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Modelling Bidding Behaviour in Electricity Auctions: Supply Function Equilibria with Uncertain Demand and Capacity Constraints

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UPPSALA
UNIVERSITET

Department of Economics, Uppsala University

Visiting address: Kyrkogårdsgatan 10, Uppsala, Sweden
Postal address: Box 513, SE-751 20 Uppsala, Sweden
Telephone: +46 18 471 11 06
Telefax: +46 18 471 14 78
Internet: http://www.nek.uu.se/

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Abstract

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In most electricity markets, producers submit supply functions to a procurement uniform-price auction under uncertainty before demand has been realized. In the Supply Function Equilibrium (SFE), every producer commits to the supply function that maximises his expected profit given the bids of competitors.

The presence of multiple equilibria is a basic weakness of the SFE framework. Essay I shows that with (i) symmetric producers, (ii) perfectly inelastic demand, (iii) a reservation price (price cap), and (iv) capacity constraints that bind with a positive probability, a unique symmetric SFE exists. The equilibrium price reaches the price cap exactly when capacity constraints bind.

Another weakness is difficulty finding a valid asymmetric SFE with non-decreasing supply functions. Essay II shows that for firms with asymmetric capacity constraints but identical constant marginal costs there exists a unique and valid SFE. Equilibrium supply functions exhibit kinks as well as vertical and horizontal segments. The price at which the capacity constraint of a firm binds is increasing in the firm's share of market capacity. The capacity constraint of the second largest firm binds when the market price reaches the price cap. Thereafter, the largest firm supplies its remaining capacity with a perfectly elastic segment at the price cap. Essay III presents a numerical algorithm that calculates a similar SFE for asymmetric firms with increasing marginal costs.

Essay IV derives the SFE of a pay-as-bid auction such as the balancing market for electric power in Britain. A unique SFE always exists if the demand's hazard rate is monotonically decreasing, as for a *Pareto distribution of the second kind*. Assuming this probability distribution, the pay-as-bid procurement auction is compared to the SFE of a uniform-price procurement auction. Two theorems in Essay V prove that the demand-weighted average price is (weakly) lower in the pay-as-bid procurement auction.

Keywords: supply function equilibrium, asymmetry, uniqueness, oligopoly, capacity constraint, uniform-price auction, pay-as-bid auction, discriminatory auction, wholesale electricity market

Pär Holmberg, *Department of Economics, Uppsala University, Box 513, SE-751 20 Uppsala, Sweden.*

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List of Papers

This thesis is comprised of the following essays which are referred to in the text by their respective Roman numerals.

I	Holmberg, Pär (2005). Unique supply function equilibrium with capacity constraints.
II	Holmberg, Pär (2005). Asymmetric supply function equilibrium with constant marginal costs.
III	Holmberg, Pär (2005). Numerical calculation of asymmetric supply function equilibrium with capacity constraints.
IV	Holmberg, Pär (2005). Comparing supply function equilibria of pay-as-bid auctions and uniform-price auctions.
V	Hästö, Peter & Holmberg, Pär (2005). Some inequalities related to the analysis of electricity auctions.

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Abbreviations

HHI	Herfindahl-Hirschman Index
ISO	Independent System Operator
MWh	Mega-Watt hour (unit of energy)
NoK	Norwegian Crowns
PABA	Pay-As-Bid Auction
PJM	Pennsylvania-New Jersey-Maryland
SFE	Supply Function Equilibrium
UPA	Uniform-Price Auction
VOLL	Value of Lost Load

List of symbols

α, β	Parameters in the *Pareto distribution of the second kind*
c	Constant marginal cost
ε	Demand
$\overline{\varepsilon}$	Upper bound of market capacity
$\underline{\varepsilon}$	Lower bound of market capacity
$\overline{\varepsilon_i}$	Capacity of firm i (upper bound)
$f(\varepsilon)$	Probability density of demand
Γ	Terminating price in the numerical calculation of SFE
N	Number of firms
$p(\cdot)$	Equilibrium price
p_i	Price at which the capacity constraint of firm i starts to bind
$\overline{p}$	Price cap, reservation price
$\underline{p}$	Price floor
q	Power contracted on forward markets
$S_i(p)$	Supply function of firm i
ΔS_N	Share of the largest firm's capacity that is offered with a perfectly elastic supply at the price cap

1 Introduction

Since the 19th century, economists have debated whether to think of firms as choosing prices or quantities as strategic variables. Nash-equilibria, from which no firm will find it profitable to deviate unilaterally, can be derived for both strategies, the Bertrand equilibrium and Cournot equilibrium respectively. The Cournot equilibrium, in which quantity is the strategic variable, has often been the first choice for electricity markets; see e.g. Bergman & Andersson [7] and Borenstein et al. [11]. This choice is motivated when the available capacity of electric power producers is viewed as a strategic variable. The Cournot case can be seen as a worst-case scenario of competition in electricity markets but is a problematic modelling device for short-term electricity markets. When demand is nearly inelastic, as in real-time markets, the equilibrium price becomes infinite.[1]

Instead of treating either price or quantity as the strategic variable, it is natural to try to find a middle ground. One such attempt is the use of conjectural variations [47]. As in the Bertrand equilibrium, price is the strategic variable but competitors are assumed to respond in accordance with a reaction function that differs from the strategic variable. This is a major drawback of conjectural variations; the timing and information structure of a one-shot (static) game implies that firms cannot react to one another [47]. Thus the methodology is not theoretically satisfactory.

Wilson [49] analysed a share auction where the object is a divisible unit whose value may be uncertain. He was the first to study games in which the strategic variable is a demand function. Similar supply function models were later presented by Grossman [24] and Hart [25], but without uncertainties. The models of Grossman and Hart yield Nash equilibria but are problematic as the resulting range of possible equilibria is enormous [31]. Without uncertainty, a firm knows its equilibrium residual demand. Thus any firm's quantity and price is determined by a single point in its supply function. As a result, firms enjoy a lot of freedom in their supply functions to support various types of equilibria.

The Supply Function Equilibrium (SFE) with uncertain demand was introduced by Klemperer & Meyer [31]. The equilibrium concept assumes that

[1] In practice, the problem may be overcome by the assumption that small firms are price-takers. The competitive fringe gives quantity-setters an elastic residual demand [12].

producers submit supply functions simultaneously to a uniform-price auction in a one-shot (static) game. In the non-cooperative Nash Equilibrium, each producer commits to the supply function that maximizes his expected profit given the bids of competitors. The set-up of the model is very similar to the organisation of most electricity markets and the equilibrium is often used when modelling bidding behaviour in uniform-price electricity auctions, an application first observed by Bolle [9] and Green & Newbery [21]. The model has also been used to analyse strategic trade policy [36] and vertically-related duopolies [35]. In general, SFE can be applied to any uniform-price auction where costs/values are certain, bidders have common knowledge, discreteness is negligible — as in a share auction — and the demand/supply of the auctioneer is uncertain.

With uncertain demand, the range of SFE diminishes drastically, but multiple equilibria remain [31]. Another basic weakness of the SFE approach is the difficulty of finding valid equilibria, i.e. with non-decreasing supply functions, in the case of asymmetric firms [6,48].[2] Furthermore, the approach is limited to uniform-price auctions. The objective of this thesis is to solve these three deficiencies.

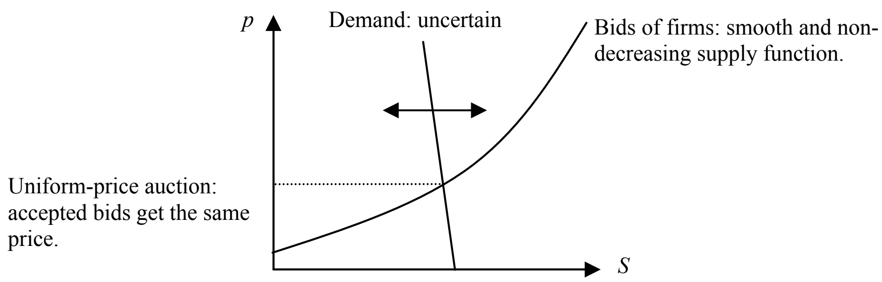

Figure 1. The set-up of the supply function equilibrium model.

[2] Decreasing supply functions are invalid in most electricity auctions.

2 The balancing market

Most electric power is sold in advance through trade on forward markets or with bilateral agreements. Production and consumption of electricity, however, are not fully predictable and the cost of storing electricity is high compared to the cost of production. In most power systems, stored electric energy is negligible. Hence, power consumption and production have to be roughly in balance at all times and to maintain balance, adjustments are made in real-time. The balancing market, also called a real-time market, is an important component in this process. It functions as an auction in which the independent system operator (ISO) can buy additional power (increments) from producers. The increment capacity $\overline{\varepsilon}$ is the part of flexible capacity that has not been sold forward with physical contracts.[3]

Each bidder submits a non-decreasing supply function to the balancing market. This is done before the start of the delivery period for which the bids are valid. Hence the imbalance — demand of the ISO — is not known when bids are submitted. The delivery period is typically an hour, as in California, Pennsylvania-New Jersey-Maryland (PJM), and the Nordic countries, or half an hour as in Britain.

Most balancing markets are organized as *uniform-price auctions* in which all accepted bids are paid the marginal bid, that is, the highest accepted bid in procurement auctions. In 2001, the balancing market of England & Wales switched to a *pay-as-bid auction*. Before the collapse of the California Power Exchange, a similar switch was also considered for this market [30]. In a pay-as-bid auction, as the name suggests, accepted bids are paid their bid. Thus the auction is discriminatory in the sense that accepted bids can receive different prices.

In the short-term, the demand for electric power is very *inelastic*. Thus perfectly inelastic demand is often assumed in models of electricity markets [1,17,20,41,I-V]. If there are significant bids from consumers in the balancing market, they can be modelled as from a producer; in the balancing market an offer of reduced consumption is equivalent to an offer of increased production.

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[3] Flexible power production can be adjusted on short notice.

Price caps, i.e. reservation prices, are employed in most balancing markets and are considered in some models of electric power markets, see e.g. [6,17,20,I-V]. One argument for the use of price caps is that consumers who do not switch off their equipment when electricity prices become extremely high do not necessarily have a high marginal benefit of power. Instead, they may not have the option to switch off or, due to long-term contracts, do not face the real-time price. Thus at some sufficiently high price, social welfare is maximized by rationing demand. The reservation price, $\overline{p}$, is typically set to an estimate of the value of lost load (VOLL) [45].

This thesis focuses on procurement auctions, but analogous results can be derived for *sales auctions*. Such an analysis is relevant for a balancing market, because the ISO sells power back to producers when production exceeds consumption. A producer can offer decrements or down-regulation if he has sold flexible power with physical contracts. The decrement capacity in the market is denoted by $\underline{\varepsilon}$. Firms will only agree to buy back power if the price is below their marginal cost and will use their market power to lower the price below the marginal cost. Thus the sales auction needs a price floor, denoted by $\underline{p}$.

3 The supply function equilibrium

The private-value, common-value and affiliated-value models are classical auction theories that have been applied to treasury auctions and auctions of rights to use natural resources [34]. Classical auction theories tend not to apply to electricity auctions for two reasons. First, the auctioneer's demand is uncertain in electricity auctions.[4] Second, it is often reasonable to assume that production costs are certain and common knowledge in electricity auctions. For these reasons, the Supply Function Equilibrium (SFE) is often better suited to modelling bidding behaviour in electricity auctions and the model has been used extensively during the last 10 years [1-6,9,20,21,38,41,48,I-V].

In the original SFE model with uncertain demand [31], firms submit supply functions simultaneously to a uniform-price auction in a one-shot (static) game. In the non-cooperative Nash Equilibrium, each firm commits to the supply function that maximizes its expected profit given the bids of competitors. Supply function equilibria of uniform-price auctions can be found by making the following observation: each producer acts as a monopolist with respect to his residual demand and submits a supply function that gives the optimal price for each demand outcome. The optimal price of a producer is given by the inverse elasticity rule of a monopolist [47]; the mark-up is inversely proportional to the elasticity of the residual demand curve for every outcome. The elasticity of residual demand comprises derivatives of the competitors' supply functions and, in equilibrium, all producers make optimal bids. Hence, the SFE is given by the solution to a system of differential equations. For symmetric producers and smooth supply functions, one can show that only symmetric equilibria exist [31], that is, the system can be reduced to a single differential equation. However, there is no end-point condition so the solution includes an undetermined integration constant.

[4] Parisio & Bosco extend a private value model to consider demand uncertainty [39].

3.1 Demand uncertainty and uniqueness of the supply function equilibrium

The integration constant in the solution of the system of differential equations allows for a continuum of symmetric supply function equilibria, bounded by the Cournot and Bertrand equilibria. The continuum can intuitively be understood from the inverse elasticity rule of a monopolist. When the supply functions of competitors are very elastic, i.e. they have low mark-ups at every supply, the best response is to have a low mark-up at every supply. When competitors' supply is very inelastic, i.e. they have high mark-ups at every supply, the best response is to have a large mark-up at every supply. Even if many possible equilibria exist, the equilibrium price always starts at marginal cost at zero supply as in Figure 2. This is also true for private-value models of uniform-price auctions [34]. An intuitive explanation is that the supply price for a firm's first unit does not affect the sale prices of any other units. Thus with certain costs, the first unit is sold under Bertrand competition.

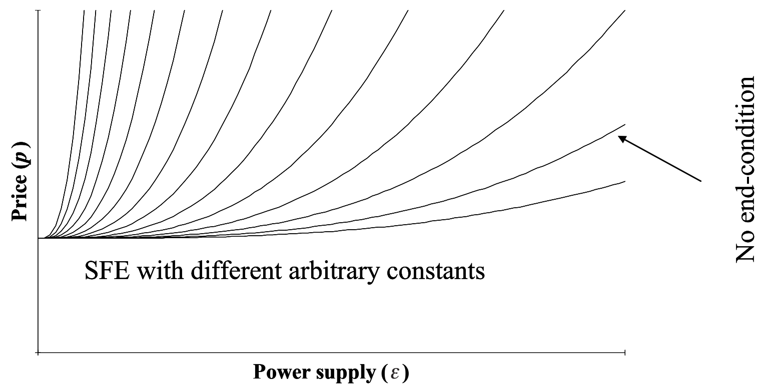

Figure 2. The undetermined integration constant allows for a continuum of SFE.

Multiple equilibria make it difficult to predict bidding behaviour in auctions. Furthermore, it is a nuisance for comparative statics and when comparing auction designs. For example, it is not clear that the integration constant associated with an equilibrium is invariant to changes in exogenous parameters or the auction design. The multiplicity of equilibria is a considerable drawback for the SFE framework and it is not surprising that many papers in the SFE literature try to single out a unique equilibrium.

Klemperer & Meyer [31] show that if outcomes with infinite demand occur with positive probability, and if demand can be met with non-binding capacity constraints, then a unique SFE exists. Unfortunately, this assumption is not realistic for the electric power market. Rudkevich et al. [41] choose the equilibrium that is least profitable for firms, which is in some sense the most robust equilibrium. Green & Newbery [21] consider the equilibrium in which firms have the highest profit, which represents the worst case for consumers. This equilibrium is unique if maximum demand could just be met at the Cournot price at full capacity. In another paper, Newbery [38] gets a unique SFE by considering entry and assuming bid-coordination; incumbent firms coordinate their bids to the most profitable equilibrium that deters entry. Anderson & Xu [2] and Baldick & Hogan [6] find a unique equilibrium in some cases by ruling out unstable equilibria. Stability is tested assuming an infinite speed of adjustment in competitor's bids when one firm has a small deviation from its best-response bid. With a sufficiently slow speed of adjustment, other equilibria might also be stable.

It is well known that capacity constraints limit the range of possible equilibria [6,21]. Genc & Reynolds have recently shown that the range of SFE candidates can be reduced even further by considering pivotal suppliers [20]. They note that global concavity of firms' profit functions, which was proven by Klemperer & Meyer [31], does not automatically apply to markets with capacity constraints.[5] Thus some candidates that were previously thought to be equilibria can now be ruled out. Essay I goes one step further by arguing that there is always a risk that demand exceeds market capacity, and that this suffices to ensure a unique equilibrium (see Figure 3). The equilibrium price reaches the price cap exactly when the market capacity binds. Baldick & Hogan [6] single out the same equilibrium but with a weaker motivation. Price caps and capacity constraints are viewed as public signals that coordinate the bids of the producers.

With perfectly inelastic demand, the uniqueness of the symmetric equilibrium can intuitively be understood from the following reasoning (see Figure 3). When demand is sufficiently high to make the capacity constraints of competitors bind, a producer faces perfectly inelastic residual demand. If such an outcome occurs with a positive probability, the producer's optimal price for this outcome should, following the inverse elasticity rule, be as high as possible, i.e. equal to the price cap. Thus the equilibrium price must reach the price cap. Furthermore, any firm would find it profitable to unilaterally deviate from equilibrium candidates hitting the price cap before the capacity constraints bind. The reason is that it is profitable to slightly undercut competitors' horizontal supply á la Bertrand.

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[5] As noted in Essay I it seems that Klemperer & Meyer prove local concavity rather global concavity.

The support of the probability density of demand is important as it determines the range of possible equilibria. The equilibrium is not otherwise sensitive to the choice of probability distribution. In addition, it is not sensitive to risk-aversion. In the supply function equilibrium of a uniform-price auction, firms choose supply functions such that profit is maximized for every demand outcome, conditional on the bids of competitors. There is no trade-off between profits for different demand outcomes. On the other hand, the learning period, i.e. the time it takes for the market to reach an equilibrium, may very well be related to the probability density. Learning may be slower for parts of the supply curve that are unlikely to be accepted.

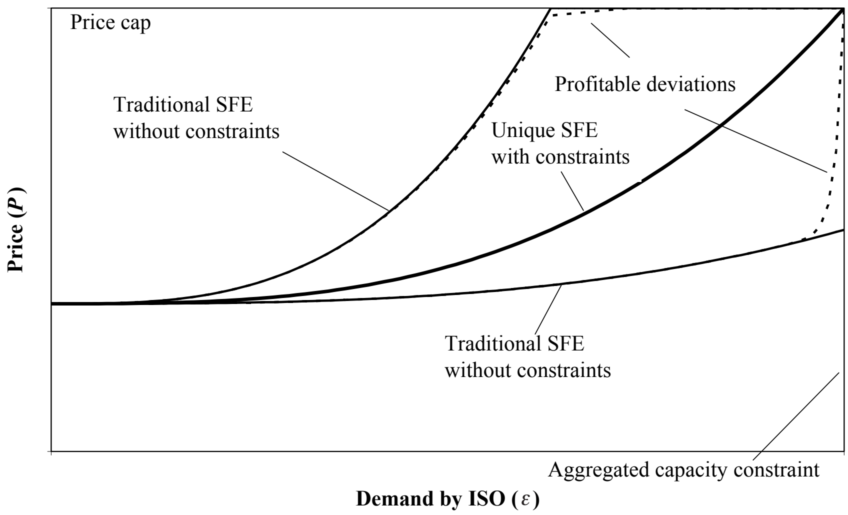

Figure 3. The risk of power shortage rules out all but one traditional SFE.

In this thesis it is argued that in balancing markets, the support of demand's probability density always includes the decrement and increment capacity. In the case of sufficiently large demand shocks or sufficiently many multiple failures in "must-run" power plants, which both occur with a very small but positive probability, the imbalance exceeds $\overline{\varepsilon}$.[6] In the case of sufficiently many multiple failures in transmission lines to power consuming cities, which also occur with a very small but positive probability, the imbalance is below or equal to $\underline{\varepsilon}$. This support of the probability density implies

[6] To avoid inconsistencies in the model, one could restrict attention to generator failures for producers who exclusively have must-run power plants that cannot be regulated in real-time, and who cannot bid strategically in real-time. Two examples of such producers in Britain are British nuclear group and British Energy, both of whom exclusively produce nuclear power.

that any point in a supply function is price-setting with some positive probability and ensures a unique equilibrium in the balancing market.

The length of the delivery period influences the standard deviation of demand. But according to the argument above, the length should neither influence the support of the probability density nor the SFE of uniform-price auctions. However, if other uncertainties exist, e.g. over production costs, then such uncertainty may dominate demand uncertainty if the latter is very small. This may occur especially for very short delivery periods. In such a situation, a private-value or common-value auction model might be preferable to the SFE model.[7] In summary, the length of the delivery period or equivalently whether bids are valid for multiple periods should not influence the SFE, but it may influence the decision whether a SFE model should be used at all.

3.2 Financial and physical contracts

Analogous to increments, the unique equilibrium price for decrements reaches the price floor exactly when the decrement capacity binds (see Figure 4).

Instead of plotting the real-time price as a function of the imbalance, it can be plotted as a function of total demand (see Figure 5). It is assumed that q units of power have been sold forward with physical contracts. Anderson & Xu derive multiple supply function equilibria for firms that have sold q units of power with financial contracts [4]. If a risk of power shortage is introduced into their model, the equilibrium would be unique and equal to the equilibrium in Figure 5. Thus the equilibrium of the real-time market is the same whether q has been sold with financial or physical contracts.

Green has suggested a two-stage model with linear SFE and linear cost functions, which endogenises contracted power, q [23]. The first-period represents a forward market and the second period is the spot market.

[7] A private-value model is suitable if firms know their own production costs but are uncertain about their competitors' costs. In a common-value model, firms are uncertain about all costs, including their own. Such a model might be applicable to an electric power market dominated by hydropower, as the opportunity cost of hydropower is estimated from a prognosis of future electricity prices.

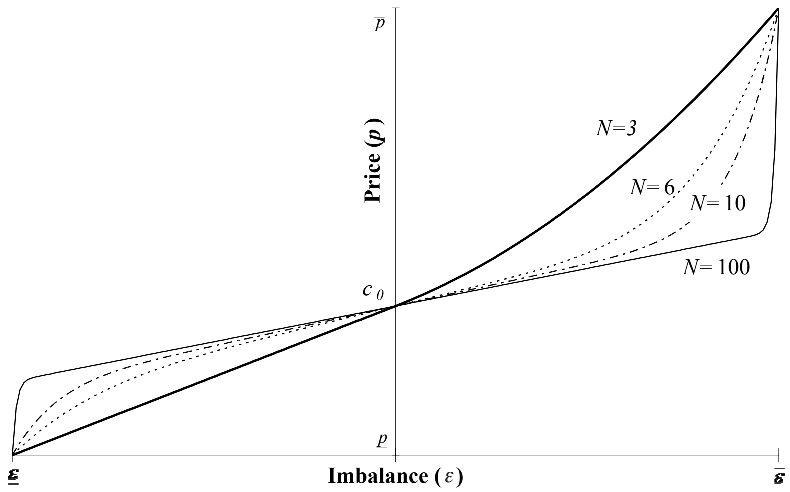

Figure 4. The unique symmetric supply function equilibrium as a function of the real-time imbalance ε. Marginal costs are linear. N=number of symmetric producers. Mark-ups decrease with more producers and are negative (mark-downs are positive) for decrements.

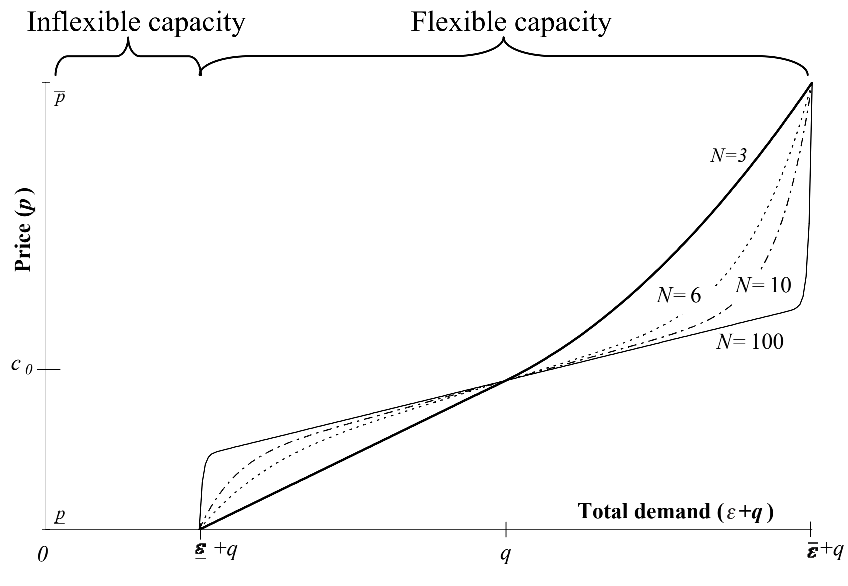

Figure 5. The unique symmetric supply function equilibrium as a function of the total demand $\varepsilon+q$, where q is power contracted on forward markets.

3.3Asymmetric firms — analytical models

The assumption of symmetric producers is convenient as it enables the SFE to be analytically derived for general cost functions as shown in [1,41,I]. However, firms in electric power markets are typically asymmetric so anti-trust policy and merger control should be based on models with asymmetric firms.

Linear SFE for asymmetric firms with linear marginal costs have been analysed by Green [22]. Baldick et al. [5] have developed this concept to piece-wise linear SFE, which can handle asymmetric firms with linear marginal costs and asymmetric intercepts. Linear and piece-wise linear SFE are both problematic when considering capacity constraints [5].

Newbery [37] and Genc & Reynolds [20] have derived SFE for two producers with identical constant marginal costs and asymmetric capacities. Essay II extends their work to multiple asymmetric producers, partly vertical and horizontal supply functions — i.e. binding slope constraints — and supply functions with kinks. There are three reasons for generalizing the supply functions. First, the supply function of a producer is vertical when his capacity constraint binds and horizontal when the price cap binds. Second, such vertical and horizontal segments are useful deviation strategies that can be used to rule out some SFE candidates. Third, in the market situation with perfectly inelastic imbalances that might be zero, excluding kinks and horizontal and vertical segments rules out all supply function equilibria. A similar observation is made by Rudkevich in his analysis of a market with zero marginal costs [42]. The extension of the strategy space complicates the analysis as more SFE candidates have to be ruled out.

Essay II shows that there is a unique SFE. It is piece-wise symmetric, as in the model of Newbery [37]. Any two producers will have the same supply function until the capacity constraint of the smaller firm binds (see Figure 6). At this price the larger firm has a kink in its supply function. The kinks ensure that firms with non-binding capacity constraints face a residual demand with a continuous elasticity. The capacity constraint of the second largest firm binds when the price reaches the price cap. Thereafter, the largest firm sells its remaining capacity with a perfectly elastic supply at the price cap.

Essay II also derives a unique supply function equilibrium for 153 firms bidding in Norway's real-time electricity market, where 99 percent of power is hydroelectric. To simplify the calculation, forward markets are neglected and it is assumed that the real-time market is isolated from power markets of the other Nordic countries, as was the case before 1996. To avoid complications with opportunity costs, Essay II considers an hour when the alternative to power production is to spill water.

The asymmetric SFE of the Norwegian market is compared to a symmetric SFE with 8 firms. The two cases have roughly the same market concentration, measured by the Herfindahl-Hirschman index (HHI) [47]. Figure 7 shows that competition is much tougher in the asymmetric market when demand is low and few asymmetric firms have binding capacity constraints. It is the other way around for large demand outcomes when the capacity constraints of most asymmetric firms bind.

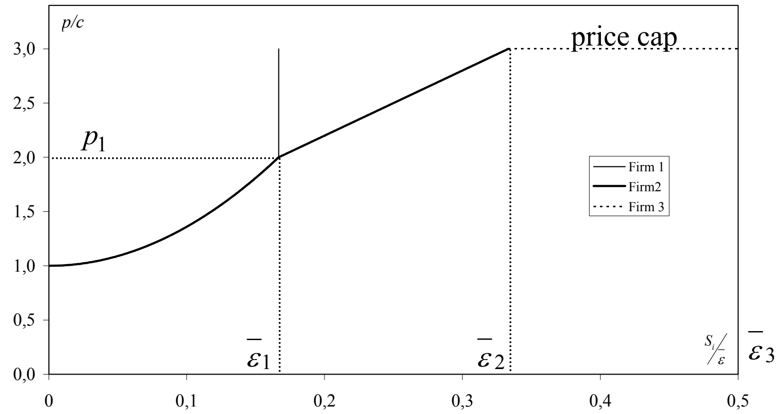

Figure 6. The unique supply function equilibrium is piece-wise symmetric if firms have identical constant marginal costs, c. The capacities of the firms are such that $\bar{\varepsilon}_1 < \bar{\varepsilon}_2 < \bar{\varepsilon}_3$.

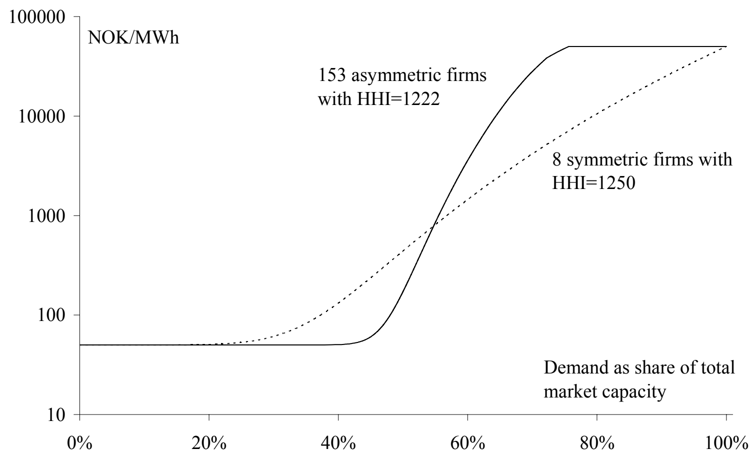

Figure 7. The unique supply function equilibrium of the real-time market in Norway compared to a case with 8 symmetric firms.

3.4 Asymmetric firms — numerical models

Asymmetric supply function equilibria with increasing marginal costs are given by a system of non-autonomous ordinary differential equations that are unlikely to have analytical solutions. Baldick & Hogan [6] observe that in general it is difficult to find valid solutions with non-decreasing supply functions by numerically integrating the system of ordinary differential equations. They identify, however, three exceptions: symmetric firms with identical cost functions, cases with affine solutions — i.e. affine marginal costs and no capacity constraints, — and small variations in demand. The symmetric and affine equilibria can often be calculated analytically.

Essay III presents a numerical algorithm that can be used to calculate a valid asymmetric SFE for more general cases than the three exceptions. The algorithm is inspired by the asymmetric SFE derived analytically in Essay II. To ensure an equilibrium with similar properties, the following two assumptions are employed. First, the larger of any two firms has weakly larger marginal cost for any percentage of the capacity.[8] Second, all firms have the same marginal cost at zero supply.[9] Let p_i be the price for which the capacity constraint of firm i starts to bind. Without loss of generality, assume that firms are ordered such that $\bar{\varepsilon}_i > \bar{\varepsilon}_j$ if $i>j$. Based on the results in Essay II, the following conjectures are made:

- all firms offer their first unit of power at the lowest marginal cost.
- $p_1 < p_2 < \ldots < p_{N-1} = \bar{p}$.
- the largest firm sells its remaining capacity, denoted by ΔS_N, with a perfectly elastic supply at the price cap.

Assuming ΔS_N and a vector $\{p_1, \ldots, p_{N-2}\}$, equilibrium candidates can be calculated. The system of non-autonomous ordinary differential equations can be solved by numerical integration, starting at the price cap and proceeding in the direction of lower prices. As shown in Figure 8, the integration terminates as soon as any supply function becomes invalid, e.g. decreasing in price. A criterion function Γ is defined by the terminating price. A SFE with the conjectured properties should fulfil $\Gamma = C'(0)$. If the equilibrium is unique, as one would expect, it can be found by choosing

[8] It should be possible to numerically calculate asymmetric SFE for even more general cost functions. However, adjustments of the conjectured SFE may be needed, such as the order in which the capacity constraints bind.

[9] With another conjecture, it should be possible to numerically calculate asymmetric SFE when firms have different marginal costs at zero supply. In this case, firms might offer their first units of power at different prices. Furthermore, the firm with the lowest marginal cost at zero supply may offer some power with a perfectly elastic segment that just undercuts the marginal costs of its competitors.

$\{p_1,\ldots,p_{N-2},\Delta S_N\}$ such that Γ is minimized. The algorithm is illustrated in Figure 9 and the solution for an example with three firms is presented in Figure 10.

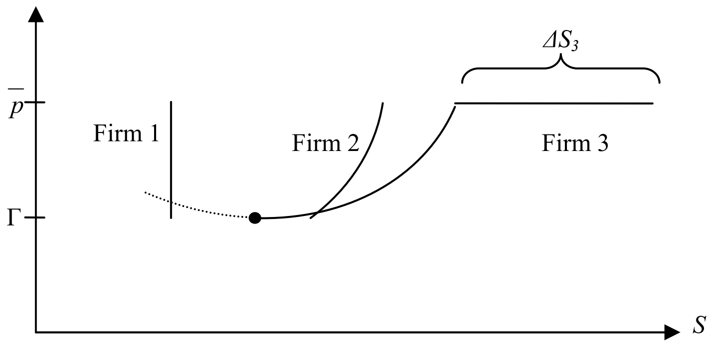

Figure 8. The integration starts at the price cap, proceeds in the direction of decreasing prices and is terminated as soon as any supply function becomes invalid. Γ is defined by the terminating price.

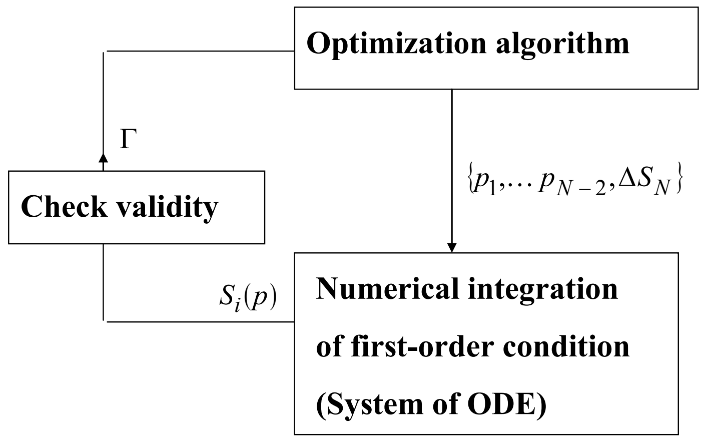

Figure 9. The numerical procedure to find a valid asymmetric SFE.

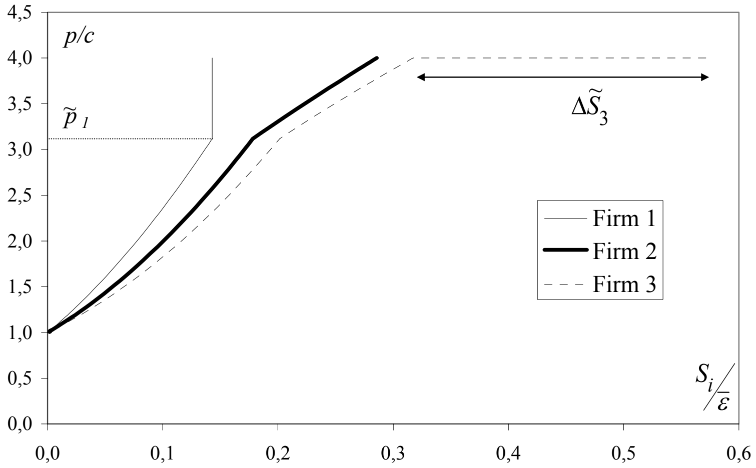

Figure 10. The unique equilibrium for an example with three firms.

3.5 Pay-As-Bid versus Uniform-Price auctions

In 2001, the balancing market of Britain switched from a uniform-price auction (UPA) to a pay-as-bid auction (PABA). It was the belief of the British regulatory authority (Ofgem) that the reform would decrease mark-ups in wholesale electricity prices. Before the collapse of the California Power Exchange, a similar switch was also considered for this market [30].

In a uniform-price procurement auction, all infra-marginal bids are accepted at a price above their bid. A first thought might be that switching to a pay-as-bid procurement auction would drastically reduce mark-ups for infra-marginal units and thereby decrease average electricity prices. But this naive reasoning does not take into account that firms will change their bidding strategy after the switch to a PABA. Using intuition and experience from classical auction theory, some papers actually argue in favour of electricity markets being organised as UPAs, see e.g. Kahn et al. [30] and Wolfram [53]. An experiment by Rassenti et al. [40] also suggests that average prices are higher in PABAs.[10] However, three theoretical studies of electricity auctions [15,44,IV] suggest that average prices are lower in PABAs and a fourth

[10] The demand in the experiment is not revealed to the players, but is certain in each period and the players can deduce it while playing. As in SFE with certain demand, this set-up would lead to an enormous range of equilibria [31]. Thus the experimental results are very much driven by the equilibrium selection process. Further, it is not certain that the experiments are long enough to allow the players find an equilibrium, especially as they have to find out the certain demand by themselves.

study has the similar conclusion that switching to a PABA would increase consumer surplus [16].

Federico and Rahman [16] compare UPAs and PABAs for two polar cases, perfect competition and monopoly, assuming that demand is elastic and follows a uniform probability distribution. They show that expected output decreases and expected consumer surplus increases after a switch to a PABA. On the other hand, welfare is reduced in the competitive case. Under monopoly bidding, welfare is larger in a PABA if and only if marginal costs are sufficiently flat and demand uncertainty sufficiently low.

Fabra et al. [15] derive a Nash equilibrium for a duopoly with constant marginal costs. In their model, each producer is required to submit a horizontal (perfectly elastic) bid for his entire capacity. Firms are asymmetric in terms of both marginal costs and capacity. Furthermore, demand is perfectly inelastic and known with certainty by the producers. Under these circumstances the authors show that average prices are lower in the PABA than in a UPA. Numerical examples show that the difference may be substantial. The implications for production efficiency are, however, ambiguous and depend on parameter values. If demand is sufficiently high, the PABA has no pure strategy equilibria, only a mixed strategy equilibrium. They present several extensions of the model, but the extensions do not lead to definite conclusions regarding the comparison of the two auction types. Son et al. [44] use a similar model as Fabra et al., but one of the two firms has two production units with different marginal costs. Son et al. also conclude that average prices are lower in the PABA than in a UPA if demand is certain and perfectly inelastic. Simulations suggest that the conclusion may hold also for elastic demand.

Essay IV employs the fundamental assumptions of the SFE for uniform-price auctions to derive a SFE for a pay-as-bid auction. In terms of the number of firms and the cost function, the comparison of the two SFE models enables more general conclusions than previous work. It is observed that for some combinations of marginal costs and probability distributions of demand, there are no pure strategy equilibria for the pay-as-bid auction. However, there is always a unique SFE if the hazard rate of demand is monotonically decreasing, as for the *Pareto distribution of the second kind* [29]. Its probability density is given by $f(x)=\beta^{\frac{1}{\alpha}}(\alpha x+\beta)^{\frac{-1}{\alpha}-1}$, which is a convex and monotonically decreasing function. The latter is realistic for the balancing market where large imbalances are less likely than small imbalances.

Two inequalities proven in Essay V show that for symmetric producers and perfectly inelastic demand that follows a *Pareto distribution of the second kind*, average mark-ups are (weakly) lower in the pay-as-bid procurement auction than in a uniform-price procurement auction. Essay IV shows that average mark-ups are equal under monopoly and perfect competition.

Figure 11 compares prices in a UPA and PABA for a duopoly market and uncertain demand given by the *Pareto distribution of the second kind*. The average price as a function of demand is called the equilibrium price and is equal to the marginal bid in a UPA. The equilibrium price is generally higher in the PABA than in the UPA for sufficiently small demand outcomes. The equilibrium price in the UPA equals *C'(0)* at zero demand, while it is generally true that the lowest bid in the PABA is higher than *C'(0)*.

For sufficiently high demand outcomes, the equilibrium price is generally lower in the PABA than in the UPA. In both auctions, only the unit with the highest marginal cost is offered at the price cap. Thus the equilibrium price as a function of demand is always below the price cap in the PABA. In the UPA, on the other hand, the equilibrium price equals the price cap when demand equals or exceeds the market capacity. Accordingly the support of the probability density of the equilibrium price is more constrained in PABAs than UPAs. This seems to be in agreement with experimental findings suggesting that price volatility is lower in PABAs [40].

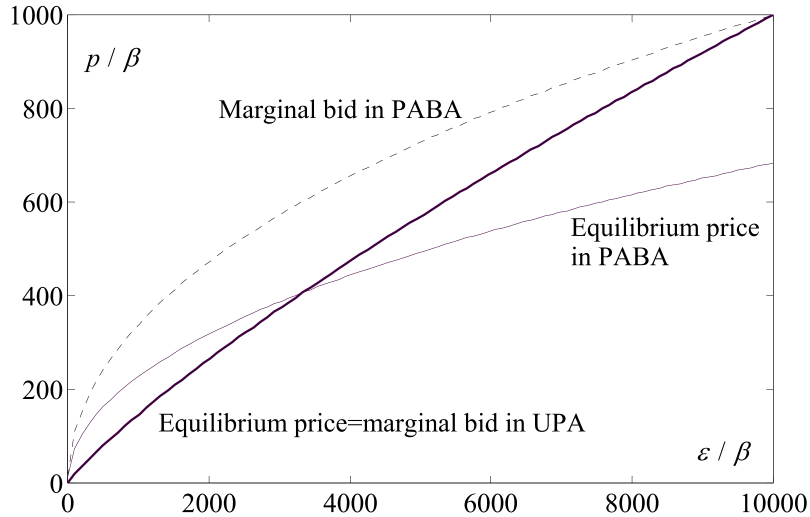

Figure 11. Example for duopoly: equilibrium prices as a function of demand (ε) are compared for the uniform-price auction and pay-as-bid auction. β is a parameter in the Pareto distribution.

3.6 Transmission lines

Most SFE models of the electricity market neglect the impact transmission lines have on bidding strategies. This drastically simplifies the SFE model and is a reasonable assumption for some electricity markets. But in the United States, most regional systems are tightly constrained by limits on transmission capacity [51]. Some simplified (linearised) versions of the SFE model have been used to simulate the influence of congestion in transmission lines [8,18,26]. Recently, Wilson has generalized the models in Essays I to V to consider a general network in which nodes have correlated demand shocks [51].

3.7 Price and quantity discreteness

As in Klemperer & Meyer [31], the greater part of the SFE literature considers smooth, twice continuously differentiable supply functions. In Essays I to III, this assumption is generalized to piece-wise smooth supply functions. In reality, however, supply functions are often given in discrete units of price and quantity [17]. The neglect of discreteness forms the main criticism of continuous SFE models.

A model incorporating quantity discreteness was originally derived by von der Fehr & Harbord [17] and subsequently used in several papers, e.g. Brunekreeft [13], Fabra et al. [15] and Son et al. [44]. Anderson & Xu have developed a corresponding model for price discreteness [3]. These models capture interesting features of electricity auctions but have numerous problems. First, they are very difficult to solve for uncertain demand, such as the case of supply functions with more than one step or more than two bidders. Second, they often do not have pure strategy equilibria. Third, in reality there is discreteness in both price and quantity. Considering both types of discreteness can yield very different results compared to models that only consider one type of discreteness [33]. Fourth, it can also be argued that the discrete SFE models put more restrictions on bids than actually prevail in most electricity markets. Even if market rules limit the number of blocks, the block size is usually flexible and firms are normally free to offer generation capacity in several blocks with different prices [6].

Discreteness in price and quantity is a feature of most markets (not only electricity markets), but is generally neglected in equilibrium models such as Bertrand and Cournot. One could argue that this approximation should be valid for the electricity market as well, e.g. due to a small and otherwise negligible degree of uncertainty in the availability and marginal cost of competitors' generators.[11] In the end, it becomes an empirical question whether continuous or discrete SFE models are best suited for modelling strategic bidding behaviour in electricity markets. It is likely that the conclusion will depend on the degree of discreteness in the market design and the size of production units in the electricity auction being studied.

3.8 Empirical support for the SFE model

Hockey-stick bidding [28] implies that some firms offer their last units of power at an extremely high price, such as the price cap. This is a problem for some US electricity auctions as it implies extreme mark-ups when demand is high. This phenomenon supports the idea in Essays I to IV that the equilibrium price should reach the price cap for sufficiently high demand.

Hortascu & Puller [27] and Sioshansi [43] empirically analyse whether bidding in the balancing service of ERCOT, a market in southern and central Texas, is consistent with the continuous SFE model of uniform-price auctions.[12] This market is especially useful for empirical tests as the bid curves of all firms are public information. Both studies conclude that the bids of the two to three largest firms match the first-order condition implied by the continuous SFE model. Interestingly, small firms, which have lower incentives to bid strategically, do not bid their marginal costs. Instead they have extremely high mark-ups and mark-downs, as if to price themselves out of the balancing market [27,43]. This is often explained as an unwillingness to deviate from their day-ahead schedules [43]. It is uncertain whether the phenomenon is due to costs associated with being flexible in real-time or merely managerial inefficiency [27].

[11] As an example, Parisio & Bosco [39] show that allowing for small uncertainties in competitors' costs may drastically change equilibria with quantity discreteness, e.g. mixed equilibria become pure equilibria.

[12] The model of Hortascu & Puller is not a true SFE model, as firms have private information. Still it is essentially a standard SFE model, as optimal bid functions are assumed to be additively-separable and linear in private information [27,43].

4 Policy implications

In most auction models, the *reservation price* influences the equilibrium so choosing the right reservation price is often an important part of auction design [34]. As shown in Essay I, a reduced price cap will lower equilibrium prices for every demand outcome, i.e. prices far below the price cap are influenced by the level of the cap (see Figure 12). This does not imply, however, that the price cap should be set as low as possible. A high price cap will increase the incentives for firms to keep and invest in reservation power that is used under extreme circumstances only. Moreover, the ISO must be prepared to ration demand, i.e. disconnect consumers, instead of accepting bids above the price cap; power producers might be inclined to test a non-credible price cap as in California [50]. To provide the right investment incentives under perfect competition, the reservation price should be set to the value of lost load (VOLL) [45]. In an oligopoly market, mark-ups will induce extra entry which can lead to overinvestment in capacity [21]. To mitigate this problem, the price cap should be set below the VOLL. The lesser the degree of competition, the lower the price cap. Analogously, the price floor relevant for decrements should be set above its perfect competition level in an oligopoly market.

The problem with hockey-stick bidding in U.S., i.e. that the last units of power are sold at an extremely high price in electricity auctions [28], lends some empirical support to the unique supply function equilibrium. However in some countries, e.g. in the European Union, abuse of market power is illegal and firms might not dare to bid with extreme mark-ups. In this case, the price cap can be interpreted more generally as the highest price acceptable without risking interference by the regulator. In this case, the regulator needs to monitor that no capacity is withheld from the market. Otherwise firms might be tempted to withhold flexible power, increasing the risk of power shortages and the probability that the market price equals the true price cap.

As illustrated in Figure 12, increased *production capacity* decreases the equilibrium price for every demand. In the balancing market, there is no difference between a producer offering increased supply and a consumer offering reduced demand. Thus increased consumer flexibility is equivalent to more flexible production. Furthermore, market capacity can be increased by co-operation with balancing markets in other countries or states.

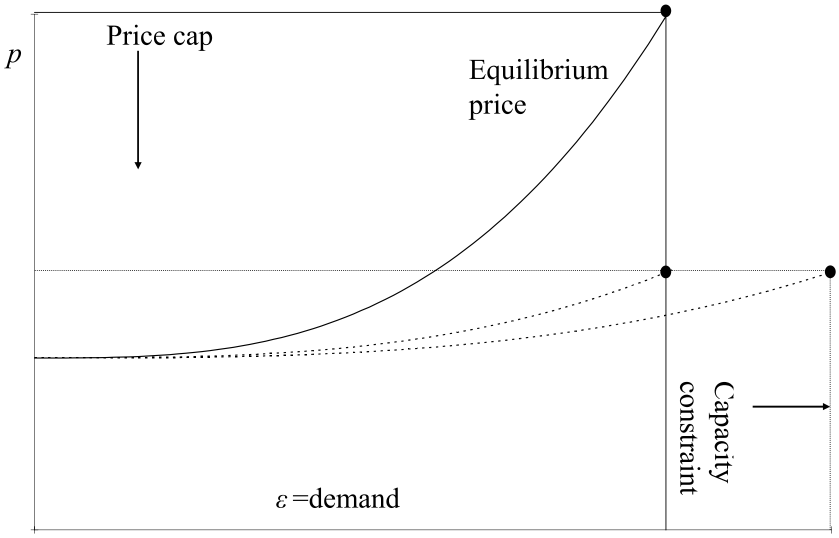

Figure 12. Reducing the price cap $\bar{p}$ and/or increasing the market capacity $\bar{\varepsilon}$ push down the equilibrium price for every demand.

Figure 4 shows, as one would expect, that increased *competition* reduces mark-ups (and mark-downs). Thus, as suggested by Green [22], it is recommended that dominant firms are split up, but politically this could be difficult. He shows that partial divestiture, which is often easier politically, can also be fairly effective. However, encouraging entry "in advance of need" tends to decrease welfare because of overinvestment [22]. More flexible consumers and co-operation with balancing markets in other countries or states also increase competition. To avoid local monopolies or oligopolies, bottle-necks in the power system should be avoided.

Not only the number of firms, but also their *asymmetry* is important for mark-ups. In Essay II, a market with symmetric firms is compared to a market with asymmetric firms. Both markets have the same market concentration, as measured by the Herfindahl-Hirschman index (HHI). For small demand outcomes, the asymmetric market has lower mark-ups whereas mark-ups are higher when demand is close to market capacity (see Figure 7).

Figure 4 shows that average mark-ups in the real-time market would be small if most power is sold in *forward markets*, i.e. the imbalance is small. This result is supported by Wolak's empirical studies [52]. The two-price settlement [46], which is used in the balancing services of Sweden and Finland, gives consumers and producers incentives to buy and sell their expected consumption and production on forward markets. This should result in small imbalances and low real-time mark-ups. On the other hand, the two-price settlement also prevents arbitrage between the real-time market and forward markets. Hence, low real-time mark-ups do not necessarily spill over to forward markets where most trade takes place.

As demand is approximately perfectly inelastic in the short-term, there are no mark-up induced *welfare losses* in the short-term apart from possible production inefficiencies. These inefficiencies occur when firms have asymmetric mark-ups and operate at different marginal costs, as in Essay III. To solve production inefficiencies, von der Fehr & Harbord [17] have suggested that electricity auctions be organized as Vickrey auctions. The advantage of this auction design is that it is optimal for producers to bid their true marginal costs as they receive an information rent. Even if there are no production inefficiencies and no welfare losses, mark-ups may lead to an undesirable redistribution of wealth from consumers of electric power to producers.

Essays IV and V show that if demand is described by a Pareto distribution of the second kind, demand-weighted average prices in a *pay-as-bid procurement auction* are (weakly) lower than in the uniform-price procurement auction. Average prices are equal under monopoly and perfect competition. These results are in line with previous theoretical studies of electricity auctions [15,16,44]. For decreasing probability density functions with low convexity, switching from a uniform-price auction to a pay-as-bid auction drastically reduces mark-ups in electric power markets. With high convexity, the change in the average price is negligible.

Risk aversion does not change the SFE of a uniform-price auction. In Essay IV it is argued that risk aversion should decrease the bids in pay-as-bid procurement auctions. Thus it seems that risk aversion would increase the advantages of pay-as-bid auctions. In addition, the uniform-price auction (weakly) facilitates *tacit collusion* compared to the pay-as-bid auction [14,32].

There are, nonetheless, disadvantages of a pay-as-bid auction relative to a uniform-price auction. First, as shown in Essay IV and by Fabra et al. [15], there is a larger risk for *non-existent pure strategy equilibria* in the pay-as-bid auction. Second, Kahn et al. [30] point out that in a UPA, it is optimal for small firms to simply bid their marginal costs while in a PABA all firms will be forced to forecast market prices if they are to receive any contributions to profits. This introduces an *additional fixed cost* for small firms, which is disadvantageous to competition in the long-run.

5 Future work

In this thesis a unique SFE is proven to exist for symmetric firms with increasing marginal costs (Essay I) and asymmetric firms with constant marginal costs (Essay II). It should be possible to prove that a SFE for asymmetric firms with increasing marginal costs must necessarily have the properties conjectured in Essay III. Many of the proofs used in Essay II are also applicable for increasing marginal costs and the proofs presented by Baldick & Hogan [6] are likely to be useful in making progress here. Proving existence and uniqueness of asymmetric SFE with increasing marginal costs are likely to be a more problematic task.

The numerical method suggested in Essay III can be employed to calculate an asymmetric SFE for an existing electricity market. The degree of forward contracts, elasticity of demand and the price cap—which can be interpreted as the maximum price allowed without risking interference by the regulator—can be calibrated to match market data.

It is argued in this thesis that the supply function equilibrium model is especially suitable for modelling bidding in balancing and real-time markets. A multi-period model is needed to improve the understanding of bidding in a forward market and its interaction with the real-time market. It would be interesting to study a two-stage game similar to that analysed by Green [23]. In place of the linear SFE used by Green, bidding in the second stage can be modelled with unique SFE as in Essays I to III.

Auction models differ in the variables treated as certain and uncertain. The dominant uncertainty influences the recommended choice of auction model. Thus estimates of the degree of demand and production uncertainty faced by bidders in electricity auctions would be highly valuable. With these estimates at hand, auction models that can handle both types of uncertainty, such as the model of Parisio & Bosco [39] and corresponding models for share auctions, should be very useful to determine the dominant uncertainty.

Essay IV compares SFE of pay-as-bid procurement auctions and uniform-price procurement auctions. It is shown that the average price in the former is lower if demand is given by a *Pareto distribution of the second kind*. Making a similar analysis for other distributions would be a valuable extension. For probability distributions with increasing hazard rate, as the normal distribution, one has to make sure that marginal costs are sufficiently steep to guarantee the existence of a SFE. Analogous to Essays II and III, it should be possible to calculate asymmetric SFE of pay-as-bid auctions.

Finally, the Vickrey auction has been suggested as an alternative to the uniform-price auction as it guarantees efficient production. Understanding how the average price in this auction compares to the average price in pay-as-bid and uniform-price auctions would be a valuable extension and contribution to the literature.

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Paper I

Unique supply function equilibrium with capacity constraints [1]

Pär Holmberg[2]

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Abstract

Consider a market where producers submit supply functions to a procurement auction with uncertain demand, e.g. an electricity auction. In the Supply Function Equilibrium (SFE), every firm commits to the supply function that maximises expected profit given the supply functions of competitors. A basic weakness of the SFE is the presence of multiple equilibria. This paper shows that with (i) symmetric producers, (ii) perfectly inelastic demand, (iii) a reservation price, and (iv) capacity constraints that bind with a positive probability, there exists a unique, symmetric SFE.

Keywords: supply function equilibrium, uniform-price auction, uniqueness, oligopoly, capacity constraint, wholesale electricity market

JEL codes: C62, D43, D44, L11, L13, L94

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[2] Department of Economics, Uppsala University, P.O. Box 513, SE-751 20 Uppsala, Sweden, phone +46 18 471 76 35, fax +46 18 471 14 78. E-mail: par.holmberg@nek.uu.se.

1. INTRODUCTION

The Supply Function Equilibrium (SFE) under uncertainty was introduced by Klemperer & Meyer [14]. The concept assumes that producers submit supply functions simultaneously to a uniform-price auction in a one-shot game. In the non-cooperative Nash Equilibrium (NE), each producer commits to the supply function that maximises expected profit given the bids of competitors and the properties of uncertain demand. The set-up of the model is similar to the organisation of most electricity auctions and the equilibrium is often used when modelling bidding behaviour in such auctions. This application was first observed by Bolle [5] and Green & Newbery [9]. More broadly, the SFE can be applied to any uniform-price auction where bidders have common knowledge, quantity discreteness is negligible — objects are divisible [23] — and the demand/supply of the auctioneer is uncertain. Multiplicity of equilibria is a basic weakness of SFE. This paper demonstrates that under certain conditions that are reasonable for electric power markets, especially balancing markets, a unique SFE exists.

Supply Function Equilibria are traditionally found by making the following observation: each producer submits a supply function such that for each demand outcome, the market price is optimised with respect to his residual demand. Each producer acts as a monopolist with respect to his residual demand and the optimal price of a producer is given by the inverse elasticity rule [22]. Hence, the mark-up percentage is inversely proportional to the elasticity of the residual demand curve for every outcome. The elasticity of residual demand is comprised of derivatives of competitors' supply functions. Thus the SFE is given by the solution to a system of differential equations. For symmetric producers with smooth supply functions, one can show that only symmetric equilibria exist [14] and the system can be reduced to a single differential equation. However, there is no end-point condition so the solution includes an integration constant.

The integration constant allows for a continuum of symmetric equilibria, bounded by the Cournot and Bertrand equilibria. The continuum can intuitively be understood by means of the inverse elasticity rule. When competitors' supply functions are highly elastic, i.e. they have low mark-ups at every supply, the best response is to have a low mark-up at every supply. When competitors' supply is inelastic, i.e. they have high mark-ups at every supply, the best response is to have a large mark-up at every supply. Multiple equilibria make it difficult to predict outcomes with SFE. Furthermore, it complicates comparative statics and comparisons of different auction designs. How can one be sure that the integration constant associated with an equilibrium does not change when the organisation of the market is changed? Thus, multiplicity of equilibria represents a considerable drawback for SFE.

I consider a market with symmetric producers, *perfectly inelastic demand* and *capacity constraints* that bind with a positive probability. I show that under these conditions, there exists a unique, symmetric SFE.[3] A *price cap*, i.e. reservation price, is needed to limit the equilibrium price and guarantee the existence of the equilibrium. The unique symmetric equilibrium price reaches the price cap precisely when the capacity constraints bind. Hence, it turns out that the integration constant in the solution of the differential equation is pinned down by the price cap and the total production capacity. The assumptions leading to uniqueness and existence are reasonable for electric power markets. In particular, *short-run demand is very inelastic* in the electric power market, and perfectly inelastic demand is often assumed in models of real-time and spot markets [2,7,8,19].

Capacity constraints reduce the set of SFE in the electric power market, as has been shown in previous research [4,9,18]. Genc & Reynolds have recently shown that the range of SFE can be reduced even further by considering pivotal suppliers [8]. Specifically, they observe that the concavity of firms' profit functions, originally proven by Klemperer & Meyer [14], does not automatically apply to markets with capacity constraints.[4] Thus some candidates that were previously thought to be SFE in markets with capacity constraints can be ruled out. The current paper goes one step further, it argues that a power shortage can occur in any delivery-period, e.g. due to demand shocks or unexpected failures in one or several power plants.[5] Further, it is shown that this risk implies a unique equilibrium. Even if power shortages are infrequent and may occur years or even decades apart, they are not zero-probability events. The support of the probability density of demand determines the set of SFE, but otherwise SFE do not depend on how likely an outcome is [14]. For this reason, even an arbitrarily small risk of power shortage is enough to yield a unique SFE.

Price caps are employed in most deregulated power markets and are considered in some previous models of electric power markets [4,7,8]. One argument for price caps is that consumers who do not switch off their equipment when electricity prices become extremely

[3] Perfectly inelastic demand and symmetry simplify the analysis, but intuitively these assumptions are not critical to get uniqueness.
[4] As noted in Section 3.6 it seems that Klemperer & Meyer prove local concavity rather global concavity.
[5] To avoid inconsistencies in the model, one could restrict attention to generator failures for producers who exclusively have must-run power plants that cannot be regulated in real-time, and who cannot bid strategically in real-time. Two examples of such producers in Britain are British nuclear group and British Energy, both of whom exclusively produce nuclear power.

high do not necessarily have a high marginal benefit of power. Instead, they may not have the option to switch off or, due to long-term contracts, do not face the real-time price. Thus at some sufficiently high price, social welfare is maximized by rationing demand.

With perfectly inelastic demand, the uniqueness of the symmetric equilibrium can intuitively be understood from the following reasoning (see Figure **1**). When demand is sufficiently high to make the capacity constraints of competitors bind, a producer faces perfectly inelastic residual demand. If such an outcome occurs with a positive probability, the producer's optimal price for this outcome should, following the inverse elasticity rule, be as high as possible, i.e. equal to the price cap. Thus the equilibrium price must reach the price cap. Furthermore, any firm would find it profitable to unilaterally deviate from equilibrium candidates hitting the price cap before the capacity constraints bind. The reason being that it is profitable to slightly undercut competitors' horizontal supply á la Bertrand.

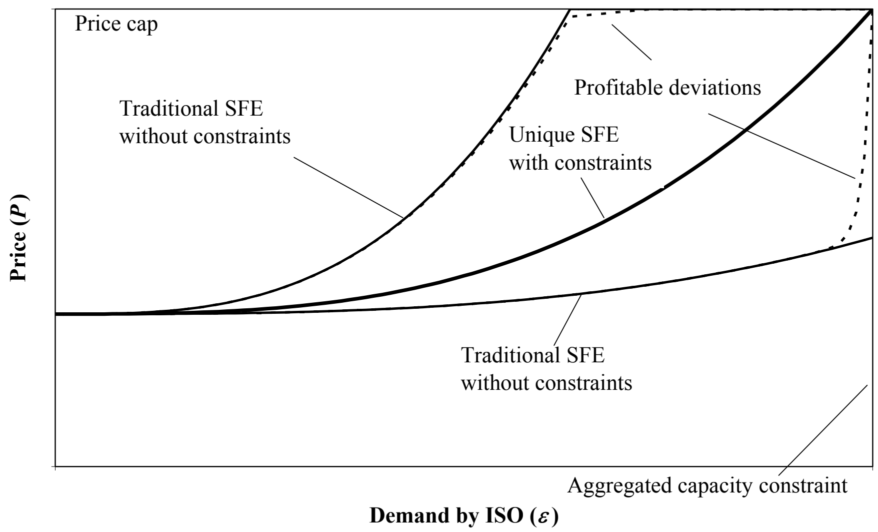

Figure 1. *Capacity constraints and a price cap rule out all but one traditional SFE.*

Many papers in the SFE literature try to single out a unique equilibrium. Klemperer & Meyer show that if outcomes with infinite demand occur with positive probability, and if an infinite demand can be met with non-binding capacity constraints — not realistic for the electric power market — then a unique SFE exists. With a price cap and capacity constraints, Baldick & Hogan [4] single out the same equilibrium as in this paper, but provide a weaker motivation for their result. In their analysis, price caps are seen as a public signal that coordinate the bids of producers. Green & Newbery [9] consider a model with linear demand and use the equilibrium in which firms have the highest profit; the worst case for consumers. This equilibrium is unique if maximum demand could just be met at the Cournot price at full capacity. In another paper, Newbery finds a unique SFE by considering entry and assuming bid-coordination; incumbent firms coordinate their bids to the most profitable equilibrium that deters entry [18]. Rudkevich et al. [19] assume that the least profitable equilibrium is most likely to approximate reality. Anderson & Xu [3] and Baldick & Hogan [4] find a unique equilibrium in some cases by ruling out unstable equilibria. Stability is tested assuming an infinite speed of adjustment when there are small deviations from best-response bids. It is possible that with a sufficiently slow speed of adjustment, other equilibria might also be stable.

In addition to considering a positive risk of power shortage, which ensures a unique SFE, this paper makes further contributions beyond the recent work of Genc & Reynolds [8]. First, their results are proven for the case of constant marginal costs and a specific load function, which corresponds to a specific probability density of demand, whereas a general cost function and a general probability density of demand is allowed in this paper. Second, this paper is the first to rule out symmetric SFE with vertical and horizontal segments. Thus extending the space of allowed strategies, as in this paper and in Genc & Reynolds' paper, does not generate any new symmetric SFE. This is a relevant contribution, as two recent papers have demonstrated that asymmetric SFE will generally include horizontal and vertical segments [10,11]. Third, because the equilibrium is unique it can be analysed with comparative statics.

The structure of the paper is as follows. Section 2 presents the notation and assumptions used in the analysis. In Section 3, the unique SFE is derived in several steps. A first-order condition is derived for smooth and monotonically increasing segments of a symmetric SFE by means of optimal control theory. The result is the first-order condition derived for unconstrained production by Klemperer & Meyer [14]. Next, symmetric equilibria with vertical or horizontal segments are ruled out by using optimal control theory with final values and their associated transversality conditions. To avoid horizontal and vertical segments in the supply, the equilibrium price must reach the price cap exactly when the capacity constraint binds. It is shown that there is exactly one symmetric SFE candidate that fulfils this end-condition and the first-order condition. It is verified that the unique candidate is an equilibrium, i.e. there are no unilateral profitable deviations.

Section 4 characterises the unique SFE. Comparative statics show that the equilibrium has intuitive properties, e.g. mark-ups are reduced if there are more competitors. Another important implication of the analysis is that the price cap and capacity constraints also affect the equilibrium price for outcomes when the constraints do not bind. The assumptions leading to the unique SFE are realistic for electric power auctions, but even more so for balancing markets. Such a market is considered in Section 5. In Section 6, the unique equilibrium is illustrated with an example of a quadratic cost function and Section 7 concludes.

2. NOTATION AND ASSUMPTIONS

Assume that there are N symmetric producers. The bid of each producer i consists of a supply function $S_i(p)$, where p is the price. $S_i(p)$ is required to be non-decreasing. Aggregate supply of the competitors of producer i is denoted $S_{-i}(p)$ and total supply is denoted $S(p)$.

In Klemperer & Meyer's [14] original work, the analysis was confined to twice continuously differentiable supply functions. In this paper the set of admissible bids is extended to include piece-wise twice continuously differentiable supply functions (see Figure **2**). The extension allows for supply functions with vertical and horizontal segments, i.e. binding slope constraints. $S_i(p)$ is not necessarily differentiable at every price, but it is required that it is differentiable on the left and right at every price. Furthermore, all supply functions are required to be left continuous.[6] From the requirements of the supply functions, it follows that all supply functions are twice continuously differentiable in the interval $\left[p_-, \bar{p}\right]$ if p_- is sufficiently large.

Denote the perfectly inelastic demand by ε and its probability density function by $f(\varepsilon)$. The density function is continuously differentiable and has a convex support set which includes ε=0. Let the capacity constraint of each producer be $\bar{\varepsilon}/N$, so that $\bar{\varepsilon}$ is the total capacity of all producers. A key assumption is that the capacity constraints of all producers will bind with a positive probability, i.e. there are extreme outcomes for which $\varepsilon > \bar{\varepsilon}$.

Above the reservation price, demand is zero. In the electricity market this is achieved by means of forced disconnection of consumers when the price threatens to rise above the price cap. Thus the market price for extreme outcomes equals the price cap. Allowing for extreme

[6] Consider a supply function $S_i(p)$ with a discontinuity at p_0. It is then assumed that firm i is willing to produce any supply in the range $\left[S_i(p_0), S_i(p_0+)\right]$ if the price is p_0. Thus the left continuous supply function is actually just a representation of a *correspondence*. The same is true for the left-continuous demand function used by Kremer & Nyborg [15] and the right continuous supply function of Genc & Reynolds [8].

outcomes and rationing is realistic, especially for real-time and balancing markets. However, it differs from the traditional SFE models which ensure market clearing by assuming that firms receive nothing if the market does not clear [8,14].

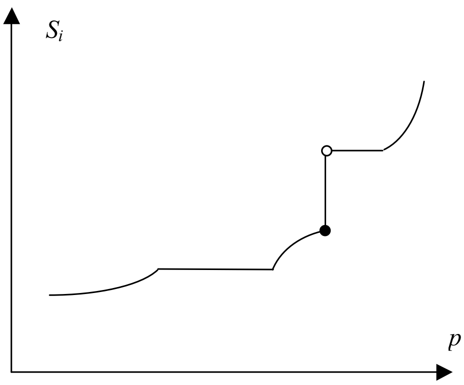

Figure 2. *Admissible supply functions are left-continuous and piece-wise twice continuously differentiable.*

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In the case where total supply has a perfectly inelastic segment that coincides with perfectly inelastic demand, it is assumed that the market design is such that the lowest price is chosen.[7] This implies that the equilibrium price as a function of demand is left continuous.

Let $q_i(\varepsilon,p)$ be the residual demand that producer i faces for $p < \bar{p}$. Provided that the supply functions of his competitors are non-horizontal at p, his residual demand is given by

[7] The same assumption is used by Baldick & Hogan [4].

$$q_i(\varepsilon, p) = \varepsilon - S_{-i}(p) \text{ if } p < \overline{p}. \tag{1}$$

All firms have identical cost functions $C(q_i)$, which are increasing, strictly convex, twice continuously differentiable, and fulfil $C'(\overline{\varepsilon}/N) < \overline{p}$. Thus marginal costs are monotonically increasing.

If more than one producer has a supply function with a perfectly elastic segment at some price p_0, supply rationing at this price is necessary for some demand outcomes. The perfectly elastic supply of producer i at this price is given by $\Delta S_i(p_0) \equiv S_i(p_0+) - S_i(p_0)$, where $S_i(p_0+) \equiv \lim_{p \downarrow p_0} S_i(p)$.[8] Similarly, the total perfectly elastic supply of his competitors at p_0 is $\Delta S_{-i}(p_0) \equiv S_{-i}(p_0+) - S_{-i}(p_0)$. I assume that the rationed supply of producer i at p_0 is given by $S_i(p_0) + R(\varepsilon - S(p_0), \Delta S_i(p_0), \Delta S_{-i}(p_0))$. In addition, it is assumed that the rationing mechanism has the following properties: $R_1 \geq 0$, $R_2 \geq 0$, $R_3 \leq 0$, and $R(0, \Delta S_i(p_0), \Delta S_{-i}(p_0)) = 0$. Furthermore, if $\Delta S_{-i}(p_0) > 0$, then

$$\begin{aligned} &0 \leq R_1 + R_2 < 1 \text{ if } S(p_0) \leq \varepsilon < S(p_0+) \\ &R_1 + R_2 = 1 \text{ if } \varepsilon = S(p_0+). \end{aligned} \tag{2}$$

The intuition for this assumption is as follows. Consider a case where rationing is needed at p_0. Assume that producer i increases the price up to p_0 for one unit that was previously offered below p_0. Then the firms' accepted supply should decrease. The assumed properties can be verified for a rationing mechanism, for example, where all producers receive a ration proportional to their perfectly elastic supply at p_0. This mechanism is called pro-rata on the margin and is used in most uniform-price auctions [8,15].

I also assume that if total supply has perfectly elastic segments at the price cap, all of these bids are accepted before demand is rationed.

3. THE UNIQUE SYMMETRIC SFE

As in the recent paper by Genc & Reynolds [8], optimal control theory is used in the derivation of Supply Function Equilibria. Allowing for vertical and horizontal segments complicates the analysis, as it requires ruling out SFE with vertical and horizontal segments to achieve a unique equilibrium. Furthermore, to ensure that optimal control theory is applicable when testing whether a supply function of a producer is the best response, one needs to ensure that the supply functions of his competitors are continuously differentiable in the integrated

[8] Recall that supply functions are left continuous.

price range. In addition, the control variable needs to be finite. These technicalities imply that supply functions of a potential equilibrium have to be studied piece-by-piece.

In Section 3.1, optimal control theory is used to derive the conditions that must be fulfilled for all *smooth and monotonically increasing* segments of a symmetric supply function equilibrium. These conditions are simplified to a differential equation which yields the standard first-order condition used in the SFE literature, for which an analytic solution exists for perfectly inelastic demand.

In Sections 3.2 to 3.4, irregular SFE are ruled out. In Section 3.2, it is proven that there are no symmetric supply function equilibria with *perfectly elastic segments*. This can be shown by means of optimal control theory with a final value.[9] The result of Section 3.2 also rules out perfectly elastic segments at the price cap. In Section 3.3, equilibria with *discontinuities in the equilibrium price* are also ruled out using optimal control theory with a final value. To avoid a discontinuity in the price when all bids have been accepted, the total supply must be elastic up to the price cap. Section 3.4 shows that no capacity is withheld in equilibrium.

The conclusion is that all supply functions of an equilibrium must fulfil the first-order condition over the whole price range. The end-condition is that the symmetric supply function must reach the price cap exactly when all capacity constraint binds. In Section 3.5 it is observed that a unique SFE candidate exists that fulfils the first-order condition and the end-condition. In Section 3.6 it is shown that it is a globally best response for a firm to follow the strategy implied by the unique candidate, given that competitors also follow the unique candidate. Thus the only remaining equilibrium candidate is a Nash-equilibrium and a SFE.

3.1. The optimal control problem for smooth segments of a SFE

In equilibrium, an arbitrary producer *i* submits his best supply function out of the class of allowed supply functions, given the bids of his competitors. Now consider *a segment* of a symmetric equilibrium candidate $\widehat{S}_i(p)$, for which supply functions are monotonically increasing and twice continuously differentiable in the range $p_- \le p \le p_+$. Assume that the competitors of producer *i* follow the equilibrium candidate. Will it be a best response of producer *i* to follow as well? In this section, only local deviations in the range $p_- \le p \le p_+$ are considered. Firm *i*'s bid outside this range is unchanged, i.e. $S_i(p_-)=\widehat{S}_i(p_-)$ and

[9] In the special case when the total supply is inelastic just below the perfectly elastic segment, this is proven with a profitable deviation.

$S_i(p_+) = \widehat{S}_i(p_+)$. Considering such deviations yields a necessary, but not sufficient, condition for SFE. Let the set $\mathbb{S}$ be defined by $\widehat{S}_i(p)$ and all considered deviations of firm *i*. Note that by choosing his supply function, producer *i* can control the total supply function, *S(p),* where

$$S(p) \equiv S_i(p) + \widehat{S}_{-i}(p) = \varepsilon \,. \tag{3}$$

As competitors follow the equilibrium candidate and $S_i(p)$ is required to be non-decreasing, the total supply, *S(p),* is monotonically increasing in the interval $p_- \le p \le p_+$. Hence, the inverse function of *S(p)* exists for this range. It is denoted *p(ε)*:

$$p(\varepsilon) \equiv S^{-1}(\varepsilon). \tag{4}$$

In terms of demand, the studied range is given by $\varepsilon_- \le \varepsilon \le \varepsilon_+$, where $\varepsilon_- = S(p_-) = \widehat{S}(p_-)$ and $\varepsilon_+ = S(p_+) = \widehat{S}(p_+)$. Hence, controlling the aggregated supply function, producer *i* effectively determines the price for each outcome in the range $\varepsilon_- \le \varepsilon \le \varepsilon_+$ under the constraints $p_- = p(\varepsilon_-)$ and $p_+ = p(\varepsilon_+)$. The optimal $p(\varepsilon)$ for this range can be calculated by solving an optimal control problem. The control variable is defined as $u = p'(\varepsilon)$, i.e. the rate of change in the price. The derivative of the inverse function in (4) can be shown to be

$$u = p'(\varepsilon) = \frac{dp}{d\varepsilon} = \left(\frac{d\varepsilon}{dp}\right)^{-1} = \frac{1}{S'(p(\varepsilon))}. \tag{5}$$

Depending on $S_i(p)$, *S(p)* is not necessarily differentiable at every price, but it is differentiable on the left and right at every price.[10] Thus *u* is piece-wise continuous. It is required that all supply functions fulfil $0 \le S_i'(\cdot)$, so the control variable is constrained by

$$0 \le u \le \frac{1}{\widehat{S}_{-i}'(p(\varepsilon))}. \tag{6}$$

In equilibrium, producer *i* submits his best allowed supply function given $\widehat{S}_{-i}(p)$, the aggregate bid of the competitors. $\widehat{S}_i(p)$ belongs to the set $\mathbb{S}$, which includes all considered deviations. Accordingly, if $\widehat{S}_i(p)$ is the globally best response, it must also be the best response in $\mathbb{S}$. Thus, given the bids of the competitors, it is necessary, but not sufficient, that the following optimal control problem returns the equilibrium candidate.

[10] Recall that all supply functions are piece-wise twice differentiable and always have left and right derivatives.

$$\underset{p(\varepsilon)}{\text{Max}} \int_{\varepsilon_-}^{\varepsilon_+} \left\{ \left[\varepsilon - \hat{S}_{-i}(p(\varepsilon)) \right] p(\varepsilon) - C\left(\varepsilon - \hat{S}_{-i}(p(\varepsilon)) \right) \right\} f(\varepsilon) d\varepsilon$$

$$\text{s.t. } u = p'(\varepsilon) \quad 0 \le u \le \frac{1}{\hat{S}_{-i}{}'(p(\varepsilon))} \qquad p(\varepsilon_-) = p_- \qquad p(\varepsilon_+) = p_+ \tag{7}$$

As a result, it is necessary that the contribution to expected profit from the demand interval $[\varepsilon_-, \varepsilon_+]$ is maximised in equilibrium, given $p(\varepsilon_+) = p_+$, $p(\varepsilon_-) = p_-$, and the bids of the competitors. The integrand of an optimal control problem should be continuously differentiable in the state variable [20], in this case *p.* This condition is fulfilled as all competitors' supply functions are twice continuously differentiable in $[p_-, p_+]$ and the cost function itself is twice continuously differentiable.

The slope constraint $0 \le u \le \frac{1}{\hat{S}_{-i}{}'(p(\varepsilon))}$ may bind if there is a profitable deviation from $\hat{S}_i(p)$, i.e. $\hat{S}_i(p)$ is not an equilibrium. However, if, as assumed, $\hat{S}_i(p)$ is to be a symmetric equilibrium with a monotonically increasing and smooth segment, i.e. $0 < \hat{S}_i{}'(p) < \infty$ for $p \in [p_-, p_+]$, then the slope constraints cannot bind in this interval. Hence, the Hamiltonian of the problem in (7) is

$$H(u, p, \lambda, \varepsilon) = \left\{ \left[\varepsilon - \hat{S}_{-i}(p(\varepsilon)) \right] p(\varepsilon) - C\left(\varepsilon - \hat{S}_{-i}(p(\varepsilon)) \right) \right\} f(\varepsilon) + \lambda(\varepsilon) u(\varepsilon), \tag{8}$$

where *λ* is a co-state or auxiliary variable of the optimal control problem [6]. The control variable *u* should be chosen such that the Hamiltonian is maximised for every *ε* [6]. Hence,

$$\frac{\partial H}{\partial u} = 0 = \lambda(\varepsilon) \tag{9}$$

and

$$\lambda(\varepsilon) \equiv \lambda'(\varepsilon) \equiv 0 \text{ for } \varepsilon \in [\varepsilon_-, \varepsilon_+]. \tag{10}$$

The following equations of motion conditions are also necessary for the optimal solution [6]:

$$p'(\varepsilon) = \frac{\partial H}{\partial \lambda} = u(\varepsilon) \tag{11}$$

and

$$\lambda'(\varepsilon) = -\frac{\partial H}{\partial p} = -\left\{ \left[\varepsilon - \hat{S}_{-i}(p(\varepsilon)) \right] + \left[C'\left(\varepsilon - \hat{S}_{-i}(p(\varepsilon)) \right) - p(\varepsilon) \right] \hat{S}_{-i}{}'(p(\varepsilon)) \right\} f(\varepsilon). \tag{12}$$

Combining (10) and (12) yields

$$0=\left[\varepsilon-\widehat{S}_{-i}(p(\varepsilon))\right]+\left[C'\left(\varepsilon-\widehat{S}_{-i}(p(\varepsilon))\right)-p(\varepsilon)\right]\widehat{S}_{-i}{}'(p(\varepsilon)) \quad \forall\varepsilon\in(\varepsilon_-,\varepsilon_+).$$

We can now use (3) to simplify the above equation,

$$S_i(p(\varepsilon))-\widehat{S}_{-i}{}'(p(\varepsilon))\left[p(\varepsilon)-C'(S_i(p(\varepsilon)))\right]=0 \quad \forall\varepsilon\in(\varepsilon_-,\varepsilon_+). \tag{13}$$

Before continuing with the analysis of this differential equation, note that by means of (1), (13) can be rewritten as

$$\frac{p(\varepsilon)-C'[S_i(p(\varepsilon))]}{p(\varepsilon)}=\frac{S_i(p(\varepsilon))/p(\varepsilon)}{\widehat{S}_{-i}{}'(p(\varepsilon))}=\frac{q_i(\varepsilon,p(\varepsilon))/p(\varepsilon)}{-\frac{\partial q_i(\varepsilon,p(\varepsilon))}{\partial p}}=-\frac{1}{\gamma_i^{res}}. \tag{14}$$

A producer maximises his profit for every outcome ε by observing the elasticity of residual demand, γ_i^{res}, and applying the inverse elasticity rule [22].

The supply functions are monotonically increasing and continuous in the price range $p_- \le p \le p_+$. Thus the equilibrium price $p(\varepsilon)$ is continuous and monotonically increasing in the demand range $\varepsilon_- \le \varepsilon \le \varepsilon_+$. Accordingly, if (13) is fulfilled for all $\varepsilon\in(\varepsilon_-,\varepsilon_+)$, then it must also be fulfilled for all $p\in(p_-,p_+)$. Moreover, the considered equilibrium candidates are symmetric, so $S_i(p)\equiv S_j(p)$, and (13) can be rewritten as

$$\widehat{S}_i(p)-(N-1)\widehat{S}_i{}'(p)\left[p-C'\left(\widehat{S}_i(p)\right)\right]=0, \quad \forall p\in(p_-,p_+). \tag{15}$$

In the subsequent two subsections, non-smooth symmetric SFE will be ruled out. Thus all SFE are given by (15).

Lemma 1 below rules out smooth symmetric transitions to a perfectly inelastic supply and isolated points in the (p,S) space, where $S_i'(p)=0$. This ensures that the control variable u is bounded and that optimal control theory can be relied upon when deriving the first-order condition for any smooth segment.[11]

Lemma 1: No symmetric equilibria exist that, for a finite positive supply bounded away from zero, have smooth symmetric transitions to a perfectly inelastic supply.

Proof: See Appendix.

[11] It follows from (5) that $\widehat{S}_i{}'(p)=0$ would violate $u<\infty$, which is required in optimal control theory [6,20].

3.2 Symmetric SFE with perfectly elastic segments do not exist

Now consider symmetric equilibrium candidates in which all producers have segments with perfectly elastic supply at some price p_0. In such a case, supply rationing is needed for some demand outcomes. In what follows I show that any producer will find it profitable to unilaterally deviate from the equilibrium candidate. He increases his expected profit by undercutting p_0 with units that, for the equilibrium candidate, are offered at p_0 (see Figure 3). The intuition is the same as for Bertrand competition, where producers undercut one another's horizontal bids down to marginal cost. As marginal costs are monotonically increasing, Bertrand equilibria can be ruled out in all price intervals. A formal proof using optimal control theory follows in Proposition 2. Optimal control theory is not applicable when the total supply is perfectly inelastic just below p_0, as the control variable $u = p'$ must be finite. This case is analysed separately in Proposition 3. Note that one implication of Propositions 2 and 3 is that symmetric SFE with perfectly elastic segments at the price cap can be excluded. Negative mark-ups are ruled out in Proposition 1. This obvious result is useful when proving Propositions 2 and 3.

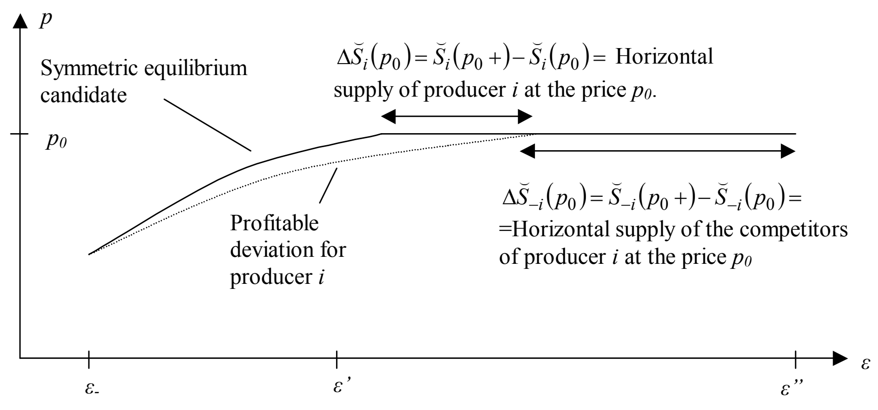

Figure 3. *Symmetric equilibria with perfectly elastic segments can be ruled out. Any producer will find it profitable to slightly undercut competitors' horizontal supply.*

Proposition 1: In equilibrium no production units are offered for sale below their marginal cost.

Proof: See Appendix.

Proposition 2: For positive supply, there are no symmetric supply function equilibria with perfectly elastic segments at p_0 when market supply is elastic just below p_0.

Proof: See Appendix.

Proposition 3: For positive supply, there are no symmetric supply function equilibria with perfectly elastic segments at p_0 when the market supply is perfectly inelastic just below p_0.

Proof: See Appendix.

3.3. The equilibrium price is not discontinuous

Assume that there is a discontinuity in the price at $\varepsilon_L \leq \bar{\varepsilon}$, at which the price jumps from p_L to p_U. This means that all producers have a perfectly inelastic supply in the interval (p_L, p_U), i.e. the slope constraint $0 \leq S_i'(p)$ binds in this price interval. As a result, any producer that bids just below p_L can increase expected profit by deviating. He can significantly increase the price for some units offered at and slightly below p_L as in Figure 4. This significantly increases the price for demand outcomes just below ε_L, while the reduction in sales is small. Thus the deviation increases expected profit. This intuition is verified in Proposition 4. This proposition also rules out discontinuities in the equilibrium price at the demand outcome for which the offered market capacity starts to bind. Thus in a symmetric equilibrium all supply functions must be elastic up to the price cap.

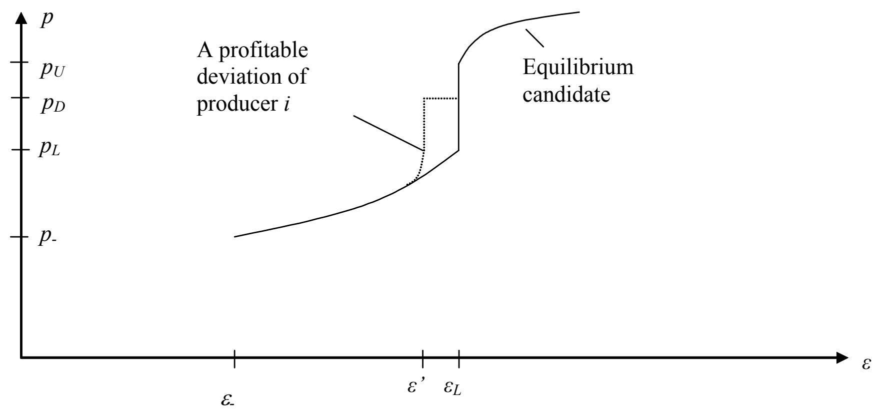

Figure 4. *Discontinuities in the equilibrium price do not exist. Any producer will find it profitable to deviate.*

Proposition 4: For symmetric equilibria there are no discontinuities in the equilibrium price.

Proof: See Appendix.

In a symmetric equilibrium, the first-order condition in (15) must be fulfilled just below and just above *p,* as equilibria with vertical and horizontal segments have been ruled out. Accordingly $S_i(p)$ is continuous at *p* and the cost function is twice continuously differentiable. Thus the first-order condition implies that $S'_i(p)$ must also be continuous at *p*, i.e. there is no kink at *p.*

3.4 No capacity is withheld in equilibrium

If producers are not required by law to offer all of their available capacity to the procurement auction, will firms withhold capacity in the equilibrium of the static game? Proposition 5 ensures that they do not. Instead of withholding some units, it is always better to offer these units at the price cap. Thus producers' bids will be exhausted exactly when the total capacity constraint binds, i.e. at $\varepsilon = \bar{\varepsilon}$.

Proposition 5: If $\bar{p} > C'\left(\frac{\bar{\varepsilon}}{N}\right)$ no capacity is withheld from supply in equilibrium.

Proof: Consider a unit that is withheld from supply by producer *i* in a potential equilibrium. Then there is a profitable deviation for producer *i* in which he offers the unit at a price equal to the price cap. This deviation strategy will not negatively affect the sales of other units or their equilibrium price. Furthermore, because $\bar{p} > C'\left(\frac{\bar{\varepsilon}}{N}\right)$ and there is a positive probability that the demand exceeds or equals the total capacity of all producers, expected profit from the previously withheld unit will be positive. Accordingly, the deviation increases the expected profit of producer *i*. Thus there are no equilibria for which units are withheld from supply.

□

3.5 There is a unique equilibrium candidate fulfilling the necessary conditions

Proposition 5 ensures that no capacity is withheld from the procurement auction in equilibrium. Sections 3.2 and 3.3 rule out all irregular symmetric equilibrium candidates. Thus symmetric equilibria must fulfil the first-order condition in (15) for $\varepsilon \in \left[0, \bar{\varepsilon}\right]$.

For $\varepsilon > \bar{\varepsilon}$, demand rationing is needed and the price will equal $\bar{p}$. According to Proposition 4 there are no discontinuities in the price for symmetric equilibria. Thus the equilibrium price must reach the price cap at $\varepsilon \le \bar{\varepsilon}$. Otherwise, discontinuity in the price will occur at $\varepsilon = \bar{\varepsilon}$. In addition, Propositions 2 and 3 ensure that the equilibrium price cannot be horizontal at the price cap. Thus the equilibrium price must reach the price cap exactly when the total capacity constraint binds, a necessary terminal condition for all symmetric SFE.

The differential equation in (15) is solved by Anderson & Philpott [2] and Rudkevich et al. [19]. There is exactly one solution that fulfils the terminal condition $p(\bar{\varepsilon}) = \bar{p}$:

$$p(\varepsilon) = \frac{\bar{p}\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}} + (N-1)\varepsilon^{N-1}\int_{\varepsilon}^{\bar{\varepsilon}} \frac{C'(x/N)dx}{x^N} \quad \forall \varepsilon \in \left[0, \bar{\varepsilon}\right]. \tag{16}$$

3.6. The unique candidate is a SFE

In Section 3.5 it was shown that there is a unique symmetric SFE candidate, given by (16), which fulfils the necessary first-order condition and end-condition. This unique candidate is denoted by $S_j^X(p)$. In this section it will be verified that the unique candidate also fulfils a second-order condition, i.e. given the residual demand $\varepsilon - S_{-i}^X(p)$, $S_i^X(p)$ is a best response for firm *i*. It is sufficient to show that this response globally maximises firm *i*'s profit for every demand outcome.

It is obvious that no firm can improve its profit in the range $\varepsilon > \bar{\varepsilon}$, as, in this range, all producers sell all of their capacity at the maximum price. For $\varepsilon \in \left[\frac{\bar{\varepsilon}}{N}, \bar{\varepsilon}\right]$, there is some price $\tilde{p}(\varepsilon)$, such that the capacity constraint of producer *i* binds if his last production unit is offered at or below $\tilde{p}(\varepsilon)$. It can never be profitable for producer *i* to push the price below $\tilde{p}(\varepsilon)$, as firm i's supply cannot be increased beyond the capacity constraint. For $\varepsilon \in \left[0, \frac{\bar{\varepsilon}}{N}\right)$, let us set $\tilde{p}(\varepsilon) = C'(0)$ in this interval, as it is never profitable to offer units for sale below marginal cost (see Proposition 1). Thus firm i's best price must be in the range $p \in [\tilde{p}(\varepsilon), \bar{p}]$ if $\varepsilon \in [0, \bar{\varepsilon}]$. Given $S_{-i}^X(p)$, neither capacity constraints nor the price cap bind in this price interval, except at the boundaries. Thus the profit of producer i for the outcome ε is given by

$$\pi_X(\varepsilon, p) = \left[\varepsilon - S_{-i}^X(p)\right]p - C\left(\varepsilon - S_{-i}^X(p)\right) \quad \forall p \in [\tilde{p}(\varepsilon), \bar{p}]$$

and

$$\frac{\partial \pi_X(\varepsilon, p)}{\partial p} = -S_{-i}'^X(p)\left[p - C'\left(\varepsilon - S_{-i}^X(p)\right)\right] + \varepsilon - S_{-i}^X(p) \quad \forall p \in [\tilde{p}(\varepsilon), \bar{p}]. \tag{17}$$

From the first-order condition in (15) it is known that

$$-S_{-i}'^X(p)\left[p - C'\left(S_i^X(p)\right)\right] + S_i^X(p) = 0 \quad \forall p \in [C'(0), \bar{p}].$$

Subtracting this expression from (17) yields:

$$\frac{\partial \pi_X(\varepsilon, p)}{\partial p} = S_{-i}'^X(p)\left[C'\left(\underbrace{\varepsilon - S_{-i}^X(p)}_{S_i}\right) - C'\left(S_i^X(p)\right)\right] + \left(\underbrace{\varepsilon - S_{-i}^X(p)}_{S_i} - S_i^X(p)\right) \quad \forall p \in [\tilde{p}(\varepsilon), \bar{p}].$$

As $\varepsilon - S_{-i}^X(p) = S_i^X(p)$ if $p = p^X(\varepsilon)$, it is straightforward to conclude that $\frac{\partial \pi_X(\varepsilon, p)}{\partial p} > 0$ $\forall p \in [\tilde{p}(\varepsilon), p^X(\varepsilon))$ and $\frac{\partial \pi_X(\varepsilon, p)}{\partial p} < 0$ $\forall p \in (p^X(\varepsilon), \bar{p}]$. Hence, given $S_{-i}^X(p)$, $p^X(\varepsilon)$ is producer i's globally optimal price for each ε. Thus, the equilibrium candidate is a SFE.[12]

[12] Klemperer & Meyer [14] present a corresponding proof for two symmetric firms without capacity constraints facing an elastic demand. In the last step of the proof it is not considered that $\varepsilon - S_{-i}^X(p)$ and $S_i^X(p)$ will generally differ for large deviations. With this simplification it can be shown that $\pi_X(\varepsilon, p)$ is locally concave in the price at $p^X(\varepsilon)$, but this is not sufficient to guarantee the existence of an equilibrium.

4. CHARACTERISING THE UNIQUE SYMMETRIC SFE

It has been shown that with reservation prices and capacity constraints, a unique, symmetric SFE exists. This is good news for comparative statics. For symmetric equilibria, equation (16) continues to be valid even when the number of firms, marginal costs, the reservation price or capacity constraints change.

4.1. Mark-ups

In a market with perfect competition, the equilibrium price is set by the marginal cost of the marginal production unit. The marginal costs of alternative cheaper or more expensive generators do not influence the price. What would happen under imperfect competition? Equation (16) shows that the equilibrium price of the unique SFE is given by a term related to the price cap and a term weighting the marginal costs of generators more expensive than the marginal unit. Thus as in the competitive case, generators cheaper than the marginal unit do not affect the equilibrium price. However, the price of the marginal unit of a producer is limited by the cost of the alternative, competitors' generators with a higher marginal cost. Thus the marginal costs of generators more expensive than the marginal unit influence the size of the mark-ups and accordingly the bid of the marginal unit. It is evident that for the term with weighted marginal costs, the weight decreases with increased demand. Furthermore, all weights are positive and integrate to less than or equal to one, as shown in the following calculation:

$$(N-1)\varepsilon^{N-1}\int_{\varepsilon}^{\bar{\varepsilon}}\frac{dx}{x^N}=\left[-\frac{\varepsilon^{N-1}}{x^{N-1}}\right]_{\varepsilon}^{\bar{\varepsilon}}=1-\frac{\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}}\leq 1.$$

According to Proposition 1, the equilibrium price never falls below the marginal cost of the marginal unit. As a result, producers will choose a positive mark-up for every positive demand. This can be shown by manipulating (16) as follows:

$$p(\varepsilon)=\varepsilon^{N-1}\left[\frac{\bar{p}}{\bar{\varepsilon}^{N-1}}+(N-1)\int_{\varepsilon}^{\bar{\varepsilon}}\frac{C'(x/N)dx}{x^N}\right]\geq\varepsilon^{N-1}\left[\frac{\bar{p}}{\bar{\varepsilon}^{N-1}}+(N-1)C'(\varepsilon/N)\int_{\varepsilon}^{\bar{\varepsilon}}\frac{dx}{x^N}\right]=$$

$$=\varepsilon^{N-1}\left\{\frac{\bar{p}}{\bar{\varepsilon}^{N-1}}-C'(\varepsilon/N)\left[\frac{1}{x^{N-1}}\right]_{\varepsilon}^{\bar{\varepsilon}}\right\}=\varepsilon^{N-1}\left[\frac{\bar{p}}{\bar{\varepsilon}^{N-1}}-C'(\varepsilon/N)\left(\frac{1}{\bar{\varepsilon}^{N-1}}-\frac{1}{\varepsilon^{N-1}}\right)\right]=$$

$$=\frac{\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}}\underbrace{\left[\bar{p}-C'(\varepsilon/N)\right]}_{>0}+C'(\varepsilon/N)>C'(\varepsilon/N).$$

4.2. Comparative statics

For any positive demand outcome, it is clear from (16) that the equilibrium price will increase if the price cap is increased. Equation (16) can also be used to investigate the effect of a symmetric change in the capacity constraints:

$$\frac{\partial p(\varepsilon)}{\partial \bar{\varepsilon}} = \frac{-(N-1)\left[\bar{p} - C'\left(\bar{\varepsilon}/N\right)\right]\varepsilon^{N-1}}{\bar{\varepsilon}^{N}} < 0, \text{ if } \varepsilon > 0.$$

That is, increased capacity decreases the price for all positive demand outcomes.

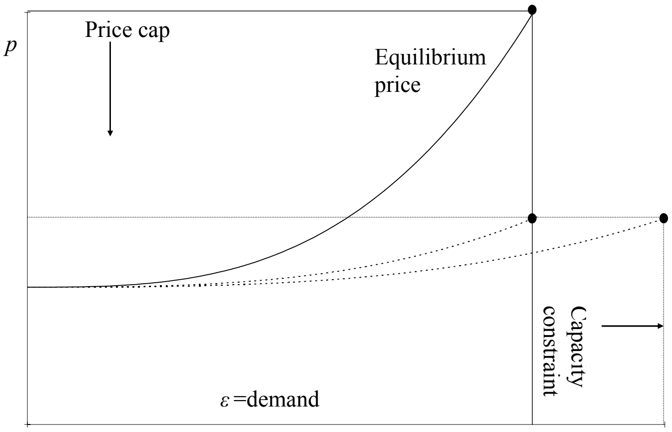

Figure 5. *Reducing the price cap* $\bar{p}$ *and/or increasing the total capacity constraint of the market* $\bar{\varepsilon}$ *push down the equilibrium price for every demand.*

What happens if the number of producers increases? Let the total capacity and aggregated cost function be fixed, i.e. independent of the number of firms. Denote the total cost to meet demand by $C_{tot}(\varepsilon)$. In the unique symmetric SFE, this total cost is N times the cost of each symmetric producer. Hence,

$$C_{tot}(\varepsilon) = NC(S_i) = NC(\varepsilon/N).$$

Thus

$$C_{tot}{}'(\varepsilon) = C'(\varepsilon/N). \tag{18}$$

Combining (16) and (18), the equilibrium price of the unique SFE can be written as

$$p(\varepsilon) = \varepsilon^{N-1}\left[\frac{\bar{p}}{\bar{\varepsilon}^{N-1}} + (N-1)\int_{\varepsilon}^{\bar{\varepsilon}} \frac{C_{tot}{}'(x)dx}{x^{N}}\right] \text{ if } \varepsilon \geq 0.$$

The cost function is twice continuously differentiable and strictly convex. Thus $C_{tot}{}''(\varepsilon) > 0$. Now, using integration by parts, the equilibrium price can be rewritten to yield

$$\begin{aligned} p(\varepsilon) &= \bar{p}\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)^{N-1} - \left[\left(\frac{\varepsilon}{x}\right)^{N-1} C_{tot}{}'(x)\right]_{\varepsilon}^{\bar{\varepsilon}} + \int_{\varepsilon}^{\bar{\varepsilon}}\left(\frac{\varepsilon}{x}\right)^{N-1} C_{tot}{}''(x)dx = \\ &= \underbrace{\left[\bar{p} - C_{tot}{}'(\bar{\varepsilon})\right]}_{>0}\underbrace{\left(\frac{\varepsilon}{\bar{\varepsilon}}\right)}_{\leq 1}^{N-1} + C_{tot}{}'(\varepsilon) + \int_{\varepsilon}^{\bar{\varepsilon}}\underbrace{\left(\frac{\varepsilon}{x}\right)}_{\leq 1}^{N-1}\underbrace{C_{tot}{}''(x)}_{>0}dx. \end{aligned} \tag{19}$$

It is evident that all terms are positive and that the first and last term decrease with *N,* unless $\varepsilon = \bar{\varepsilon}$. The middle term is not influenced by *N* at all. Hence, for every positive demand below $\bar{\varepsilon}$, the equilibrium price of the unique SFE decreases when the number of symmetric producers increases. From equation (19), it can also be noted that the equilibrium price approaches the marginal cost of the marginal unit as the number of symmetric producers approaches infinity.

What happens if entrants increase total capacity? This can be viewed as a combination of an increase in the number of producers and an increase in the total capacity and it has been established that both decrease the equilibrium price for every positive demand.

From equation (19) it is easy to verify that $p(0) = C_{tot}{}'(0) = C'(0)$, which is also proven by Klemperer & Meyer [14]. Hence, the equilibrium price equals the marginal cost of the marginal unit for zero demand. The intuition behind this is that firms' first units are not price-setting for any other units. Thus the first unit is sold under Bertrand competition. The first unit in private value models of uniform-price auctions is also offered at marginal cost [16].

4.3. The slope of the equilibrium price

Using (16) and integration by parts it can be shown that

$$\begin{aligned} \frac{\partial p(\varepsilon)}{\partial \varepsilon} &= \frac{(N-1)\bar{p}\varepsilon^{N-2}}{\bar{\varepsilon}^{N-1}} + (N-1)^2 \varepsilon^{N-2}\int_{\varepsilon}^{\bar{\varepsilon}}\frac{C'(x/N)dx}{x^N} - (N-1)\varepsilon^{N-1}\frac{C'(\varepsilon/N)}{\varepsilon^N} = \\ &= \frac{(N-1)\varepsilon^{N-2}\left(\bar{p} - C'(\bar{\varepsilon}/N)\right)}{\bar{\varepsilon}^{N-1}} + \frac{(N-1)\varepsilon^{N-2}}{N}\int_{\varepsilon}^{\bar{\varepsilon}}\frac{C''(x/N)dx}{x^{N-1}} \quad \forall \varepsilon \in (0, \bar{\varepsilon}). \end{aligned} \tag{20}$$

Thus $0 < \frac{\partial p(\varepsilon)}{\partial \varepsilon} < \infty$ for $\varepsilon \in \left(0, \bar{\varepsilon}\right)$. What happens if $\varepsilon \to 0+$? For N=2, $\lim_{\varepsilon \downarrow 0} \frac{\partial p(\varepsilon)}{\partial \varepsilon} = \infty$, as $C''(\varepsilon) > 0$ and $\int_0^{\bar{\varepsilon}} \frac{dx}{x} \to \infty$. For N>2, the limit is of the type $0 \cdot \infty$, but it can be written in the form $\frac{\infty}{\infty}$. Hence, the limit can be calculated by means of l'Hospital's rule [1]:

$$\begin{aligned}
\lim_{\varepsilon \to 0+} \frac{\partial p(\varepsilon)}{\partial \varepsilon} &= \lim_{\varepsilon \to 0+} \frac{(N-1)\varepsilon^{N-2}}{N} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C''(x/N)dx}{x^{N-1}} = \lim_{\varepsilon \to 0+} \frac{(N-1)}{N\varepsilon^{2-N}} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C''(x/N)dx}{x^{N-1}} = \\
&= \lim_{\varepsilon \to 0+} \frac{(N-1)\frac{-C''(\varepsilon/N)}{\varepsilon^{N-1}}}{(2-N)N\varepsilon^{1-N}} = \frac{(N-1)C''(0)}{(N-2)N}.
\end{aligned} \tag{21}$$

Thus $0 < \frac{\partial p(\varepsilon)}{\partial \varepsilon} < \infty$ for $N>2$, as the cost function is strictly convex and twice continuously differentiable by assumption.

5. BALANCING MARKETS

Relative to production costs, storage of electric energy is expensive. As a result, stored electric energy is negligible in most power systems and power consumption and production must be roughly in balance at all times. Because most electric power is sold on forward markets or with long-term agreements but neither consumption nor production is fully predictable, adjustments have to be made in real-time in order to maintain balance. The balancing market is an important component in this process. It is an auction in which the independent system operator (ISO) can buy additional power (increments) from producers or sell power back (decrements). The latter occurs if contracted production exceeds the realised total demand. A producer can offer decrements if his contracted production is larger than his current inflexible power production, which can not be regulated in real-time.

That the ISO's demand can be both negative and positive in the balancing market ensures that the support of demand's probability density includes zero demand, which was assumed in Section 2. Moreover, the demand is particularly inelastic in real-time. Further, unexpected failures in power plants may result in real-time demand exceeding the supply, especially as only a fraction of the power production is sufficiently flexible to be regulated in real-time. Thus the derived unique SFE should be of particular relevance to real-time and balancing markets.

Decrements can be analysed analogously to increments, but require slightly different assumptions. Namely, $S_i(p)$ must be right continuous for negative supply. In addition, in the case where a perfectly inelastic segment of total (negative) supply coincides with negative perfectly inelastic demand, the highest price will be chosen, i.e. the best price of the ISO. The total decrement capacity is denoted by $\underline{\varepsilon}$. This reflects contracted flexible production which can be bought back and turned off. Producers will not buy back power if the price exceeds the marginal cost. Instead they will use their market power to lower price below marginal cost. As a result, a price floor, $\underline{p}$, is needed for decrements. It is assumed that $\underline{p} \le C'\left(\frac{\underline{\varepsilon}}{N}\right)$. One can use arguments analogous to the increment case to show that a unique, symmetric SFE also exists for decrements.

$$p(\varepsilon)=\begin{cases} \dfrac{\overline{p}\varepsilon^{N-1}}{\overline{\varepsilon}^{N-1}}+(N-1)\varepsilon^{N-1}\displaystyle\int_{\varepsilon}^{\overline{\varepsilon}}\dfrac{C'(x/N)dx}{x^N} & \text{if } \varepsilon \ge 0 \\ \dfrac{\underline{p}\varepsilon^{N-1}}{\underline{\varepsilon}^{N-1}}+(N-1)\varepsilon^{N-1}\displaystyle\int_{\varepsilon}^{\underline{\varepsilon}}\dfrac{C'(x/N)dx}{x^N} & \text{if } \varepsilon < 0 \end{cases} \tag{22}$$

In Section 4.2 it was verified that $p(0)=C'(0)$ for all symmetric SFE. Thus the equilibrium price is continuous at $\varepsilon=0$.

6. A NUMERICAL ILLUSTRATION OF THE UNIQUE SFE

When the cost function is polynomial in form, it is straightforward to analytically calculate the equilibrium price as a function of the demand by means of (22). Here the equilibrium is illustrated with a simple example of a quadratic cost function, i.e. linear marginal costs:

$$C'_{tot}(x)=c_0+kx.$$

The result for $N>2$ is

$$p(\varepsilon)=\begin{cases} c_0+(\overline{p}-c_0)\dfrac{\varepsilon^{N-1}}{\overline{\varepsilon}^{N-1}}+\dfrac{k(N-1)\varepsilon}{(N-2)}\left(1-\dfrac{\varepsilon^{N-2}}{\overline{\varepsilon}^{N-2}}\right) & \text{if } \varepsilon \ge 0 \\ c_0+(\underline{p}-c_0)\dfrac{\varepsilon^{N-1}}{\underline{\varepsilon}^{N-1}}+\dfrac{k(N-1)\varepsilon}{(N-2)}\left(1-\dfrac{\varepsilon^{N-2}}{\underline{\varepsilon}^{N-2}}\right) & \text{if } \varepsilon < 0. \end{cases} \tag{23}$$

Demand is negative in sales auctions and positive in procurement auctions. Both are relevant for balancing markets. Equation (23) is used in Figure 6 to illustrate the effect of the number of producers on the equilibrium price. For positive (negative) demand, more producers implies reduced mark-ups (mark-downs). In Section 4.2, this was proven for all strictly convex and twice continuously differentiable cost functions. For N=100, the market price is very close to marginal cost, except near the capacity constraints. In general, residual demand is less elastic and mark-ups more extreme close to the capacity constraints. This is in agreement with the inverse elasticity rule in (14). For negative demand (decrements), the market price is below marginal cost. Oligopoly producers use their market power to buy back power at a price below their marginal cost. Note also that in all cases, price equals marginal cost at zero supply/demand.

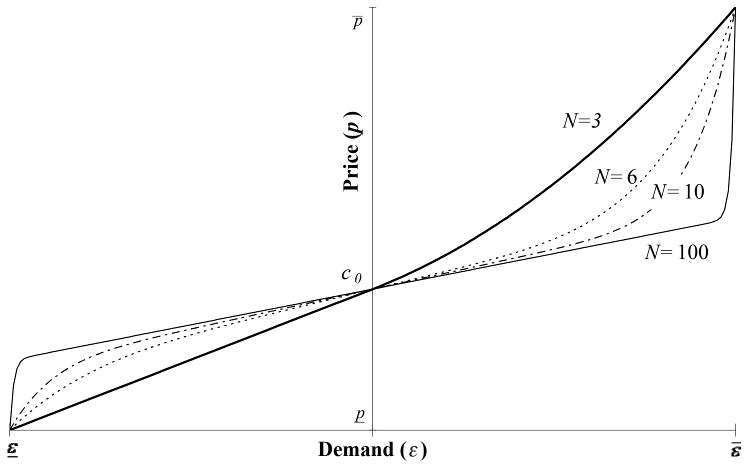

Figure 6. *The unique symmetric supply function equilibrium for linear marginal costs (N= number of symmetric producers). Demand is negative in sales auctions and positive in procurement auctions.*

7. CONCLUSIONS

Multiplicity of equilibria is one basic criticism of the supply function equilibrium (SFE), an established model of strategic bidding in electricity markets. It is well known that capacity constraints reduce the set of SFE [4,9,18]. Genc & Reynolds [8] have recently shown that the range of SFE can be reduced even further by considering pivotal suppliers [8], at least for perfectly inelastic demand, symmetric firms, constant marginal costs, and a specific load function (which corresponds to a specific probability density of demand). This paper allows general cost functions, a general probability density of demand, and constrains the range of SFE even further. It is argued that there is always a risk of power shortage, and it is shown that this leads to a unique SFE. The uniqueness result is sensitive to the support of the probability distribution of the demand, but the unique equilibrium is otherwise insensitive to the probability density. An arbitrarily small risk of power shortage is enough to yield uniqueness. In real-time, demand may exceed the market capacity in any delivery period due to demand shocks or sufficiently many unexpected simultaneous failures in power plants. Whilst such events are very unlikely, they are not zero-probability events. As some demand outcomes are very unlikely, this may result in a long learning period before the market finds the unique SFE.

Reservation prices, i.e. price caps, are used in most electric power markets. The market price in the unique supply function equilibrium reaches the price cap, i.e. reservation price, exactly when the capacity constraints bind. Hockey-stick bidding in some US electricity auctions [13] lends empirical support to the end-condition of the unique equilibrium. This phenomenon means that some firms offer their last units of power at an extremely high price, such as the price cap. In the European Union, abuse of market power is illegal and it is plausible that firms do not dare to bid with extreme mark-ups. However, the price cap can be interpreted more generally, for example, as the highest price acceptable without risking interference by the regulator. In this case, monitoring by the regulator is needed to ensure that no capacity is withheld from the market. Otherwise, firms might be tempted to withhold power, increasing the risk of power shortages and the probability that the market price equals the true price cap.

If the price cap is decreased or capacity constraints increased, the equilibrium price decreases for each positive demand outcome. That is, changing these constraints affects prices also for outcomes when the constraints are non-binding. Increasing the number of producers also decreases the equilibrium price for every level of positive demand. Mark-ups are zero at zero supply and positive for every positive supply.

Perfectly inelastic demand is a realistic assumption for real-time markets and helps simplify the analysis. However, the uniqueness result is expected to hold also for elastic demand. Symmetry is not required to achieve a unique equilibrium, but as two recent papers demonstrate, asymmetry is likely to change the characteristics of the equilibrium [10,11]. In particular, asymmetric equilibria will typically include supply functions with kinks and vertical and horizontal segments. This paper rules out such irregularities for symmetric firms with smooth cost functions.

Continuous SFE, analysed in this paper, has been criticized for not considering the quantity discreteness of real electricity auctions. To address this issue, von der Fehr and Harbord introduced an alternative model with stepped supply functions [7]. Whilst their critique is partly justified, empirical studies are needed to determine which model is actually best suited to represent strategic bidding in electricity auctions. The conclusion might very well depend on the auction design. Two recent empirical studies of ERCOT (a balancing market in Texas) suggest that the bids of the two to three largest firms do indeed match the first-order condition of the continuous SFE [12,21].

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APPENDIX

Proof of Lemma 1:

The result follows from the symmetric first-order condition in (15). If $\varepsilon > A > 0$ — i.e. $\widehat{S}_i(p) > \frac{A}{N} > 0$ — then $0 < m \le \widehat{S}_i{}'(p)$, where *m* is a number independent of *ε*. Thus $\frac{1}{\widehat{S}_i{}'(p)}$ is bounded for positive supply bounded away from zero.

If transition to $\widehat{S}_{il}{}'(p_0) = 0$ is smooth from the left, then $\widehat{S}_i{}'(p)$ is twice continuously differentiable and monotonically increasing in some interval below, but arbitrarily close to p_0. From the argument above it follows that $\frac{1}{\widehat{S}_i{}'(p)}$ is bounded for *p* arbitrarily close to p_0. Thus a smooth transition to $\widehat{S}_i{}'(p_0) = 0$ from the left can be ruled out. With similar reasoning it can be shown that there are no smooth transitions from the right to a perfectly inelastic supply if the positive supply is bounded away from zero. □

Proof of Proposition 1:

Assume that there is an equilibrium $S_i^Z(p)$, in which producer *i* offers production for sale below marginal cost, i.e. there are some prices *p*, for which $C'\left[S_i^Z(p)\right] > p$. Denote this set of prices by $\mathbb{P}$. Then there exists a profitable deviation for producer *i*. Adjust the supply of producer *i* such that units previously offered below their marginal cost are now offered at their marginal cost. Formally, $C'[S_i(p)] = p$ for all $p \in \mathbb{P}$, where $S_i(p)$ is the adjusted supply function. The supply is unchanged for all other p, i.e. $S_i(p) = S_i^Z(p)\ \forall p \notin \mathbb{P}$. $S_i(p)$ is non-decreasing like $S_i^Z(p)$, as *C()* is strictly convex and increasing. The contribution to expected profits from units that are offered at or above their marginal costs are not negatively affected by the deviation. Their contribution might even increase as the equilibrium price increases for some imbalance outcomes. Now consider a unit that was previously offered below its marginal cost c_0. Let ε_0 denote the imbalance for which the market price reaches c_0 in the assumed equilibrium. After the deviation, the price will reach c_0 at an imbalance $\varepsilon \leq \varepsilon_0$. Moreover, market prices will not decrease for any positive imbalances. Thus the positive contribution of the considered unit to the expected profit is either increased or unchanged. Furthermore, after the deviation, the unit is never sold below marginal cost. The same reasoning is true for all units offered below their marginal cost. Thus the deviation increases the expected profit of producer *i* and, in equilibrium, no production is offered below its marginal cost. □

Proof of Proposition 2

Consider a symmetric SFE candidate with perfectly elastic segments at $p_0 \leq \overline{p}$. Denote supply functions following the equilibrium candidate by $\breve{S}_i$. Thus $\Delta\breve{S}_i(p_0) = \breve{S}_i(p_0+) - \breve{S}_i(p_0) > 0$ and $\Delta\breve{S}_{-i}(p_0) > 0$. All considered supply functions are twice continuously differentiable in some price interval $[p_-, p_0]$, see Section 2. The market supply is elastic just below p_0. Thus $0 < \breve{S}_{il}{}'(p_0) < \infty$. Further, a sufficiently large p_- can be chosen such that $0 < \breve{S}_i{}'(p) < \infty$ for all $p \in [p_-, p_0)$.

Now consider unilateral deviations $S_i(p)$ of player *i*, where all his bids above p_0 and below p_- are unchanged. Let $\varepsilon_- = S(p_-) = \breve{S}(p_-)$, $\varepsilon' = S(p_0)$ and $\varepsilon'' = S(p_0+) = \breve{S}(p_0+)$. For demand outcomes $\varepsilon \in (\varepsilon', \varepsilon'')$, the supply at p_0 has to be rationed somehow. The accepted ration of the perfectly elastic supply of producer *i* is given by $R\left(\varepsilon - \varepsilon', \Delta S_i(p_0), \Delta\breve{S}_{-i}(p_0)\right)$, where $\Delta S_i(p_0) = \varepsilon'' - \varepsilon' - \Delta\breve{S}_{-i}(p_0)$.

To keep the equilibrium candidate, $\breve{S}_i(p_0)$ must be the best response of the considered deviation strategies. The best response can be derived from

$$\begin{aligned}
&\underset{p(\varepsilon)}{\text{Max}} \int_{\varepsilon_-}^{\varepsilon'} \left\{ \left[\varepsilon - \breve{S}_{-i}(p(\varepsilon))\right] p(\varepsilon) - C\left(\varepsilon - \breve{S}_{-i}(p(\varepsilon))\right) \right\} f(\varepsilon) d\varepsilon + F(\varepsilon') \\
&\text{s.t. } u = p'(\varepsilon) \quad 0 \le u \le \frac{1}{\breve{S}_{-i}{}'(p(\varepsilon))} \qquad (24) \\
&p(\varepsilon_-) = p_- \qquad p(\varepsilon') = p_0 .
\end{aligned}$$

The final value of the optimal control problem, *F(),* returns the contribution to the expected profit from the rationed supply at p_0.

$$\begin{aligned}
F(\varepsilon') = \int_{\varepsilon'}^{\varepsilon''} &\left\{ \left[\varepsilon' - \breve{S}_{-i}(p_0) + R\left(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta\breve{S}_{-i}, \Delta\breve{S}_{-i}\right)\right] p_0 + \right. \\
&\left. - C\left[\varepsilon' - \breve{S}_{-i}(p_0) + R\left(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta\breve{S}_{-i}, \Delta\breve{S}_{-i}\right)\right] \right\} f(\varepsilon) d\varepsilon
\end{aligned} \qquad (25)$$

The slope constraints $0 \le u \le \dfrac{1}{\breve{S}_{-i}{}'(p(\varepsilon))}$ might bind for $\varepsilon \in [\varepsilon', \varepsilon_-]$ if there is a profitable deviation from $\breve{S}_i(p)$, i.e. $\breve{S}_i(p)$ is not an equilibrium component. However, as in Section 3.1, the slope constraints can be disregarded when a necessary condition for $\breve{S}_i(p)$ is derived under the assumption that $\breve{S}_i(p)$ is a SFE.

The Hamiltonian, the Max H condition and the equations of motion are the same as for the optimal control problem in (7) [17]. In particular, $\lambda(\varepsilon) \equiv 0$ for $\varepsilon \in [\varepsilon_-, \varepsilon']$ as in (10). The transversality condition associated with the terminal constraint at the right end-point is [17]

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} = 0. \qquad (26)$$

The first term is the marginal value of increasing ε'. The second term, which is negative, represents the marginal loss in final value. It is known that $p(\varepsilon') = p_0$ and $R\left(0, \Delta S_i, \Delta\breve{S}_{-i}\right) = 0.$ These relations, combined with (8), (10), (25) and (26), imply[13]

$$\begin{aligned}
H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} &= \int_{\varepsilon'}^{\varepsilon''} \left\{ \left(p_0 - C'(\cdot)\right) \left[1 + \frac{dR\left(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta\breve{S}_{-i}, \Delta\breve{S}_{-i}\right)}{d\varepsilon'}\right] \right\} f(\varepsilon) d\varepsilon = \\
&= \int_{\varepsilon'}^{\varepsilon''} \left\{ \left(p_0 - C'(\cdot)\right) \left[1 - R_1(\cdot) - R_2(\cdot)\right] \right\} f(\varepsilon) d\varepsilon.
\end{aligned} \qquad (27)$$

[13] Note that the first term in (8) cancels out one of the terms given by Leibniz' theorem [1] when differentiating the integral in (25).

Costs are strictly convex and Proposition 1 ensures that there are no equilibria with negative mark-ups. Thus $p_0 > C'$ for $\varepsilon \in [\varepsilon', \varepsilon'')$ and $p_0 \geq C'$ for $\varepsilon = \varepsilon''$. Thus the combination of (2) and (27) implies that

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} > 0 . \tag{28}$$

For the equilibrium candidate, the marginal value of continuing, i.e. increasing *ε'*, is larger than the marginal loss in final value. The reason is that by slightly undercutting p_0, as in Figure 3, producer *i* can sell significantly more. The relation in (28) is true as long as producer *i* has a perfectly elastic supply remaining at p_0. Hence, equilibria of the type $\breve{S}_i(p)$ can be excluded. □

Proof of Proposition 3

Use the same notation as in Proposition 2, but now consider the case when aggregate supply is perfectly inelastic just below p_0. Denote supply functions following the equilibrium candidate by the superscript *H*. Isolated perfectly inelastic points are ruled out by Lemma 1. Thus the equilibrium supply must be perfectly inelastic in a price interval below p_0. Consider the following deviation: producer *i* can offer some units previously offered at p_0 at the price p_0-η, where η is positive and infinitesimally small. As in Proposition 2, the perfectly elastic aggregate supply starts at *ε'*. Let $\varepsilon' = \varepsilon'_H$ in the potential equilibrium. The optimal $\varepsilon' \in [\varepsilon'_H, \varepsilon'']$ is then given by

$$\begin{aligned}
\operatorname*{Max}_{\varepsilon'} \Omega = & \int_{\varepsilon'_H}^{\varepsilon'} \left\{ \left[\varepsilon - S_{-i}^H(p_0)\right](p_0 - \eta) - C\left[\varepsilon - S_{-i}^H(p_0)\right] \right\} f(\varepsilon) d\varepsilon + \\
& + \int_{\varepsilon'}^{\varepsilon''} \left\{ \left[\varepsilon' - S_{-i}^H(p_0) + R\left(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta S_{-i}^H, \Delta S_{-i}^H\right)\right] p_0 + \right. \\
& \left. - C\left[\varepsilon' - S_{-i}^H(p_0) + R\left(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta S_{-i}^H, \Delta S_{-i}^H\right)\right] \right\} f(\varepsilon) d\varepsilon.
\end{aligned}$$

Thus[14]

$$\begin{aligned}
\frac{\partial \Omega}{\partial \varepsilon'} = & \left\{ \left[\varepsilon' - S_{-i}^H(p_0)\right](p_0 - \eta) - C\left(\varepsilon' - S_{-i}^H(p_0)\right) \right\} f(\varepsilon') + \\
& - \left\{ \left[\varepsilon' - S_{-i}^H(p_0)\right] p_0 - C\left[\varepsilon' - S_{-i}^H(p_0)\right] \right\} f(\varepsilon') + \\
& + \int_{\varepsilon'}^{\varepsilon''} \left[1 + \frac{dR\left(\varepsilon - \varepsilon', \varepsilon'' - \varepsilon' - \Delta S_{-i}^H, \Delta S_{-i}^H\right)}{d\varepsilon'} \right] \left[p_0 - C'(\cdot)\right] f(\varepsilon) d\varepsilon.
\end{aligned}$$

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[14] Recall that $R(0, \Delta S_i(p_0), \Delta S_{-i}(p_0)) = 0$.

In order to keep the potential equilibrium with discontinuous supply functions at p_0, $\varepsilon'=\varepsilon'_H$ must be optimal.

$$\left.\frac{\partial\Omega}{\partial\varepsilon'}\right|_{\varepsilon'=\varepsilon_H{}'} = -\left\{\varepsilon'_H - S_{-i}^H(p_0)\right\}\eta f(\varepsilon'_H) + \int_{\varepsilon'_H}^{\varepsilon''} \{[1 - R_1(\cdot) - R_2(\cdot)][p_0 - C'(\cdot)]\} f(\varepsilon) d\varepsilon \qquad (29)$$

The first term is negative but infinitesimally small, as η is infinitesimally small. It is known from the proof of Proposition 2 that the second term is positive and bounded away from zero. Thus $\left.\frac{\partial\Omega}{\partial\varepsilon'}\right|_{\varepsilon'=\varepsilon'_H} > 0$. Hence, producer *i* will find it profitable to deviate by slightly reducing the price of his perfectly elastic supply at p_0, i.e. $\varepsilon'>\varepsilon'_H$ for the optimal ε'. Accordingly, symmetric supply function equilibria with perfectly elastic segments can be ruled out when supply functions are perfectly inelastic just below p_0. □

Proof of Proposition 4

Consider a symmetric equilibrium candidate with a discontinuity in the price at $\varepsilon_L > 0$. Denote its upper price by p_U and its lower by p_L. Denote the equilibrium candidate by $\widetilde{S}_i$. All considered supply functions are twice continuously differentiable in some price interval $[p_-, p_L]$, see Section 2. Equilibria with perfectly elastic segments are ruled out in Section 3.2. Furthermore, smooth transitions to a perfectly inelastic supply are ruled out in Lemma 1. Thus $0 < \widetilde{S}_{il}{}'(p_L) < \infty$. Further, a sufficiently large p_- can be chosen such that $0 < \widetilde{S}_i{}'(p) < \infty$ for all $p \in [p_-, p_L)$, i.e. neither of the slope constraints bind just below p_L.

Now consider the following deviation strategy for producer *i*: leave the supply above p_U and below p_- unchanged, increase the bids for the production units offered at and just below p_L and offer them at a price $p_D \in (p_L, p_U)$ instead. If it is optimal to change the bids for a positive number of units, the deviation is more profitable than the equilibrium strategy and the equilibrium can be knocked out. Whether this occurs can be investigated using an optimal control problem similar to (7) but with an added final value. The final value considers the contribution to expected profit from units sold at the price p_D.

$$\underset{p(\varepsilon)}{\text{Max}} \int_{\varepsilon_-}^{\varepsilon'} \left\{ \left[\varepsilon - \tilde{S}_{-i}(p(\varepsilon)) \right] p(\varepsilon) - C\left(\varepsilon - \tilde{S}_{-i}(p(\varepsilon)) \right) \right\} f(\varepsilon) d\varepsilon + F(\varepsilon')$$

$$\text{s.t. } u = p'(\varepsilon) \quad 0 \le u \le \frac{1}{\tilde{S}_{-i}{}'(p(\varepsilon))} \tag{30}$$

$$p(\varepsilon_-) = p_- \quad p(\varepsilon') = p_L$$

where

$$F(\varepsilon') = \int_{\varepsilon'}^{\varepsilon_L} \left\{ \left(\varepsilon - \tilde{S}_{-i}(p_L) \right) p_D - C\left(\varepsilon - \tilde{S}_{-i}(p_L) \right) \right\} f(\varepsilon) d\varepsilon. \tag{31}$$

The slope constraints $0 \le u \le \frac{1}{\tilde{S}_{-i}{}'(p(\varepsilon))}$ may bind for $\varepsilon \in [\varepsilon_-, \varepsilon']$ if there is a profitable deviation from $\tilde{S}_i(p)$, i.e. $\tilde{S}_i(p)$ is not an equilibrium. However, as in Section 3.1, the slope constraints can be disregarded when a necessary condition for $\tilde{S}_i(p)$ is derived under the assumption that $\tilde{S}_i(p)$ is a SFE.

The Hamiltonian, the Max H condition and the equations of motion are the same as for the optimal control problem in (7) [17]. In particular $\lambda(\varepsilon) \equiv 0$ for $\varepsilon \in [\varepsilon_-, \varepsilon']$. The transversality condition associated with the terminal constraint at the right end-point is [17]

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} = 0.$$

From (8), (10), and (31) we get

$$H(u, p, \lambda, \varepsilon') + \frac{\partial F(\varepsilon')}{\partial \varepsilon'} = \left\{ \left[\varepsilon' - \tilde{S}_{-i}(p_L) \right] p_L - C\left(\varepsilon' - \tilde{S}_{-i}(p_L) \right) \right\} f(\varepsilon') +$$
$$- \left\{ \left(\varepsilon' - \tilde{S}_{-i}(p_L) \right) p_D - C\left(\varepsilon' - \tilde{S}_{-i}(p_L) \right) \right\} f(\varepsilon').$$

The relation must be true for $\varepsilon' = \varepsilon_L$, otherwise $\tilde{S}_i(p)$ cannot be part of an equilibrium:

$$H(u, p, \lambda, \varepsilon_L) + \left. \frac{\partial F(\varepsilon')}{\partial \varepsilon'} \right|_{\varepsilon' = \varepsilon_L} = \underbrace{\left[\varepsilon_L - \tilde{S}_{-i}(p_L) \right]}_{= \tilde{S}_i(p_L) > 0} \underbrace{(p_L - p_D)}_{-} f(\varepsilon_L) < 0.$$

Thus the transversality condition cannot be fulfilled for equilibrium candidates with a discontinuity in the price. The marginal value of continuing, i.e. increasing ε', is less than the marginal loss in final value. Thus, as in Figure 4, any producer will find it profitable to decrease ε' and raise the price for some production units offered just below p_L. □

Paper II

Asymmetric supply function equilibrium with constant marginal costs[1]

Pär Holmberg[2]

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Abstract

In a real-time electricity auction, the bids of producers consist of committed supply as a function of price. Bids are submitted under uncertainty, that is, before demand of the system operator has been realised. In the Supply Function Equilibrium (SFE), every producer chooses the supply function that maximises expected profit given residual demand. I consider a uniform-price auction with a reservation price. Demand is perfectly inelastic and exceeds the market capacity with a positive probability. Firms have identical constant marginal costs but asymmetric capacities. I show that under these conditions, a unique SFE exists which is piece-wise symmetric.

Keywords: supply function equilibrium, uniform-price auction, uniqueness, asymmetry, oligopoly, capacity constraint, wholesale electricity market

JEL codes: C62, D43, D44, L11, L13, L94

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[2] Department of Economics, Uppsala University, P.O. Box 513, SE-751 20 Uppsala, Sweden, phone +46 18 471 76 35, fax +46 18 471 14 78. E-mail: par.holmberg@nek.uu.se.

1. INTRODUCTION

Electric energy is expensive to store compared to its production cost. As a result, stored energy is negligible in most electricity power systems and power consumption and production have to be roughly in balance at all times. Most electric power produced is traded on forward markets or with long-term agreements. However, because neither consumption nor production is fully predictable, adjustments need to be made in real-time to maintain balance. The real-time market, also called a balancing market, is an important component in this process. The market functions as an auction — often a uniform-price auction — in which the independent system operator (ISO) can buy additional power (increments) from power producers.[3] Each bidder submits a non-decreasing supply function to the real-time market before the start of the delivery period for which bids are valid. Hence, the imbalance —demand of the ISO — is not known when bids are submitted. The delivery period is typically an hour, as in California, Pennsylvania-New Jersey-Maryland (PJM), and the Nordic countries, or half an hour as in Britain.

The Supply Function Equilibrium (SFE) under uncertainty was introduced by Klemperer & Meyer [10]. The set-up of the model is similar to the organisation of most electricity markets, and SFE is now an established model of bidding behaviour in electricity auctions [1-9,11-13]. In the non-cooperative Nash equilibrium of the static game, each producer commits to the bid that maximises his expected profit given the bids of competitors. Klemperer & Meyer show that all smooth supply function equilibria are characterised by a differential equation, which in this paper is called the KM first-order condition

In general, a continuum of possible equilibria exists. But the presence of capacity constraints can often drastically reduce the set of SFE candidates, at least when demand is perfectly inelastic [5]. If extreme demand outcomes are allowed for, i.e. demand exceeds total capacity with a positive probability, then there is a unique, symmetric equilibrium for symmetric producers with strictly convex cost functions [8]. A reservation price, i.e. a price cap, is needed to limit the equilibrium price and clear the market in the case of a power shortage. Risk of extreme demand outcomes, perfectly inelastic demand and reservation prices are all realistic assumptions for a real-time market [8].

[3] If power production is too high, the ISO sells back power to the producers (decrements). The analysis in this paper focuses on increments, but as in [8] analogous results can be derived for decrements.

The assumption of symmetric producers is convenient as it allows straightforward calculation of SFE for general cost functions [1,5,12]. However, firms in electric power markets are typically asymmetric. In order to assess efficient antitrust policy and merger control, models that can analyse asymmetric markets are important.

The case of linear SFE for asymmetric firms with linear marginal costs was analysed by Green [7]. Baldick et al. [2] extended this concept to piece-wise linear SFE that can be used to analyse asymmetric firms with asymmetric intercepts. However, both linear and piece-wise linear SFE are problematic in the presence of capacity constraints [2], an important feature of electricity markets.

Newbery [11] and Genc & Reynolds [5] derived SFE for two producers with identical constant marginal costs but asymmetric capacities. This essay extends the framework to multiple asymmetric producers. In addition, the model is generalised to consider partly vertical and horizontal supply functions—i.e. binding slope constraints—and supply functions with kinks. There are three reasons for this extension. First, the supply function of a producer is vertical when his capacity constraint binds and horizontal when the price cap binds. Second, such segments are useful deviation strategies that can be used to rule out some SFE candidates. Third, given the market structure in this paper of perfectly inelastic imbalances that might be zero, excluding kinks and horizontal and vertical segments would rule out all SFE candidates. Rudkevich makes a similar observation in his analysis of a market with zero marginal costs [13]. The extension of the strategy space in this paper does, however, complicate the analysis as more SFE candidates have to be ruled out to yield a unique equilibrium.

To avoid horizontal and vertical segments in the supply function of a firm with non-binding capacity constraint, its residual demand must be smooth.[4] In the case of a duopoly with elastic demand and no price cap, as studied by Newbery [11], this implies that the capacity constraint of the smaller firm must bind after a smooth transition to a perfectly inelastic demand. In markets with more than two firms, such transitions are not necessary. This paper shows that a continuous elasticity of residual demand for firms with non-binding capacity constraints can also be guaranteed if all of these firms discontinuously increase the elasticity of their supply at the price for which the capacity constraint of another firm starts to bind.

[4] This follows from the inverse elasticity rule of a monopolist; the mark-up is inversely proportional to the elasticity of the (residual) demand [14]. Thus a continuous mark-up requires a continuous elasticity of the (residual) demand

Genc & Reynolds' framework considers a duopoly facing perfectly inelastic demand and a price cap. Kinks are avoided with the assumption that demand is always above some positive level, providing an extra free parameter. The assumption is reasonable for forward markets, but not for balancing markets where imbalances are generally close to zero.

The model in this paper assumes that extreme demand outcomes occur with a positive probability. I show that a unique SFE exists for multiple producers with identical and constant marginal costs but asymmetric capacities, and that the unique SFE is piece-wise symmetric. Any two producers will have the same supply function until the capacity constraint of the smaller firm binds. At this price, the larger firm has a kink in its supply function. The capacity constraint of the second largest firm binds when the price reaches the price cap. Thereafter, the largest firm sells its remaining capacity at a price equal to the price cap. The piece-wise symmetric nature of the equilibrium is a specific result of piece-wise symmetric costs. With this exception, the derived properties of the equilibrium are believed to be true for a general class of cost functions. By means of this conjecture, asymmetric SFE can be numerically calculated for increasing marginal costs [9].

The structure of the paper is as follows. Notation and assumptions are introduced in Section 2 and the unique SFE is derived in Section 3. In equilibrium, a firm has perfectly inelastic supply segments only when its capacity constraint binds, and a perfectly elastic segment only when the price cap binds. All producers will offer their first unit at marginal cost. Thus, in equilibrium, the supply function of a producer must fulfil the KM first-order condition from marginal cost up to the price where either the capacity constraint or price cap binds. It is also shown that an equilibrium must have the following properties: (i) the price must be a continuous function of demand up to the price cap, (ii) no producer can face a perfectly inelastic residual demand, and (iii) only one producer can have a perfectly elastic supply at the price cap. These properties yield the end-condition of the system of differential equations, namely, the capacity constraint of the second largest producer starts to bind at the price cap.

A unique SFE candidate exists that satisfies the end-condition and the KM first-order condition. I verify that the candidate is an equilibrium, i.e. no firm will find it profitable to deviate. In Section 4, the unique SFE is numerically illustrated for the case of three asymmetric producers. In Section 5, the unique SFE is calculated for 153 firms in the Norwegian real-time market and Section 6 concludes.

2. NOTATION AND ASSUMPTIONS

The analysis in this paper is similar to that in my previous paper [8]. However, symmetric producers with strictly convex cost functions are replaced by producers with identical constant marginal costs c and asymmetric capacities. The analysis is confined to real-time and balancing markets with positive imbalances but corresponding results can be readily derived for negative imbalances, as in [8].

There are $N \geq 2$ producers, all of whom have different production capacities. The bid of an arbitrary producer i consists of a non-decreasing supply function $S_i(p)$. Aggregate supply of his competitors is denoted $S_{-i}(p)$ and total aggregate supply is denoted $S(p)$. In the original work by Klemperer & Meyer [10], analysis was confined to twice continuously differentiable supply functions. Here, as in [8], the set of admissible supply functions is extended to allow for price intervals with perfectly inelastic supply and discontinuities in the supply function $S_i(p)$, i.e. perfectly elastic segments. Kinks in the piece-wise twice continuously differentiable supply functions are also allowed. It is required that all supply functions are left continuous.[5]

Let $\bar{\varepsilon}_i$ be the capacity constraint of producer i. Without loss of generality, firms can be ordered according to their capacity, i.e. $\bar{\varepsilon}_1 < \bar{\varepsilon}_2 < \ldots < \bar{\varepsilon}_N$. Total capacity is designated by $\bar{\varepsilon}$, i.e. $\bar{\varepsilon} = \sum_{i=1}^{N} \bar{\varepsilon}_i$. Denote demand by ε and its probability density function by $f(\varepsilon)$. The density function is continuously differentiable and has a convex support set that includes zero demand and $\bar{\varepsilon}$.[6] By means of forced disconnection of consumers, the ISO ensures that demand is zero above the price cap. Thus, in the case of $\varepsilon > \bar{\varepsilon}$, the market price equals the price cap. The market design is such that when total supply is partially perfectly inelastic and coincides with perfectly inelastic demand, then the best price for the ISO is chosen, i.e. the lowest price.

Let $q_i(p,\varepsilon)$ be the residual demand that firm i faces for $p < \bar{p}$. As long as the supply functions of his competitors are non-horizontal at p, firm i's residual demand is given by

$$q_i(p,\varepsilon) = \varepsilon - S_{-i}(p) \quad \text{if } p < \bar{p}. \tag{1}$$

[5] Consider a supply function $S_i(p)$ with a discontinuity at p^*. It is assumed that firm i is willing to produce any supply in the range $\left[S_i\left(p^*\right), S_i\left(p^*+\right)\right]$ if the price is p^*. Thus the left continuous supply function is actually just a representation of a *correspondence*. The same is true for the right continuous supply function of Genc & Reynolds [5].

[6] To get a unique SFE, it should be enough to assume that the support set includes zero demand and $\sum_{i=1}^{N-1} \bar{\varepsilon}_i + \bar{\varepsilon}_{N-1}$. In equilibrium, this would imply that the capacity constraints of the N-1 smallest firms bind with a positive probability. This corresponds to the uniqueness criterion used by Newbery in his study of two asymmetric firms [11].

If more than one producer has a supply function with a perfectly elastic segment at some price p_0, supply rationing at this price is necessary for some demand outcomes. This possibility is addressed in a previous paper [8].

3. THE UNIQUE ASYMMETRIC SFE

The KM first-order condition is a differential equation whose derivation can be found in the existing literature [5,10]. It must be fulfilled by all elastic supply functions in any price interval in which all supply functions are smooth, i.e. twice continuously differentiable. Irregular supply functions do not necessarily fulfil the KM first-order condition and have perfectly elastic segments even if the price cap does not bind or perfectly inelastic segments even if the capacity constraint does not bind.

In Section 3.1, the system of differential equations is solved for a price interval where all supply functions are smooth. The solution contains some undetermined integration constants but is nonetheless useful when irregular SFE candidates are ruled out in Sections 3.2 to 3.5. It is a long proof that involves the seven propositions illustrated in Figure 1.

In Section 3.2, it is proven that no power is offered below marginal cost or withheld from the auction (Proposition i). The latter means that all market capacity is offered at or below the price cap. It is further shown that no firm will have an elastic supply in a price interval for which residual demand is perfectly inelastic (Proposition ii).

Perfectly elastic segments below the price cap (Proposition iii) and discontinuities in the equilibrium price (Proposition iv) are ruled out in Section 3.3. Proposition iii also proves that at most one firm can have a perfectly elastic segment at the price cap. Section 3.4 shows that all producers will offer their first unit of power at marginal cost (Proposition v) which provides an initial condition for the system of differential equations given by the KM first-order condition. Thus, in equilibrium, the supply function of each firm must fulfil the KM first-order condition from marginal cost up to the price at which either the capacity constraint or the price cap bind.

Section 3.5 proves that any two supply functions of an equilibrium are identical up to the price at which one supply function becomes perfectly inelastic (Proposition vi). Using this result it is possible to show that firms with non-binding capacity constraints do not have vertical segments in equilibrium (Proposition vii).

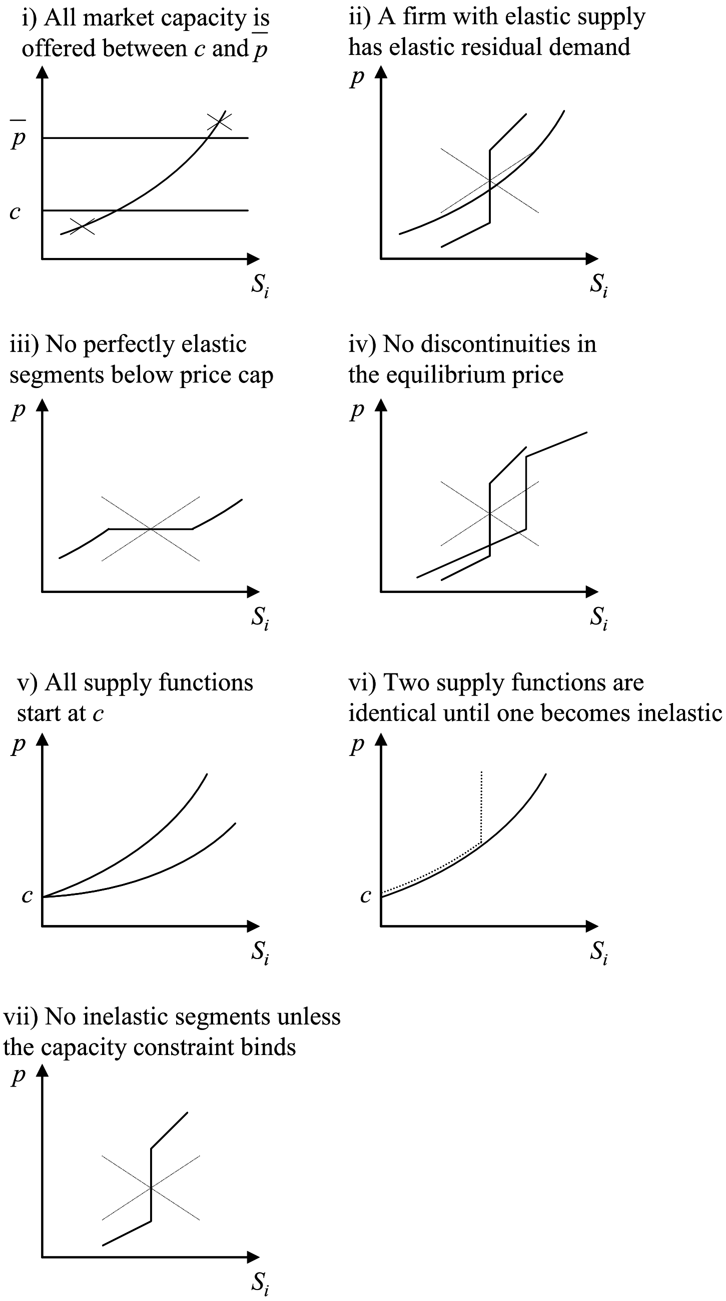

Figure 1. *Graphical illustration of Propositions i-vii, the necessary properties of a supply function equilibrium with constant marginal costs.*

The integration constants in the solution of the first-order condition are determined in Section 3.6. Any two producers will have the same supply function until the capacity constraint of the smaller firm binds (see Proposition vi and Proposition vii). At least two firms must have non-binding capacity constraints up to the price cap (see Proposition ii and iv) and only one firm can have a perfectly elastic segment at the price cap (see Proposition iii). Thus the end-condition is that the capacity constraint of the second largest firm must start to bind at the price cap. The remaining capacity of the largest firm is sold with perfectly elastic supply at the price cap. Only one SFE candidate satisfies both the KM first-order condition and the end-condition.

Section 3.7 verifies that the unique candidate is indeed an equilibrium. If the competitors of firm *i* follow the SFE candidate, then the profit of firm *i* is globally maximised for every demand outcome if it also follows the SFE candidate. This is a sufficient condition for a SFE.

3.1. The first-order condition for smooth parts of the supply functions

Given the bids of competitors, each producer submits his best supply function out of the class of admissible supply functions. Now consider a price interval $[p_-, p_+]$, in which all supply functions $S_i(p)$ of an equilibrium are twice continuously differentiable, i.e. no firm has kinks or perfectly elastic supply in the interval. Denote equilibrium supply functions by $\hat{S}_i(p)$. Assume that there are $M \geq 2$ firms with elastic supply in the interval. Let ***E*** be a set of all firms belonging to this group. Firms not belonging to ***E*** have $\hat{S}_j{}'(p) = 0 \;\; \forall \;\; p \in [p_-, p_+]$. It can be shown that within the price interval, the supply function of each producer with elastic supply is given by the KM first-order condition [8][7]

$$\hat{S}_i(p) - \hat{S}_{-i}{}'(p)(p-c) = 0. \tag{2}$$

Thus the supply functions of firms belonging to ***E*** are given by a system of differential equations. The system can be solved with a two-step approach. First, the total supply function of the *M* producers, $\hat{S}_e(p)$, is calculated. Second, individual supply functions can be derived. We start with the first step. The supply function of each firm in the set ***E*** follows a differential equation as in (2). Summing these equations yields

$$\sum_{i \in E} \hat{S}_i(p) - \sum_{i \in E} \hat{S}_{-i}{}'(p)(p-c) = \hat{S}_e(p) - (M-1)(p-c)\hat{S}_e{}'(p) = 0.$$

[7] The first-order condition derived by Klemperer & Meyer is more general, as it allows for general cost functions and elastic demand [10].

The differential equation is separable and has the following solution,

$$S_e = \beta(p-c)^{1/(M-1)}, \tag{3}$$

where $\beta > 0$ is an integration constant. Now, we can proceed with the second step. For any producer i belonging to the set $\boldsymbol{E}$, (2) can be written in the following form:

$$\hat{S}_i(p) - (p-c)\left(\hat{S}_e{}'(p) - \hat{S}_i{}'(p)\right) = 0.$$

Rewriting it by means of (3) yields

$$\hat{S}_i(p) + \hat{S}_i{}'(p)(p-c) = \frac{\beta}{M-1}(p-c)^{1/(M-1)}.$$

Thus, it follows from the product rule of differentiation that

$$\frac{d}{dp}\left\{\hat{S}_i(p)(p-c)\right\} = \frac{\beta}{M-1}(p-c)^{1/(M-1)}.$$

Integrating both sides yields

$$\hat{S}_i(p) = \frac{\beta}{M}(p-c)^{1/(M-1)} + \frac{\gamma_i}{p-c}. \tag{4}$$

Note that all of the M firms with elastic supply have the same β in the interval $[p_-, p_+]$. On the other hand, they have individual specific constants γ_i. Nevertheless, the individual solutions in (4) must add up to the aggregate solution in (3). Thus

$$\sum_{i=1}^{M} \gamma_i = 0. \tag{5}$$

It follows from (3) that the slope of the aggregate supply function in the interval $[p_-, p_+]$ is given by

$$\hat{S}_e{}'(p) = \frac{\beta}{M-1}(p-c)^{(2-M)/(M-1)}.$$

Thus

$$p'(\varepsilon) = \frac{1}{\frac{d\varepsilon}{dp}} = \frac{1}{\hat{S}_e{}'(p)} = \frac{M-1}{\beta}(p-c)^{(M-2)/(M-1)}. \tag{6}$$

3.2. Basic properties of the supply function equilibrium

Proposition i: In equilibrium, no units are offered below marginal cost or withheld. The latter means that all units are offered at a price equal to or below the price cap.

Proof: See proof of Proposition 1 and 5 in [8].

No power is offered below its marginal cost in equilibrium. A profitable deviation is always to offer this power at its marginal cost. The deviation would cut losses without reducing the contribution from profitable outcomes. Similarly, no power is withheld in equilibrium. It is better to offer this power at the price cap. The deviation will not affect the market price negatively and the firm can increase supply for some demand outcomes.

Proposition ii: In equilibrium, there is no price interval (p_-, p_+), in which a producer with elastic supply faces perfectly inelastic residual demand.

Proof: Assume that firm *i* has an elastic supply in (p_-, p_+), while its residual demand is perfectly inelastic in this interval. Without losing accepted supply, all units previously offered in the range (p_-, p_+) could be offered at a price arbitrarily close to, but still below, p_+. Thus firm *i* would deviate. □

Lemma i below proves a technicality that will be useful in later proofs.

Lemma i: An equilibrium with a smooth transition to a perfectly inelastic or perfectly elastic aggregate supply is not possible if the price p^* is above marginal cost *c*. In the perfectly inelastic case, the result applies also for $p^* \geq c$. In the perfectly elastic case, the result is valid both for individual firms and in aggregate.

Proof: See Appendix.

Given the left continuous and piece-wise smooth properties of supply functions, Proposition ii and Lemma i together imply the following corollary.

Corollary i: For every $p>c$, one can find a sufficiently high $p_-<p$ such that all supply functions are twice continuously differentiable in the interval $[p_-, p]$. Either no firm or at least two firms have elastic supply in the interval.

3.3. Ruling out SFE with perfectly elastic segments and discontinuities in the equilibrium price

Proposition iii: In equilibrium, no firm has a perfectly elastic segment below the price cap. At most, one firm can have a perfectly elastic segment at the price cap.

Proof: See Appendix.

For $p>c$, the intuition of the proof is the same as for the Bertrand equilibrium in which producers undercut each others' perfectly elastic bids as long as price exceeds marginal cost. If $p=c$, it can be shown that a firm will gain by increasing the price for some of the units offered at c.[8]

Lemma i shows that there are no smooth transitions to perfectly elastic supply for $p>c$. Thus it follows from Proposition iii that $S_i'(p)=\infty$ can be ruled out for every $p \in (c, \overline{p})$. This leads to the following corollary.

Corollary ii: For every $p \in (c, \overline{p})$ one can, in equilibrium, find a sufficiently large p_- and a sufficiently low p_+ such that all supply functions $S_i(\cdot)$ are twice continuously differentiable in the intervals $[p_-, p]$ and $[p, p_+]$. Furthermore, all supply functions are continuous at p.

Proposition iv: There are no discontinuities in the equilibrium price.

Proof: Analogous to the proof of Proposition 4 in [8].

Assume that there is a discontinuity in the equilibrium price at $\varepsilon_L \le \overline{\varepsilon}$, at which the price jumps from p_L to p_U, that is, aggregate supply is perfectly inelastic in this price interval. Then any producer with bids just below p_L can increase expected profit by deviating. With a slight sales reduction, the price for units offered at and just below p_L can be significantly increased to just below p_U instead. Isolated prices for which all supply functions are perfectly inelastic are ruled out by Lemma i. Thus Propositions ii and iv imply the following:

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[8] Note that equilibria with $\lim_{p\to c} S_i'(p)=\infty$ cannot be ruled out if $\lim_{p\to c} S_i(p)=0$.

Corollary iii: From the lowest bid to the price cap, there must be at least two firms with elastic supply in each price interval. Moreover, $p'(\varepsilon)$ is finite for $\varepsilon \in (0, \bar{\varepsilon})$.

3.4. Every firm offers its first unit of power at marginal cost

In Lemma ii (below) it is shown that at least two producers will offer their first unit of power at $p=c$. This result is employed in Proposition v to prove that all producers must offer their first unit of power at $p=c$. The intuition underlying this result is that the bid of the first unit is never price-setting for other units of a firm. Thus the first unit is sold under Bertrand competition.

Lemma ii: In equilibrium, the first unit of power is offered at c.

Proof: Denote the lowest offer in the total supply by p^*. Assume that $p^*>c$. According to Proposition iii, at most one producer has a supply function with a perfectly elastic segment at the price cap. Thus $c< p^*< \bar{p}$. There are three additional implications from Section 3.3. First, perfectly elastic segments at p^* can be excluded, according to Proposition iii. Second, Corollary ii implies that a price p_+ can be found such that all producers have twice continuously differentiable supply functions in the interval $[p^*, p_+]$. Third, according to Corollary iii, at least two firms have elastic supply in the whole interval if p_+ is sufficiently small. The aggregate supply of the firms with elastic supply is given by (3). Thus $p^*>c$ can be excluded as it would imply $S_e(p^*) > 0$. Thus $p^*=c$ according to Proposition i. □

Proposition v: In equilibrium, there must be some $p_+>c$ such that all producers have elastic supply functions in the interval $[c, p_+]$.

Proof: Assume that there is a potential equilibrium in which producer i offers his first unit at the price $p^*>c$. Denote the aggregate supply and the equilibrium price of the potential equilibrium by $S^A(p)$ and $p^A(\varepsilon)$. It follows from Lemma ii that $p^A(0)=c$. Now consider the following deviation. Producer i reduces the price for an infinitesimally small unit of power so that his first unit of power is offered at the price $p \in (c, p^*)$. For each unit of deviated power, the deviation leads to the following marginal change in expected profit:

$$\int_{S^A(p)}^{S^A(p^*)} \left[p^A(\varepsilon)-c\right]f(\varepsilon)d\varepsilon. \tag{7}$$

It follows from Corollary iii that $S^{A'}(p)>0$ for $p \in \left[p, p^*\right]$. Thus $S^A(p^*)-S^A(p)>0$ and the deviation is profitable. Accordingly, equilibria where producer i offers his first unit of power at a price $p^*>c$ can be eliminated. Supply functions with a perfectly elastic segment at c are excluded by Proposition iii. Thus the first unit of power of producer i must be offered with an elastic supply function in some interval $[c, p_+]$, which is true for all producers. □

3.5. Each producer has an elastic supply, unless his capacity constraint binds

In this section, it is shown that a firm has perfectly inelastic segments in its supply function only when its capacity constraint is binding.[9] The intuition behind this result is somewhat involved. Assume that no producer with a non-binding capacity constraint has perfectly inelastic supply below p_L. Assume further that supply of producer i and possibly some of his competitors becomes perfectly inelastic just above p_L. According to Corollaries ii and iii, there must be at least two producers with elastic supply functions that follow the KM first-order condition just above p_L. Denote this set of elastic supply functions by $\boldsymbol{S}$. It follows from the KM first-order condition in (2) that any supply function in $\boldsymbol{S}$ must face a continuous derivative of its residual demand at the price p_L. Thus to compensate for the switch to perfectly inelastic supply by producer i, the elasticity of supply functions in $\boldsymbol{S}$ must increase discontinuously at p_L. Likewise, this increases the elasticity of the residual demand of producer i discontinuously at p_L. As a result, the producer wants to sell more units just above p_L. Accordingly, he deviates unless his capacity constraint binds.

To accomplish the proof, it is first shown that any two firms have identical supply functions up to the price at which either has a perfectly inelastic segment.

Proposition vi: In equilibrium, any two producers have identical supply functions in the interval $[c, p]$, where $p<\overline{p}$, if neither of them have supply functions with perfectly inelastic segments in this interval.

[9] A similar result is proven by Baldick & Hogan for strictly elastic demand and general cost functions [3]. However, they only consider cases where the residual demand is smooth.

Proof: Without loss of generality, denote the two producers by 1 and 2. According to Proposition iii, neither producer has a supply function with perfectly elastic segments in the interval $\left[c, p\right]$. As it is also assumed that neither have perfectly inelastic segments in the interval, the supply functions of both producers follow the KM first-order condition in the whole interval. The number of competitors with elastic supply may change in the interval, but the supply functions of firm 1 and 2 continue to be piece-wise solutions of the type in (4) over the whole interval $\left[c, p\right]$. Proposition v implies that the two firms have the same initial condition $S_i(c) = 0$ and according to Corollary ii both supply functions are continuous. Thus firm 1 and 2 will have identical γ_i over the whole interval $\left[c, p\right]$. □

Proposition vii: There is no equilibrium for which the supply function of a producer is perfectly inelastic in an interval $\left[p_L, p_U\right]$, where $c \le p_L < p_U \le \overline{p}$, unless his capacity constraint is binding.

Proof: Denote the potential equilibrium by the superscript *B*. Let ***I*** be a set with firms that have supply functions with perfectly inelastic segments before their capacity constraints bind. Define p_L by the following: no firm in ***I*** has a perfectly inelastic segment below p_L, but at least one firm in ***I*** starts being inelastic at p_L. Assume that producer *i* is one of these firms and that its supply is perfectly inelastic for $p \in \left[p_L, p_U\right]$. Proposition vi implies that all producers with a non-binding capacity constraint must have identical supply functions in the interval $\left[c, p_L\right]$.

According to Corollary iii, there must, for every price $p \in \left[p_L, p_U\right]$, be at least two producers — not necessarily the same over the whole interval — with elastic supply. Thus for a subinterval $\left[p_I, p_{II}\right] \subseteq \left[p_L, p_U\right]$, where a firm *j≠i* has an elastic supply function, its supply function must satisfy (2), i.e.

$$S_j^B(p) - S_{-j}^{B'}(p)(p-c) = 0 \ \forall \ p \in \left[p_I, p_{II}\right].$$

Because producer *i* has perfectly inelastic supply in the interval $\left[p_L, p_U\right]$, it follows that $S_{-i}^{B'}(p) > S_{-j}^{B'}(p)$ for $\forall \ p \in \left(p_I, p_{II}\right)$. Furthermore, as $S_j^B(p_L) = S_i^B(p_L)$, $S_j^B(p) > S_i^B(p)$ for $\forall \ p \in \left(p_I, p_{II}\right)$. Thus

$$S_i^B(p) - S_{-i}^{B'}(p)(p-c) < 0 \ \forall \ p \in \left(p_I, p_{II}\right]. \tag{8}$$

Now consider the following deviation. Producer *i* decreases the price for an infinitesimally small unit of power previously offered at the price p_U and offers it at p_L instead. The supply function is unchanged above p_U and below p_L. Per marginal unit of deviated power, the deviation leads to the following marginal change in firm *i*'s expected profit:

$$\Delta E(\pi_i) = \int_{S^B(p_L)}^{S^B(p_U)} \left\{ \left[p^B(\varepsilon) - c \right] - S_i^B(p_L) p^{B'}(\varepsilon) \right\} f(\varepsilon) d\varepsilon. \tag{9}$$

The first term is due to increased sales and the second term due to the reduced price in the demand interval. As the supply of producer *i* is perfectly inelastic in the interval under consideration, (9) can be rewritten in the following way:

$$\Delta E(\pi_i) = \int_{S^B(p_L)}^{S^B(p_U)} \left\{ \left[p^B(\varepsilon) - c \right] - \frac{S_i^B(p_L)}{S_{-i}^{B'}(p_L)} \right\} f(\varepsilon) d\varepsilon.$$

There is a producer $j \neq i$ with an elastic supply at each $p \in [p_L, p_U]$. Hence, it follows from (8) that $\Delta E(\pi_i) > 0$, and a profitable deviation exists. In equilibrium, firm *i* cannot have a perfectly inelastic segment in the interval $[p_L, p_U]$ unless its capacity constraint binds. □

Proposition vii rules out perfectly inelastic segments in the supply of a firm unless its capacity constraint binds so the following corollary can be concluded by means of Propositions iii, v and vi.

Corollary iv: The supply of each firm is (i) elastic, (ii) has no perfectly elastic segments, (iii) follows the KM first-order condition in (2), and (iv) is identical to the supply of the largest firm from marginal cost up to the price at which either its capacity constraint or the price cap binds.

3.6. A unique SFE candidate that satisfies the necessary conditions

Let p_i denote the price at which the capacity constraint of firm *i* starts to bind. Recall that $\bar{\varepsilon}_1 < \bar{\varepsilon}_2 < \ldots < \bar{\varepsilon}_N$. Thus Proposition iii and Corollaries iii and iv together imply the following:

Corollary v: The supply function of the largest firm has a perfectly elastic segment at the price cap and $c < p_1 < \ldots < p_{N-1} = \bar{p}$.

An intuitive explanation of the result that the capacity constraints of small firms bind at lower prices is that small firms have less market-power and lower mark-ups for any percentage of their capacity, including full capacity.

Consider the first price-interval $[c, p_1]$, where all producers have non-binding capacity constraints. As all have identical supply functions within the interval, it follows from (4) and (5) that all firms have $\gamma_i = 0$ in the interval. Thus (4) yields

$$S_j(p) = \frac{\beta_1 (p-c)^{1/(N-1)}}{N} \qquad \text{for } j=1, 2, \ldots, N. \tag{10}$$

The subscript 1 on β is used to indicate that the constant is valid for the first price interval. In the next price-interval $[p_1, p_2]$, there are *N-1* remaining producers with non-binding capacity constraints. Following the line of argument used for the first interval, one can conclude that $\gamma_i = 0$ also in this interval. With *M=N-1* it follows (4) from that

$$S_j(p) = \frac{\beta_2 (p-c)^{1/(N-2)}}{N-1}, \text{if } p \in [p_1, p_2] \text{ and } j = 2...N.$$

Analogously, the solution for the price interval $[p_{n-1}, p_n]$ is

$$S_j(p) = \frac{\beta_n (p-c)^{1/(N-n)}}{N-n+1}, \text{if } p \in [p_{n-1}, p_n] \text{ and } j = n...N, \tag{11}$$

where $n = 1 \ldots N-1$ and $p_0=c$. The latter is relevant for n=1. Combining the end-condition $p_{N-1} = \overline{p}$ with (11) yields

$$\beta_{N-1} = \frac{2\overline{\varepsilon}_{N-1}}{\overline{p}-c}. \tag{12}$$

Thus β_{N-1} can be uniquely determined. To avoid discontinuities in the supply functions and equilibrium price — which would violate Corollaries ii and iii, respectively — the following relations must be fulfilled at the boundary between two price intervals:

$$S_j(p_n) = \frac{\beta_n (p_n - c)^{1/(N-n)}}{N-n+1} = \frac{\beta_{n+1} (p_n - c)^{1/(N-n-1)}}{N-n}.$$

Thus

$$p_n = \left(\frac{(N-n)\overline{\varepsilon}_n}{\beta_{n+1}} \right)^{N-n-1} + c \tag{13}$$

and

$$\beta_n = \frac{(N-n+1)\beta_{n+1} (p_n - c)^{1/(N-n-1)-1/(N-n)}}{N-n}. \tag{14}$$

Accordingly, starting with (12), all β_n can be uniquely determined by iterative use of (13) and (14). Thus there is only one candidate that satisfies the necessary conditions for a SFE.

3.7. The only remaining equilibrium candidate is a SFE

Consider an arbitrary producer *i*. Assume that his competitors follow the only remaining SFE candidate, i.e. their total supply equals $\breve{S}_{-i}(p)$. Then it must be a best response of producer *i* to follow the equilibrium candidate, otherwise the candidate is not an equilibrium. To show best response, it is sufficient to show that $\breve{S}_i(p)$ maximises firm *i*'s profit for every demand outcome.

For $\varepsilon > \bar{\varepsilon}$, producer *i* will sell all capacity at the maximum price, as long as no power is withheld. Thus, for these outcomes, no profitable deviation from the equilibrium candidate exists. Given $\breve{S}_{-i}(p)$, for $\varepsilon \in [\bar{\varepsilon}_i, \bar{\varepsilon}]$ there is some sufficiently low price $\tilde{p}(\varepsilon)$ such that the capacity constraint of producer *i* binds if his last unit is offered at or below $\tilde{p}(\varepsilon)$. It is never a profitable deviation for producer *i* to push down the price below $\tilde{p}(\varepsilon)$, as the firm's supply cannot increase beyond the capacity constraint. Thus for $\varepsilon \in [\bar{\varepsilon}_i, \bar{\varepsilon}]$, the optimal market price for producer *i* must be in the range $p \in [\tilde{p}(\varepsilon), \bar{p}]$. For $\varepsilon \in [0, \bar{\varepsilon}_i]$, one can set $\tilde{p}(\varepsilon) = c,$ as it is never optimal to offer power below its marginal cost (see Proposition i).

Competitors follow the only remaining equilibrium candidate. Hence, according to Corollary iv, competitors do not have supply functions with perfectly elastic segments below the price cap. Thus for $p \in [\tilde{p}(\varepsilon), \bar{p})$, residual demand of an arbitrary producer *i* is given by (1). Hence, for given demand and price, the profit of producer *i* is

$$\pi_i(\varepsilon, p) = \underbrace{[\varepsilon - \breve{S}_{-i}(p)]}_{S_i}(p - c), \quad \text{if } p \in [\tilde{p}(\varepsilon), \bar{p}). \tag{15}$$

Thus

$$\frac{\partial \pi_i(\varepsilon, p)}{\partial p} = \underbrace{[\varepsilon - \breve{S}_{-i}(p)]}_{S_i} - (p - c)\breve{S}_{-i}{}'(p), \text{if } p \in [\tilde{p}(\varepsilon), \bar{p}). \tag{16}$$

With left-hand derivatives, the result is also valid for $p = \bar{p}$. It follows from (2) that

$$\breve{S}_N(p) - \breve{S}_{-N}{}'(p)(p-c) = 0, \text{if } p \in \left[c, \bar{p}\right). \tag{17}$$

$\breve{S}_N(p) = \breve{S}_i(p)$ if $p \in [c, p_i]$ and $\breve{S}_i{}'(p) = 0$ if $p \in \left(p_i, \bar{p}\right]$. Thus subtracting (17) from (16) yields:

$$\frac{\partial \pi_i(p,\varepsilon)}{\partial p} = \begin{cases} \underbrace{\left[\varepsilon - \breve{S}_{-i}(p)\right]}_{S_i} - \breve{S}_N(p), \text{if } p \in \left[\tilde{p}(\varepsilon), p_i\right) \\ \underbrace{\left[\varepsilon - \breve{S}_{-i}(p)\right]}_{S_i} - \breve{S}_N(p) - (p-c)\breve{S}_N{}'(p), \text{if } p \in \left(p_i, \bar{p}\right]. \end{cases} \tag{18}$$

As supply functions are piece-wise symmetric, it follows from (17) that

$$(p-c)\breve{S}'_N(p) = \frac{\breve{S}_N(p)}{M-1},$$

where $M \geq 2$ is the number of elastic producers in the interval. Thus (18) can be written

$$\frac{\partial \pi_i(p,\varepsilon)}{\partial p} = \begin{cases} \underbrace{\left[\varepsilon - \breve{S}_{-i}(p)\right]}_{S_i} - \breve{S}_N(p), \text{if } p \in \left[\tilde{p}(\varepsilon), p_i\right) \\ \underbrace{\left[\varepsilon - \breve{S}_{-i}(p)\right]}_{S_i} - \breve{S}_N(p) - \frac{\breve{S}_N(p)}{M-1}, \text{if } p \in \left(p_i, \bar{p}\right]. \end{cases} \tag{19}$$

Accordingly, $\frac{\partial \pi_i(p,\varepsilon)}{\partial p}$ is monotonically decreasing in p within the interval $\left[\tilde{p}(\varepsilon), \bar{p}\right]$. Thus for outcomes $\varepsilon \leq \breve{S}(p_i)$, producer i maximises profit by choosing the price such that $\frac{\partial \pi_i(p,\varepsilon)}{\partial p} = 0$. This is achieved by following $\breve{S}_i(p)$, see (19), as $\breve{S}_i(p) = \breve{S}_N(p)$ for $p \leq p_i$.

For $\varepsilon > \breve{S}(p_i)$, it follows that $\tilde{p}(\varepsilon) > p_i$. In this case, $\frac{\partial \pi_i(p,\varepsilon)}{\partial p} < 0$, as $\breve{S}_N(p) > \bar{\varepsilon}_i$ for $p \in \left[\tilde{p}(\varepsilon), \bar{p}\right]$. Then producer i maximises profit by maximising $\frac{\partial \pi_i(p,\varepsilon)}{\partial p}$. According to (19), this is achieved by following $\breve{S}_i(p)$, as $\breve{S}_i(p) = \bar{\varepsilon}_i$ if $p > p_i$.

Now, consider producer N and the case $\varepsilon \in \left(\breve{S}\left(\bar{p}\right), \bar{\varepsilon}\right]$.[10] It follows from (19) that $\frac{\partial \pi_N(p,\varepsilon)}{\partial p} > 0$ for $p \in \left[\tilde{p}(\varepsilon), \bar{p}\right)$ and that $\frac{\partial \pi_{Nl}\left(\bar{p},\varepsilon\right)}{\partial p} > 0$ (the left derivative). However,

---

[10] Recall that supply functions are left continuous. Thus $\breve{S}\left(\bar{p}\right)$ does not include the largest firm's perfectly elastic supply at the price cap.

profit cannot be raised as price cannot be increased beyond the price cap. Therefore, producer N cannot do better than follow $\breve{S}_N(p)$.

The conclusion is that given $\breve{S}_{-i}(p)$, where i=1,2,...,N, the unique equilibrium candidate $\breve{S}_i(p)$ globally maximises the profit for every demand outcome ε. Thus the unique equilibrium candidate is a SFE.

4. EXAMPLE 1 — THREE ASYMMETRIC PRODUCERS

Consider a market with three producers. The producers have identical and constant marginal cost c. Assume that the producers have $\frac{1}{6}$, $\frac{1}{3}$ and $\frac{1}{2}$ of total capacity $\bar{\varepsilon}$ and that $\bar{p} = 3c$. Order the producers according to their production capacity so that producer 1 has the smallest capacity.

Firms 2 and 3 are symmetric up to the price cap, at which the capacity constraint of firm 2 starts to bind, i.e. $S_2(\bar{p}) = S_3(\bar{p}) = \frac{\bar{\varepsilon}}{3}$. The remaining capacity of firm 3 is sold at the price cap. Thus

$$p = \bar{p} = 3c, \text{ when } \frac{\bar{\varepsilon}}{3} \le S_3 \le \frac{\bar{\varepsilon}}{2}. \tag{20}$$

β_2 can be calculated from (12):

$$\beta_2 = \frac{2\bar{\varepsilon}/3}{2c} = \frac{\bar{\varepsilon}}{3c}.$$

Thus according to (11),

$$S_2(p) = S_3(p) = \frac{\bar{\varepsilon}(p-c)}{6c}, \text{ if } p \in [p_1, \bar{p}), \tag{21}$$

where p_1 is the price at which the capacity constraint of firm 1 binds. By means of (13) it can be shown that $p_1 = 2c$. Now (14) can be used to calculate β_1:

$$\beta_1 = \frac{3\beta_2 c^{1/2}}{2} = \frac{\bar{\varepsilon}}{2c^{1/2}}.$$

According to (11)

$$S_1(p) = S_2(p) = S_3(p) = \frac{\bar{\varepsilon}}{6}\left(\frac{p-c}{c}\right)^{1/2}, \text{ if } p \in [c, p_1]. \tag{22}$$

The unique SFE, which is characterised by equations (20) to (22), is presented in Figure 2. All supply functions are symmetric up to the price p_1, at which point the capacity constraint of the smallest firm binds. Above this price, firms 2 and 3 have symmetric supply functions up to $\bar{p} = 3c$, at which point the capacity constraint of producer 2 binds. The remaining supply of producer 3 is offered with perfect elasticity at $p = \bar{p}$. The equilibrium is piece-wise symmetric as in the SFE derived by Newbery for a duopoly facing linear demand [11].

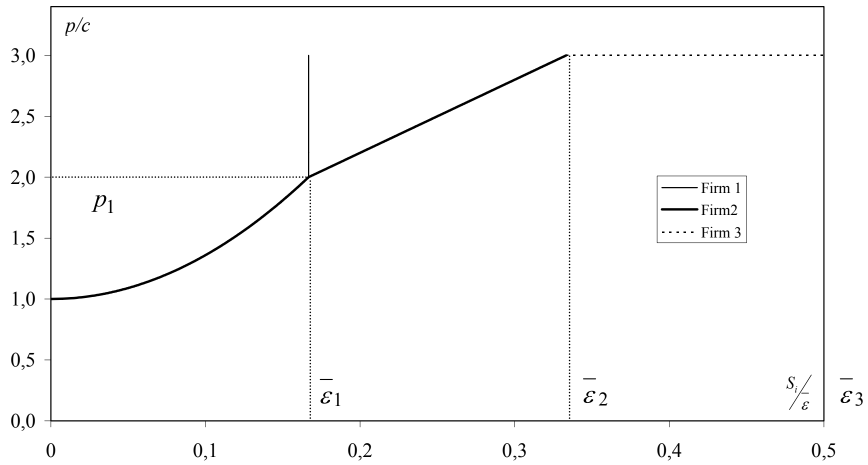

Figure 2. *The unique supply function equilibrium is piece-wise symmetric if firms have identical constant marginal costs c. The capacities of the firms are such that* $\bar{\varepsilon}_1 < \bar{\varepsilon}_2 < \bar{\varepsilon}_3$.

The supply functions of firm 2 and 3 have kinks at p_1; their elasticities increase discontinuously at this price. This compensates the kink in the supply of firm 1 and ensures that the elasticity of the residual demand of both producer 2 and 3 is continuous at p_1.

In Figure 2 we can note that the supply functions of all producers become perfectly elastic at the point where the price approaches c in the limit. By differentiating (10), it is straight-forward to verify that this is a general result for identical and constant marginal costs. The intuition for this result is that when the producers sell their first unit, its price will not influence the profit from other units. Thus, competitors undercut each others' bids for the first unit down to marginal cost, as if under Bertrand competition.

5. EXAMPLE 2 — THE NORWEGIAN REAL-TIME MARKET

More than 99 percent of electric power production in Norway is hydroelectric. Because nearly all power is produced with the same technology, the Norwegian real-time market is a suitable application of the model in this essay. At present, Norway shares a common electric power market with the other Nordic countries, but in this simplified example it is assumed that Norway has a power market of its own, as it did before 1996. It is also assumed that all power is sold in real-time, i.e. forward and futures markets are ignored.

The marginal cost of hydropower is very small, roughly c=50 NOK/MWh (≈6 €/MWh). But water is a limited resource. Thus hydropower bids are often driven by opportunity cost, the revenue from selling a unit of hydropower at a later day/hour. To avoid this complication, we consider an hour in the late spring when the alternative to power production is to spill water. The price cap is 50 000 NOK/MWh (≈6000 €/MWh). The calculation of the SFE is based on the installed capacity of the 153 largest hydropower producers in Norway. The remaining firms and non-hydroelectric power are not included in the sample. The 10 largest producers in Norway and their share of the installed capacity are listed in Table 1.

Table 1: *Share of installed hydroelectric capacity of Norway's 10 largest power producers*

Company	Share of installed capacity	HHI
Statkraft	31,7%	1001,986
E-CO	7,3%	53,02816
Lyse	5,6%	31,04265
BKK	5,6%	30,92214
Norsk Hydro	4,8%	23,3467
Agder Energi	4,3%	18,37795
Skagerak Kraft	3,8%	14,52085
Otra Kraft	3,1%	9,856047
Trondheim Energiverk	2,7%	7,246727
Nord-Trøndelag Elverk	2,0%	4,20108
Rest (143 firms)	29,1%	27,2
Total	100%	1222

Source: Data provided by the *Norwegian Water Resources and Energy Directorate (NVE)*.

Cross-ownership in Norway is extensive. Statkraft has a majority stake in Skagerak Kraft and Trondheim Energiverk, Agder Energi has a majority stake in Otra kraft, and Norsk Hydro has a majority stake in Røldal-Suldal Kraft. As there is no available theory for the treatment of cross-ownership in a SFE analysis, all cross-ownerships are disregarded even though they may be relevant for mark-ups.

The Herfindahl-Hirschman index (HHI) is a commonly used measure of market concentration and is given by the sum of market shares (in percentages) squared [14]. The Norwegian electric power market has HHI=1222 which corresponds to slightly less than 8 symmetric firms (HHI=1250). In a traditional Cournot model with certain demand, the market mark-up would be the same for 8 symmetric firms as for the 153 asymmetric firms [14]. Thus it is interesting to compare the supply function equilibria of these two cases. By means of the formulae in Section 3.6, it is straightforward to calculate the supply of all asymmetric firms and the 8 symmetric firms. In the symmetric case, the capacity constraint of all firms will start to bind at the price cap [8].

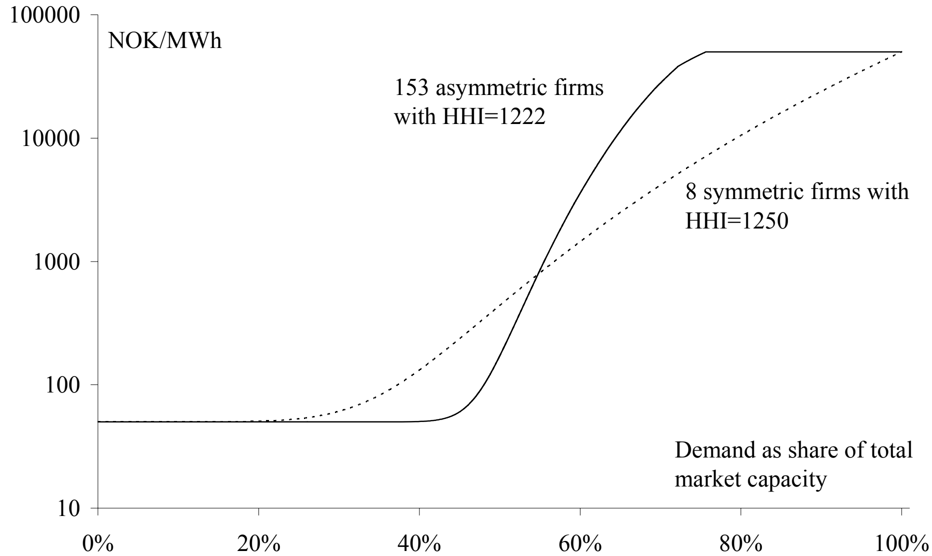

Figure 3. *The unique supply function equilibrium of the real-time market in Norway compared to a case with 8 symmetric firms.*

For the asymmetric model, Figure 3 shows that mark-ups are modest, less than 10 percent for the first 44 percent of capacity. For the last 52 percent of the capacity, mark-ups are excessive, larger than 100 percent. For small demands, competition is much tougher in the asymmetric case than the symmetric case, as most asymmetric firms have non-binding capacity constraints. On the other hand, the competition is tougher in the symmetric case for large demands when most asymmetric firms have binding capacity constraints. Assuming that this intuition also holds for non-constant marginal costs, it is likely that the symmetric model will overestimate mark-ups during a summer night, when demand in Norway is, on average, 30 to 35 percent of installed capacity. On the other hand, it is likely that the symmetric model will underestimate mark-ups during a winter day, when demand is, on average, 60 to 70 percent of installed capacity.

6. CONCLUSIONS

A unique Supply Function Equilibrium (SFE) is derived for an electric power market in which producers have identical and constant marginal costs and production capacities are asymmetric. I assume that there is a positive probability that demand exceeds total market capacity.[11] The unique equilibrium is piece-wise symmetric; two arbitrary producers have the same supply function until the capacity constraint of the smaller firm binds. The constraint of the producer with the second largest capacity starts to bind at the price cap, while the capacity constraints of smaller firms bind below the price cap. The largest firm offers its remaining capacity as a perfectly elastic supply at the price cap.

At the price at which the capacity constraint of a small firm starts to bind, the elasticity of the supply of larger firms will increase discontinuously. This ensures that larger firms face a continuous elasticity of their residual demand. The unique equilibrium has supply functions with kinks and vertical and horizontal segments. This implies that one cannot limit attention to smooth supply functions, as is routinely done in the SFE literature.

The equilibrium will only be piece-wise symmetric if costs are. However, in other aspects the qualitative properties of the asymmetric SFE are conjectured to hold for a more general class of cost functions [9]. This understanding can be used to numerically calculate valid SFE — with non-decreasing supply functions — for asymmetric firms with increasing marginal costs as in [9]. Without this conjecture it has proven to be difficult to numerically calculate asymmetric SFE [3].

Compared to a symmetric SFE with the same Herfindahl-Hirschman Index (HHI), the asymmetric SFE is much more competitive for small demand outcomes when the capacity constraints of few asymmetric firms bind. On the other hand, the asymmetric SFE is less competitive for large demand outcomes when the capacity constraints of many asymmetric firms bind. Intuitively, this result should also hold for increasing cost functions.

[11] It is enough to assume that the capacity constraints of all but the largest firm bind with a positive probability.

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APPENDIX

Proof of Lemma i: Consider a smooth transition from the left. According to the assumed properties of the supply functions, a sufficiently large p_- can be chosen such that all supply functions are twice continuously differentiable in the range $\left[p_-, p^*\right)$. In addition, p_- can be chosen such that each supply function is either monotonically increasing in the whole range or perfectly inelastic in the whole range. If all supply functions are perfectly inelastic in the interval $\left[p_-, p^*\right)$, smooth transitions are not possible. Thus according to Proposition ii there must be $M \geq 2$ producers with monotonically increasing supply functions in the range $p \in \left[p_-, p^*\right)$.

It follows from (6) that one can find numbers $\underline{m} > 0$ and $\overline{m} < \infty$ such that $\underline{m} \leq p'(\varepsilon) \leq \overline{m}$, for every $p(\varepsilon) \in \left[p_-, p^*\right)$ if $p_- > c$. For the upper boundary this is true also when $p_- \geq c$. Thus smooth transitions from the left to a perfectly elastic aggregate supply are not possible if $p^* > c$. This is valid for individual producers as well, as $p'(\varepsilon) = 0$ if the marginal bidder has a perfectly elastic supply. Similarly, smooth transitions from the left to a perfectly inelastic aggregate supply are not possible if $p^* \geq c$.

Analogous proofs can be performed to rule out smooth transitions from the right. □

Proof of Proposition iii. The proposition can be divided into three claims:

a) In equilibrium, two or more firms cannot have supply functions with perfectly elastic segments at the same price $p^* \in \left(c, \overline{p}\right]$

b) In equilibrium, no producer has a supply function with a perfectly elastic segment at the marginal cost *c*.

c) In equilibrium, one firm cannot have a perfectly elastic segment at $p^* \in \left(c, \overline{p}\right)$.

Proof of a) Due to Corollary i, the proof is almost identical to the proofs of Proposition 2 and 3 in [8].

Proof of b) Assume that in equilibrium, producer *i* offers $S_i^Q(c+) = \lim_{p \to c+} S_i^Q(p) > 0$ units of power with perfectly elastic supply at the constant marginal cost *c*. The aggregate supply of his competitors at this price is denoted by $S_{-i}^Q(c+) \geq 0$. Thus producer *i* may be the only firm with a perfectly elastic segment at *c*.

Assume first that aggregate supply is elastic just above *c*. Then Corollary i implies that $M \geq 2$ firms are elastic just above *c*. The assumed properties of the supply functions ensure that a sufficiently low p_+ can be chosen so that in equilibrium, all supply functions are twice continuously differentiable in the interval $\left(c, p_+\right)$, and the same $M \geq 2$ producers have an elastic supply in the price interval. The other *N-M* producers have a perfectly inelastic supply over the whole interval. The supply functions of the *M* producers are given by (2). The aggregate supply of this group is denoted by $S_e^Q(p)$. It follows from (3) that

$$S_e^Q(p) = \beta(p - c)^{1/(M-1)}. \tag{23}$$

Thus $S_e^Q(p)$ approaches zero as the price approaches *c*. Accordingly producer *i* and any other producers with a perfectly elastic segment at *c* cannot belong to the group with elastic supply in the interval $\left(c, p_+\right)$. Hence, their supply is perfectly inelastic in this interval.

Now consider the following marginal deviation of producer *i*. The price of an infinitesimally small unit, previously offered at *c,* is increased to the price $p^* \in \left(c, p_+\right]$. The marginal change in expected profit is given by

$$\Delta E(\pi_i)=\int_{S^Q(c+)}^{S^Q(p^*)}\left\{S_i{}^Q(c+)p^{Q'}(\varepsilon)-\left[p^Q(\varepsilon)-c\right]\right\}f(\varepsilon)d\varepsilon.$$

The first term reflects an increased price and the second term reflects reduced sales in the demand interval. By means of (6), the integral can be written as

$$\Delta E(\pi_i)=\int_{S^Q(c+)}^{S^Q(p^*)}\left\{S_i{}^Q(c+)\frac{M-1}{\beta}\left(p^Q(\varepsilon)-c\right)^{(M-2)/(M-1)}-\left[p^Q(\varepsilon)-c\right]\right\}f(\varepsilon)d\varepsilon=$$

$$=\int_{S^Q(c+)}^{S^Q(p^*)}\left[p^Q(\varepsilon)-c\right]\left\{S_i{}^Q(c+)\frac{M-1}{\beta}\left(p^Q(\varepsilon)-c\right)^{(M-2)/(M-1)-1}-1\right\}f(\varepsilon)d\varepsilon.$$

The value $\left(p^Q(\varepsilon)-c\right)^{(M-2)/(M-1)-1}$ can be made arbitrarily large for sufficiently small $p^Q>c$. Thus $\Delta E(\pi_i)>0$ for a sufficiently small p^*. Accordingly, there are profitable deviations. Equilibria where $S_i^Q(c+)>0$ can be ruled out if aggregate supply is elastic just above *c*.

Next consider the case where total supply is perfectly inelastic just above *c.* Smooth transitions to a perfectly inelastic supply are ruled out by Lemma i. Thus if p_+ is chosen sufficiently small, the bids of all producers are now perfectly inelastic over the whole price range $(c,p_+]$. Denote by ε_0 the demand outcome for which the last unit offered at the price *c* is sold. For the assumed equilibrium there is no contribution to expected profit from demands below ε_0. Now consider the following unilateral deviation of producer *i.* Increase the price for the $S_i^Q(c+)$ units to $p^*\in(c,p_+)$. This action increases the expected profit of producer *i* for demand outcomes below ε_0. Supply above p_+ is not affected, nor is the contribution to expected profit from demands above ε_0. As a result, the deviation increases expected profit of producer *i*. Accordingly, equilibria where a producer offers $S_i^Q(c+)>0$ units of power at the constant marginal cost *c* can also be excluded if aggregate supply is perfectly inelastic just above *c*. □

Proof of c: Denote the potential equilibrium by the superscript *W*. It is assumed that $p^W(\varepsilon)=p^*$, if and only if $\varepsilon\in[\varepsilon',\varepsilon'']$. Let firm *i* be the producer with the perfectly elastic segment.

All firms cannot have perfectly inelastic supply just above p^*, as, in that case, firm *i* would deviate. The price of the perfectly elastic segment can be increased without losing sales. Furthermore, smooth transitions to an aggregate perfectly inelastic supply are excluded by

Lemma i. Thus it follows from Corollary i that at least two producers have an elastic supply just above p^*, i.e. $S_{-ir}^{W\,\prime}\left(p^*\right) > 0$ (the right-hand derivative). However, any competitor $j \neq i$ with an elastic bid just above p^* would find it profitable to deviate. He can slightly reduce the price of his units offered just above p^* and instead offer them just below p^*. The marginal change in expected profit of producer j from deviating by one, infinitesimally small unit is

$$\int_{\varepsilon'}^{\varepsilon''} \left(p^* - c\right) f(\varepsilon) d\varepsilon.$$

The deviation is profitable because $p^* > c$. □

Paper III

Numerical Calculation of an Asymmetric Supply Function Equilibrium with Capacity Constraints[1]

Pär Holmberg[2]

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Abstract

Producers submit supply functions to a procurement auction, e.g. an electricity auction, before uncertain demand has been realised. In the Supply Function Equilibrium (SFE), every firm commits to the bid that maximises its expected profit, given the bids of competitors. In the case of asymmetric producers with general cost functions, previous work has shown that it is very difficult to find valid SFE candidates. This paper presents a new numerical procedure that provides a solution. It is comprised of numerical integration and an optimisation algorithm that searches for an end-condition. The procedure is illustrated by an example with three asymmetric firms.

Keywords: supply function equilibrium, uniform-price auction, numerical integration, oligopoly, asymmetry, capacity constraint, wholesale electricity market

JEL codes: C61, D43, D44, L11, L13, L94

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[2] Department of Economics, Uppsala University, P.O. Box 513, SE-751 20 Uppsala, Sweden, phone +46 18 471 76 35, fax +46 18 471 14 78. E-mail: par.holmberg@nek.uu.se.

1. INTRODUCTION

Klemperer & Meyer introduced the Supply Function Equilibrium (SFE), in which producers submit bids to a uniform-price auction in a one-shot game [8]. In the non-cooperative Nash Equilibrium, each producer commits to the supply function that maximises his expected profit given the bids of competitors and the properties of uncertain demand. Since Bolle [2] and Green & Newbery [5] observed that the framework is similar to the organisation of most electricity markets, the equilibrium has often been applied to model bidding behaviour in electric power auctions. More broadly, the SFE can be applied to any uniform-price auction where costs/valuations are certain and common knowledge among bidders, quantity discreteness is negligible — that is, objects are divisible [11] — and the demand/supply of the auctioneer is uncertain.

Klemperer & Meyer showed that all smooth SFE are characterised by a differential equation, in this paper labelled the KM first-order condition. In the general case, there exists a continuum of possible supply function equilibria that fulfil this first-order condition [8]. However, by considering capacity constraints, the set of SFE candidates can be drastically reduced [4], at least in the presence of perfectly inelastic demand. The set of SFE can be further reduced by allowing for the risk of extreme demand outcomes, i.e. situations when the capacity constraints of all but the largest firm bind with a positive probability. It is then possible to show analytically that a unique equilibrium exists in at least two specific cases. One such case is that of symmetric producers with strictly convex cost functions [6]. The other involves producers with identical constant marginal costs but asymmetric capacities [7]. A reservation price, or price cap, is needed to limit the equilibrium price. Perfectly inelastic demand, a reservation price and the possibility of extreme demand outcomes are all realistic assumptions for electric power markets, especially so for balancing markets [6].

In reality, however, firms typically have both non-constant marginal costs and asymmetric production capacities. In this general case, the KM first-order conditions — one for each firm — constitute a system of non-autonomous ordinary differential equations. To solve this system analytically is not only very difficult, but likely to be impossible. Baldick & Hogan [1] calculate approximate asymmetric SFE by numerically integrating the system of ordinary differential equations. They note that it is difficult to find solutions that do not violate the requirement that supply functions must be non-decreasing.[3] Three exceptions to this are as follows: symmetric firms with identical cost functions, cases with affine solutions — i.e. affine marginal costs and no capacity constraints — and small variations in demand.

In this paper, I suggest a new numerical algorithm to find a valid SFE. It is intended for $N \geq 2$ asymmetric firms and cost functions more general than the three special cases identified by Baldick & Hogan. The equilibrium consists of piece-wise smooth supply functions and is inspired by the unique equilibrium derived for asymmetric producers with constant marginal costs [7]. Several of the analytically derived properties are conjectured to also be valid for increasing marginal costs. First, large firms have more market power and larger mark-ups for any percentage of total capacity. Hence, capacity constraints of smaller firms bind at lower prices. Second, the capacity constraint of the second largest firm starts to bind at the price cap, $\overline{p}$. Let p_i be the price at which the capacity constraint of firm i starts to bind. Arranging the producers according to size, starting with the smallest firm, these two properties can be stated as $C'(0) < p_1 < \ldots < p_{N-1} = \overline{p}$, where $C()$ is the aggregate cost function. Third, the largest producer offers its remaining capacity ΔS_N with perfectly elastic supply at the price cap. Lastly, all firms offer their first unit of power at marginal cost, which is in agreement with general results for uniform-price auctions [9].

To ensure an equilibrium with the conjectured properties, the following two assumptions are made. First, the larger of any two firms has weakly larger marginal cost for any share of capacity.[4] Second, all firms have the same marginal cost at zero supply.[5] The constants $\Delta S_N, p_1, p_2, \ldots p_{N-2}$ are so far unknown. Given these constants, the end-conditions of the system of KM first-order conditions are known and the supply functions of all firms can be solved by numerical integration. The numerical integration starts at the price cap and proceeds in the direction of decreasing prices. The integration is terminated as soon as any supply function violates the two requirements that a supply function must be non-decreasing and non-negative. The criterion function $\Gamma(p_1, \ldots p_{N-2}, \Delta S_N)$ is equal to the terminated price. In theory, all considered SFE candidates should fulfil $\Gamma(p_1 \ldots p_{N-2}, \Delta S_N) = C'(0)$. In practice, however, one has to be somewhat forgiving due to numerical errors. If a unique equilibrium exists,

[3] Non-decreasing supply functions are required by most electricity auctions.
[4] It should be possible to numerically calculate asymmetric SFE for more general cost functions. However, then adjustments of the conjecture might be needed, i.e. the order in which the capacity constraints bind.
[5] It should also be possible to numerically calculate asymmetric SFE when firms have different marginal costs at zero supply, but in this case firms may offer their first units of power at different prices. Further, the firm with the lowest marginal cost at zero supply is expected to have a supply function with a perfectly elastic segment at the second lowest marginal cost at zero supply.

which would be expected from the previous analytical models in [6,7], it can be found by an optimisation algorithm minimising Γ.

The structure of the paper is as follows. Section 2 introduces the notation and assumptions used in the analysis. Section 3 presents a set of systems of differential equations and a numerical algorithm that can be employed to calculate the conjectured SFE. In Section 4, the numerical algorithm is applied to an example with three firms. The algorithm returns one solution that approximately fulfils the first-order condition and the non-decreasing requirement. It is graphically verified that no firm will find it profitable to deviate from the equilibrium candidate. The accepted production of the equilibrium is inefficient because mark-ups are asymmetric. The paper is concluded in Section 5.

2. NOTATION AND ASSUMPTIONS

Except for firms' capacities and costs, the notation and market assumptions are the same as in previous papers by Holmberg [6,7]. There are N asymmetric producers. The bid of each firm i consists of a piece-wise smooth — i.e. piece-wise twice continuously differentiable — non-decreasing and left continuous supply function $S_i(p)$.[6] The aggregate supply of firm i's competitors is denoted $S_{-i}(p)$ and total supply is denoted $S(p)$.

Let $\bar{\varepsilon}_i$ be the capacity constraint of producer i. Without loss of generality, firms can be ordered according to their capacity, i.e. $\bar{\varepsilon}_1 < \bar{\varepsilon}_2 < \ldots < \bar{\varepsilon}_N$. Total capacity is designated by $\bar{\varepsilon}$, i.e. $\bar{\varepsilon} = \sum_{i=1}^{N} \bar{\varepsilon}_i$. Let p_i denote the price at which firm i chooses to offer its last unit, i.e. $S_i(p_i) = \bar{\varepsilon}_i$.

Denote the perfectly inelastic demand by ε and its probability density function by $f(\varepsilon)$. I assume that demand is always non-negative.[7] The density function is continuously differentiable and has a convex support set that includes zero demand. To yield a unique equilibrium, extreme demand outcomes are permitted, i.e. ε such that $\varepsilon > S(\bar{p})$ occurs with

[6] Consider a supply function $S_i(p)$ with a discontinuity at p_0. It is then assumed that firm i is willing to produce any supply in the range $[S_i(p_0), S_i(p_0+)]$, if the price is p_0. Thus the left continuous supply function is actually just a representation of a *correspondence*.

[7] As in [6], it is straightforward to extend the analysis to negative demand, which is relevant for balancing markets.

positive probability.[8] In equilibrium, this implies that the capacity constraints of the N-1 smallest firms bind with a positive probability. The reservation price $\overline{p}$ ensures that demand is zero above the price cap. Accordingly, the market price equals the price cap when $\varepsilon > \overline{\varepsilon},$ if such demand outcomes occur.

All firms have increasing, strictly convex and twice continuously differentiable cost functions. Denote the aggregated cost function of all firms by $C(S)$. For the cost functions of the individual firms, it is assumed that $C_i^{'}(S_i) \geq C_j^{'}(S_j)$ if $\frac{S_i}{\overline{\varepsilon}_i} = \frac{S_j}{\overline{\varepsilon}_j}$ and $i>j$. Furthermore, $C_i^{'}(0) = C_j^{'}(0)$. These assumptions are made to ensure an equilibrium with the conjectured properties. For more general cost functions, adjustments of the conjecture might be necessary.

Residual demand of an arbitrary producer i is denoted by $q_i(p,\varepsilon)$. As long as the supply functions of his competitors are not perfectly elastic at p, residual demand is

$$q_i(\varepsilon, p) = \varepsilon - S_{-i}(p). \tag{1}$$

3. THE CONJECTURED SFE

For symmetric producers and producers with asymmetric capacities and identical constant marginal costs, it has been shown that there exists a unique equilibrium with the following properties:

- All producers offer their first units of power at the price $C'(0)$.
- All supply functions are twice continuously differentiable, except at points where the capacity constraint of at least one producer starts to bind.
- There are no supply functions with perfectly elastic segments below the price cap and only firm N (the largest firm) can have a perfectly elastic segment at the price cap. This implies that all supply functions $S_i(p)$ are continuous below the price cap.
- A firm's supply function does not have perfectly inelastic segments below p_i, where the firm's capacity constraint starts to binds.
- Below the price cap, all supply functions with non-binding capacity constraints fulfil the KM first-order condition.
- $C'(0) < p_1 < p_2 < \ldots < p_{N-1} = \overline{p}$.

[8] Note that $S(\overline{p})$ does not include ΔS_N, as supply functions are left continuous.

It is conjectured that these properties are true also for the asymmetric firms studied in this paper.[9] The conjecture is the basis of the numerical algorithm developed below.

3.1. Necessary conditions

Assuming that competitors do not have perfectly elastic supply functions below the price cap, the residual demand of an arbitrary producer *i* is given by (1). Hence, for given demand and price, the profit of producer *i* is

$$\pi_i(\varepsilon, p) = \underbrace{[\varepsilon - S_{-i}(p)]}_{S_i} p - C_i[\varepsilon - S_{-i}(p)] \text{ if } S_i \le \bar{\varepsilon}_i \text{ and } p < \bar{p}. \quad (2)$$

In the traditional SFE literature, see Klemperer & Meyer [8] for example, the KM first-order condition is derived by simply differentiating (2) with respect to *p*. That is,

$$S_i(p) - S_{-i}'(p)[p - C_i'(S_i(p))] = 0. \quad (3)$$

Below the price cap, all supply functions with non-binding capacity constraints fulfil the KM first-order condition. This implies that all SFE candidates are given by *N*-1 systems of differential equations. The first system has *N* differential equations and is valid for the price interval $(C'(0), p_1)$. The second system has *N*-1 differential equations and is valid for the price interval (p_1, p_2) and so on. The continuity assumption links the end-conditions of the systems of differential equations. Including the end-conditions, the *N*-1 systems of differential equations are as follows:

$$\begin{array}{lll}
p \in (C'(0), p_1) & \begin{cases} S_1(p) - S'_{-1}[p - C'_1(S_1(p))] = 0 \\ \vdots \\ S_N(p) - S'_{-N}[p - C'_N(S_N(p))] = 0 \end{cases} & \begin{cases} S_1(p_1) = \bar{\varepsilon}_1 \\ \vdots \\ S_N(p_1 -) = S_N(p_1 +) \end{cases} \\
\vdots & \vdots & \vdots \\
p \in (p_{N-3}, p_{N-2}) & \begin{cases} S_{N-2}(p) - S'_{-(N-2)}[p - C'_{N-2}(\cdot)] = 0 \\ S_{N-1}(p) - S'_{-(N-1)}[p - C'_{N-1}(\cdot)] = 0 \\ S_N(p) - S'_{-N}[p - C'_N(S_N(p))] = 0 \end{cases} & \begin{cases} S_{N-2}(p_{N-2}) = \bar{\varepsilon}_{N-2} \\ S_{N-1}(p_{N-2} -) = S_{N-1}(p_{N-2} +) \\ S_N(p_{N-2} -) = S_N(p_{N-2} +) \end{cases} \\
p \in (p_{N-2}, \bar{p}) & \begin{cases} S_{N-1}(p) - S'_{-(N-1)}[p - C'_{N-1}(\cdot)] = 0 \\ S_N(p) - S'_{-N}[p - C'_N(S_N(p))] = 0 \end{cases} & \begin{cases} S_{N-1}(\bar{p}) = \bar{\varepsilon}_{N-1} \\ S_N(\bar{p}) = \bar{\varepsilon}_N - \Delta S_N \end{cases}
\end{array} \quad (4)$$

Given a set of values $\{p_1, p_2, \ldots p_{N-2}, \Delta S_N\}$, the *N*-1 systems of differential equations can be solved backwards. One must start with the price interval $(p_{N-2}, \bar{p})$, for which all end-conditions are known, i.e. $S_{N-1}(\bar{p}) = \bar{\varepsilon}_{N-1}$ and $S_N(\bar{p}) = \bar{\varepsilon}_N - \Delta S_N$. Thus $S_N(p)$ and $S_{N-1}(p)$ can

[9] It is likely that they can be proven by means of the analytical tools used in previous work [1,6,7].

be calculated for $(p_{N-2}, \bar{p})$. This solution can then be used to determine the end-conditions for the price interval (p_{N-3}, p_{N-2}). After solving the system of differential equations associated with this price interval, one can proceed recursively to the interval (p_{N-4}, p_{N-3}) and so on.

The integration of the systems of ordinary differential equations starts at the price cap and proceeds in the direction of decreasing prices. It terminates as soon as any supply function violates the non-decreasing and non-negativity constraints. The criterion function $\Gamma(p_1, p_2, \ldots p_{N-2}, \Delta S_N)$ returns the terminating price. According to the conjecture, all producers will, in equilibrium, offer their first unit of power at $C'(0)$. Thus, theoretically all SFE candidates must fulfil $\Gamma[p_1, p_2, \ldots p_{N-2}, \Delta S_N] = C'(0)$.

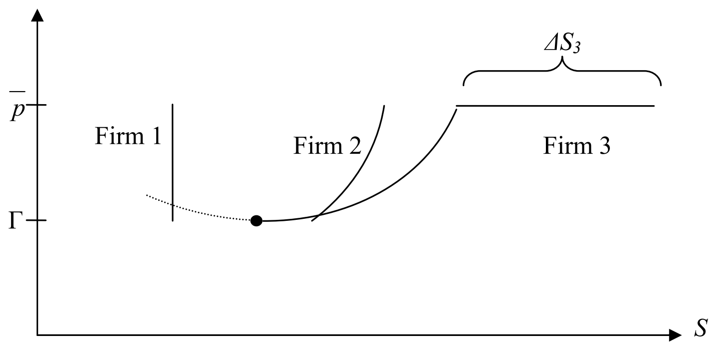

Figure 1. *The integration starts at the price cap, proceeds in the direction of decreasing prices and is terminated as soon as any supply function becomes invalid. Γ is defined by the terminating price.*

3.2. A sufficient condition

The first-order condition and $\Gamma[p_1, p_2, \ldots p_{N-2}, \Delta S_N] = C'(0)$ are necessary conditions for a SFE, as they ensure a local extremum. When competitors follow strategies implied by the candidate, it is still not clear that the globally best response for a producer is to follow the SFE candidate. A sufficiently strong second-order condition is that the market price of the equilibrium candidate globally maximises $\pi_i(\varepsilon, p)$ for every ε, given that the competitors follow the candidate.

3.3. The numerical algorithm

For asymmetric producers with general cost functions, it is difficult and likely impossible to calculate SFE analytically. Nevertheless, the system of differential equations in (4) can be solved by numerical integration, given $\{p_1, p_2, \dots p_{N-2}, \Delta S_N\}$. By gridding the space and/or employing optimisation algorithms, values $\{p_1, p_2, \dots p_{N-2}, \Delta S_N\}$ that (nearly) fulfil $\Gamma = C'(0)$ can be found. In practice, because of numerical errors, SFE candidates that almost fulfil $\Gamma[p_1, p_2, \dots p_{N-2}, \Delta S_N] = C'(0)$ can not be ruled out. The second-order condition can be checked graphically or numerically.

4. AN EXAMPLE WITH THREE ASYMMETRIC FIRMS

The numerical procedure to find valid SFE is illustrated by an example with three firms. Their production capacities are: $\bar{\varepsilon}_1 = \frac{\bar{\varepsilon}}{7}$, $\bar{\varepsilon}_2 = \frac{2\bar{\varepsilon}}{7}$ and $\bar{\varepsilon}_3 = \frac{4\bar{\varepsilon}}{7}$. The marginal cost function of all firms is linear, $C_i' = c\left(1 + \frac{S_i}{\bar{\varepsilon}_i}\right)$, up to the capacity constraint. Assume further that the price cap is $\bar{p} = 4c$.

4.1. Necessary conditions

The KM first-order conditions of the SFE candidates corresponding to (4) are given by the following set of two systems of differential equations:

$$p \in (c, p_1) \quad \begin{cases} S_1(p) - S'_{-1}(p)\left[p - c\left(1 + \frac{7S_1}{\bar{\varepsilon}}\right)\right] = 0 \\ S_2(p) - S'_{-2}(p)\left[p - c\left(1 + \frac{7S_2}{2\bar{\varepsilon}}\right)\right] = 0 \\ S_3(p) - S'_{-3}(p)\left[p - c\left(1 + \frac{7S_3}{4\bar{\varepsilon}}\right)\right] = 0 \end{cases} \quad \begin{cases} S_1(p_1) = \frac{\bar{\varepsilon}}{7} \\ S_2(p_1 -) = S_2(p_1 +) \\ S_3(p_1 -) = S_3(p_1 +) \end{cases}$$

$$p \in (p_1, 4c) \quad \begin{cases} S_2(p) - S'_{-2}(p)\left[p - c\left(1 + \frac{7S_2}{2\bar{\varepsilon}}\right)\right] = 0 \\ S_3(p) - S'_{-3}(p)\left[p - c\left(1 + \frac{7S_3}{4\bar{\varepsilon}}\right)\right] = 0 \end{cases} \quad \begin{cases} S_2(4c) = \frac{2\bar{\varepsilon}}{7} \\ S_3(4c) = \frac{4\bar{\varepsilon}}{7} - \Delta S_3 \end{cases}$$

The variables can be normalised such that $p = c\tilde{p}$ and $S_i(p) = \bar{\varepsilon}\tilde{S}_i(\tilde{p})$, so that

$$
\begin{aligned}
&\tilde{p} \in (1, \tilde{p}_1) \quad
\begin{cases}
\tilde{S}_1(\tilde{p}) - \tilde{S}'_{-1}(\tilde{p})\left[\tilde{p} - \left(1 + 7\tilde{S}_1\right)\right] = 0 \\
\tilde{S}_2(\tilde{p}) - \tilde{S}'_{-2}(\tilde{p})\left[\tilde{p} - \left(1 + \dfrac{7\tilde{S}_2}{2}\right)\right] = 0 \\
\tilde{S}_3(\tilde{p}) - \tilde{S}'_{-3}(\tilde{p})\left[\tilde{p} - \left(1 + \dfrac{7\tilde{S}_3}{4}\right)\right] = 0
\end{cases}
\quad
\begin{cases}
\tilde{S}_1(\tilde{p}_1) = \dfrac{1}{7} \\
\tilde{S}_2(\tilde{p}_1 -) = \tilde{S}_2(\tilde{p}_1 +) \\
\tilde{S}_3(\tilde{p}_1 -) = \tilde{S}_3(\tilde{p}_1 +)
\end{cases} \\
&\tilde{p} \in (\tilde{p}_1, 4) \quad
\begin{cases}
\tilde{S}_2(\tilde{p}) - \tilde{S}'_{-2}(\tilde{p})\left[\tilde{p} - \left(1 + \dfrac{7\tilde{S}_2}{2}\right)\right] = 0 \\
\tilde{S}_3(\tilde{p}) - \tilde{S}'_{-3}(\tilde{p})\left[\tilde{p} - \left(1 + \dfrac{7\tilde{S}_3}{4}\right)\right] = 0
\end{cases}
\quad
\begin{cases}
\tilde{S}_2(4) = \dfrac{2}{7} \\
\tilde{S}_3(4) = \dfrac{4}{7} - \Delta\tilde{S}_3
\end{cases}
\end{aligned}
\tag{5}
$$

Given a set of values $\{\tilde{p}_1, \Delta\tilde{S}_3\}$, the system representing the price interval $(\tilde{p}_1, 4)$ can be solved by numerical integration (see the Appendix for details of the numerical integration). This solution yields end-conditions for the system of differential equations valid for $\tilde{p} \in (1, \tilde{p}_1)$, which in turn can be solved.

The next step is to check whether the calculated supply functions violate the non-decreasing and non-negativity requirements. The criterion function $\tilde{\Gamma}(\tilde{p}_1, \Delta\tilde{S}_3)$ returns the first price (starting from the price cap) for which any supply function violates any of the requirements.

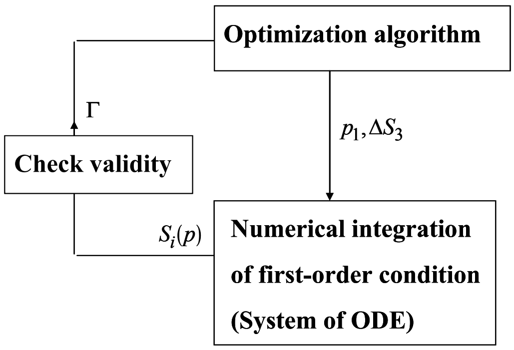

Figure 2. *The numerical procedure to find valid SFE candidates.*

An example of a parameter set that generates a non-valid SFE is $\tilde{p}_1 = 4$ and $\Delta\tilde{S}_3 = 0$. This is the boundary condition if the price cap is viewed as a public signal that coordinates bids.[10] The equilibrium was suggested by Baldick & Hogan, as it yields a unique SFE for symmetric

[10] All supply functions are smooth up to the price cap, at which point all capacity constraints bind.

producers [1]. They observe, however, that for asymmetric producers the public signal assumption often leads to invalid equilibria as in Figure 3. In this example, the supply functions violate both the non-decreasing and non-negativity requirements.

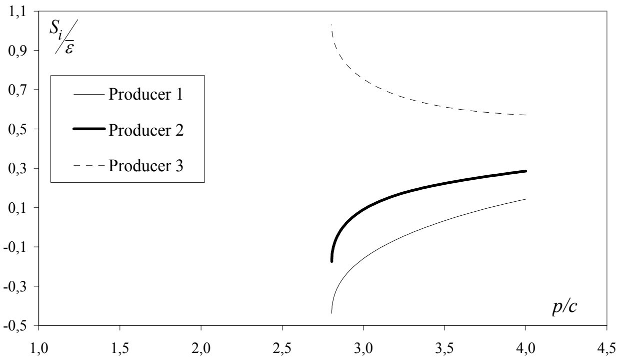

Figure 3. *The parameter set* $\tilde{p}_1 = 4$ *and* $\Delta\tilde{S}_3 = 0$ *generates invalid supply functions. This is the boundary condition when the price cap is viewed as a public signal that coordinates bids.*

To get an idea of the parameter space for which $\tilde{\Gamma}(\tilde{p}_1, \Delta\tilde{S}_3) = 1$, $\tilde{\Gamma}$ is calculated for a grid with 400x400 points in the space $(\tilde{p}_1, \Delta\tilde{S}_3) \in [2,4] \times \left[0, \frac{4}{7}\right]$. The result is presented as a contour plot in Figure 4, which indicates that $\tilde{\Gamma}(\tilde{p}_1, \Delta\tilde{S}_3)$ has a minimum around $\tilde{p}_1 \approx 3$ and $\Delta\tilde{S}_3 \approx 0.25$. By means of an optimisation algorithm, the minimum of $\tilde{\Gamma}(\tilde{p}_1, \Delta\tilde{S}_3)$ is located at $\tilde{p}_1 \approx 3.117$ and $\Delta\tilde{S}_3 \approx 0.2541$, and the min-value is approximately 1.005.[11] $\tilde{\Gamma} \approx 1.005$ is close to, but still above, $\tilde{\Gamma} = 1$, which is theoretically necessary for the conjectured SFE. The difference can be explained by the numerical sensitivity of the solution. If a unique SFE exists, as should intuitively be expected based on previous SFE studies [6,7], then a unique set of $\{\tilde{p}_1, \Delta\tilde{S}_3\}$ exists that yields valid supply functions. Thus there is a valid, unique triple of trajectories associated with this set that fulfils the systems of KM first-order conditions in (5). The slightest deviation from this triple, due to a small numerical error, will lead to

[11] The optimisation calculation employs the fminsearch algorithm, a simplex search method of Matlab. Estimation of the min-value depends on tolerances used in the numerical integration.

$\widetilde{\Gamma}\left[\widetilde{p}_1, \Delta\widetilde{S}_3\right] > 1$. The calculated supply functions for the set $\left\{\widetilde{p}_1 \approx 3.117, \Delta\widetilde{S}_3 \approx 0.2541\right\}$ are plotted in Figure 5.

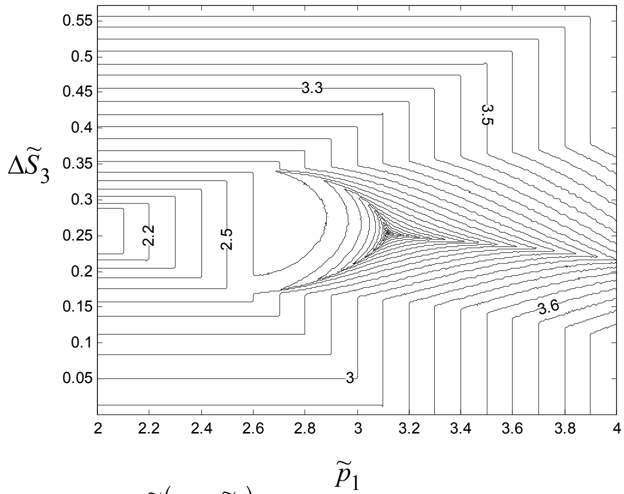

Figure 4. *Contour plot of* $\widetilde{\Gamma}\left(\widetilde{p}_1, \Delta\widetilde{S}_3\right)$*. A minimum exists around* $\widetilde{p}_1 \approx 3.1$ *and* $\Delta\widetilde{S}_3 \approx 0.25$.

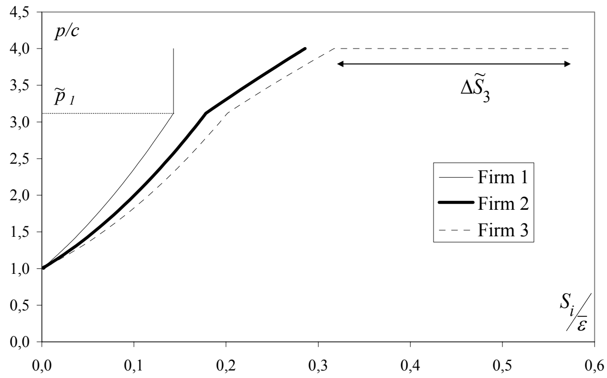

Figure 5. *The equilibrium candidate*.

As shown in a previous paper, the unique asymmetric equilibrium for constant marginal costs is piece-wise symmetric [7]. Two arbitrary producers have the same supply function unless the capacity constraint of one binds. With the strictly convex cost functions assumed in this paper, it will be more expensive for a smaller firm to produce a given supply compared to a larger firm. Thus it is expected that producers with more capacity sell more at every price, as in

Figure 5. Still it is apparent that the largest firm uses its market power extensively. More than 40 percent of the capacity of producer 3 is not offered below the price cap.

Note that firms 2 and 3 have a kink in their supply functions at $\tilde{p}_1$, which compensates firm 1's switch from elastic supply to perfectly inelastic supply. A discontinuous increase in the elasticity of the supplies of firms 2 and 3 ensures that the elasticity of their residual demand is continuous. It follows from the KM first-order condition in (3) that this is necessary if the supply functions of firms 2 and 3 are to be continuous at p_1.

4.2. The second-order condition

Does the candidate fulfil the sufficient second-order condition? Denote the supply functions of the SFE candidate in Figure 5 by $S_i^X(p)$ and denote its market price by $p^X(\varepsilon)$. Given $S_{-i}^X(p)$, does $p^X(\varepsilon)$ globally maximise $\pi_i(\varepsilon, p)$ for every ε? To indicate this, the isoprofit lines of all producers are plotted in Figures 6 to 8 together with $p^X(\varepsilon)$. For a local extremum, a vertical line, corresponding to a constant ε, should have a tangency point with a isoprofit line at $p^X(\varepsilon)$. This corresponds to the KM first-order condition and appears to be true for every demand level for all producers with non-binding capacity constraints. For such firms, one can also deduce from the shape of the isoprofit lines that profit is globally maximised at $p^X(\varepsilon)$.

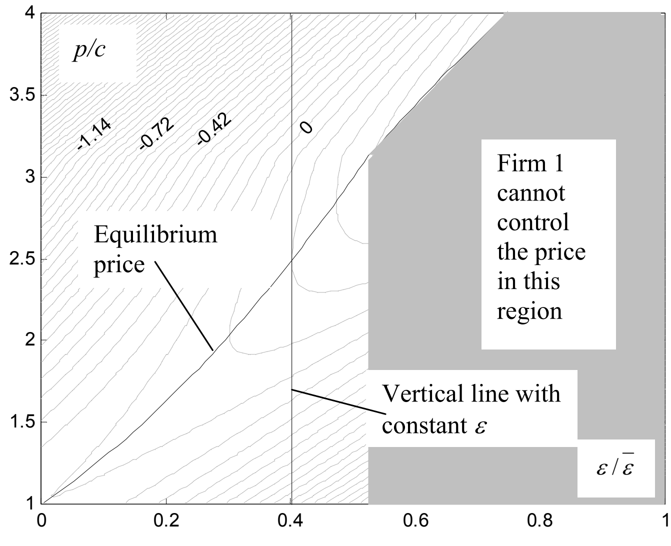

Figure 6. *Isoprofit lines of firm 1.*

In regions where producers cannot control the price, either due to a binding capacity constraint or a binding price cap, the tangency condition is not necessarily fulfilled. For example, due to its capacity constraint, firm 1 cannot unilaterally push the price below $p^X(\varepsilon)$ for $\varepsilon > S^X(p_1)$. By increasing mark-ups, the firm is still able to increase the market price. However, according to Figure 6, such deviations decrease profits. Neither firm 1 nor 2 can control the price for $\varepsilon > S^X(\bar{p})$.[12] Their capacity constraints prevent them from reducing the price and the price cap prevents them from increasing the price. Firm 3 could reduce the price for $\varepsilon > S^X(\bar{p})$, but according to Figure 8 it would not be profitable. Thus it appears that $p^X(\varepsilon)$ globally maximises $\pi_i(\varepsilon, p)$ for every ε when competitors' aggregate supply is given by $S_{-i}^X(p)$.

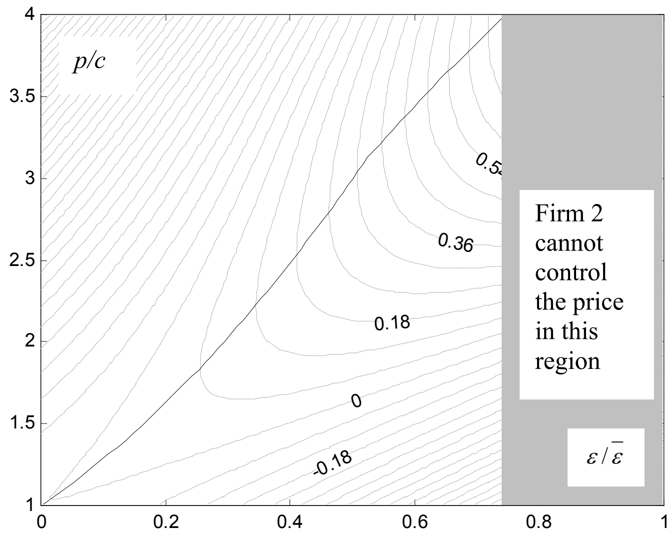

Figure 7. *Isoprofit lines of firm 2.*

[12] Recall that $S(\bar{p})$ does not include ΔS_N, as supply functions are left continuous.

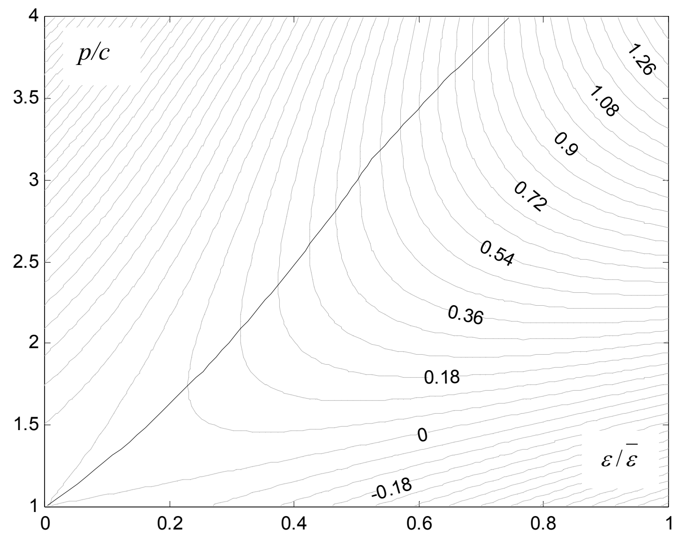

Figure 8. *Isoprofit lines of firm 3.*

4.3. Welfare loss

With symmetric cost functions as in [6], or asymmetric capacities and identical constant marginal costs as in [7], there are no inefficiencies because demand is perfectly inelastic and all firms operate at the same marginal cost. However, when marginal costs are increasing and large firms have larger mark-ups for every marginal cost, as in Figure 5, there is a welfare loss. Production is inefficient as some units with a high marginal cost are accepted from small firms instead of cheaper production from larger firms. For the example with three firms, the welfare loss is illustrated in Figure 9.

In this example, the relative production inefficiency is largest for the demand outcome $\varepsilon = S^X(p_1)$, when the capacity constraint of firm 1 starts to bind. For higher demand, additional production from firms 2 and 3 is accepted, which is cheaper than the most expensive generator of firm 1, and the cost ratio decreases. Another kink in the cost ratio occurs when the capacity constraint of firm 2 binds. Production is optimal when the whole capacity is utilised, i.e. $\varepsilon \geq \bar{\varepsilon}$.

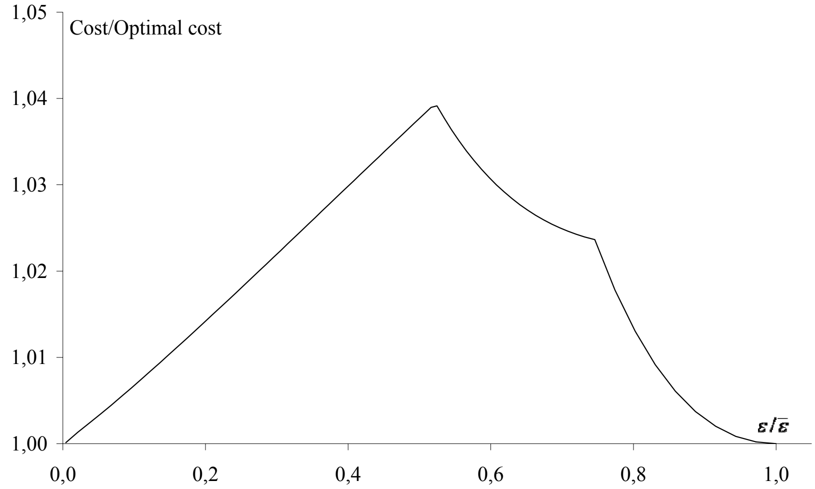

Figure 9. *Total production cost relative to optimal cost for the example with three firms.*

The problem of inefficient production has lead von der Fehr & Harbord [3] to suggest that electric power markets should consider Vickrey auctions instead of uniform-price auctions. The main advantage of the Vickrey auction is that it is optimal for producers to bid their true marginal costs because they are offered an information rent to do so.

5. CONCLUSIONS

Firms typically have non-constant marginal costs and asymmetric production capacities. In this general case, the first-order conditions of a Supply Function Equilibrium (SFE) constitute a system of non-autonomous ordinary differential equations. Solving such a system analytically is very difficult and likely to be impossible. Nevertheless, it can be solved by numerical integration. One problem, however, is that electricity auctions normally require non-decreasing supply functions and Baldick & Hogan have observed that numerically calculated asymmetric supply function equilibria tend to violate this restriction [1]. The three exceptions are: symmetric firms with identical cost functions, cases with affine solutions — i.e. affine marginal costs and no capacity constraints — and when there are small variations in demand.

This paper presents a numerical procedure that can solve the problem of invalid asymmetric supply function equilibria. It is conjectured that the general asymmetric SFE has properties similar to those found in the case of constant marginal costs, analysed in [7]. All supply functions fulfil the first-order condition from the lowest marginal cost up to the price at which either the capacity constraint or price cap binds. The capacity constraints of small firms bind at lower prices compared to firms with larger capacity. The capacity constraint of the second largest firm starts to bind at the price cap. In turn, the largest firm has a perfectly elastic supply ΔS_N at the price cap. Except for the two largest firms, the prices p_i at which the capacity constraints of firms bind are unknown constants, as is ΔS_N. The first-order conditions of this assumed equilibrium yield *N*-1 systems of non-autonomous ordinary differential equations. Given $\{p_1, p_2, \dots p_{N-2}, \Delta S_N\}$, the set of systems can be solved by means of numerical integration starting at the price cap and proceeding in the direction of a decreasing price. When any of the supply functions violate the restrictions that a supply function must be increasing and non-negative, the integration is terminated. The criterion function $\Gamma(p_1, \dots p_{N-2}, \Delta S_N)$ returns the price at which the integration terminates. For a valid SFE candidate, Γ must in theory equal the marginal cost of the cheapest unit. Based on the results for asymmetric producers with constant marginal costs one would intuitively expect a unique SFE, which can be found by an optimisation algorithm minimising Γ.

The procedure for finding asymmetric SFE candidates is illustrated by an example with three firms and linear marginal costs. A contour plot of Γ and an optimisation algorithm indicate that it has a unique minimum just above the marginal cost of the cheapest unit. This slight deviation can be expected as numerical errors would force the unique triple of SFE trajectories slightly off their track. Furthermore, numerically calculated isoprofit lines indicate that no producer will find it profitable to unilaterally deviate from the SFE candidate.

At the price p_i, for which the capacity constraint of firm *i* starts to bind, the elasticity of supply will increase discontinuously for each firm with a non-binding capacity constraint. This ensures that the elasticity of residual demand of all firms with non-binding capacity constraints is continuous at p_i. Thus in equilibrium, all but the smallest firm will have kinks in their supply functions below their capacity constraint.

The numerical procedure can be generalised to elastic demand, any increasing and convex cost function and, with enough computer power, any number of firms. Thus the algorithm presented should be general enough to calculate supply function equilibria of real electricity markets. The procedure is more likely to generate a valid SFE with the conjectured properties if (i) the larger of any two firms has weakly larger marginal cost for any share of capacity and (ii) if all firms have the same marginal cost at zero supply. With adjustments in the conjectured properties, for example, the order in which firms' capacities bind, it should be possible to apply the method to even more general cost functions. In addition, if firms have different marginal costs at zero supply, the supply function of the firm with the lowest marginal cost (at zero supply) is expected to have a perfectly elastic segment at the second lowest marginal cost. Given sufficiently elastic demand, a price cap is not needed in the model. In this case, the remaining capacity of the largest firm will be sold along the Cournot schedule, as in the asymmetric duopoly studied by Newbery [10].

For asymmetric firms with increasing marginal costs, asymmetric mark-ups imply inefficient production. The reason for this is that large firms have greater market power. At every marginal cost, small firms have lower mark-ups compared to large firms. Hence, some output from costly generators of small firms will be accepted instead of cheaper production from larger firms.

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APPENDIX

The numerical integration is performed in Matlab. It has been observed by Newbery that the coupled differential equations associated with SFE are stiff and highly sensitive to the starting point chosen for the numerical integration [10]. The example studied in this paper has the same problem. Thus a robust solver is employed, the ode15s of Matlab with the backward differentiation option.

When using numerical integration algorithms, it is often necessary to rewrite the system of differential equations in the standard form $x'(t) = f(x)$. This transformation is illustrated for the system of differential equations below. The first-order condition is

$$\begin{cases} S_1(p) - S'_{-1}(p)\left[p - C_1'(S_1(p))\right] = 0 \\ \vdots \\ S_N(p) - S'_{-N}(p)\left[p - C_N'(S_N(p))\right] = 0\,. \end{cases}$$

The system can be rewritten in the following form:

$$\begin{cases} \dfrac{S_1(p)}{p - C_1'(S_1(p))} = S'_{-1}(p) \\ \vdots \\ \dfrac{S_N(p)}{p - C_N'(S_N(p))} = S'_{-N}(p). \end{cases} \tag{6}$$

Summing over all equalities yields

$$\sum_{j=1}^{N} \frac{S_j(p)}{p - C_j'(S_j(p))} = (N-1)S'(p).$$

As $S_{-i}'(p) = S'(p) - S_i'(p)$, the system in (6) can now be rewritten as

$$\begin{cases} S_1'(p) = \dfrac{1}{N-1} \displaystyle\sum_{j=1}^{N} \frac{S_j(p)}{p - C_j'(S_j(p))} - \frac{S_1(p)}{p - C_1'(S_1(p))} \\ \vdots \\ S_N'(p) = \dfrac{1}{N-1} \displaystyle\sum_{j=1}^{N} \frac{S_j(p)}{p - C_j'(S_j(p))} - \frac{S_N(p)}{p - C_N'(S_N(p))}, \end{cases}$$

which has the standard form $x'(t) = f(x)$. A more general expression, which also considers elastic demand, has been derived by Baldick & Hogan [1].

Paper IV

Comparing Supply Function Equilibria of Pay-as-Bid and Uniform-Price Auctions[1]

Pär Holmberg[2]

May 16, 2005

Revised August 3, 2005

Abstract

This paper derives a Supply Function Equilibrium (SFE) of a pay-as-bid auction (discriminatory auction), such as the balancing market for electric power in Britain. It is shown that a SFE always exists if the hazard rate of the perfectly inelastic demand is monotonically decreasing and marginal costs are non-decreasing. With demand following a Pareto distribution of the second kind, the SFE of a pay-as-bid auction is compared to the SFE of a uniform-price auction, the auction form in most electricity markets. The demand-weighted average price in the former is found to be (weakly) lower than in the latter.

Keywords: supply function equilibrium, pay-as-bid auction, uniform-price auction, discriminatory auction, oligopoly, capacity constraint, wholesale electricity market

JEL codes: C62, D43, D44, L11, L13, L94

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[2] Department of Economics, Uppsala University, P.O. Box 513, SE-751 20 Uppsala, Sweden, phone +46 18 471 76 35, fax +46 18 471 14 78. E-mail: par.holmberg@nek.uu.se.

1. INTRODUCTION

Most electric power markets are organised as uniform-price auctions (UPAs). One exception, however, is the balancing market for electricity trade in England and Wales, which in 2001 switched from a UPA to a pay-as-bid auction (PABA), also known as a discriminatory auction. It was the belief of the British regulatory authority (Ofgem) that the reform would decrease mark-ups in wholesale electricity prices. Before the collapse of the California Power Exchange, a similar switch was also considered for this market [17].

The balancing market allows the system operator to buy or sell last-minute power from power producers to keep a continuous balance of demand and supply. This paper focuses on market situations where more supply is needed, i.e. the system operator buys power as in a procurement auction. It is straightforward nonetheless to draw analogous conclusions for market situations where less supply is needed and the system operator sells power in the balancing market, as in a sales auction.

In a UPA, all accepted bids are paid the marginal bid. Thus in its procurement version, all infra-marginal bids are accepted at a price above their bid. In a PABA, all accepted bids are paid their bid. A natural, but naive, first thought is that switching to a pay-as-bid auction would drastically reduce mark-ups for infra-marginal units and thereby decrease the average electricity price. However, firms change their bidding strategy after switching to a PABA. Based on intuition and experience from classical auction theory, many researchers actually argue in favour of electricity markets being organized as UPAs, see Kahn et al. [17] and Wolfram [27] for example. An experiment by Rassenti et al. [22] also suggests that average prices are higher in PABAs.[3]

In classical auction theory, comprising the *private-value*, *common-value* and *affiliated-value* models, demand of the auctioneer is certain whereas uncertainties relate to costs [20]. In contrast, electricity markets are generally characterised by known production costs. Moreover, the system operator's demand is uncertain when bids are submitted as it depends on unexpected temperature variations as well as unexpected outages in generators, machines and transmission-lines. Thus in models of strategic bidding in electricity procurement auctions, costs are often assumed to be certain and demand uncertain. To date, most theoretical studies of electric power auctions have been devoted to the UPA, see e.g. [1-3,9,11,13-15,23]. Three

[3] The demand in the experiment is not revealed to the players, but is certain in each period and the players can deduce it while playing. As in SFE with certain demand, this set-up would lead to an enormous range of equilibria [18]. Thus the experimental results are very much driven by the equilibrium selection process. Further, it is not certain that the experiments are long enough to allow the players find an equilibrium, especially as they have to find out the certain demand by themselves.

recent exceptions [6,8,24] study bidding behaviour in electric power markets organised as PABAs and compare prices and welfare in PABAs and UPAs. All of these studies indicate that electricity consumers should prefer PABA. This paper comes to the same conclusion with a model that is more general in terms of number of firms and production costs.

The model developed in this paper is very much related to the Supply Function Equilibrium (SFE) under uncertainty, which was introduced by Klemperer & Meyer [18]. In the non-cooperative Nash equilibrium of the static game, each producer commits to the supply function that maximises his expected profit given the bids of competitors. The set-up of their model is similar to the organisation of most electricity markets, with firms submitting supply functions to a uniform-price auction with uncertain demand, and SFE is an often used model of strategic bidding in electric power markets organised as UPA [1-3,11,13-15,23]. In this paper, the fundamental assumptions of the SFE model are employed to derive a similar model for a pay-as-bid auction. It is assumed that demand is perfectly inelastic and that there is a risk of power shortage, which is allowed to be arbitrarily small. Both assumptions are realistic for balancing markets [13]. As in [13], the risk of power shortage ensures a unique equilibrium. To facilitate an analytical solution, only symmetric equilibria are considered. As for UPAs, it should be possible to extend the analysis to consider asymmetric producers [14,15].

Another contribution of this paper is the comparison of the two SFE models for procurement auctions. When demand follows the Pareto distribution of the second kind [16], the demand-weighted average price is weakly lower in PABA than in UPA.[4] This probability distribution is not unreasonable for the balancing market, for which large imbalances are less likely than small imbalances. In a one-shot game with perfectly inelastic demand and symmetric firms, mark-ups have no implications for social efficiency. Large mark-ups do, however, imply substantial redistribution of income from power consumers to power producers, itself of social interest. Furthermore, large mark-ups lead to welfare losses in the long term as firms entering the market invest unnecessarily in additional capacity [11].

In a previous paper, Federico and Rahman [8] compare a UPA and PABA for two polar cases, perfect competition and monopoly, assuming that demand is elastic and follows a uniform probability distribution. They show that expected output decreases and expected consumer surplus increases after switching to a PABA. On the other hand, welfare is reduced in the competitive case. Under monopoly bidding, welfare is larger in PABAs if and only if marginal costs are sufficiently flat and demand uncertainty sufficiently low.

[4] Analogously, demand-weighted average prices in sales auctions would be higher in a PABA than a UPA.

Fabra et al. [6] derive a Nash equilibrium for a duopoly with constant marginal costs. The two firms are asymmetric in terms of both marginal costs and capacity. In their model, each producer must submit a horizontal (perfectly elastic) bid for its entire capacity. Demand is perfectly inelastic and known with certainty by the producers. Under these circumstances they show that average prices are lower in the PABA than in a UPA, and numerical examples suggest that the difference might be substantial. If demand is sufficiently high, the PABA has no pure strategy equilibria and only a mixed strategy equilibrium. The authors offer several extensions of the model, but the extensions do not lead to any definite conclusions regarding the comparison of the two auction types. Son et al. [24] use a similar model as Fabra et al., but one of the two firms has two production units with different marginal costs. Son et al. also conclude that average prices are lower in the PABA than in a UPA if demand is certain and perfectly inelastic. Simulations suggest that the conclusion may hold also for elastic demand.

This paper is structured as follows. Notation and assumptions are presented in Section 2 and the unique SFE of a PABA is derived in Section 3. It is shown that the first-order condition implies that the bid of each production unit is chosen to maximise the unit's expected profit, given the bids of competitors. The risk of power shortage provides an end-condition for the supply functions. A unique equilibrium candidate exists that satisfies both the first-order condition and the end-condition. Next, a second-order condition is derived. A unique equilibrium always exists if the demand's probability distribution has a downward sloping hazard rate and marginal costs are non-decreasing. In Section 4, average prices in the two procurement auctions are compared. Section 5 illustrates the two supply function equilibria with a simple example and Section 6 presents the conclusions.

2. NOTATION AND ASSUMPTIONS

Assume that there are $N \geq 2$ symmetric producers. The bid of each producer i consists of a monotonically increasing supply function $S_i(p)$, where p is the price.[5] The inverse of the supply function is denoted by $p_i(S_i)$. $S_{-i}(p)$ and $S(p)$ denote the combined supply of firm i's competitors' and total supply in the marketplace, respectively. The aggregate bid as function of total supply is denoted $p(S)$. The average price as a function of supply, $\hat{p}(S)$, is called the equilibrium price. In a UPA, all accepted bids are paid the marginal bid, i.e. $\hat{p}_U(S) = p_U(S)$,

[5] Electricity auctions do not normally accept decreasing supply functions.

while $\hat{p}(S) = \int_0^S p(x)dx / S$ in the PABA. As in Klemperer & Meyer's original work [18], only equilibria with twice continuously differentiable supply functions are considered. Thus in a symmetric equilibrium, $p_i(S_i)$ is smooth for $S_i \in (0, \varepsilon^* / N)$, where ε^* is the total offered supply of all producers. If firms are withholding capacity ε^* is less than $\bar{\varepsilon}$, the total capacity of all producers.

Denote perfectly inelastic demand by ε, its probability density function by $f(\varepsilon)$ and its distribution function by $F(\varepsilon)$. The density function is continuously differentiable and has support on the interval $[0, \hat{\varepsilon}]$, where $\hat{\varepsilon}$ is maximum demand. It is assumed that $\hat{\varepsilon} \geq \bar{\varepsilon}$, i.e. the capacity constraints of all producers will bind with a positive probability, which is allowed to be arbitrarily small. Demand is zero above the reservation price (price cap) $\bar{p}$. Therefore, in the extreme case where demand exceeds market capacity, the market price of the uniform-price auction equals the price cap.

All firms have identical cost functions $C(S_i)$, which are increasing, convex, twice continuously differentiable, and fulfil $C'(\bar{\varepsilon} / N) < \bar{p}$. R denotes the sum of firms' expected revenues and π_i denotes the expected profit of firm i.

3. THE UNIQUE SYMMETRIC SFE OF A PAY-AS-BID AUCTION

In the SFE of a UPA, a firm chooses, conditional on residual demand, a supply function that maximises profit for each demand outcome [18]. This section derives necessary and sufficient conditions for the SFE of a PABA. In Section 3.1, the first-order condition of a PABA is derived. It implies that each firm chooses a supply function that maximises expected profit for each of its production units, given residual demand. The first-order condition is a differential equation which can be solved for general cost functions and the solution has one integration constant. In Section 3.2, this constant is identified by considering the risk of power shortage; the symmetric equilibrium bids of all firms must reach the price cap exactly when the aggregate capacity constraint binds. This forms the end-condition.

The first-order condition and the end-condition must necessarily be satisfied in equilibrium. In Section 3.3, a sufficient second-order condition is also derived. Unlike a UPA, it is not possible to prove that all increasing, smooth supply functions satisfying the necessary conditions are supply function equilibria of a PABA. In particular, it turns out that if the hazard rate of demand is locally increasing and marginal costs are sufficiently flat, then pure strategy equilibria of a PABA do not exist. On the other hand, I show that a pure strategy equilibrium always exists if the hazard rate is monotonically decreasing and marginal costs are non-decreasing. This is fulfilled by the Pareto distribution of the second kind, which is used in the comparison of the UPA and PABA. Moreover, this choice simplifies the algebraic manipulations, as the inverse of its hazard rate is linear.

3.1. The first-order condition

It is assumed that firm *i*'s competitors follow a symmetric equilibrium candidate. The first-order condition derived below must necessarily be fulfilled if the strategy implied by the symmetric equilibrium candidate locally maximises firm *i*'s expected profit. To avoid differentiability problems, all considered deviations of firm *i* satisfy $p_i(0) = p_j(0)$ and $p_i(\varepsilon^* / N) = p_j(\varepsilon^* / N)\ \forall j \neq i.$ The profit from an accepted bid of an infinitesimally small unit is $\left[p(S_i) - C'(S_i)\right]dS_i.$ Thus the expected profit of firm *i* is

$$\pi_i = \int_0^{\varepsilon^*} f(\varepsilon) \int_0^{\varepsilon - S_{-i}(p(\varepsilon))} [p_i(S_i) - C'(S_i)] dS_i d\varepsilon + \int_{\varepsilon^*}^{\hat{\varepsilon}} f(\varepsilon) \int_0^{\varepsilon^*/N} [p_i(S_i) - C'(S_i)] dS_i d\varepsilon.$$

The second term of this expression represents the contribution from demand outcomes exceeding market supply. By changing the order of integration the following can be shown:[6]

$$\begin{aligned}
\pi_i[p_i(S_i)] &= \int_0^{\varepsilon^*/N} [p_i(S_i) - C'(S_i)] \int_{S_i + S_{-i}[p_i(S_i)]}^{\varepsilon^*} f(\varepsilon)\, d\varepsilon dS_i + \\
&+ \int_0^{\varepsilon^*/N} [p_i(S_i) - C'(S_i)] \int_{\varepsilon^*}^{\hat{\varepsilon}} f(\varepsilon) d\varepsilon dS_i = \int_0^{\varepsilon^*/N} [p_i(S_i) - C'(S_i)] \int_{S_i + S_{-i}[p_i(S_i)]}^{\hat{\varepsilon}} f(\varepsilon)\, d\varepsilon dS_i = \qquad (1) \\
&= \int_0^{\varepsilon^*/N} \underbrace{[p_i(S_i) - C'(S_i)][1 - F(S_i + S_{-i}[p_i(S_i)])]}_{\varphi_i[S_i, p_i(S_i)]} dS_i.
\end{aligned}$$

Firm *i* chooses the bid function $p_i(S_i)$ such that its expected profit is maximised. That is, the firm faces a calculus of variation problem with the fixed terminal points $p_i(0) = p_j(0)$ and

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[6] Note that the limits of the integrals may change when the order of integration is changed [25]. It is straightforward to verify the new integration limits by plotting the integrated area.

$p_i\left(\varepsilon^* / N\right) = p_j\left(\varepsilon^* / N\right)$. As $p_i{}'(S_i)$ does not enter the integral, the Euler equation degenerates to the following equation [5]:

$$\begin{aligned}&\frac{\partial \varphi_i}{\partial p_i} = 1 - F[S_{-i}(p_i(S_i)) + S_i] - S_{-i}{}'(p_i(S_i))(p_i(S_i) - C'(S_i))f[S_{-i}(p_i(S_i)) + S_i] = 0,\\ &\forall\, S_i \in \left[0, \varepsilon^* / N\right].\end{aligned} \tag{2}$$

The functional $\varphi_i[S_i, p_i(S_i)]$ represents the contribution to expected profit from an infinitesimally small unit. Thus, the Euler equation implies that expected profit from each unit is maximised, conditional on residual demand. Because only equilibria with smooth and increasing supply functions are considered, (2) can be written as follows:

$$\begin{aligned}&1 - F[S_{-i}(p) + S_i(p)] - S_{-i}{}'(p)(p - C'(S_i(p)))f[S_{-i}(p) + S_i(p)] = 0,\\ &\forall\, p : S_i(p) \in \left(0, \varepsilon^* / N\right).\end{aligned} \tag{3}$$

In addition, only symmetric SFE are considered, i.e. $S_{-i}(p) \equiv (N-1)S_i(p)$. Thus (3) can be further simplified to:

$$1 - F[NS_i(p)] - (N-1)S_i{}'(p)(p - C'(S_i(p)))f[NS_i(p)] = 0, \forall\, p : S_i(p) \in \left(0, \varepsilon^* / N\right). \tag{4}$$

This first-order condition corresponds to the first-order condition of a UPA derived by Klemperer & Meyer [18]. Note that (4) implies that *p>C'* if and only if $S_i'(p)>0$.

In order to solve the differential equation, it is transformed into an equation in terms of $p(\varepsilon)$, the price of the marginal unit as a function of the demand, instead of $S_i(p)$. The same transformation is applied when solving the differential equation associated with the SFE of a UPA [1,23]. In the symmetric equilibrium, $\varepsilon = NS_i(p(\varepsilon))$ and $S_i{}' = \dfrac{1}{Np'(\varepsilon)}$, if $\varepsilon \le \varepsilon^*$. Thus

$$1 - F(\varepsilon) - \frac{(N-1)}{Np'(\varepsilon)}[p(\varepsilon) - C'(\varepsilon / N)]f(\varepsilon) = 0 \quad \forall \varepsilon \in \left[0, \varepsilon^*\right].$$

and

$$\frac{(N-1)p(\varepsilon)f(\varepsilon)}{N} - p'(\varepsilon)[1 - F(\varepsilon)] = \frac{(N-1)C'(\varepsilon / N)f(\varepsilon)}{N}.$$

This differential equation can be solved by means of the integrating factor, $[1 - F(\varepsilon)]^{\frac{N-1}{N}-1}$, to yield

$$p(\varepsilon) = \frac{A - \int^{\varepsilon}(N-1)C'(u/N)f(u)[1 - F(u)]^{\frac{N-1}{N}-1}\,du}{N[1 - F(\varepsilon)]^{\frac{N-1}{N}}} \quad \forall \varepsilon \in \left[0, \varepsilon^*\right], \tag{5}$$

where A is an integration constant.

3.2. Determining the integration constant

Analogous to the derivation of the SFE of a UPA, the integration constant A allows for a continuum of potential equilibria [18]. In this section, I argue that the integration constant can be uniquely determined if, as in [13], the capacity constraint binds with a positive probability. This can occur if there is a risk of large demand shocks and/or unexpected multiple generator failures.[7] In this case, the price of the marginal unit must reach the price cap exactly when the capacity constraint starts to bind.

If $\varepsilon^* < \bar{\varepsilon},$ some capacity is withheld from the auction. However, it cannot be optimal to withhold power. A producer will find it profitable to offer previously withheld units at or just below the price cap. Bidding with his whole capacity will increase the contribution to expected profit of demand outcomes $\varepsilon > \varepsilon^*,$ while the possible profit reduction associated with demand outcomes $\varepsilon \leq \varepsilon^*$ can be made arbitrarily small.

The highest bid in the auction must equal the price cap. Otherwise, the highest bid could be increased without lowering the probability that its associated unit is accepted. Moreover, as noted in Section 2, the analysis is confined to equilibria with twice continuously differentiable supply functions. Hence $S_i'(p) < \infty,$ implying that $p'(\varepsilon) > 0$.[8] Thus by construction supply functions cannot have horizontal segments at the price cap.

In summary, the price of the marginal unit must reach the price cap but not before the capacity constraint binds. Hence, the integration constant can be pinned down by the end-condition $p(\bar{\varepsilon}) = \bar{p}$.[9] It follows from (5) that

$$p(\varepsilon) = \frac{N\left[1-F(\bar{\varepsilon})\right]^{\frac{N-1}{N}}\bar{p} + \int_{\varepsilon}^{\bar{\varepsilon}} (N-1)C'(u/N)f(u)\left[1-F(u)\right]^{\frac{N-1}{N}-1}du}{N\left[1-F(\varepsilon)\right]^{\frac{N-1}{N}}} \quad \forall \varepsilon \in \left[0, \bar{\varepsilon}\right]. \tag{6}$$

[7] To avoid inconsistencies in the model, one can limit attention to production uncertainties for firms who exclusively have must-run production. These firms cannot bid strategically in the balancing market. Two examples of such firms in the British market are British Energy and British nuclear group, both of whom exclusively produce nuclear power.

[8] The assumption simplifies the proof but is not critical. Allowing for perfectly elastic bids does not change the result because, as in a Bertrand game, it is profitable to slightly undercut competitors' horizontal bids [13].

[9] The same end-condition is used to derive a unique SFE for UPAs [13]. Baldick & Hogan have suggested the same end-condition for UPAs but offer a weaker motivation; the price cap and capacity constraints can be viewed as public signals that coordinate the bids of producers [2].

3.3. The second-order condition

The only remaining equilibrium candidate is given by (6) which fulfils both the first-order condition and the end-condition of the PABA. For this candidate, let $\breve{p}(\varepsilon)$ be the price of the marginal unit as a function of demand. The symmetric supply functions of the candidate are designated by $\breve{S}_i(p)$. If the aggregate supply of competitors equals $\breve{S}_{-i}$, then — as shown by (2) — the expected profit of any unit of firm i is at a local extremum. By studying the second-order condition, it can be verified that, under certain conditions, expected profit is globally maximised for each production unit of firm i. Because firms and the equilibrium candidate are both symmetric, this argument is true for any firm and offers a sufficient condition for a SFE.

It follows from (1) that for a given S_i, the expected profit from the marginal unit of firm i is

$$\varphi_i(S_i, p)dS_i = [p - C'(S_i)][1 - F(S_i + S_{-i}(p))]dS_i, \tag{7}$$

where $p=p_i(S_i)$ is the bid of the marginal unit. Because competitors follow $\breve{S}_i$, (2) can be rewritten as

$$\frac{\partial \varphi_i(S_i, p)}{\partial p} = 1 - F\left[\breve{S}_{-i}(p) + S_i\right] - \breve{S}_{-i}{}'(p)(p - C'(S_i))f\left[\breve{S}_{-i}(p) + S_i\right]$$

or

$$\frac{\partial \varphi_i(S_i, p)}{\partial p} = f\left[\breve{S}_{-i}(p) + S_i\right]\underbrace{\left\{\overbrace{\left(\frac{1 - F\left[\breve{S}_{-i}(p) + S_i\right]}{f\left[\breve{S}_{-i}(p) + S_i\right]}\right)}^{G\left[\breve{S}_{-i}(p)+S_i\right]=1/H\left[\breve{S}_{-i}(p)+S_i\right]} - \breve{S}_{-i}{}'(p)(p - C'(S_i))\right\}}_{\eta(p,S_i)}, \tag{8}$$

where $G(x)$ is the inverse of the hazard rate $H(x)$. Let $p^* = \breve{p}_i(S_i)$. The first-order condition of the PABA in (4) ensures that $\left.\frac{\partial \varphi_i(S_i, p)}{\partial p}\right|_{p=p^*} = 0$. As $f>0$ and $\eta(p^*, S_i) = 0$, the following two conditions would ensure that $\varphi_i(S_i, p)$ is globally maximised at the price p^*:

$\eta(p,S_i) > 0$ for $p \in [\breve{p}(0), p^*)$[10] and $\eta(p,S_i) < 0$ for $p \in (p^*, \bar{p}]$. A SFE is guaranteed if they are both fulfilled for all $S_i \in [0, \bar{\varepsilon}/N]$. On the other hand, if $\left.\frac{\partial \eta(p,S_i)}{\partial p}\right|_{p=p^*} > 0$ for some S_i, then $\varphi_i(S_i, p)$ is locally minimised at the price $p^* = \breve{p}_i(S_i)$ and there exists a profitable deviation. The conditions for the global maximum and local minimum can be used to show the following theorem:

Theorem 1.

i) If $G'(\varepsilon) > 0 \, \forall \varepsilon \in (0, \bar{\varepsilon})$, then the equilibrium candidate in (6) is a SFE.

ii) If $[p(\varepsilon) - C'(\varepsilon/N)]G'(\varepsilon) + G(\varepsilon)C''(\varepsilon/N) < 0$ for some $\varepsilon \in (0, \bar{\varepsilon})$, then the equilibrium candidate in (6) is not a SFE, and a smooth, symmetric SFE does not exist.

Proof: See Appendix.

As $H'(x) = \frac{-G'(x)}{G^2(x)}$, it follows from Theorem 1 that a downward sloping hazard function ensures a SFE. On the other hand, if the hazard function is locally upward sloping and marginal costs sufficiently flat, then smooth symmetric equilibria can be ruled out. There is some intuition behind the non-existent equilibria. In the case of a monopolist or Cournot player, a similar problem occurs when demand or residual demand is sufficiently convex [10]. In the PABA, it follows from (7) that $1 - F[\breve{S}_{-i}(p) + S_i]$ can be interpreted as the residual demand of the marginal unit when firm i supplies S_i units of power.[11] Now, differentiate $1 - F[\breve{S}_{-i}(p) + S_i]$ twice with respect to p. Eliminate $\breve{S}''_{-i}(p)$ by differentiating the first-order condition of the PABA in (4). Consider the case $f'(\varepsilon) \geq 0$, which implies an upward sloping hazard rate (see (8)). Then it can be shown that the residual demand, $1 - F[\breve{S}_{-i}(p) + S_i]$, is convex, if marginal costs are sufficiently flat.

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[10] It is never profitable to offer a unit below $\breve{p}(0)$, as the unit is always accepted at this price.

[11] Previously Bulow and Klemperer [4] have noted that the probability that a bid is accepted (1-F) can be interpreted as residual demand.

The existence of equilibria is easier to guarantee in UPAs, as supply function equilibria of UPAs are independent of $f(\varepsilon)$. A symmetric SFE of a UPA exists as long as the demand function is concave [18].

3.4. The Pareto distribution of the second kind

As shown below, the Pareto distribution of the second kind has $G'(x) = const > 0,$ which according to Theorem 1 guarantees a SFE. Moreover, due to the linearity of the inverse hazard rate, it will turn out that the first-order condition becomes particularly simple. The Pareto distribution of the second kind has the probability distribution

$$F(x) = 1 - \beta^{\frac{1}{\alpha}}(\alpha x + \beta)^{\frac{-1}{\alpha}} \tag{9}$$

and the probability density

$$f(x) = \beta^{\frac{1}{\alpha}}(\alpha x + \beta)^{\frac{-1}{\alpha}-1}. \tag{10}$$

Hence, the inverse of its hazard rate is

$$G(x) = \frac{1 - F(x)}{f(x)} = \alpha x + \beta, \tag{11}$$

where $\alpha,\beta>0$.

The parameter β determines $f(0)$ as illustrated in Figure 1. When α is large, f has a steep negative slope for small arguments and a thick tail for large arguments, vice versa for small α. The density function is decreasing and strictly convex for all $\alpha,\beta>0$. Thus the Pareto distribution of the second kind captures the important characteristic that small imbalances are more likely than large imbalances in a balancing market.

With the Pareto distribution of the second kind, the first-order condition of the PABA in (4) can be simplified to

$$\alpha N S_i(p) + \beta - (N-1)S_i'(p)(p - C'(S_i(p))) \equiv 0,$$

which resembles the first-order condition of the UPA [18],

$$S_i(p) - (N-1)S_i'(p)(p - C'(S_i(p))) \equiv 0.$$

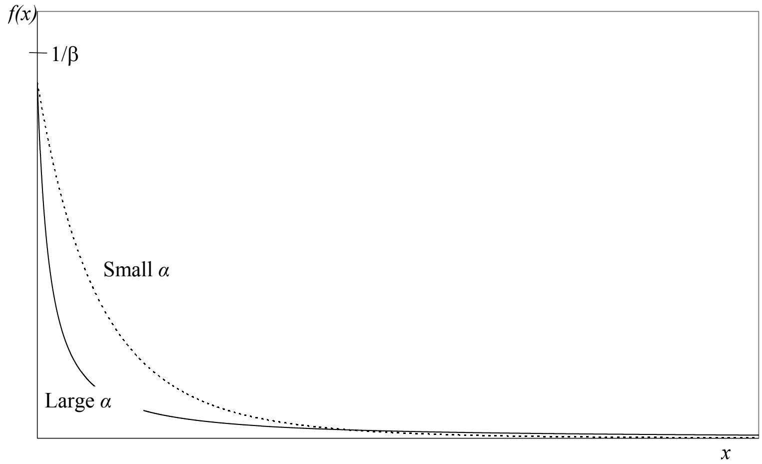

Figure 1. *The effect of α and β on the probability density function f(x).*

It follows from (6) that the equilibrium marginal bid in a PABA with a Pareto distribution of the second kind is

$$p(\varepsilon) = \frac{N\left(\alpha\bar{\varepsilon} + \beta\right)^{\frac{1-N}{\alpha N}}\bar{p} + \int_{\varepsilon}^{\bar{\varepsilon}}(N-1)C'(u/N)(\alpha u + \beta)^{\frac{1-N}{\alpha N}-1}\,du}{N\left(\alpha\varepsilon + \beta\right)^{\frac{1-N}{\alpha N}}}. \tag{12}$$

4. Comparing pay-as-bid and uniform-price auctions

Demand is assumed to be perfectly inelastic. As a result, total production in a pay-as-bid and uniform-price auction are equivalent. Furthermore, only symmetric equilibria are considered. This means that for every demand outcome, the most cost-effective generators will be accepted in both procurement auctions. Hence, production costs are also the same in both auctions for all outcomes. Average prices and mark-ups will differ, however, and the extent of the difference is investigated by comparing firms' total expected revenue in the two auctions. To ensure a SFE in both procurement auctions, demand is assumed to follow the Pareto distribution of the second kind. In the first subsection, expected revenues in the PABA are shown to be equal to or lower than expected revenues in the UPA when marginal costs are constant. This result is then used in the next subsection to prove the same inequality for non-decreasing marginal costs.

4.1. Constant marginal costs

It follows from (1) that total expected revenue for all firms in a PABA is

$$R_p = \int_0^{\bar{\varepsilon}} (1 - F(S)) p(S) dS = \int_0^{\bar{\varepsilon}} (1 - F(\varepsilon)) p(\varepsilon) d\varepsilon. \tag{13}$$

By means of (9) and (12) it can be shown that

$$R_P = \int_0^{\bar{\varepsilon}} \frac{\beta^{\frac{1}{\alpha}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} \left\{ \bar{p} N (\alpha\bar{\varepsilon} + \beta)^{\frac{1-N}{\alpha N}} + (N-1) \int_{\varepsilon}^{\bar{\varepsilon}} C'(u/N)(\alpha u + \beta)^{\frac{1-N}{\alpha N} - 1} du \right\}}{N (\alpha\varepsilon + \beta)^{\frac{1-N}{\alpha N}}} d\varepsilon \,. \tag{14}$$

Constant marginal costs are assumed in this section, i.e. $C'(u/N) \equiv c.$ For this case, straightforward integration yields

$$\begin{aligned} R_P &= \frac{\beta^{\frac{N-1}{\alpha N}} (\bar{p} - c)}{(\alpha\bar{\varepsilon} + \beta)^{\frac{N-1}{\alpha N}}} \int_0^{\bar{\varepsilon}} \left(1 + \frac{\alpha\varepsilon}{\beta}\right)^{\frac{-1}{\alpha N}} d\varepsilon + \beta^{\frac{1}{\alpha}} c \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} d\varepsilon = \\ &= (\bar{p} - c)\bar{\varepsilon} g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) + \beta^{\frac{1}{\alpha}} c \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} d\varepsilon, \end{aligned} \tag{15}$$

where

$$g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) = \frac{\left(1 + \frac{\alpha\bar{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha N} + 1} - 1}{\left(1 - \frac{1}{\alpha N}\right) \frac{\alpha\bar{\varepsilon}}{\beta} \left(\frac{\alpha\bar{\varepsilon}}{\beta} + 1\right)^{\frac{N-1}{\alpha N}}}. \tag{16}$$

The following can be shown by means of integration by parts:

$$\beta^{\frac{1}{\alpha}} \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} d\varepsilon = \beta^{\frac{1}{\alpha}} \left[(\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} \varepsilon \right]_0^{\bar{\varepsilon}} + \beta^{\frac{1}{\alpha}} \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha} - 1} \varepsilon d\varepsilon = \bar{\varepsilon}\left[1 - F(\bar{\varepsilon})\right] + \int_0^{\bar{\varepsilon}} f(\varepsilon) \varepsilon d\varepsilon.$$

Thus the second term in (15), $\beta^{\frac{1}{\alpha}} c \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} d\varepsilon$, is the expected production cost, and the first term, $(\bar{p} - c)\bar{\varepsilon} g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)$, is a measure of the mark-up.

The equilibrium marginal bid for symmetric firms in a UPA can be calculated from [13],

$$p_U(\varepsilon) = \frac{\bar{p}\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}} + (N-1)\varepsilon^{N-1} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C'(u/N) du}{u^N} \quad \forall [0, \bar{\varepsilon}]. \tag{17}$$

The total expected revenue for firms in a UPA is[12]

$$R_U = \int_0^{\bar{\varepsilon}} f(\varepsilon)\varepsilon p_U(\varepsilon) d\varepsilon + \left(1 - F\left(\bar{\varepsilon}\right)\right)\bar{\varepsilon}\bar{p}.$$

The second term is the contribution from demand outcomes exceeding market capacity. Combining (9), (10) and (17) yields

$$R_U = \int_0^{\bar{\varepsilon}} \beta^{\frac{1}{\alpha}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}-1} \varepsilon \left[\frac{\bar{p}\varepsilon^{N-1}}{\bar{\varepsilon}^{N-1}} + (N-1)\varepsilon^{N-1} \int_{\varepsilon}^{\bar{\varepsilon}} \frac{C'(u/N) du}{u^N} \right] d\varepsilon + \beta^{\frac{1}{\alpha}} \left(\alpha\bar{\varepsilon} + \beta\right)^{\frac{-1}{\alpha}} \bar{\varepsilon}\bar{p}. \quad (18)$$

Assuming constant marginal costs, the expression can be simplified by means of integration by parts:

$$\begin{aligned} R_U &= \frac{N(\bar{p}-c)}{\bar{\varepsilon}^{N-1}} \int_0^{\bar{\varepsilon}} \left(\frac{\alpha\varepsilon}{\beta} + 1 \right)^{\frac{-1}{\alpha}} \varepsilon^{N-1}\, \mathrm{d}\varepsilon + \beta^{\frac{1}{\alpha}} c \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} d\varepsilon = \\ &= (\bar{p}-c)\bar{\varepsilon} g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta} \right) + \beta^{\frac{1}{\alpha}} c \int_0^{\bar{\varepsilon}} (\alpha\varepsilon + \beta)^{\frac{-1}{\alpha}} d\varepsilon, \end{aligned} \quad (19)$$

where

$$g_U\left(\alpha, N, \tfrac{\alpha\bar{\varepsilon}}{\beta}\right) = \frac{N}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}\right)^N} \int_0^{\frac{\alpha\bar{\varepsilon}}{\beta}} (1+t)^{\frac{-1}{\alpha}} t^{N-1}\, \mathrm{d}t. \quad (20)$$

The integral can be solved by repeated use of integration by parts. Subtracting (15) from (19) yields the following:

$$R_U - R_P = (\bar{p}-c)\bar{\varepsilon}\left[g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta} \right) - g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta} \right) \right].$$

The contour plot of $\dfrac{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}$ in Figure 2 illustrates the relative decrease of mark-ups when switching from a UPA to a PABA. The plot is not very sensitive to the number of firms. As the ratio is positive over a wide range of parameters, it seems that $R_U - R_P \geq 0$. This inequality can indeed be proven mathematically [12].

---

[12] Analogously $R_p = \int_0^{\bar{\varepsilon}} f(\varepsilon)\varepsilon\hat{p}(\varepsilon) d\varepsilon + \left(1 - F\left(\bar{\varepsilon}\right)\right)\bar{\varepsilon}\hat{p}\left(\bar{\varepsilon}\right)$. By means of integration by parts and the definition of $\hat{p}(\varepsilon)$ it is straightforward to verify that this expression is equal to (13)

Theorem 2. With perfectly inelastic demand given by the Pareto distribution of the second kind and constant marginal costs, the expected revenue of symmetric firms in a pay-as-bid procurement auction is weakly lower than their expected revenue in a uniform-price procurement auction.

Figure 2 shows that switching from a UPA to a PABA almost eliminates mark-ups in the area, for which both $\bar{\varepsilon} >> \beta$ and $\alpha < 1$. As can be seen in Figure 3, this area correspond to a very low risk of power shortage. In contrast, mark-ups are nearly unchanged for either large α (fat tail of the probability density function) or small $\bar{\varepsilon}$ (small capacity), both of which imply a high risk of power shortage.

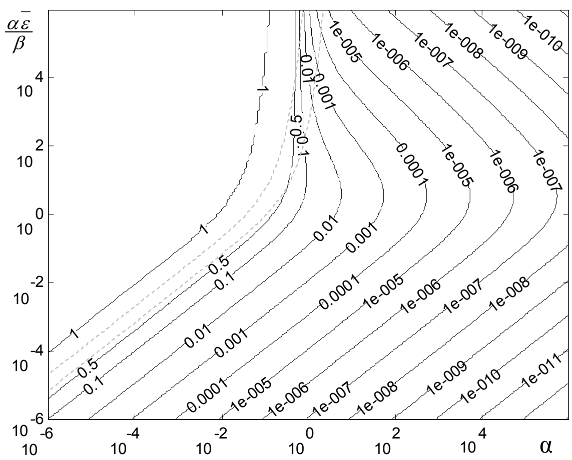

Figure 2. *Contour plot of* $\dfrac{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}{g_U\left(\alpha, N, \frac{\alpha\bar{\varepsilon}}{\beta}\right)}$ *when N=2. The grey dotted lines indicate a region with a risk of power shortage realistic for electric power markets.*

In most electric power markets, reasonable assumptions for the likelihood of power shortages range from once every hundred years to 100 times per year. This range roughly correspond to the per hour probability of a power shortage being 10^{-6} to 0.01; one hour is the normal length of the delivery period. This region is indicated in Figure 2. Switching to a pay-as-bid auction in an electric power market can reduce average mark-ups by 60 to 99 percent if $\alpha < 0.1$ and N=2; the lower the risk of power shortage, the larger the impact. The impact is somewhat reduced if the number of symmetric firms increases. For N=10 and $\alpha < 0.1$, switching to a pay-as-bid auction reduces average mark-ups between 20 and 90 percent. For $\alpha > 1$, which corresponds to a more convex probability density function, there is little gain from switching to a pay-as-bid auction in the electric power market, regardless of the number of symmetric firms.

The intuition underlying the role of α in the comparison of PABAs and UPAs is as follows. Equilibrium bids in UPAs are not influenced by the probability distribution of demand [13,18]. Bids in PABAs are, however, sensitive to α. In particular, a smaller α makes low demand outcomes more likely. Intuitively, this increases the elasticity of residual demand for small S_i. [13] Thus mark-ups are lower for these units in a PABA in accordance with the inverse elasticity rule [26]. For small values of α, two effects drive down firms' expected revenues in the PABA; (i) lower mark-ups for low demand outcomes and (ii) an increased probability of low demand outcomes. In the UPA, only the second effect drives down firms' expected revenues. The same intuition may also explain why firms' expected revenues are lower in PABAs than UPAs for the Pareto distribution of the second kind, which is characterised by a decreasing probability density.

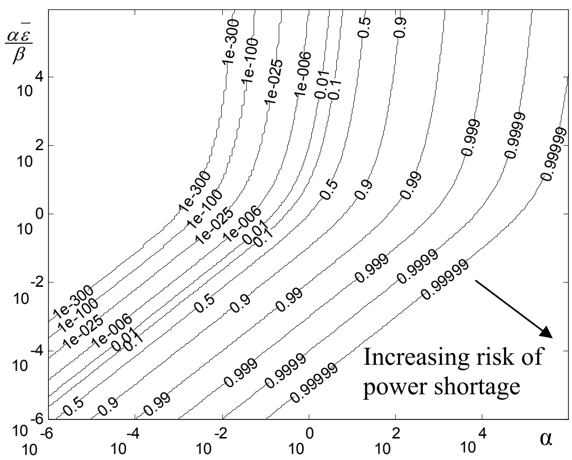

Figure 3. *Contour plot of 1-* $F(\bar{\varepsilon}) = \left(\frac{\alpha\bar{\varepsilon}}{\beta}+1\right)^{\frac{-1}{\alpha}}$, *the probability of a power shortage.*

[13] Recall that $1 - F\left[\breve{S}_{-i}(p) + S_i\right]$ can be interpreted as residual demand of the marginal unit when firm i supplies S_i units of power.

4.2 Non-decreasing marginal costs

In Section 4.1 it was shown that switching from a UPA to PABA reduces firms' revenues if marginal costs are constant and demand follows a Pareto distribution of the second kind. Using Theorem 2, this section demonstrates that the conclusion can be generalised to non-decreasing marginal costs.

The expected revenue in (14) is valid for non-decreasing marginal costs. The term related to the price cap can be rewritten in the same manner as (15):

$$R_P = \overline{p}\overline{\varepsilon} g_P\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + \int_0^{\overline{\varepsilon}} \frac{\beta^{\frac{1}{\alpha}}(\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}}(N-1)\int_{\varepsilon}^{\overline{\varepsilon}} C'(u/N)(\alpha u+\beta)^{\frac{1-N}{\alpha N}-1}du}{N(\alpha\varepsilon+\beta)^{\frac{1-N}{\alpha N}}}d\varepsilon.$$

By reversing the order of integration [25], it can be shown that

$$\begin{aligned} R_p &= \overline{p}\overline{\varepsilon} g_P\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + \frac{\beta^{\frac{1}{\alpha}}(N-1)}{N}\int_0^{\overline{\varepsilon}} C'(u/N)(\alpha u+\beta)^{\frac{1-N}{\alpha N}-1}\int_0^u(\alpha\varepsilon+\beta)^{\frac{-1}{\alpha N}}d\varepsilon du = \\ &= \overline{p}\overline{\varepsilon} g_P\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + (N-1)\int_0^{\overline{\varepsilon}} C'(u/N)\underbrace{\left(\frac{\alpha u}{\beta}+1\right)^{\frac{1-N}{\alpha N}-1}\frac{\left(\frac{\alpha u}{\beta}+1\right)^{1-\frac{1}{\alpha N}}-1}{\alpha N-1}}_{h_P\left(\alpha, N, \frac{\alpha u}{\beta}\right)}du. \end{aligned} \quad (21)$$

Similarly it follows from (18) and (19) that

$$\begin{aligned} R_U &= \overline{p}\overline{\varepsilon} g_U\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + \beta^{\frac{1}{\alpha}}(N-1)\int_0^{\overline{\varepsilon}}(\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}-1}\varepsilon^N\int_{\varepsilon}^{\overline{\varepsilon}}\frac{C'(u/N)du}{u^N}d\varepsilon = \\ &= \overline{p}\overline{\varepsilon} g_U\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + \beta^{\frac{1}{\alpha}}(N-1)\int_0^{\overline{\varepsilon}}\frac{C'(u/N)}{u^N}\int_0^u(\alpha\varepsilon+\beta)^{\frac{-1}{\alpha}-1}\varepsilon^N d\varepsilon du = \\ &= \overline{p}\overline{\varepsilon} g_U\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) + (N-1)\int_0^{\overline{\varepsilon}} C'(u/N)\underbrace{\left(\frac{\alpha u}{\beta}\right)^{-N}\int_0^{\frac{\alpha u}{\beta}}\frac{(t+1)^{\frac{-1}{\alpha}-1}t^N}{\alpha}dt}_{h_U\left(\alpha, N, \frac{\alpha u}{\beta}\right)}du. \end{aligned} \quad (22)$$

The integral in $h_U\left(\alpha, N, \frac{\alpha u}{\beta}\right)$ can be solved analytically by repeated integration by parts. Let $h_{\alpha N}(x) = h_U(\alpha, N, x) - h_P(\alpha, N, x)$. Equations (21) and (22) now imply that

$$\Delta R = R_U - R_P = \overline{p}\overline{\varepsilon}\left[g_U\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right) - g_P\left(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\right)\right] + (N-1)\int_0^{\overline{\varepsilon}} C'(u/N)h_{\alpha N}\left(\frac{\alpha u}{\beta}\right)du. \quad (23)$$

Figure 4 presents a contour plot of $h_{\alpha N}(x)$. The levels in the contour plot are very sensitive to *N,* whilst the pattern is comparatively stable. The function $h_{\alpha N}(x)$ appears to have profile -|+ for *N*≥*2* and *x*≥0, which is verified mathematically in [12]. If $h_{\alpha N}\left(\frac{\alpha u}{\beta}\right)$ changes sign for $u < \bar{\varepsilon},$ let this point be denoted u^*, otherwise set $u^* = \bar{\varepsilon}.$ Use (23) to calculate ΔR_1 for the non-decreasing cost function $C_1(\varepsilon / N)$. Next calculate ΔR_2 for the constant marginal cost $c_2 = C_1'(u^* / N)$. Compared to $C_1'(\varepsilon / N),$ c_2 puts a (weakly) higher weight on negative $h_{\alpha N}(x)$ and a (weakly) lower weight on positive $h_{\alpha N}(x)$. Thus

$$\Delta R_1 \geq \Delta R_2.$$

From Theorem 2 it follows that $\Delta R_2 \geq 0.$ Thus $\Delta R_1 \geq 0$ and $R_U \geq R_P$ is true also for non-decreasing marginal costs. From the reasoning above we can also conclude that ΔR_1 increases if the slope of C_1'' is increased while $C_1'(u^* / N)$ is kept constant.

Theorem 3. With perfectly inelastic demand given by the Pareto distribution of the second kind and non-decreasing marginal costs, the expected revenue of symmetric firms in a pay-as-bid procurement auction is weakly lower than their expected revenue in a uniform-price procurement auction.

Proof: See [12].

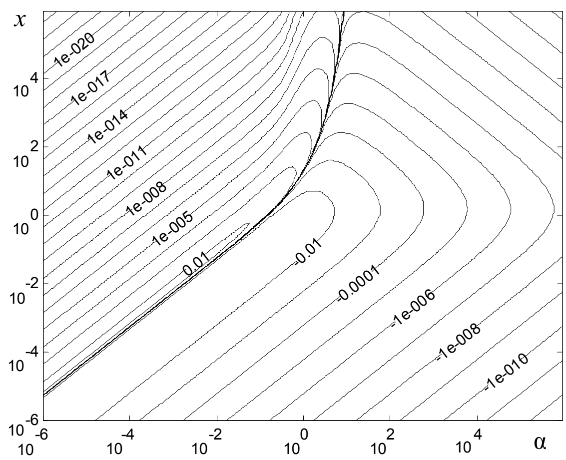

Figure 4. *Contour plot of* $h_{\alpha N}(x)$ for *N*=2.

Recall that demand is assumed to be perfectly inelastic and, accordingly, independent of the auction design. Thus Theorem 3 implies that the demand-weighted average price is weakly lower in PABAs than in UPAs. Furthermore, because only symmetric equilibria are considered, the most cost-effective generators will be accepted in both auctions for any level of demand. Thus production costs are the same in both procurement auctions and average mark-ups are weakly lower in PABAs than UPAs.

It is obvious that R_U=R_P=0 when $\bar{\varepsilon} = 0$, i.e. when market capacity is zero. It can also be shown that firms' total expected revenues are the same in both auctions under perfect competition and monopoly.

Theorem 4. With perfectly inelastic demand given by the Pareto distribution of the second kind, non-decreasing marginal costs, and $N \to \infty$ or N=1, the expected revenue of symmetric firms in a pay-as-bid auction is identical to their expected revenue in a uniform-price auction.

Proof: See Appendix.

5. EXAMPLE

Assume N=2, α=1, and linear marginal costs, $C'(x) \equiv \gamma x$. The marginal bid in the PABA for these parameter values can be calculated by means of integration by parts and (12):

$$\frac{p(\varepsilon)}{\beta} = \frac{\left(\frac{2\bar{p}}{\beta} + \frac{\gamma\bar{\varepsilon}}{\beta} + 2\gamma\right)\left(\frac{\bar{\varepsilon}}{\beta} + 1\right)^{-1/2}}{2\left(\frac{\varepsilon}{\beta} + 1\right)^{-1/2}} + \frac{\gamma\varepsilon}{2\beta} - \gamma\left(\frac{\varepsilon}{\beta} + 1\right)$$

The demand and price are normalised with respect to β. In the PABA, the average price as a function of demand (the equilibrium price) is:

$$\frac{\hat{p}(\varepsilon)}{\beta} = \frac{\int_0^{\varepsilon} p(x)dx}{\beta\varepsilon} = \left(\frac{2\bar{p}}{\beta} + \frac{\gamma\bar{\varepsilon}}{\beta} + 2\gamma\right)\left(\frac{\bar{\varepsilon}}{\beta} + 1\right)^{-1/2} \frac{\left(\frac{\varepsilon}{\beta} + 1\right)^{3/2} - 1}{3\varepsilon/\beta} + \frac{\gamma\varepsilon}{4\beta} - \gamma\left(\frac{\varepsilon}{2\beta} + 1\right).$$

The equilibrium price in the UPA can be calculated by means of (17):

$$p_U(\varepsilon) = \frac{\bar{p}\varepsilon}{\bar{\varepsilon}} + \frac{\varepsilon\gamma}{2}\ln\left(\frac{\bar{\varepsilon}}{\varepsilon}\right) = \beta\left[\frac{\bar{p}\varepsilon/\beta^2}{\bar{\varepsilon}/\beta} + \frac{\varepsilon\gamma/\beta}{2}\ln\left(\frac{\bar{\varepsilon}/\beta}{\varepsilon/\beta}\right)\right].$$

Figure 5 shows $p_U(\varepsilon)$, $p(\varepsilon)$, and $\hat{p}(\varepsilon)$ for γ=0.02, $\frac{\overline{p}}{\beta} = 10^3$ and $\frac{\overline{\varepsilon}}{\beta} = 10^4$. The latter corresponds to a risk of power shortage $\approx 10^{-4}$. The equilibrium price in the uniform-price auction equals *C'(0)* at zero demand. This is true in general for symmetric SFE of UPAs [13]. The unit with the lowest marginal cost still contributes to profits, as it is paid the marginal bid for ε>0. It is also true in general that the lowest bid in the pay-as-bid auction is higher than *C'(0)*. If not, then the cheapest unit would not contribute to profits because accepted bids are always paid their bid. Therefore, the equilibrium price is higher in the PABA when demand is sufficiently small. In both procurement auctions, all units except for the one with the highest bid are offered below the price cap. Thus the equilibrium price in a PABA is always below the price cap. In the uniform-price auction, on the other hand, the equilibrium price equals the price cap when demand equals or exceeds the market capacity. Hence, the equilibrium price is lower in the PABA when demand is sufficiently high.

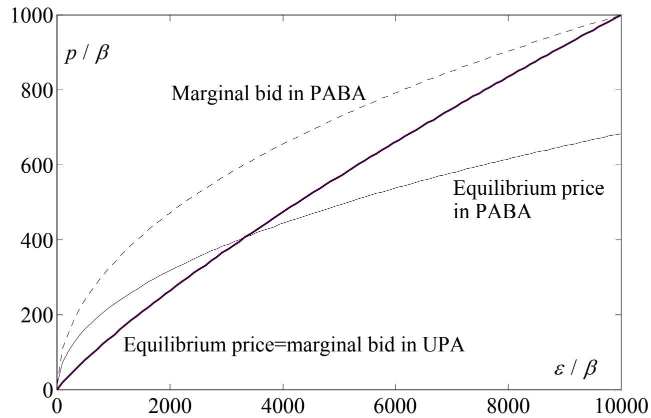

Figure 5. *Example for duopoly: prices as a function of demand are compared for the uniform-price auction (UPA) and pay-as-bid auction (PABA).*

6. CONCLUSIONS

The supply function equilibrium (SFE) framework for uniform-price auctions (UPAs) is similar to the organisation of most electricity markets and is often employed to model strategic bidding in such markets. This paper derives a SFE of a pay-as-bid auction (PABA), the auction used in the balancing market of Britain. In the analysis, demand is assumed to be perfectly inelastic and firms symmetric. Demand is also assumed to exceed market capacity with a positive probability, which is allowed to be arbitrarily small. This assumption rules out multiple equilibria. Unlike supply function equilibria of UPAs, pure strategy equilibria of PABAs do not always exist. In particular, it can be shown that a SFE of a PABA does not exist if there is a demand interval in which the hazard rate is locally upward sloping and marginal costs are sufficiently flat. However, a SFE always exists if the hazard rate of demand is monotonically decreasing and marginal costs are non-decreasing.

The equilibrium of a PABA is compared to the SFE of a UPA. It is assumed that demand is given by the Pareto distribution of the second kind, for which the inverse of the hazard rate is linear and increasing. It can then be shown that the demand-weighted average price in the PABA is equal to or lower than the price in the UPA.[14] Equality occurs in the cases of a monopoly or perfect competition. For a probability density function with a low degree of convexity, switching from a UPA to a PABA will substantially reduce the average mark-up in electricity procurement auctions. With a high degree of convexity, the change in the average mark-up is negligible. That mark-ups are lower and consumer surplus higher in PABAs is in line with previous theoretical studies based on other assumptions [8,9,24]. The result contradicts the findings of an experimental study [22]. However, that study did not consider uncertain demand.

The equilibrium price — the average price as a function of demand — is higher in PABAs compared to UPAs when demand is sufficiently low, but lower when demand is sufficiently high. This seems to be in agreement with the experimental finding that price volatility is lower in PABAs than UPAs [22].

A general assumption of the analysis is that firms are risk-neutral. Introducing risk aversion does not change the SFE of a UPA, as firms receive the best price for every demand outcome, given the bids of competitors. A risk-averse firm in a PABA, however, would put less weight on high-demand outcomes when profits are high and more weight on low-demand outcomes when profits are low. Hence, given the bids of competitors, risk-averse firms decrease their bids to increase profits for low-demand outcomes. Intuitively this would also be true in equilibrium. It appears that with risk-averse bidders, the advantages of PABAs are

[14] An analogous calculation would show that the demand-weighted average price in a pay-as-bid sales auction, in which the supply of the auctioneer follows a Pareto distribution of the second kind, is (weakly) higher compared to a uniform-price sales auction. Thus the auctioneer would prefer the pay-as-bid auction for positive as well as negative demand.

likely to increase.[15] Another advantage of PABAs is that the risk for tacit collusion is lowered compared to UPAs. This is shown by both Fabra [6] and Klemperer [19].

The larger risk for non-existent pure strategy equilibria in the PABA, shown in this paper and by Fabra et al. [9], could be a disadvantage. Kahn et al. [17] also point out that in a UPA it is optimal for small firms to simply bid their marginal costs while in a PABA, all firms must forecast market prices if they are to receive any contributions to profits. This introduces an additional fixed cost for small firms which could be disadvantageous to competition in the long-run.

As small imbalances are more likely than large imbalances, the Pareto distribution of the second kind is a more reasonable representation of the uncertain demand in balancing markets than the uniform distribution employed by Federico & Rahman [8]. Nonetheless, an interesting topic for future research is to compare PABAs and UPAs for other distributions. The normal distribution is a natural choice. However, as its hazard rate is increasing, one has to make sure that marginal costs are sufficiently steep to ensure the existence of a SFE.

Lastly, this paper focuses on the case of symmetric firms. Analogous to [14], it should be possible to analytically derive supply function equilibria of PABAs for firms with identical constant marginal costs but asymmetric capacities. The unique equilibrium is expected to be piece-wise symmetric. For more general cost functions, asymmetric equilibria could be calculated numerically as in [15].

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APPENDIX

Proof of Theorem 1

It follows from (8) that

$$\eta(p, S_i) = G\left[\breve{S}_{-i}(p) + S_i\right] - \breve{S}_{-i}{}'(p)\left(p - C'(S_i)\right). \tag{24}$$

The equality in (4) is valid for an interval of prices. Thus

$$G\left[\breve{S}(p)\right] - \breve{S}_{-i}{}'(p)\left[p - C'\left(\breve{S}_i(p)\right)\right] \equiv 0.$$

The expression above can be used to eliminate $\breve{S}_{-i}{}'(p)$ from (24). Accordingly, (24) can be written on the following form:

$$\begin{aligned}\eta(p, S_i) &= \frac{\left(p - C'\left(\breve{S}_i(p)\right)\right)G\left[\breve{S}_{-i}(p) + S_i\right] - G\left[\breve{S}(p)\right]\left(p - C'(S_i)\right)}{p - C'\left(\breve{S}_i(p)\right)} = \\ &= \frac{\left[p - C'\left(\breve{S}_i(p)\right)\right]\left\{G\left[\breve{S}_{-i}(p) + S_i\right] - G\left[\breve{S}(p)\right]\right\} + G\left[\breve{S}(p)\right]\left(C'(S_i) - C'\left(\breve{S}_i(p)\right)\right)}{p - C'\left(\breve{S}_i(p)\right)}.\end{aligned} \tag{25}$$

It follows from (4) that $p > C'\left(\breve{S}_i\right)$ for increasing supply functions. Further, marginal costs are non-decreasing and *G(x)*>0, as the hazard rate is never negative. To prove claim i) consider the case when *G(x)* is monotonically increasing. If $p \in \left[\breve{p}(0), p^*\right)$ then $S_i > \breve{S}_i(p) \Rightarrow$

$C'(S_i) > C'\left(\breve{S}_i(p)\right)$ and $G\left(\breve{S}_{-i}(p) + S_i\right) > G\left(\breve{S}(p)\right)$. Thus it follows from (25) that $\eta(p, S_i) > 0$ for $p \in \left[\breve{p}(0), p^*\right)$. Analogously, it can be proven that $\eta(p, S_i) < 0$ for $p \in \left(p^*, \overline{p}\right]$. The two conditions are fulfilled for all $S_i \in \left[0, \overline{\varepsilon}/N\right]$, which proves claim i.

There is a local minimum if $\eta\left(p^*-,S_i\right)<0$ and $\eta\left(p^*+,S_i\right)>0$. It follows from (25) that a local minimum occurs if $\left[p-C'\left(\breve{S}_i(p)\right)\right]G'\left[\breve{S}(p)\right]+G\left[\breve{S}(p)\right]C''\left(\breve{S}_i(p)\right)<0$, which proves claim ii). □

Proof of Theorem 4

It is known from (23) that

$$R_U-R_P=\bar{p}\bar{\varepsilon}\left[g_U\left(\alpha,N,\frac{\alpha\bar{\varepsilon}}{\beta}\right)-g_P\left(\alpha,N,\frac{\alpha\bar{\varepsilon}}{\beta}\right)\right]+$$
$$+(N-1)\int_0^{\bar{\varepsilon}}C'(u/N)\left[h_U\left(\alpha,N,\tfrac{\alpha u}{\beta}\right)-h_P\left(\alpha,N,\tfrac{\alpha u}{\beta}\right)\right]du.$$

Thus the auctions have the same expected revenue under perfect competition, if

$\lim_{N\to\infty}g_U\left(\alpha,N,\frac{\alpha\bar{\varepsilon}}{\beta}\right)=\lim_{N\to\infty}g_P\left(\alpha,N,\frac{\alpha\bar{\varepsilon}}{\beta}\right)$ and $\lim_{N\to\infty}(N-1)h_U\left(\alpha,N,\frac{\alpha u}{\beta}\right)=\lim_{N\to\infty}(N-1)h_P\left(\alpha,N,\frac{\alpha u}{\beta}\right)$.

These two equalities are proven below.

It follows from (16) that

$$\lim_{N\to\infty}g_P\left(\alpha,N,\frac{\alpha\bar{\varepsilon}}{\beta}\right)=\lim_{N\to\infty}\frac{\left(1+\frac{\alpha\bar{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha N}+1}-1}{\left(1-\frac{1}{\alpha N}\right)\frac{\alpha\bar{\varepsilon}}{\beta}\left(\frac{\alpha\bar{\varepsilon}}{\beta}+1\right)^{\frac{N-1}{\alpha N}}}=\frac{1}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}+1\right)^{\frac{1}{\alpha}}}.$$

Now consider $\lim_{N\to\infty}g_U\left(\alpha,N,\frac{\alpha\bar{\varepsilon}}{\beta}\right)$. It is known that $\lim_{x\to\infty}\frac{x^p}{a^x}=0$ if $a>1$ [21], which implies that $\lim_{N\to\infty}\frac{Nt^{-1}}{\left(\frac{\alpha\bar{\varepsilon}}{\beta t}\right)^N}=0$ if $t<\frac{\alpha\bar{\varepsilon}}{\beta}$. Thus only t infinitesimally close to $\frac{\alpha\bar{\varepsilon}}{\beta}$ will contribute to the value of the integral in (20).

$$\lim_{N\to\infty}g_U\left(\alpha,N,\tfrac{\alpha\bar{\varepsilon}}{\beta}\right)=\lim_{N\to\infty}\frac{N\int_0^{\frac{\alpha\bar{\varepsilon}}{\beta}}(1+t)^{\frac{-1}{\alpha}}t^{N-1}dt}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}\right)^N}=\lim_{N\to\infty}\frac{N\left(1+\frac{\alpha\bar{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha}}}{\left(\frac{\alpha\bar{\varepsilon}}{\beta}\right)^N}\int_0^{\frac{\alpha\bar{\varepsilon}}{\beta}}t^{N-1}dt=\left(1+\tfrac{\alpha\bar{\varepsilon}}{\beta}\right)^{\frac{-1}{\alpha}}=$$
$$=\lim_{N\to\infty}g_P\left(\alpha,N,\tfrac{\alpha\bar{\varepsilon}}{\beta}\right).$$

Using (21) it can also be shown that

$$\lim_{N\to\infty}(N-1)h_P\left(\alpha,N,\frac{\alpha u}{\beta}\right)=\lim_{N\to\infty}(N-1)\left(\frac{\alpha u}{\beta}+1\right)^{\frac{1-N}{\alpha N}-1}\frac{\left(\frac{\alpha u}{\beta}+1\right)^{1-\frac{1}{\alpha N}}-1}{\alpha N-1}=\frac{\left(\frac{\alpha u}{\beta}+1\right)^{\frac{-1}{\alpha}-1}u}{\beta}.$$

Now consider $\lim_{N\to\infty}(N-1)h_U\left(\alpha,N,\frac{\alpha u}{\beta}\right)$. With the same argument as above,

$\lim_{N\to\infty}\frac{(N-1)t^N}{\left(\frac{\alpha u}{\beta}\right)^N}=0$ if $t<\frac{\alpha u}{\beta}$. Thus only t infinitesimally close to $\frac{\alpha u}{\beta}$ will contribute to the value of the integral in (22).

$$\lim_{N\to\infty}(N-1)h_U\left(\alpha,N,\frac{\alpha u}{\beta}\right)=\lim_{N\to\infty}(N-1)\left(\frac{\alpha u}{\beta}\right)^{-N}\int_0^{\frac{\alpha u}{\beta}}\frac{(t+1)^{\frac{-1}{\alpha}-1}t^N\,dt}{\alpha}=$$

$$=\lim_{N\to\infty}(N-1)\left(\frac{\alpha u}{\beta}+1\right)^{\frac{-1}{\alpha}-1}\left(\frac{\alpha u}{\beta}\right)^{-N}\int_0^{\frac{\alpha u}{\beta}}\frac{t^N\,dt}{\alpha}=\frac{\left(\frac{\alpha u}{\beta}+1\right)^{\frac{-1}{\alpha}-1}u}{\beta}=\lim_{N\to\infty}(N-1)h_p\left(\alpha,N,\frac{\alpha u}{\beta}\right).$$

This analysis has assumed that $N\geq 2$, which excludes the case of a monopoly. In such a case, the inelastic auctioneer must buy from the monopolist who will offer all units at the price cap in both auctions (or arbitrarily close to the price cap).[16] Thus, expected revenue is identical in the two auctions. □

[16] Supply functions are assumed to fulfil $S'(p)<\infty$.

Paper V

Some inequalities related to the analysis of electricity auctions

Peter Hästö*

Department of Mathematical Sciences,

Norwegian University of Science and Technology,

7491 Trondheim, Norway

peter.hasto@helsinki.fi

Pär Holmberg†

Department of Economics,

Uppsala University,

SE-751 20 Uppsala, Sweden

par.holmberg@nek.uu.se

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Abstract

Most balancing markets of electric power are organized as uniform-price auctions. In 2001, the balancing market of England and Wales switched to a pay-as-bid auction with the intention of reducing wholesale electricity prices. Numerical simulations of an electricity auction model have indicated that this should lead to decreased average prices. In this article we prove two inequalities which give an analytic proof of this claim in the same model.

Keywords: uniform-price auctions, pay-as-bid auctions, wholesale electricity markets, inequalities.

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1 Introduction

In a balancing market, the system operator buys last-minute power from electric power producers. Most balancing markets are organized as uniform-price auctions (UPAs), in which all accepted bids receive the same price and the market price is determined by the marginal bid, i.e. the highest accepted bid. Following research by Klemperer & Meyer [1], Bolle [2] and Green & Newbery [3], bidding behaviour in electricity UPAs is often modelled with Supply Function Equilibria (SFE). The concept assumes that firms submit smooth supply functions simultaneously to a UPA in a one-shot game. In the non-cooperative Nash equilibrium, each firm commits to the supply function that maximizes expected profit given the bids of competitors and the properties of uncertain demand.

In 2001, electricity trading in the balancing market of England and Wales switched from a UPA to a pay-as-bid auction (PABA). As the name suggests, all accepted bids in PABAs are paid their bid. It was the belief of the British regulatory authority (Ofgem) that the reform would decrease wholesale electricity prices. Before the collapse of the California Power Exchange, a similar switch was considered also for that market [4].

It is not straightforward to establish whether prices will be lower or higher in a PABA, as firms will change their bidding strategy after a switch from a UPA to a PABA [4]. Federico and Rahman [5] compare the auction forms for two polar cases, perfect competition ($N \to \infty$) and monopoly (N=1), where N is the number of firms. In the competitive case, average prices are lower in the PABA when demand is elastic (price dependent). The same is true for the monopoly case, except when demand uncertainty is high. Fabra et al. [6] derive a Nash equilibrium for an asymmetric duopoly model (N=2) with single units, that is, marginal costs are constant and producers must submit a single price offer for their entire capacity. Fabra et al. show that for perfectly inelastic and certain demand, average prices are lower in a PABA than in a UPA. Numerical examples suggest that the difference could be substantial.

Holmberg [7] has derived a unique SFE for a PABA with symmetric firms and perfectly inelastic demand and shown that an equilibrium always exists if demand follows a *Pareto Distribution of the second kind*. Numerical calculations indicate that for this probability distribution, average prices are lower in a PABA compared to a UPA.

In this essay we prove two inequalities which provide an analytic proof of this claim. The inequalities are integral inequalities of rational functions and the proofs are based on investigating the derivatives of the functions involved. This robust method is often appreciated, perhaps due to the influence of Hardy, Littlewood and Pólya [9], and has been used widely by researchers specializing in inequalities. For instance, we draw upon studies of inequalities of the gamma and poly-gamma functions, e.g. [10, 11]. Before beginning the proofs, we provide a more detailed description of the context in which the inequalities arise.

2 Comparing pay-as-bid and uniform-price auctions

In the presence of perfectly inelastic demand and a risk of power shortage, both of which are realistic assumptions for balancing markets, it can be shown that the SFE of the pay-as-bid and uniform-price auctions are unique [7, 12]. To do so, it is assumed that the cost function C is convex, increasing and twice continuously differentiable. In this paper, we work with the derivative C' , marginal cost, and assume additionally that $C'(\overline{\varepsilon}/N) < \overline{p}$, where N is the number of symmetric firms, $\overline{\varepsilon}$ is total production capacity and $\overline{p}$ is the reservation price (the price cap). The unique equilibrium price of a UPA with symmetric firms, p_U, is given by

$$p_U(\varepsilon) = \frac{\overline{p}\varepsilon^{N-1}}{\overline{\varepsilon}^{N-1}} + (N-1)\varepsilon^{N-1} \int_{\varepsilon}^{\overline{\varepsilon}} \frac{C'(x/N)}{x^N}\, dx,$$

where $\varepsilon \geqslant 0$ is the demand outcome.

If a SFE of a PABA exists, the marginal bid as a function of demand is given by [7]

$$p_P(\varepsilon) = \frac{N[1-F(\overline{\varepsilon})]^{\frac{N-1}{N}}\overline{p} + \int_{\varepsilon}^{\overline{\varepsilon}} (N-1)C'(u/N)f(u)\,[1-F(u)]^{\frac{N-1}{N}-1}\, du}{N[1-F(\varepsilon)]^{\frac{N-1}{N}}}$$

where $f = F'$ is the probability density of demand. In some cases a pure strategy equilibrium does not exist, e.g. if there is some interval $\varepsilon \in [\varepsilon_-, \varepsilon_+]$ where $C'(\varepsilon/N)$ is constant and $f'(\varepsilon) > 0$ [7]. A decreasing density function does not guarantee the existence of a pure strategy equilibrium but one can show that a pure strategy equilibrium always exists if f has the form [7]

$$\beta^{\frac{1}{\alpha}}(\alpha x + \beta)^{-\frac{1}{\alpha}-1} \quad \alpha \geqslant 0, \beta \geqslant 0,$$

which implies that F is a *Pareto distribution of the second kind* [8] (the case $\alpha = 0$ is by continuous extension). Henceforth, only auctions in which demand follows a *Pareto distribution of the second kind* are considered.

The total expected revenue for symmetric firms bidding in a pay-as-bid auction is [7]

$$R_P = \int_0^{\overline{\varepsilon}} \big(1-F(\varepsilon)\big)p_p(\varepsilon)\, d\varepsilon = \overline{p}\,\overline{\varepsilon} g_P\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big) + (N-1)\int_0^{\overline{\varepsilon}} C'\Big(\frac{u}{N}\Big) h_P\Big(\alpha, N, \frac{\alpha u}{\beta}\Big)\, du,$$

where we have denoted

$$g_P(\alpha, N, x) = \frac{(1+x)^{-\frac{1}{\alpha N}+1} - 1}{(1-\frac{1}{\alpha N})x(1+x)^{\frac{N-1}{\alpha N}}} \quad \text{and} \quad h_P(\alpha, N, x) = (x+1)^{\frac{1-N}{\alpha N}-1}\frac{(x+1)^{1-\frac{1}{\alpha N}} - 1}{\alpha N - 1}.$$

It was further found in [7] that this can be simplified to

$$R_P = (\overline{p} - c)\overline{\varepsilon} g_P\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big) + \beta^{\frac{1}{\alpha}} c \int_0^{\overline{\varepsilon}} (\alpha\varepsilon + \beta)^{-\frac{1}{\alpha}} d\varepsilon$$

for constant marginal costs, $C' \equiv c$.

The total expected revenue for symmetric firms bidding in a UPA is [7]

$$\begin{aligned} R_U &= \int_0^{\overline{\varepsilon}} f(\varepsilon)\varepsilon p_U(\varepsilon)\, d\varepsilon + (1 - F(\overline{\varepsilon}))\overline{\varepsilon}\,\overline{p} \\ &= \overline{p}\,\overline{\varepsilon} g_U\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big) + (N-1)\int_0^{\overline{\varepsilon}} C'\Big(\frac{u}{N}\Big) h_U\Big(\alpha, N, \frac{\alpha u}{\beta}\Big)\, du, \end{aligned}$$

where we used the functions

$$g_U(\alpha, N, x) = \frac{N}{x^N}\int_0^x (1+t)^{-\frac{1}{\alpha}} t^{N-1} dt \quad \text{and} \quad h_U(\alpha, N, x) = \frac{1}{\alpha x^N}\int_0^x (1+t)^{-\frac{1}{\alpha}-1} t^N dt.$$

Continuing to follow [7], this simplifies to the following expression for the case of constant marginal costs:

$$R_U = (\overline{p} - c)\overline{\varepsilon} g_U\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big) + \beta^{\frac{1}{\alpha}} c \int_0^{\overline{\varepsilon}} (\alpha\varepsilon + \beta)^{-\frac{1}{\alpha}} d\varepsilon.$$

We conclude this section by stating the implications of the model described so far for the inequalities in the next section.

Theorem 1. *For non-decreasing marginal costs, $R_P \leqslant R_U$. Equality occurs for $N = 1$.*

Proof. We denote $G = g_U - g_P$ and $H = h_U - h_P$. It follows directly from Theorem 5 that H has profile $-$ or $-|+$. From the formulae for R_P and R_U we find that

$$R_U - R_P = \overline{p}\,\overline{\varepsilon} G\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big) + (N-1)\int_0^{\overline{\varepsilon}} C'\Big(\frac{u}{N}\Big) H\Big(\alpha, N, \frac{\alpha u}{\beta}\Big)\, du.$$

If H changes sign below $\overline{\varepsilon}$, then we define x^* to be the point where the sign-change occurs, otherwise we set $x^* = \overline{\varepsilon}$. Since C' is non-decreasing, we find that $C'(u/N) - C'(x^*/N)$ is non-negative when H is non-negative and non-positive when H is non-positive. Hence

$$\int_0^{\overline{\varepsilon}} C'\Big(\frac{u}{N}\Big) H\Big(\alpha, N, \frac{\alpha u}{\beta}\Big)\, du \geqslant C'\Big(\frac{x^*}{N}\Big)\int_0^{\overline{\varepsilon}} H\Big(\alpha, N, \frac{\alpha u}{\beta}\Big)\, du.$$

Suppose our initial data gave us R_P and R_U. Now we keep all the data fixed, except marginal cost which is set to the constant $C'(x^*/N)$ and leads to the revenues R'_P and R'_U. Then we have shown that $R_U - R_P \geqslant R'_U - R'_P$. It now suffices to show that $R'_U \geqslant R'_P$ to prove our claim. In this case we use the formulae for the constant marginal cost case to calculate

$$R'_U - R'_P = (\overline{p} - c)\overline{\varepsilon} G\Big(\alpha, N, \frac{\alpha\overline{\varepsilon}}{\beta}\Big).$$

It follows from Theorem 3 that G is a positive function, so that $R'_U \geqslant R'_P$. □

Recall that demand is assumed to be perfectly inelastic and accordingly independent of auction design. Thus Theorem 1 implies the following result:

Corollary 2. *Average prices are weakly lower in PABAs than in UPAs.*

3 The mathematical treatment of the inequalities

In order to state the proofs succinctly, we introduce some new notation in this section. We use $a = 1/\alpha$, and multiply the functions G and H (from the proof of Theorem 1) by suitable powers of x as this does not affect their sign.

Theorem 3. *Let $x, a \in (0, \infty)$ and $N \geqslant 1$. The inequality*

$$\frac{1}{N-a} x^{N-1}(x+1)^{(1-N)a/N}\big[(1+x)^{1-a/N} - 1\big] \leqslant \int_0^x (1+t)^{-a} t^{N-1} dt$$

holds when $N \neq a$. Corresponding to $N = a$, we also have

$$\frac{1}{N}\frac{x^{N-1}}{(1+x)^{N-1}}\log(x+1) \leqslant \int_0^x (1+t)^{-N} t^{N-1} dt.$$

Proof. The second inequality follows from the first as $a \to N$. We will use the short-hand notation $b = a/N$. To prove the first inequality for $b \neq 1$ we define

$$\begin{aligned} g(x) &= \int_0^x (1+t)^{-a} t^{N-1} dt - \frac{x^{N-1}}{N-a}(1+x)^{(1-N)b}\big[(1+x)^{1-b} - 1\big] \\ &= \int_0^x (1+t)^{-a} t^{N-1} dt - \frac{x^{N-1}}{N-a}\big[(1+x)^{1-a} - (1+x)^{(1-N)b}\big]. \end{aligned}$$

Then the claim is that $g \geqslant 0$. Since $g(0) = 0$, it suffices to show that $g'(x) \geqslant 0$ for all x. We find that

$$
\begin{aligned}
(N-a)g'(x) &= (N-a)(1+x)^{-a}x^{N-1} - (N-1)x^{N-2}\big[(1+x)^{1-a} - (1+x)^{(1-N)b}\big] \\
&\quad - x^{N-1}\big[(1-a)(1+x)^{-a} - (1-N)b(1+x)^{(1-N)b-1}\big]. \\
&= (1+x)^{-a}x^{N-2}\big\{(N-a)x - (N-1)\big[1+x-(1+x)^{b}\big] \\
&\quad - x\big[1-a+(N-1)b(1+x)^{b-1}\big]\big\} \\
&= (N-1)(1+x)^{-a}x^{N-2}\big\{-1+(1+x)^{b} - bx(1+x)^{b-1}\big\} \\
&= (N-1)b(1-b)(1+x)^{-a}x^{N-2}\int_0^x t(1+t)^{b-2}dt.
\end{aligned}
\tag{4}
$$

By the definition of b, we have $\frac{(N-1)b(1-b)}{N-a} = (N-1)N^{-2}a \geqslant 0$, so our expression for g' directly implies that g is increasing. $\square$

We define the function $h\colon \mathbb{R}_+ \to \mathbb{R}$ by

$$h(x) = \int_0^x (1+t)^{-a-1}t^N dt - \frac{x^N}{N-a}\big[(1+x)^{-a} - (1+x)^{(1-N)a/N-1}\big]$$

for $N \geqslant 1$ and $a \in (0,\infty)$ with $N \neq a$. For $N = a$ we define h by the corresponding limit:

$$h(x) = \int_0^x (1+t)^{-N-1}t^N dt - \frac{x^N}{N}(1+x)^{-N}\log(1+x).$$

Theorem 5. *If $N > 1$, then there exists a value $x^* \in (0,\infty)$ such that $h \leqslant 0$ on $(0,x^*)$ and $h \geqslant 0$ on (x^*,∞). If $N = 1$, then $h \leqslant 0$ on $[0,\infty)$.*

Proof. We start by assuming that $N \neq a$ and again employ the notation $b = a/N$. Differentiating h we find that

$$
\begin{aligned}
(N-a)h'(x) &= (N-a)(1+x)^{-a-1}x^N - Nx^{N-1}\big[(1+x)^{-a} - (1+x)^{b-a-1}\big] \\
&\quad - x^N\big[-a(1+x)^{-a-1} - \big(b-a-1\big)(1+x)^{b-a-2}\big] \\
&= x^{N-1}(1+x)^{-a-1}\big\{(N-a)x - N\big[1+x-(1+x)^{b}\big] \\
&\quad + x\big[a+\big(b-a-1\big)(1+x)^{b-1}\big]\big\} \\
&= x^{N-1}(1+x)^{-a-1}\big\{-N + N(1+x)^{b} + \big(b-a-1\big)x(1+x)^{b-1}\big\}.
\end{aligned}
$$

We ignore the factor $x^{N-1}(1+x)^{-a-1}$ which is not relevant for the sign, so h' is positive if and only if

$$
\begin{aligned}
q(x) &= (N-a)^{-1}\big[-N + N(1+x)^{b} + \big((1-N)b-1\big)x(1+x)^{b-1}\big] \\
&= (N-a)^{-1}\big[-N + (1+x)^{b-1}\big(N+(N-1)(1-b)x\big)\big]
\end{aligned}
$$

is also positive. Using the second expression we find the formula

$$q'(x) = (N-a)^{-1}(b-1)(1+x)^{b-2}\big[1-(N-1)bx\big] = -\frac{1}{N}(1+x)^{b-2}\big[1-(N-1)bx\big]$$

for the derivative. The claim regarding the case $N = 1$ follows directly from this, so from now on we assume that $N > 1$. Then q is initially decreasing and then increasing. Since $q(0) = 0$, this means that q and hence h' has the profile $-$ or $-|+$. By continuity, we see that this conclusion holds also for $N = a$.

Since $h(0) = 0$, this means that h has profile $-$ or $-|+$. Hence we need to investigate $\lim_{x\to\infty} h(x)$. For $N < a$ we go back to the definition of h and note that the second term tends to zero as $x \to \infty$. Therefore h is the integral of a positive function, hence positive. For $N > a$ we use the inequality $(1+t)^N - t^N \le N(1+t)^{N-1}$ (to derive this inequality divide by t^N, set $s = 1/t$, and combine the two last lines of (4)) to derive

$$\begin{aligned}\int_0^x (1+t)^{-a-1}t^N dt &= \int_0^x (1+t)^{N-a-1}dt + O\Big(\int_0^x (1+t)^{N-a-2}dt\Big)\\ &= \frac{(1+x)^{N-a}}{N-a} + O\Big(\frac{(1+x)^{N-a-1}-1}{N-a-1}\Big)\end{aligned} \tag{6}$$

provided $N \ne a+1$. Using this we conclude that

$$\begin{aligned}(N-a)h(x) &= (1+x)^{N-a} + O\big(x^{N-a-1}-1\big) - x^N\big[(1+x)^{-a} - (1+x)^{(1-N)b-1}\big]\\ &= ((1+x)^N - x^N)(1+x)^{-a} + O\big(x^{N-a-1}-1\big) + x^N(1+x)^{(1-N)b-1}\\ &= O\big(x^{N-a-1}-1\big) + x^N(1+x)^{(1-N)b-1}.\end{aligned}$$

Since $N + (1-N)b - 1 > \max\{0, N-a-1\}$, we see that the second term is dominant. In the case $N - a = 1$, we have to adjust (6) accordingly with a logarithmic term but the conclusion still holds. Thus $h(\infty) = \infty$ if $N > a$.

The borderline case, $N = a$, remains to be investigated. We have from (6) that

$$\int_0^x (1+t)^{-N-1}t^N dt = \log(1+x) + O(1).$$

So it follows that

$$h(x) = \log(1+x) + O(1) - \frac{1}{N}\frac{x^N}{(1+x)^N}\log(1+x) = \Big(1 - \frac{1}{N}\frac{x^N}{(1+x)^N}\Big)\log(1+x) + O(1).$$

Since $N > 1$, the first term is unbounded and hence dominant. $\square$

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