Stratified algebras and classification of tilting modules

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Abstract

This thesis contains three papers in representation theory of algebras. It mainly studies two types of algebras; quasi-hereditary algebras and standardly stratified algebras.

Paper I provides a classification of generalized tilting modules and full exceptional sequences for a family of quasi-hereditary algebras and for another related family of algebras. These algebras are referred to as leaf quotients of type A zig-zag algebras. We also give a characterization of the first family of algebras as quasi-hereditary algebras with a simple preserving duality, where exactly one indecomposable projective module is not injective.

Paper II proves uniqueness of the essential order for standardly stratified algebras having a simple preserving duality. We use this result to classify, up to equivalence, regular blocks of S-subcategories in the BGG category O. We also establish some derived equivalences between blocks in type A. Additionally, the paper provides explicit formulas for the projective dimension of certain structural modules in S-subcategories of O and for the finitistic dimension of these subcategories.

Paper III provides a classification of generalized tilting modules and full exceptional sequences for a family of quasi-hereditary algebras. These algebras are examples of dual extension algebras. For the classification of generalized tilting modules we develop a combinatorial model for the poset of indecomposable self-orthogonal modules with standard filtration, with respect to the relation arising from higher extensions.

Keywords: Representation theory, Generalized tilting module, Exceptional sequence, Quasi-hereditary algebra, Standardly stratified algebra, Essential order, Category O, S-subcategory, Projective dimension, Finitistic dimension

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I am so clever that sometimes I don’t understand a single word of what I am saying.
List of papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.


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Additional papers

The following papers are not included in this thesis.

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1. Introduction

The subject of this thesis is representation theory. Representation theory is the art of representing a difficult algebraic structure by a simpler structure. In algebra the difficult structure can for example be an algebra, a group or a Lie algebra, and the simpler structure is usually a vector space together with a linear operator.

The same idea occurs in many different areas, even outside mathematics. For example, the internal structure of a crystal can be studied by letting the crystal act on a beam of light by dispersing it, and then analyzing the diffraction patterns. Here the crystal is an object which is difficult to study, but it is easy to see how it effects the beam of light. Another example is found in quantum mechanics where a physical system can be represented by the corresponding density matrix. The density matrix contains almost all information about the system, but is an object that is a lot easier to study than the physical system itself. We also find examples in art, where we can represent abstract concepts in the form of a painting or a piece of clothing. Our interactions with the representations allow us explore these concepts.

So why do we want to represent our algebraic structure by a vector space together with a linear operator? Vector spaces are well-known structures that have been extensively studied and we therefore have a lot of knowledge about their properties. For example, every linear transformation on a finite-dimensional vector space over a field \( k \) can be represented by a matrix. If the field \( k \) is algebraically closed, there exists a basis for every linear transformation such that the corresponding matrix is of a special form. This is called the Jordan normal form and looks as follows:

\[
\begin{pmatrix}
J_1 & & \\
& \ddots & \\
& & J_n
\end{pmatrix},
\]

where each block \( J_i \) is of the form

\[
\begin{pmatrix}
\lambda_i & 1 & & \\
& \lambda_i & \ddots & \\
& & \ddots & 1 \\
& & & \lambda_i
\end{pmatrix}.
\]
There are two parallel concepts in representation theory: representations and modules. A representation is a homomorphism from the object we are studying, for example an algebra, a group or a Lie algebra, to the set of linear operators on a vector space $V$. The kind of homomorphism we have of course depends on the algebraic structure we are studying. If we have a representation of an algebra, we should have an algebra homomorphism, and so on. An important fact here is that the set of linear maps has the structure of both an algebra and a Lie algebra, and if we restrict to invertible linear maps, this set has the structure of a group.

There is also the notion of a module. With this concept we focus on what happens on the level of elements. A module over, for example, an algebra is a vector space. For every element in the algebra we can act on an element in the vector space to get a new element in the vector space. This action should satisfy certain properties to be compatible with the operations of the algebra. Modules over other algebraic structures work in a similar way. Even though these concepts seem different at first sight, there is a one-to-one correspondence between representations of an algebra and modules over this algebra.

The idea of algebra is to study properties of mathematical objects, rather than the actual objects themselves. For example, a ring is a set together with two operations, addition and multiplication, that has similar properties as the integers. When studying different mathematical objects it is important to be able to determine when two structures are the same or not. This is in general a very difficult task. Mathematicians therefore look at different kinds of invariants, that is, properties that are preserved if two objects are deemed to be the same. For example, we can all agree that a point and a line in a vector space are different things. One way of telling them apart is to look at their dimension. The point has dimension zero and can therefore never be the same as a line, which has dimension one. Here the dimension is an example of an invariant.

The notion of “being the same” changes depending on what kind of mathematical structure you are studying. For example, if you are building something using Lego pieces you may, or may not, care about the color of the pieces. Are two Lego pieces the same if they have the same shape, or do they also need to have the same color? This might depend on the situation (or who you ask).

For algebras, we say that two algebras are “the same”, or isomorphic, if there exists a bijective homomorphism between the two algebras. In representation theory, we mainly care about the modules over an algebra, and it is therefore natural to say that two algebras are “the same” if their module categories are equivalent. If this is the case the two algebras are said to Morita equivalent. Of course, if two algebras are isomorphic, they are also Morita equivalent.

We can take this one step further by looking at the derived category of an algebra. If the derived category of two algebras are equivalent, we say that the
algebras are derived equivalent. The derived category is defined using the module category of an algebra, so if two algebras are Morita equivalent, they are also derived equivalent. A generalized tilting module over an algebra \( A \) is a module with a certain set of properties. These modules are important since their endomorphism algebra is derived equivalent to the original algebra \( A \). By classifying all generalized tilting modules for an algebra we therefore indirectly gain knowledge about algebras that are derived equivalent to the original algebra. In Paper I and Paper III we classify generalized tilting modules for three different families of algebras. There is a generalization of this notion, called tilting complexes. It has been shown that two algebras are derived equivalent if and only if one of them is the endomorphism algebra of a tilting complex of the other algebra.

When comparing two algebraic structures it is often convenient to break these down into smaller pieces and comparing these pieces instead. For example, if we have a module over a finite-dimensional we can decompose this into indecomposable direct summands. This decomposition is unique up to isomorphism, so the problem of classifying all modules, or determining whether two modules are isomorphic, reduces to studying indecomposable modules.

This can also be done for categories, and in particular for category \( \mathcal{O} \) and the related \( \mathcal{S} \)-subcategories. These categories can be decomposed into (indecomposable) blocks which can be described by some combinatorial information. To be exact, indecomposable blocks in the two categories are parameterized by a Weyl group and one or two parabolic subgroups. However, different choices of Weyl groups and parabolic subgroups may lead to isomorphic blocks, and this is what is studied in Paper II.
2. Preliminaries

2.1 Algebras and modules

In this section we recall the most important definitions and results about the main objects of study in this thesis, which are algebras and their modules. For more details, see e.g. [1].

**Definition.** Let $k$ be a field. An associative unital $k$-algebra is a $k$-vector space equipped with a bilinear operation $A \times A \to A$, $(a, b) \mapsto ab$, such that $(ab)c = a(bc)$ for all $a, b, c \in A$. We also require the existence of an element $1 \in A$, such that $1a = a = a1$, for all $a \in A$.

The bilinear operation is often referred to as the multiplication of the algebra, while the operation coming from the vector space structure is referred to as scalar multiplication. As we mainly deal with associative unital algebras, we will refer to them simply as algebras from now on. Another way of describing an algebra is to say that it is a ring with a vector space structure that is compatible with the operations of the ring.

Examples of $k$-algebras include the field $k$ itself, but also $n \times n$-matrices with entries in $k$. Another important example is the path algebra of a quiver. A quiver $Q = (Q_0, Q_1)$ is a directed graph with $Q_0$ the set of vertices and $Q_1$ the set of arrows, e.g.

$$
\begin{array}{c}
1 \\
\end{array} \begin{array}{c} \\
\end{array} \begin{array}{c}
\begin{array}{c}
\end{array}
\end{array} \begin{array}{c} \\
\end{array} \begin{array}{c}
2 \xrightarrow{a} 3 \\
\end{array}
\end{array}
$$

where $Q_0 = \{1, 2, 3\}$ and $Q_1 = \{a, b\}$. We say that a quiver is finite if both $Q_0$ and $Q_1$ are finite. A path in $Q$ is a sequence of arrows $a_n a_{n-1} \cdots a_2 a_1$ such that the target $t(a_i)$ of $a_i$ is equal to the source $s(a_{i+1})$ of $a_{i+1}$. For example, $ba$ is a path in the quiver above since $a$ has target 2 and $b$ has source 2. For every vertex $i$ there is also the empty path $e_i$.

Given a quiver $Q$ and a field $k$ we can define the path algebra of $Q$, denoted $kQ$, as the vector space with a basis consisting of all paths of $Q$. For the quiver above, a basis of $kQ$ would be $\{e_1, e_2, e_3, a, b, ba\}$. The multiplication of the basis elements is given by concatenation of paths, provided that the first path ends at the vertex where the next path begins. If not, then the product is defined to be zero. This multiplication is then extended to arbitrary elements of $kQ$ linearly. The identity element of $kQ$ is the sum of all empty paths $e_i$. 

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Given a quiver we can impose relations on the quiver. A relation on a quiver is a \( \mathbb{k} \)-linear combination of paths of length at least two, all of which having the same source and the same target. Given a set of relations we can look at the ideal in \( \mathbb{k}Q \) generated by these relations. Such an ideal will be an admissible ideal, meaning that it contains all paths of length at least \( r \), for some \( r \geq 2 \), and does not contain any linear combinations of paths of length less than two. The quotient algebra \( \mathbb{k}Q/I \), where \( I \) is an admissible ideal, is called a bound quiver algebra. If \( Q \) is a finite quiver, then \( \mathbb{k}Q/I \) is a finite-dimensional algebra.

An element \( e \) in an algebra \( A \) is called an idempotent if \( e^2 = e \). Two idempotents \( e_1, e_2 \) are said to be orthogonal if \( e_1 e_2 = e_2 e_1 = 0 \). An idempotent \( e \) is called primitive if it cannot be written as a sum \( e = e_1 + e_2 \), where \( e_1 \) and \( e_2 \) are two non-zero orthogonal idempotents. A set of primitive pairwise orthogonal idempotents \( \{e_1, \ldots, e_n\} \) is called complete if \( e_1 + \cdots + e_n = 1 \). An algebra \( A \) is called basic if there is a complete set of primitive idempotents \( \{e_1, \ldots, e_n\} \) such that \( Ae_i \neq Ae_j \) for all \( i \neq j \), and it is called connected if it is not a direct product of two algebras.

We see the importance of bound quiver algebras in the following theorem.

**Theorem.** Let \( A \) be a basic and connected finite-dimensional \( \mathbb{k} \)-algebra. There exists a connected quiver \( Q \) and an admissible ideal \( I \) of \( \mathbb{k}Q \) such that

\[
A \cong \mathbb{k}Q/I.
\]

An important tool to study algebras are modules. These objects let us apply all the knowledge we have about linear algebra and vector spaces to better understand the more complicated structures of algebras. If \( A \) is a \( \mathbb{k} \)-algebra, then a left \( A \)-module is a \( \mathbb{k} \)-vector space \( M \) together with a bilinear operation \( \cdot : A \times M \rightarrow M \), \((a, m) \mapsto a \cdot m\), such that \((ab) \cdot m = a \cdot (b \cdot m)\), for all \( m \in M \) and \( a, b \in A \). There is also the dual notion of a right \( A \)-module. We say that a module is finite-dimensional if it is finite-dimensional as a vector space.

Given two left \( A \)-modules \( M \) and \( N \), we say that a \( \mathbb{k} \)-linear map \( f : M \rightarrow N \) is an (left) \( A \)-module homomorphism if \( f(a \cdot m) = a \cdot f(m) \) for all \( a \in A \) and \( m \in M \). We define \( A \)-Mod to be the category of all left \( A \)-modules and \( A \)-mod to be the category of all finite-dimensional left \( A \)-modules. The morphisms in both these categories are \( A \)-module homomorphisms between left \( A \)-modules.

Given a finite quiver \( Q \) and an admissible ideal \( I \) of \( \mathbb{k}Q \) we can define a representation \( M \) of \((Q, I)\) in the following way. To each vertex \( a \) in \( Q_0 \) we associate a \( \mathbb{k} \)-vector space \( M_a \) and to each arrow \( \alpha : a \rightarrow b \) in \( Q_1 \) we associate a \( \mathbb{k} \)-linear map \( \varphi_{\alpha} : M_a \rightarrow M_b \). These maps should satisfy the corresponding
relations as those that generate $I$. For example, let $Q$ be the quiver

$$1 \xrightarrow{a} 2 \xrightarrow{b} 3$$

and let $I$ be the ideal generated by the path $ba$. Then for a representation $M$, the maps $\varphi_a$ and $\varphi_b$ must satisfy $\varphi_b \circ \varphi_a = 0$. A representation $M$ of $(Q, I)$ is said to be finite-dimensional if each vector space $M_a$ is finite-dimensional. We sometimes write $M = (M_a, \varphi_a)$ to emphasize the notation of the vector spaces and linear maps.

Given two representations $M = (M_a, \varphi_a)$ and $N = (N_a, \psi_a)$ of $(Q, I)$, a morphism $f : M \to N$ is a collection of maps $\{f_a : M_a \to N_a\}_{a \in Q_0}$, such that, for each arrow $\alpha : a \to b$ in $Q_1$, we have

$$\psi_\alpha \circ f_a = f_b \circ \varphi_\alpha.$$  

We define $\text{Rep}_k(Q, I)$ to be the category of all representations of $(Q, I)$ and $\text{rep}_k(Q, I)$ to be the category of all finite-dimensional representations of $(Q, I)$. The morphisms in both these categories are morphisms between representations of $(Q, I)$.

The following theorem tells us that we may look at quiver representations instead of $A$-modules. Quiver representations are sometimes easier to work with as these can be more concretely defined compared to modules.

**Theorem.** Let $A = \mathbb{k}Q/I$, where $Q$ is a finite connected quiver and $I$ is an admissible ideal of $\mathbb{k}Q$. There exists an equivalence of categories

$$F : A \text{- Mod} \to \text{Rep}_k(Q, I)$$

that restricts to an equivalence of categories $F : A \text{- mod} \to \text{rep}_k(Q, I)$.

There are a few special modules that we want to define. First of all, a submodule of an $A$-module $M$ is a subspace $N \subseteq M$ such that $N$ is closed under the action of $A$. A module $L$ is called simple if it is non-zero and any submodule is either the zero-module $\{0\}$ or $L$ itself. If we have a bound quiver algebra over a finite quiver, there is a one-to-one correspondence between the simple modules and the vertices of the quiver.

Next we have the projective modules. An $A$-module $P$ is projective if for every surjective $A$-module homomorphism $f : M \to N$ and every $A$-module homomorphism $g : P \to N$, there exists an $A$-module homomorphism $h : P \to M$, such that $g = f \circ h$. We also have the dual notion of an injective module. An $A$-module $I$ is injective if for every injective $A$-module homomorphism $f : M \to N$ and every $A$-module homomorphism $g : M \to I$, there exists an $A$-module homomorphism $h : N \to I$, such that $g = h \circ f$.  

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We can form new modules by taking quotients and direct sums of $A$-modules, and we call an $A$-module indecomposable if it is not isomorphic to a direct sum of two non-zero $A$-modules. For a finite-dimensional algebra $A$, every module in $A$-mod can be written as a direct sum of indecomposable modules in a unique way (up to isomorphism). We call an $A$-module basic if its indecomposable direct summands are pair-wise non-isomorphic. Given an $A$-module $M$ we define $\text{Add } M$ as the full subcategory of $A\text{-Mod}$ consisting of direct summands of $M$. Similarly, given $M$ in $A\text{-mod}$, we define $\text{add } M$ as the full subcategory of $A\text{-mod}$ consisting of finite direct sums of direct summands of $M$.

If $A$ is a finite-dimensional algebra and $M$ is a module in $A\text{-mod}$, then there exists a composition series

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_n = M,$$

where each quotient $M_{i+1}/M_i$ is simple. These simple quotients are called composition factors. This composition series is not unique. However, all composition series have the same length, and each simple module always occurs as a composition factor the same number of times. We denote by $[M : L]$ the number of composition factors of a module $M$ that are isomorphic to $L$. The structure of an indecomposable module and its composition factors can, under certain conditions, be depicted in a Loewy diagram. Below you find an example of a representation of a quiver with relations to the left, and to the right the corresponding Loewy diagram.

A simple preserving duality on a module category is an involutive contravariant autoequivalence which preserves the isomorphism classes of simple modules. Importantly, a simple preserving duality sends projectives to injectives and vice versa.

A warning to the reader: in Paper I we work with right modules and use the convention that we write $\alpha\beta$ for a path where $\alpha$ comes before $\beta$, contrary to the definition made above. In Paper II and Paper III we instead work with left modules, and therefore use the convention that $\beta\alpha$ denotes a path where $\alpha$ comes before $\beta$, as above.
2.2 Homological algebra

In this section we recall the most important definitions and results in the area of homological algebra. For more details see e.g. [1, 32].

Let $A$ be an algebra. A complex in $A$-Mod is a sequence

$$M_{\bullet}: \ldots \rightarrow M_{k-1} \xrightarrow{d_{k-1}} M_k \xrightarrow{d_k} M_{k+1} \rightarrow \ldots$$

of left $A$-modules with $A$-module homomorphisms $d_k$ such that $d_k \circ d_{k-1} = 0$. Given a complex $M_{\bullet}$, we define $M[i]_{\bullet}$ as the complex with $M[i]_j = M_{i+j}$ and differential $d_j^{M[i]} = (-1)^j d_{i+j}^{M}$. For each $k \in \mathbb{Z}$ we define the $k$th homology of a complex $M_{\bullet}$ as

$$H^k(M_{\bullet}) = \text{Ker} d_k / \text{Im} d_{k-1}.$$ 

An exact sequence is a complex with trivial homology, in other words, a complex such that $\text{Ker} d_k = \text{Im} d_{k-1}$ for all $k \in \mathbb{Z}$. An exact sequence

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

with only three non-zero modules is called a short exact sequence.

Given an algebra $A$ we can look at the category of complexes, $\text{Kom}(A)$, where morphisms are given by collection of maps $\{f_k : M_k \rightarrow N_k\}_{k \in \mathbb{Z}}$, such that, for each $k \in \mathbb{Z}$, the following diagram commutes.

$$
\begin{array}{ccc}
M_{k-1} & \xrightarrow{d_k^M} & M_k \\
\downarrow f_{k-1} & & \downarrow f_k \\
N_{k-1} & \xrightarrow{d_k^N} & N_k
\end{array}
$$

We also have the full subcategories of bounded complexes, $\text{Kom}^b(A)$, complexes bounded from above, $\text{Kom}^{-}(A)$, and complexes bounded from below, $\text{Kom}^{+}(A)$. Apart from these, we also have the homotopy category, $K(A)$, where morphisms are equivalence classes of morphisms of complexes up to homotopy, and the derived category $D(A)$, which is the localization of $K(A)$ with respect to quasi-isomorphisms.

A projective resolution $P_{\bullet}$ of a module $M$ is a complex

$$P_{\bullet}: \ldots \rightarrow P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \rightarrow 0$$

such that each $P_k$ is projective and $P_{\bullet} \rightarrow M$ is an exact sequence. Such a projective resolution always exists. We define the projective dimension of $M$,
denoted by \( \text{proj. dim}(M) \), to be the smallest non-negative integer \( m \) such that there exists a projective resolution

\[
0 \longrightarrow P_m \xrightarrow{d_m} P_{m-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_1} P_0 \longrightarrow 0
\]

of length \( m \), if such an integer exists. If not, then we say that the module has infinite projective dimension. We define the global dimension of an algebra \( A \) as the supremum of the projective dimensions of all \( A \)-modules, and the finitistic dimension as the supremum of the projective dimensions of all finitely generated \( A \)-modules with finite projective dimension.

Finally, for two \( A \)-modules \( M, N \) we define, for each \( k \geq 0 \), the functor

\[
\text{Ext}_A^k(M, N) = H^k(\text{Hom}_A(P_\bullet, N)),
\]

where \( P_\bullet \) is a projective resolution of \( A \)-module \( M \). Note that the result does not depend on the choice of a projective resolution of \( M \). From the definition, one can deduce that \( \text{Ext}_A^k(M, N) = 0 \) for all \( k > \text{proj. dim}(M) \). We say that an \( A \)-module \( M \) is self-orthogonal if \( \text{Ext}_A^k(M, M) = 0 \) for all \( k > 0 \).

\[ \text{2.3 Tilting theory} \]

In this section we recall the definition of a generalized tilting module and some important results about these modules. The notion of a generalized tilting module was defined by Miyashita in [25]. This is a generalization of classical tilting modules, which were introduced by Brenner and Butler in [4]. However, the definition of a classical tilting module most commonly used today is due to Happel and Ringel, see [19].

**Definition.** [25] Let \( A \) be an algebra and let \( T \) be an (left) \( A \)-module. Then, \( T \) is called a generalized tilting module if

1. **(T1)** \( T \) has finite projective dimension;
2. **(T2)** \( \text{Ext}_A^k(T, T) = 0 \) for all \( k > 0 \);
3. **(T3)** there is an exact sequence

\[
0 \longrightarrow A A \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow \cdots \longrightarrow Q_r \longrightarrow 0
\]

such that \( Q_i \in \text{add} T \) for all \( 0 \leq i \leq r \).

This generalizes the notion of a classical tilting module in the sense that a module \( T \) is a classical tilting module if and only if it is a generalized tilting module and the projective dimension is at most 1.

The third property in the definition of a generalized tilting module is usually the most difficult one to verify. Luckily, in the case where we have only finitely
many isomorphism classes of indecomposable modules, there is an equivalent property which is considerably easier to verify.

**Theorem.** [27] Let $A$ be an algebra of finite representation type and let $T$ be an $A$-module satisfying:

(T1) $T$ has finite projective dimension;

(T2) $\text{Ext}_A^n(T, T) = 0$, for all $k > 0$;

(T3') $T$ has $n$ non-isomorphic indecomposable direct summands, where $n$ is the number of isomorphism classes of simple modules.

Then, $T$ is a generalized tilting module.

Given a generalized tilting $A$-module $T$ we can look at its endomorphism algebra $B := \text{End}_A(T)$. Miyashita showed that $T$ is also a generalized tilting $B$-module and that $A \cong \text{End}_B(T)$. In [18], Happel showed that if $A$ is a finite-dimensional algebra of finite global dimension and $T$ is a classical tilting module, then $D^b(A\text{-Mod})$ and $D^b(B\text{-Mod})$ are equivalent as triangulated categories. This result was then generalized by Cline, Parshall and Scott in [5] to algebras of infinite global-dimension and $T$ a generalized tilting module.

Another generalization of tilting modules are so-called tilting complexes, defined in [26]. A tilting complex is an object $T$ in $K^b(A\text{-proj})$, the homotopy category of bounded complexes of finitely generated projective modules, such that $\text{Hom}(T, T[i]) = 0$, for all $i \neq 0$, and such that $\text{add}(T)$ generates $K^b(A\text{-proj})$. Importantly for us, a projective resolution of a generalized tilting module is an example of a tilting complex. We have the following important theorem about tilting complexes, due to Rickard. The theorem is stated for rings, but it implies the corresponding result for finite-dimensional algebras.

**Theorem.** [26] Let $A$ and $B$ be two rings. The following conditions are equivalent.

1. $K^-(A\text{-Proj})$ and $K^-(B\text{-Proj})$ are equivalent as triangulated categories;

2. $D^b(A\text{-Mod})$ and $D^b(B\text{-Mod})$ are equivalent as triangulated categories;

3. $K^b(A\text{-Proj})$ and $K^b(B\text{-Proj})$ are equivalent as triangulated categories;

4. $K^b(A\text{-proj})$ and $K^b(B\text{-proj})$ are equivalent as triangulated categories;

5. $B$ is isomorphic to $\text{End}(T)$, where $T$ is a tilting complex in $K^b(A\text{-proj})$. 

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2.4 Exceptional sequences

The study of exceptional sequences goes back to exceptional vector bundles, studied in for example [17]. Shortly after the introduction of exceptional vector bundles, the notion of exceptional objects and exceptional sequences were generalized to arbitrary triangulated categories, see [3]. The definition used in Paper I and Paper III is a bit more restrictive and focuses on $A$-modules, rather than arbitrary objects in the (triangulated) category $D^b(A\text{-mod})$.

**Definition.** Let $A$ be a finite-dimensional $\mathbb{k}$-algebra. An indecomposable $A$-module $M$ is called exceptional provided that

1. $\text{End}_A(M) \cong \mathbb{k}$;
2. $\text{Ext}^k_A(M, M) = 0$, for all $k > 0$.

A sequence $(M_1, \ldots, M_n)$ of $A$-modules is called an exceptional sequence provided that

1. each $M_i$ is exceptional;
2. $\text{Ext}^k_A(M_x, M_y) = 0$, for all $1 \leq y < x \leq n$ and all $k \geq 0$.

An exceptional sequence is called full (or complete) if it generates the derived category $D^b(A\text{-mod})$ as a triangulated category.

Exceptional sequences of modules have been studied in the special case of hereditary algebras, see [10, 29]. In this case, there is a transitive action of the braid group on the set of exceptional sequences of a hereditary algebra.

2.5 Quasi-hereditary algebras

In this section we recall the definition of a quasi-hereditary algebra and a few important result about these. Quasi-hereditary algebras is a class of algebras that were introduced by Cline, Parshall and Scott, see [6, 30]. The algebras studied in Paper I and Paper III are examples of quasi-hereditary algebras.

**Definition.** [6] Let $A$ be a finite-dimensional algebra. Let $\{1, \ldots, n\}$ be an indexing set for the isomorphism classes of simple $A$-modules and let $<$ be a partial order on $\{1, \ldots, n\}$. The algebra $A$ is said to be quasi-hereditary with respect to $<$ if there exist modules $\Delta(i)$, where $i \in \{1, \ldots, n\}$, called standard modules, satisfying the following.

1. There is a surjection $P(i) \twoheadrightarrow \Delta(i)$ whose kernel admits a filtration with subquotients $\Delta(j)$, where $j > i$. 

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2. There is a surjection $\Delta(i) \to L(i)$ whose kernel admits a filtration with subquotients $L(j)$, where $j < i$.

This is equivalent to the existence of modules $\nabla(i)$, where $i \in \{1, \ldots, n\}$, called costandard modules, satisfying the following.

1. There is an injection $\nabla(i) \hookrightarrow I(i)$ whose cokernel admits a filtration with subquotients $\nabla(j)$, where $j > i$.

2. There is an injection $L(i) \hookrightarrow \nabla(i)$ whose cokernel admits a filtration with subquotients $L(j)$, where $j < i$.

Let $(A, <)$ be a quasi-hereditary algebra and $e$ the idempotent associated to a maximal element with respect to $<$. For such an idempotent $e$, the algebra $A/AeA$ will again be quasi-hereditary together with the restriction of the partial order $<$. Furthermore, for all modules $M, N \in A/AeA$-mod and all $k \geq 0$, we have

$$\operatorname{Ext}^k_A(M, N) = \operatorname{Ext}^k_{A/AeA}(M, N),$$

see [12]. We can use this to prove that if $k > 0$ and $i \leq j$, then

$$\operatorname{Ext}^k_A(\Delta(j), \Delta(i)) = 0 \text{ and } \operatorname{Ext}^k_A(\nabla(i), \nabla(j)) = 0.$$

To see this, we can repeatedly use the result above to assume that $j$ is maximal. In this case, $\Delta(j)$ is projective, and $\nabla(j)$ is injective. It now follows that the two extensions are zero. In particular, this means that all standard modules $\Delta(i)$ and all costandard modules $\nabla(i)$ are self-orthogonal. Furthermore, for every quasi-hereditary algebra, the sequence $(\Delta(1), \Delta(2), \ldots, \Delta(n))$ containing all standard modules and the sequence $(\nabla(n), \ldots, \nabla(2), \nabla(1))$ containing all costandard modules always form two exceptional sequences.

We denote by $\mathcal{F}(\Delta)$ the full subcategory of $A$-mod consisting of modules admitting a filtration with subquotients isomorphic to standard modules. Similarly, we denote by $\mathcal{F}(\nabla)$ the full subcategory of $A$-mod consisting of modules admitting a filtration with subquotients isomorphic to costandard modules.

**Proposition.** [28] Let $A$ be a quasi-hereditary algebra, and let $X \in \mathcal{F}(\Delta)$ and $Y \in \mathcal{F}(\nabla)$. Then $\operatorname{Ext}^k_A(X, Y) = 0$ for all $k > 0$.

In [28], Ringel shows that if $A$ is quasi-hereditary, then there exists a module $T = \bigoplus_{i=1}^n T(i)$ such that $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \text{add}(T)$, where $\text{add}(T)$ denotes the full subcategory of $A$-mod consisting of all finite direct sums of direct summands of $T$. This module is called the characteristic tilting module. Ringel also shows that $T$ is, in fact, a generalized tilting module.
2.6 Standardly stratified algebras

Around ten years after the introduction of quasi-hereditary algebras by Cline, Parshall and Scott, the same authors introduced a generalization of these, which are called standardly stratified algebras. In Paper II we study properties of these algebras under certain assumptions.

**Definition.** [7] Let $A$ be a finite-dimensional algebra. Let $\{1, \ldots, n\}$ be an indexing set for the isomorphism classes of simple $A$-modules and let $<$ be a partial order on $\{1, \ldots, n\}$. The algebra $A$ is said to be standardly stratified with respect to $<$ if there exist modules $\Delta(i)$, where $i \in \{1, \ldots, n\}$, called standard modules, satisfying the following.

1. There is a surjection $P(i) \twoheadrightarrow \Delta(i)$ whose kernel admits a filtration with subquotients $\Delta(j)$, where $j > i$.

2. If $[\Delta(i) : L(j)] \neq 0$, then $j \leq i$.

In [7], the authors show that $\Delta(i)$ is the largest quotient of $P(i)$ such that $\Delta(i)$ only has composition factors $L(j)$ satisfying $j \leq i$. This is the quotient of $P(i)$ by the trace of all $P(j)$ in $P(i)$, where $j \not< i$. Recall that the trace of a module $N$ in a module $M$ is the submodule of $M$ spanned by the images of all homomorphisms from $N$ to $M$. If we define the standard modules in this way, then an algebra $A$ is standardly stratified if and only if $AA \in \mathcal{F}(\Delta)$.

Apart from the standard module $\Delta(i)$, we can also define the proper standard module $\overline{\Delta}(i)$ as the maximal quotient of $\Delta(i)$ such that $L(i)$ occurs as a composition factor exactly once. Note that a standardly stratified algebra is quasi-hereditary if and only if $\Delta(i) = \overline{\Delta}(i)$ for all $i$.

We also have the dual notion of a costandard module $\nabla(i)$, which is defined as the maximal submodule of $I(i)$ such that $\nabla(i)$ only has composition factors $L(j)$ with $j \leq i$. As before, we define the proper costandard module $\overline{\nabla}(i)$ as the largest submodule of $\nabla(i)$ such that $L(i)$ occurs as a composition factor exactly once.

Let the standard modules $\Delta(i)$ and the proper costandard modules $\overline{\nabla}(i)$ be defined as above. It was shown by Dlab in [11], see also [15], that for a finite-dimensional algebra $A$, we have $AA \in \mathcal{F}(\Delta)$ if and only if $D(AA) \in \mathcal{F}(\overline{\nabla})$. However, this is not equivalent to $D(AA) \in \mathcal{F}(\nabla)$, as in the case with quasi-hereditary algebras.

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2.7 Category $\mathcal{O}$

Other important algebraic structures are Lie algebras and the associated category $\mathcal{O}$. For more details about Lie algebras and category $\mathcal{O}$, see for example [14, 22, 23].

A *Lie algebra* $\mathfrak{g}$ over a field $\mathbb{k}$ is a vector space together with a bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, satisfying certain relations. Given a semi-simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ and a *Cartan subalgebra* $\mathfrak{h}$ of $\mathfrak{g}$ we have the *root space decomposition*

$$\mathfrak{g} = \bigoplus_{\alpha \in \mathfrak{h}^*} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{ x \in \mathfrak{g} \mid [h,x] = \alpha(h)x \text{ for all } h \in \mathfrak{h} \}$. The non-zero elements in $\mathfrak{h}^*$ with non-zero root-spaces $\mathfrak{g}_\alpha$ are called *roots*, and can be divided into positive and negative roots. This partition into positive and negative roots depends on a choice of *simple roots*. We can now fix a *triangular decomposition*

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+,$$

where $\mathfrak{n}_- = \bigoplus_{\alpha < 0} \mathfrak{g}_\alpha$ and $\mathfrak{n}_+ = \bigoplus_{\alpha > 0} \mathfrak{g}_\alpha$. A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is called a *parabolic subalgebra* if it contains the Borel subalgebra $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}_+$.

For every root $\alpha$ we can define a *reflection* $s_\alpha : \mathfrak{h}^* \to \mathfrak{h}^*$. These reflections $s_\alpha$ have the property that they permute the set of roots and that $s_\alpha(\alpha) = -\alpha$. The *Weyl group* $W$ corresponding to $(\mathfrak{g}, \mathfrak{h})$ is defined as the group generated by all reflections $s_\alpha$. A *parabolic subgroup* of $W$ is a subgroup generated by a subset of the simple reflections, that is, reflections corresponding to simple roots. We define the *dot-action* of $w \in W$ on an element $\lambda \in \mathfrak{h}^*$ as $w \cdot \lambda = w(\lambda + \rho) - \rho$, where $\rho$ denotes the half of the sum of all positive roots.

Given a triangular decomposition we can now define the associated *Bernstein-Gelfand-Gelfand (BGG) category* $\mathcal{O}$ as the full subcategory of $U(\mathfrak{g}) \cdot \text{Mod}$, where $U(\mathfrak{g})$ denotes the universal enveloping algebra of $\mathfrak{g}$, consisting of modules satisfying the following.

1. $M$ is a finitely generated $U(\mathfrak{g})$-module.
2. $M$ is a weight module, that is, $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$, where
   $$M_\lambda = \{ v \in M \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h} \}.$$
3. $M$ is locally $\mathfrak{n}_+$-finite, that is, for each $v \in M$, the subspace $U(\mathfrak{n}_+) \cdot v$ of $M$ is finite-dimensional.

This category was first defined by Bernstein, Gelfand and Gelfand in [2]. It contains, for example, all finite-dimensional $\mathfrak{g}$-modules.
Given a weight module \( M \), we say that \( \lambda \in \mathfrak{h}^* \) is a weight of \( M \) if \( M_\lambda \) is non-zero. A weight \( \lambda \) is an integral weight if and only if there exists a finite-dimensional module \( M \) such that \( M_\lambda \) is non-zero. We can define a partial order on \( \mathfrak{h}^* \) given by \( \lambda \leq \mu \) if and only if \( \mu - \lambda \) is a \( \mathbb{Z}_{\geq 0} \)-linear combination of positive roots. A weight \( \lambda \) of \( M \) is called a highest weight if \( \mu \leq \lambda \) for all weights \( \mu \) of \( M \). Note that if \( \lambda \) is a highest weight for a module \( M \), then \( n_+ \cdot M_\lambda = 0 \). Indeed, if \( \alpha \) is a positive root, then \( M_{\lambda + \alpha} = 0 \) since \( \lambda < \lambda + \alpha \). The claim now follows from the fact that \( g_\alpha \cdot M_\lambda \subseteq M_{\lambda + \alpha} = 0 \) and \( n_+ = \bigoplus_{\alpha > 0} g_\alpha \). A module \( M \) is called a highest weight module if it is generated (as a \( U(\mathfrak{g}) \)-module) by a non-zero element in \( M_\lambda \), where \( \lambda \) is a highest weight of \( M \).

Category \( \mathcal{O} \) also contains a set of special modules, called Verma modules which we will now define. First, let \( \lambda \in \mathfrak{h}^* \) and define \( C_\lambda \) to be the one-dimensional \( \mathfrak{b} \)-module with action \( h \cdot v = \lambda(h)v \), for all \( v \in C_\lambda \) and \( h \in \mathfrak{h} \), and \( n_+ \cdot C_\lambda = 0 \). We can now define the Verma module corresponding to \( \lambda \in \mathfrak{h}^* \) as

\[
\Delta(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} C_\lambda.
\]

This is a \( U(\mathfrak{g}) \)-module and it can be shown that it indeed belongs to category \( \mathcal{O} \). The module \( \Delta(\lambda) \) has a unique simple quotient which we denote by \( L(\lambda) \), and every simple module in category \( \mathcal{O} \) is isomorphic to \( L(\lambda) \), for some \( \lambda \in \mathfrak{h}^* \).

Category \( \mathcal{O} \) is a very large category and can be difficult to work with. For example, as we just saw, there are infinitely many simple objects in this category. However, category \( \mathcal{O} \) can be split into indecomposable direct summands, which are called blocks. Each of these blocks turns out to be equivalent to the category of modules over some finite-dimensional algebra. In particular, a block contains only finitely many simple modules, up to isomorphism. In fact, from the description of the center of \( U(\mathfrak{g}) \) due to Harish-Chandra, see [20], it follows that two simple modules \( L(\lambda) \) and \( L(\mu) \) lie in the same block only if \( \mu = w \cdot \lambda \), for some \( w \in W \).

Category \( \mathcal{O} \) is an example of a highest weight category. Comparing the definition below with that of a quasi-hereditary algebra, and combining this with the facts above, one can conclude that each (indecomposable) block of category \( \mathcal{O} \) is equivalent to the module category of a quasi-hereditary algebra.

**Definition.** [6] A locally Artinian category \( \mathcal{C} \) over \( \mathbb{k} \) is called a highest weight category if there exists a poset \( \Lambda \) satisfying the following conditions:

1. For every \( \mu \leq \lambda \) in \( \Lambda \), the “interval” \([\mu, \lambda] = \{ \tau \in \Lambda \mid \mu \leq \tau \leq \lambda \}\) is finite.

2. There is a complete collection \( \{L(\lambda)\}_{\lambda \in \Lambda} \) of non-isomorphic simple objects of \( \mathcal{C} \) indexed by the set \( \Lambda \).
3. There is a collection \( \{ \nabla(\lambda) \}_{\lambda \in \Lambda} \) of objects of \( C \) and, for each \( \lambda \), an embedding \( L(\lambda) \subset \nabla(\lambda) \) such that all composition factors \( L(\mu) \) of \( \nabla(\lambda)/L(\lambda) \) satisfy \( \mu < \lambda \).

4. For \( \lambda, \mu \in \Lambda \), we have that \( \dim_k \text{Hom}_C(\nabla(\lambda), \nabla(\mu)) \) and \( [\nabla(\lambda) : L(\lambda)] \) are finite.

5. Each simple object \( L(\lambda) \) has an injective envelope \( I(\lambda) \) in \( C \). Also, \( I(\lambda) \) has a “good filtration” which begins with \( \nabla(\lambda) \) - namely, an increasing (finite or infinite) filtration \( 0 = F_0(\lambda) \subset F_1(\lambda) \subset \cdots \) such that
   
   a) \( F_1(\lambda) \cong \nabla(\lambda) \);
   
   b) for \( n > 1 \), \( F_n(\lambda)/F_{n-1}(\lambda) \cong \nabla(\mu) \) for some \( \mu = \mu(n) > \lambda \);
   
   c) for a given \( \mu \in \Lambda \), \( \mu(n) = \mu \) for only finitely many \( n \);
   
   d) \( \bigcup F_i(\lambda) = I(\lambda) \).

For each block in category \( O \) we can look at the set of the highest weights corresponding to the simple modules in this block. Exactly one of these will be a dominant weight, which means that the corresponding Verma module is projective. Soergel showed that this block can then be described by \( (W', S) \), where \( W' \) is a subgroup of \( W \), and \( S \) is a parabolic subgroup of \( W' \). Let \( \lambda \) be the (unique) dominant highest weight for our block. The subgroup \( W' \) is the subgroup of \( W \) generated by all reflections \( s_\alpha \) such that \( \lambda - s_\alpha(\lambda) \) is an integer multiple of \( \alpha \). We usually call \( W' \) the integral Weyl group of \( \lambda \). The parabolic subgroup \( S < W' \) is the dot-stabilizer

\[
W'_\lambda = \{ w \in W' \mid w \cdot \lambda = \lambda \}
\]

of the (unique) dominant weight \( \lambda \).

Let \( \lambda \) be a dominant integral weight and let \( O_\lambda \) denote the block that contains \( L(\lambda) \) (and the Verma module \( \Delta(\lambda) \)). We call such a block an integral block. In this case the subgroup \( W' \) is equal to \( W \). Using the fact that each block in \( O \) is described by an integral Weyl group \( W' \) and a parabolic subgroup \( S < W' \), Soergel showed in [31] that each (indecomposable) block of category \( O \) is equivalent to an integral block of category \( O \) for some Lie algebra \( g' \), not necessarily equal to \( g \). We can therefore restrict our study to integral blocks.

Integral blocks are particularly nice; for example, the simple module \( L(\mu) \) is contained in the block \( O_\lambda \) if and only if \( \mu = w \cdot \lambda \), for some \( w \in W \). This implies that the simple modules in \( O_\lambda \) are parameterized by the longest coset representatives in \( W/W_\lambda \).

On every Weyl group \( W \) we can define a partial order, called the Bruhat order, given by \( u \leq_B w \) if some substring of a reduced expression of \( w \) is a reduced expression of \( u \). We can restrict the Bruhat order to a partial order of the set
of longest coset representatives in $W/S$, where $S$ is a parabolic subgroup of $W$. In [8] it was shown that two blocks corresponding to $(W, S)$ and $(W', S')$, respectively, are equivalent if and only if the partially ordered sets $(W/S; \leq_B)$ and $(W'/S'; \leq_B)$ are isomorphic.

For a Weyl group $W$, corresponding to $(\mathfrak{g}, \mathfrak{h})$, we can define the associated coinvariant algebra $\mathcal{C}_W$ as the quotient of $\mathbb{C}[\mathfrak{h}]$ modulo the ideal generated by the homogeneous $W$-invariant polynomials of non-zero degree. This is a finite-dimensional algebra of dimension $|W|$. However, this algebra can be realized in a different way. Let $w_0$ denote the longest element in $W$, then the projective module $P(w_0 \cdot 0)$, contained in the block $\mathcal{O}_0$, is the unique projective-injective module in this block. Soergel showed that the coinvariant algebra $\mathcal{C}_W$ is isomorphic to the endomorphism algebra of $P(w_0 \cdot 0)$, see [31, Endomorphismensatz].

### 2.8 $S$-subcategories

In paper II we are interested in a certain kind of subcategories of category $\mathcal{O}$ called $S$-subcategories, introduced in [16]. To define an $S$-subcategory we first fix an integral block $\mathcal{O}(W, S)$, which corresponds to the pair $(W, S)$, where $W$ is a Weyl group and $S$ is a parabolic subgroup. Next, we fix another parabolic subgroup $G < W$, unrelated to $S$.

Let $\lambda$ be the dominant integral weight corresponding to $\mathcal{O}(W, S)$. Recall that the set $W \cdot \lambda$ parameterizes the simple modules in the block $\mathcal{O}(W, S)$. We now split this set into two parts, the set $X$ consisting of all $\mu \in W \cdot \lambda$ such that $w \cdot \mu \geq \mu$, for any $w \in G$, and the complement $Y = (W \cdot \lambda) \setminus X$. Let $S(G)$ denote the Serre subcategory of $\mathcal{O}(W, S)$ generated by all $L(\mu)$, where $\mu \in Y$. We can now define the $S$-subcategory corresponding to $(W, S, G)$, denoted by $S(W, S, G)$, as the abelian quotient

$$S(W, S, G) := \mathcal{O}(W, S)/S(G).$$

The simple objects in $S(W, S, G)$ are parameterized by the double cosets in $G \backslash W/S$. By choosing the longest element in each double coset as a representative, we can restrict the Bruhat order on $W$ to $G \backslash W/S$.

There is also an alternative way of defining the same subcategory. Let $A$ be the up to isomorphism unique basic algebra such that $A$ - mod is equivalent to $\mathcal{O}(W, S)$. If $e \in A$ is the sum of the primitive idempotents corresponding to all $\mu \in X$, then

$$eAe \cong \text{End}_\mathcal{O} \left( \bigoplus_{\mu \in X} P(\mu) \right)^{\text{op}}.$$ 

The $S$-subcategory $S(W, S, G)$ is then equivalent to $eAe$ - mod.
As we mentioned before, each indecomposable block $\mathcal{O}(W, S)$ is equivalent to the module category for a quasi-hereditary algebra. This is no longer true for the $S$-subcategory $S(W, S, G)$. However, it is equivalent to the module category of a standardly stratified algebra, see [16]. This important fact is used in Paper II to classify equivalences of regular blocks in $S$-subcategories.
3. Summary of papers

3.1 Paper I

Paper I, “Tilting modules and exceptional sequences for leaf quotients of type $A$ zig-zag algebras”, contains a classification of generalized tilting modules and full exceptional sequences for two families of algebras. It also contains a characterization of one of these families.

The algebras studied in Paper I are quotients of zig-zag algebras. To define a zig-zag algebra, we start with a finite connected unoriented graph $Q$ without loops and with at least one edge. Let $\tilde{Q}$ denote the quiver obtained from $Q$ via substituting every edge $i \to j$ in $Q$ by two oriented edges $i \xrightarrow{\alpha} j$. We denote by $A_Q$ the quotient of the path algebra $k\tilde{Q}$ of $\tilde{Q}$ by the ideal generated by the following relations:

- any path of length three is zero;
- any path of length two which is not a cycle is zero;
- for any vertex $v$, all length two cycles which start and terminate at $v$ are equal.

The algebra $A_Q$ is usually called the zig-zag algebra associated with $Q$, see [13, 21].

For $n \in \mathbb{Z}_{>1}$, we denote by $A_n$ the algebra $A_Q$, where $Q$ is the following Dynkin diagram of type $A$:

$$
\begin{array}{c}
1 \quad 2 \quad 3 \quad \cdots \quad n
\end{array}
$$

We denote by $B_n$ the quotient of $A_n$ by the additional relation that the length two loop at the vertex $n$ is zero. We denote by $C_n$ the quotient of $B_n$ by the additional relation that the length two loop at the vertex 1 is zero. The vertices 1 and $n$ are both leaves in the Dynkin diagram of type $A$. We therefore refer to these quotients as leaf quotients of type $A$ zig-zag algebras.

The algebra $B_n$ is given by the quiver

$$
\begin{array}{c}
1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \xrightarrow{\cdots} n-1 \xrightarrow{\alpha_{n-1}} n
\end{array}
$$
and the relations
\[ \alpha_i \alpha_{i+1} = 0 = \beta_{i+1} \beta_i, \quad \beta_i \alpha_i = \alpha_{i+1} \beta_{i+1} \quad \text{and} \quad \beta_{n-1} \alpha_{n-1} = 0. \]

Note that in this paper we use the convention that we write \( \alpha \beta \) for a path where \( \alpha \) comes before \( \beta \), and all modules are assumed to be right modules. The algebras \( B_n \) are quasi-hereditary and have a simple preserving duality induced by the involutive anti-automorphism given by swapping the arrows \( \alpha_i \) and \( \beta_i \) in the quiver above.

The first main result of Paper I is the following.

**Theorem 1.** Let \( \Lambda \) be a basic, connected, quasi-hereditary algebra with respect to some order \( L_1 < L_2 < \cdots < L_n \), where \( n \in \mathbb{Z}_{>0} \). Assume the following:

1. Exactly \( n - 1 \) of the indecomposable projective \( \Lambda \)-modules are injective.
2. The algebra \( \Lambda \) has a simple preserving duality.

Then \( n > 1 \) and \( \Lambda \cong B_n \).

To classify generalized tilting modules for \( B_n \), we first prove that the only self-orthogonal modules are the projective-injective modules as well as the standard and costandard modules. This is then used to prove the second main result of Paper I, which is the following.

**Theorem 2.** Basic generalized tilting \( B_n \)-modules are, up to isomorphism, exactly the following modules:

\[ P(1) \oplus P(2) \oplus \cdots \oplus P(n-1) \oplus X, \]

where \( X \) is a standard or costandard module.

For the classification of full exceptional sequences, we show that the only exceptional \( B_n \)-modules are the standard and costandard modules.

**Theorem 3.** There are exactly \( 2^{n-1} \) full exceptional sequences of \( B_n \)-modules and they are all of the form

\[ (\nabla(i_1), \ldots, \nabla(i_k), L(1), \Delta(j_1), \ldots, \Delta(j_l)), \]

where

- \( k + l = n - 1 \), where \( 0 \leq k, l \leq n - 1 \);
- \( \{i_1, \ldots, i_k, j_1, \ldots, j_l\} = \{2, \ldots, n\} \).
\[ i_s > i_t \text{ and } j_s < j_t \text{ for } s < t. \]

The paper ends with similar classification results for the family of algebras \( C_n \). We first show that the only indecomposable self-orthogonal modules are the indecomposable projective-injective modules, and also the modules \( N(i, i+1) \) and \( S(i, i+1) \), for \( i = 1, 2, \ldots, n-1 \), where the Loewy diagram of the modules \( N(i, i+1) \) and \( S(i, i+1) \) looks as follows.

\[
\begin{array}{ccc}
N(i, i+1) : & i+1 & i \\
\downarrow & i & \downarrow \\
S(i, i+1) : & i & i+1 \\
\end{array}
\]

We are then able to show that the basic generalized tilting \( C_n \)-modules are, up to isomorphism, exactly the following modules:

\[
P(2) \oplus P(3) \oplus \cdots \oplus P(n-1) \oplus S(i, i+1) \oplus N(j, j+1),
\]

where \( i, j \in \{1, 2, \ldots, n-1\} \).

Finally, we show that there are no full exceptional sequences of \( C_n \)-modules.

### 3.2 Paper II

Paper II, “Essential orders on stratified algebras with duality and \( S \)-subcategories in \( \mathcal{O} \)”, consists of three parts. In the first part, we look at the essential order of a standardly stratified algebra \((A, \prec)\). The essential order, \( \prec_e \), is defined as the minimal partial order on an indexing set of isomorphism classes of simple modules, such that \( \lambda \prec_e \mu \) if \( L_\lambda \) is a composition factor of the standard module \( \Delta_\mu \) or a composition factor of the proper costandard module \( \nabla_\mu \). If \( A \) is standardly stratified for two different partial orders, then the essential orders coincide if and only if the standard modules coincide and the proper costandard modules coincide for both partial orders. The first main result of the paper is the following, which is a generalization of one of the main results of [8].

**Theorem 2.** Let \((A, \prec^1)\) and \((A, \prec^2)\) be standardly stratified. Assume that \( A \text{-mod} \) has a simple preserving duality. Then \( \prec^1_e = \prec^2_e \).

In the second part of this paper, we use the result about uniqueness of the essential order to classify, up to equivalence, regular blocks of \( S \)-subcategories in category \( \mathcal{O} \), for a simple finite-dimensional complex Lie algebra. We can parameterize (blocks of) \( S \)-subcategories using triples \((W, S, G)\), where \( W \) is a Weyl group and \( S \) and \( G \) are two parabolic subgroups. A block is regular
if and only if $S = \{e\}$. The simple objects in $S(W, \{e\}, G)$ are naturally indexed by the set $X_G$ of the longest coset representatives in $G \setminus W$. We say that $(W, S, G)$ and $(W', S', G')$ are isomorphic if there is an isomorphism of Coxeter groups from $W$ to $W'$ which sends $S$ to $S'$ and $G$ to $G'$.

The second main result of the paper is the following.

**Theorem 6.** We have $S(W, \{e\}, G) \cong S(W', \{e\}, G')$ if and only if $(W, \{e\}, G)$ and $(W', \{e\}, G')$ are isomorphic.

To prove this, we first show that $C_W \cong C_{W'}$ if and only if $W$ and $W'$ are isomorphic as Coxeter groups, where $C_W$ is the coinvariant algebra corresponding to the Coxeter group $W$. We also need the fact that the Bruhat order on $X_G$ coincides with the essential order for $B$, where $B$ is the (up to isomorphism unique) basic algebra such that $B$-mod is equivalent to $S(W, \{e\}, G)$. Since $B$ is standardly stratified and has a simple preserving duality, we know that the essential order is unique.

We also establish various derived equivalences between blocks in type $A$, that is, when $W = S_n$. A parabolic subgroup $S$ of $S_n$ is, by definition, generated by a set of transpositions $(i, i+1)$. The parabolic subgroup $S$ defines a set partition of $\{1, 2, \ldots, n\}$ by saying that $i$ and $i + 1$ are in the same set if $(i, i + 1) \in S$. The sizes of each set in the set partition defines a partition of the integer $n$. Note that different parabolic subgroups can define the same partition of $n$.

This notion allows us to describe some cases when there is a derived equivalence between two $S$-subcategories. Below, when we say that two categories are derived equivalent we mean that there is a triangular equivalence of their corresponding right bounded derived categories.

**Proposition 10.** Assume that $W$ is the symmetric group $S_n$ and that $(W, S, G)$ and $(W, S', G')$ are such that $S$ and $S'$ correspond to the same partition of $n$. Then the category $S(W, S, G)$ is derived equivalent to the category $S(W, S', G')$.

**Proposition 11.** Assume that $W$ is the symmetric group $S_n$ and that $(W, S, G)$ and $(W, S, G')$ are such that $G$ and $G'$ correspond to the same partition of $n$. Then the category $S(W, S, G)$ is derived equivalent to the category $S(W, S, G')$.

Next, we define an invariant of equivalent $S$-subcategories. Let $(A, \prec)$ be a standardly stratified algebra with a simple preserving duality. Define the weight function $\nu : \Lambda \to \mathbb{Z}_{>0}$ as follows:

$$\nu(\lambda) := \dim_k \mathrm{End}(\Delta_\lambda).$$
From the uniqueness of the essential order it follows that the weighted poset $(\Lambda, \prec, \nu)$ is an invariant of the Morita equivalence class of $A$. By looking at the poset $G\backslash W/S$ together with the Bruhat order and labeling each vertex with the weight function $\nu$ gives a strong invariant for telling different blocks $S(W, S, G)$ apart. We study these weighted posets in some small rank cases.

However, it is not true that $S(W, S, G)$ and $S(W', S', G')$ are equivalent if and only if their weighted posets are isomorphic. A very degenerate counterexample would be the one element poset of weight 6. This can be realized, on the one hand, by the coinvariant algebra of type $A_2$ and, on the other hand, by the algebra of $A_1$-invariants in the coinvariant algebra of type $G_2$. These two algebras are not isomorphic (the first one has two generators while the second one has only one generator).

In the third part of the paper, we describe objects in $S(W, \{e\}, G)$ that have finite projective dimension. Denote by $a_G$ the semi-simple part of the parabolic subalgebra of $\mathfrak{g}$ corresponding to $G$. Each $M$ in the category $\mathcal{O}$ for $\mathfrak{g}$, when considered as an $a_G$-module, is a (possibly infinite) direct sum of modules from the category $\mathcal{O}$ for $a_G$. We say that $M$ is admissible provided that all summands of this latter decomposition are projective-injective (in the category $\mathcal{O}$ for $a_G$).

**Theorem 15.** An object $N \in S(W, \{e\}, G)$ has finite projective dimension if and only if there is an admissible $M \in \mathcal{O}_0$ such that $N \cong \Phi(M)$, where $\Phi$ is the quotient functor. Moreover, the minimal projective resolution of $N$ in $S(W, \{e\}, G)$ is obtained from a minimal projective resolution of $M$ in $\mathcal{O}_0$ by applying $\Phi$ and the projective dimensions of $N$ (in $S(W, \{e\}, G)$) and $M$ (in $\mathcal{O}_0$) coincide.

We also find explicit formulas for the projective dimension of certain structural modules in $\mathcal{S}$-subcategories of $\mathcal{O}$ and for the finitistic dimension of these subcategories. To be exact, the finitistic dimension of $S(W, \{e\}, G)$ equals $2a(w_0^G w_0)$, where $a : W \to \mathbb{Z}_{\geq 0}$ denotes Lusztig’s $a$-function, see [24]. For $w \in X_G$, we have

$$\text{proj.dim}(T^S_w) = a(w_0^G w), \quad \text{proj.dim}(I^S_w) = 2a(w_0 w)$$

and

$$\text{proj.dim}(\Delta^S_w) = d_\lambda(w^{-1}),$$

where $\lambda$ is an dominant integral weight such that its dot-stabilizer coincides with $G$ and the function $d_\lambda$ assigns to $w \in W$ the projective dimension of the (potentially singular) Verma module $\Delta(w \cdot \lambda)$, see [9, Page 1084].
3.3 Paper III

Paper III, “Tilting modules and exceptional sequences for a family of dual extension algebras”, classifies generalized tilting modules and exceptional sequences for a family of algebras which are examples of dual extension algebras, first introduced by Xi in [33]. The algebras in this family are index by the number of isomorphism classes of simple modules. If \( n > 1 \), then the algebra \( \Lambda_n \) is given by the quiver

\[
\begin{array}{c}
1 \xrightarrow{\alpha_1} 2 \xrightarrow{\alpha_2} 3 \cdots n \xrightarrow{\alpha_{n-1}} n \\
\alpha'_1 & \alpha'_2 & \alpha'_3 & \cdots & \alpha'_{n-1}
\end{array}
\]

subject to the relations

\[
\alpha_{i+1}\alpha_i = 0, \quad \alpha'_j\alpha'_{j+1} = 0 \quad \text{and} \quad \alpha_i\alpha'_i = 0.
\]

Note that in this paper we use the convention that we write \( \beta\alpha \) for a path where \( \alpha \) comes before \( \beta \), and all modules are assumed to be left modules. The algebras \( \Lambda_n \) are quasi-hereditary and have a simple preserving duality induced by the involutive anti-automorphism given by swapping the arrows \( \alpha_i \) and \( \alpha'_i \) in the quiver above.

The classification of generalized tilting modules in this case is a lot more involved compared to Paper I, as the algebras in this paper only have one indecomposable projective-injective module, instead of \( n - 1 \), as in the case of Paper I. However, for the classification of full exceptional sequences, we show that these sequences are of exactly the same form as in Paper I.

The first step towards classifying all generalized tilting modules is to classify all indecomposable modules and then determining which of these are self-orthogonal. We define \( \Omega(i, j, k) \), where \( i, j \leq k \), to be the (up to isomorphism unique) indecomposable \( \Lambda_n \)-module with the following Loewy diagram.

1. If \( k \equiv i \mod 2 \) and \( k \equiv j \mod 2 \):

\[
\begin{array}{c}
i \xrightarrow{i+1} \cdots \xrightarrow{k-1} k \\
\alpha_i & \cdots & \alpha_j
\end{array}
\]

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2. If \( k \equiv i \mod 2 \) and \( k \not\equiv j \mod 2 \):

\[
\begin{array}{c}
i \quad i+1 \quad \ldots \quad k-1 \\
\downarrow \quad \downarrow \quad \ldots \quad \downarrow \\
j \quad j+1 \quad \ldots \quad k-1
\end{array}
\]

3. If \( k \not\equiv i \mod 2 \) and \( k \equiv j \mod 2 \):

\[
\begin{array}{c}
i \quad i+1 \quad \ldots \quad k-1 \\
\downarrow \quad \downarrow \quad \ldots \quad \downarrow \\
j \quad j+1 \quad \ldots \quad k-1
\end{array}
\]

4. If \( k \not\equiv i \mod 2 \) and \( k \not\equiv j \mod 2 \):

\[
\begin{array}{c}
i \quad i+1 \quad \ldots \quad k-1 \\
\downarrow \quad \downarrow \quad \ldots \quad \downarrow \\
j \quad j+1 \quad \ldots \quad k-1
\end{array}
\]

The set \( \{ \Omega(i, j, k) \mid 1 \leq i, j, k \leq n \} \) is a complete and irredundant list of isomorphism classes of indecomposable \( \Lambda_n \)-modules. There are, in total, \( \frac{n(n+1)(2n+1)}{6} \) isomorphism classes of indecomposable \( \Lambda_n \)-modules.

We show that it is possible to determine whether or not a module has a standard or costandard filtration merely by looking at the parities of the indices \( i, j, k \). Using this, we are able to prove the following result.

**Proposition 21.** If an indecomposable \( \Lambda_n \)-module \( M \) has neither a standard filtration nor a costandard filtration, then \( M \) is not self-orthogonal.

However, not all modules with a standard filtration or costandard filtration are self-orthogonal. To be exact, let \( \Omega(i, j, k) \in F(\Delta) \cup F(\nabla) \) be such that \( \Omega(i, j, k) \not\in F(\Delta) \cap F(\nabla) \), then \( \Omega(i, j, k) \) is self-orthogonal if and only if \( |i - j| = 1 \).

We also prove that there is a non-zero extension of positive degree between any self-orthogonal module in \( F(\Delta) \) and any self-orthogonal module in \( F(\nabla) \),
assuming that neither of these modules belong to $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$. These results imply that it is enough to classify all generalized tilting modules contained in $\mathcal{F}(\Delta)$. The rest of the classification follows using the simple preserving duality.

To classify all generalized tilting modules contained in $\mathcal{F}(\Delta)$, we provide a very transparent combinatorial description of the poset of self-orthogonal indecomposable modules in $\mathcal{F}(\Delta)$ with respect to the relation arising from higher extensions. The indecomposable self-orthogonal modules contained in $\mathcal{F}(\Delta)$ can be encoded by pairs $(i, k)$, with $1 \leq i \leq k \leq n$. We define the module $M(i, k)$ as follows:

$$M(i, k) := \begin{cases} 
\Omega(1, 1, k), & \text{if } i = 1; \\
\Omega(i, i - 1, k), & \text{if } i > 1 \text{ and } i \equiv k \pmod{2}; \\
\Omega(i - 1, i, k), & \text{if } i > 1 \text{ and } i \not\equiv k \pmod{2}.
\end{cases}$$

For $n = 4$ we have the following diagram depicting the relation given by higher extensions. There is a directed path starting from a module $M$ and ending in a module $N$ if and only if there is a non-zero extension of positive degree from $M$ to $N$.

```
  M(1, 1) -- M(1, 2) -- M(1, 3) -- M(1, 4)  
  |        |        |        |        
  |        v        |        v        |
  |        M(2, 2)   |        M(2, 3)   |
  |                |        |        |
  |                v        v        |
  |                M(3, 3)   |        M(3, 4)   |
  |                        |        |        |
  |                        v        v        |
  |                        M(4, 4)  |
```

The classification of generalized tilting modules is then obtained by considering maximal anti-chains in this poset. In order to formulate the result, we need the module $Z(i, x)$, which is defined as follows:

$$Z(i, x) := \bigoplus_{j=1}^{i-1} P(j) \bigoplus_{k \in X_1} M(i, k) \bigoplus_{\ell \in X_2} M(i + 1, \ell)$$

$$\bigoplus M(i, x) \oplus M(i + 1, x)$$

where

- $X_1 = \{ k \mid i + 1 \leq k \leq x - 1 \text{ and } k \equiv i + 1 \pmod{2} \}$;
- $X_2 = \{ k \mid i + 2 \leq k \leq x - 1 \text{ and } k \equiv i + 2 \pmod{2} \}$. 

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Theorem 44. Assume that $T \in \mathcal{F}(\Delta)$ is a basic generalized tilting $\Lambda_n$-module which is not equal to the characteristic tilting module. Then, there exists an integer $i$, where $1 \leq i \leq n - 1$, and an integer $x$, where $i < x \leq n$, such that:

$$T = Z(i, x) \oplus \bigoplus_{k=x+1}^{n} M(i, k),$$

if $x \equiv i \mod 2$, and

$$T = Z(i, x) \oplus \bigoplus_{k=x+1}^{n} M(i + 1, k),$$

if $x \not\equiv i \mod 2$.

In particular, there are $\frac{n(n-1)}{2}$ basic generalized tilting $\Lambda_n$-modules in $\mathcal{F}(\Delta)$ which are not equal to the characteristic tilting module. In total, there are $n(n - 1) + 1$ basic generalized tilting $\Lambda_n$-modules.

The paper ends with the following classification of full exceptional sequences of $\Lambda_n$-modules.

Theorem 46. Let $M$ be a full exceptional sequence of $\Lambda_n$-modules. Then $M$ is of the form

$$(\nabla(m_1), \nabla(m_2), \ldots, \nabla(m_i), L(1), \Delta(n_1), \Delta(n_2), \ldots, \Delta(n_j))$$

where

- $i + j = n - 1$;
- $\{m_1, m_2, \ldots, m_i, n_1, n_2, \ldots, n_j\} = \{2, 3, \ldots, n\}$;
- $m_1 > m_2 > \cdots > m_i$ and $n_1 < n_2 < \cdots n_j$. 
4. Summary in Swedish - Sammanfattning på svenska

Den här avhandlingen handlar om representationsteori. Det är ett område inom matematiken som studerar abstrakta algebraiska strukturer genom att representera dem med hjälp av något “enklare” objekt. De algebraiska strukturer vi framförallt studerar i denna avhandling är algebror. En algebra (över en kropp \( k \)) är en mängd med element som vi kan addera och multiplicera, precis som vi kan göra med heltalen, men vi kan även multiplicera varje element med en skalär från kroppen \( k \). Exempel på algebror (över \( \mathbb{C} \)) är mängden av alla komplexa polynom och mängden av komplexa \( n \times n \)-matrizer. Man kan på ett kortfattat sätt beskriva en algebra som en ring som samtidigt är ett vektorrum.


En generalisering av kvasiärfliga algebror är så kallade standardstratifierade algebror. De definieras på liknande sätt med hjälp av en partiell ordning på
mängden av isomorfiklasser av enkla moduler. Men i detta fall tittar vi på två olika typer av moduler, nämligen standardmoduler och äkta standardmoduler, där den senare är en kvot av den första. Vi har även en dual definition som använder sig av kostandardmoduler och äkta kostandardmoduler. Givet en standardstratifierad algebra $A$ tillsammans med en partiell ordning kan vi definiera den essentiella ordningen. Om en algebra är standardstratifierad med avseende på två olika partiella ordningar så är de essentiella ordningarna samma om och endast om standardmodulerna sammanfaller och de äkta kostandardmodulerna sammanfaller för de båda partiella ordningarna.

Kvasiärftliga och standardstratifierade algebror dyker även upp när man studerar BGG-kategorin $\mathcal{O}$ och $\mathcal{S}$-delkategorier för en ändligdimensionell halvenkel komplex Liealgebra. En kategori kan delas upp i mindre delar som kallas block. Ett odelbart block i kategori $\mathcal{O}$ för en ändligdimensionell halvenkel komplex Liealgebra är ekvivalent med modulkategorin av en kvasiärftlig algebra, och ett odelbart block i en $\mathcal{S}$-delkategori är ekvivalent med modulkategorin av en standardstratifierad algebra.

4.1 Artikel I

Artikel I, ”Tilting modules and exceptional sequences for leaf quotients of type A zig-zag algebras”, behandlar två klassifikationsproblem. Den första är klassifikationen av generaliserade tiltingmoduler för två familjer av algebror. Det andra problemet är klassifikationen av exceptionella följer av samma familjer av algebror. En av dessa familjer av algebror består av kvasiärftliga algebror. Vi betecknar algebrorna i denna familj med $B_n$, där $n$ är antalet enkla moduler. Förutom de två klassifikationsproblemen innehåller även Artikel I en karakterisering av dessa kvasiärftliga algebror, närmare bestämt bevisar vi följande resultat.

**Sats 1.** Låt $\Lambda$ vara en basal, sammanhängande, kvasiärftlig algebra med avseende på någon ordning $L_1 < L_2 < \cdots < L_n$, där $n \in \mathbb{Z}_{>0}$. Antag följande:

1. Exakt $n - 1$ av de odelbara projektiva $\Lambda$-modulerna är injektiva.
2. Algebran $\Lambda$ har en enkelbevarande dualitet.

Då gäller $n > 1$ och $\Lambda \cong B_n$.

För att klassificera generaliserade tiltingmoduler för $B_n$ visar vi först att de enda självortogonala modulerna är de projektiva-injektiva modulerna, samt standard- och kostandardmodulerna. Vi kan sedan använda detta för att bevisa det andra huvudresultatet i artikeln, nämligen att varje basal generaliserad
tiltingmodul är en direkt summa av de \( n-1 \) odelbara projektiva-injektiva modularerna och antingen en standardmodul eller en kostandardmodul.

För klassifikationen av fullständiga exceptionella följer av \( B_n \)-moduler visar vi att de enda exceptionella modularerna är standardmodulerna och kostandardmodulerna. Vi visar sedan att de fullständiga exceptionella följderna är precis följer på formen \( (\nabla(i_1), \ldots, \nabla(i_k), L(1), \Delta(j_1), \ldots, \Delta(j_t)) \), där indexen \( i_x \) är avtagande, indexen \( j_x \) är ökande, \( i_x \neq j_y \) för alla \( x, y \) och där det finns precis \( n \) stycken moduler i följen.

Artikeln avslutas med liknande klassifikationsresultat för en särskild kvot \( C_n \) av algebra \( B_n \).

4.2 Artikel II

Artikel II, "Essential orders on stratified algebras with duality and \( S \)-subcategories in \( O \)”, består av tre delar. I den första delen tittar vi på den essentiella ordningen av en standardstratifierad algebra \( (A, \prec) \). Det första huvudresultatet i denna artikel säger att om algebra \( A \) har en enkelbevarande dualitet så sammanfaller de essentiella ordningarna för varje partiell ordning som gör \( A \) standardstratifierad. Detta är en generalisering av ett av huvudresultaten i [8].

I den andra delen av artikeln använder vi resultatet om unikhet av den essentiella ordningen för att (upp till ekvivalens) klassificera reguljära block av \( S \)-delkategorier i kategori \( O \), för en enkel ändligdimensionell komplex Liealgebra. Ekvivalensklasserna av sådana block parametreras av par \( (W, S, G) \) där \( W \) är en Weylgrupp och \( S, G \) är två olika paraboliska delgrupper av \( W \). Fallet med reguljära block motsvarar \( S = \{e\} \). Vi visar att två block som motsvarar \( (W, \{e\}, G) \) och \( (W', \{e\}, G') \) är ekvivalenta om och endast om det finns en isomorf av Coxetergrupper från \( W \) till \( W' \) som avbildar \( G \) på \( G' \). Vi fastställer också olika härledda ekvivalenser mellan block i typ \( A \).

I den tredje delen av artikeln karakteriseras vi objekt av ändlig projektiv dimension i \( S \)-delkategorier av \( O \) via Serre-kvotfunktion. Vi hittar också explicita formler för den projektiva dimensionen av vissa strukturmoduler i \( S \)-delkategorier av \( O \) och för den finitistiska dimensionen av dessa delkategorier.

4.3 Artikel III

Artikel III, "Tilting modules and exceptional sequences for a family of dual extensions algebras”, klassificerar generaliserade tiltingmoduler och exceptionella följer för en familj av algebror som är kvasiärförs och har en enkelbevarande dualitet. Klassifikationen av generaliserade tiltingmoduler är i detta fall mycket mer involverad jämfört med Artikel I, eftersom algebrorna i denna artikel
endast har en odelbar projektiv-injektiv modul, jämfört med \( n - 1 \) som var fallet i Artikel I.

Det första steget mot att klassificera alla generaliserade tiltingmoduler är att klassificera alla odelbara moduler och sedan avgöra vilka av dessa som är självvortogonala. Lät \( \mathcal{F}(\Delta) \) beteckna delkategorin av moduler med en standardfiltrering och \( \mathcal{F}(\nabla) \) delkategorin av moduler med en kostandardfiltrering. Vi visar att alla självvortogonala moduler antingen har en standardfiltrering eller en kostandardfiltrering. Därefter visar vi att det finns en nollskild utvidgning av positiv grad mellan varje självvortogonal modul i \( \mathcal{F}(\nabla) \) och varje självvortogonal modul i \( \mathcal{F}(\Delta) \), med antagandet att ingen av modulerna ligger i snittet \( \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) \). Dessa resultat implicerar att det är tillräckligt att klassificera alla generaliserade tiltingmoduler i \( \mathcal{F}(\Delta) \). Resten av klassifikationen följer genom att använda den enkelbevarande dualiteten.

För att klassificera alla generaliserade tiltingmoduler i \( \mathcal{F}(\Delta) \) konstruerar vi en väldigt transparent kombinatorisk beskrivning av den partiellt ordnade mängden av självvortogonala moduler i \( \mathcal{F}(\Delta) \) med avseende på ordningen som uppkommer från nollskilda utvidgningar av positiv grad.

Avslutningsvis visar vi att de exceptionella \( \Lambda_n \)-modulerna är precis standardmodulerna och kostandardmodulerna, samt att de fullständiga exceptionella följderna är precis följder på formen

\[
(\nabla(i_1), \ldots, \nabla(i_k), L(1), \Delta(j_1), \ldots, \Delta(j_\ell)),
\]

där indexen \( i_x \) är avtagande, indexen \( j_x \) är ökande, \( i_x \neq j_y \) för alla \( x,y \) och där det finns precis \( n \) stycken moduler i följen. Observera att de fullständiga exceptionella följderna av \( B_n \)-moduler i Artikel I är på precis samma form.
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