K}_0 = \mathcal{K}_1 = P_n(\mathcal{H}_0) = P_n(\mathcal{H}_1)$ where the norm in $\mathcal{K}_i$ is defined as the restriction of the norm of $\mathcal{H}_i$, $i = 0, 1$. By Lemma 5.1 we can assume that $f^0, g^0 \in \mathcal{K}_0 \cap \mathcal{K}_1$. Since
\[
K_2(t, f; \mathcal{K}) = K_2(t, f; \mathcal{H}), \quad f \in \mathcal{K}_0 + \mathcal{K}_1,
\]
we can (replacing $\mathcal{H}$ by $\mathcal{K}$ if necessary) assume that the norms of $\mathcal{H}_0$ and $\mathcal{H}_1$ are equivalent. Our next task is to approximate the problem by a finite-dimensional one.
5. Solution of the Problem

Lemma 5.2. Given \( f^0, g^0 \in P_n(\mathcal{H}_0) \) and a number \( \varepsilon > 0 \), there exists a finite-dimensional Hilbert subcouple \( \overline{\mathcal{V}} \subset \overline{\mathcal{H}} \) such that \( f^0, g^0 \in \mathcal{V}_0 + \mathcal{V}_1 \) and

\[
(5.4) \quad (1-\varepsilon)K_2(t, f; \overline{\mathcal{H}})^2 \leq K_2(t, f; \overline{\mathcal{V}})^2 \leq (1+\varepsilon)K_2(t, f; \overline{\mathcal{H}})^2, \quad t > 0, \ f \in \mathcal{V}_0 + \mathcal{V}_1.
\]

Moreover, \( \overline{\mathcal{V}} \) can be chosen so that all eigenvalues of the associated operator \( A_{\overline{\mathcal{V}}} \) are of multiplicity 1.

Proof. By the foregoing remarks, we can assume that the norms of \( \mathcal{H}_0 \) and \( \mathcal{H}_1 \) be equivalent. Then the associated operator \( A \) is bounded and bounded below. Take \( \bar{\varepsilon} > 0 \) and let \( \{\lambda_i\}_{i=1}^n \) be a finite subset of \( \sigma(A) \) such that \( \sigma(A) \subset \bigcup_{i=1}^n (\lambda_i - \bar{\varepsilon}/2, \lambda_i + \bar{\varepsilon}/2) \). Let \( \{E_i\}_{i=1}^n \) be a covering of \( \sigma(A) \) consisting of disjoint intervals of length at most \( \bar{\varepsilon} \) such that \( \lambda_i \in E_i \). Define a Borel function \( w : \sigma(A) \rightarrow \sigma(A) \) by \( w(\lambda) = \lambda_i \) on \( E_i \cap \sigma(A) \); then \( \|w(A) - A\|_{L(\mathcal{H}_0)} \leq \bar{\varepsilon} \). The Lipschitz constants of the restrictions of the functions \( h_t \) (cf. (3.1)) to \( \sigma(A) \) are bounded above by \( C_0 \min(1, t) \) where \( C_0 \) is independent of \( t \), whence

\[
\|h_t(w(A)) - h_t(A)\|_{L(\mathcal{H}_0)} \leq C_0 \bar{\varepsilon} \min(1, t).
\]

Thus by Schwarz' inequality and the assumption on \( \overline{\mathcal{H}} \),

\[
(5.5) \quad \|(h_t(w(A)) - h_t(A))f, f\|_0 \leq C_0 \bar{\varepsilon} \min(1, t)\|f\|_0^2 \leq C_1 \bar{\varepsilon} \min(1, t) \max(\|f\|_0^2, \|f\|_1^2), \quad f \in \mathcal{H}_0 + \mathcal{H}_1.
\]

Now let \( c > 0 \) be such that \( A \geq c \). Using that \( h_t(c) \geq (1/2) \min(1, ct) \), we get

\[
(h_t(A)f, f)_0 \geq h_t(c)\|f\|_0^2 \geq C_2 \min(1, t) \max(\|f\|_0^2, \|f\|_1^2), \quad f \in \mathcal{H}_0 + \mathcal{H}_1.
\]

This and (5.5) yields

\[
(5.6) \quad \|(h_t(w(A))f, f)_0 - (h_t(A)f, f)_0\| \leq C_3 \bar{\varepsilon} (h_t(A)f, f)_0, \quad f \in \mathcal{H}_0 + \mathcal{H}_1.
\]

Choose unit vectors \( e_i, f_i \) supported by the spectral sets \( E_i \) such that \( f^0 \) and \( g^0 \) belongs to the space \( \mathcal{V} \) spanned by \( \{e_i, f_i\}_{i=1}^n \). Put \( \mathcal{W}_0 = \mathcal{W}_1 = \mathcal{V} \) (as sets) and define the norms by

\[
\|f\|_{\mathcal{W}_0}^2 = \|f\|_{\mathcal{H}_0}^2 \quad \text{and} \quad \|f\|_{\mathcal{W}_1}^2 = (w(A)f, f)_{\mathcal{H}_0} \quad f \in \mathcal{V}.
\]

The operator associated with \( \overline{\mathcal{V}} \) is then the compression \( A_{\overline{\mathcal{V}}} \) of \( w(A) \) to \( \mathcal{V}_0 \), i.e.

\[
(5.7) \quad \|f\|_{\overline{\mathcal{V}_1}}^2 = (A_{\overline{\mathcal{V}}}f, f)_{\mathcal{V}_0} = (w(A)f, f)_{\mathcal{H}_0}, \quad f \in \mathcal{V}.
\]

Let \( \varepsilon = 2C_3 \bar{\varepsilon} \) and observe that (5.6) and (5.7) yield

\[
(5.8) \quad |K_2(t, f; \overline{\mathcal{V}})^2 - K_2(t, f; \overline{\mathcal{H}})^2| \leq (\varepsilon/2)K_2(t, f; \overline{\mathcal{H}})^2, \quad f \in \mathcal{V}.
\]

In general the eigenvalues of the operator \( A_{\overline{\mathcal{V}}} \) have multiplicity 2. To remedy this, perturb \( A_{\overline{\mathcal{V}}} \) slightly to a positive matrix \( A_{\overline{\mathcal{V}}} \), all of whose eigenvalues have multiplicity 1, such that \( \|A_{\overline{\mathcal{V}}} - A_{\overline{\mathcal{V}}}\|_{L(\mathcal{H}_0)} < \varepsilon/(2C_3) \). Let \( \overline{\mathcal{V}} \) be the couple associated with \( A_{\overline{\mathcal{V}}} \), i.e.

\[
\|f\|_{\overline{\mathcal{V}_1}}^2 = \|f\|_{\overline{\mathcal{V}_0}}^2 \quad \text{and} \quad \|f\|_{\overline{\mathcal{V}_1}}^2 = (A_{\overline{\mathcal{V}}}f, f)_{\mathcal{V}_0}, \quad f \in \mathcal{V}.
\]

By a calculation analogous to the one leading to (5.8), one gets without difficulty

\[
|K_2(t, f; \overline{\mathcal{V}})^2 - K_2(t, f; \overline{\mathcal{V}})^2| \leq (\varepsilon/2)K_2(t, f; \overline{\mathcal{H}})^2, \quad f \in \mathcal{V}.
\]

Together with (5.8), this yields (5.4). \( \square \)
II. THE CALDERÓN PROBLEM

The following lemma finishes the reduction to the finite-dimensional case.

**Lemma 5.3.** Verifying (5.2) one can besides (5.1) w.l.o.g. assume that \( \mathcal{H} \) is finite-dimensional and all eigenvalues of the associated operator are of unit multiplicity.

**Proof.** Let \( \mathcal{H}, \varrho, g^0, f^0 \) fulfill our basic assumption (5.1). By Lemma 5.1 we can assume that \( g^0, f^0 \in P_n(\mathcal{H}_0) \) for large enough \( n \). In this case, for a given \( \varepsilon > 0 \) (to be fixed later) Lemma 5.2 provides us with couple \( \mathcal{V} \) of the desired form such that

\[
K_2(t, g^0; \mathcal{V})^2 \leq (1 + \varepsilon)K_2(t, g^0; \mathcal{H})^2 < \varepsilon K_2(t, f^0; \mathcal{V})^2 + \varepsilon(K_2(t, f^0; \mathcal{H})^2 + K_2(t, g^0; \mathcal{H})^2).
\]

Choosing \( \varepsilon > 0 \) sufficiently small in this inequality we can arrange that

\[
K_2(t, g^0; \mathcal{V})^2 < \varrho_0^{-1}K_2(t, f^0; \mathcal{V})^2, \quad t > 0,
\]

where \( \varrho_0 \) is any number in the interval \( 1 < \varrho_0 < \varrho \). By using (5.9) instead of our basic assumption (5.1) and the hypothesis that the conclusion (5.2) holds true with respect to the couple \( \mathcal{V} \), we infer that for each \( \varrho_1 \) in the interval \( 1 < \varrho_0 < \varrho_0 \) there exists an operator \( T \in \mathcal{L}_{1/\varrho_1}(\mathcal{V}) \) such that \( T f^0 = g^0 \). Denote the canonical inclusion and projections

\[
I : \mathcal{V}_0 + \mathcal{V}_1 \to \mathcal{H}_0 + \mathcal{H}_1, \quad P : \mathcal{H}_0 + \mathcal{H}_1 \to \mathcal{V}_0 + \mathcal{V}_1,
\]

then by (5.4)

\[
\|I\|\mathcal{L}(\mathcal{V},\mathcal{H}) \leq (1 - \varepsilon)^{-1}, \quad \|P\|\mathcal{L}(\mathcal{H},\mathcal{V}) \leq 1 + \varepsilon.
\]

Let \( \varepsilon > 0 \) be sufficiently small that

\[
\frac{1}{\varrho_1\left(\frac{1 + \varepsilon}{1 - \varepsilon}\right)^{1/2}} \leq 1.
\]

and put \( S = ITP \in \mathcal{L}(\mathcal{H}) \). Then \( S f^0 = g^0 \) and (5.10), (5.11) yields \( S \in \mathcal{L}_1(\mathcal{H}) \). Thus (5.2) is satisfied with respect to \( \mathcal{H} \) and the operator \( S \). \( \square \)

**5.2. The finite-dimensional case.** Let \( \mathcal{H} \) be of the type described in Lemma 5.3. Henceforth we shall assume complex scalars (the real case is postponed to the end of the proof). Let \( A \in \mathcal{L}(\mathcal{H}_0) \) be the operator associated with \( \mathcal{H} \) and \( \lambda = (\lambda_i)_{i=1}^n \) its distinct eigenvalues, ordered in increasing order. Denote by \( (e_i)_{i=1}^n \) the corresponding orthonormal basis of \( \mathcal{H}_0 \) consisting of eigenvectors of \( A \). Then for a generic vector \( f = \sum_{i=1}^n f_i e_i \in \mathcal{H}_0 \)

\[
\|f\|_0^2 = \sum_{i=1}^n |f_i|^2, \quad \|f\|_1^2 = \sum_{i=1}^n \lambda_i |f_i|^2.
\]

Working in the co-ordinate system \( e_i \) the couple \( \mathcal{H} \) becomes identified with the weighed \( n \)-dimensional \( \ell_2 \)-couple \( (\ell_2^n, (\ell_2^n(\lambda))) \) defined by (5.12) for \( f = (f_i)_{i=1}^n \in \ell_2^n \). Put (see (3.1))

\[
K_{2,\lambda}(t, f)^2 = K_2(t, f; (\ell_2^n, (\ell_2^n(\lambda))))^2 = \sum_{i=1}^n |f_i|^2 \frac{t \lambda_i}{1 + t \lambda_i}.
\]

Let \( \varrho > 1 \) be given such that

\[
\varrho \lambda_i < \lambda_{i+1}, \quad i = 1, \ldots, n - 1.
\]
The problem then becomes the following: given \( f^0, g^0 \in \mathcal{L}_n \) such that
\[
K_{2,\lambda}(t^{-1}, g^0)^2 < g^{-1} K_{2,\lambda}(t^{-1}, f^0)^2, 
\]
t \geq 0,
we must verify the existence of a matrix \( T = T_{f^0, g^0} \in M_n(\mathbb{C}) := \mathcal{L}(\ell^n_2) \) such that
\[
T f^0 = g^0 \quad \text{and} \quad K_{2,\lambda}(t, T f) \leq K_{2,\lambda}(t, f), 
\]
t \geq 0, f \in \ell^n_2.
To simplify the problem, let us first suppose that our problem is soluble with respect to the elements \( |f^0| = (|f_i^0|)_{i=1}^n, |g^0| = (|g_i^0|)_{i=1}^n \), i.e. there exists \( T_0 \in M_n(\mathbb{C}) \) such that
\[
T_0 |f^0| = |g^0| \quad \text{and} \quad K_{2,\lambda}(t, T_0 f) \leq K_{2,\lambda}(t, f), 
\]
t \geq 0, f \in \ell^n_2.
Choosing for each \( k \) numbers \( \theta_k, \varphi_k \in \mathbb{R} \) such that \( f^0_k = e^{i\theta_k} |f_k^0| \) and \( g^0_k = e^{i\varphi_k} |g_k^0| \), we infer that (5.15) holds with respect to the matrix
\[
T = \text{diag}(e^{i\varphi_k}) T_0 \text{diag}(e^{-i\theta_k}),
\]
whence there is no loss of generality in assuming that the entries \( f^0_k, g^0_k \) be non-negative. Moreover, replacing \( g^0_k \) by \( f^0_k \) and \( f^0_k \) by \( g^0_k \) by small perturbations if necessary, we can besides (5.14), assume
\[
(5.16) \quad f^0_i > 0, \quad g^0_i > 0.
\]
Put \( \beta_i = \lambda_i, \alpha_i = \varphi_i \); then (5.13) becomes
\[
(5.17) \quad 0 < \beta_1 < \alpha_1 < \cdots < \beta_n < \alpha_n.
\]
It is a simple matter to check that
\[
(5.18) \quad K_{2,\beta}(t, f)^2 \leq K_{2,\alpha}(t, f)^2 \leq \varrho K_{2,\beta}(t, f)^2, 
\]
t \geq 0, f \in \ell^n_2,
whence (5.14) yields
\[
K_{2,\alpha}(t^{-1}, g^0) < K_{2,\beta}(t^{-1}, f^0), 
\]
t \geq 0.
Moreover, (5.18) yields that, verifying (5.15), it suffices to verify the existence of a matrix \( T = T_{g^0, f^0} \in M_n(\mathbb{C}) \) such that
\[
(5.20) \quad T f^0 = g^0 \quad \text{and} \quad K_{2,\alpha}(t^{-1}, T f) \leq K_{2,\beta}(t^{-1}, f), 
\]
t \geq 0, f \in \ell^n_2.
Starting the construction of \( T \in M_n(\mathbb{C}) \) fulfilling (5.20), we put
\[
L_\beta(t) = \prod_{i=1}^n (t + \beta_i) \quad L_\alpha(t) = \prod_{i=1}^n (t + \alpha_i) \quad L(t) = L_\beta(t) L_\alpha(t),
\]
and note that (5.17) yields
\[
(5.21) \quad L'(-\beta_i) > 0, \quad L'(-\alpha_i) < 0.
\]
We now define an important polynomial \( P \in \mathcal{P}_{2n-1}(\mathbb{R}) \) by
\[
(5.22) \quad \frac{P(t)}{L(t)} = \frac{K_{2,\beta}(t^{-1}, f^0)^2 - K_{2,\alpha}(t^{-1}, g^0)^2}{\sum_{i=1}^n (f_i^0)^2 \frac{\beta_i}{t + \beta_i} - \sum_{i=1}^n (g_i^0)^2 \frac{\alpha_i}{t + \alpha_i}}.
\]
By (5.19) we have \( P(t) > 0 \) for \( t \geq 0 \). Moreover, \( P \) is uniquely determined by the \( 2n \) conditions
\[
(5.23) \quad P(-\beta_i) = (f_i^0)^2 \beta_i L'(-\beta_i), \quad P(-\alpha_i) = -(g_i^0)^2 \alpha_i L'(-\alpha_i).
\]
We note that (5.16), (5.21) and (5.23) yields
\[
(5.24) \quad P(-\beta_i) > 0, \quad P(-\alpha_i) > 0.
\]
We claim that it suffices to prove (5.20) in the case when
\[
(5.25) \quad P \text{ has exact degree } 2n - 1 \text{ and all zeros of } P \text{ have multiplicity } 1.
\]
To prove this, we note that polynomials fulfilling (5.25) constitute an open, dense subset \( G \) of polynomials \( P \in \mathcal{P}_{2n-1}(\mathbb{R}) \) fulfilling \( P(-\beta_i) > 0 \), \( P(-\alpha_i) > 0 \) and \( P(t) > 0 \) for \( t \geq 0 \). Since the formulas (5.23) for the coefficients \( f^0, g^0 \) define continuous (positive) functions of \( P \in C \), we can, replacing \( f^0, g^0 \) by small perturbations if necessary, assure that (5.25) holds. The set \( G \) shall henceforth be referred to as the set of generic polynomials.

Let \( P \in G \), i.e. \( P \) fulfills (5.24) and (5.25) and \( P(t) > 0 \) for \( t \geq 0 \). We split the zeros of \( P \) according to
\[
P^{-1}(\{0\}) = \{-r_i\}_{i=1}^{2m-1} \cup \{-c_i\}_{i=1}^{n-m} \cup \{-\bar{c}_i\}_{i=1}^{n-m},
\]
where the \( r_i \)'s are positive, and the \( c_i \) are in the open upper half plane. We have the following lemma.

**Lemma 5.4.** The following inequalities hold,
\[
(5.26) \quad L'(-\beta_i)P(-\beta_i) > 0 , \quad L'(-\alpha_i)P(-\alpha_i) < 0,
\]
and there is a splitting \( \{r_i\}_{i=1}^{2m-1} = \{\delta_i\}_{i=1}^{m} \cup \{\gamma_i\}_{i=1}^{m-1} \), where
\[
(5.27) \quad L(-\delta_j)P'(-\delta_j) > 0 , \quad L(-\gamma_k)P'(-\gamma_k) < 0.
\]

**Proof.** The inequalities (5.26) follow immediately from (5.21) and (5.24), so it remains only to prove (5.27). Let \(-h \) denote the leftmost real zero of the polynomial \( LP \). We claim that \(-h \) is a zero of \( P \). In order to prove this, we assume the contrary, i.e. \( h = \alpha_n \). Since the degree of \( P \) is odd, and \( P(t) > 0 \) for \( t > 0 \), this polynomial must be negative for large negative values of \( t \), which implies \( P(-\alpha_n) < 0 \), contradicting (5.24) and proving our claim. It follows that \( L(-h)P'(-h) = (LP)'(-h) > 0 \), which justifies the notation \( h = \delta_m \). Putting \( P_*(t) = P(t)/(t + \delta_m) \) and noting that \( t + \delta_m > 0 \) for \( t \in \{-\alpha_i\}_{i=1}^{n} \cup \{-\beta_i\}_{i=1}^{n} \), (5.24) yields
\[
P_*(-\beta_i) > 0 , \quad P_*(-\alpha_i) > 0.
\]
Thus \( P_* \) has either 0 or an even number of zeros between each pair of zeros of \( L \).

Let \( \{-r_j^*\}_{j=1}^{2m-2} \) denote the real zeros of \( P_* \). Since the degree of \( LP_* \) is even, and the polynomial \( (LP_*)' \) has alternating signs in the set \( \{-\alpha_i\}_{i=1}^{n} \cup \{-\beta_i\}_{i=1}^{n} \cup \{-r_j^*\}_{j=1}^{2m-2} \), this yields that we may split the zeros of \( P_* \) as \( \{-\delta_i\}_{i=1}^{m} \cup \{-\gamma_i\}_{i=1}^{m-1} \), where
\[
(5.28) \quad L(-\delta_i)P_*'(-\delta_i) > 0 , \quad L(-\gamma_i)P_*'(-\gamma_i) < 0.
\]
Since \( P'(r_j^*) = (\delta_m - r_j^*)P_*'(r_j^*) \) and \( \delta_m > r_j^* \), we get that \( \text{sign}(P'(r_j^*)) = \text{sign}(P_*'(r_j^*)) \) for all \( j \), and (5.27) follows from (5.28). \( \square \)

Recall that \( \{-c_i\}_{i=1}^{n-m} \) denotes those zeros of \( P \) which are in the open upper half-plane, and put
\[
L_\delta(t) = \prod_{i=1}^{m}(t + \delta_i) , \quad L_\gamma(t) = \prod_{i=1}^{m-1}(t + \gamma_i) , \quad L_c(t) = \prod_{i=1}^{n-m}(t + c_i).
\]
(If \( m = 1 \), we define \( L_\alpha = 1 \), and if \( n = m \), we define \( L_c = 1 \).) Then \( P \) belongs to the \( n \)-dimensional space \( V \subset \mathcal{P}_{2n-1}(\mathbb{C}) \) defined by
\[
(5.29) \quad V = \{L_tL_\delta q : q \in \mathcal{P}_{n-1}(\mathbb{C})\}.
\]
Indeed, we have that $P = L_ε L_δ \cdot (a L_γ L_ε)$ where $a$ is the leading coefficient of $P$. For $Q \in V$, we now put

$$\frac{|Q(t)|^2}{L(t)P(t)} = \sum_{i=1}^{n} |f_i|^2 \frac{\beta_i}{t + \beta_i} - \sum_{i=1}^{n} |g_i|^2 \frac{\alpha_i}{t + \alpha_i} - \sum_{i=1}^{m-1} |h_i|^2 \frac{\gamma_i}{t + \gamma_i},$$

where

$$f_i = \frac{Q(-\beta_i)}{(\beta_i L'(-\beta_i)P(-\beta_i))^{\frac{1}{2}}}, \quad g_i = \frac{Q(-\alpha_i)}{(-\alpha_i L'(-\alpha_i)P(-\alpha_i))^{\frac{1}{2}}}, \quad h_i = \frac{Q(-\gamma_i)}{(-\gamma_i L(-\gamma_i)P'(-\gamma_i))^{\frac{1}{2}}}.$$  

By (5.31) are defined linear maps

$$\ell_2^0 \to V : f \mapsto Q \quad V \to \ell_2^0 \oplus \ell_2^{m-1} : Q \mapsto g \oplus h.$$  

Their composite is thus a matrix $\overline{T} \in M_{n \times (n+m-1)}(\mathbb{C})$. Putting $Q = P$ in (5.30), we see that $\overline{T} f_0^0 = g_0^0 \oplus 0$. Let $T \in M_n(\mathbb{C})$ be defined by $T = E \overline{T}$ where $E$ is the orthogonal projection onto the first $n$ co-ordinates of $\ell_2^0 \oplus \ell_2^{m-1}$. Then $T f_0^0 = g_0^0$ and by (5.30)

$$K_{2,\beta}(t^{-1}, f)^2 - K_{2,\alpha}(t^{-1}, T f)^2 = \sum_{i=1}^{n} |f_i|^2 \frac{\beta_i}{t + \beta_i} - \sum_{i=1}^{n} |g_i|^2 \frac{\alpha_i}{t + \alpha_i} \geq \sum_{i=1}^{n} |f_i|^2 \frac{\beta_i}{t + \beta_i} - \sum_{i=1}^{n} |g_i|^2 \frac{\alpha_i}{t + \alpha_i} - \sum_{i=1}^{m-1} |h_i|^2 \frac{\gamma_i}{t + \gamma_i} = \frac{|Q(t)|^2}{L(t)P(t)} \geq 0, \quad t > 0, \ f \in \ell_2^0.$$  

Thus $T$ satisfies (5.20), and Lemma 2.2 is proved in the case of complex scalars. In the case of real scalars, the proof must be modified by showing that any matrix $T$ satisfying (5.20) can be replaced by a real matrix fulfilling the same condition. Hereby the argument is as follows: since the complex matrix $T \in M_n(\mathbb{C})$ satisfies (5.20), so does the complex conjugate $\overline{T}$, and therefore, so does any convex combination of $T$ and $\overline{T}$. In particular, (5.20) is satisfied by the real matrix $\Re(T) = (T + \overline{T})/2$. This finishes the proof of Lemma 2.2. q.e.d.

## 6. Computations

In this section, we shall consider an arbitrary increasing sequence $\lambda = (\lambda_i)_{i=1}^{n} \subset \mathbb{R}_+$ and two non-negative vectors $f^0, g^0 \in \ell_2^0(= \ell_2^0(\mathbb{C}))$ fulfilling $f_i^0 \geq 0, g_i^0 \geq 0$ and

$$K_{2,\lambda}(t, f^0) \leq K_{2,\lambda}(t, 0), \quad t > 0.$$  

We shall explain how the material from the previous section can be used to explicitly calculate good approximations of a real matrix $T = T_{f^0, g^0} \in M_n(\mathbb{R}) := \mathcal{L}(\ell_2^0(\mathbb{R}))$ fulfilling

$$T f^0 = g^0 \quad \text{and} \quad K_{2,\lambda}(t, T f) \leq K_{2,\lambda}(t, f), \quad t > 0, \ f \in \ell_2^0.$$  

For $g > 1$ fulfilling (5.13) we perturb $f^0, g^0$ slightly to vectors $f^e, g^e$ such that the following conditions are satisfied

(i) $f_i^e > 0$ and $g_i^e > 0$ for all $i$,

(ii) $f^e \to f^0$ and $g^e \to g^0$ as $g \to 1$,

(iii) $K_{2,\lambda}(t^{-1}, g^e) < g^{-1} K_{2,\lambda}(t^{-1}, f^e), \ t \geq 0$,

(iv) the polynomial $P(t) = L(t)(K_{2,\beta}(t^{-1}, f^e)^2 - K_{2,\alpha}(t^{-1}, g^e)^2)$ satisfies (5.25) (i.e. $P$ belongs to the set $G$ of generic polynomials).
We shall first consider the problem of constructing for each \( g, f^e, g^e \) as above, a matrix \( T_g = T_{g,f^e,g^e} \) such that

\[
T_g f^e = g^e \quad \text{and} \quad K_{2,\alpha}(t, T_g f) \leq K_{2,\beta}(t, f), \quad t > 0, \ f \in \ell^2_0,
\]

where it is understood that \( \beta_i = \lambda_i \) and \( \alpha_i = \gamma_i \lambda_i \) for all \( i \). As \( g, f^e \) and \( g^e \) approach \( 1, f^0 \) and \( g^0 \) respectively, it is clear that any cluster point of the corresponding set \( T_g \) of matrices fulfilling (6.3) must be a matrix fulfilling (6.2).

**Theorem 6.1.** The matrix \( T_g = T_{g,f^e,g^e} \) defined by

\[
T_g = \left( \Re \left[ \frac{1}{\alpha_i - \beta_k} \frac{f_k^e \beta_k L_\delta(-\alpha_i) L_\gamma(-\alpha_i) L_{\alpha_i}(-\beta_k) L_{\beta_i}(-\beta_k)}{g_k^e \alpha_i L_\delta(-\beta_k) L_{\alpha_i}(-\beta_k) L_{\beta_i}(-\alpha_i)} \right] \right)_{i,k=1}^{n}
\]

satisfies (6.3). Moreover, each accumulation point \( T \) of the \( T_g \)'s as \( g \downarrow 1 \) satisfies (6.2).

**Remark 6.1.** We emphasize that each of the quantities \( \alpha_i, \delta_i, \gamma_i \) and \( c_i \) appearing in the formula (6.4) for \( T_g \) depend in an essential way on the parameter \( g \), whereas the vectors \( f^e, g^e \) depend on \( g \) only in the pathological cases when \( f^0, g^0 \) fail to satisfy (i), (iii) and (iv).

**Proof of Theorem 6.1.** We have the following polynomials

\[
L_{\beta}(t) = \prod_{i=1}^{n}(t + \beta_i) \quad L_\alpha(t) = \prod_{i=1}^{n}(t + \alpha_i) \quad L(t) = L_{\beta}(t)L_\alpha(t)
\]

\[
L_\delta(t) = \prod_{i=1}^{m}(t + \delta_i) \quad L_\gamma(t) = \prod_{i=1}^{m}(t + \gamma_i) \quad L_c(t) = \prod_{i=1}^{m}(t + c_i)
\]

\[
P(t) = \sum_{i=1}^{n} (\beta_i(f_i^0)^2 - \alpha_i(g_i^0)^2)L_\delta(t)L_\gamma(t)L_c(t)^2.
\]

(To verify the expression for the leading coefficient of \( P \) in the last formula, multiply (5.22) by \( t \), then let \( t \to \infty \).) We introduce the following basis of the space \( V \) (cf. (5.29)):

\[
Q_k(t) = \frac{L_\delta(t)L_c(t)L_{\beta}(t)}{t + \beta_k} \left( \frac{(\beta_kL'(-\beta_k)P(-\beta_k))^{\frac{3}{2}}}{L_\delta(-\beta_k)L_\gamma(-\beta_k)L_{\beta}'(-\beta_k)} \right), \quad k = 1, \ldots, n.
\]

Then

\[
\frac{Q_k(-\beta_i)}{(\beta_iL'(-\beta_i)P(-\beta_i))^{\frac{3}{2}}} = \delta_{ik} \quad \text{(Kronecker delta)}.
\]

Denoting by \( (e_i)_{i=1}^{n} \) be the canonical basis in \( \ell^2_0 \) and using (5.31), we get

\[
(T_g)e_k = (T_{g,f^e,g^e})e_k = \frac{Q_k(-\alpha_i)}{(\alpha_iL'(-\alpha_i)P(-\alpha_i))^{\frac{3}{2}}}
\]

\[
= \frac{1}{\beta_k - \alpha_i} \frac{L_\delta(-\alpha_i)L_c(-\alpha_i)L_{\beta}(-\alpha_i)}{\beta_kL'(-\beta_k)P(-\beta_k)} \left( \frac{(\beta_kL'(-\beta_k)P(-\beta_k))^{\frac{1}{2}}}{L_\delta(-\beta_k)L_\gamma(-\beta_k)L_{\beta}'(-\beta_k)} \right), \quad 1 \leq i, k \leq n.
\]
Tucking in the expressions (5.23) for \( P(-\beta_k), P(-\alpha_i) \) into (6.5) we thus get
\[
(T_{\rho})_{ik} = \frac{1}{\beta_k - \alpha_i} \frac{L_\delta(-\alpha_i)L_c(-\alpha_i)L_\beta(-\alpha_i)}{L_\delta(-\beta_k)L_c(-\beta_k)L_\beta(-\beta_k)} \left( \left( \frac{f_\delta^n \beta_k L'(-\beta_k)}{-g_\delta^n \alpha_i L'(-\alpha_i)} \right)^2 \right)^{\frac{1}{2}} = \frac{1}{\alpha_i - \beta_k} \frac{f_\delta^n \beta_k L_\delta(-\alpha_i)L_c(-\alpha_i)L_\alpha(-\beta_k)}{g_\delta^n \alpha_i L_c(-\beta_k)L_\alpha(-\alpha_i)}.
\]
This matrix solves (6.3) in the case of complex scalars. In order to get the solution in the form (6.4), we recall from the concluding remarks of the proof of Lemma 2.2, that (6.3) remains valid if we replace \( T_{\rho} \) by its real part.

**Example 6.1.** Let \( n = 5 \) and
\[
\lambda = (1, 2, 4, 5, 6), \quad f^0 = \left(0, \frac{5}{\sqrt{24}}, 0, \frac{4}{\sqrt{3}}, 0\right)^t, \quad g^0 = \left(\frac{2}{\sqrt{15}}, 0, \frac{7}{\sqrt{12}}, 0, \frac{9}{\sqrt{40}}\right)^t.
\]
Then, letting \( L_\lambda(t) = \prod_1^5 (t + \lambda_i) \), it follows that
\[
K_{2,\lambda}(t^{-1}, f^0)^2 - K_{2,\lambda}(t^{-1}, g^0)^2 = \frac{t(t-3)^2}{L_\lambda(t)} \geq 0, \quad t > 0,
\]
and hence there exists a matrix \( T \) such that \( T f^0 = g^0 \) and \( \| T \|_{L_2(\mathcal{H}_i)} \leq 1 \). In fact, \( \| f^0 \|_i = \| g^0 \|_i, \quad i = 0, 1 \), so we must have \( \| T \|_{L_2(\mathcal{H}_i)} = 1, \quad i = 0, 1 \). To get a good approximation \( T_{\rho} \) of such a matrix, we made use of a Matlab-program based on formula (6.4) with the following data
\[
(6.6) \quad \rho = 1.0001, \quad f^e = \rho f^0 + (\rho - 1) u, \quad g^e = \rho^{-1} g^0 + (\rho - 1) u, \quad u = (1, 1, 1, 1, 1)^t.
\]
For \( T_{\rho} \), we obtained the following matrix:
\[
(6.7) \quad \begin{pmatrix}
0.0002 & 0.8299 & 0.0000 & -0.1432 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.4683 & 0.0000 & 0.6678 & 0.0000 \\
0.0000 & 0.0000 & 0.0000 & -0.0013 & 0.0000 \\
0.0000 & -0.2531 & 0.0000 & 0.7279 & 0.0000
\end{pmatrix}
\]
We also found that
\[
T_{\rho} f^0 - g^0 = -\left(0.0001, 0.0000, 0.0004, 0.0031, 0.0002\right)^t,
\]
\[
\| T_{\rho} \|_{L_2(\mathcal{H}_2)} = 0.9999, \quad \| T_{\rho} \|_{L_2(\mathcal{H}_2(\lambda))} = 0.9998.
\]
However, in this case it is possible to calculate exactly a solution matrix, because the problem is of “type 2”, cf. Appendix A. It yields
\[
\begin{pmatrix}
0 & \frac{4\sqrt{2}}{3\sqrt{5}} & 0 & -\frac{1}{3\sqrt{5}} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & \frac{\sqrt{2}}{3} & 0 & \frac{2}{3} & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & -\frac{1}{\sqrt{15}} & 0 & \frac{4}{\sqrt{30}} & 0
\end{pmatrix}
\]
and we note that (6.7) is indeed a good approximation of this matrix.

**Example 6.2.** Let \( n = 5 \) and
\[
\lambda = (1, 2, 4, 5, 6), \quad f^0 = \left(1, \frac{7}{\sqrt{24}}, 1, \sqrt{\frac{19}{3}}, 1\right)^t, \quad g^0 = \left(\frac{19}{15}, 1, \frac{61}{12}, 1, \frac{121}{40}\right)^t.
\]
As in our first example,
\[ K_{2,\lambda}(t^{-1}, f^0)^2 - K_{2,\lambda}(t^{-1}, g^0)^2 = \frac{t(t-3)^2}{L_\lambda(t)} \geq 0, \quad t > 0, \]
but this problem is of none of the types 1 or 2 of Appendix A. We must thus rely on Theorem 6.1 to compute a matrix \( T \) such that \( Tf^0 = g^0 \) and \( \|T\|_{L^2} \leq 1 \). We used the same data (6.6) as in the first example and obtained the matrix
\[
\begin{pmatrix}
0.8888 & 0.2685 & 0.0000 & -0.0585 & 0.0000 \\
0.0000 & 0.6997 & 0.0000 & 0.0000 & 0.0000 \\
0.0000 & 0.2989 & 0.4437 & 0.5497 & 0.0002 \\
0.0000 & 0.0000 & 0.0000 & 0.3973 & 0.0000 \\
0.0000 & -0.1462 & 0.0000 & 0.5456 & 0.5749 \\
\end{pmatrix}
\]
as is to be expected, the diagonal elements of this matrix are considerably larger than those of (6.7) whereas the off-diagonal elements are much smaller. We obtained
\[
g^0 - Tg^0 = (0.0002, 0.0002, 0.0005, 0.0001, 0.0003)^t,
\|T\|_{L^2} = 0.9999 = \|T\|_{L^2(\lambda)}.
\]

**Example 6.3.** As before, let \( n = 5 \) but
\[
\lambda = (1, 2, 4, 5, 6), \quad f^0 = \left( \frac{1}{\sqrt{30}}, 0, \sqrt{\frac{17}{48}}, 0, \sqrt{\frac{37}{240}} \right)^t, \quad g^0 = \left( 0, \sqrt{\frac{5}{48}}, 0, \sqrt{\frac{13}{30}} \right)^t.
\]
This time, one finds (with \( L_\lambda(t) = \prod_{i=1}^5 (t + \lambda_i) \))
\[
\frac{t^2 + 1}{L_\lambda(t)} = K_{2,\lambda}(t^{-1}, f^0)^2 - K_{2,\lambda}(t^{-1}, g^0)^2 \geq 0, \quad t > 0.
\]
Even though the vectors \( f^0, g^0 \) have “disjoint supports”, the problem is of none of the types 1 or 2 of Appendix A, because the polynomial \( t^2 + 1 \) is neither of the form \( Q(t)^2 \), nor \( tQ(t)^2 \). Again, we must thus rely on Theorem 6.1 and our computer algebra. We used the following data
\[
g = 1.0001, \quad f^e = f^0 + (g - 1)u, \quad g^e = g^0 + (g - 1)u, \quad u = (1, 1, 1, 1, 1)^t.
\]
For \( T_g \), we obtained the following matrix
\[
\begin{pmatrix}
0.0005 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
0.5654 & 0.0003 & 0.6504 & 0.0000 & -0.4266 \\
0.0000 & 0.0000 & -0.0002 & 0.0000 & 0.0000 \\
-0.0055 & 0.0000 & 0.5956 & 0.0002 & 0.7994 \\
0.0000 & 0.0000 & 0.0000 & 0.0000 & 0.0000 \\
\end{pmatrix}
\]
It yields
\[
g^0 - T_g f^0 = (0.0001, 0.0001, -0.0002, -0.0001, 0.0000)^t,
\|T\|_{L^2} = 0.9999 = \|T\|_{L^2(\lambda)}.
\]

**Example 6.4.** Let \( L_2 \) be the Lebesgue space on \( \mathbb{R}^d \), \( m \) a positive integer and \( H^m \) the Sobolev space on \( \mathbb{R}^d \), i.e. the set of tempered distributions \( g \) on \( \mathbb{R}^d \) for which all derivatives \( D^\alpha \) of order \( |\alpha| \leq m \) belong to \( L_2 \). It is well-known that \( H^m \) is a Hilbert space under the norm (cf. [11], p. 5)
\[
(6.8) \quad \|g\|_{H^m} = \left( \int_{\mathbb{R}^d} (1 + |y|^2)^m |\hat{g}(y)|^2 dy \right)^{1/2}
\]
Thus Theorem 2.2 yields that the inequality (7.1)

\[ \|g\|_A \leq \|f^0\|_A \]

implies \( (g^0, f^0) \in L_2 \). Hence (3.1), (6.8) and Plancherel’s theorem we have

\[ K_2(t, g; L_2, H^m)^2 = \int_{\mathbb{R}^d} \frac{t(1 + |y|^2)^m}{1 + t(1 + |y|^2)^m} |\hat{g}(y)|^2 dy, \quad g \in L_2. \]

Thus Theorem 2.2 yields that the inequality \((g^0, f^0) \in L_2\)

\[ \int_{\mathbb{R}^d} \frac{t(1 + |y|^2)^m}{1 + t(1 + |y|^2)^m} |\hat{g}(y)|^2 dy \leq \int_{\mathbb{R}^d} \frac{t(1 + |y|^2)^m}{1 + t(1 + |y|^2)^m} |\hat{f^0}(y)|^2 dy, \quad t > 0 \]

is sufficient to guarantee the existence of a linear operator \( T \) such that \( Tf^0 = g^0 \) and

\[ \int_{\mathbb{R}^d} \frac{t(1 + |y|^2)^m}{1 + t(1 + |y|^2)^m} |\hat{Tf}(y)|^2 dy \leq \int_{\mathbb{R}^d} \frac{t(1 + |y|^2)^m}{1 + t(1 + |y|^2)^m} |\hat{f}(y)|^2 dy, \quad f \in L_2. \]

In particular, \( \|g^0\|_A \leq \|f^0\|_A \) whenever \( A \) is an exact interpolation space with respect to the couple \((L_2, H^m)\).

### 7. Concluding remarks

Let us review some results from the theory of Calderón couples in the case of weighted Lebesgue spaces. We shall initially restrict the discussion to the diagonal case.

Two seemingly different characterizations of the exact interpolation spaces with respect to the couple \((L_1, L_{\infty})\) were independently found by A. P. Calderón [7] and B. S. Mityagin [13] in the mid-1960’s. The approaches, which have both proven worthwhile for the further development of the theory turn out to be equivalent, and moreover equivalent to exact \(K\)-monotonicity, cf. [21], pp. 233-234. Hence our terminology “Calderón couple” is somewhat biased, but is advantageous due to its brevity.

Around 1965 was also known another (rather more trivial) Calderón couple, namely \((L_{\infty}(a_0), L_{\infty}(a_1))\), \(a_0, a_1\) being positive measurable weight functions on some measure space \((X, \mu)\). Less obvious is the case \((L_1(a_0), L_1(a_1))\), which was discovered in 1971 by A. A. Sedaev and E. M. Semenov [18].

The early 1970’s saw considerable progress in the field, as several authors tried to generalize the above results to arbitrary weighted \(L_p\)-spaces. Before we describe the outcome of this work, let us remind of some notions from the corresponding non-exact theory.

We say that intermediate spaces \( A, B \) are (non-exact) \( K \)-monotonic with respect to \( A, B \) iff there exists \( C \) such that

\[ f \in A \text{ and } K(t, g; B) \leq K(t, f; A) \text{ implies } g \in B \text{ and } \|g\|_B \leq C \|f\|_A. \]  

(7.1) By Lemma 2.1, (7.1) implies that \( A, B \) are \( C \)-interpolation spaces with respect to \( A, B \). Let us denote \( A, B \) non-exact Calderón couples if (c-Int) implies (7.1) with some \( C \geq c \). If this is the case, we have a characterization of the class of interpolation spaces, but in general not of the subclass of exact interpolation spaces.
Couples may fail to be Calderón even in the non-exact sense. This happens e.g. in the case $A = B = (L^p(R), W^{1,p}(R)), p \neq 2$, cf. [8] ($W^{1,p}$ is Sobolev space). It is interesting to compare with the Hilbert case $p = 2$, which is a case of Calderón couples by Theorem 2.1, cf. Example 6.4.

Below is stated the answer to the non-exact problem for weighted $L^p$-spaces, which was announced by Gunnar Sparr in 1974 [20], and completed with proofs and details in a remarkable paper in 1978 [21].

**Theorem S.** Let $1 \leq p_0, p_1 \leq \infty$. Let $A = (L^{p_0}(a_0), L^{p_1}(a_1))$ and $B = (L^{p_0}(b_0), L^{p_1}(b_1))$ be the $L^p$-spaces associated with some (possibly different) measure spaces. Then, for all exact interpolation spaces $A, B$ relative $A, B$ we have that (7.1) holds with some constant $C = C(p_0, p_1) \leq 2$. In the special case $p_0 = p_1 = p$ moreover holds $C \leq 2^{1/p'}$ for the best $C$.

As a complement to the theorem, Sparr also gives a counterexample disproving the possibility of a general exact theorem for weighted $L^p$-spaces. More precisely is shown that the couple $(L^1, L^p)$ fails to be Calderón for all $1 < p < \infty$ ([21], Example 5.2). However, restricting to the case $p_0 = p_1 = p$ the question of exactness is still open, and shall be commented on below.

Relative to couples of the form $A = (L_p, L_p(\lambda))$ it is interesting besides $K$ to study the functional $K_p$ given by

$$K_p(t, f) = K_p(t, f; A) = \inf_{f = f_0 + f_1} (\|f_0\|_p^p + t\|f_1\|_p^p)^{1/p}. $$

Defining the corresponding quasi-order $g \leq f[K_p]$ in the obvoius way, we have (cf. Sparr [21], Lemma 3.3)

$$g \leq f[K] \iff g \leq f[K_p].$$

It is not hard to verify the following formula for $K_p$ ($1 < p < \infty$)

$$K_p(t, f; L_p, L_p(\lambda))^p = \int_X \frac{t\lambda(x)}{(1 + (t\lambda(x))^{p-1})^{p-1}}|f(x)|^p d\mu(x), \quad f \in L_p + L_p(\lambda),$$

which yields an analytic expression for the quasi-order (7.2) all $1 < p < \infty$, resembling in several aspects the case $p = 2$. This circumstance and the presence of exact results in the cases $p \in \{1, 2, \infty\}$ inspires us to round off the exposition with the following conjecture.

**Conjecture 7.1.** Generally holds that the couples $A = (L_p(a_0), L_p(a_1))$, $B = (L_p(b_0), L_p(b_1))$ are Calderón, $1 \leq p \leq \infty$. 

---

[4] Sparr also includes the quasi-Banach case $0 < p_0, p_1 \leq 1$ with some restrictions on the underlying measure spaces. We prefer here this more compact version of the theorem.

[5] This estimate was first obtained in the diagonal case by Sedaev [17].
APPENDIX A

Two matrices associated with the couple \( (\ell_2^n, \ell_2^0(\lambda)) \).

In this appendix, we associate to the couple \( (\ell_2^n, \ell_2^0(\lambda)) \) two (real) matrices, one of which generalizes Löwner’s matrix (known from the theory of monotone matrix functions, cf. [9], [12]) the other one seems to have been introduced by the author in [2]. Our construction owes much to Sparr [19], indeed (for even \( n \)) our construction of Löwner’s matrix is the same as Sparr’s. In the following, we will assume real scalars, but with minor modifications, one may include complex scalars into the setting.

Suppose that \( f^0, g^0 \in \ell_2^n(\mathbb{R}) \) fulfill
\[
K_{2,\lambda}(t^{-1}, f^0) \leq K_{2,\lambda}(t^{-1}, g^0), \quad t > 0.
\]
Let \( L_\lambda(t) = \prod_{i=1}^n (t + \lambda_i) \) and \( P \in \mathcal{P}_{n-1}(\mathbb{R}) \) the polynomial fulfilling
\[
\frac{P(t)}{L_\lambda(t)} = K_{2,\lambda}(t^{-1}, f^0)^2 - K_{2,\lambda}(t^{-1}, g^0)^2 = \sum_{i=1}^n ((f_i^0)^2 - (g_i^0)^2) \frac{\lambda_i}{t + \lambda_i}.
\]
Here \( P \) is uniquely determined by the \( n \) conditions \( P(-\lambda_i) = ((f_i^0)^2 - (g_i^0)^2)/(-\lambda_iL'_\lambda(-\lambda_i)) \).
Let \( (u_1, v_1, u_2, v_2, \ldots) \) denote the ordered canonical basis of \( \ell_2^n \) and let \( \ell_2^0 = V_o \oplus V_e \) be the corresponding splitting, i.e.
\[
V_o = \text{span}\{u_i\} \quad V_e = \text{span}\{v_i\}.
\]
Note that
\[
\dim(V_o) = \lceil (n - 1)/2 \rceil + 1 \quad \dim(V_e) = \lceil (n - 2)/2 \rceil + 1.
\]
Here \( \lceil r \rceil \) denotes the integer part of \( r \), a real number. Below are constructed matrices \( T \in M_n(\mathbb{R}) \) such that
\[
(T f^0, f^0) \quad \text{and} \quad K_{2,\lambda}(t, T f) \leq K_{2,\lambda}(t, f), \quad t > 0, f \in \ell_2^n.
\]
in the following special cases: \[1\]
\begin{enumerate}
\item \( P(t) = Q^0(t)^2, \quad Q^0 \in \mathcal{P}_{(n-1)/2}(\mathbb{R}), f^0 \in V_o, g^0 \in V_e, \)
\item \( P(t) = tQ^0(t)^2, \quad Q^0 \in \mathcal{P}_{(n-2)/2}(\mathbb{R}), f^0 \in V_e, g^0 \in V_o. \)
\end{enumerate}
Hereby, it will be convenient to rewrite the \( \lambda_i \)’s as \( \lambda_i = \xi_i \) when \( i \) is odd and \( \lambda_i = \eta_i \) when \( i \) is even. Let us put
\[
L_\xi(t) = \prod_{i \text{ odd}} (t + \xi_i) \quad L_\eta(t) = \prod_{i \text{ even}} (t + \eta_i) \quad L_\lambda = L_\xi L_\eta
\]
and note that \( L'_\lambda(-\xi_i) > 0, L'_\lambda(-\eta_i) < 0. \)
Type 1: \( P(t) = Q^0(t)^2, Q^0 \in \mathcal{P}_{(n-1)/2}(\mathbb{R}), f^0 \in V_o, g^0 \in V_e. \) In this case,
\[
\frac{Q^0(t)^2}{L_\lambda(t)} = \sum_{k \text{ odd}} \frac{(f_k^0)^2}{t + \xi_k} - \sum_{i \text{ even}} \frac{(g_i^0)^2}{t + \eta_i},
\]
\[1\] In connection with this problem it is interesting to note that any polynomial \( P \) such that \( P(t) \geq 0 \) for \( t \geq 0 \) can be written \( P(t) = Q_0(t)^2 + tQ_1(t)^2 \) for some polynomials \( Q_0, Q_1 \) with real coefficients, cf. [1], p. 77.
where
\begin{equation}
(A.2) \quad f_0^0 = \frac{\varepsilon_k Q^0(-\xi_k)}{(\xi_k L'_\lambda(-\xi_k))^\frac{1}{2}}, \quad g_i^0 = \frac{\zeta_i Q^0(-\eta_i)}{(-\eta_i L'_\lambda(-\eta_i))^\frac{1}{2}},
\end{equation}
for some choice of sign \( \varepsilon_k, \zeta_i \in \{\pm 1\} \). By (A.2) are defined linear maps
\[
V_o \rightarrow \mathcal{P}_{(n-1)/2} : f \mapsto Q ; \quad \mathcal{P}_{(n-1)/2} \rightarrow \mathcal{V}_e : Q \mapsto g,
\]
and their composite is thus a linear map
\[
T_0 : \mathcal{V}_o \rightarrow \mathcal{V}_e : f \mapsto g.
\]
We now define \( T \in M_n(\mathbb{R}) \) by
\[
T : \mathcal{V}_o \oplus \mathcal{V}_e \rightarrow \mathcal{V}_o \oplus \mathcal{V}_e : f \oplus v \mapsto 0 \oplus T_0 f.
\]
Evidently \( T f^0 = g_0^0 \) and
\begin{equation}
(A.3) \quad K_{2,\lambda}(t^{-1}, f \oplus v)^2 - K_{2,\lambda}(t^{-1}, T(f \oplus v))^2 \geq 0,
\end{equation}
and (A.1) is verified. A computation similar to the proof of Theorem 6.1 yields that (with respect to the bases \( u_k \) and \( v_i \))
\[
(T_0)_{ik} = \frac{\varepsilon_k \zeta_i}{\xi_k - \eta_i} \frac{L_\lambda(-\eta_i)}{L'_\lambda(-\xi_k)} \left( \frac{\xi_k L'_\lambda(-\xi_k) L_\lambda(-\xi_k)}{(-\eta_i L_\lambda(-\eta_i) L'_\lambda(-\eta_i))} \right)^\frac{1}{2}.
\]
It is noteworthy that multiplying (A.3) by \( t \), then letting \( t \rightarrow \infty \) yields
\[
\sum_{k \text{ odd}} f_k^2 \xi_k - \sum_{i \text{ even}} (T_0 f)^2 \eta_i = 0,
\]
that is, \( T \) is a partial isometry from \( \mathcal{V}_o \) to \( \mathcal{V}_e \) with respect to the norm of \( \ell^2_2(\lambda) \).

Type 2: \( P(t) = t Q^0(t)^2, \) \( Q^0 \in \mathcal{P}_{(n-2)/2}(\mathbb{R}) \), \( f^0 \in \mathcal{V}_e, \) \( g^0 \in \mathcal{V}_o \).

In this case,
\[
\frac{t Q^0(t)^2}{L_\lambda(t)} = - \sum_{i \text{ odd}} (g_i^0)^2 \frac{\xi_i}{t + \xi_i} + \sum_{k \text{ even}} (f_k^0)^2 \frac{\eta_k}{t + \eta_k},
\]
where
\begin{equation}
(A.4) \quad g_i^0 = \frac{\varepsilon'_i Q^0(-\xi_i)}{L'_\lambda(-\xi_i)^\frac{1}{2}}, \quad f_k^0 = \frac{\zeta'_k Q^0(-\eta_k)}{(-L'_\lambda(-\eta_i))^\frac{1}{2}}
\end{equation}
for some \( \varepsilon'_i, \zeta'_k \in \{\pm 1\} \). By (A.4) are defined linear maps
\[
\mathcal{V}_e \rightarrow \mathcal{P}_{(n-2)/2}(\mathbb{R}) : f \mapsto Q ; \quad \mathcal{P}_{(n-2)/2}(\mathbb{R}) \rightarrow \mathcal{V}_o : Q \mapsto g,
\]
and their composite
\[
T_1 : \mathcal{V}_e \rightarrow \mathcal{V}_o : f \mapsto g.
\]
Defining \( T \in M_n(\mathbb{R}) \) by
\[
T : \mathcal{V}_o \oplus \mathcal{V}_e \rightarrow \mathcal{V}_o \oplus \mathcal{V}_e : u \oplus f \mapsto T_1 f \oplus 0
\]
we thus get
\[
-K_{2,\lambda}(t^{-1}, T(u \oplus f))^2 + K_{2,\lambda}(t^{-1}, u \oplus f)^2 \geq 0,
\]
and
\[
-K_{2,\xi}(t^{-1}, T_1 f)^2 + K_{2,\eta}(t^{-1}, f)^2 = \frac{t Q(t)^2}{L_\lambda(t)} \geq 0, \quad t > 0, \quad u \in \mathcal{V}_o, \quad f \in \mathcal{V}_e,
\]
and (A.1) is verified also in this case. A computation yields that (with respect to the bases $v_k$ and $u_i$)

$$(T_1)_{ik} = \frac{\varepsilon_i' \zeta_k'}{\eta_k - \xi_i} \frac{L_{\eta_i}(-\xi_i)}{L'_{\eta_i}(-\eta_k)} \left( \frac{-L_{\xi_i}(-\eta_k)L'_{\eta_i}(-\eta_k)}{L'_{\xi_i}(-\xi_i)L_{\eta_i}(-\xi_i)} \right)^{\frac{1}{2}}. $$

In the case of even $n$ (and $\varepsilon_i' = \zeta_k' = 1$), the matrix $T_1$ coincides with Löwner’s matrix. In any case, $T$ is a partial isometry from $V_e$ to $V_o$ with respect to the norm of $\ell_2^n$. 

References

CHAPTER III

On some classes of matrix functions related to interpolation of Hilbert spaces

Abstract. A new proof is given to a theorem of W. F. Donoghue which characterizes certain classes of functions whose domain of definition are finite sets, and which are subject to certain matrix inequalities. The result generalizes the classical Löwner theorem on monotone matrix functions, and also yields some new information with respect to the finer study of monotone functions of finite order.

1. Introduction

A real function $h$ on $\mathbb{R}_+$ is said to be monotone of order $n$ and to belong to the class $P_n$ iff for any positive definite matrices $A, B \in M_n(\mathbb{C})$

$$A \leq B \quad \text{implies} \quad h(A) \leq h(B).$$

A function is matrix monotone iff it is monotone of all finite orders.

This definition along with the principal theorem (stated below) appeared in a remarkable paper [13] of Karl Löwner in the year 1934.

Theorem 1.1. For a positive function $h$ to be matrix monotone, it is necessary and sufficient that $h$ be representable in the form

$$h(\lambda) = \int_{[0, \infty]} \frac{(1 + t)\lambda}{1 + t\lambda} d\rho(t), \quad \lambda > 0,$$

with a positive Radon measure $\rho$ on the compactified half-line $[0, \infty]$.

The near inaccessibility of Löwner’s text to the non-experts has inspired several authors to find simplifications and new proofs cf. [4], [11], [15], [17], [10], cf. also Donoghue’s book [5] which contains a fairly comprehensive exposition of the theory up until 1974.

In our approach, we shall obtain Löwner’s theorem as a corollary of a different theorem which is equivalent to any of the following theorems of William Donoghue: [7], Theorem III and [6], Theorem 1. We note that Donoghue’s proof is rather complicated and relies on his (non-trivial) extension [7] of Löwner’s solution of a problem of interpolation by rational functions of a certain class.

Our method of proof is based on the theorem previously established by the author in [2] that all regular Hilbert couples are “Calderón couples”. We shall also in several instances benefit from ideas from Gunnar Sparr’s proof of Löwner’s theorem [17]. (A different proof of Donoghue’s theorem was previously given by the author in [1], where it was also noted that Löwner’s theorem is a consequence thereof.)

We introduce some notions from the theory of interpolation spaces. (For a detailed study of that theory we refer to [3]).
Let $\ell^2_0 = \ell^2_0(\mathbb{C})$ be the usual $n$-dimensional $\ell_2$-space. Given a positive definite matrix $A \in M_n(\mathbb{C}) := L(\ell^2_0)$ let a Hilbert couple $\mathcal{H} = (\mathcal{H}_0, \mathcal{H}_1)$ be defined by $\mathcal{H}_0 = \ell^2_0$ and
\[
\|f\|^2_0 = (Af, f)_0, \quad f \in \ell^2_0.
\]
(A is the associated operator relative to $\mathcal{H}$, cf. [2].)

It will be convenient to use the following alternative notations for the operator norms with respect to $\mathcal{H}_0$ and $\mathcal{H}_1$ ($T \in M_n(\mathbb{C})$)
\[
\|T\| = \sup_{(f, f)\leq 1} (T^* T f, f)_0, \quad \|T\|_A^2 = \sup_{(Af, f)\leq 1} (T^* A T f, f)_0.
\]

A positive function $h$ defined on $\sigma(A)$ is said to be interpolation with respect to $A$ and to belong to the set $C_A$ iff
\[
(2.1) \quad \|T\|_{h(A)} \leq \max(\|T\|, \|T\|_A), \quad T \in M_n(\mathbb{C}),
\]
where, naturally
\[
\|T\|_{h(A)}^2 = \sup_{(h(A)f, f)\leq 1} (T^* h(A) T f, f)_0.
\]

The following is our principal result; it is equivalent to Donoghue’s theorems [7], Theorem III and [6], Theorem 1.

THEOREM 1.2. Let $A \in M_n(\mathbb{C})$ be a positive definite matrix. For a positive function $h$ defined on $\sigma(A)$ to belong to the class $C_A$, it is necessary and sufficient that $h$ be representable in the form
\[
(1.3) \quad h(\lambda) = \int_{[0,\infty]} \frac{(1 + t)\lambda}{1 + t\lambda} d\rho(t), \quad \lambda \in \sigma(A),
\]
for some positive Radon measure $\rho$ on $[0, \infty]$. (The measure is not unique.)

2. Proof of Theorem 1.2

Let us introduce some further notation with some simplifying remarks.

Let $A \in M_n(\mathbb{C})$ be positive definite and put $\sigma(A) = \{\lambda_i\}_{i=1}^n \subset \mathbb{R}_+$. Let $(\zeta_i)_{i=1}^n$ be an orthonormal basis of $\ell^2_0$ consisting of eigenvectors of $A$ corresponding to the eigenvalues $\lambda_i, \ i = 1, \ldots, n$; then for a generic vector $f = \sum_{i=1}^n \zeta_i f_i \in \ell^2_0$,
\[
(2.1) \quad \|f\|_0^2 = \sum_{i=1}^n |f_i|^2, \quad \|f\|_1^2 = \sum_{i=1}^n \lambda_i |f_i|^2, \quad f \in \ell^2_0.
\]

Working in the co-ordinate system $(\zeta_i)_{i=1}^n$ yields a canonical isomorphism identifying the couple $(\mathcal{H}_0, \mathcal{H}_1)$ with the weighted couple $(\ell^2_0, \ell^2_0(\lambda))$ defined by (2.1). Using this isomorphism, one easily sees that, for functions $h$ defined on $\sigma(A)$, the membership of $h$ in the cone $C_A$ is equivalent to that the space $\ell^2_0(h(\lambda))$ is exact interpolation with respect to the couple $(\ell^2_0, \ell^2_0(\lambda))$ i.e.
\[
(2.2) \quad \|T\|_{\ell^2_0(h(\lambda))} \leq \max(\|T\|_{\ell^2_0}, \|T\|_{\ell^2_0(\lambda)}), \quad T \in M_n(\mathbb{C}).
\]

Our problem is thus to show that for a function to satisfy (2.2), it is necessary and sufficient that it be representable in the form (1.3) with some positive Radon measure $\rho$ on $[0, \infty]$.

Sufficiency. Let $h$ be of the form (1.1), and
\[
K_{2,\lambda}(t, f) = K_2(t, f; \ell^2_0, \ell^2_0(\lambda)) = \inf_{f = f_0 + f_1} (\|f_0\|_0^2 + t\|f_1\|_1^2)^{1/2}.
\]
It is immediate from the definition that, for any \( t \), the function \( K_{2, \lambda}(t, \cdot) \) is an exact interpolation norm with respect to \((\ell_2^n, \ell_2^n(\lambda))\), in the sense that
\[
K_{2, \lambda}(t, Tf) \leq \max(\|T\|, \|T\|_A) K_{2, \lambda}(t, f), \quad T \in M_n(\mathbb{C}), \ f \in \ell_2^n.
\]
A calculation shows that (cf. [2]),
\[
K_{2, \lambda}(t, f)^2 = \sum_{i=1}^n |f_i|^2 \frac{t \lambda_i}{1 + t \lambda_i}.
\]
Thus
\[
(h(A)f, f)_0 = \sum_{i=1}^n |f_i|^2 \left( \int_{[0, \infty]} \frac{(1 + t) \lambda_i}{1 + t \lambda_i} d\rho(t) \right) = 
\]
\[
= \int_{[0, \infty]} (1 + t^{-1}) \left( \sum_{i=1}^n |f_i|^2 \frac{t \lambda_i}{1 + t \lambda_i} \right) d\rho(t) = \int_{[0, \infty]} (1 + t^{-1}) K_{2, \lambda}(t, f)^2 d\rho(t).
\]
By (2.3), the latter expression is the square of an exact interpolation norm with respect to \((\ell_2^n, \ell_2^n(\lambda))\), i.e. (2.2) holds.

**Necessity:** Let \( h \) satisfy (2.2). Denote by \( C \) the unital \( C^* \)-algebra of continuous complex functions on \([0, \infty]\) with the sup-norm \( \|u\|_\infty = \sup_{t \geq 0} |u(t)| \). Put
\[
e_i(t) = \frac{(1 + t) \lambda_i}{1 + t \lambda_i}, \quad i = 1, \ldots, n.
\]
Then (2.4)
\[
K_{2, \lambda}(t, f)^2 = (1 + t^{-1})^{-1} \sum_{i=1}^n |f_i|^2 e_i(t).
\]
Let \( V \subset C \) be the linear span of the \( e_i \)'s, and let a linear functional on \( V \) be defined by
\[
\phi : V \to \mathbb{C} : \sum_{i=1}^n a_i e_i \mapsto \sum_{i=1}^n a_i h(\lambda_i).
\]
We have the following lemma.

**Lemma 2.1.** Let \( h \in C_A \). Then \( \phi \) is a positive functional on \( V \), viz. the conditions \( u \in V \) and \( u(t) \geq 0, \ t > 0 \) implies \( \phi(u) \geq 0 \).

**Proof.** Let \( u(t) = \sum_{i=1}^n a_i e_i(t) \geq 0, \ t > 0 \) and put \( a_i = |f_i|^2 - |g_i|^2 \) with some \( f, g \in \ell_2^n \). Then by (2.5),
\[
(1 + t^{-1}) K_{2, \lambda}(t, f)^2 = \sum_{i=1}^n |f_i|^2 e_i(t) \geq \sum_{i=1}^n |g_i|^2 e_i(t) = (1 + t^{-1}) K_{2, \lambda}(t, g)^2.
\]
Dividing (2.6) by the positive number \( (1 + t^{-1}) \) yields \( K_{2, \lambda}(t, f) \geq K_{2, \lambda}(t, g) \), \( t > 0 \). Because \( \ell_2^n(h(\lambda)) \) is exact interpolation with respect to \((\ell_2^n, \ell_2^n(\lambda))\), we are in a position to use the fact (cf. [2], Theorem 2.1) that this couple is a *Calderón couple*. It yields \( \|f\|_{\ell_2^n(h(\lambda))} \geq \|g\|_{\ell_2^n(h(\lambda))} \), or
\[
\phi(u) = \sum_{i=1}^n (|f_i|^2 - |g_i|^2) h(\lambda_i) = \|f\|_{\ell_2^n(h(\lambda))}^2 - \|g\|_{\ell_2^n(h(\lambda))}^2 \geq 0.
\]
\[\square\]
Replacing $\lambda_i$ by $c \lambda_i$ with a suitable constant $c > 0$, we can w.l.o.g. assume that $1 \in \sigma(A)$, i.e. that the unit $1 \in C$ belongs to $V$. By the positivity of $\phi$ (Lemma 2.1)

$$
\|\phi\| = \sup_{u \in V : \|u\|_{\infty} \leq 1} |\phi(u)| = \phi(1).
$$

Let $\Phi : C \to \mathbb{C}$ be a Hahn–Banach extension of $\phi$; then

$$
\|\Phi\| = \|\phi\| = \phi(1) = \Phi(1).
$$

Hence $\Phi$ is a positive functional on $C$ (cf. [14], Corollary 3.3.4), and the Riesz representation theorem yields a positive Radon measure $\rho$ on $[0, \infty]$ such that

$$
\Phi(u) = \int_{[0, \infty]} u(t)d\rho(t), \quad u \in C.
$$

In particular,

$$
h(\lambda_i) = \phi(e_{\lambda_i}) = \Phi(e_{\lambda_i}) = \int_{[0, \infty]} \frac{(1 + t)\lambda_i}{1 + t\lambda_i}d\rho(t), \quad i = 1, \ldots, n.
$$

Thus $h$ has the required representation (1.3) and the proof of Theorem 1.2 is finished. \qed

3. The theorems of Löwner, Kraus and Foiaș–Lions

In this section, we give new proofs and a unified treatment of the theorems stated in the headline, based on Theorem 1.2, and on a characterization of the positive matrix monotone functions on $\mathbb{R}_+$ due to Frank Hansen [9]. In detail, Hansen’s theorem states that a continuous function $h$ defined on $\mathbb{R}_+$ is matrix monotone if and only if for any $n \in \mathbb{N}$ and any positive definite matrix $A \in M_n(\mathbb{C})$, we have the following inequality

$$
T^*h(A)T \leq h(T^*AT), \quad T \in M_n(\mathbb{C}), \quad T^*T \leq 1.
$$

(It is well-known that matrix monotone functions are continuous, i.e. (3.1) is valid in general. We shall not use this fact.)

Let us now define some function-classes.

A real function $h$ defined on $\mathbb{R}_+$ is matrix concave iff for all $n \in \mathbb{N}$, any positive definite matrices $A, B \in M_n(\mathbb{C})$ and any $\lambda \in [0, 1]$, we have the following “Jensen inequality”

$$
\lambda h(A) + (1 - \lambda)h(B) \leq h(\lambda A + (1 - \lambda)B).
$$

We shall say that a positive function $h$ defined of $\mathbb{R}_+$ is interpolation in the sense of Foiaș–Lions iff for every $n \in \mathbb{N}$, and every positive definite matrix $A \in M_n(\mathbb{C})$, we have the following implication holds

$$
\tag{3.2}
\|T\|_{h(A)} \leq \max(\|T\|, \|T\|_A), \quad T \in M_n(\mathbb{C}).
$$

By homogeneity of the norms it is clear that (3.2) is equivalent to the following: for every positive definite matrix $A \in M_n(\mathbb{C})$, the following implication holds

$$
\tag{3.3}
\|T\| \leq 1, \quad \|T\|_A \leq 1 \quad \text{implies} \quad \|T\|_{h(A)} \leq 1.
$$

Restating (3.3) in a rather more operator-theoretic way yields the implication

$$
\tag{3.4}
T^*T \leq 1, \quad T^*AT \leq A \quad \text{implies} \quad T^*h(A)T \leq h(A),
$$

which is to hold for every positive definite matrix $A \in M_n(\mathbb{C})$. This latter form (3.4) will turn out to be convenient for the applications we have in mind.
3. THE THEOREMS OF LÖwner, KRAUS AND FOIAȘ–LIONS

Following [6], we shall denote the cone of functions representable in the form (1.1) by the letter \( P' \).

**Theorem 3.1.** Let \( h \) be a positive function defined on \( \mathbb{R}^+ \). The following are equivalent

1. \( h \in P' \),
2. \( h \) is interpolation in the sense of Foiaș–Lions,
3. \( h \) is matrix monotone,
4. \( h \) is matrix concave.

**Proof.** (1) \( \Leftrightarrow \) (2): This follows immediately from Theorem 1.2 and the following observation: for a function to belong to \( P' \) it is necessary and sufficient that its restriction to every finite subset of \( \mathbb{R}^+ \) coincide on that set with a \( P' \) function. This latter property is a consequence of the well-known fact that the cone \( P' \) is compact relative to the topology of pointwise convergence on \( \mathbb{R}^+ \) (use Helly’s theorem, cf. [7], top of p. 154).

(2) \( \Rightarrow \) (3): Let \( h \) fulfill (3.4), and let \( A, B \in M_n(\mathbb{C}) \) be positive definite matrices such that \( A \leq B \). Form the \( 2n \times 2n \)-matrices

\[
\tilde{A} = \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix}, \quad T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.
\]

Evidently \( \tilde{A} \) is positive definite, and

\[
T^*T = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \leq 1, \quad T^*\tilde{A}T = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \leq \tilde{A}.
\]

Hence (3.4) yields \( T^*h(\tilde{A})T \leq h(\tilde{A}) \), or

\[
\begin{pmatrix} h(A) & 0 \\ 0 & 0 \end{pmatrix} \leq \begin{pmatrix} h(B) & 0 \\ 0 & h(A) \end{pmatrix}.
\]

Comparing the elements in the upper-left corners now yield the desired conclusion, \( h(A) \leq h(B) \).

(3) \( \Rightarrow \) (2): Let \( h \) be positive and matrix monotone on \( \mathbb{R}^+ \). Let us further assume that \( h \) be continuous; this extra assumption will be removed at the end of the proof. Then, given matrices \( A, T \in M_n(\mathbb{C}) \) such that \( A \) is positive definite and \( T^*T \leq 1, T^*AT \leq A \), using (3.1) and the monotonicity of \( h \),

\[
T^*h(A)T \leq h(T^*AT) \leq h(A), \quad \text{i.e. } h \text{ fulfills (3.4)}.
\]

Now let \( h \) be an arbitrary (not necessarily continuous) matrix monotonic function on \( \mathbb{R}^+ \). Let \( \varphi \) be a smooth non-negative function on \( \mathbb{R}^+ \) such that \( \int_0^\infty \varphi(t)dt/t = 1 \) and define a sequence \( h_k \) by

\[
h_k(\lambda) = \frac{1}{k} \int_0^\infty \varphi\left(\frac{\lambda^k}{t}\right)h(t)\frac{dt}{t}.
\]

Since the class of matrix monotone functions is a convex cone, closed under pointwise convergence (cf. [5], p. 68), the \( h_k \)'s are matrix monotone for all \( k \in \mathbb{N} \). Since they are evidently positive and continuous, the first part of the proof yields that they satisfy (3.4). Since the property (3.4) is finitary in nature, it is easy to see (the same proof as for monotone functions) that the set of interpolation functions in the sense of Foiaș–Lions is closed under pointwise convergence on \( \mathbb{R}^+ \). Hence the limiting function \( h = \lim h_k \) is also in that set, as desired.
(2) \Rightarrow (4): Let \( h \) satisfy (3.4). Let positive definite matrices \( A, B \in M_n(\mathbb{C}) \) and a number \( 0 \leq \lambda \leq 1 \) be given. Form the \( 3n \times 3n \)-matrices
\[
\tilde{A} = \begin{pmatrix}
\lambda A + (1 - \lambda)B & 0 & 0 \\
0 & A & 0 \\
0 & 0 & B \\
\end{pmatrix}
\quad T = \begin{pmatrix}
0 & 0 & 0 \\
\lambda^2 & 0 & 0 \\
(1 - \lambda)^2 & 0 & 0 \\
\end{pmatrix}.
\]
Then \( \tilde{A} \) is positive definite
\[
T^*T = \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \leq 1 \quad T^*\tilde{A}T = \begin{pmatrix}
\lambda A + (1 - \lambda)B & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \leq \tilde{A}.
\]
Thus (3.4) yields \( T^*h(\tilde{A})T \leq h(\tilde{A}) \), or
\[
(3.5) \quad \begin{pmatrix}
\lambda h(A) + (1 - \lambda)h(B) & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0 \\
\end{pmatrix} \leq \begin{pmatrix}
h(\lambda A + (1 - \lambda)B) & 0 & 0 \\
0 & h(A) & 0 \\
0 & 0 & h(B) \\
\end{pmatrix}.
\]
Comparing the elements in the upper left corners in (3.5) yields the matrix concavity of \( h \).

(4) \Rightarrow (3): Let \( h \) be positive and matrix concave on \( \mathbb{R}_+ \). Let \( A, B \in M_n(\mathbb{C}) \) be positive definite with \( A \leq B \), and take \( 0 < \lambda < 1 \). Since
\[
\lambda B = \lambda A + (1 - \lambda)(\lambda(1 - \lambda)^{-1}(B - A)),
\]
the concavity of \( h \) yields
\[
h(\lambda B) \geq \lambda h(A) + (1 - \lambda)h(\lambda(1 - \lambda)^{-1}(B - A)) \geq \lambda h(A).
\]
Letting \( \lambda \to 1 \) yields \( h(B) \geq h(A) \), i.e. \( h \) is matrix monotone. \( \square \)

**Remark 3.1.** The above proof contains ideas due to several authors, which we hereby want to give credit. The implication “(2) \Rightarrow (3)” goes back to W. Donoghue, [6], pp. 266–267 (there stated differently). Our proof of the implication “(4) \Rightarrow (3)” is literally the same as the argument by Hansen–Pedersen [10], top of p. 233. Finally, our proof of the implication “(2) \Rightarrow (4)” is essentially the same as the trick used by Peetre in [23], top of p. 170.

Historically, the biimplication “(1)\Leftrightarrow(2)” is due to Foiaş and Lions [8], the biimplication “(1)\Leftrightarrow(4)” to Kraus [12], whereas “(1)\Leftrightarrow(3)” is of course Löwner’s theorem.

### 4. A closer look at monotone and interpolation functions

This final section comprises a finer study of the relation between interpolation functions and matrix monotone functions of finite order. (Recall that a function \( h \) is said to belong to the class \( P_n \) of matrix monotone functions of order \( n \) if \( A, B \in M_n(\mathbb{C}), 0 < A \leq B \) yields \( h(A) \leq h(B) \).) In the following, it will be convenient to use the letter \( P_n^\prime \) to denote the set of positive functions in the class \( P_n \) (\( n \in \mathbb{N} \)). A further scale of function classes, which we shall denote \( C_n \), \( n = 1, 2, \ldots \) is obtained by the definition
\[
\|T\|_{h(A)} \leq \max(\|T\|, \|T\|_A), \quad A, T \in M_n(\mathbb{C}), A > 0.
\]
It is fitting to refer to elements of \( C_n \) as **interpolation functions of order \( n \).**
It is easy to see that \( C_n \) and \( P'_n \) are convex cones, and that \( C_n \supset C_{n+1} \) and \( P'_n \supset P'_{n+1} \) for all \( n \). Moreover, Theorem 3.1 shows that

\[
\bigcap_{n=1}^{\infty} C_n = \bigcap_{n=1}^{\infty} P'_n = P'.
\]

A closer look at the proof yields that \( P'_{2n} \subset C_n \) and \( C_{2n} \subset P'_n \) for all \( n \). In this section, we shall refine that result by showing that \( \{C_n\} \) is a “finer scale” than \( \{P'_n\} \).

**Theorem 4.1.** Generally holds

\[
P'_{n+1} \subset C_{2n} \subset P'_n.
\]

**Proof.** The inclusion \( C_{2n} \subset P'_n \) is contained in the proof of “(2) \( \Rightarrow \) (3)” of Theorem 3.1. It remains to prove \( P'_{n+1} \subset C_{2n} \). In order to accomplish this, we need to invoke the class \( P \) of Pick functions on \( \mathbb{R}_+ \). This is by definition the class of functions \( h \) having a representation of the form

\[
h(\lambda) = \alpha \lambda + \beta + \int_{-\infty}^{0} \left[ \frac{1}{t - \lambda} - \frac{t}{t^2 + 1} \right] d\mu(t),
\]

with some \( \alpha \geq 0, \beta \in \mathbb{R} \) and some positive Borel measure \( \mu \) on \((-\infty, 0)\) such that \( \int (t^2 + 1)^{-1} d\mu(t) \) is finite. It is easy to see that \( P' = \{h \in P : h \geq 0\} \). Recalling that \( P_n \) denotes the class of general (not necessarily positive) monotone functions of order \( n \) and using Theorem 3.1, it is not hard to prove that

\[
\bigcap_{n=1}^{\infty} P_n = P,
\]

which result is closer to Löwner’s “original” form of the theorem. (We will not need this fact.) In [7], pp. 153–154 was noted that the class \( P \) has an important compactness property: if \( h_k(\lambda) \) is a sequence in \( P \) such that for a pair of distinct points \( \lambda', \lambda'' \in \mathbb{R}_+ \) the sequences \( h_k(\lambda') \) and \( h_k(\lambda'') \) remain bounded, then there exists a subsequence of those functions converging uniformly on closed subintervals of \( \mathbb{R}_+ \) to a function \( h \) in the class \( P \).

For our purposes, the usefulness of the class \( P \) depends on the following interpolation theorem due to Löwner, cf. [5], Theorem I, p. 128.

**Lemma L.** Let \( h \in P_{n+1} \) and let \( S \subset \mathbb{R}_+ \) consist of \( 2n + 1 \) points. There then exists a function \( \tilde{h} \in P \) such that \( \tilde{h} = h \) on \( S \).

Assume that \( h \in P'_{n+1} \) and take \( A \in M_{2n}(\mathbb{C}) \) positive definite. By the lemma, there exists a sequence \( \tilde{h}_k \) of Pick functions such that for each \( k \), \( \tilde{h}_k = h \) on the set \( \sigma(A) \cup \{1/k\} \). Thus by compactness, there exists a subsequence of the \( \tilde{h}_k \)'s converging uniformly on compact subsets of \( \mathbb{R}_+ \) to a function \( \tilde{h} \) in the class \( P \). Since \( h \) is positive, moreover

\[
\tilde{h}(0) \geq \liminf_{k \to \infty} h(1/k) \geq 0.
\]

Together with the simple fact that Pick functions are increasing, this yields that \( \tilde{h} \in P' \). Finally, since \( \tilde{h} = h \) on \( \sigma(A) \), the implication “(1) \( \Rightarrow \) (2)” of Theorem 3.1 yields that \( h \) is exact interpolation with respect to \( A \). Since \( A \) was arbitrary, we obtain \( h \in C_{2n} \) as desired. \( \square \)
Remark 4.1. A closer study shows that Theorem 4.1 yields a strengthening of Sparr [17], Lemma 1, p. 267. It is not hard to prove that Sparr’s lemma implies $P_{n+2}^\prime \subset C_{2n} \subset P_n^\prime$ but not the sharper statement of Theorem 4.1.
References

CHAPTER IV

Note on a theorem of Sparr

Abstract. We show that for every positive concave function \( \psi \) there exists a function \( h \), positive and regular on \( \mathbb{R}^+ \) and admitting of analytic continuation to the upper half-plane and having positive imaginary part there, such that \( h \leq \psi \leq 2h \). This fact is closely related to a theorem of Foiaş, Ong and Rosenthal, which states that regardless of the choice of a concave function \( \psi \), and a “weight function” \( \lambda \), the weighted \( \ell_2 \)-space \( \ell_2(\psi(\lambda)) \) is \( c \)-interpolation with respect to the couple \((\ell_2, \ell_2(\lambda))\), where we have \( c \leq \sqrt{2} \) for the best \( c \). It turns out that the value \( c = \sqrt{2} \) is best possible in this theorem; a fact which is implicit in the work of G. Sparr. (This answers the question (i) on p. 811 of Foiaş, Ong and Rosenthal [6].)

A lemma on Pick functions. Of general interest in the theory of interpolation spaces is the class \( P' \) of functions representable in the form

\[
(1) \quad h(\lambda) = \int_{[0,\infty]} \frac{(1 + t)\lambda}{1 + t\lambda} d\rho(t), \quad \lambda \in \mathbb{R}^+,
\]

where \( \rho \) is some positive Radon measure on the compactified halfline \([0, \infty]\). This class is usually referred to as the set of positive Pick functions on \( \mathbb{R}^+ \) (cf. [2] or [5].) It is easy to see that \( P' \) constitutes a subcone of the convex cone of positive concave functions on \( \mathbb{R}^+ \).

In the following, it will be convenient besides (1) to work with a modified representation for \( P' \)-functions (cf. [6], p. 266)

\[
(2) \quad h(\lambda) = \alpha + \beta\lambda + \int_0^\infty \frac{\lambda t}{\lambda + t} d\nu(t),
\]

where \( \alpha \geq 0, \beta \geq 0 \) and \( \nu \) is a positive measure on \( \mathbb{R}^+ \) such that \( \int_0^\infty d\nu(t)/(1 + t^{-1}) < \infty \).

We have the following basic lemma.

Lemma 1. Let \( \psi \) be a positive concave function on \( \mathbb{R}^+ \). There then exists a function \( h \in P' \) such that \( h \leq \psi \leq 2h \).

Proof. It is well-known that an arbitrary positive, concave function can be represented in the form (cf. [4], Lemma 1, p. 46)

\[
(3) \quad \psi(\lambda) = \alpha + \beta\lambda + \int_0^\infty \min(\lambda, t)d\nu(t),
\]

where \( \alpha \geq 0, \beta \geq 0 \) and \( \nu \) a positive measure on \( \mathbb{R}^+ \) such that \( \int_0^\infty d\nu(t)/(1 + t^{-1}) < \infty \). Next observe that for \( \lambda, t > 0 \),

\[
\frac{\lambda t}{\lambda + t} \leq \min(\lambda, t) \leq 2 \frac{\lambda t}{\lambda + t}.
\]

The lemma now follows from (2) and (3) on integration with respect to \( \nu \). \( \square \)
Remark 1. The class $P'$ coincides with the set of positive and regular functions on $\mathbb{R}_+$ which prolong to the upper half-plane and have positive imaginary parts there, cf. [5], sect. 2.

The Foiaş–Ong–Rosenthal question. As we shall see presently, Lemma 1 is closely related to an interpolation theorem of Foiaş, Ong and Rosenthal [8], which goes back to the work of Jaak Peetre [9]. Before we formulate this theorem, let us remind of some notions from the theory of interpolation spaces. (For a detailed exposition of this theory, we refer to [3]).

Relative to a Hilbert couple $\mathcal{H} = (\mathcal{H}_0, \mathcal{H}_1)$ we have the $K_2$-functional

$$K_2(t, f) = K_2(t, f; \mathcal{H}_0, \mathcal{H}_1) = \inf_{f = f_0 + f_1} (\|f_0\|_0^2 + t\|f_1\|_1^2)^{1/2}.$$  

Suppose now that $\mathcal{H}$ be regular, in the sense that $\mathcal{H}_0 \cap \mathcal{H}_1$ is dense in $\mathcal{H}_0$ and in $\mathcal{H}_1.$ The basic fact for $K_2$ is the following (cf. [1]). Denote by $A$ the unbounded, densely defined, positive, injective operator in $\mathcal{H}_0$ such that

$$\|f\|_1^2 = (Af, f)_0, \quad f \in \mathcal{H}_0 \cap \mathcal{H}_1,$$

then

$$K_2(t, f)^2 = K_2(t, f; \mathcal{H})^2 = \left(\frac{tA}{1 + tA} f, f\right)_0.$$  

With respect to $\mathcal{H}_0$ and $\mathcal{H}_1$ it will be advantageous to make use of several notations for the operator norms.

$$\|T\|_T^2 = \sup_{(f, f_0) \leq 1} \|T^* T f, f\|_0, \quad \|T\|_A^2 = \sup_{(Af, f_0) \leq 1} \|T^* A T f, f\|_0.$$  

Let $\mathcal{L}(\mathcal{H})$ be the set of linear operators on $\mathcal{H}_0 + \mathcal{H}_1$ such that the restriction of $T$ to $\mathcal{H}_i$ belongs to $\mathcal{L}(\mathcal{H}_i)$, $i = 0, 1$. A Banach space norm on $\mathcal{L}(\mathcal{H})$ is defined by

$$\|T\|_{\mathcal{L}(\mathcal{H})} = \max(\|T\|_{\mathcal{L}(\mathcal{H}_0)}, \|T\|_{\mathcal{L}(\mathcal{H}_1)}) = \max(\|T\|, \|T\|_A).$$  

We note that the function $K_2(t, \cdot)$ is an exact interpolation norm with respect to $\mathcal{H}$, i.e. (for any $t$)

$$K_2(t, T f) \leq \|T\|_{\mathcal{L}(\mathcal{H})} K_2(t, f), \quad T \in \mathcal{L}(\mathcal{H}), \quad f \in \mathcal{H}_0 + \mathcal{H}_1,$$

which property is immediate from the definition of $K_2$. Given a positive concave function $\psi$ on $\mathbb{R}_+$ let an intermediate Hilbert space $\mathcal{H}_*$ be defined as the completion of $\mathcal{H}_0 \cap \mathcal{H}_1$ under the norm

$$\|f\|_*^2 = (\psi(A)f, f)_0.$$  

In accordance with (5) we shall use different notations for the operator norms

$$\|T\|_{\psi(A)} = \|T\|_{\mathcal{L}(\mathcal{H}_*)}^2 = \sup_{(\psi(A)f, f_0) \leq 1} (T^* \psi(A)T f, f)_0.$$  

By a theorem of Peetre [9] it is known that every positive concave function $\psi$ on $\mathbb{R}_+$ is an interpolation function of power 2 meaning that (for any $A$)

$$\max(\|T\|, \|T\|_A) < \infty \quad \text{implies} \quad \|T\|_{\psi(A)} < \infty, \quad T \in \mathcal{L}(\mathcal{H}).$$  

However, it does not follow from Peetre’s theorem that there exists a constant $c \geq 1$ such that $\mathcal{H}_*$ is a $c$-interpolation space with respect to $\mathcal{H}$ in the sense that

$$\|T\|_{\psi(A)} \leq c \max(\|T\|, \|T\|_A), \quad T \in \mathcal{L}(\mathcal{H}).$$
That this is indeed the case was first observed by C. Foiaş [7], who also stated the upper bound \( c \leq 2 \) for the best \( c \). In a later paper, Foiaş, Ong and Rosenthal improved that bound to \( c \leq \sqrt{2} \), and also posed the question whether the constant \( \sqrt{2} \) is best possible, cf. [8], question (i), p. 811. It is shown below that the answer to this question is affirmative.

**Theorem 1.** The best \( c \) in (9) fulfills \( c = \sqrt{2} \).

**Remark 2.** This theorem is implicit in the work of Gunnar Sparr, cf. [10], Lemma 5.1. We shall presently give a partially new proof, based on Lemma 1 and the following lemma.

**Lemma 2.** Every function \( h \) in the class \( P' \) is exact interpolation in the sense that

\[
\left\| T \right\|_{h(A)} \leq \max(\left\| T \right\|, \left\| T \right\|_A), \quad T \in \mathcal{L}(\mathcal{H}).
\]

**Proof.** Denote by \( E \) the spectral measure of \( A \) and let \( \rho \) be the measure associated with \( h \) as in (1). Then by (4)

\[
\left\| f \right\|_0^2 = (h(A)f, f)_0 = \int_{0,\infty} (\int_{[0,\infty]} \frac{(1 + t)\lambda}{1 + t\lambda} d\rho(t)) d\rho(t) = \int_{0,\infty} (1 + t^{-1})(\int_{0,\infty} t\lambda d\rho(t)) d\rho(t) = \int_{0,\infty} (1 + t^{-1})K_2(t, f; \mathcal{H})^2 d\rho(t), \quad f \in \mathcal{H}_0 \cap \mathcal{H}_1.
\]

It is easy to see that the latter expression extends to an exact interpolation norm with respect to \( \mathcal{H} \), viz. (10) holds (use (6) and integrate with respect to \( d\rho(t) \)). \( \square \)

**Proof of Theorem 1.** Referring to the smallest constant in (9), we first show that \( c \leq \sqrt{2} \). Given an arbitrary concave positive function \( \psi \) on \( \mathbb{R}_+ \), let \( h \in P' \) be such that \( h \leq \psi \leq 2h \); then by Lemma 2,

\[
\left\| T \right\|_{h(A)}^2 = \sup_{(\psi(A)f, f)_0 \leq 1} (T^*\psi(A)Tf, f)_0 \leq \sup_{(h(A)f, f)_0 \leq 1} 2(T^*h(A)Tf, f)_0 = 2\left\| T \right\|_{h(A)}^2 \leq 2 \max(\left\| T \right\|_A^2, \left\| T \right\|_A^2), \quad T \in \mathcal{L}(\mathcal{H}),
\]

and the estimate \( c \leq \sqrt{2} \) follows incidentally. Proving \( c \geq \sqrt{2} \) is more subtle; we shall require a clever three-dimensional argument due to G. Sparr, cf. [10], Example 5.3. Let \( \mathcal{H}_0 = \ell_2^3 \) be the three-dimensional \( \ell_2 \)-space. For \( n \in \mathbb{N} \) let us put

\[
A_n = \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
0 & 0 & 4n^2
\end{pmatrix}, \quad g = \begin{pmatrix}
0 \\
1 \\
0
\end{pmatrix}, \quad f^n = \begin{pmatrix}
n \\
0 \\
\frac{1}{2}
\end{pmatrix}, \quad T_n = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}
\]

then \( T_nf^n = g \) and direct calculation yields that

\[
\left\| T_n \right\| = \left\| T_n \right\|_{A_n} = \sqrt{1 + 1/(4n^2)}, \quad n \in \mathbb{N}.
\]

On the other hand, letting \( \psi(\lambda) = \min(1, \lambda) \), we have

\[
\left\| T_nf^n \right\|_{\psi(\lambda)}^2 = \left\| g \right\|_{\psi(\lambda)}^2 = (\psi(A_n)g, g)_0 = 1, \quad n \in \mathbb{N},
\]

whereas

\[
\left\| f^n \right\|_{\psi(\lambda)}^2 = (\psi(A_n)f^n, f^n)_0 = n^2 \min(1, 1/(4n^2)) + (1/4) \min(1, 4n^2) = 1/2, \quad n \in \mathbb{N},
\]

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and it follows that
\[ c \geq \frac{\|T_n\|_{\psi(A_n)}}{\sqrt{1 + 1/(4n^2)}} \geq \sqrt{\frac{2}{1 + 1/(4n^2)}} \geq \sqrt{2}, \quad n \to \infty. \]

We also note the following, sharp version of Lemma 1.

**Theorem 2.** The constant \( c = 2 \) is smallest possible with respect to the property that for any positive concave function \( \psi \) on \( \mathbb{R}_+ \), there exists \( h \in \mathcal{P}' \) such that \( h \leq \psi \leq ch \).

**Proof.** Referring to the least constant, we have \( c \leq 2 \) by Lemma 1, and as in the proof of Theorem 1, one shows that for any positive concave \( \psi \), any \( A, T \)
\[ \|T\|_{\psi(A)} \leq \sqrt{c \max(\|T\|, \|T\|_A)}. \]
By Theorem 1, the smallest possible constant in the latter inequality is \( \sqrt{c} = \sqrt{2} \). \( \square \)

**Remark 3.** The above investigations also gives a necessary condition for interpolation (8) namely that \( \psi \) be equivalent to a concave function. This observation is due to Peetre [9].

**A note on \( K_2 \)-functors.** We consider an application of Lemma 1 to the more functorial aspects of the theory. Given a positive Radon measure \( \rho \) on \([0, \infty]\), let an interpolation functor \( K_2(\rho) \) be defined on the category of Banach couples by
\[ \|f\|_{K_2(\rho)(\mathcal{H})} = \left( \int_{[0, \infty]} (1 + t^{-1})K_2(t, f; \mathcal{A})^2 d\rho(t) \right)^{1/2}. \]
(Here the function \( k : t \mapsto (1 + t^{-1})K_2(t, f)^2 \) is defined by continuity at the points 0 and \( \infty \), \( k(0) = \|f\|^2 \) and \( k(\infty) = \|f\|^2_0 \), where we have used the convention: \( \|f\|_i = \infty \) if \( f \notin \mathcal{H}_i, i = 0, 1 \).)

**Corollary 1.** Let \( \mathcal{H} \) be a regular Hilbert couple with associated operator \( A \). Then, given any positive concave function \( \psi \) on \( \mathbb{R}_+ \), there exists a positive Radon measure \( \rho \) on \([0, \infty]\) such that
\[ (1/\sqrt{2})\|f\|_{K_2(\rho)(\mathcal{H})} \leq \|f\|_{\psi(A)} \leq \sqrt{2}\|f\|_{K_2(\rho)(\mathcal{H})}, \quad f \in K_2(\rho)(\mathcal{H}), \]
where the constant \( \sqrt{2} \) cannot be improved.

**Proof.** This follows easily from Theorem 2 and (11). \( \square \)

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References

CHAPTER V

A new proof of Donoghue’s interpolation theorem

Abstract. We give a new proof and new interpretation of Donoghue’s interpolation theorem; for an intermediate Hilbert space \( H^* \) to be exact interpolation with respect to a regular Hilbert couple \( H \) it is necessary and sufficient that the norm in \( H^* \) be representable in the form

\[
\|f\|_*= (\int_{[0,\infty]} (1 + t^{-1})K_2(t, f; \overline{H})^2 d\rho(t))^{1/2}
\]

with some positive Radon measure \( \rho \) on the compactified half-line \([0, \infty)\).

1. Introduction

In [3] the author established that regular Hilbert couples are Calderón couples; a fact which allowed us to describe all exact interpolation (Banach-) spaces with respect to such couples in terms of Peetre’s \( K \)-functional. By a theorem due to William Donoghue ([7], notably Theorem 1) a complete description is available for the exact interpolation Hilbert spaces. In this paper, we want to show how the theory of the Calderón problem in a natural way leads to Donoghue’s theorem.

Amongst the somewhat more technical aspects we note that we rely on the core result [2], Theorem 1.2 which yields a representation formula for “interpolation functions”. This formula is readily seen to be equivalent to Donoghue’s theorem in the case of finite-dimensional spaces. In order to pass to the infinite-dimensional case it becomes necessary to recast portion of the arguments used by Donoghue in a form which is adapted to our setting. We take this opportunity to remark that Donoghue’s setting is advantageous over ours in the following two ways: (i) it treats spaces in a symmetric way, and (ii) it is well-suited for generalization to the non-regular case.

In this note we will only consider the case of regular couples (recall that \((H_0, H_1)\) is regular if \( H_0 \cap H_1 \) is dense in \( H_0 \) and in \( H_1 \)) and make the further restriction that the spaces involved be separable.

2. Donoghue’s Theorem 1

Let \( \overline{H} = (H_0, H_1) \) be a Hilbert couple over \( \mathbb{C} \). We will assume that \( \overline{H} \) is separable and regular: \( H_0, H_1 \) are separable and \( H_0 \cap H_1 \) is dense in \( H_0, H_1 \). In the following, all involutions and inner products will be taken with respect to the norm of \( H_0 \).

The squared norm \( \| \cdot \|_2^2 \) is an unbounded, densely defined quadratic form on \( H_0 \) and can hence be expressed as

\[
\|f\|_2^2 = (Af, f)_0
\]

where \( A \) is a densely defined, positive injective operator in \( H_0 \) (the domain of \( A^{1/2} \) is \( H_0 \cap H_1 \)). A useful tool in the study of interpolation of Hilbert spaces is the functional \( K_2 \), given by

\[
K_2(t, f) = K_2(t, f; \overline{H}) = \inf_{f=f_0+f_1} (\|f_0\|^2_0 + t\|f_1\|^2_1)^{1/2}.
\]

\(^1\)With minor modifications, one can include the case of real scalars in the theory.
It will be convenient to use the following alternative notations for the operator
(2.2) (1/r
r>1 with the property
Given an intermediate Hilbert space \( H \), it has the property that \( \text{meas}(\cdot) \) is the spectral measure of \( E \). This measure is defined on Borel subsets \( \mathcal{B} \) of \( \mathcal{H} \). It follows from the basic formula for (ExInt)

\[
\| T \|_B = 0 \iff T f, f \in H_0 \cap H_1.
\]

Recall that \( H_\ast \) is exact interpolation with respect to \( \mathcal{H} \) if \( T \in \mathcal{L}(\mathcal{H}) \),

\[
\| T \|_B \leq \max(\| T \|_A, \| T \|_A), \quad T \in \mathcal{L}(\mathcal{H}),
\]

where \( \mathcal{L}(\mathcal{H}) \) denotes the set of linear operators \( T \) on \( H_0 + H_1 \) such that the restriction of \( T \) to \( H_i \) belongs to \( \mathcal{L}(H_i), i = 0, 1 \).

Our main result is the following.

**Donoghue’s Theorem 1.** For a Hilbert space \( H_\ast \) to satisfy (ExInt), it is necessary and sufficient that there exist a positive Radon measure \( \rho \) on the compactified half line \([0, \infty]\) such that

\[
\| f \|_\ast = \left( \int_{[0, \infty]} (1 + t^{-1}) K_2(t, \cdot)^2 d\rho(t) \right)^{1/2}, \quad f \in H_\ast.
\]

(Here the function \( k: t \mapsto (1 + t^{-1}) K_2(t, \cdot)^2 \) is defined by continuity at the points \( t = 0 \) and \( t = \infty \), i.e. \( k(0) = \| f \|_0^2 \) and \( k(\infty) = \| f \|_0^2 \), where we have used the convention: \( \| f \|_\ast^2 = \infty \) if \( f \notin H_i, i = 0, 1 \).)

**Proof.** Sufficiency: This is straightforward and left to the reader.

Necessity: Following Donoghue, we start with the following lemma.

**Lemma 2.1.** (Cf. [7], Lemma 1 and Lemma 2)

1. If \( H_\ast \) fulfills (ExInt) then the corresponding operator \( B \) fulfills \( B A = A B \) and \( B = h(A) \) for some Borel measurable real function \( h \) defined on \( \sigma(A) \).

2. Moreover, \( h \) can be modified on a null-set for the scalar-valued spectral measure \(^2\) of \( A \) to a quasi-concave function:

\[
h(\lambda) \leq \max(1, \lambda/\mu) h(\mu), \quad \lambda, \mu \in \sigma(A) \cap \mathbb{R}_+.
\]

In particular, this version of \( h \) is locally Lipschitzian on \( \sigma(A) \cap \mathbb{R}_+ \).

\(^2\)This measure is defined on Borel subsets \( \omega \) of \( \sigma(A) \) in the following way: \( \text{meas}(\omega) = \sum 2^{-k}(E(\omega)f_k, f_k) \), where \( E \) is the spectral measure of \( A \) and \( f_k \) is an orthonormal basis of \( H_0 \); it has the property that \( \text{meas}(\omega) = 0 \) iff \( E(\omega) = 0 \), cf. [6].
PROOF. (1): We first note that, by (2.3), (ExInt) can be rephrased in the following way
\[ (2.6) \quad T^*BT \leq B \text{ for all } T \in \mathcal{L}(\mathcal{H}_0) \text{ such that } T^*T \leq 1 \text{ and } T^*AT \leq A. \]
Since for orthogonal projections \( E \),
\[ EAE \leq A \iff EA = AE, \]
we infer that (2.6) yields that \( B \) belongs to the bicommutator algebra of the von-Neumann algebra generated by the spectral projections of \( A \). The result now follows from the bicommutant theorem.

(2): We shall show that the function \( h \) from part (1) can be modified on a null set to satisfy (2.5). To simplify the problem, we note that it suffices for every compact subset \( K \subset \mathbb{R}_+ \cap \sigma(A) \) to show that the restriction \( h|K \) is quasi-concave on \( K \). Now fix \( K \) and let \( \mathcal{K}_0 = \mathcal{K}_1 = \mathcal{K}_* = E_K(\mathcal{H}_0) \), where \( E \) is the spectral measure of \( A \), and the norms are defined by restriction,
\[ \|f\|_{\mathcal{K}_0} = \|f\|_{\mathcal{H}_0}, \quad \|f\|_{\mathcal{K}_1} = \|f\|_{\mathcal{H}_1}, \quad \|f\|_{\mathcal{K}_*} = \|f\|_{\mathcal{H}_*}, \quad f \in E_K(\mathcal{H}_0). \]
It is plain that the operator \( A_K \) corresponding to \( \mathcal{K} \) is the compression of \( A \) to \( E_K(\mathcal{H}_0) \), and likewise, the operator \( B_K \) corresponding to \( \mathcal{K}_* \) is the compression of \( B \) to \( E_K(\mathcal{H}_0) \). Moreover, \( \mathcal{K}_* \) is exact interpolation with respect to \( \mathcal{K} \), and it is easy to see that \( B_K = (h|K)(A_K) \). By these considerations, if necessary by replacing \( \mathcal{K} \) by \( \mathcal{K}_* \), we can w.l.o.g. assume that the operator \( A \) is bounded above and below. By (2.2), then also \( B \) is bounded above and below.

Let \( c < 1 \) be a positive number such that \( \sigma(A) \subset (c, c^{-1}) \). Take \( \lambda_0, \lambda_1 \in \sigma(A) \) with \( \lambda_0 \leq \lambda_1 \). For fixed \( \varepsilon > 0 \), \( \varepsilon < c/2 \) we consider the functions
\[ m_\varepsilon(\lambda) = \text{ess inf } h \text{ over } \sigma(A) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \]
\[ M_\varepsilon(\lambda) = \text{ess sup } h \text{ over } \sigma(A) \cap (\lambda - \varepsilon, \lambda + \varepsilon), \]
the essential inf and sup being taken with respect to the scalar-valued spectral measure of \( A \). Take \( \varepsilon' > 0 \) and a Borel set \( E_0 \subset \sigma(A) \cap (\lambda_0 - \varepsilon, \lambda_0 + \varepsilon) \) supporting a unit vector \( e_0 \) (for the norm of \( \mathcal{H}_0 \)) such that \( h(\lambda) > M_\varepsilon(\lambda) - \varepsilon' \) on \( E_0 \). Likewise, there exists another Borel set \( E_1 \subset \sigma(A) \cap (\lambda_1 - \varepsilon, \lambda_1 + \varepsilon) \) supporting a unit vector \( e_1 \) such that \( h(\lambda) < m_\varepsilon(\lambda) + \varepsilon' \) on \( E_1 \). Let \( \mathcal{M} \) be the reducing subspace for \( A \) corresponding to the spectral set \( E_0 \cup E_1 \) and consider the map \( P \) defined on \( \mathcal{M} \) by \( Pf = (f, e_1)_0e_0 \). Then
\[ (P^*APf, f)_0 = |(f, e_1)_0|^2(Ae_0, e_0)_0 \leq \|f\|_{\mathcal{H}_0}^2(\lambda_0 + \varepsilon) \leq \frac{\lambda_0 + \varepsilon}{\lambda_0 - \varepsilon}(Af, f)_0, \quad f \in \mathcal{M}. \]
Extending \( P \) to \( \mathcal{H}_0 \) by \( P = 0 \) on the orthogonal complement of \( \mathcal{M} \), and observing that \( \frac{\lambda_0 + \varepsilon}{\lambda_0 - \varepsilon} \leq 1 + 4\varepsilon/c \), we obtain the bound \( \|P\|_A \leq 1 + 4\varepsilon/c \). Hence \( \|P\|_B \leq 1 + 4\varepsilon/c \), so that
\[ M_\varepsilon(\lambda_0) - \varepsilon' \leq (B\varepsilon_0, e_0)_0 = (P^*BP e_1, e_1)_0 \leq (1 + 4\varepsilon/c)(B\varepsilon_1, e_1)_0 \leq (1 + 4\varepsilon/c)(m_\varepsilon(\lambda_1) + \varepsilon'). \]
Since \( \varepsilon' > 0 \) was arbitrary, it follows that
\[ (2.7) \quad M_\varepsilon(\lambda_0) - m_\varepsilon(\lambda_1) \leq 4\varepsilon m_\varepsilon(\lambda_1)/c \leq 4\|B\|\varepsilon/c. \]
As \( \varepsilon \) approaches 0, the functions \( M_\varepsilon(\lambda) \) diminish monotonically, converging uniformly to a function \( h_*(\lambda) \) which is also the uniform limit of the increasing family \( m_\varepsilon(\lambda) \). Clearly, \( h_* \) is continuous on \( \sigma(A) \), and by (2.7), it is increasing. Moreover, since \( m_\varepsilon(\lambda) \leq h_*(\lambda) \leq M_\varepsilon(\lambda) \) holds almost everywhere with respect to the
spectral measure of $A$, $h_\ast$ is equivalent to $h$ for that measure. To see that $h_\ast$ is quasi-concave, we make use of the fact that the space $\mathcal{H}_\ast$ is exact interpolation with respect to the reversed couple $(\mathcal{H}_1, \mathcal{H}_0)$. The latter couple has corresponding operator $A^{-1}$, and we have the relation

$$\|f\|_\ast^2 = (h_\ast(A)f, f)_0 = (A^{-1}h_\ast(A)f, f)_1 = (h_\ast^r(A^{-1})f, f)_0$$

where $h_\ast^r(\lambda) = \lambda h_\ast(\lambda^{-1})$.

Thus the first part of the proof can be applied to show that the function $h_\ast^r$ is increasing on $\sigma(A)^{-1}$. Hence $h_\ast$ is quasi-concave on $\sigma(A)$. \hfill \Box

To obtain an operator-theoretic interpretation of our problem we will need another lemma:

**Lemma 2.2.** For a space $\mathcal{H}_\ast$ to be of the form (2.4), it is necessary and sufficient that $B = h(A)$ where $h$ is of the form

$$h(\lambda) = \int_{[0, \infty]} \frac{(1 + t)\lambda}{1 + t\lambda} d\rho(t) \quad \text{a.e. on } \sigma(A),$$

with $\rho$ is some positive Radon measure on $[0, \infty]$ (“a.e.” denotes a.e. with respect to the scalar-valued spectral measure of $A$).

**Proof.** Let $E$ denote the spectral measure of $A$. Using a change of the order of integration and the formula (2.1) for $K_2$, we infer that $\mathcal{H}_\ast$ is of the form (2.4) if and only if

$$\|f\|_\ast^2 = \int_{[0, \infty]} (1 + t^{-1}) \left( \frac{tA}{1 + tA} f, f \right)_0 d\rho(t) =$$

$$= \int_{[0, \infty]} (1 + t^{-1}) \left( \int_0^\infty \frac{t\lambda}{1 + t\lambda} d(E_\lambda f, f)_0 \right) d\rho(t) =$$

$$= \int_0^\infty \left( \int_{[0, \infty]} \frac{(1 + t)\lambda}{1 + t\lambda} d\rho(t) \right) d(E_\lambda f, f)_0 = (h(A)f, f)_0.$$  \hfill \Box

By Lemma 2.2, our problem reduces to showing that a function $h$ corresponding to an exact interpolation space $\mathcal{H}_\ast$ is necessarily of the form (2.8). We will need some facts about the convex cone functions of the form (2.8), which is henceforth denoted by the letter $P'$ (cf. [4]).

**Lemma 2.3.**

1. Let $S$ be a nonvoid subset of $\mathbb{R}_+$. Then the convex cone $P'|S$ of restrictions of $P'$-functions to $S$ is closed under pointwise convergence.

2. In particular, a continuous function $h$ belongs to the set $P'|\sigma(A) \cap \mathbb{R}_+$ iff the restriction $h|\Lambda \in P'|\Lambda$ for every finite subset $\Lambda \subset \sigma(A) \cap \mathbb{R}_+$.

**Proof.** Let $h_n$ be a sequence of $P'$-functions converging pointwise on $S$. By (2.8) the boundedness of the numbers $h_n(\lambda)(1 + \lambda^{-1})$ imply the boundedness of the total masses of the corresponding measures $\rho_n$, whence a subsequence of $\rho_n$ converges weak* to a positive Radon measure $\rho$ on $[0, \infty]$, by Helly's theorem. Putting $h(\lambda) = \int_{[0, \infty]} \frac{(1 + \lambda)\lambda}{1 + t\lambda} d\rho(t) \in P'$, we infer that the corresponding subsequence of the $h_n$ converges pointwise on $\mathbb{R}_+$ to $h$, proving (1). To prove (2), let $\Lambda_n$ be an increasing sequence of finite subsets of $\sigma(A)$, whose union is dense. Put $h_n = h|\Lambda_n$,
and assume that $h_n \in P'|\Lambda_n$ for all $n$. Since $h_n$ converges pointwise to $h$ on $\bigcup_n \Lambda_n$, and $h$ is continuous, we obtain $h \in P'\sigma(A) \cap \mathbb{R}_+$ from part (1). \hfill \Box

Fix a finite subset $\Lambda = \{\lambda_i\}_{i=1}^n \subset \sigma(A) \cap \mathbb{R}_+$ and put $\lambda = (\lambda_i)_{i=1}^n$ (increasing order). Let $(\ell_2^0, \ell_2^n(\lambda))$ denote the following couple

$$\|f\|_0^2 = \sum_{i=1}^n |f_i|^2, \quad \|f\|_1^2 = \sum_{i=1}^n \lambda_i |f_i|^2, \quad f \in \ell_2^n.$$ 

The following lemma gives a useful characterization of the set $P'|\Lambda$.

**Lemma 2.4.** For a function $h$ to belong to $P'|\Lambda$ it is necessary and sufficient that the space $\ell_2^n(h(\lambda))$ is exact interpolation with respect to $(\ell_2^n, \ell_2^n(\lambda))$.

**Proof.** This is Theorem 1.2 of [2]. \hfill \Box

Recall that $\mathcal{H}_e$ is exact interpolaton with respect to $\mathcal{H}$ and $B = h(A)$ where $h$ is quasi-concave on $\sigma(A)$. Multiplying $h$ by a positive number, we can assume that its Lipshitz constant is 1. By the above lemmata, Donoghue’s Theorem 1 reduces to showing that $\ell_2^n(h(\lambda))$ is exact interpolation with respect to $(\ell_2^n, \ell_2^n(\lambda))$ for arbitrary sequences $\lambda = (\lambda_i)_{i=1}^n \subset \sigma(A)$. This is done below.

Take $\varepsilon > 0$ and let $E_i = [\lambda_i - \varepsilon/2, \lambda_i + \varepsilon/2] \cap \sigma(A)$, where $\varepsilon < 2\lambda_i$ is sufficiently small that the $E_i$'s be disjoint. Let $\mathcal{M}$ be the reducing subspace of $\mathcal{H}_e$ corresponding to the spectral set $\bigcup_{i=1}^n E_i$, and let $\widetilde{A}$ be the compression of $A$ to $\mathcal{M}$. Define a Borel function $g$ on $\sigma(\widetilde{A}) = \bigcup_i^n E_i$ by $g(\lambda) = \lambda_i$ on $E_i$. Then, since $|g(\lambda) - \lambda| < \varepsilon$ on $\sigma(\widetilde{A})$,

$$\|\widetilde{A} - g(\widetilde{A})\| \leq \varepsilon, \quad \|h(\widetilde{A}) - h(g(\widetilde{A}))\| \leq \varepsilon.$$ 

**Lemma 2.5.** (Cf. [7], Lemma 4) Suppose that $A', A'' \in \mathcal{L}(\mathcal{M})$ satisfy $A', A'' \geq c > 0$ and $\|A' - A''\| \leq \varepsilon$. Then, for $T \in \mathcal{L}(\mathcal{M})$, $\|T\|_{A''} \leq \sqrt{1 + 2\varepsilon c^{-1}} \max(\|T\|, \|T\|_{A'})$.

**Proof.** By definition, $\|T\|_{A'}$ is the smallest number $C \geq 0$ such that $T^* A'T \leq C^2 A'$. Thus

$$T^* A'' T = T^* (A'' - A') T + T^* A' T \leq \|T\|^2 \varepsilon + \|T\|^2_{A'} (A'' + (A' - A'')) \leq \leq 2\varepsilon \max(\|T\|^2, \|T\|^2_{A'}) + \|T\|^2_{A'} A'' \leq \max(\|T\|^2, \|T\|^2_{A'}) (1 + 2\varepsilon c^{-1}) A''.$$

We can find $c > 0$ such that $\widetilde{A}, g(\widetilde{A}), h(\widetilde{A})$ and $h(g(\widetilde{A}))$ are $\geq c$. Hence, repeated use of Lemma 2.5 and (2.9) yield

$$\|T\|_{h(g(\widetilde{A}))} \leq \sqrt{1 + 2\varepsilon c^{-1}} \max(\|T\|, \|T\|_{h(\widetilde{A})}) \leq \leq \sqrt{1 + 2\varepsilon c^{-1}} \max(\|T\|, \|T\|_{\widetilde{A}}) \leq \leq (1 + 2\varepsilon c^{-1}) \max(\|T\|, \|T\|_{g(\widetilde{A})}), \quad T \in \mathcal{L}(\mathcal{M}).$$

Let $e_i$ be a unit vector in $\mathcal{M}$ supported by the spectral set $E_i$ and define a subspace $\mathcal{V} \subset \mathcal{M}$ as the space spanned by the $e_i$'s, $i = 1, \ldots, n$. Let $A^0$ be the compression of $g(\widetilde{A})$ to $\mathcal{V}$; then

$$\|T\|_{h(A^0)} \leq (1 + 2\varepsilon c^{-1}) \max(\|T\|, \|T\|_{A^0}), \quad T \in \mathcal{L}(\mathcal{V}).$$
Identifying $\mathcal{V}$ with $\ell_2^n$ and $A^\circ$ with the matrix diag$(\lambda_i)$, we infer that (2.10) is independent of $\varepsilon$. Letting $\varepsilon$ diminish to 0 now gives that $\ell_2^n(h(\lambda))$ is exact interpolation with respect to $(\ell_2^n, \ell_2^n(\lambda))$, which finishes the proof of Donoghue’s Theorem 1. □

3. Donoghue’s Theorem 2

We shall now draw attention to the second theorem of Donoghue, [7], Theorem 2, in which the quadratic interpolation methods are characterized. These methods were defined by Donoghue to be those exact interpolation methods $F$ which have the property that for each regular Hilbert couple $\mathcal{H}$, the space $F(\mathcal{H})$ is a Hilbert space. A direct consequence of Donoghue’s Theorem 1, is that to a quadratic method $F$ and a regular Hilbert couple $\mathcal{H}$ there corresponds a positive Radon measure $\rho$ such that

\begin{equation}
(3.1) \quad \|f\|_F^2(\mathcal{H}) = \int_{[0,\infty]} (1 + t^{-1})K_2(t, f; \mathcal{H})^2 d\rho(t), \quad f \in F(\mathcal{H}).
\end{equation}

A priori $\rho$ may depend not only on $F$ but also on the particular $\mathcal{H}$. That this is not the case is the content of Donoghue’s second theorem. (For a given $\rho$ we define $K_2(\rho)$ be the quadratic interpolation method $F$ defined by (3.1).)

**Donoghue’s Theorem 2.** The assignment $\rho \mapsto K_2(\rho)$ yields a one-one correspondence from the set of positive Radon measures on $[0, \infty]$ onto the set of quadratic interpolation methods.

**Proof.** This follows in the same way as [7], Theorem 2 follows from [7], Theorem 1, notably pp. 255-256 (cf. sect. 4 to translate between the articles.) □

4. Connection to Donoghue’s versions

In this section, we explain how the results of the previous section are related to Donoghue’s original versions of the theorems in [7]. Consider a regular Hilbert couple $\mathcal{H}$. In Donoghue’s setting, the principal object is the space $\Delta = \mathcal{H}_0 \cap \mathcal{H}_1$ with norm defined by

$$
\|f\|_\Delta^2 = \|f\|_0^2 + \|f\|_1^2.
$$

In the following, all involutions are understood as being taken with respect to the norm of $\Delta$, and we express the inner products of the spaces $\mathcal{H}_i$ in the following way:

$$
\|f\|_0^2 = (Hf, f)_\Delta, \quad \|f\|_1^2 = ((1 - H)f, f)_\Delta.
$$

Here $H \in B(\Delta)$ satisfies $0 \leq H \leq 1$, and the regularity of $\mathcal{H}$ yields that $H$ has no eigenvalues in the set $\{0, 1\}$. Likewise, if $\mathcal{H}_*$ is an intermediate Hilbert space, we have

$$
\|f\|_*^2 = (Kf, f)_\Delta
$$

for some positive operator $K \in B(\Delta)$. It is easy to see that $\mathcal{H}_*$ fulfills (ExInt) iff $T^*KT \leq K$ for all $T \in B(\Delta)$ such that $T^*HT \leq H$ and $T^*(1 - H)T \leq 1 - H$.

The relations between $H, K$ and the operators $A, B$ of the previous section can then be expressed in the following way

$$
H = \frac{1}{1 + A}, \quad A = \frac{1 - H}{H}, \quad K = \frac{B}{1 + A}, \quad B = \frac{K}{H}.
$$
Since $\mathcal{H}_*$ fulfills (ExInt) iff $B = h(A)$ where $h \in P'$, we obtain the following: $\mathcal{H}_*$ fulfills (ExInt) iff $K = k(H)$ where

$$K = k(H) = \frac{1}{1 + A} h(A) = H h \left( \frac{1 - H}{H} \right),$$

and thus

$$k(\lambda) = \lambda \left( \int_{[0, \infty]} \frac{(1 + t)(1 - \lambda)/\lambda}{1 + t(1 - \lambda)/\lambda} d\rho(t) \right) = \int_{[0, \infty]} \frac{(1 + t)\lambda(1 - \lambda)}{\lambda + t(1 - \lambda)} d\rho(t), \quad \lambda \in [0, 1].$$

Applying the change of variables $s = t/(1 + t)$ to the latter integral, and defining a positive Radon measure $\nu$ on $[0, 1]$ by $d\nu(s) = d\rho(t)$, we arrive at the following formula

$$k(\lambda) = \int_{[0, 1]} \frac{\lambda(1 - \lambda)}{(1 - s)(1 - \lambda) + s\lambda} d\nu(s), \quad \lambda \in [0, 1],$$

which is the form of functions used by Donoghue in [7], bottom of p. 253.

**Remark 4.1.** The results of sect. 2 can be extended to non-regular couples by using a device of Donoghue, notably the remarks on p. 264 and theorems 1' and 2' on p. 269. The transition is straight-forward, and we refrain from stating the so-extended results here.

**Remark 4.2.** As we have seen, expressing all involutions and inner products with respect to the inner product of $\mathcal{H}_0$ yields precisely the $K_2$-method whereas Donoghue’s choice $\Delta$ gives an alternative description of the interpolation spaces which is advantageous for other purposes. Similarly, one can, starting from any exact interpolation Hilbert space $\mathcal{H}_*$ obtain a different, but equivalent, description.

### 5. Remarks on a theorem of Foiaş and Lions

In this section, we comment on a theorem of C. Foiaş and J.-L. Lions, which preceded Donoghue’s theorem, and was obtained by quite different methods. It is shown below that this difference is essentially the same as the difference between the $J$ and $K$ methods of Peetre. As a result, we obtain a quantitatively stronger version of Peetre’s equivalence theorem, valid on the category of Hilbert couples.

Let the functional $J_2$ be defined by

$$J_2(t, f; \mathcal{H})^2 = \|f\|_2^2 + t\|f\|_1^2 = ((1 + tA)f, f)_0, \quad f \in \mathcal{H}_0 \cap \mathcal{H}_1.$$

Given a positive Radon measure $\nu$ on $[0, \infty]$, let a space $J_2(\nu)(\mathcal{H})$ be defined to consist of the elements $f \in \mathcal{H}_0 + \mathcal{H}_1$ such that there exists a $\nu$-measurable function $u(t) \in \mathcal{H}_0 \cap \mathcal{H}_1$ such that

$$f = \int_{[0, \infty]} u(t) d\nu(t) \quad \text{in } \mathcal{H}_0 + \mathcal{H}_1 \quad \text{and} \quad \int_{[0, \infty]} (1 + t)^{-1} J_2(t, u(t))^2 d\nu(t) < \infty. \quad (5.1)$$

The norm in the space $J_2(\nu)(\mathcal{H})$ is defined by

$$\|f\|_{J_2(\nu)(\mathcal{H})}^2 = \inf_u \int_{[0, \infty]} (1 + t)^{-1} J_2(t, u(t); \mathcal{H})^2 d\nu(t) \quad \text{(over } u(t) \text{ fulfilling (5.1)).} \quad (5.2)$$

We have the following version of the Foiaş–Lions theorem.
Let $h$ be defined by

$$
(5.3) \quad h(\lambda)^{-1} = \int_{[0, \infty]} \frac{1 + t}{1 + t\lambda} d\nu(t).
$$

Then $\|f\|_{J_2(\nu) / (\mathbb{R})}^2 = (h(A)f, f)_0$. Moreover, every quadratic interpolation method is representable in this form (for some positive Radon measure $\nu$ on $[0, \infty)$).

Remark 5.1. A nice feature of the Foiaş–Lions proof is that it yields an explicit minimizer $u$ of (5.2):

$$
u(t) = \phi_t(A)f \quad \text{where} \quad \phi_t(\lambda) = \frac{1 + t}{1 + t\lambda} \left( \int_{[0, \infty]} \frac{1 + s}{1 + s\lambda} d\nu(s) \right)^{-1}.
$$

We have the following proposition.

Proposition 1. For an intermediate Hilbert space $\mathcal{H}_*$ to be exact interpolation with respect to a regular Hilbert couple $\mathcal{H}$, it is necessary and sufficient that there exist a positive Radon measure $\nu$ such that $\mathcal{H}_* = J_2(\nu)(\mathcal{H})$ (isometrically).

Proof. We use the fact that the set of functions of the form (5.3) coincides with the cone $P'$ (use Löwner’s theorem (cf. [2]), stating that $P'$ equals the set of matrix monotone functions on $\mathbb{R}^+$ and the simple fact that a function $h$ is matrix monotone iff the function $h(\lambda^{-1})^{-1}$ is matrix monotone). The proposition is now immediate from Donoghue’s Theorem 1, Lemma 2.2 and Theorem FL.

We finish by giving two examples on the not-so trivial relation between the $J_2$, $K_2$ and complex methods.

Example 5.1. Let $K_{2, \theta, 2}$ be the functor

$$
\|f\|_{K_{2, \theta, 2}(\mathcal{H})}^2 = \int_{0}^{\infty} t^{-\theta} K_2(t, f; \mathcal{H})^2 \frac{dt}{t},
$$

and let $J_{2, \theta, 2}$ be defined by

$$
\|f\|_{J_{2, \theta, 2}(\mathcal{H})}^2 = \inf_u \int_{0}^{\infty} J_2(t, u(t); \mathcal{H})^2 \frac{dt}{t} \quad \text{(over $u$ such that $f = \int_{0}^{\infty} u(t) \frac{dt}{t}$)}.
$$

By the elementary identity (cf. [10], integral 3.222(2))

$$
(5.4) \quad \int_{0}^{\infty} (1 + t)\lambda c^{\theta} t^{-\theta} \frac{dt}{1 + t\lambda} = \left( \int_{0}^{\infty} \frac{1 + t}{1 + t\lambda} c^{\theta} t^{1 + t\lambda} \frac{dt}{t} \right)^{-1} = \lambda^{\theta},
$$

where $c_\theta = \pi / \sin(\theta\pi)$, $0 < \theta < 1$, it is now easy to see that $K_{2, \theta, 2}(\mathcal{H}) = J_{2, \theta, 2}(\mathcal{H})$ isometrically. This result should be compared with the “equivalence theorem” of Peetre, cf. [5], p. 44, which yields that $K_{2, \theta, 2}(\mathcal{X}) = J_{2, \theta, 2}(\mathcal{X})$ isomorphically for all Banach couples $\mathcal{X}$.

Example 5.2. Given a Hilbert couple $\mathcal{H}$, we have the following definition (cf. [11]) of the complex interpolation spaces $\mathcal{H}_{[\theta]}$, $0 < \theta < 1$,

$$
(5.5) \quad \|f\|_{[\theta]}^2 = (A^{\theta} f, f)_0, \quad f \in \mathcal{H}_{[\theta]}.
$$

It can be shown that the definition (5.5) is consistent (with equality of norms) with the standard definition of the complex interpolation spaces as stated in e.g. [5], ch. 4; details are found in [1], sect. 5.3 (modulo some evident misprints).
Now (5.4) and (5.5) yield
\[
\|f\|_{[\theta]}^2 = (A^\theta f, f)_0 = \int_0^\infty (1 + t^{-1}) \left( \frac{tA}{1 + tA} f, f \right)_0 \frac{c_0 t^{-\theta}}{1 + t} dt = \]
\[
c_0 \int_0^\infty t^{-\theta} K_2(t, f, \overline{\mathcal{H}})^2 \frac{dt}{t} = c_0 \|f\|_{K_2,2(\overline{\mathcal{H}})}^2,
\]
i.e. the complex \( \theta \)-method is (on the category of Hilbert couples) proportional to the \( K_{2,2,2} \)-method.
References

CHAPTER VI

Interpolation functions of several matrix variables

ABSTRACT. The interpolation theorem of W. F. Donoghue is extended to interpolation of tensor products. The result is related to Korányi’s work on monotone matrix functions of several variables.

Statement and proof of main result. Let us recall the definition of an interpolation function (of one variable) cf. \[1\], \[6\]. Let \( A \in M_n(\mathbb{C}) \) be a positive definite matrix. A real function \( h \) defined on \( \sigma(A) \) is said to belong to the class \( C_A \) of interpolation functions with respect to \( A \) if

\[
T \in M_n(\mathbb{C}) \quad T^*T \leq 1 \quad T^*AT \leq A
\]

(1)

implies

\[
T^*h(A)T \leq h(A).
\]

By Donoghue’s theorem (cf. \[6\], Theorem 1, or \[1\], Theorem 1.1), the functions in \( C_A \) are precisely those which are representable in the form

\[
h(\lambda) = \int_{[0, \infty]} \frac{(1 + t)\lambda}{1 + t\lambda} d\rho(t), \quad \lambda \in \sigma(A),
\]

(3)

with some positive Radon measure \( \rho \) on the compactified half-line \([0, \infty]\). Thus, by Löwner’s theorem (see \[7\] or \[5\]) \( C_A \) is precisely the set of restrictions to \( \sigma(A) \) of the positive matrix monotonic functions on \( \mathbb{R}_+ \), in the sense that \( A, B \in M_n(\mathbb{C}) \) positive definite and \( A \leq B \) imply \( h(A) \leq h(B) \). Before we proceed, it is important to note that

\[
h \in C_A \quad \text{implies} \quad h^{\frac{1}{2}} \in C_A.
\]

(4)

because the function \( \lambda \mapsto \lambda^{\frac{1}{2}} \) is matrix monotone and the class of matrix monotone functions is a semi-group under composition.

Given two positive definite matrices \( A_i \in M_n(\mathbb{C}) \), let us define the class \( C_{A_i}^{A_2} \) of interpolation functions with respect to \( A_1, A_2 \) as the set of functions \( h \) defined on \( \sigma(A_1) \times \sigma(A_2) \) having the following property:

\[
T_i \in M_n(\mathbb{C}) \quad T_i^*T_i \leq 1 \quad T_i^*A_iT_i \leq A_i, \quad i = 1, 2
\]

(5)

imply

\[
(T_1 \otimes T_2)^*h(A_1, A_2)(T_1 \otimes T_2) \leq h(A_1, A_2).
\]

(6)

(Here cf. \[11\])

\[
h(A_1, A_2) = \sum_{(\lambda_1, \lambda_2) \in \sigma(A_1) \times \sigma(A_2)} h(\lambda_1, \lambda_2)E_{\lambda_1} \otimes F_{\lambda_2},
\]

where \( E, F \) are the spectral resolutions of \( A_1, A_2 \).)
Note that if \( h = h_1 \otimes h_2 \) is an elementary tensor where \( h_i \in C_{A_i} \), then \( h \in C_{A_1,A_2} \), because then (5) yields
\[
(T_1 \otimes T_2)^* h(A_1, A_2)(T_1 \otimes T_2) =

= (T_1^* h_1(A_1)T_1) \otimes (T_2^* h_2(A_2)T_2) \leq h_1(A_1) \otimes h_2(A_2) = h(A_1, A_2),
\]

viz. (6) holds. It is a simple fact (see e.g. [1], [6]) that the functions
\[
\lambda \mapsto \frac{(1 + t_1)\lambda}{1 + t_1\lambda}
\]
where \( t \in [0, \infty] \) are interpolation functions of one variable, viz. are in \( C_\lambda \) for all \( A \). Since the class \( C_{A_1,A_2} \) is a convex cone, which is closed under pointwise convergence, it follows that functions of the type
\[
h(\lambda_1, \lambda_2) = \int_{[0,\infty]^2} \frac{(1 + t_1)\lambda_1 (1 + t_2)\lambda_2}{1 + t_1\lambda_1 + t_2\lambda_2} d\rho(t_1, t_2),
\]
where \( \rho \) is a positive Radon measure on \([0, \infty]^2\) are in \( C_{A_1,A_2} \) for all \( A_1, A_2 \). We have thus proved the easy part of our main theorem:

**Theorem 3.** Let \( h \) be a real function defined on \( \sigma(A_1) \times \sigma(A_2) \). Then \( h \in C_{A_1,A_2} \) iff \( h \) is representable in the form (7) with some positive Radon measure \( \rho \).

It remains to prove “\( \Rightarrow \)”.

Let us make some preliminary observations:

(i) ([3], Lemma 2.2) The class \( C_{A_1,A_2} \) is unitarily invariant in the sense that if \( A_1 \) and \( A_2 \) are unitarily equivalent to \( A'_1 \) and \( A'_2 \) respectively, then \( h \in C_{A_1,A_2} \) implies \( h \in C_{A'_1,A'_2} \). (Indeed,
\[
h(U_1^* A_1 U_1, U_2^* A_2 U_2) = (U_1 \otimes U_2)^* h(A_1, A_2)(U_1 \otimes U_2)
\]
for all unitaries \( U_1, U_2 \).)

(ii) ([3], Lemma 2.1) The class \( C_{A_1,A_2} \) respects compressions to invariant subspaces in the sense that if \( f \in C_{A_1,A_2} \) and \( A'_1, A'_2 \) are compressions of \( A_1, A_2 \) respectively to invariant subspaces, then \( h \in C_{A'_1,A'_2} \). (Indeed,
\[
(E \otimes F)h(A_1, A_2)(E \otimes F) = (E \otimes F)h(EA_1, FA_2)(E \otimes F)
\]
whenever \( E, F \) are orthogonal projections commuting with \( A_1, A_2 \) respectively.)

(iii) If \( \lambda \) is any (fixed) eigenvalue of \( A_2 \) and the function \( h(\lambda) : \sigma(A_1) \to \mathbb{R} \) is defined by \( h(\lambda) = h(\lambda_1, \lambda_2) \), then
\[
h(A_1, \lambda^*_2 F \lambda) = \sum_{\lambda_1 \in \sigma(A_1)} h(h(\lambda_1, \lambda^*_2), E_{\lambda_1} \otimes F \lambda) =

= \left( \sum_{\lambda_1 \in \sigma(A_1)} h(\lambda_1, \lambda_1^*) E_{\lambda_1} \right) \otimes F \lambda = h(\lambda_1, \lambda_2) \otimes F \lambda.
\]

(\text{By symmetry, of course (with fixed } \lambda^*_1 \in \sigma(A_1) \text{ and } h(\lambda^*_1, \lambda_2) = h(\lambda^*_1, \lambda_2))
\[
h(h(\lambda^*_1, \lambda_2), A_2) = E_{\lambda^*_1} \otimes h(\lambda^*_1, \lambda_2).)
\]

**Lemma 3.** Let \( h \in C_{A_1,A_2} \), \( \lambda^*_1, \lambda^*_2 \) fixed eigenvalues of \( A_1 \) and \( A_2 \) respectively. Then \( h^\frac{1}{2}_{\lambda^*_1} \in C_{A_2} \) and \( h^\frac{1}{2}_{\lambda^*_2} \in C_{A_1} \).
VI. INTERPOLATION OF TENSOR PRODUCTS

Let $h \in C_{A_1,A_2}$; by (iii),

$$h(A_1, \lambda_2 F_{\lambda_2}) = h_{\lambda_2}(A_1) \otimes F_{\lambda_2}.$$

Let $f_2^{\ast}$ be a fixed non-zero vector in the range of $F_{\lambda_2}$ and put $c = (F_{\lambda_2} f_2^{\ast}, f_2^{\ast}) > 0$. Put $T_2 = F_{\lambda_2}$ and let $T_1$ be any matrix fulfilling $T_1^* T_1 \leq 1$ and $T_1^* A_1 T_1 \leq A_1$; then plainly $T_1, T_2$ satisfy the condition (5). So, since $h \in C_{A_1, \lambda_2 F_{\lambda_2}}$, (6) yields

$$((T_1 \otimes F_{\lambda_2})^* h(A_1, \lambda_2^* F_{\lambda_2})(T_1 \otimes F_{\lambda_2})(f_1 \otimes f_2^{\ast}), f_2 \otimes f_2^{\ast}) - (h(A_1, \lambda_2^* F_{\lambda_2})(f_1 \otimes f_2^{\ast}), f_1 \otimes f_2^{\ast}) =$$

$$= c((T_1^* h_{\lambda_2^*}(A_1) T_1 f_1, f_1) - (h_{\lambda_2^*}(A_1) f_1, f_1)) \leq 0, \quad f_1 \in M_{n_1}(\mathbb{C}).$$

This yields $T_1^* h_{\lambda_2^*}(A_1) T_1 \leq h_{\lambda_2^*}(A_1), T_1 \in M_{n_1}(\mathbb{C})$, i.e. $h_{\lambda_2^*} \in C_{A_1}$. In view of the observation (4), then $h_{\lambda_2^*} \in C_{A_1}$.

Let $h$ be a fixed function in the class $C_{A_1,A_2}$. Replacing the matrices $A_1, A_2$ by $c_1 A_1, c_2 A_2$ for suitable constants $c_1, c_2 > 0$, we can without loss of generality assume that

$$\tag{8} (1, 1) \in \sigma(A_1) \times \sigma(A_2).$$

Let $C$ be the $C^\ast$-algebra of continuous functions $[0, \infty] \to \mathbb{C}$ with the supremum norm, and denote (for fixed $\lambda \in \mathbb{R}_+$) by $e_\lambda$ the function

$$e_\lambda(t) = \frac{(1 + t)\lambda}{1 + t\lambda} \in C, \quad t \in [0, \infty].$$

Let two finite-dimensional subspaces $V_1, V_2$ be defined by

$$V_i = \text{span}\{e_{\lambda_i} : \lambda_i \in \sigma(A_i)\} \subset C, \quad i = 1, 2.$$

Then (8) yields that the unit $1 = e_1(t) \in C$ belongs to $V_1 \cap V_2$. Let now for fixed $\lambda_i^* \in \sigma(A_i)$ two linear functionals

$$\phi_{\lambda_1^*} : V_2 \to \mathbb{C}, \quad \phi_{\lambda_2^*} : V_1 \to \mathbb{C}$$

be defined by

$$\phi_{\lambda_1^*} \left( \sum_{\lambda_2 \in \sigma(A_2)} a_{\lambda_2} e_{\lambda_2} \right) = \sum_{\lambda_2 \in \sigma(A_2)} a_{\lambda_2} h_{\lambda_1^*}(\lambda_2)^{\frac{1}{2}},$$

$$\phi_{\lambda_2^*} \left( \sum_{\lambda_1 \in \sigma(A_1)} a_{\lambda_1} e_{\lambda_1} \right) = \sum_{\lambda_1 \in \sigma(A_1)} a_{\lambda_1} h_{\lambda_2^*}(\lambda_1)^{\frac{1}{2}}.$$

We have the following lemma:

**Lemma 4.** The functional $\phi_{\lambda_1^*}$ is positive on $V_2$ in the sense that if $u \in V_2$ satisfies $u(t) \geq 0$ for all $t > 0$, then $\phi_{\lambda_1^*}(u) \geq 0$. Similarly, $\phi_{\lambda_2^*}$ is a positive functional on $V_1$.

**Proof.** This follows from Lemma 3 and Lemma 2.1 of [1].

Consider now the bilinear form

$$\phi : V_1 \times V_2 \to \mathbb{C}$$
The Hahn–Banach theorem yields an extension \( Φ \) of the same norm. Hence the positivity of \( \tilde{A} \). A simple rewriting yields that (9) equals

\[
\phi \left( \sum_{\lambda_1 \in \sigma(A_1)} a_{\lambda_1} e_{\lambda_1}, \sum_{\lambda_2 \in \sigma(A_2)} a_{\lambda_2} e_{\lambda_2} \right) =
= \sum_{(\lambda_1^*, \lambda_2^*) \in \sigma(A_1) \times \sigma(A_2)} \phi_{\lambda_1^*} \left( \sum_{\lambda_2 \in \sigma(A_2)} a_{\lambda_2} e_{\lambda_2} \right) \phi_{\lambda_2^*} \left( \sum_{\lambda_1 \in \sigma(A_1)} a_{\lambda_1} e_{\lambda_1} \right).
\]

By Lemma 4, we infer that \( \phi \) is a positive bilinear form on \( V_1 \times V_2 \) in the sense that \( u_i \in V_i, u_i \geq 0 \) implies \( \phi(u_1, u_2) \geq 0 \). Hence (since the \( V_i \)'s contain the function 1)

\[
\|\phi\| = \sup \{ |\phi(u_1, u_2)| : u_i \in V_i, \|u_i\|_\infty \leq 1, i = 1, 2 \} = \phi(1, 1).
\]

Now \( \phi \) lifts to a linear functional

\[
\tilde{\phi} : V_1 \otimes V_2 \to \mathbb{C},
\]

which is positive on \( V_1 \otimes V_2 \), because

\[
\|\tilde{\phi}\| = \|\phi\| = \phi(1, 1) = \tilde{\phi}(1).
\]

The Hahn–Banach theorem yields an extension \( \Phi : C \otimes C = C([0, \infty]^2) \to \mathbb{C} \) of \( \tilde{\phi} \) of the same norm. Hence the positivity of \( \tilde{\phi} \) yields

\[
\|\Phi\| = \|\tilde{\phi}\| = \tilde{\phi}(1) = \Phi(1),
\]

i.e. \( \Phi \) is a positive functional on \( C([0, \infty]^2) \). Hence, the Riesz representation theorem provides us with a positive Radon measure \( \rho \) on \( [0, \infty]^2 \) such that

\[
\Phi(u) = \int_{[0,\infty]^2} u(t_1, t_2) d\rho(t_1, t_2), \quad u \in C([0, \infty]^2).
\]

A simple rewriting yields that (9) equals

\[
\sum_{(\lambda_1^*, \lambda_2^*) \in \sigma(A_1) \times \sigma(A_2)} \left( a_{\lambda_1^*} a_{\lambda_2^*} \overline{h(\lambda_1^*, \lambda_2^*)} + \sum_{(\lambda_1, \lambda_2) \neq (\lambda_1^*, \lambda_2^*)} a_{\lambda_1} a_{\lambda_2} \overline{h(\lambda_1^*, \lambda_2^*)} \right).
\]

Tucking in the latter expression in (11) yields

\[
h(\lambda_1^*, \lambda_2^*) = \phi(\lambda_1^*, \lambda_2^*) = \Phi(e_{\lambda_1^*} \otimes e_{\lambda_2^*}) =
= \int_{[0,\infty]^2} \frac{(1 + t_1) \lambda_1^* (1 + t_2) \lambda_2^*}{1 + t_1 \lambda_1^* + t_2 \lambda_2^*} d\rho(t_1, t_2).
\]

Since \( \lambda_1^*, \lambda_2^* \) were arbitrary, the theorem is proved. \( \square \)

**Remark 4.** It is easy to modify the above proof to obtain a representation theorem for interpolation functions of more than two matrix variables. (Where the latter set of functions is interpreted in the obvious way.)

**Korányi’s theorem.** Consider the class of functions which are monotonic according to the definition of Korányi [11] \(^1\) \( A_1 \leq A'_1 \) and \( A_2 \leq A'_2 \) imply

\[
h(A'_1, A'_2) - h(A_1, A'_2) - h(A_1, A'_2) - h(A_1, A_2) \geq 0.
\]

The functions

\[
h_t(\lambda) = \frac{(1 + t)\lambda}{1 + t\lambda}
\]

\(^1\) A different definition of monotonicity of several matrix variables was recently given by Frank Hansen in [10].
are monotonic of one variable \((0 \leq t \leq \infty)\), whence with \(h_{t_1t_2} = h_{t_1} \otimes h_{t_2}\) (cf. [11], p. 544),
\[
h_{t_1t_2}(A'_1, A'_2) - h_{t_1t_2}(A'_1, A_2) - h_{t_1t_2}(A_1, A'_2) + h_{t_1t_2}(A_1, A_2) =
\]
\[
= (h_{t_1}(A'_1) - h_{t_1}(A_1)) \otimes (h_{t_2}(A'_2) - h_{t_2}(A_2)) \geq 0,
\]
i.e. \(h_{t_1t_2}\) is monotonic. Since the class of monotonic functions of two variables is closed under pointwise convergence, the latter inequality can be integrated, which yields that all functions of the form (7) are monotonic. Hence we have proved the easy half of the following theorem of A. Korányi, cf. [11], Theorem 4, cf. also [12].

**Theorem 4.** Let \(h\) be a positive function on \(\mathbb{R}^2_+\). Assume that (a) the first partial derivatives and the mixed second partial derivatives of \(h\) exist and are continuous. Then \(h\) is monotonic iff \(h\) is representable in the form (7) with some positive Radon measure \(\rho\) on \([0, \infty]^2\).

**Remark 5.** According to Korányi the differentiability condition (a) was imposed “in order to avoid lengthy computations which are of no interest for the main course of our investigation” ([11], bottom of p. 541).

Let us denote a function \(h\) defined on \(\mathbb{R}^2_+\) an interpolation function if \(h \in C_{A_1, A_2}\) for any positive matrices \(A_1, A_2\). Theorem 3 and Theorem 4 then yield the following corollary, which nicely generalizes the one-variable case.

**Corollary 2.** The set of interpolation functions coincides with the set of monotonic functions satisfying (a).
References