Comprehensive Summaries of Uppsala Dissertations
from the Faculty of Science and Technology 916

Fast Numerical Techniques for Electromagnetic Problems in Frequency Domain

BY

MARTIN NILSSON

ACTA UNIVERSITATIS UPSALIENSIS
UPPSALA 2003

Dissertation at Uppsala University to be publicly examined in room 1211, Building 1, Polacksbacken, Friday, January 30, 2004 at 10:15 for the Degree of Doctor of Philosophy in Numerical Analysis. The examination will be conducted in English

Abstract

Nilsson, M. 2003. Fast Numerical Techniques for Electromagnetic Problems in Frequency Domain. Acta Universitatis Upsaliensis. *Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology* 916. 38 pp. Uppsala. ISBN 91-554-5827-0

The Method of Moments is a numerical technique for solving electromagnetic problems with integral equations. The method discretizes a surface in three dimensions, which reduces the dimension of the problem with one. A drawback of the method is that it yields a dense system of linear equations. This effectively prohibits the solution of large scale problems.

Papers I-III describes the Fast Multipole Method. It reduces the cost of computing a dense matrix vector multiplication. This implies that large scale problems can be solved on personal computers. In Paper I the error introduced by the Fast Multipole Method is analyzed. Paper II and Paper III describe the implementation of the Fast Multipole Method.

The problem of computing the monostatic Radar Cross Section involves many right hand sides. Since the Fast Multipole Method computes a matrix times a vector, iterative techniques are used to solve the linear systems. It is important that the solution time for each system is as low as possible. Otherwise the total solution time becomes too large. Different techniques for reducing the work in the iterative solver are described in Paper IV-VI. Paper IV describes a block Quasi Minimal Residual method for several right hand sides and a Sparse Approximate Inverse preconditioner that reduce the number of iterations significantly. In Paper V and Paper VI a method based on linear algebra called the Minimal Residual Interpolation method is described. It reduces the work in an iterative solver by accurately computing an initial guess for the iterative method.

In Paper VII a hybrid method between Physical Optics and the Fast Multipole Method is described. It can handle large problems that are out of reach for the Fast Multipole Method.

Keywords: Fast Multipole Method, Minimal Residual Interpolation, Sparse Approximate Inverse preconditioning, Method of Moments, fast solvers, iterative methods, multiple right hand sides, error analysis

Martin Nilsson, Department of Information Technology, Scientific Computing.
Uppsala University. Box 337, SE-751 05 Uppsala, Sweden

ISBN 91-554-5827-0
ISSN 1104-232X
urn:nbn:se:uu:diva-3884 (http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-3884)

To my family

List of Papers

The thesis is based on the following papers, which will be referred to in the text by Roman numerals I-VII

[I] M. Nilsson, Stability of the Fast Multipole Method for Helmholtz Equation in Three Dimensions, *IT Scientific Report 2003-054*, ISSN 1404-3203, Department of Information Technology, Uppsala University, November 2003. Downloadable from Web site
http://www.it.uu.se/research/reports/2003-054/. Submitted for publication.

[II] M. Nilsson, A Fast Multipole Method Solver for Large Scale Scattering Problems, In Fredrik Edelvik et al. editors, *EMB 01 – Electromagnetic Computations – Methods and Applications*, ISBN 91-631-1629-4, pages 148–155, Uppsala University, November 2001. SNRV.

[III] M. Nilsson, A Parallel Shared Memory Implementation of the Fast Multipole Method for Electromagnetics, *IT Scientific Report 2003-049*,
ISSN 1404-3203, Department of Information Technology, Uppsala University, October 2003. Downloadable from Web site
http://www.it.uu.se/research/reports/2003-049/. Submitted for publication.

[IV] M. Nilsson, A Fast Multipole Accelerated Block Quasi Minimum Residual Method for Solving Scattering from Perfectly Conducting Bodies, In Magdy F. Iskander editor, *Proceedings of IEEE Antennas and Propagation Society International Symposium*, ISBN 0-7803-6369-8, volume 4, pages 1848-1851, Salt Lake City, Utah, USA, July 2000. [1]

[V] P. Lötstedt and M. Nilsson, A Minimal Residual Interpolation Method for Linear Equations with Multiple Right Hand Sides. This is a preprint of a paper that has been accepted and will appear in SIAM Journal of Scientific Computing. [2]

[1] © 2003 IEEE. Reprinted, with permission.
[2] © 2003 SIAM. Reprinted, with permission.

[VI] M. Nilsson, Rapid Solution of Parameter-dependent Linear Systems for Electromagnetic Problems in the Frequency Domain, *IT Scientific Report 2003-055*, ISSN 1404-3203, Department of Information Technology, Uppsala University, November 2003. Downloadable from Web site http://www.it.uu.se/research/reports/2003-055/.

[VII] M. Nilsson, The Minimum Residual Interpolation Method Applied to Multiple Scattering in MM-PO, In Jonathan D. Young and John L. Volakis editors, *Proceedings of IEEE Antennas and Propagation Society International Symposium*, ISBN 0-7803-7846-6, volume 3, pages 828-831, Columbus, Ohio, USA, June 2003. [3]

[3] © 2003 IEEE. Reprinted, with permission.

Contents

Introduction

Products that depend on electromagnetics surround us. Everything from mobile telephones and televisions to large aircrafts and satellites use electricity. Many of these products have antennas that emit electromagnetic radiation when they are used. It is important to understand how the emitted radiation can affect other products and most of all how they can affect us. The development and construction of new devices that depend on electricity are expensive because a lot of time has to be spent on measuring the devices. This is because one wants to check that the device works within specified parameters and that it meets all requirements.

One way of reducing the development time is to use computers to predict the electromagnetic fields. From the predicted fields an engineer can see where a model should be changed in order to achieve a specified level of performance. Computer modeling can be used to give a first clue to which of the best models that should be measured more closely. One can then proceed by constructing the models to see which one performs best. Earlier the engineers had to construct a new model every time they came up with a new idea. This is not necessary with computers that test the models first.

Computer models are used for predicting the electromagnetic environment in problems varying from Electromagnetic Compatibility (EMC), which deals with the electromagnetic interference between different electric devices, to electromagnetic scattering by large objects, which deals with how much an object reflects from an incident plane wave.

The increased capacity of modern day computers means that real life problems can be simulated on a regular PC. However, it is important to note that the simulation time still is several days with standard methods for the type of problems we want to solve here. This is a bottleneck in the design stage of an electrical device. To address this problem one can either wait for a better computer or develop new methods that reduce the computational time.

1.1 Numerical methods for electromagnetic problems

There exist two main types of numerical methods for solving electromagnetic problems. The first type is Time-Domain methods (TD) [CJMS01, EL00, EL02, Ede02, Jin93, RT97, Ryl02, Taf95]. They solve Maxwell's equations

in the time domain and are suitable for broadband applications. The solution is obtained by stepping forward in time. Solutions for all frequencies that can be resolved by the geometry are obtained at once. Time-domain methods usually suffer from dispersion errors which arise because different frequencies move at different speeds on the computational grid. Methods in time-domain include the Finite-Difference Time-Domain method (FDTD) [Taf95], Finite-Volume Time-Domain methods (FVTD) [EL00, Ede00, Ede02, RT97] and Finite-Element Time-Domain methods (FETD) [EL02, Ede02, Jin93, Ryl02] and the integral equation methods in time called Marching-On-in-Time methods (MOT) [CJMS01].

The second type of methods is Frequency-Domain methods (FD) [CJMS01, Edl01, Kel62, Nil02, PK74, PRM98]. They solve the time-harmonic Maxwell equations for one frequency at a time. Therefore, they are better suited for narrow band applications. Frequency-domain methods are divided in low frequency, mid frequency and high frequency methods. The low frequency and mid frequency methods include the Method of Moments (MM). It is a method that given an external field finds the currents on the surface of an electromagnetic scatterer and then computes the field from the known surface currents. The Method of Moments defines an integral equation which is discretized and solved. The solution converges to the exact solution if the resolution is refined. A drawback with this method is that the memory requirements effectively prohibits the possibility of solving the equations for high frequencies. This is what makes the Method of Moments a low frequency to mid frequency method. The high frequency methods are based on asymptotic expansions of the solution and are often of ray-tracing type. They include Physical Optics (PO) [Edl01, ELS03], which can be viewed as an asymptotic approximation to the Method of Moments. Other methods are Geometrical Optics (GO), Geometrical Theory of Diffraction (GTD) [Kel62] and Uniform Theory of Diffraction (UTD) [PK74], which are ray-based methods. The expansions are usually valid when the object is much larger than the wavelength. Between the high frequency methods and the mid frequency methods there is a gap which one can not handle with the Method of Moments because of memory requirements. It is not possible to use the high frequency methods either because the asymptotic expansions are too inaccurate. Hybrid methods try to cover this intermediate frequency range. Volume integral methods [CJMS01, PRM98] and Finite Element methods [CJMS01, PRM98] are also used in frequency domain.

1.2 Content of the thesis

This thesis deals with ways of shortening the solution time for the Method of Moments and reducing its memory requirements. Instead of solving the equations by explicitly generating the system of linear equations we can use

approximations of them. The approximations are then solved with iterative methods. The advantages with this approach are that we do not require the same amount of memory and that the method is faster provided that the number of iterations in the iterative method is low. This implies that the frequency range in which the Method of Moments can be used is increased.

The approximation to the linear system is achieved through the multilevel Fast Multipole Method for Helmholtz' equation presented in Paper I, Paper II and Paper III. It can reduce the time of computing a matrix vector multiplication from $O\left(N^2\right)$ for a standard method to $O(N\log N)$, where N is the number of unknowns. It also reduces the memory requirement from $O\left(N^2\right)$ to $O(N\log N)$. The reason is that the multilevel Fast Multipole Method computes the action of the matrix on a vector and therefore it does not need to store the matrix in memory.

The Fast Multipole Method reduces the time for solving a linear system from $O\left(N^3\right)$, the time if Gaussian elimination is used, to $O(KMN\log N)$. Here, M is the average number of iterations per right hand side in an iterative method and K is the number of right hand sides. Clearly, the total number of matrix vector multiplications KM should be low in order for the Fast Multipole Method to be effective. In Paper IV, Paper V and Paper VI several methods are described that reduce KM.

For very large problems or problems involving a large number of right hand sides, not even the Fast Multipole Method is fast enough. In Paper VII a hybrid method between Physical Optics and the Fast Multipole Method is described.

The outline of this thesis is as follows. In Chapter 2 we discuss the Maxwell equations in frequency domain and derive the integral equations that are used in the Method of Moments. Chapter 3 discusses discretization of the integral equations into systems of linear equations. The Fast Multipole Method is described in Chapter 4, where a summary of Paper I, Paper II and Paper III is given. Chapter 5 gives a summary of the methods for reducing the number of matrix vector multiplications in Paper IV, Paper V and Paper VI and Chapter 6 explains the hybrid method in Paper VII. We end with some conclusions in Chapter 7.

1.3 A note on GEMS and SMART

This project has obtained its financial support from the Computational Electro-Magnetics program at the Parallel and Scientific Computing Institute (PSCI), which is a competence center supported by VINNOVA, The Swedish Agency for Innovation Systems.

The project was first part of the research activities in the General Electro-Magnetic Solver code project (GEMS), funded by VINNOVA and the National Aeronautical Research Program (NFFP). The aim of GEMS was to develop an

industrial class suite of hybrid electromagnetic solvers in both frequency and time domains. Later the development was carried out in the project Signature Modeling and Reduction Tools (SMART), which was funded by NFFP via PSCI. Here, the aim was to develop tools for analyzing and reducing the Radar Cross Section. GEMS has continued in the projects GEMS2 and GEMS3.

In frequency domain the aim was to develop a hybrid between MM and PO and GTD. In order to speed up the initial development phase a MM code called CESC was bought from CERFACS, Toulouse, France [BF99] and a GTD code called FASANT was bought from Cantabria University, Spain [PSG+99]. These codes were used as a basis for the design of the new codes, which were developed in FORTRAN 90. Some of the progress is summarized in [Edl01, Nil02, Sef03, Hag03].

In time domain a hybrid method between FDTD and FVTD and FETD was developed. The hybrid method can handle arbitrarily oriented thin wires. FVTD or FETD is used close to a surface in order to get a smooth description of the surface. Far from the surface the more efficient FDTD is used. Due to problems with late time instabilities in FVTD and the development of a stable hybrid between FDTD and FETD in [RB00, RB02] the emphasis of the development was on a hybrid between FDTD and FETD. Some of the progress is summarized in [And01, Led01, Ede02, Joh03, Atl03].

Maxwell's equations

In this chapter the Maxwell equations are formulated. They are solved using integral equation formulations. The integral equations are used in Chapter 3 to formulate boundary element methods for Maxwell's equations. The emphasis of this thesis is on perfect electric conductors, so formulas are derived for them.

2.1 Helmholtz' equation

In order to understand the nature of the integral equations for Maxwell's equations one solution of the Helmholtz equation is formulated. The inhomogeneous Helmholtz' equation is

$$\Delta\Psi(\mathbf{x}) + \kappa^2\Psi(\mathbf{x}) = -f(\mathbf{x}) \tag{2.1}$$

where Ψ is the unknown function, κ is the wavenumber and $f(\mathbf{x})$ is a source term. The function

$$G(\mathbf{x},\mathbf{x}') = \frac{e^{\imath\kappa|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \tag{2.2}$$

where $\mathbf{x}$ and $\mathbf{x}'$ are two points in space, is a fundamental solution to Helmholtz' equation called Green's function. It represents outgoing solutions or solutions satisfying the radiation condition of Helmholtz' equation given by [Néd01]

$$\left|\frac{\partial\Psi}{\partial r} - \imath\kappa\Psi\right| \leq \frac{C}{|\mathbf{x}|^2} \quad r = |\mathbf{x}| \to \infty \tag{2.3}$$

The Green's function satisfies

$$\Delta G(\mathbf{x},\mathbf{x}') + \kappa^2 G(\mathbf{x},\mathbf{x}') = -\delta(\mathbf{x}-\mathbf{x}') \tag{2.4}$$

where δ is the Dirac measure. The solution to Helmholtz' equation can according to Huygens' principle be expressed as a superposition of waves from individual point sources. The properties of the Green's function implies that the solution to the inhomogeneous Helmholtz' equation can be written

$$\Psi(\mathbf{x}) = \int_{\mathbb{R}^3} f(\mathbf{x}')\, G(\mathbf{x},\mathbf{x}')\, d\mathbf{x}' \tag{2.5}$$

The fundamental solution (2.2) is used later as the basis for the fundamental solutions of Maxwell's equations.

2.2 The Maxwell equations in a homogenous dielectric

In this section the Maxwell equations are solved based on integral equation formulations. To obtain an integral equation, the concept of ficticious equivalent currents will be used. The theory is applicable to homogenous dielectric bodies and homogenous perfect electric conductors. The reader is referred to [HK97, Néd01, PRM98, Rum54, BFG99, Nil02] for more information.

Let the electric and magnetic fields be time-harmonic in a homogenous dielectric with the time dependence $e^{-\imath\omega t}$. The dielectric domain $\Omega \in \mathbb{R}^3$ is either an interior or an exterior domain. It is characterized by the relative permittivity ε and relative permeability μ. The wavenumber is $\kappa = \omega c^{-1}$, where $c = c_0 n^{-1}$. Here, $c_0 = \sqrt{(\varepsilon_0 \mu_0)^{-1}}$ is the speed of light and $n = \sqrt{\varepsilon\mu}$ is the absolute index of refraction. The impedance of the medium is $Z = Z_0\sqrt{\mu\varepsilon^{-1}}$, where $Z_0 = 120\pi$ in SI-units is the impedance in vacuum. Applied magnetic currents, denoted by $\mathbf{M}_a$, and applied electric currents, denoted by $\mathbf{J}_a$ describe the effect of sources in the domain Ω. The total electromagnetic field in domain Ω is then governed by the Maxwell equations

$$\begin{cases} \nabla \times \mathbf{E} - \imath\kappa Z\mathbf{H} = -\mathbf{M}_a & \mathbf{x} \in \Omega \\ \nabla \times \mathbf{H} + \imath\kappa Z^{-1}\mathbf{E} = \mathbf{J}_a & \mathbf{x} \in \Omega \end{cases} \tag{2.6}$$

Define the normal $\hat{\mathbf{n}}$ to be the one pointing outwards from region Ω. By introducing the ficticious equivalent currents $\mathbf{J} = -\hat{\mathbf{n}} \times \mathbf{H}$ and $\mathbf{M} = \hat{\mathbf{n}} \times \mathbf{E}$ on the boundary Γ as in Figure 2.1, the equations are extended to $\mathbb{R}^3$, assumed to be filled by the same dielectric, by the equation

$$\begin{cases} \nabla \times \mathbf{E} - \imath\kappa Z\mathbf{H} = -\mathbf{M} - \mathbf{M}_a & \mathbf{x} \in \mathbb{R}^3 \\ \nabla \times \mathbf{H} + \imath\kappa Z^{-1}\mathbf{E} = \mathbf{J} + \mathbf{J}_a & \mathbf{x} \in \mathbb{R}^3 \end{cases} \tag{2.7}$$

The ficticious surface currents replace the fields in $\mathbb{R}^3 \setminus \Omega$ by zero fields so that $\mathbf{E} = \mathbf{0}$ and $\mathbf{H} = \mathbf{0}$ there.

The ficticious currents that are defined on the boundary Γ of Ω are called the electric current $\mathbf{J}$ and magnetic current $\mathbf{M}$ respectively. Sometimes it is more convenient to use the normal pointing into region Ω. In that case the equations are adjusted appropriately.

Let $\mathbf{x}$ be a point in space. In the case of an exterior domain it is assumed that the field satisfies the Silver-Müller radiation condition [Néd01]

$$\left|\sqrt{\varepsilon}\mathbf{E} - \sqrt{\mu}\mathbf{H} \times \hat{\mathbf{n}}\right| \leq \frac{C}{|\mathbf{x}|^2} \quad |\mathbf{x}| \to \infty \tag{2.8}$$

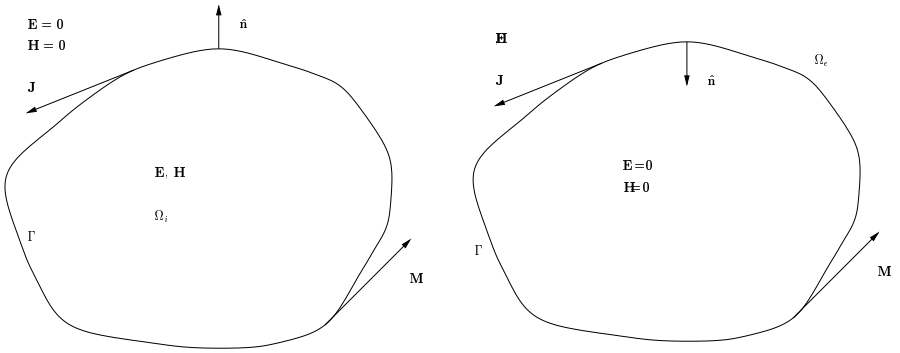

Figure 2.1: The principle of ficticious equivalent currents applied to an interior domain Ω_i (left) and an exterior domain Ω_e (right).

This condition ensures that the fields are bounded when $|\mathbf{x}| \to \infty$ [Néd01]

$$\begin{cases} |\mathbf{E}\cdot\mathbf{x}| \leq \frac{C}{|\mathbf{x}|} \\ |\mathbf{H}\cdot\mathbf{x}| \leq \frac{C}{|\mathbf{x}|} \qquad |\mathbf{x}| \to \infty \\ |\mathbf{H}\cdot\mathbf{E}| \leq \frac{C}{|\mathbf{x}|^3} \end{cases} \tag{2.9}$$

The Silver-Müller radiation condition is taken into account by the fundamental solution constructed from the Green's function in (2.2) [Néd01].

The currents are related to charges through the assumption of conservation of charge. This assumption leads to a relationship between $\nabla\cdot\mathbf{M}$, $\nabla\cdot\mathbf{M}_a$, $\nabla\cdot\mathbf{J}$, $\nabla\cdot\mathbf{J}_a$ and the respective charges ρ_m, $(\rho_m)_a$, ρ_e, $(\rho_e)_a$ of the form

$$\nabla\cdot\mathbf{J} + i\omega\rho_e = 0 \tag{2.10}$$

Charges are only present on the boundaries between different regions. These observations can be used to rewrite the Maxwell equations in a form similar to Helmholtz equation

$$\begin{cases} -\left(\Delta\mathbf{E}+\kappa^2\mathbf{E}\right) = \imath\kappa Z\left(\frac{1}{\kappa^2}\nabla\nabla\cdot\mathbf{J}+\mathbf{J}\right) \\ \qquad\qquad + \imath\kappa Z\left(\frac{1}{\kappa^2}\nabla\nabla\cdot\mathbf{J}_a+\mathbf{J}_a\right) \qquad \mathbf{x}\in\mathbb{R}^3 \\ \qquad\qquad - \nabla\times\mathbf{M} - \nabla\times\mathbf{M}_a \\ -\left(\Delta\mathbf{H}+\kappa^2\mathbf{H}\right) = \imath\kappa Z^{-1}\left(\frac{1}{\kappa^2}\nabla\nabla\cdot\mathbf{M}+\mathbf{M}\right) \\ \qquad\qquad + \imath\kappa Z^{-1}\left(\frac{1}{\kappa^2}\nabla\nabla\cdot\mathbf{M}_a+\mathbf{M}_a\right) \qquad \mathbf{x}\in\mathbb{R}^3 \\ \qquad\qquad + \nabla\times\mathbf{J} + \nabla\times\mathbf{J}_a \end{cases} \tag{2.11}$$

Assuming that the applied sources do not intersect the boundary Γ, one can express the applied currents $\mathbf{J}_a$ and $\mathbf{M}_a$ in the fields $\mathbf{E}_a$ and $\mathbf{H}_a$. From the properties of the Green's function associated with Helmholtz' equation in (2.4) it follows that

$$\begin{cases} \mathbf{E} = \mathbf{E}_a + \imath\kappa Z \mathcal{T}\mathbf{J} + \mathcal{K}\mathbf{M} & \mathbf{x} \notin \Gamma \\ \mathbf{H} = \mathbf{H}_a + \imath\kappa Z^{-1} \mathcal{T}\mathbf{M} - \mathcal{K}\mathbf{J} & \mathbf{x} \notin \Gamma \end{cases} \tag{2.12}$$

where the three operators defined in [HK97] have been introduced in order to simplify the notation

$$\begin{cases} \mathcal{V}\mathbf{J} = \int_\Gamma G(\mathbf{x},\mathbf{x}')\,\mathbf{J}(\mathbf{x}')\,d\Gamma(\mathbf{x}') & \mathbf{x} \notin \Gamma \\ \mathcal{T}\mathbf{J} = \left(\frac{1}{\kappa^2}\nabla\nabla\cdot + \mathbf{1}\right)\mathcal{V}\mathbf{J} & \mathbf{x} \notin \Gamma \\ \mathcal{K}\mathbf{J} = -\nabla\times\mathcal{V}\mathbf{J} & \mathbf{x} \notin \Gamma \end{cases} \tag{2.13}$$

2.3 Scattering from metallic objects

A nice theory that can be used to describe electromagnetic interactions between complicated metallic and dielectric objects is the Rumsey reaction principle [Rum54]. For details on how it works, see for instance [BFG99, Nil02].

Although the methods can be extended to dielectric materials, this thesis mainly concerns fast methods for scattering from metallic objects. Hence, the focus is on integral equations for metallic surfaces. The ability of metals to conduct electricity is good. This implies that a good approximation to many metals is the perfect electric conductor, which is characterized by the fact that $\mathbf{E} = \mathbf{0}$ inside it. The boundary condition for the perfect electric conductor is then $\mathbf{E}_t = \hat{\mathbf{n}} \times \mathbf{E} = \mathbf{M} = \mathbf{0}$.

Let $\mathbf{J}'$ be a tangential testing current on the surface Γ. Multiplying the first equation in (2.12) with $\mathbf{J}'$ and using the fact that $\mathbf{M} = \mathbf{0}$ yields the Electric Field Integral Equation (EFIE)

$$\int_\Gamma T\mathbf{J}\cdot\mathbf{J}'d\Gamma = -\frac{1}{\imath\kappa Z}\int_\Gamma \mathbf{E}_a\cdot\mathbf{J}'d\Gamma \tag{2.14}$$

Here, T is the limits of $\mathcal{T}$ as the boundary is approached in Cauchy principal value sense

$$\begin{aligned} T\mathbf{J} = &\frac{1}{\kappa^2}\nabla_\Gamma\int_\Gamma G(\mathbf{x},\mathbf{x}')\,\nabla_\Gamma\cdot\mathbf{J}(\mathbf{x}')\,d\Gamma(\mathbf{x}') \\ &+ \left(\int_\Gamma G(\mathbf{x},\mathbf{x}')\,\mathbf{J}(\mathbf{x}')\,d\Gamma(\mathbf{x}')\right)_t \end{aligned} \tag{2.15}$$

where ∇_Γ denotes the surface divergence of a vector field tangent to Γ and $()_t$ denotes the tangential part. The hypersingularity in the first part of T is usually

handled by moving one derivative from the $\mathbf{x}'$-variable to the $\mathbf{x}$-variable by using partial integration on the variational formulation (2.14) yielding

$$\begin{aligned}\int_\Gamma T\mathbf{J}\cdot\mathbf{J}'d\Gamma = &-\int_\Gamma\int_\Gamma G\left(\mathbf{x},\mathbf{x}'\right)\frac{1}{\kappa^2}\nabla_\Gamma\cdot\mathbf{J}\left(\mathbf{x}'\right)\nabla_\Gamma\cdot\mathbf{J}'\left(\mathbf{x}\right)d\Gamma\left(\mathbf{x}'\right)d\Gamma\left(\mathbf{x}\right)\\ &+\int_\Gamma\int_\Gamma G\left(\mathbf{x},\mathbf{x}'\right)\mathbf{J}\left(\mathbf{x}'\right)\mathbf{J}'\left(\mathbf{x}\right)d\Gamma\left(\mathbf{x}'\right)d\Gamma\left(\mathbf{x}\right)\end{aligned} \tag{2.16}$$

Another integral equation, which is called the Magnetic Field Integral Equation (MFIE), can be derived from the second equation in (2.12). In the case of a scattering problem the normal is defined as pointing into the region. The definition of the surface current is then $\mathbf{J}=\hat{\mathbf{n}}\times\mathbf{H}$. When the boundary is approached in the Cauchy principal value sense, equation (2.12) together with the boundary limit of $\mathcal{K}$ yields

$$\mathbf{J}=\hat{\mathbf{n}}\times\mathbf{H}=\hat{\mathbf{n}}\times\mathbf{H}_a-\left(-\frac{1}{2}\mathbf{J}+\hat{\mathbf{n}}\times K\mathbf{J}\right) \tag{2.17}$$

where K is defined by

$$K\mathbf{J}=\left(\int_\Gamma\nabla_{\mathbf{x}'}G\left(\mathbf{x},\mathbf{x}'\right)\times\mathbf{J}\left(\mathbf{x}'\right)d\Gamma\left(\mathbf{x}'\right)\right)_t \tag{2.18}$$

Multiplying with a test current $\mathbf{J}'$ defined as before and integrating yield the MFIE equation

$$\frac{1}{2}\int_\Gamma\mathbf{J}\cdot\mathbf{J}'d\Gamma+\int_\Gamma\hat{\mathbf{n}}\times K\mathbf{J}\cdot\mathbf{J}'d\Gamma=\int_\Gamma\hat{\mathbf{n}}\times\mathbf{H}_a\cdot\mathbf{J}'d\Gamma \tag{2.19}$$

The EFIE (2.14) and the MFIE (2.19) both have different advantages and disadvantages. The EFIE has spurious solutions if κ^2 is a corresponding eigenvalue to the interior problem with perfect electric conducting walls. In case κ^2 is an eigenvalue, the spurious modes will not radiate in the exterior. Thus, the field is not corrupted outside of the object. The MFIE can also have nullspace solutions in the interior. However, the spurious solutions of the MFIE do radiate in the exterior domain and corrupt the field. The EFIE can also handle the case of an open object which the MFIE is not able to handle. Therefore, the EFIE would be the best choice, when solving the perfect electric conductor case. The EFIE is an Fredholm integral equation of the first kind, while the MFIE is a Fredholm integral equation of the second kind [HK97]. Hence, the EFIE suffers from ill-conditioning, while the MFIE is better conditioned thanks to the term $\frac{1}{2}\mathbf{J}$ in (2.17). Thus, the MFIE is more appropriate for an iterative solution method.

The solution to these problems is to combine the two formulations into one called the Combined Field Integral Equation (CFIE). The CFIE is a linear combination of the EFIE (2.14) and the MFIE (2.19) defined as

$$\text{CFIE} = \alpha\text{EFIE} + (1-\alpha)\tfrac{\imath}{\kappa}\text{MFIE} \quad 0 < \alpha < 1 \tag{2.20}$$

With this choice there are no spurious modes. From experience it turns out that this equation has better conditioning than both EFIE and MFIE [CJMS01]. Since the formulation involves MFIE it can not be used for open objects. Thus, the CFIE is used for closed objects while EFIE is used for open objects involving perfect electric conductors.

2.4 The far field

The behavior of the electromagnetic field far from the object is often important in electromagnetic problems. Let $\mathbf{x} = r\hat{\kappa}$, where $\hat{\kappa}$ is a unit vector, and consider the case when $r \to \infty$. Through Taylor expansion

$$\begin{aligned} \left|\mathbf{x}-\mathbf{x}'\right| &= \sqrt{r^2 - 2r\hat{\kappa}\cdot\mathbf{x}' + \left|\mathbf{x}'\right|^2} \\ &= r\sqrt{1 - \frac{2\hat{\kappa}\cdot\mathbf{x}'}{r} + \frac{\left|\mathbf{x}'\right|^2}{r^2}} \\ &\approx r - \hat{\kappa}\cdot\mathbf{x}' \end{aligned} \tag{2.21}$$

Applying this procedure to the operator $\mathcal{V}\mathbf{J}$ yields

$$\begin{aligned} \mathcal{V}\mathbf{J} &= \int_\Gamma G\left(\mathbf{x},\mathbf{x}'\right)\mathbf{J}\left(\mathbf{x}'\right)d\Gamma\left(\mathbf{x}'\right) \\ &\approx \frac{e^{\imath\kappa r}}{4\pi r}\int_\Gamma e^{-\imath\kappa\cdot\mathbf{x}'}\mathbf{J}\left(\mathbf{x}'\right)d\Gamma\left(\mathbf{x}'\right) \\ &\equiv \frac{e^{\imath\kappa r}}{r}F\mathbf{J}\left(\kappa\right) \end{aligned} \tag{2.22}$$

Here, κ is the wavenumber and $\kappa = \kappa\hat{\kappa}$ is a vector of length κ pointing in the direction of $\hat{\kappa}$.

Using the relation $\nabla_{\mathbf{x}'}e^{-\imath\kappa\cdot\mathbf{x}'} = -\imath\kappa e^{-\imath\kappa\cdot\mathbf{x}'}$ together with the definition of $\mathcal{T}$ gives the far field for $\mathcal{T}$

$$\mathcal{T}\mathbf{J} \approx \frac{e^{\imath\kappa r}}{r}\left(F\mathbf{J}\left(\kappa\right) - \hat{\kappa}\left(\hat{\kappa}\cdot F\mathbf{J}\left(\kappa\right)\right)\right) = \frac{e^{\imath\kappa r}}{r}\hat{\kappa}\times\left(F\mathbf{J}\left(\kappa\right)\times\hat{\kappa}\right) \tag{2.23}$$

The same relationship applied to $\mathcal{K}$ gives the far field generated by $\mathcal{K}$

$$\mathcal{K}\mathbf{M} \approx \frac{e^{\imath\kappa r}}{r}\left(-\imath\kappa\times F\mathbf{M}\left(\kappa\right)\right) \tag{2.24}$$

The equations (2.23) and (2.24) appear in the expressions for the Fast Multipole Method.

Consider the case of electromagnetic scattering from a plane wave with wavenumber κ described by

$$\begin{cases} \mathbf{E}_a(\mathbf{x}, \hat{\kappa}_a) = \mathbf{E}_0 e^{-\imath\kappa\hat{\kappa}_a\cdot\mathbf{x}} \\ \mathbf{H}_a(\mathbf{x}, \hat{\kappa}_a) = \mathbf{H}_0 e^{-\imath\kappa\hat{\kappa}_a\cdot\mathbf{x}} \end{cases} \tag{2.25}$$

The wave is traveling in the direction given by the unit vector $-\hat{\kappa}_a$. Let $\mathbf{E}_s(\mathbf{x}, \hat{\kappa}_a)$ and $\mathbf{H}_s(\mathbf{x}, \hat{\kappa}_a)$ be the corresponding scattered electromagnetic field. Then the bistatic Radar Cross Section (RCS) $\sigma(\hat{\kappa}, \hat{\kappa}_a)$, in Figure 2.2, is defined as

$$\sigma(\hat{\kappa}, \hat{\kappa}_a) = \lim_{r\to\infty} 4\pi r^2 \frac{|\mathbf{E}_s(r\hat{\kappa}, \hat{\kappa}_a)|^2}{|\mathbf{E}_a(r\hat{\kappa}, \hat{\kappa}_a)|^2} \tag{2.26}$$

The special case $\sigma(\hat{\kappa}_a, \hat{\kappa}_a)$ is called the monostatic RCS. The RCS is often computed in decibels (dB) by the relation $\sigma_{\text{dB}}(\hat{\kappa}, \hat{\kappa}_a) = 10\log_{10}\sigma(\hat{\kappa}, \hat{\kappa}_a)$.

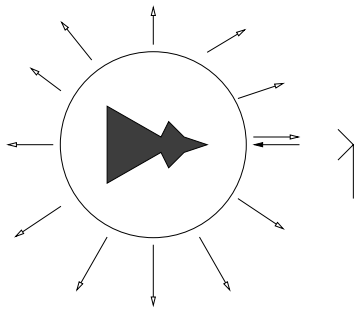

Figure 2.2: The scattered field from a source. The return direction is called the monostatic direction. The other directions are called bistatic directions.

The Radar Cross Section is often used as a measure of the numerical errors from different methods. It is the parameter that is minimized in the construction of stealth targets.

The definition of the far field imply that one can express the scattered field as

$$\mathbf{E}_s(r\hat{\kappa}, \hat{\kappa}_a) \approx \frac{e^{\imath\kappa r}}{r}\mathbf{A}(\hat{\kappa}, F\mathbf{J}) \quad r \to \infty \tag{2.27}$$

Here, $\mathbf{A}(\hat{\kappa}, F\mathbf{J})$ is defined by the behavior in the far field. It is derived from the equations (2.12), (2.23) and (2.24), where it is assumed that the electric surface current is known. Thus, the RCS is given by

$$\sigma(\hat{\kappa},\hat{\kappa}_a) = \lim_{r\to\infty} 4\pi r^2 \frac{|\mathbf{E}_s(r\hat{\kappa},\hat{\kappa}_a)|^2}{|\mathbf{E}_a(r\hat{\kappa},\hat{\kappa}_a)|^2} = 4\pi \frac{|\mathbf{A}(\hat{\kappa},F\mathbf{J})|^2}{|\mathbf{E}_0|^2} \tag{2.28}$$

The function $\mathbf{A}(\kappa,F\mathbf{J})$ is often referred to as the far field pattern.

Boundary Element Method

Integral equations can be solved using Boundary Element Methods. Boundary Element Methods discretize the integral equations so that a dense linear systems of equations is obtained. The discretization is achieved by a Finite Element discretization of the surface integrals involved. In the case of the Maxwell equations, Boundary Element Methods are usually called Method of Moments (MM).

An advantage with the Method of Moments is that because of their formulation they are free from dispersion errors, no outer artificial boundary is needed for exterior problems and only surfaces have to be discretized in $\mathbb{R}^3$, i.e. the dimension of the problem is reduced by one. The price of the dimensional reduction is a dense linear system matrix. Computationally this is not a big issue, because the computers of today are equipped with cache memory hierarchy's that are very effective for computing dense matrix vector multiplications. However, the available memory limits the size of the problems that can be solved.

3.1 Discretization

The integral equations EFIE (2.14), MFIE (2.19) and CFIE (2.20) are approximated numerically. The numerical approximation gives a dense complex system of linear equations in the unknown currents. The approximation requires a suitable representation of the geometry. Here, the surfaces are partitioned into triangles, as in Figure 3.1. This gives a perfect approximation of flat surfaces, but introduces a geometrical error of $O\left(h^2\right)$ on curved surfaces, where h is in the order of the length of the longest edge among all triangles.

The so called Rao-Wilton-Glisson (RWG) basis functions [RWG82] can be used to approximate the surface currents of the triangulated object. A RWG basis function is defined over the common edge of two adjacent triangles. If the two triangles are denoted by T_+ and T_-, then the RWG basis function is defined as

$$\mathbf{j}(\mathbf{x}) = \begin{cases} \frac{1}{2A_+}(\mathbf{x}-\mathbf{x}_+) & \mathbf{x} \in T_+ \\ -\frac{1}{2A_-}(\mathbf{x}-\mathbf{x}_-) & \mathbf{x} \in T_- \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{3.1}$$

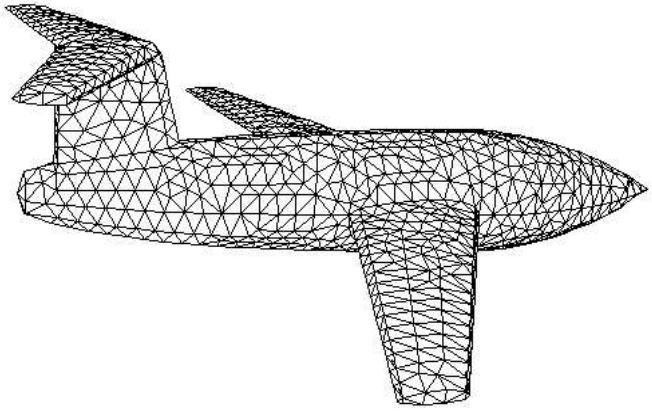

Figure 3.1: An example of an object surface represented by triangles. In this case it is a generic aircraft model called RUND.

Here, $\mathbf{x}_+$ and $\mathbf{x}_-$ are the two corners of the triangles not shared by the common edge, as in Figure 3.2. The RWG basis functions ensure that there is no accumulation of line-charges at the outer boundaries of the triangle pair. The scaling makes the normal continuous across the common edge. Also, the divergence on the triangle surface satisfies

$$\nabla_\Gamma \cdot \mathbf{j} = \begin{cases} \frac{1}{A_+} & \mathbf{x} \in T_+ \\ -\frac{1}{A_-} & \mathbf{x} \in T_- \\ \mathbf{0} & \text{otherwise} \end{cases} \tag{3.2}$$

To discretize the integral equations RWG basis functions are used to expand the currents as

$$\mathbf{J}(\mathbf{x}) = \sum_{l=1}^{N} I_l \mathbf{j}_l(\mathbf{x}) \tag{3.3}$$

Here, $\mathbf{j}_l(\mathbf{x})$ is an electric current basis function and N is the number of electric basis functions. The same basis functions are used in the testing procedure, resulting in a Galerkin method.

The expansion ends up in an equation $\mathbf{ZI} = \mathbf{V}$, where Z_{kl} is given by inserting (3.3) and letting $\mathbf{J}' = \mathbf{j}_l$ in one of the equations EFIE (2.14), MFIE (2.19) or CFIE (2.20). The element Z_{kl} is computed from the integral

$$Z_{kl} = \alpha \int_\Gamma T \mathbf{j}_l \cdot \mathbf{j}'_k d\Gamma + (1-\alpha)\frac{\imath}{\kappa} \int_\Gamma \hat{\mathbf{n}} \times K\mathbf{j}_l \cdot \mathbf{j}'_k d\Gamma \tag{3.4}$$

Here, $\alpha = 1$ is the special case EFIE and $\alpha = 0$ is the special case MFIE. The

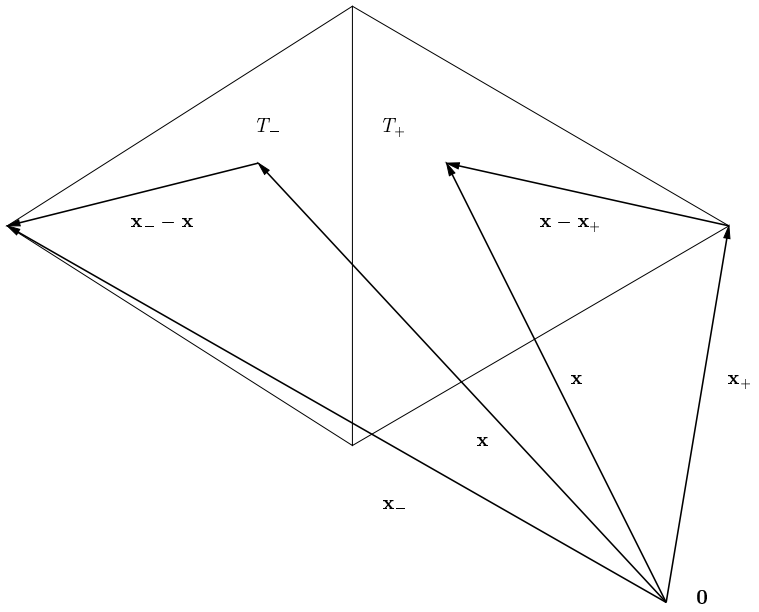

Figure 3.2: The Rao-Wilton-Glisson basis function (RWG).

same testing procedure applies to the element V_k defined by the equation

$$V_k = -\alpha \frac{1}{\imath \kappa Z} \int_\Gamma \mathbf{E}_a \cdot \mathbf{j}'_k d\Gamma + (1-\alpha) \frac{\imath}{\kappa} \int_\Gamma \hat{\mathbf{n}} \times \mathbf{H}_a \cdot \mathbf{j}'_k d\Gamma \tag{3.5}$$

For different applied electromagnetic fields at a fixed wavenumber the only difference is in the right hand side **V**. The currents are given by the unknown vector **I** and the matrix **Z** is called an impedance matrix. In order for the approximation to have any accuracy at least 2 edges per wavelength are needed. This is related to the ability of a general basis function to describe the function $\sin(x)$. Experiments reveal that at least 5 edges per wavelength are needed to get any accuracy at all with RWG basis functions. At least 10 edges per wavelength are usually recommended. Since the object is a surface the requirement on the discretization density implies that $N = O\left(f^2\right)$ where f is the frequency.

The assembly of the matrix **Z** is carried out by calculating the contributions from basis functions associated with each triangle and adding the contributions to the impedance matrix. The simplest way to calculate the individual contributions from the basis functions is by numerical integration. Because of the singular nature of the integrals associated with integral equations, numerical integration is difficult for triangles close to each other. This problem can be handled in several ways. One way is to use analytical treatment of the singular integrals to extract the singularity [YT03]. Another way is to transform the singular integrals using Duffy's transformation to remove the singularity [ES98].

Experiments in [YT03] show that this procedure is more sensitive to the shape of the triangles.

The Fast Multipole Method

The equation from the Method of Moments discretization $\mathbf{ZI} = \mathbf{V}$ is an $N \times N$ dense system of linear equations with $\mathbf{Z}$ complex. In the case of EFIE the matrix is complex symmetric, that is $\mathbf{Z} = \mathbf{Z}^T$. The storage of the system requires $O\left(N^2\right)$ memory positions. Computing the solution requires $O\left(N^3\right)$ arithmetic operations when Gaussian elimination is used. Instead one can use iterative methods like the Generalized Minimal RESidual method (GMRES) [SS86]. Iterative methods require $O\left(MN^2\right)$ arithmetic operations, where M is the number of matrix vector multiplications required to meet some convergence criteria. The number of iterations depends on the condition number of the problem. For ill-conditioned problems an iterative method can be as slow as a direct method or worse. A good preconditioner can reduce the number of iterations.

For the integral equations it is usually the memory that limits the size of the problem one can solve. With about 10000 unknowns the storage requirement is larger than the memory size of most computers. One can resort to parallel computers to solve this problem, but in the order of 100000 unknowns the memory requirement is again prohibiting. In [RBBD01] it is reported that 100000 unknowns can be solved with Gaussian elimination and requires 180 Gb of memory and 43 hours of computing time on a supercomputer.

A parallel solver for boundary integral methods is described in [Edl99]. It uses a dense block LDL^T-solver to factorize the impedance matrix. This means that it only works for EFIE. A parallel iterative solver is described in [Nil99]. It is also only applicable to EFIE and is based on a block symmetric matrix vector multiplication algorithm. It uses the symmetric Quasi-Minimal Residual method (QMR) to solve the linear system and is preconditioned by a Sparse Approximate Inverse preconditioner (SPAI) which is developed further in Paper IV and in [Nil02].

4.1 Paper I

It is evident that faster methods are needed in order to solve large scale problems in reasonable time. The Fast Multipole Method [CRW93] is one such fast method. It is based on a diagonal approximation of the Green's function

$$\frac{e^{\imath\kappa|\mathbf{x}-\mathbf{x}'|}}{4\pi|\mathbf{x}-\mathbf{x}'|} \approx \sum_{k=1}^{K} e^{\imath\kappa_k\cdot(\mathbf{x}-\mathbf{X}_m)} \mathcal{T}_k^L\left(\kappa, \mathbf{X}_m - \mathbf{X}'_m\right) e^{\imath\kappa_k\cdot(\mathbf{X}'_m-\mathbf{x}')} \tag{4.1}$$

where $\mathcal{T}_k^L(\kappa, \mathbf{X})$ is called the translation operator and is defined by

$$\mathcal{T}_k^L(\kappa, \mathbf{X}) = \frac{\imath\kappa}{16\pi^2} w_k \sum_{l=0}^{L} \imath^l (2l+1) h_l^{(1)}(\kappa X) P_l\left(\hat{\kappa}_k \cdot \hat{\mathbf{X}}\right) \tag{4.2}$$

Here, w_k is the quadrature weight associated with plane wave direction $\hat{\kappa}_k$, the function $h_l^{(1)}(x)$ is a spherical Hankel function and $P_l(x)$ is a Legendre polynomial of order l. The quadrature weights w_k are usually chosen such that they integrate the $2L$ first spherical harmonics exactly. This implies that error in (4.1) only depends on the truncation number L in exact arithmetic. A problem with (4.1) is that the approximation becomes numerically unstable for large values on L. This is due to the finite precision in computer arithmetic and the divergence of the spherical Hankel function for large order l and constant argument.

The Fast Multipole Method divides an object into a number of equally sized boxes. Equation (4.1) approximates the Green's function for two points $\mathbf{x}$ and $\mathbf{x}'$, when the respective boxes they are in are far apart. The boxes that must be between the two points in order for them to be considered far apart are called buffer boxes.

In Paper I error estimates are given. Bounds on L where the approximation of the Green's function is stable are also given. When the side length of a box is a and the number of buffer boxes are n it is found that for $\sqrt{3}\kappa a \leq L < (n+1)\kappa a$ the relative error ε can be obtained if

$$L \geq \sqrt{3}\kappa a + 1.8\left(-\log_{10}\varepsilon\right)^{\frac{2}{3}} \left(\sqrt{3}\kappa a\right)^{\frac{1}{3}} \tag{4.3}$$

When $L > (n+1)\kappa a$ one should choose the truncation number as

$$L \geq (n+1)\kappa a + \frac{\log_{10}(\varepsilon) - \left(\frac{\left(n+1-\sqrt{3}\right)\kappa a}{1.8\left(\sqrt{3}\kappa a\right)^{\frac{1}{3}}}\right)^{\frac{3}{2}}}{\log_{10}\frac{\sqrt{3}}{n+1}} \tag{4.4}$$

It is however important to note that the error level can only be achieved if

$$L < (n+1)\kappa a + 1.8\left(\log_{10}\left(\frac{\varepsilon}{C_n C_\varepsilon \kappa a}\right)\right)^{2/3} \left((n+1)\kappa a\right)^{1/3} \tag{4.5}$$

where $C_n = 2\left(n+1+\sqrt{3}\right)$ and C_ε is related to the relative error in the approximations of the different parts in the Fast Multipole Method. Numerical experiments validate the theory.

4.2 Paper II

The Fast Multipole Method for electromagnetics uses the expansion in (4.1) to compute the interactions between the different basis functions. The interactions are divided into near field interactions and far field interactions. The far field interactions are computed from

$$Z_{kl} \approx \sum_{j=1}^{K} \mathbf{R}_k(\hat{\kappa}_j) \cdot \mathcal{T}_j^L\left(\kappa, \mathbf{X}_m - \mathbf{X}'_m\right) \mathbf{F}_l(\hat{\kappa}_j) \tag{4.6}$$

where, $\mathbf{F}_l(\hat{\kappa})$ is called the far field pattern and $\mathbf{R}_k(\hat{\kappa})$ the receiving pattern. They are defined by

$$\begin{cases} \mathbf{F}_l(\hat{\kappa}) = \hat{\kappa} \times \int_\Gamma e^{\imath\kappa\cdot(\mathbf{X}'_m - \mathbf{x}')} \mathbf{j}_l d\Gamma(\mathbf{x}') \times \hat{\kappa} \\ \mathbf{R}_k(\hat{\kappa}) = \alpha\hat{\kappa} \times \displaystyle\int_\Gamma e^{\imath\kappa\cdot(\mathbf{x} - \mathbf{X}_m)} \mathbf{j}'_k d\Gamma(\mathbf{x}) \times \hat{\kappa} \\ \qquad\qquad + (1-\alpha) \displaystyle\int_\Gamma e^{\imath\kappa\cdot(\mathbf{x} - \mathbf{X}_m)} \mathbf{j}'_k \times \hat{\mathbf{n}}(\mathbf{x}) d\Gamma(\mathbf{x}) \times \hat{\kappa} \end{cases} \tag{4.7}$$

The far field pattern and the receiving pattern can either be stored or computed when they are needed.

The object is enclosed in a box which is then recursively divided into eight smaller boxes until the smallest boxes have a prescribed size. Boxes that contain basis functions are stored in a tree structure together with the parents of the boxes. Interactions between basis functions that are close to each other is computed by the near field formula (3.4). The near field interactions are usually stored in a sparse matrix structure. The far field interactions are computed from (4.6) on the level of the tree where the interactions are considered near interactions on the higher levels.

The Fast Multipole reduces the computational time for a matrix vector multiplication from $O\left(N^2\right)$ to $O(N \log N)$. The memory consumption is reduced from $O\left(N^2\right)$ to $O(N \log N)$. In practice the memory consumption is reduced to $O(N)$ because the memory usage on the lowest level is dominant.

In Paper II a brief description of the Fast Multipole Method is given and some experiments are performed to validate the accuracy and efficiency of the Fast Multipole Method.

4.3 Paper III

In an implementation of the Fast Multipole Method there are several potential bottle necks that must be handled appropriately if one wants to solve large problems.

The multilevel version of the Fast Multipole Method requires that interpolation is performed between levels. The interpolation can be made spectrally

accurate using the filter in [JCA97]. For low accuracy settings it is enough to use Lagrange interpolation on the sphere. This is more efficient than a spectral filter. In Paper III it is shown how the arithmetic complexity of the interpolation can be reduced from $O\left(Kp^2\right)$ to $O(Kp)$, where p is the number of interpolation points.

The values of the translation operator that should be stored is a memory bottle neck. A naive way of storing the values is to store them separate for each box. Using symmetries the storage requirement can be reduced dramatically. The first symmetry comes from the fact that the same combination of $(\kappa, \mathbf{X})$ appears several times. Hence, it is better to store the possible combinations of $(\kappa, \mathbf{X})$ in a table and let a box that needs a certain combination of $(\kappa, \mathbf{X})$ fetch it from there. The second symmetry has to do with the fact that the only important values in (4.2) are κX and $\hat{\kappa} \cdot \hat{\mathbf{X}}$. Since the $\hat{\kappa}$-values are picked from certain quadrature rules with symmetries on the sphere and the $\hat{\mathbf{X}}$-values comes from a Cartesian grid, the same combinations of κX and $\hat{\kappa} \cdot \hat{\mathbf{X}}$ appear for different $\mathbf{X}$. This is explained in Paper III. An algorithm that uses this fact is given in [Nil02]. These symmetries can also be used to reduce the memory for the exponential translation operator, which is used in the multilevel version of FMM.

Another way to save memory is to use the symmetries for the far field and radiation patterns given in Paper III. They are present when the wavenumber is real.

A third way of saving memory is to prescribe the size of the box on the lowest level and construct a tree in an upward pass. The effect is that the near memory, which is the dominant memory consumption is reduced. This is because the smallest box size is not always achieved in a downward pass. A simple estimate of the impact on the memory consumption is given in Paper III and validated by experiments. The price is a slight reduction in speed.

The parallelization of the Fast Multipole Method can be performed in two ways. Parallelization over boxes and parallelization over quadrature points. The parallelization over boxes is more advantageous on the lower levels because the number of boxes is large, while the number of quadrature points is low. On the highest levels the situation is the other way around, so it is better to parallelize over quadrature points there. A hybrid between the two methods, based on blocking, is proposed in Paper III.

A parallel shared memory implementation is described in Paper III. It combines the parallel strategies described. The experiments in Paper III show that the hybrid method improves the scalability of the parallel method.

Acceleration techniques for iterative methods

The total computational cost for solving one right hand side in an iterative method that uses the Fast Multipole Method is $O(MN\log N)$, where M is the number of iterations. Gaussian elimination requires $O\left(N^3\right)$ arithmetic operations for the factorization phase, but only $O\left(N^2\right)$ arithmetic operations for the substitution phase. This implies that the advantage of any fast method that uses iterative techniques diminishes as the number of right hand sides is increased.

The computational cost of the iterative solver depends on the number of matrix vector multiplications. In order for the fast methods to compete with Gaussian elimination the number of matrix vector multiplications for each right hand side must be small.

5.1 Paper IV

Since the EFIE is a Fredholm integral equation of the first kind it is not well conditioned. This implies that the convergence rate of an iterative method like GMRES or QMR is slow, so M is large in this case. An obvious solution is to use CFIE instead, but CFIE can only be used when the object is a closed surface.

Block iterative methods are one way of reducing the number of iterations per right hand side. They are based on block Krylov subspaces rather than Krylov subspaces. In Paper IV a block Quasi-Minimal Residual (block-QMR) algorithm is used. It is demonstrated that for EFIE it can be very effective in reducing the work for each right hand side. The symmetric version without preconditioner was given in [SG93]. The improvement is the ability to use a symmetric preconditioner. The theoretical proof of the correctness of the algorithm is given in [Nil02]. The algorithm is repeated in Algorithm 2 for the reader. It solves the system $\mathbf{ZX} = \mathbf{B}$, where $\mathbf{Z}$ is complex symmetric $N \times N$ matrix. As input it takes $\mathbf{Z}$, the complex symmetric $N \times N$ preconditioner $\mathbf{M}$, the $N \times s$ block of right hand side vectors $\mathbf{B}$ and the initial guess $\mathbf{X}$. The approximate solution $\mathbf{X} \approx \mathbf{Z}^{-1}\mathbf{B}$ is returned as output. In the algorithm the modified Gram-Schmidt procedure in Algorithm 1 (gram_M_sym) and the block version of Givens rotations in [SG93, Nil02] (givens) are required. The modified Gram-Schmidt procedure computes biorthogonal $N \times s$ vectors $\mathbf{Q}^T\mathbf{V} = \mathbf{I}_s$, where $\mathbf{I}_s$ is the $s \times s$ identity matrix, given a block of vectors $\tilde{\mathbf{V}}$ and the complex

Algorithm 1 A Gram-Schmidt process modified to handle the case of constructing **M**-biorthogonal vectors.

Require: $\tilde{\mathbf{V}}$, $\mathbf{M}$
Ensure: $\mathbf{V}$, $\mathbf{Q}$, ρ, ψ, β
  $\rho = \mathbf{0}$, $\psi = \mathbf{0}$, $\beta = \mathbf{0}$
  $\mathbf{V} = \tilde{\mathbf{V}}$
  $\mathbf{Q} = \mathbf{M}\tilde{\mathbf{V}}$
  for $j = 1 : s-1$ **do**
    $\delta_j = ||\mathbf{V}_{:,j}||_2$
    $\mathbf{V}_{:,j} = \frac{\mathbf{V}_{:,j}}{\delta_j}$
    $\mathbf{Q}_{:,j} = \frac{\mathbf{Q}_{:,j}}{\delta_j}$
    $\rho_{j:s,j} = \mathbf{Q}_{:,j:s}^T \mathbf{V}_{:,j}$
    $\beta_{j,j} = \rho_{j,j}$
    $\psi_{j,j+1:s} = \rho_{j+1:s,j}$
    $\rho_{j,j} = \delta_j \rho_{j,j}$
    $\psi_{j,j} = \delta_j$
    $\mathbf{V}_{:,j} = \frac{\mathbf{V}_{:,j}}{\beta_{j,j}}$
    $\mathbf{V}_{:,j+1:s} = \mathbf{V}_{:,j+1:s} - \mathbf{V}_{:,j}\rho_{j+1:s,j}^T$
    $\mathbf{Q}_{:,j+1:s} = \mathbf{Q}_{:,j+1:s} - \mathbf{Q}_{:,j}\psi_{j,j+1:s}$
  end for
  $\delta_s = ||\mathbf{V}_{:,s}||_2$
  $\mathbf{V}_{:,s} = \frac{\mathbf{V}_{:,s}}{\delta_s}$
  $\mathbf{Q}_{:,s} = \frac{\mathbf{Q}_{:,s}}{\delta_s}$
  $\beta_{s,s} = \mathbf{Q}_{:,s}^T \mathbf{V}_{:,s}$
  $\rho_{s,s} = \delta_s \beta_{s,s}$
  $\psi_{s,s} = \delta_s$
  $\mathbf{V}_{:,s} = \frac{\mathbf{V}_{:,s}}{\beta_{s,s}}$

symmetric $N \times N$ matrix $\mathbf{M}$. The output of the algorithm are the block vectors $\mathbf{Q}$ and $\mathbf{V}$ and the $s \times s$ block matrices ρ, ψ and β. They are related to each other by $\mathbf{V}\rho^T = \tilde{\mathbf{V}}$, $\mathbf{Q}\psi = \mathbf{M}\tilde{\mathbf{V}}$ and $\psi^T = \rho\beta^{-1}$.

Another way of reducing the number of iterations is to use preconditioners to improve the condition number. Consider iterative solution of the system $\mathbf{ZX} = \mathbf{V}$. A natural idea is to solve an equation on the form $\mathbf{M}_1\mathbf{ZM}_2\tilde{\mathbf{X}} = \mathbf{M}_1\mathbf{V}$, where $\mathbf{X} = \mathbf{M}_2\tilde{\mathbf{X}}$. If $\mathbf{M}_2\mathbf{M}_1 \approx \mathbf{Z}^{-1}$ the new problem is well-conditioned and converges quickly since $\mathbf{M}_1\mathbf{ZM}_2 \approx \mathbf{I}$, where $\mathbf{I}$ is the identity matrix. The problem is now to construct $\mathbf{M}_1$ and $\mathbf{M}_2$. The preconditioners are called left and right preconditioners respectively. Some natural requirements on a preconditioner is that it should not be more expensive to apply the preconditioner than

Algorithm 2 The Block-QMR algorithm for several right-hand sides.

Require: $\mathbf{B}$, $\mathbf{X}$, $\mathbf{Z}$ $\mathbf{M}$
Ensure: $\mathbf{X} \approx \mathbf{Z}^{-1}\mathbf{B}$
  $\tilde{\mathbf{V}}_0 = \mathbf{B} - \mathbf{Z}\mathbf{X}$
  $\mathbf{X}_0 = \mathbf{X}$, $\mathbf{R}_0 = \tilde{\mathbf{V}}_0$
  $\mathbf{D}_0 = \mathbf{0}$, $\mathbf{U}_0 = \mathbf{0}$
  $\mathbf{G}_0 = \mathbf{I}$
  $[\mathbf{V}_0, \rho_0, \mathbf{Q}_0, \psi_0, \beta_0] = \text{gram\_M\_sym}(\tilde{\mathbf{V}}_0, \mathbf{M})$
  $\mathbf{P}_0 = \mathbf{Q}_0$
  $\tilde{\tau}_0 = \psi_0$
  for $k = 1 : \mathit{maxiter}$ **do**
    $\mathbf{L}_k = \mathbf{Z}\mathbf{P}_{k-1}$
    $\varepsilon_k = \mathbf{P}_{k-1}^T \mathbf{L}_k$
    $\phi_k = \beta_{k-1}^{-1} \varepsilon_k$
    $\tilde{\mathbf{V}}_k = \mathbf{L}_k - \mathbf{V}_{k-1} \varepsilon_k$
    $[\mathbf{V}_k, \rho_k, \mathbf{Q}_k, \psi_k, \beta_k] = \text{gram\_M\_sym}(\tilde{\mathbf{V}}_k, \mathbf{M})$
    $\zeta_k = \mathbf{G}_{k-1}(1 : s, s+1 : 2s)\, \phi_k$
    $\chi_k = \mathbf{G}_{k-1}(s+1 : 2s, s+1 : 2s)\, \phi_k$
    $[\mathbf{G}_k, \tilde{\tau}_k, \tau_{k-1}, \gamma_k] = \text{givens}(\chi_k, \psi_k, \tilde{\tau}_{k-1})$
    $\mathbf{D}_k = (\mathbf{P}_{k-1} - \mathbf{D}_{k-1}\zeta_k)\, \gamma_k^{-1}$
    $\mathbf{X}_k = \mathbf{X}_{k-1} + \mathbf{D}_k \tau_{k-1}$
    $\mathbf{U}_k = (\mathbf{L}_k - \mathbf{U}_{k-1}\zeta_k)\, \gamma_k^{-1}$
    $\mathbf{R}_k = \mathbf{R}_{k-1} - \mathbf{U}_k \tau_{k-1}$
    if Convergence is achieved **then**
      $\mathbf{X} = \mathbf{X}_k$
      Break
    end if
    $\delta_k = \varepsilon_k^{-1} \rho_k$
    $\mathbf{P}_k = \mathbf{Q}_k - \mathbf{P}_{k-1} \delta_k$
  end for

to compute the original matrix vector product, it should not be more expensive to store than the original matrix and the construction time of the preconditioner should be low compared to the work reduction in the iterative solver. Another natural requirement is that the construction of the preconditioner should parallelize easily, since a parallel solver is used.

In Paper IV a modified SParse Approximate Inverse preconditioner (SPAI) is used. The preconditioner is used as a right preconditioner, i.e. $\mathbf{M}_1 = \mathbf{I}$, because the residual of the original problem remains unchanged in this case. The preconditioner was originally suggested in [Nil99] and is further analyzed in [Nil02]. It should be mentioned that a similar preconditioner have been developed in [Car02].

Let $\mathbf{M}$ be the preconditioner generated by SPAI. It is the solution of a Frobenius norm minimization problem

$$\min_{\mathbf{M}\in\mathcal{A}} \|\mathbf{I}-\mathbf{ZM}\|_F^2 = \sum_{i=1}^{N} \min_{\mathbf{M}\in\mathcal{A}} \|(\mathbf{I}-\mathbf{ZM})\,\mathbf{e}_i\|_2^2 \tag{5.1}$$

where $\mathcal{A}$ is a constraint on the sparsity pattern of $\mathbf{M}$. If the sparsity pattern is chosen so that all entries in $\mathbf{M}$ are allowed to be nonzero $\mathbf{M} = \mathbf{Z}^{-1}$. The problem is now to find a sparsity pattern that makes the minimization problem cheap to solve and gives a matrix $\mathbf{M} \approx \mathbf{Z}^{-1}$. The method is attractive since the problem naturally reduces to N uncoupled problems, which makes parallelization simple.

The Fast Multipole Method only stores the near field of the matrix $\mathbf{Z}$ so one natural restriction on $\mathcal{A}$ is that the nonzero entries should be restricted to the near field entries in $\mathbf{Z}$. The number of nonzero entries can be made even smaller by considering some distance measure to the center of the boxes in the Fast Multipole Method.

Equation (5.1) is still difficult to solve. It requires that all values in $\mathbf{Z}$ from the columns needed in each minimization problem (5.1) are known. Since $G(\mathbf{x},\mathbf{x}') \to 0$ when $|\mathbf{x}-\mathbf{x}'| \to \infty$ a simple modification is to only consider the values in the near field part instead. Some justification for this choice is given in [Nil02]. With this choice one can prove that the construction time of the preconditioner is $O\left(Nk^2\right)$ and that the application time is $O(Nk)$, where k is the average number of non zero values in each column of the matrix $\mathbf{M}$ [Nil02].

The results in Paper IV show that for the EFIE a block-QMR method together with SPAI can have a huge impact on the number of iterations for each right hand side.

5.2 Paper V

The solution time can be greatly reduced if the right hand sides are linearly dependent on each other. In this case it is enough to solve for vectors that span the space of the right hand sides and construct the solution to the right hand sides from them. In Paper V it is shown that if the vectors depend smoothly but nonlinearly on a parameter it is possible to find a subspace that accurately predicts the solutions.

Consider the problem $\mathbf{Ax}_i = \mathbf{b}_i$ for $i = 1 \ldots M$. Assume $m < M$ right hand sides have been solved by an iterative method such that $\mathbf{r}_i = \mathbf{b}_i - \mathbf{Ax}_i = \mathbf{b}_i - \mathbf{s}_i$. Compute a guess to the next right hand side based on a linear combination of the previous right hand sides $\mathbf{x}_{m+1}^0 = \sum_{i=1}^{m} y_i \mathbf{x}_i = \mathbf{X}_m \mathbf{y}_m$. If $[\mathbf{s}_1 \ldots \mathbf{s}_m] = \mathbf{Q}_{\mathbf{S}_m}\mathbf{R}_{\mathbf{S}_m}$,

one can prove that the choice

$$\begin{aligned} \mathbf{x}_{m+1}^{(0)} &= \mathbf{X}_m \mathbf{R}_{\mathbf{S}_m}^{-1} \mathbf{Q}_{\mathbf{S}_m}^H \mathbf{b}_{m+1} \\ \mathbf{r}_{m+1}^{(0)} &= (\mathbf{I} - \mathbf{Q}_{\mathbf{S}_m} \mathbf{Q}_{\mathbf{S}_m}^H) \mathbf{b}_{m+1} \end{aligned} \tag{5.2}$$

minimize $\|\mathbf{r}_{m+1}^{(0)}\|$ the residual of the initial guess.

In Paper V we propose a simple strategy for picking the right hand sides when the dependence on the smoothness parameter is known. The right hand sides are picked such that the space in the smoothness parameter is divided in a binary tree like fashion. The method is called the Minimal Residual Interpolation method (MRI). The analysis in Paper V show that it is an optimal method, in a certain sense. The main theorem in Paper V explains why this method works

Theorem 1 *Assume that the components in the right hand side vectors* $\mathbf{b}_i = \mathbf{b}(\phi_i)$ *have p continuous derivatives in* ϕ, $\|\mathbf{r}_i\| \leq \varepsilon_I$, *and that an approximation to* $\mathbf{b}_\alpha$ *at* ϕ_α *is computed at level l by the minimization*

$$\min_{\mathbf{y}} \|\mathbf{b}_\alpha - \sum_{i=1}^{p} \mathbf{s}_i y_i\|.$$

Then

$$\|\mathbf{b}_\alpha - \sum_{i=1}^{p} \mathbf{s}_i y_i\| \leq \min(\sqrt{N} b_{max}^{(p)} \Delta\phi_{l-1}^p + \sqrt{p}\|\mathbf{l}\|\varepsilon_I, \|\mathbf{b}_\alpha\|),$$

where $b_{max}^{(p)} = \max_j \max_\phi |b_j^{(p)}(\phi)|$, $b_j^{(p)}$ *is the p:th derivative of* b_j, *and* $\mathbf{l}$ *consists of the coefficients of the Lagrange polynomial at the point* ϕ_α.

In fact, Theorem 1 suggests that if the right hand sides in a problem depend smoothly on a parameter, a Singular Value Decomposition (SVD) [GvL96] can be used to find a subspace that approximates the right hand sides. However, computing a SVD is expensive and not really necessary if the strategy suggested here is used instead.

In Paper V we also consider the special case of computing monostatic RCS at a fixed frequency. Although an approximate bound is given on the number of right hand sides that span the subspace, a better estimate based on multipole theory is given in Paper VI.

Several experimental results are presented that justify the method proposed. Specifically the convergence rate shows excellent agreement with the prediction in Theorem 1 when $\Delta\phi \to 0$.

5.3 Paper VI

All aspects of the proposed methods in Paper V were not satisfactorily addressed. In Paper VI some of these issues are dealt with in more detail.

A technique for handling the case when the matrix $\mathbf{A}$ depends on a smooth parameter was presented in Paper V. In paper VI the proposed method is implemented and tested on the case of electromagnetic scattering. Consider the problem $\mathbf{A}_i\mathbf{x}_i = \mathbf{b}_i$ for $i = 1 \ldots M$ and assume that $m < M$ problems have been solved. The change from Paper V is that we consider $\mathbf{s}_{i,m+1}$ defined by

$$\mathbf{s}_{i,m+1} = \mathbf{A}_{m+1}\mathbf{x}_i = \mathbf{b}_i - \mathbf{r}_i + (\mathbf{A}_{m+1} - \mathbf{A}_i)\,\mathbf{x}_i,\ i = 1 \ldots m \tag{5.3}$$

and compute the initial guess and residual from

$$\begin{aligned} \mathbf{x}_{m+1}^{(0)} &= \mathbf{X}_m \mathbf{R}_{\mathbf{S}_{m,m+1}}^{-1} \mathbf{Q}_{\mathbf{S}_{m,m+1}}^{H} \mathbf{b}_{m+1} \\ \mathbf{r}_{m+1}^{(0)} &= (\mathbf{I} - \mathbf{Q}_{\mathbf{S}_{m,m+1}} \mathbf{Q}_{\mathbf{S}_{m,m+1}}^{H}) \mathbf{b}_{m+1} \end{aligned} \tag{5.4}$$

A theorem related to the convergence rate is given in Paper VI. It is an extension of Theorem 1.

Theorem 2 *Assume that the components in the right hand side vectors* $\mathbf{b}_i = \mathbf{b}(\phi_i)$ *and the matrices* $\mathbf{A}_i = \mathbf{A}(\phi_i)$ *have* p *continuous derivatives in* ϕ *and let* $\mathbf{A} = \mathbf{A}(\phi)$ *and* $\mathbf{b} = \mathbf{b}(\phi)$, $\|\mathbf{r}_i\| \leq \varepsilon_I$, $|\det(\mathbf{A}(\phi))| \geq C > 0$ *for some constant* C *and* $\phi_{\min} \leq \phi \leq \phi_{\max}$, *and that an approximation to* $\mathbf{b}_\alpha$ *at* ϕ_α *is computed by the minimization*

$$\min_{\mathbf{y}} \|\mathbf{b}_\alpha - \sum_{i=1}^{p} \mathbf{s}_{i,\alpha} y_i\|.$$

Then

$$\begin{aligned} \|\mathbf{b}_\alpha - \sum_{i=1}^{p} \mathbf{s}_{i,\alpha} y_i\| \leq \min(&\sqrt{N}\left(\mathbf{A}_\alpha \mathbf{A}^{-1}\mathbf{b}\right)_{max}^{(p)} \Delta\phi_{l-1}^{p} \\ &+ \sqrt{p}\|\mathbf{l}\| \max\|\mathbf{A}_\alpha \mathbf{A}^{-1}\|\varepsilon_I, \|\mathbf{b}_\alpha\|), \end{aligned}$$

where $\left(\mathbf{A}_\alpha \mathbf{A}^{-1}\mathbf{b}\right)_{max}^{(p)} = \max_i \max_\phi |\sum_{j=1}^{N} (A_\alpha)_{ij} \tilde{x}_j^{(p)}(\phi)|$, $\tilde{x}_j^{(p)}$ *is the p:th derivative of* $\tilde{x}_j = D_j/D$ *where* $D = \det(\mathbf{A})$ *and* D_j *is the determinant of the matrix with the* j*:th column of* $\mathbf{A}$ *replaced by* $\mathbf{b}$, *and* $\mathbf{l}$ *consists of the coefficients of the Lagrange polynomial at the point* ϕ_α.

In the case of electromagnetic scattering the matrix $\mathbf{A}$ is the impedance matrix, which depends on the frequency. Usually one is interested in the monostatic RCS in a frequency band. In Paper VI, MRI is used to compute the initial guesses in the frequency sweep. It should be noted that since the Fast Multipole Method is used to compute the solution, the matrix is not completely

smooth. This is because the number of terms used in the expansion varies with frequency according to the estimates in Paper I.

Paper VI also discusses the possibility of using a version of the block-seed GMRES method to compute initial guesses. It is noted that if enough right hand sides are solved in a block, the initial guesses for the other right hand sides will be accurate enough. However, one can only be certain that this occurs if the block size is as large as the number of vectors required in MRI to accurately predict the remaining solutions. Since the Krylov subspace in GMRES is larger than the block this strategy requires a lot of memory compared to MRI.

For the case of monostatic RCS computations a formula is given to estimate the number of right hand sides that must be solved before the remaining right hand sides can be predicted by MRI. It is $K = O\left(L^2\right)$ when the entire sphere of monostatic directions is considered and $K = O(L)$ when a plane of monostatic directions is considered. Here, the number L is computed from the formula

$$L \approx \kappa R + 1.8\left(-\log_{10}\varepsilon_I\right)^{\frac{2}{3}}(\kappa R)^{\frac{1}{3}} \tag{5.5}$$

where R is the radius of the smallest sphere enclosing the object.

There are still open issues from Paper V. The analyses there implies that local interpolation can be used instead of global. A way of computing the bistatic RCS for all incident directions is also presented. These issues will be investigated in a separate paper.

A hybrid method

The Physical Optics approximation (PO) is a high frequency asymptotic approximation of the solution to the MFIE. Consider the surface of an infinite perfect electric conducting ground plane. On the surface of the plane the integral operator $K\mathbf{J}$ satisfies $\hat{\mathbf{n}} \times K\mathbf{J} = \mathbf{0}$. Equation (2.17) implies that the electric surface current $\mathbf{J} = 2\hat{\mathbf{n}} \times \mathbf{H}_a$. For smooth electrically large surfaces a reasonable approximation is then $\mathbf{J} \approx 2\hat{\mathbf{n}} \times \mathbf{H}_a$, which can also be stated in a Galerkin form as in [Edl01, ELS03]

$$\int_\Gamma \mathbf{J} \cdot \mathbf{J}' d\Gamma \approx 2 \int_\Gamma \hat{\mathbf{n}} \times \mathbf{H}_a \cdot \mathbf{J}' d\Gamma \tag{6.1}$$

This is the PO formulation used here. Discretizing with RWG-basis functions leads to a sparse system matrix $\mathbf{Z}_{PO}$. It is positive definite and real since it is a mass matrix. This implies that the Conjugate Gradient method (CG) can be used to solve the problem iteratively. The experience in [Edl01, ELS03] and also Paper VII is that the solution is obtained in a few iterations.

One way of improving the result from (6.1) is to apply shadowing. The assumption is that the electric current $\mathbf{J} \approx \mathbf{0}$ if the current is not directly visible to the source of the applied field. This implies that $\hat{\mathbf{n}} \times \mathbf{H}_a \approx \mathbf{0}$, which is used in the right hand side of (6.1). A simple way to determine if a patch is visible to a plane wave traveling in direction $-\hat{\boldsymbol{\kappa}}_a$ is to use the normal test

$$\hat{\mathbf{n}} \times \mathbf{H}_a = \begin{cases} \hat{\mathbf{n}} \times \mathbf{H}_a & \text{if } \hat{\mathbf{n}} \cdot \hat{\boldsymbol{\kappa}}_a \geq 0 \\ \mathbf{0} & \text{if } \hat{\mathbf{n}} \cdot \hat{\boldsymbol{\kappa}}_a < 0 \end{cases} \tag{6.2}$$

More refined methods are described in [Sef03].

The PO method can be combined with the Method of Moments in a hybrid method following [Edl01, ELS03]. This method is called the MM-PO hybrid method. The surface is divided into two parts and solved by a Galerkin formulation. One part, which is considered smooth is solved with the PO-method. The other part, which yields a matrix $\mathbf{Z}_{MM}$ is solved with the Method of Moments. There are also matrices describing the couplings from PO to MM and MM to PO denoted by $\mathbf{Z}_{POMM}$ and $\mathbf{Z}_{MMPO}$. Since an approximation to MFIE is solved in the PO-part, it is natural to use MFIE in the Method of Moments part as well. From experience CFIE or EFIE can also be used in the Method

of Moments part. The resulting system that should be solved is

$$\begin{pmatrix} \mathbf{Z}_{MM} & \mathbf{Z}_{POMM} \\ \mathbf{Z}_{MMPO} & \mathbf{Z}_{PO} \end{pmatrix} \begin{pmatrix} \mathbf{I}_{MM} \\ \mathbf{I}_{PO} \end{pmatrix} = \begin{pmatrix} \mathbf{V}_{MM} \\ \mathbf{V}_{PO} \end{pmatrix} \tag{6.3}$$

The advantage of the Galerkin formulation is that all the integrals appearing in (6.3) are already implemented in the Method of Moments part. It does not require any special treatment of the singularities either. On the other hand when the MM and PO parts are distant from each other, the use of such rigorous approximations in the coupling seem less motivated.

Shadowing can improve the solution and can be used on $\mathbf{V}_{PO}$, but also on $\mathbf{Z}_{MMPO}$. An entry in $\mathbf{Z}_{MMPO}$ is then approximated to zero if the triangles are determined not to be visible to each other.

6.1 Paper VII

The matrices $\mathbf{Z}_{MM}$, $\mathbf{Z}_{MMPO}$ and $\mathbf{Z}_{POMM}$ can be approximated by the Fast Multipole Method. In Paper VII one such implementation is described. Equation (6.3) is solved by iterative block Gauss-Seidel. The subsystems in iteration k, $\mathbf{Z}_{MM}\mathbf{I}_{MM}^{(k)} = \mathbf{V}_{MM} - \mathbf{Z}_{POMM}\mathbf{I}_{PO}^{(k)}$ and $\mathbf{Z}_{PO}\mathbf{I}_{PO}^{(k)} = \mathbf{V}_{PO} - \mathbf{Z}_{MMPO}\mathbf{I}_{MM}^{(k-1)}$, are solved by iterative methods, where the initial guess is taken from the previous iteration. Assume that the Gauss-Seidel method is required to fulfill the convergence criteria $\|\mathbf{ZI} - \mathbf{V}\| \leq \varepsilon\|\mathbf{V}\|$. This is ensured if

$$\begin{aligned} \|\mathbf{Z}_{MM}\mathbf{I}_{MM} + \mathbf{Z}_{POMM}\mathbf{I}_{PO} - \mathbf{V}_{MM}\| &\leq 0.5\varepsilon\|\mathbf{V}\| \\ \|\mathbf{Z}_{PO}\mathbf{I}_{PO} + \mathbf{Z}_{MMPO}\mathbf{I}_{MM} - \mathbf{V}_{PO}\| &\leq 0.5\varepsilon\|\mathbf{V}\| \end{aligned} \tag{6.4}$$

In Paper VII it is proven that when the convergence rate of the Gauss-Seidel method is fast, the method is faster than if the Fast Multipole Method is used on the entire object.

For multiple right hand sides MRI in Paper V and Paper VI is used. The experiments in Paper VII show that MRI can reduce the solution time for sparse Finite Element type approximations like the PO approximation. Also, the hybrid method benefits from MRI and a significant reduction of the solution time is obtained. This demonstrates that MRI is independent of the iterative method.

Conclusions

This thesis presents fast methods for solving large scale electromagnetic problems.

The Fast Multipole Method computes a dense matrix vector multiplication in $O(N \log N)$ arithmetic operations and only requires $O(N \log N)$ memory. This increases the frequency range where the Method of Moments can be used. A scalable parallel implementation is described. It allows the solution of a 1 million problem to be computed in a few minutes. A 5 million unknown problem was solved on 24 processors of a SUNFire 15k server in 50 minutes with this implementation. It required 27 Gb of memory. The error in the Fast Multipole Method is analyzed and error estimates are given. A stability criterion is also provided.

The convergence rate of iterative methods is improved by using block methods and appropriate preconditioning. Minimal Residual Interpolation is a linear algebra method for reducing the work needed in the iterative solver by finding an appropriate subspace that accurately spans the space of right hand sides and uses it as an initial guess. The method can also be used in frequency sweeps to compute an initial guess.

For very large problems where the Fast Multipole Method falls short, a hybrid method with Physical Optics is presented. The method is faster than the Fast Multipole Method. Minimal Residual Interpolation can be used in this case as well. This demonstrates its versatility.

Several open questions remain. Some of them are: How to pick the right hand sides in an optimal way in MRI. How to decide the areas that should be approximated by Physical Optics in the MM-PO hybrid. How close to optimal speed the implementation of the Fast Multipole Method is. How to reduce the constant in front of the scaling in the Fast Multipole Method.

Acknowledgments

I wish to thank my supervisor Per Lötstedt for all his valuable advice during this research and always keeping an open door. Especially for the work on Paper V.

I would also like to thank Bo Strand at Aerotech Telub who has helped me a lot over the years. He is the one who initiated the project and has always been supportive of my work.

I also thank my previous colleague Johan Edlund. He has developed a large part of the Method of Moments code and the Physical Optics code which my solver is based on.

I would like to thank Fredrik Edelvik and Gunnar Ledfelt for helping me out from time to time and providing time domain results for comparison and Anders Ålund for all the important debugging work.

I am thankful to all the people involved in GEMS and SMART that have contributed to this work. A special thanks to Jonas Hamberg who provided several of the geometries.

I am grateful to Prof. Abderrahmane Bendali and Dr. M'Barek Fares at CERFACS, Toulouse, who taught us integral equation methods and provided the original Method of Moments code. Without their initial help it is doubtful that I would have come this far.

Thanks to all the people at TDB. You make this department fun to work at.

Finally, I want to thank my family for there love and support. A special thanks to Helén and William. I love you.

This work was supported by the CEM program at the Parallel and Scientific Computing Institute (PSCI) through the General ElectroMagnetic Solvers project (GEMS) and the Signature Modeling And Reduction Tools project (SMART). Financial support was provided by the National Aeronautical Research Program (NFFP), Vinnova, Aerotech Telub and Ericsson Microwave Systems AB. During my research I have been able to participate in a number of conferences and the Zürich Summer School thanks to: PSCI, a scholarship from Prof. Stefan Sauter at Universität Zürich, and Rektors reseanslag från Wallenbergstiftelsen at Uppsala University.

References

[And01] Ulf Andersson. *Time-Domain Methods for the Maxwell Equations*. PhD thesis, KTH, February 2001.

[Atl03] Andreas Atle. *Numerical approximations of time domain boundary integral equation for wave propagation*. Licentiate thesis, NADA, Stockholm University, October 2003.

[BF99] Abderrahmane Bendali and M'Barek Fares. EMC project electromagnetism, main features of the Cerfacs electromagnetics solver code. Technical Report /TR/EMC/99/52, CERFACS, Toulouse, France, 1999.

[BFG99] Abderrahmane Bendali, M'Barek Fares, and Jean Gay. A boundary–element solution of the Leontovitch problem. *IEEE Transactions on Antennas and Propagation*, 47(10):1597–1605, October 1999.

[Car02] Bruno Carpentieri. *Sparse Preconditioners for Dense Linear Systems from Electromagnetic Applications*. PhD thesis, CERFACS, Toulouse, France, April 2002. TH/PH/02/48.

[CJMS01] Weng Cho Chew, Jian-Ming Jin, Eric Michielssen, and Jiming Song. *Fast and Efficient Algorithms in Computational Electromagnetics*. Artech House, Inc., Norwood, 2001.

[CRW93] Ronald Coifman, Vladimir Rokhlin, and Stephen Wandzura. The fast multipole method for the wave equation: A pedestrian prescription. *IEEE Transactions on Antennas and Propagation*, 35(3):7–12, June 1993.

[Ede00] Fredrik Edelvik. *Finite Volume Solvers for the Maxwell Equations in Time Domain*. Licentiate thesis No. 2000-005, Department of Information Technology, Uppsala University, October 2000.

[Ede02] Fredrik Edelvik. *Hybrid Solvers for the Maxwell Equations in Time-Domain*. PhD thesis, Department of Information Technology, Uppsala University, 2002.

[Edl99] Johan Edlund. Solving the electric field integral equation using a block LDLT method. MSc thesis UPTEC F99 007, Uppsala University, February 1999.

[Edl01] Johan Edlund. *A Parallel, Iterative Method of Moments and Physical Optics Hybrid Solver for Arbitrary Surfaces*. Licentiate thesis No. 2001-010, Department of Information Technology, Uppsala University, August 2001.

[EL00] Fredrik Edelvik and Gunnar Ledfelt. Explicit hybrid solver for the Maxwell equations in 3D. *Journal of Scientific Computing*, 15(1), December 2000.

[EL02] Fredrik Edelvik and Gunnar Ledfelt. A comparison of time-domain hybrid solvers for complex scattering problems. *International Journal of Numerical Modeling*, 15:475–487, 2002.

[ELS03] Johan Edlund, Per Lötstedt, and Bo Strand. Iterative solution of a hybrid method for Maxwell's equations in the frequency domain. *International Journal for Numerical Methods in Engineering*, 54(1):1755–1770, March 2003.

[ES98] Stefan Erichsen and Stefan A. Sauter. Efficient automatic quadrature in 3-d Galerkin BEM. *Computer Methods in Applied Mechanics and Engineering*, 157(3–4):215–224, May 1998.

[GvL96] Gene H. Golub and Charles F. van Loan. *Matrix Computations*. The Johns Hopkins University Press, Baltimore, London, 3rd edition, 1996.

[Hag03] Stefan Hagdahl. *Hybrid methods for computational electromagnetics in the frequency domain*. Licentiate thesis No. 2003-018, KTH, October 2003.

[HK97] George C. Hsiao and Ralph E. Kleinman. Mathemathical foundations for error estimation in numerical solutions of integral equations in electromagnetics. *IEEE Transactions on Antennas and Propagation*, 45(3):316–328, March 1997.

[JCA97] Rüdiger Jakob-Chien and Bradley K. Alpert. A fast spherical filter with uniform resolution. *Journal of Computational Physics*, 136(2):580–584, September 1997.

[Jin93] Jian-Ming Jin. *The Finite Element Method in Electromagnetics*. Wiley, New York, 1993.

[Joh03] Christer Johansson. *Numerical Methods for Ports in Closed Waveguides*. Licentiate thesis TRITA-NA-0319, NADA, Stockholm University, October 2003.

[Kel62] Joseph B. Keller. Geometrical theory of diffraction. *Journal of the Optical Society of America*, 52(2):116–130, February 1962.

[Led01] Gunnar Ledfelt. *Hybrid Time-Domain Methods and Wire Models for Computational Electromagnetics*. PhD thesis, KTH, March 2001.

[Néd01] Jean-Claude Nédélec. *Acoustic and Electromagnetic Equations*. Springer-Verlag, New York, Berlin, Heidelberg, 2001.

[Nil99] Martin Nilsson. Solving the electric field integral equation using a sparse approximate inverse preconditioned iterative method. MSc thesis UPTEC F99 004, Uppsala University, February 1999.

[Nil02] Martin Nilsson. *Iterative solution of Maxwell's equations in frequency domain*. Licentiate thesis No. 2002-004, Department of Information Technology, Uppsala University, May 2002. Available at: http://www.it.uu.se/research/reports/lic/2002-004/.

[PK74] P. H. Pathak and R. G. Kouyoumjian. A uniform geometrical theory of diffraction for an edge in a perfectly conducting surface. *Proceedings of the IEEE*, 62(11):1448–1461, November 1974.

[PRM98] Andrew F. Peterson, Scott L. Ray, and Raj Mittra. *Computational Methods for Electromagnetics*. IEEE Press and Oxford University Press, New York, Oxford, Tokyo, Melbourne, 1998.

[PSG+99] J. Perez, F. Saez, J. O. Gutirrez, I. Gonzalez, M. F. Catedra, I. Montiel, and J. Guzman. FASANT: Fast computer tool for the analysis of on-board antennas. *IEEE Antennas and Propagation Magazine*, 41(2):94–98, April 1999.

[RB00] Thomas Rylander and Anders Bondeson. Stable FEM-FDTD hybrid method for Maxwell's equations. *Computer Physics Communications*, 125(1–3):75–82, March 2000.

[RB02] Thomas Rylander and Anders Bondeson. Stability of explicit-implicit hybrid time-stepping schemes for Maxwell's equations. *Journal of Computational Physics*, 179(2):426–438, July 2002.

[RBBD01] P. Ratajczak, R. Bills, P. Brachat, and C. Dedeban. Rigorous approach applied to large EM problems. In Fredrik Edelvik et al., editors, *EMB 01 – Electromagnetic Computations – Methods and Applications*, pages 19–34, Uppsala, Sweden, November 2001. SNRV.

[RT97] D. J. Riley and C. D. Turner. VOLMAX: A solid-model-based, transient volumetric Maxwell solver using hybrid grids. *IEEE Antennas Propagat. Magazine*, 39(1):20–33, 1997.

[Rum54] V. H. Rumsey. Reaction concept in electromagnetic theory. *Physical Review*, 94(6):1483–1491, June 1954.

[RWG82] Sadasiva. M. Rao, Donald. R. Wilton, and Allen. W. Glisson. Electromagnetic scattering by surfaces of arbitrary shape. *IEEE Transactions on Antennas and Propagation*, 30(3):409–418, May 1982.

[Ryl02] Thomas Rylander. *Stable FEM-FDTD hybrid method for Maxwell's equations*. PhD thesis, Department of Electromagnetics, Chalmers University of Technology, May 2002. Technical report no. 424.

[Sef03] Sandy Sefi. *Ray Tracing Tools for High Frequency Electromagnetics Simulations*. Licentiate thesis TRITA-NA-0314, Department of Numerical Analysis and Computer Science, Royal Institute of Technology, May 2003.

[SG93] V. Simoncini and E. Gallopoulos. Iterative methods for complex symmetric systems with multiple right–hand sides. Technical Report 1322, Center for Supercomputing Research and Development, University of Illinois at Urbana–Champaign, Dec. 1993.

[SS86] Youcef Saad and Martin H. Schultz. GMRES: A generalized minimal residual algorithm for solving nonsymmetric linear systems. *SIAM J. Sci. Comput.*, 7:856–869, 1986.

[Taf95] A. Taflove. *Computational Electrodynamics: The Finite-Difference Time-Domain Method*. Artech House, Boston, MA, 1995.

[YT03] Pasi Ylä and Matti Taskinen. Calculation of CFIE impedance matrix elements with RWG and $n\times$RWG functions. *IEEE Transactions on Antennas and Propagation*, 51(8):1837–1846, August 2003.

Acta Universitatis Upsaliensis

Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology

Editor: The Dean of the Faculty of Science and Technology

A doctoral dissertation from the Faculty of Science and Technology, Uppsala University, is usually a summary of a number of papers. A few copies of the complete dissertation are kept at major Swedish research libraries, while the summary alone is distributed internationally through the series *Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology*. (Prior to October, 1993, the series was published under the title "Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science".)

Distribution:
Uppsala University Library
Box 510, SE-751 20 Uppsala, Sweden
www.uu.se, acta@ub.uu.se

ISSN 1104-232X
ISBN 91-554-5827-0