Linear and non-linear deformations of stochastic processes

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ABSTRACT


This thesis consists of three papers on the following topics in functional analysis and probability theory: Riesz bases and frames, weakly stationary stochastic processes and analysis of set-valued stochastic processes.

In the first paper we investigate Uniformly Bounded Linearly Stationary processes from the point of view of the theory of Riesz bases. By regarding these stochastic processes as generalized Riesz bases we are able to gain some new insight into their structure. Special attention is paid to regular UBLS processes as well as perturbations of weakly stationary processes.

An infinite sequence of subspaces of a Hilbert space is called regular if it is decreasing and zero is the only vector in its intersection. In the second paper we ask for conditions under which the regularity of a sequence of subspaces is preserved when the sequence undergoes a deformation by a linear and bounded operator. Linear, bounded and surjective operators are closely linked with frames and we also investigate when a frame is a regular sequence of vectors.

A multiprocess is a stochastic process whose values are compact sets. As a generalization of the class of subharmonic processes and the class of subholomorphic processes as introduced by Thomas Ransford, in the third paper we introduce the general notion of a gauge of processes and a multigauge of multiprocesses. Compositions of multiprocesses with multifunctions are discussed and the boundary crossing property, related to the intermediate-value property, is investigated for general multiprocesses. Time changes of multiprocesses are investigated in the environment of multigauges and we give a multiprocess version of the Dambis-Dubins-Schwarz Theorem.

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Introduction

1 Riesz bases, frames and weakly stationary processes

The purpose of this section is to give some background material and motivate, as well as briefly outline, the content of the first two papers of this thesis.

At the very heart of Hilbert space theory and its applications lies the notion of an orthonormal sequence, i.e. a sequence of pairwise orthogonal elements of norm one from the underlying space. We will begin by discussing two different directions, followed to great length in the existing literature, along which the notion of an orthonormal sequence has been generalized.

1.1 Riesz bases and frames

In what follows $\mathbb{H}$ will always denote a separable complex Hilbert space. With $\{x_n\}_{n \in I}$ we will mean a sequence of elements from $\mathbb{H}$ indexed by a countable index set $I$. (The set $I$ will most of the time be $\mathbb{N}$ or $\mathbb{Z}$).

A basis for $\mathbb{H}$ (with the index set $\mathbb{N}$) is a sequence $\{x_n\}_{n \in \mathbb{N}}$ such that for each $x \in \mathbb{H}$ there is a unique sequence of scalars $\{a_n\}_{n \in \mathbb{N}}$ with

$$x = \lim_{m \to \infty} \sum_{n=1}^{m} a_n x_n \equiv \sum_{n \in \mathbb{N}} a_n x_n,$$

(with convergence in the norm of $\mathbb{H}$). A general basis like this is sometimes called a Schauder basis.

The first type of basis a student of functional analysis (or linear algebra for that matter) encounters, is the orthonormal basis. An orthonormal basis $\{e_n\}_{n \in I}$ for $\mathbb{H}$ is simply an orthonormal sequence which is total in $\mathbb{H}$, i.e. the only element of $\mathbb{H}$ orthogonal to all $e_n$'s is the zero element. Orthonormal bases have many nice features and for future comparison we recall some of these here. First we have the Pythagorean property: for any finite sequence of scalars $\{a_n\}_{n \in J}$,

$$\| \sum_{n \in J} a_n x_n \|^2 = \sum_{n \in J} |a_n|^2.$$

Also of importance is Parseval’s relation: for any $x \in \mathbb{H}$,

$$\sum_{n \in I} |\langle x, e_n \rangle|^2 = \|x\|^2.$$
Series expansions with respect to orthonormal bases take on a particularly nice form: for any $x \in \mathbb{H}$,
\[ x = \sum_{n \in I} \langle x, e_n \rangle e_n. \]

To demand orthogonality between the elements of a basis can sometimes be very restrictive. Luckily, there is a family of non-orthogonal bases which are almost as nice as orthogonal bases. These bases are called Riesz bases. We define the concept of a Riesz basis by relaxing the Pythagorean property. A sequence $\{x_n\}_{n \in I}$ is called a Riesz basis if it is total in $\mathbb{H}$ and there are constants $0 < A \leq B$ such that
\[ A \sum_{n \in J} |a_n|^2 \leq \| \sum_{n \in J} a_n x_n \|^2 \leq B \sum_{n \in J} |a_n|^2, \]
for every finite sequence of scalars $\{a_n\}_{n \in J}, J \subset I$. We mention also that a sequence in $\mathbb{H}$ is called a Bessel sequence if the upper inequality above holds for every finite sequence of scalars.

A Riesz basis is indeed a Schauder basis. Moreover, it is possible to prove that a sequence is a Riesz basis if and only if it is the image under a linear, bounded and invertible operator of an orthonormal basis for $\mathbb{H}$. Parallel with the fact that each such operator has an invertible adjoint, Riesz bases appear naturally in biorthogonal pairs: for every Riesz basis $\{x_n\}_{n \in I}$ there is a unique Riesz basis $\{y_n\}_{n \in I}$ for which
\[ \langle x_n, y_m \rangle = \delta_{nm} \quad \forall \, n, m \in I. \]

This duality also appears in series expansions with respect to Riesz bases. For any $x \in \mathbb{H}$ we can write
\[ x = \sum_{n \in I} \langle x, y_n \rangle x_n = \sum_{n \in I} \langle x, x_n \rangle y_n. \]

Further afield along the same direction we find frames. Although a Riesz basis is always a frame, a frame is not necessarily a Schauder basis. The definition of a frame is related to Parseval’s relation. We say that a sequence $\{x_n\}_{n \in I}$ is a frame for $\mathbb{H}$ if there are constants $0 < A \leq B$ such that for every $x \in \mathbb{H}$,
\[ A \|x\|^2 \leq \sum_{n \in I} |\langle x, x_n \rangle|^2 \leq B \|x\|^2. \]


Although not necessarily a basis, a frame has many basis-like features and can be used for series expansions. This works as follows. The following characterization result for frames is due to Holub (see [15] from 1994):

A sequence $\{x_n\}_{n=1}^\infty$ in a Hilbert space $\mathbb{H}$ is a frame for $\mathbb{H}$ if and only if there exists a linear and bounded operator $Q$ which maps $\ell^2(\mathbb{N})$ onto $\mathbb{H}$ for which $Q \delta_n = x_n$ for each $n \geq 1$. 

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The operator $T$ is called the pre-frame operator. A quick calculation shows that the adjoint of this operator, $T^* : \mathbb{H} \rightarrow \ell^2(I)$ is given by
\[ T^* x = \{ \langle x, x_n \rangle \}_{n \in I}, \]
and hence that
\[ TT^* x = \sum_{n \in I} \langle x, x_n \rangle x_n. \]
Since $T$ is surjective, the operator $S = TT^*$, being self adjoint, is invertible. The operator $S$ is called the frame operator and for any $x \in \mathbb{H}$ we can now write
\[ x = SS^{-1} x = \sum_{n \in I} \langle S^{-1} x, x_n \rangle x_n = \sum_{n \in I} \langle x, S^{-1} x_n \rangle x_n \]
and this is our series representation.

One major difference between Riesz bases and frames is that a Riesz basis is minimal, in the sense that removing an element from a Riesz basis leaves a sequence which is no longer total in $\mathbb{H}$, whereas a frame can contain redundant elements. Often in applications of frames, this redundancy is exactly what one is after. We end this subsection by giving an example, which we believe is new to the literature, of a frame consisting of unit length elements which in one sense contains great redundancy. First we need a lemma.

**Lemma 1.1.** For every $0 < \epsilon < 1$ there is $0 < A < 1$ such that
\[ |z + w|^2 + \epsilon |w|^2 \geq A(|z|^2 + |w|^2), \quad \forall z, w \in \mathbb{C}. \]  

**Proof.** For $0 < \epsilon < 1$ we put $A = \frac{2 + \epsilon - \sqrt{4 + \epsilon^2}}{2}$. Then $0 < A < 1$ and
\[ (1 - A)(1 - A + \epsilon) = \left( \frac{\sqrt{4 + \epsilon^2} - \epsilon}{2} \right) \left( \frac{\sqrt{4 + \epsilon^2} + \epsilon}{2} \right) = 1. \]
Since
\[ |z + w|^2 + \epsilon |w|^2 - A(|z|^2 + |w|^2) = (1 - A)|z|^2 + (1 - A + \epsilon)|w|^2 + 2 Re z \bar{w} = |(\sqrt{1 - A})z + (\sqrt{1 - A + \epsilon})w|^2 \geq 0, \]
the inequality (2) follows. \hfill \Box

**Example 1.2.** There is a frame for $\mathbb{H}$, consisting of unit length elements, which can be written as a disjoint union of infinitely many Riesz bases for $\mathbb{H}$.

**Proof.** Let $\{e_{m,n}\}_{m,n=1}^{\infty}$ be an orthonormal basis for a Hilbert space $\mathbb{H}$ indexed by $\mathbb{N}^2$. Let $\{\epsilon_k\}_{k=1}^{\infty}$ be a sequence of numbers such that $0 < \epsilon_k < \frac{1}{2}$ for every $k$ and $\sum_{k=1}^{\infty} \epsilon_k^2 < \infty$. For a fixed $m'$ we let $\{g_{m',k}\}_{k=1}^{\infty}$ be an enumeration of the set $\{e_{m,n}\}_{m,n=1,m' \neq m}^{\infty}$ and we define $\{f_{m',n}\}_{n=1}^{\infty}$ by
\[ f_{m',n} = \begin{cases} e_{m',n + \frac{1}{2}}, & \text{if } n \text{ is an odd number} \\ e_{m',n + \frac{1}{2}} + \epsilon_{m'} g_{m',n}, & \text{if } n \text{ is an even number} \end{cases}, \]
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if $n$ is an even number. It is easy to see that $\{f_{m',n}\}_{n=1}^{\infty}$ is total in $\mathbb{H}$ and that it is a Bessel sequence (we will actually prove this later). We want to check the lower Riesz basis condition. Let $\{a_k\}_{k=1}^p$ be a sequence of complex numbers where we assume that $p$ is an even number. From Lemma 1.1 we know that there is $0 < A < 1$ such that

$$\|\sum_{k=1}^{p} a_k f_{m',k}\|^2 = \|\sum_{k=1}^{p} a_{2k-1} e_{m',k} + \sum_{k=1}^{p} a_{2k} (e_{m',k} + \epsilon_m g_{m',k})\|^2 =$$

$$\|\sum_{k=1}^{p} (a_{2k-1} + a_{2k}) e_{m',k} + \sum_{k=1}^{p} \epsilon_m a_{2k} g_{m',k}\|^2 = \sum_{k=1}^{p} |a_{2k-1} + a_{2k}|^2 + \sum_{k=1}^{p} \epsilon_m^2 |a_{2k}|^2 =$$

$$\sum_{k=1}^{p} (|a_{2k-1} + a_{2k}|^2 + \epsilon_m^2 |a_{2k}|^2) \geq \sum_{k=1}^{p} A(|a_{2k-1}|^2 + |a_{2k}|^2) = A \sum_{k=1}^{p} |a_k|^2.$$

This proves that $\{f_{m',n}\}_{n=1}^{\infty}$ is a Riesz basis for $\mathbb{H}$ for any $m'$.

We now want to show that $\{f_{m,n}\}_{m,n=1}^{\infty}$ is a frame for $\mathbb{H}$. Obviously we do not have to worry about the lower frame condition. Let $x \in \mathbb{H}$. Then

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\langle x, f_{m,n}\rangle|^2 =$$

$$\sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} |\langle x, e_{m,n}\rangle|^2 + \sum_{n=1}^{\infty} |\langle x, e_{m,n} + \epsilon_m g_{m,n}\rangle|^2 \right) \leq$$

$$\sum_{m=1}^{\infty} \left( \sum_{n=1}^{\infty} |\langle x, e_{m,n}\rangle|^2 + 4 \sum_{n=1}^{\infty} |\langle x, e_{m,n}\rangle|^2 + 4\epsilon_m^2 \sum_{n=1}^{\infty} |\langle x, g_{m,n}\rangle|^2 \right) \leq$$

$$5 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |\langle x, e_{m,n}\rangle|^2 + 4\epsilon_m^2 \sum_{m=1}^{\infty} |x||^2 = 5 |x||^2 + 4 |x||^2 \sum_{m=1}^{\infty} \epsilon_m^2 =$$

$$(5 + 4 \sum_{m=1}^{\infty} \epsilon_m^2)|x||^2.$$

Thus $\{f_{m,n}\}_{m,n=1}^{\infty}$ is a frame for $\mathbb{H}$. Since $1 \leq ||f_{m,n}|| \leq \sqrt{1 + \frac{1}{2}}$ for every $m$ and $n$ we can normalize this frame and the normalized Riesz bases will still be Riesz bases so we get a frame consisting of unit length elements which is an infinite union of Riesz bases for $\mathbb{H}$.

### 1.2 Weakly stationary stochastic processes

Before we go into the subject of weakly stationary stochastic processes we mention some notions from general linear theory of stochastic processes. In linear theory of stochastic processes (as it is presented for example in Chapter IV of the book [14] by Gihman and Skorohod from 1972) one studies certain stochastic processes with emphasis on the fact that the variables of the process lie in a linear space. Therefore, inner-products, projections and linear spans often enter the discussion.

For now we will only consider discrete time processes $\{x_n\}_{n \in \mathbb{Z}}$ whose variables lie
in a Hilbert space $\mathbb{H}$. Usually in the literature, $\mathbb{H}$ is assumed to be the Hilbert space $L_0^2(P)$, consisting of all random variables on a probability space $(\Omega, \mathcal{F}, P)$ with zero expectation and a finite second moment, but for the discussion we have in mind any complex separable Hilbert space will do. Thus we will not distinguish between a sequence and a process.

We will regard $\mathbb{Z}$ as infinite discrete time and with any process $\{x_n\}_{n \in \mathbb{Z}}$, we will associate a number of “information spaces”. First, we write $\mathbb{H}(x)$ for the closed linear span of all variables in the process. Secondly, for any $t \in \mathbb{Z}$ we write $\mathbb{H}_t(x)$ for the closed linear span of all $x_n$’s for which $n \leq t$. This represents the information generated by the process up to and including time $t$. Finally, we write $\mathbb{H}_{-\infty}(x)$ for the intersection of all $\mathbb{H}_t(x)$’s. This can be regarded as the information the process carries with it since the beginning of time. If a process has no long-time memory, i.e. $\mathbb{H}_{-\infty}(x) = \{0\}$ then the process is called regular, and if no new information is ever generated by the process, i.e. $\mathbb{H}_{-\infty}(x) = \mathbb{H}_t(x) \forall \ t \in \mathbb{Z}$, then the process is called deterministic.

To exemplify how these spaces are interpreted, suppose that we are at time $t$, and we wonder where the process will be at time $t + n$. Then with the information we have at our disposal, $\mathbb{H}_t(x)$, our best guess (best linear prediction) is given by the unique element in $\mathbb{H}_t(x)$ closest to $x_{t+n}$, namely the orthogonal projection of $x_{t+n}$ onto $\mathbb{H}_t(x)$.

In analysis of a process $\{x_n\}_{n \in \mathbb{Z}}$ its covariance function is a very valuable source of information. In fact, since we usually don’t assume knowledge of probability distributions, this might be our only source of information. The covariance function of $\{x_n\}_{n \in \mathbb{Z}}$ is defined as the function $K : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ for which

$$K(n, m) = \langle x_n, x_m \rangle \quad \forall \ n, m \in \mathbb{Z}.$$ 

It is well known that a function $K : \mathbb{Z} \times \mathbb{Z} \to \mathbb{C}$ is the covariance function of a process $\{x_n\}_{n \in \mathbb{Z}}$ in $L_0^2(P)$ if and only if it is symmetric and non-negative definite. This can be proved by use of the Kolmogorov extension theorem or by using reproducing kernel Hilbert spaces. However, in Paper 1 Proposition 2.1 we give a direct proof of a slightly more general result.

The simplest process as far as covariance function goes and also a basic building block in the whole theory is the orthonormal sequence or the white noise process as it is often called in this context. That is, the process with the covariance function $K(n, m) = \delta_{nm}$. A white noise process is a very special example of a so called weakly stationary process:

A stochastic process $\{x_n\}_{n \in \mathbb{Z}}$ is called weakly stationary if it has a covariance function of the form $K(n, m) = K(n - m)$, i.e. if the correlation between any two variables in the process depends only on the time difference between them and not on where in the process they are seated.

Because of their nice covariance functions, weakly stationary processes have many nice features. One of them is the existence of a shift operator. For any weakly stationary process $\{x_n\}_{n \in \mathbb{Z}}$ there is a unitary operator $S : \mathbb{H}(x) \to \mathbb{H}(x)$ such that $Sx_n = x_{n+1}$ for each $n$. This implies that each weakly stationary process is of the form $\{S^nx_0\}_{n \in \mathbb{Z}}$ for some unitary operator $S$ and some $x_0 \in \mathbb{H}$. Also, since $S$ is unitary, prediction errors with best linear prediction depends only on how far into the future we want to look and not on where we are at the moment.
The most important decomposition result for weakly stationary processes is the Wold decomposition theorem. It works as follows. For a weakly stationary process \( \{ x_n \}_{n \in \mathbb{Z}} \) we define two new processes \( \{ x_n^{(1)} \}_{n \in \mathbb{Z}} \) and \( \{ x_n^{(2)} \}_{n \in \mathbb{Z}} \) by,
\[
x_n^{(1)} = P_1 x_n \quad \text{and} \quad x_n^{(2)} = P_2 x_n,
\]
where \( P_1 \) denotes the orthogonal projection onto \( H_{-\infty}(X) \) and \( P_2 \) denotes the orthogonal projection onto \( H_{-\infty}(X)^\perp \). It turns out that both these processes are weakly stationary. Moreover, \( \{ x_n^{(1)} \}_{n \in \mathbb{Z}} \) is deterministic and \( \{ x_n^{(2)} \}_{n \in \mathbb{Z}} \) is regular. Clearly we can now write
\[
x_n = x_n^{(1)} + x_n^{(2)} \quad \forall \ n \in \mathbb{Z},
\]
and this is the Wold decomposition theorem.

A regular weakly stationary process can be written in a very nice way. For any regular weakly stationary process \( \{ x_n \}_{n \in \mathbb{Z}} \) there is a white noise process \( \{ e_n \}_{n \in \mathbb{Z}} \) with \( \mathbb{H}_t(x) = \mathbb{H}_t(e) \) for every \( t \) (i.e. the two processes generate exactly the same information) and a sequence \( \{ a_n \}_{n \in \mathbb{Z}} \) of scalars such that
\[
x_n = \sum_{k=0}^{\infty} a_k e_{n-k} \quad \forall \ n \in \mathbb{Z}.
\]
In other words, every regular weakly stationary sequence admits a representation as a one-sided moving average sequence. A regular weakly stationary sequence can in general be written as a moving average sequence in many different ways. However, every regular weakly stationary sequence has a unique one sided moving average representation like the one above such that \( \mathbb{H}_t(x) = \mathbb{H}_t(e) \) for every \( t \in \mathbb{Z} \) and such that \( a_0 > 0 \). We will call this the canonical moving average representation of the sequence.

Weakly stationary stochastic processes have found applications in a large number of areas (see for example [23]).

1.3 Paper 1: Riesz bases and Uniformly Bounded Linearly Stationary (UBLS) sequences

In the last two subsections we have tried to suggest that both Riesz bases and weakly stationary sequences are children of the same mother, the orthonormal sequence. The first objective of the first paper of this thesis is to adapt a unified approach to these sequences and hence define a class of sequences in \( \mathbb{H} \) containing all Riesz bases for \( \mathbb{H} \) as well as all weakly stationary sequences in \( \mathbb{H} \). This is achieved as follows:

It is easy to see that a sequence \( \{ x_n \}_{n \in \mathbb{Z}} \) in \( \mathbb{H} \) is weakly stationary if and only if there is a symmetric and non-negative definite function \( K : \mathbb{Z} \to \mathbb{C} \) such that
\[
\| \sum_{n=p}^{q} a_n x_n \|^2 = \sum_{k=p}^{q} \sum_{l=p}^{q} a_k \overline{a_l} K(k - l),
\]
for every finite sequence of scalars \( \{ a_n \}_{n=p}^{q} \). Looking back at the definition of a Riesz basis it is now natural to say that a sequence \( \{ x_n \}_{n \in \mathbb{Z}} \) belongs to our class
if and only if there is a symmetric and non-negative definite function $K: \mathbb{Z} \to \mathbb{C}$ and two constants $0 < A \leq B$ such that

$$A \sum_{k=p}^{q} \sum_{l=p}^{q} a_k \overline{a_l} K(k-l) \leq \left\| \sum_{n=p}^{q} a_n x_n \right\|^2 \leq B \sum_{k=p}^{q} \sum_{l=p}^{q} a_k \overline{a_l} K(k-l), \quad (3)$$

for every finite sequence of scalars $\{a_n\}_{n=p}^{q}$. With the theory of Riesz bases in mind, this characterization turns out to be the right one, at least in the sense that it is equivalent with the statement that $\{x_n\}_{n \in \mathbb{Z}}$ is the image of a weakly stationary sequence in $\mathbb{H}$ under a linear and bounded operator. This is proved in Theorem 2.4 of Paper 1.

Interestingly enough, although the approach to the class of sequences characterized by (3) presented in this paper is entirely new, the class in itself has been studied before, but from a different angle. The first attempt at a rigorous treatment of so called Uniformly Bounded Linearly Stationary (UBLS) processes can be found in the paper [38] by Tjøstheim and Thomas from 1975, but the roots of the notion go back to the paper [13] by Getoor from 1956. UBLS processes were then studied further by a number of authors especially during the 70’s and 80’s (see [26] by Niemi from 1976, [37] by Tjøstheim from 1980, [1] by Abreu and Fetter from 1985, [16] by Kakihara from 1985, [10] by Dehay from 1990 and also the book [17] by Kakihara from 1997). A UBLS process is defined to be a stochastic process which has a uniformly bounded shift operator (hence the name), i.e. in discrete time a process $\{x_n\}_{n \in \mathbb{Z}}$ is called UBLS if there is a constant $M$ such that

$$\left\| \sum_{n=p}^{q} a_n x_{n+k} \right\|^2 \leq M \left\| \sum_{n=p}^{q} a_n x_n \right\|^2,$$

for every finite sequence of scalars $\{a_n\}_{n=p}^{q}$ and every $k \in \mathbb{Z}$. In Theorem 2.4 of Paper 1 we prove that our class of sequences, when we choose $\mathbb{H} = L^0_0(P)$, is exactly the class of discrete time UBLS processes.

Loosely speaking, the existing literature on UBLS processes deals with questions about how close to, and sometimes how far from, the class of weakly stationary processes the class of UBLS processes is. However, the connection to Riesz bases (or frames) have up until now not been investigated so our main source of inspiration for new results in this paper comes from comparing UBLS sequences (we only work in discrete time) with Riesz bases. (In fact, the connection to Riesz bases and its literature were hinted at, but not followed through, by Tjostheim and Thomas in the original paper [38]. However, as the field developed, interest in continuous time processes took over completely and the connection was lost.)

Seen from our characterization (3) a of a UBLS sequence, to tell if a given sequence is a Riesz basis (for its closed linear span) we only have to compare the sequence at hand with the covariance function $\delta_{nm}$. One interesting phenomenon that appears when we move from Riesz bases to UBLS sequences, is that a given UBLS sequence can satisfy the inequalities in (3) for a number of different covariance functions $K$ (with $A$ and $B$ depending on $K$). This is analogous to the statement that a given UBLS sequence can be described as the image of several different weakly stationary sequences under several different invertible operators, as observed already by Niemi in [26]. The inherited relation among weakly stationary sequences is an equivalence relation and we say that two weakly stationary
sequences are similar if there is a linear, bounded and invertible operator mapping one onto the other. In Section 3 and Section 5 of Paper 1, we characterize for some different classes of weakly stationary sequences the equivalence classes generated by this relation, we believe that this has not been done before. Implicitly, for a given UBLS sequence we find exactly for which covariance functions $K$ the inequalities in (3) are satisfied. We will return to this topic in a moment.

In Theorem 2.8 of Paper 1 we show that for any UBLS sequence $\{x_n\}_{n \in \mathbb{Z}}$ and any associated covariance function $K$ (as discussed above) there is a unique UBLS sequence $\{y_n\}_{n \in \mathbb{Z}}$ such that,

$$\langle x_n, y_n \rangle = K(n - m).$$

In analogy with the notion of biorthogonality, in this situation we say that $\{x_n\}_{n \in \mathbb{Z}}$ and $\{y_n\}_{n \in \mathbb{Z}}$ are bistationary with covariance function $K$. To make the connection to the theory of Riesz bases even more explicit, we also show in this theorem how the above duality can be used (in some cases) for expansions of elements of $\mathbb{H}$ using UBLS sequences.

In Section 3 we take a short look at UBLS sequences $\{x_n\}_{n \in \mathbb{Z}}$ for which $\mathbb{H}(x)$ is finite dimensional, and in Section 5 we investigate regular UBLS sequences. We first note that a regular UBLS sequence is always the image of a regular weakly stationary sequence $\{s_n\}_{n \in \mathbb{Z}}$ under a linear, bounded and invertible operator $T$. Thus, by writing $\{s_n\}_{n \in \mathbb{Z}}$ as a one sided moving average sequence and then applying $T$, we find that each regular UBLS sequence admits a representation as a one sided moving average sequence on a Riesz basis instead of an orthonormal basis. See Theorem 5.2 in Paper 1 for more details. This observation also strengthens Theorem 3 of [38]. We then investigate those equivalence classes (generated by the equivalence relation discussed above) which contain a regular weakly stationary sequence. In them main result of Section 5 in Paper 1, Theorem 5.4, given the canonical moving average representation of a regular weakly stationary sequence, we find exactly which weakly stationary sequences are similar to the given one. All of these sequences are regular and our characterization displays each sequence in its canonical representation. Applying this result to an orthonormal basis for $\mathbb{H}$ yields a characterization of all sequences in $\mathbb{H}$ which are on one hand weakly stationary and on the other hand Riesz bases for $\mathbb{H}$. This is done (although slightly differently) in Corollary 5.5 of Paper 1.

The above mentioned Theorem 5.4 is stated in terms of so called pseudomeasures (on the integers). In Section 4 we present several different characterizations of these sequences of complex numbers. Most of these already exist in the literature, see for example the book [25] by Larsen from 1971. However, we believe that Theorem 4.4 in Paper 1 which relates pseudomeasures and invertible pseudomeasures via the shift operator to Riesz bases and Bessel sequences is new.

In Section 6 we consider two closely related questions. We first ask if a sequence which is close, in a certain geometric sense, to a weakly stationary sequence must be a UBLS sequence? To answer this question, in Theorem 6.2 of Paper 1 we generalize the Paley-Wiener Theorem about stability of bases to fit our purposes (in fact, the Paley-Wiener Theorem in its original form was mentioned but not generalized in [38]). Next we ask if a UBLS sequence can be used to model a weakly stationary sequence which has been disturbed by a white noise? The case of a white noise sequence lying in the orthogonal complement of the subspace
spanned by the weakly stationary sequence is treated in [38]. However, without this assumption the problem becomes much harder. By answering a number of questions about moment sequences we are able to give conditions in Theorem 6.6 of Paper 1 under which a finite white noise sequence can be added to a weakly stationary sequence to give a UBLS sequence.

1.4 Paper 2: Frames, covariance functions and regular sequences

Paper 2 of this thesis is mainly a paper about frames and operators but the inspiration for the questions we ask comes directly from linear theory of stochastic processes. For example, one of our main objectives is to investigate conditions under which a frame is a regular sequence. In connection with this question, bearing in mind the close link between frames and surjective operators, we also investigate when the regularity of a regular sequence (of vectors or subspaces) is preserved when the sequence undergoes a deformation by a linear and bounded operator.

We begin however with a question more related to those we asked in Paper 1. What kind of sequence do we get if we apply a linear, bounded and surjective operator $T : \mathbb{H} \rightarrow \mathbb{H}$ to a linearly independent weakly stationary sequence total in the space $\mathbb{H}$? The answer is given in Section 3 of Paper 2 and because of Holub’s theorem on the pre-frame operator (see Subsection 1.1 of this introduction) the resulting sequences will be called frame-stationary sequences. To make the connection to frames even more explicit we give a definition of this family of sequences which is very similar to the definition of a frame and for this we need a generalization of Holub’s theorem. In Theorem 3.1 of Paper 2 we find conditions, involving two positive constants and a covariance function $K$, which are satisfied by a sequence $\{x_n\}_{n \in \mathbb{Z}}$ in $\mathbb{H}$ if and only if there is a sequence $\{y_n\}_{n \in \mathbb{Z}}$ with covariance function $K$ total in $\mathbb{H}$ and a linear, bounded and surjective operator $T : \mathbb{H} \rightarrow \mathbb{H}$ such that $Ty_n = x_n$ for each $n \in \mathbb{Z}$. Using this theorem we define the concept of a frame stationary sequence as follows:

We say that a sequence $\{x_n\}_{n \in \mathbb{Z}}$ in $\mathbb{H}$ is frame stationary if there is a covariance function $K$ belonging to a linearly independent weakly stationary sequence and two constants $0 < A \leq B$ such that

$$A\|x\|^2 \leq \sup_{p \in \mathbb{N}} \sum_{n=-p}^{p} \sum_{m=-p}^{p} \langle x, x_n \rangle \langle x, x_m \rangle \tilde{K}_p(m, n) \leq B\|x\|^2,$$

for every $x \in \mathbb{H}$, where

$$[\tilde{K}_p(m, n)]_{m,n=-p,...,p} = ([K_p(m, n)]_{m,n=-p,...,p})^{-1}.$$

In Section 3 of Paper 2 we also prove some elementary properties of frame stationary sequences. The relationship between frame stationary sequences and UBLS sequences is roughly the same as the relationship between frames and Riesz bases. This is demonstrated for example in Proposition 3.1 where we give conditions under which a frame stationary sequence is in fact a UBLS sequence (there
is a very similar result stating when a frame is a Riesz basis). However, we do not claim to give a thorough exposition of frame stationary sequences. The main reason why we introduce this family of sequences is the fact that this is a family where regular frames appear naturally, to fill the role played by white noise sequences in the context of regular weakly stationary sequences. In Theorem 3.2 of Paper 2 we prove that a regular frame stationary sequence always admits a representation as a one sided moving average process on a regular frame. (The notion of a frame stationary sequence is in fact not entirely new. Considered as a stochastic process a frame stationary sequence is a special kind of weakly harmonizable process. However, frames and Riesz bases have not been considered in connection to these processes before. For information about weakly harmonizable processes see the book [17] by Kakihara from 1997).

In analogy with the notion of a regular sequence of elements in a Hilbert space $\mathbb{H}$, we say that a sequence $\{A_n\}_{n \in \mathbb{N}}$ of subspaces of $\mathbb{H}$ is regular if

$A_1 \supset A_2 \supset A_3 \supset \ldots$ and $\bigcap_{n=1}^{\infty} A_n = \{0\}.$

We begin Section 4 of Paper 2 with the following question: if $T : \mathbb{H} \to \mathbb{H}$ is a linear and bounded operator and $\{A_n\}_{n \in \mathbb{N}}$ is a regular sequence of subspaces of $\mathbb{H}$, when does it hold that

$\bigcap_{n=1}^{\infty} T(A_n) = \{0\}?$

If this holds, we say that $T$ preserves the regularity of $\{A_n\}_{n \in \mathbb{N}}$. In Proposition 4.1 we prove that the only aspect of $T$ that comes into play in this question is its kernel. If the operator $T$ is injective then it clearly preserves the regularity of a sequence of subspaces and in the main result of this section, Theorem 4.1, we prove that an operator preserves the regularity of any sequence of subspaces if and only if it has a finite dimensional kernel (i.e. if and only if it is almost injective). We also include an example, Example 4.1, of a regular sequence of subspaces and a linear and bounded operator which maps each subspace in the sequence surjectively onto the whole space.

We then turn to more direct questions about regular frames. A frame is called a Riesz frame if each subset of the frame is a frame for its closed linear span with a common lower bound $A$ (see the papers [7] and [6] by Christensen and Christensen and Casazza from 1996). In the elementary Proposition 4.2 of Paper 2 we describe a class of regular frames which includes all Riesz frames. As a corollary to this proposition we also show that if a frame $\{x_n\}_{n \geq 0}$ is weakly stationary in the sense that $x_n = U^ny$ for some surjective isometry $U : \mathbb{H} \to \mathbb{H}$, every $n \geq 0$ and some $0 \neq y \in \mathbb{H}$, then $\{x_n\}_{n \geq 0}$ is regular. A near Riesz basis is a frame which can be shrunk down to a Riesz basis by removal of a finite number of elements. Near Riesz bases are studied by Holub in [15] and they are closely related to bounded linear operators with finite dimensional kernels. In the final part of Section 4, we use Theorem 4.1 to describe a peculiar relationship between regular frames and near-Riesz bases, see Paper 2 Theorem 4.3.
2 Multifunctions and multiprocesses. Gauges and multigauges

In this section we give some background material and motivate, as well as briefly outline the content of the third paper of this thesis.

2.1 Analytic multifunctions

In this subsection we give a short overview of the basic theory of analytic multifunctions. The original definition of an analytic multifunction was stated by Oka already in the 1930’s but the fire spread slowly and the Klondike of the concept did not occur until the early 1980’s. Once considered merely as an interesting curiosity, the analytic multifunction has today found its way into crucial parts of a vast number of papers and results and the theory has been applied in areas such as general spectral theory, the Corona problem, interpolation theory and complex dynamical systems.

The original definition of an analytic multifunction is stated using concepts from pluricomplex analysis. We begin this subsection with a quick reminder of the notions needed. If $X$ is a metric space, a function $u : X \to [-\infty, \infty)$ is called upper semicontinuous if for each real number $c$, $\{x \in X : u(x) < c\}$ is open in $X$. Let $U$ be an open subset of $\mathbb{C}^n$. An upper semicontinuous function $u : U \to [-\infty, \infty)$ which is not identically $-\infty$ on any connected component of $U$ is called plurisubharmonic if for each $a \in U$ and $b \in \mathbb{C}^n$, the function $\lambda \to u(a + \lambda b)$ is subharmonic or identically $-\infty$ on every component of the set $\{\lambda \in \mathbb{C} : a + \lambda b \in U\}$. If $K$ is a compact subset of $U$, the plurisubharmonically convex hull of $K$ in $U$ is defined as

$$\{z \in U : u(z) \leq \sup u(K) \text{ for every } u \text{ plurisubharmonic on } U\}.$$  

Note that replacing plurisubharmonic functions with linear functionals in the definition of the set above gives the ordinary convex hull of $K$. Finally, the open set $U$ is said to be pseudoconvex if the plurisubharmonically convex hull of each compact subset of $U$ is relatively compact in $U$. For more information about plurisubharmonic functions and pseudoconvex sets see for example the book [19] by Klimek from 1991.

In order to discuss analytic multifunctions we first have to know what a multifunction is. By a multifunction we mean a function whose values are compact sets, i.e. for topological Hausdorff spaces $X$ and $Y$ a multifunction $K$ is a function $K : X \to \kappa(Y)$, where $\kappa(Y)$ denotes the collection of all compact subsets of $Y$. The topological theory of set valued functions was created in Poland in the 1930’s just a few years before the birth of the analytic multifunction in Japan, (see the paper [24] by Kuratowski from 1932). We will need only one topological property: a multifunction $K : X \to \kappa(Y)$ is said to be upper semicontinuous if the set

$$\{x \in X \mid K(x) \subset U\},$$  

is open in $X$ for every open set $U \subset Y$. Note that if $[-\infty, \infty]$ is treated as a compactification of $\mathbb{R}$, then a function $u : X \to [-\infty, \infty]$ is upper semicontinuous on the set $\{x \in X : u(x) < \infty\}$ which is open in $X$ if and only if the multifunction
$x \to [-\infty, u(x)]$ is upper semicontinuous. This motivates the definition of upper semicontinuity in the case of multifunctions. On the other hand if a multifunction $K$ is of the form $K(x) = \{f(x)\}$, where $f : X \to Y$ is a function, then $K$ is upper semicontinuous if and only if $f$ is continuous.

A continuous function $f : \mathbb{C} \to \mathbb{C}$ is holomorphic if and only if the complement of its graph in $\mathbb{C}^2$ is an open pseudoconvex set. This is one of several important results due to Hartogs. In the paper [28] from 1934, inspired by Hartogs, Oka gave the first definition of an analytic multifunction, or a pseudoconcave function as he then called it. Expressed in modern terms, Oka’s definition looks as follows:

An upper semicontinuous multifunctions $K : U \to \kappa(\mathbb{C})$, where $U$ is an open subset of $\mathbb{C}$, is called an analytic multifunction if

$$U \times \mathbb{C} \setminus graph(K)$$

is pseudoconvex,

where the graph of $K$ is defined as the set

$$graph(K) = \{(z, w) \in X \times Y \mid w \in K(z)\}.$$

Besides giving the definition, on the last page of this paper Oka stated a number of theorems about analytic multifunctions but he did not include proofs. Proofs as well as generalizations of Oka’s results were later supplied by Nishino in the paper [27] from 1962 and Yamaguchi in [39] from 1973.

Oka was way ahead of his time in many respects and one might argue that his paper [28] was written almost 50 years to early. Apart from the papers by Nishino and Yamaguchi, the ball thrown by Oka in [28] was not picked up for many years. In fact, when the theory finally came into fashion again, the discovery of Oka’s work must have come as a big surprise for the participating mathematicians who thought that the analytic multifunction were there own invention. During the 60’s and 70’s questions arose in spectral theory which provoked a number of mathematicians into the search for a good set valued analog of a holomorphic function. Both Wermer and Aupetit contributed to this search, but it was Słodkowski who during a stay in Canada 1978-1979 was able to independently re-give Oka’s definition. This was described and expanded in the paper [34] from 1981. In this paper Słodkowski also proved the following useful characterization of analytic multifunctions:

If $U$ is an open subset of $\mathbb{C}$ and $K : U \to \kappa(\mathbb{C})$ is an upper semicontinuous multifunction, then the following statements are equivalent:

(i) $K$ is an analytic multifunction,

(ii) for any relatively compact open subset $U'$ of $U$ and any function $v$ plurisubharmonic on a neighborhood of $graph(K) \cap (U' \times \mathbb{C})$, the function

$$u(z) \equiv \max v(\{z\} \times K(z))$$

is subharmonic on $U'$,

(iii) the function

$$(z, w) \to -\log dist(w, K(z))$$

is plurisubharmonic on $(U \times \mathbb{C}) \setminus graph(K)$.
Anyone of the statements above can be found in the literature as the definition of an analytic multifunction and \( \mathbb{C} \) is sometimes conveniently replaced by the Riemann sphere \( \mathbb{C}_\infty \). Also, if we in (ii) let \( U \subset \mathbb{C}^n, K : U \to \kappa(\mathbb{C}^n) \) be an upper semicontinuous multifunction and we replace subharmonic with plurisubharmonic, then we end up with the definition of an analytic multifunction \( K : U \to \kappa(\mathbb{C}^n) \).

The family of analytic multifunctions contains many interesting and pleasant individuals. In this family we can find a copy of each holomorphic function. In fact, a function \( f : U \to \mathbb{C} \) is holomorphic if and only if \( \lambda \to \{ f(\lambda) \} \) defines an analytic multifunction. Subharmonic functions are also well represented: if \( u : U \to [-\infty, \infty) \) is subharmonic then
\[
K(z) = \{ w \in \mathbb{C} | \log |w| \leq u(z) \},
\]
is an analytic multifunction. Other basic example include the \( k \)-th-root multifunction \( K : \mathbb{C} \setminus \{0\} \to \kappa(\mathbb{C}) \):
\[
K(z) = \{ w \in \mathbb{C} | w^k = z \},
\]
and the logarithm multifunction \( K : \mathbb{C} \to \kappa(\mathbb{C}_\infty) \):
\[
K(z) = \{ w \in \mathbb{C} | \exp w = z \} \cup \{ \infty \},
\]
(\( \infty \) is added to make \( K(z) \) compact in \( \mathbb{C}_\infty \)).

We now turn to some more advanced examples. Two main classes of examples of analytic multifunctions are related to the theory of Banach algebras and to complex dynamics, respectively.

Let \( A^n \) denote the \( n \)-th Cartesian power of a complex Banach algebra \( A \) and let \( \sigma(a) \) denote the spectrum of \( a \in A^n \) (or some type of joint spectrum if \( n > 1 \)). If \( f \) is a holomorphic function with values in \( A^n \), then the mapping \( z \mapsto \sigma(f(z)) \) is an analytic multifunction. This has been shown by Słodkowski [34] for \( n = 1 \) and — for different types of joint spectra — by Klimek [18] and Słodkowski [33] for \( n \geq 1 \). The converse statement is true locally in the one-dimensional case (see [34]), but is not completely clear in higher dimensions (see [35], [33]). The original motivation for analytic multifunctions of this type came from the theory of Banach algebras (Vesentini, Aupetit, Wermer) and spectral theory in \( C^* \)-algebras (Pełczyński, Semadeni).

The complex dynamics examples essentially assert that the filled-in Julia set depends analytically on the coefficients of the generating polynomials. In the one-dimensional case this follows from a more general result due to Baribeau and Ransford [3]. This can be considerably strengthened, and also extended to higher dimensions, if the generating polynomial can change during the process of iteration and if some set-theoretic operations are allowed in the process of formation of the Julia sets. Results of this type, based on techniques borrowed from pluripotential theory, have been obtained by Klimek and Kosek in [20], [21] and [22]. A simple case can be described as follows. If \( p : \mathbb{C} \to \mathbb{C} \) is a polynomial of degree \( d \geq 2 \), let \( p^0 = \text{Id} \) and \( p^n \) denote the \( n \)-th iterate of \( p \). We say that a point \( z \in \mathbb{C} \) escapes to infinity if \( p^n(z) \to \infty \) as \( n \to \infty \). The filled-in Julia set \( K_+(p) \) associated with \( p \) is defined as the set of points in the plane which do not escape to infinity. We can identify \( p \) with an element of \( \mathbb{C}^{d+1} \). If
\[
\Omega = \{ p \in \mathbb{C}^{d+1} : \text{the leading coefficient of } p \neq 0 \},
\]
then

\[ K_+ : \Omega \longrightarrow \kappa(\mathbb{C}), \quad p \mapsto K_+(p), \]

is an analytic multifunction (see [20] of [21] for details).

Many theorems in single valued complex analysis have analogs for analytic multifunctions. Among these are the Open Mapping Theorem, Picard’s Theorem, Schwarz’s Lemma, Rouche’s Theorem, Principle of the Argument, Perron’s Theorem etc. There are several good survey papers written on the subject, for example the paper [2] by Aupetit and the paper [29] by Ransford.

2.2 Holomorphic, subharmonic and subholomorphic processes

The close relationship between probability theory, with emphasis on martingale methods, and potential theory has been known for a long time and has been studied by many authors. For example, in many ways a martingale is the stochastic analog of a harmonic function. The depth of this relationship is well represented in the classic monograph by Doob [9]. However, in complex analysis in several variables, or in the theory of analytic multifunctions probabilistic methods have been used relatively rarely. Some of the most interesting and innovative contributions in this direction can be found in the papers [30] from 1990 and [31] from 1994 by Ransford. In this subsection we will discuss what we believe are the main objectives and results in these papers.

If a martingale can be seen as the stochastic analog of a harmonic function, what does the stochastic analog of a subharmonic function look like? What about the stochastic analog of a holomorphic function or an analytic multifunction? Answers to these questions are given in [30]. The resulting processes are interesting in themselves but they also provide the basis for the answer to another interesting question. According to a famous theorem by Lévy, if we apply a holomorphic function to a complex Brownian motion, the resulting process can be transformed back to a complex Brownian motion by a stochastic change of time scale (see for example [4] by Bernard, Campbell and Davie from 1979). Suppose now that we apply an analytic multifunction to a complex Brownian motion. What can be said about the resulting set-valued process? Using the notions developed in [30], Ransford answers this question in [31] in the very general setting of a Brownian motion taking values in a complex Banach space.

The topic of the next subsection of this introduction will be another paper by Ransford, [32] from 1999, in which he defines the abstract notions of a gauge of functions and a multigauge of multifunctions. Very loosely speaking, a gauge (and also a multigauge) is an object which in a certain sense contains both its lower and its upper closure (to be specified later). In the description of [30] and [31] that now follows, we will try to emphasize the fact that although gauges and multigauges (of processes and multiprocesses instead of functions and multifunctions) are not explicitly present in full generality in these papers, they are implicitly indicated by the methods used.

It seems unavoidable to include a few technical details. By a filtration \( \mathcal{F} \) we will mean a complete, right continuous filtration on complete probability space \((\Omega, \Sigma, P)\). All stochastic processes in this subsection are assumed to have index set \([0, \infty]\). If \( S \) and \( T \) are random times we define the stochastic interval \(((S, T])\)
by
\[(S,T) = \{(t, \omega) \in [0, \infty] \times \Omega \mid S(\omega) < t \leq T(\omega)\} \].

Other intervals are defined similarly. If \(\Phi\) is a stochastic process we put
\[\Phi_\chi[[S,T]](t, \omega) = \begin{cases} \Phi(t, \omega) & \text{if } (t, \omega) \in [[S, T]], \\ 0 & \text{if } (t, \omega) \notin [[S, T]]. \end{cases} \]

Finally, we denote by \(\mathcal{M}(\mathcal{F})\) the set of all martingales \(\Phi\) with respect to \(\mathcal{F}\) which has a last element \(\Phi_\infty \in L^2(\mathcal{F}_\infty, \mathbb{P})\) such that \(\Phi_t = E[\Phi_\infty \mid \mathcal{F}_t]\) for every \(t \in [0, \infty]\).

In Section 1 of [30], Ransford gives the definition of a so called holomorphic process, corresponding to a holomorphic function. The same class of processes was presented before Ransford by Føllmer in [12] from 1974 via the notion of a conformal basis, but we will concentrate on Ransford’s description. A holomorphic function is in particular a complex valued harmonic function which when squared is still a harmonic function. Therefore, Ransford begins with the notion of a conformal martingale:

A martingale in \(\mathcal{M}(\mathcal{F})\) is called \textit{conformal} if it is continuous and its square is also a martingale. The set of all conformal martingales in \(\mathcal{M}(\mathcal{F})\) is denoted by \(\mathcal{C}(\mathcal{F})\).

However, the property of staying harmonic when squared is not exclusive to holomorphic functions in the family of harmonic functions. It is also satisfied by all anti-holomorphic functions. Thus, it is not difficult to see that the class of conformal martingales is not closed under addition and is too big to form a good analog of the class of holomorphic functions. Another important property of the family of holomorphic functions is the fact that each harmonic function can be written as the sum of a holomorphic function and the conjugate of a holomorphic function. Taking all this into account, Ransford gives the following definition:

A \textit{holomorphic atlas} for \(\mathcal{F}\) is a vector subspace \(\mathcal{H}\) of \(\mathcal{C}(\mathcal{F})\), which contains all constant martingales in \(\mathcal{M}(\mathcal{F})\) and has the property that each continuous martingale in \(\mathcal{M}(\mathcal{F})\) can be written as a sum of an element of \(\mathcal{H}\) and the conjugate of an element of \(\mathcal{H}\). Elements of \(\mathcal{H}\) are then called \textit{holomorphic processes}.

A natural example of a holomorphic atlas can be found in Theorem 1.12 of [30] (it is a bit too long to be included here) and an indication that a holomorphic process is a good analog of a holomorphic function is given in Theorem 1.9 of [30] where it is proved that the composition of a bounded holomorphic function in \(n\) variables with \(n\) holomorphic processes gives a bounded holomorphic process.

What is the stochastic analog of a subharmonic function? What kind of stochastic process do we get if we compose a plurisubharmonic function in \(n\) variables with \(n\) holomorphic processes? It seems reasonable that these questions should have the same answer and the answer to the second question, a subharmonic process, is given in Theorem 2.12 of [30]. A subharmonic process is defined as follows:

A process \(\Phi\) is called \textit{subharmonic} if it is bounded above, predictable, right continuous and satisfies
\[E[\Phi_t \mid \mathcal{F}_s] \geq \Phi_s \text{ a.s.}\]
whenever $0 \leq s \leq t \leq \infty$. The family of all subharmonic processes is denoted by $\mathcal{S}$.

Subharmonic processes are related to submartingales in several different ways. A rather deep connection is proved in Theorem 2.2 of [30]: a process $\Phi$ is subharmonic if and only if it is the limit of a decreasing sequence of bounded and continuous submartingales $\Psi^{(n)}$ with bounded first elements $\Psi_0^{(n)}$.

In order to prove the above mentioned Theorem 2.12, in a couple of lemmas Ransford investigates what we in this introduction call the lower and the upper closure of $\mathcal{S}$ (Ransford does not use these terms). The lower closure is implicitly defined through downwards convergence:

If $(\Phi^{(n)})_{n \geq 1}$ is a decreasing sequence of subharmonic processes and $\Phi^{(n)}$ converges almost surely to $\Phi$, then $\Phi$ is a subharmonic process.

The upper closure of $\mathcal{S}$ is a bit more complicated. Roughly, Ransford proves that if a process $\Phi$ can be approximated locally and from below by sequences of subharmonic processes then $\Phi$ is subharmonic:

Let $\Phi : [0, \infty] \to [-\infty, \infty)$ be a process which is bounded above, predictable, right continuous and upper semicontinuous. Suppose also that given any predictable time $R \not= \infty$, there exist processes $\Psi^{(n)} \in \mathcal{S}$ ($n \geq 1$) and a predictable time $R' \geq R$ with $R' \not= R$, such that

$$\Phi_R \leq \sup_{n \geq 1} \Psi^{(n)}_R \text{ on } \{R < R'\},$$

and such that, for all $n \geq 1$,

$$\Psi^{(n)} \chi_{[[R,R']]} \leq \Phi \chi_{[[R,R']]} \text{ on } \{R < R'\}.$$

Then $\Phi \in \mathcal{S}$.

In Section 3 of [30], Ransford turns to the definition of a subholomorphic process. This definition, which will be presented below, is rather complicated and it could be helpful for the reader to bear in mind that Ransford has several different objectives with his definition. A subholomorphic process is intended partly as a stochastic analog of an analytic multifunction, partly as a generalization of both a holomorphic and a subharmonic process and partly as a type of process that can be used to describe the image of a Brownian motion under an analytic multifunction. Thus (although this implication is far from trivial), bearing subharmonic and holomorphic processes in mind, this family of set valued stochastic processes is defined as a family of processes which contains both its lower and its upper closure, namely the smallest such family which also contains all holomorphic processes.

By a stochastic multiprocess we mean a stochastic process which takes compact sets as values, i.e. for a metric space $Y$, a stochastic multiprocess is a map $K : [0, \infty] \times \Omega \to \kappa(Y)$ which satisfies a certain measurability condition (see the beginning of Section 3 in [30]). Given an open subset $U$ of $\mathbb{C}_\infty$ we denote by $\mathcal{K}(U)$ the family of all multiprocesses with values in $\kappa(U)$ which are bounded, predictable, upper semicontinuous and weakly right lower semicontinuous (for the
definition of these adaptability and smoothness conditions, see Definition 3.1 of [30].

As mentioned above, for the definition of a subholomorphic process we need to specify the lower and the upper closure of a family of multiprocesses. This is done in direct analogy with how it was done for subharmonic processes. Thus we need a concept of downward convergence for multiprocesses as well as a definition of what it means for a multiprocess to be approximated locally from below by sequences of multiprocesses:

For multiprocesses $K^{(n)}$ and $K$ we write $K^{(n)} \downarrow K$ if for all $(t, \omega) \in [[0, \infty]]$,

$$K^{(1)}_t(\omega) \supset K^{(2)}_t(\omega) \supset \ldots \text{ and } \bigcap_{n \geq 1} K^{(n)}_t(\omega) = K_t(\omega).$$

If $K$ is a set valued process and $S$ and $T$ are random times we put

$$K\chi_{[[S,T]]}(t, \omega) := \begin{cases} K(t, \omega) & \text{if } (t, \omega) \in [[S, T]], \\ \emptyset & \text{if } (t, \omega) \notin [[S, T]]. \end{cases}$$

The upper closure of a family of multiprocesses is defined through the following:

If $K$ is a multiprocess and $D$ is any collection multiprocesses, then $K$ has local $D$-selections if it satisfies the following condition: given any predictable time $R \neq \infty$, there exist multiprocesses $L^{(n)} \in D$ ($n \geq 1$), and a predictable time $R' \geq R$ with $R' \neq R$, such that

$$\partial K_R \subset \bigcup_{n \geq 1} L^{(n)}_R \text{ on } \{R < R'\},$$

and such that, for all $n \geq 1$,

$$L^{(n)}_R \chi_{[[R,R']]} \subset K\chi_{[[R,R']]} \text{ on } \{R < R'\}.$$

We are now almost ready to state the definition of a subholomorphic process. We just have to deal with one more technical issue. Let $H$ be a holomorphic atlas with respect to the filtration $F$. For a predictable time $S$, $^S H$ denotes the corresponding delayed atlas, i.e. the holomorphic atlas with respect to the delayed filtration $^S F$ which more then any other atlas with respect to $^S F$ resembles $H$ (see Definition 1.17 and Theorem 1.18 of [30]). Also, for an open subset $U$ of $\mathbb{C}_\infty$, $(^S H)^\infty(U)$ denotes the family of all bounded processes in $^S H$ whose values are restricted to $U$.

Let $U$ be open in $\mathbb{C}_\infty$. Then $K H(U)$ is the smallest subclass $D$ of $K(U)$ which satisfies the following three conditions:

(i) if $S$ is a predictable time and $Z \in (^S H)^\infty(U)$, then $\{Z\} \chi_{[[S, \infty]]} \in D$,

(ii) if $K \in K(U)$ and there exists a sequence $(K^{(n)})_{n \geq 1}$ in $D$ such that $K^{(n)} \downarrow K$, then $K \in D$.

(iii) if $K \in K(U)$ and $K$ has local $D$-selections, then $K \in D$. 

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Elements of $\mathcal{KH}(U)$ are called subholomorphic processes.

We end this subsection with Ransford’s version of Lévy’s theorem from [31]. We let $E$ denote a complex separable Banach space:

A stochastic process $B : [0, \infty) \times \Omega \to E$ is an $E$-valued Brownian motion if $\xi(B)$ is a complex Brownian motion for every bounded linear functional $\xi$ on $E$.

Theorem 4.4 of [31] says more or less that the image of an $E$-valued Brownian motion under a bounded analytic multifunction is a subholomorphic process. The picture is made slightly more complicated by the fact that subholomorphic processes are derived from martingales in $\mathcal{M}(\mathcal{F})$, and these have a last element which is not the case for a Brownian motion. However, a Brownian motion can be stopped:

Let $U$ be an open subset of $E$ containing $0 \in E$, and let $F : U \to \kappa(\mathbb{C})$ be an analytic multifunction. Let $B$ be an $E$-valued Brownian motion, and set

$$T = \inf\{t \geq 0 \mid B_t \notin U\}.$$  

Then, there exist predictable times $(T_k)_{k \geq 1}$ with $T_k$ converging strictly from below to $T$ such that $F(B^{T_k})$ is a subholomorphic process for each $k$, where $B^{T_k}$ denotes $B$ stopped at $T_k$.

2.3 Gauges and multigauges of functions and multifunctions

In the next subsection we will continue the discussion of processes and multiprocesses initiated in the last subsection, but before we do that we return for a moment to analytic multifunctions. There are some drawbacks with the definitions and characterizations of analytic multifunctions presented in Subsection 2.1. The basic idea of an analytic multifunction as a set valued analog of a single valued analytic function can be presented to and be grasped by anyone with a basic training in complex analysis. However, as plurisubharmonic functions and pseudoconvex sets appears already in the definitions, non-experts in several complex variables and pluripotential theory might be scared away from the subject. Also, single valued analytic functions are only implicitly present in these definitions and characterizations. Thus it seems natural to ask if it is possible to avoid this detour and construct each analytic multifunction from single valued analytic functions using some basic operations.

The answer to this question is yes and it was first delivered by Słodkowski in the paper [36] from 1989. In this paper Słodkowski works with analytic multifunctions in several complex variables under a definition which is slightly different than the definition we use in this introduction. However, the two definitions are equivalent for multifunctions $K : U \to \kappa(\mathbb{C})$ where $U \subset \mathbb{C}$. In the deep and very useful main result of this paper, Theorem 1.3, Słodkowski proves that every analytic multifunction can be obtained as a decreasing limit of locally trivial analytic multifunctions, i.e. multifunctions which locally behave as unions of single valued analytic functions.
Onwards to 1999 and the paper [32] by Ransford, a paper which is warmly recommended to anyone looking for a nicely written, self contained and above all different introduction to analytic multifunctions as well as more general families of multifunctions. In a different way than in Słodkowski’s paper, Ransford here introduces the family of analytic multifunctions \( K : U \rightarrow \kappa (C_{\infty}) \) where \( U \subset \mathbb{C} \), by relating them directly to ordinary holomorphic functions. As we shall see, to achieve this Ransford takes the methods he developed in [30] and [31] to treat stochastic analogs of subharmonic functions and analytic multifunctions and turn them back onto the theory of analytic multifunctions (the hunter becomes the hunted).

This paper is not about analytic multifunctions only. In terms very similar to those used in [30] to analyze the lower and the upper closure of the class \( \mathcal{S} \) of subharmonic processes, Ransford gives the definition of an abstract gauge of functions:

Let \( X \) be a Hausdorff topological space, and let \( \mathcal{G} \) be a family of upper semicontinuous functions from \( X \) to \( [-\infty, \infty] \). We define two new such families \( \mathcal{G} \downarrow \) and \( \mathcal{G} \uparrow \) as follows. Given an upper semicontinuous function \( u : X \rightarrow [-\infty, \infty] \), we say that:

(i) \( u \in \mathcal{G} \downarrow \) if there exists a decreasing sequence \( (u_n) \) in \( \mathcal{G} \) which converges pointwise to \( u \).

(ii) \( u \in \mathcal{G} \uparrow \) if \( u \) has local \( \mathcal{G} \)-support: this means that for each \( x_0 \in X \) with \( u(x_0) < \infty \) there exists \( v \in \mathcal{G} \) such that \( v(x_0) = u(x_0) \) and \( v \leq u \) on a neighborhood of \( x_0 \).

We call \( \mathcal{G} \) a gauge if both \( \mathcal{G} \downarrow = \mathcal{G} \) and \( \mathcal{G} \uparrow = \mathcal{G} \).

The most important example of a gauge is the following: If \( X \) is a Riemann surface (for example an open subset of \( \mathbb{C} \)) then the family of all upper semicontinuous functions \( u : X \rightarrow [-\infty, \infty] \) which are subharmonic on \( \{ u < \infty \} \) constitutes a gauge (see [32] Proposition 1.2).

Next, Ransford turns to the definition of an abstract multigaue of multifunctions.

Let \( X, Y \) be Hausdorff topological spaces, and let \( \mathcal{M} \) be a family of upper semicontinuous multifunctions from \( X \) to \( \kappa(Y) \). We define two new such families \( \mathcal{M} \downarrow \) and \( \mathcal{M} \uparrow \) as follows. Given an upper semicontinuous multifunction \( K : X \rightarrow \kappa(Y) \), we say that:

(i) \( K \in \mathcal{M} \downarrow \) if there exists a decreasing sequence \( (K_n) \) in \( \mathcal{M} \) such that \( \bigcap_n K_n(x) = K(x) \) for all \( x \in X \),

(ii) \( K \in \mathcal{M} \uparrow \) if \( K \) has local \( \mathcal{M} \)-supports: this means that for each \( x_0 \in X \) and each \( y_0 \in \partial K(x_0) \), there exists \( L \in \mathcal{M} \) such that \( y_0 \in L(x_0) \) and \( L(x) \subset K(x) \) for all \( x \) in a neighborhood of \( x_0 \).

We call \( \mathcal{M} \) a multigaue if both \( \mathcal{M} \downarrow = \mathcal{M} \) and \( \mathcal{M} \uparrow = \mathcal{M} \).
The multigauge generated by a family of upper semicontinuous multifunctions is the smallest multigauge which contains the family.

Very similar to how the class of subholomorphic processes was introduced in [30], the class of analytic multifunctions (of one variable and with values in $\kappa(\mathbb{C}_\infty)$) is now defined as follows:

Let $U$ be an open subset of $\mathbb{C}_\infty$.

(i) We write $R(U)$ for the family of all multifunctions $K : U \to \kappa(\mathbb{C}_\infty)$ of the form

$$K(z) = \{q(z)\}$$

where $q$ is a rational function.

(ii) We write $A(U)$ for the multigauge generated by $R(U)$. Elements of $A(U)$ are called analytic multifunctions on $U$.

In Theorem 4.6 of [32] Ransford proves that his definition of an analytic multifunction is equivalent with the definitions given in this introduction. Besides giving the definition above, as well as a rapid development of the theory of analytic multifunctions using this definition, Ransford treats a general multigauge of multifunctions as an object which is of interest in itself. A number of fundamental results about multigauges are proved (see for example the Pull-Back and the Push-Forward lemma in [32]), and in Section 5 of [32] the intermediate-value property for general multifunctions in general multigauges is investigated.

2.4 Paper 3: Gauges and multigauges of processes and multiprocesses

In the third paper of this thesis, inspired by Ransford’s papers [30] and [31] and guided by his work in [32], we define and investigate the abstract concept of a gauge of processes and a multigauge of multiprocesses because we believe that these concepts are of interest in themselves. These definitions are of course intended to correspond to Ransford’s definitions of a gauge of functions and a multigauge of multifunctions, but we like to point out that although in many respects similar, none of the two theories is included in the other.

Our definitions are set up in such a way that a gauge or a multigauge can be generated from any collection of measurable and smooth processes or multiprocesses. Thus, special attention is paid throughout the paper to questions about how properties of multiprocesses are inherited from a family of multiprocesses to the multigauge generated by the family.

In Section 3 Definition 3.6 we define a gauge of stochastic processes with index sets either $[0, \infty)$ or $[0, \infty]$ and with values in $[-\infty, \infty)$ to be a family of processes which contains both its lower and its upper closure. The gauge generated by a collection of processes is then defined as the smallest gauge containing this collection. Besides working out some technical details (Proposition 3.4), in this section we also give two examples of gauges. In the first of these, we prove the non-trivial fact that if $S$ and $T$ are predictable times with $S \leq T$ and $S < \infty$ and $G$ is the collection of all predictable, right continuous and upper semicontinuous
processes which are zero on the stochastic interval \([S, T]\), then \(G\) is a gauge. This example shows that the gauge concept is correctly tuned in the sense that when a gauge is generated, zero processes can not generate non zero processes. In the second example we point out that Ransford’s class of subharmonic processes indeed forms a gauge.

In Section 4 we turn to multiprocesses and multigauges. Similar to how a gauge was defined, in Definition 4.8 we define a multigauge to be a family of multiprocesses which contains both its lower and its upper boundary. The multigauge generated by a collection of multiprocesses is defined as the smallest multigauge containing this collection. In the main result of this section, Theorem 4.13, we prove that for processes and multiprocesses without a last element, a gauge is indeed a special case of a multigauge. It works as follows: if \(\Phi\) is a stochastic process with index set \([0, \infty)\) taking values in \([-\infty, \infty)\) we define a multiprocess \(G_\Phi\) who’s values are compact subsets of \([-\infty, \infty)\) by

\[
G_\Phi(t, \omega) = [-\infty, \Phi(t, \omega)], \quad \forall (t, \omega) \in [0, \infty)).
\]

In Proposition 4.10 we prove that \(\Phi\) is a predictable, right continuous and upper semicontinuous process if and only if \(K_\Phi\) is a predictable, weakly right lower semicontinuous and upper semicontinuous multiprocess and then Theorem 4.13 says that if \(G\) is a family of such processes, then \(G\) is a gauge if an only if \(\tilde{G} = \{G_\Phi : \Phi \in G\}\) is a multigauge.

In the short Section 5 we look at compositions of multiprocesses with multifunctions. Compositions of subharmonic subharmonic processes and subholomorphic processes with analytic functions, plurisubharmonic functions and analytic multifunctions are considered in depth by Ransford in [30] and [31], and our results in this section are merely easy translations of his results into our more abstract setting. In Theorem 5.2 we prove that if a family of multiprocesses is invariant under composition with a given continuous multifunction, then the multigauge generated by the family is also invariant under composition with this multifunction.

In Section 6 we discuss the so called boundary crossing property for multiprocesses. A multiprocess \(K\) whose values are compact subsets of a metric space \(Y\) is said to have the boundary crossing property if it cannot leave an open subset \(U\) of \(Y\) without crossing its boundary. More precisely, \(K\) has the boundary crossing property if for each predictable time \(\sigma\) and every open set \(U \subset Y\), the functions \(S_\sigma\) and \(T_\sigma\) defined by

\[
S_\sigma = \inf\{t \geq \sigma \mid K_t \cap \overline{U} \neq \emptyset\},
\]

\[
T_\sigma = \inf\{t \geq \sigma \mid K_t \cap \partial U \neq \sigma\},
\]

are predictable times and \(K \cap U \neq \emptyset\) on \([S_\sigma, T_\sigma)\). In [30], Ransford proves that every subholomorphic process satisfies a weaker form of the boundary crossing property. In the main result of this section, Theorem 6.6 we prove that if every multiprocess in a collection of multiprocesses have the boundary crossing property, then every multiprocess in the multigauge generated by the collection also have the boundary crossing property.

There are a number of important theorems in the literature on continuous time stochastic processes that involve random time changes. Levy’s theorem on the holomorphic image of a Brownian motion is one example. In the final section of the third paper of this thesis we investigate time changes of multiprocesses.
in multigauges. In the literature, time changes of stochastic processes are traditionally considered for adapted process in the environment of stopping times and therefore, in this section, we approach multiprocesses and multigauges from the same direction. We rewrite the definition of a multigauge to make it compatible with this environment and we discuss how smoothness and adaptability of a multiprocess is preserved under change of time. In the main result of this section, Theorem 7.8, we prove that if the operation of changing time, with a given time change, maps a collection of multiprocesses into a given multigauge, then this operation maps the multigauge generated by the family into the given multigauge.

According to the Dambis-Dubins-Schwarz Theorem a local martingale can always be represented as a time changed Brownian motion. In Theorem 7.9, using Theorem 7.8 we translate the Dambis-Dubins-Schwarz Theorem into our multiprocess setting. Theorem 7.9 says roughly that each “multi-local martingale” with a given bracket process starting at zero can be represented as a time changed “multi-Brownian motion”.

References


