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Effective Domains and Admissible Domain Representations

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Abstract

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This thesis consists of four papers in domain theory and a summary. The first two papers deal with the problem of defining effectivity for continuous cpos. The third and fourth paper present the new notion of an admissible domain representation, where a domain representation D of a space X is λ-admissible if, in principle, all other λ-based domain representations E of X can be reduced to X via a continuous function from E to D.

In Paper I we define a cartesian closed category of effective bifinite domains. We also investigate the method of inducing effectivity onto continuous cpos via projection pairs, resulting in a cartesian closed category of projections of effective bifinite domains.

In Paper II we introduce the notion of an almost algebraic basis for a continuous cpo, showing that there is a natural cartesian closed category of effective consistently complete continuous cpos with almost algebraic bases. We also generalise the notion of a complete set, used in Paper I to define the bifinite domains, and investigate what closure results that can be obtained.

In Paper III we consider admissible domain representations of topological spaces. We present a characterisation theorem of exactly when a topological space has a λ-admissible and κ-based domain representation. We also show that there is a natural cartesian closed category of countably based and countably admissible domain representations.

In Paper IV we consider admissible domain representations of convergence spaces, where a convergence space is a set X together with a convergence relation between nets on X and elements of X. We study in particular the new notion of weak κ-convergence spaces, which roughly means that the convergence relation satisfies a generalisation of the Kuratowski limit space axioms to cardinality κ. We show that the category of weak κ-convergence spaces is cartesian closed. We also show that the category of weak κ-convergence spaces that have a dense, λ-admissible, κ-continuous and α-based consistently complete domain representation is cartesian closed when $\alpha \leq \lambda \geq \kappa$. As natural corollaries we obtain corresponding results for the associated category of weak convergence spaces.

Keywords: domain theory, admissible domain representation, cartesian closure, effective domains, κ-sequential space, limit space

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List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I	Hamrin, G., Stoltenberg-Hansen V. (2002) Cartesian closed categories of effective domains. In H. Schwichtenberg and R. Steinbrüggen, editors, *Proof and System-Reliability*, 1-20, Kluwer Academic Publisher
II	Hamrin. G. (2005) Two categories of effective continuous cpos. Technical Report U.U.D.M. Report 2005:21
III	Hamrin. G. (2005) Admissible domain representations of topological spaces. Technical Report U.U.D.M. Report 2005:16
IV	Hamrin. G. (2005) Admissible domain representations of convergence spaces. Technical Report U.U.D.M. Report 2005:22

Contents

1 Introduction

In this introductory chapter we give some background material, necessary for the understanding of the results in this thesis. The results will be summarised in Chapter 2.

1.1 Background

The mathematical theory of domains started with the works of D. S. Scott [27, 28] and Y. L. Ershov [6, 8]. It has by now developed into a rich subject with applications in many fields of science. Thus we restrict this background to the particular aspects of domain theory we study in the thesis. We will focus upon domain theory as a theory of approximation and as a theory of computability on mathematical structures via approximations. Foundational work in these areas have been made by V. Stoltenberg-Hansen and J.V. Tucker. For references, see [36, 33, 37, 31] and the two handbook chapters [34, 35].

1.1.1 Background on approximations

In this section we give a background to domain theory as a theory of approximations. A reference for this material is [30].

We first consider the problem of approximation abstractly. Let P be a set of approximations of elements of a structure X. Suppose that for each $x \in X$ we have a unique set of approximations $\mathrm{approx}(x)$ in P, and that there is a unique element $\perp \in P$ such that $\perp \in \mathrm{approx}(x)$ for each $x \in X$. Suppose further that we have a partial ordering $\sqsubseteq$ on P such that for all $a, b \in \mathrm{approx}(x)$ there is a better approximation $c \in \mathrm{approx}(x)$ in the sense that $a \sqsubseteq c$ and $b \sqsubseteq c$. Then $(P, \sqsubseteq)$ is called an *approximation structure* for X.

We now "complete" P by adding the points of X. Each point $x \in X$ is added so as to respect the approximation ordering $\sqsubseteq$. As a result we obtain the *ideal completion* D of $(P, \sqsubseteq)$. That is, D is the set of ideals of $(P, \sqsubseteq)$, ordered by the inclusion ordering. Then D is a directed complete partial order or a *domain*. We see that there is both an injection of X into D and an injection of P into D. Thus D can be considered as a topological space which admits a very natural notion of approximation in that it contains each $x \in X$, together with the subset $\mathrm{approx}(x)$.

Now consider the problem of approximating a topological space. Recall that a topology on a set X is a family of subsets $\tau \subseteq 2^X$, characterised by the closure under union and finite intersections of elements from τ. Suppose that τ is a T_0-topology. A domain D approximating (X,τ) is then obtained by letting D be the ideal completion of the approximation structure $(B_\tau, \supseteq)$, where B_τ is a topological base for τ.

A topology on X can also be described by a so-called *convergence class* of nets on X, since there is a natural one-to-one correspondence between the topologies and the convergence classes on X. (See Chapter 2 of [16] for a detailed treatment.) Let $S : (\Sigma, \leq) \to (X,\tau)$ be a net and recall that S *converges* to $x \in X$ with respect to τ if the image of S is eventually in each $O \in \tau$ such that $x \in O$.

Let $\to$ be a binary relation between nets on X and elements of X. Then $\to$ is a *convergence class for* X if it, besides a property that guarantees the existence of iterated limits, satisfies the following generalisation [5] of the Kuratowski limit space axioms [18], for each net S on X and each $x \in X$:

1. $S \to x$, if S is constantly equal to x.
2. If $S \to x$ and S' is a subnet of S then $S' \to x$.
3. If $S \not\to x$ then there is a subnet S' of S such that for each subnet S'' of S' we have that $S'' \not\to x$.

It is easy to see that convergence in a topological space generates a convergence class for X. To each convergence class we associate a topology in the following way. For each subset $A \subseteq X$ we define the closure $\bar{A} \subseteq X$ of A by $x \in \bar{A}$ if and only if there is a net S on A such that $S \to x$. This describes a closure operator on X, and hence there is a unique topology τ on X associated with X such that $S \to x$ if and only if S converges to x with respect to τ. Thus it is a natural approach to the problem of approximating (X,τ) to consider the convergence class associated with τ.

In Paper III we in particular focus on the case when the convergence class can be characterised by a collection of nets for which there is some cardinality κ such that all index sets have cardinality less than or equal to κ. These spaces we call *κ-net spaces*. We investigate the κ-net spaces via a new notion of an *admissible* domain representation. We present different admissible domain representations and attempt to characterise the topological spaces that can be represented by them. We also show that there is a natural cartesian closed category of T_0-spaces that have a countably based and countably admissible domain representation.

Our goal to construct cartesian closed categories of spaces with different types of admissible domain representations leads us to also consider non-topological spaces. In Paper IV we consider a generalisation of the Kuratowski limit spaces to a class of spaces called *weak κ-convergence spaces*. We prove that the category of weak κ-convergence spaces is cartesian closed.

We also show that there is a natural cartesian closed subcategory of weak κ-convergence spaces having an admissible domain representation. Analogous results are obtained for the associated cartesian closed categories of κ-convergence spaces and weak convergence spaces.

1.1.2 Background on computability

In this section we give a background to domain theory as a theory of computability.

A fundamental question in mathematics is when a function on the natural numbers is "computable". The answer is intuitively clear: $f: \mathbb{N} \to \mathbb{N}$ is computable if we for all inputs $n \in \mathbb{N}$ can compute the result $f(n)$ in finite time, using some mechanical device as an existing computer.

To make a mathematically precise definition of the term *computable function* is more troublesome. The different sensible suggestions made all define the same class of functions as computable, namely the *recursive functions*. This lead to the generally accepted Church-Turing thesis: *Every computable function belongs to the class of recursive functions*. Thus the study of computability on $\mathbb{N}$ is the study of the recursive functions [24, 21].

The Church-Turing thesis tells us what computability theory we should use on the natural numbers, and hence what can in principle be computed with help of a digital machine. We wish to extend this computability notion to uncountable mathematical structures. Consider an algebraic structure such as a ring. Suppose that we have an indexing or *numbering* of the elements of the ring with elements in $\mathbb{N}$. Then we transform questions of computability of functions on and between rings to questions of existence of recursive functions on and between the subsets of $\mathbb{N}$ that index the rings. This approach was first used by Fröhlich and Shepherdson [12] and later developed by Rabin [23] and Mal'cev [19]. The theory of numberings has then been thoroughly developed by Ershov (in for example [7], [9] and [11]).

We now apply the theory of numberings on domain theory. Let X be a structure and let D be the ideal completion of an approximation structure P for X. Suppose that all the basic relations on P, such as $\sqsubseteq$, is at least recursively enumerable. If P is countable then we encode the finite elements of the domain and the relations between them with help of a numbering and use recursive functions to represent operations on and between domains. Then D is an *effective domain*. As an example of an effective domain for the real numbers we can take the ideal completion of the set of rational intervals, partially ordered by reverse inclusion. The basic relations are recursive, since they are expressed in terms of comparison of rational numbers.

One reason this approach to computability theory on uncountable structures is useful is that there are many natural cartesian closed categories of domains. If we consider a cartesian closed category of effective domains we can lift the notion of computability to higher orders. An early example of this is Ershov's representation [10] of the Kleene-Kreisel functionals [17] from constructive mathematics by the pure type structure over the flat domain of natural numbers. In Paper I and Paper II we consider the problem of defining large categories of effective continuous cpos that are cartesian closed.

1.2 Domain theory

In this section we recall some preliminary notions of domain theory. Standard references for this material are the textbook [32] and [1].

1.2.1 Basic definitions

Let $D = (D; \sqsubseteq, \perp)$ be a partially ordered set with least element $\perp$. A non-empty set $A \subseteq D$ is *directed* if for each $x, y \in A$ there is $z \in A$ such that $x, y \sqsubseteq z$. D is a *complete partial order* (abbreviated *cpo*) if whenever $A \subseteq D$ is directed then $\bigsqcup A$ (the least upper bound or supremum of A) exists in D. A function $f\colon D \to E$ between cpos is *continuous* if f is monotone and for each directed set $A \subseteq D$

$$f(\textstyle\bigsqcup_D A) = \bigsqcup_E \{f(x) : x \in A\}.$$

For cpos D and E we define the *function space* $[D \to E]$ of D and E by

$$[D \to E] = \{f\colon D \to E \mid f \text{ continuous}\}.$$

We order $[D \to E]$ by

$$f \sqsubseteq g \iff (\forall x \in D)(f(x) \sqsubseteq g(x)).$$

Then $[D \to E]$ is a cpo where, for a directed set $\mathscr{F} \subseteq [D \to E]$ and $x \in D$,

$$(\textstyle\bigsqcup \mathscr{F})(x) = \bigsqcup_E \{f(x)\colon f \in \mathscr{F}\}.$$

We form a category whose objects are cpos and whose morphisms are continuous functions between cpos. It is well-known and easy to prove that this category is cartesian closed, where the exponent is the function space and the product of cpos D and E is given by

$$D \times E = \{(x, y)\colon x \in D, y \in E\}$$

and ordered by

$$(x, y) \sqsubseteq (z, w) \iff x \sqsubseteq_D z \text{ and } y \sqsubseteq_E w.$$

Definition 1.2.1. Let $D = (D; \sqsubseteq, \bot)$ be a cpo.

1. For $x, y \in D$ we say that x is *way below* y, denoted $x \ll y$, if for each directed set $A \subseteq D$,
$$y \sqsubseteq \bigsqcup A \Longrightarrow (\exists z \in A)(x \sqsubseteq z).$$
2. $a \in D$ is said to be *compact* if $a \ll a$. The set of compact elements in D is denoted D_c.

It is easily verified that $x \ll y \implies x \sqsubseteq y$, and that $z \sqsubseteq x \ll y \sqsubseteq w \implies z \ll w$.

Definition 1.2.2. Let $D = (D; \sqsubseteq, \bot)$ be a cpo. Then D is *continuous* if
1. the set $\{y \in D : y \ll x\}$ is directed (w.r.t. $\sqsubseteq$); and
2. $x = \bigsqcup\{y \in D : y \ll x\}$.

We use the notation $\twoheaddownarrow x = \{y \in D : y \ll x\}$ and $\twoheaduparrow x = \{y \in D : x \ll y\}$. Similarly we let $\downarrow x = \{y \in D : y \sqsubseteq x\}$ and $\uparrow x = \{y \in D : x \sqsubseteq y\}$.

As observed above, the way below relation $\ll$ is reflexive only for compact elements. However, for continuous cpos it satisfies the following crucial *interpolation property*.

Lemma 1.2.3. *Let D be a continuous cpo. Let $M \subseteq D$ be a finite set and suppose that $M \ll y$. Then there is $x \in D$ such that $M \ll x \ll y$.*

It follows that if D is a continuous cpo then $\twoheaddownarrow y$ is directed with respect to $\ll$ for each $y \in D$.

Let $D = (D; \sqsubseteq, \bot)$ be a cpo. A subset $B \subseteq D$ is a *base* for D if for each $x \in D$,
$$\mathrm{approx}_B(x) = \{a \in B : a \ll x\}$$
is directed and $\bigsqcup \mathrm{approx}_B(x) = x$. Thus all information about the cpo D is contained in a base.

Proposition 1.2.4. *A cpo is continuous if, and only if, it has a base.*

Also continuous functions between continuous cpos are determined by their behaviour on the bases.

Proposition 1.2.5. *Let D and E be continuous cpos with bases B_D and B_E respectively. A function $f : D \to E$ is continuous if, and only if, f is monotone and for each $x \in D$,*
$$(\forall b \in \mathrm{approx}_{B_E}(f(x)))(\exists a \in \mathrm{approx}_{B_D}(x))(b \ll f(a)).$$

Definition 1.2.6. A cpo D is *algebraic* if the set D_c of compact elements is a base for D.

Thus the algebraic cpos make up a subclass of the continuous cpos. An algebraic cpo is in general a simpler structure to deal with than a continuous cpo, since the way below relation $\ll$ coincides with $\sqsubseteq$ on its canonical base of compact elements. This is particularly useful when dealing with effectivity. Nonetheless, for each continuous cpo D there is an algebraic cpo E such that D is a projection of E.

Let D and E be cpos. Then a pair of functions $e{:}D \to E$ and $p{:}E \to D$ is a *projection pair* from D to E if they are continuous and

$$p \circ e = \mathrm{id}_D \quad \text{and} \quad e \circ p \sqsubseteq \mathrm{id}_E$$

where id is the identity function.

Let $P = (P, \leq)$ be a preorder. A set $I \subseteq P$ is an *ideal* if directed and if $x \in I$ and $y \leq x$ then $y \in I$. Let $\mathrm{Idl}(P, \leq)$ be the set of ideals ordered under inclusion. It is easily verified that $\mathrm{Idl}(P, \leq)$ is an algebraic cpo.

Let D be a continuous cpo with a base B and let $E = \mathrm{Idl}(B; \sqsubseteq)$. Define $e{:}D \to E$ and $p{:}E \to D$ by

$$e(x) = \mathrm{approx}_B(x) = \{a \in B{:} a \ll x\} \quad \text{and} \quad p(I) = \bigsqcup_D I.$$

Proposition 1.2.7. *The pair (e, p) is a projection pair from D to E.*

1.2.2 The function space

In this section we review the fact that the categories of consistently complete continuous cpos and consistently complete algebraic cpos are cartesian closed.

Definition 1.2.8. A cpo $D = (D; \sqsubseteq, \perp)$ is *consistently complete* if whenever $x, y \in D$ is bounded from above (or consistent) then $x \sqcup y$, the supremum of x and y, exists in D.

Given cpos D and E with bases B_D and B_E we want to construct a base for the function space $[D \to E]$. It turns out that such a base, under appropriate conditions, can be taken as finite suprema of step functions determined from B_D and B_E. Here is the definition of a step function.

Definition 1.2.9. Let $D = (D; \sqsubseteq, \perp)$ and $E = (E; \sqsubseteq, \perp)$ be cpos. For $a \in D$ and $b \in E$, define $\langle a; b\rangle{:} D \to E$ by

$$\langle a; b\rangle(x) = \begin{cases} b & \text{if } a \ll x \\ \perp & \text{otherwise.} \end{cases}$$

It is easily verified that each step function is continuous. Recall that if a is compact then $a \ll x \Longleftrightarrow a \sqsubseteq x$.

Proposition 1.2.10. *Let D and E be cpos and let $a \in D$ and $b \in E$.*

1. *Suppose $f\colon D \to E$ is continuous. Then*

$$b \ll f(a) \Longrightarrow \langle a;b\rangle \ll f.$$

2. *If D and E are continuous cpos with bases B_D and B_E and $f\colon D \to E$ is continuous then*

$$f = \bigsqcup\{\langle a;b\rangle\colon a \in B_D,\ \ b \in B_E,\ \langle a;b\rangle \ll f\}.$$

In important cases (i) is an equivalence. For example, if a and b are compact then $\langle a;b\rangle$ is compact and $\langle a;b\rangle \sqsubseteq f \Longleftrightarrow b \sqsubseteq f(a)$.

The characterisation in the following proposition is important when we consider the effectivity of the function space construction.

Proposition 1.2.11. *Let D be a continuous cpo, E a consistently complete cpo, and let $a_1,\dots,a_n \in D$ and $b_1,\dots,b_n \in E$. Then*

$$\{\langle a_1;b_1\rangle,\dots,\langle a_n;b_n\rangle\} \quad \textit{is consistent in} \quad [D \to E]$$

if, and only if,

$$\forall I \subseteq \{1,\dots,n\}(\bigcap_{i\in I} {\uparrow\!\!\uparrow} a_i \neq \emptyset \Longrightarrow \{b_i\colon i \in I\} \ \ \textit{consistent}).$$

Using Proposition 1.2.10 (ii) it is straightforward to prove that the categories of consistently complete continuous and algebraic cpos are cartesian closed.

Theorem 1.2.12. *Let D and E be continuous cpos with bases B_D and B_E. If E is consistently complete then $[D \to E]$ is continuous and consistently complete. A base $B_{[D\to E]}$ for $[D \to E]$ is*

$$\{\bigsqcup_{i=1}^{n}\langle a_i;b_i\rangle : a_i \in B_D,\ b_i \in B_E,\ \{\langle a_1;b_1\rangle,\dots,\langle a_n;b_n\rangle\} \text{ consistent}\}.$$

For consistently complete algebraic cpos we let the bases be D_c and E_c. For $a \in D_c$ and $b \in E_c$ the step function $\langle a;b\rangle$ is compact. It follows that $B_{[D\to E]}$ is a base for $[D \to E]$ consisting only of compact elements. This shows that $[D \to E]$ is a consistently complete algebraic cpo.

1.2.3 Bifinite domains and the Plotkin power domain

In this section we review bifinite domains and the Plotkin power domain construction. Let $\mathscr{P}_f(A)$ denote the set of finite subsets of a set A and let $\mathscr{P}_f{}^*(A)$ denote the set of non-empty finite subsets of A.

Definition 1.2.13. Let D be a continuous cpo and let B_D be a basis for D. The *Plotkin power domain* $P_P(D)$ of D is defined as $\mathrm{Idl}(\wp_f^*(B_D), \ll_{EM})$, where $\ll_{EM}$ is the *continuous Egli-Milner* preorder defined by

$$
\begin{aligned}
A \ll_{EM} B \quad :\Leftrightarrow \quad & \forall b \in B\ \exists a \in A\ a \ll b \\
\wedge \quad & \forall a \in A\ \exists b \in B\ a \ll b.
\end{aligned}
$$

The structure $(P_P(D); \subseteq, \perp)$ is a continuous cpo. It has a basis consisting of principal ideals of the form $[A] = \{B \in \wp_f^*(B_D) : B \ll_{EM} A\}$, for $A \in \wp_f^*(B_D)$. If D is an algebraic cpo then $P_P(D)$ is an algebraic cpo, but not necessarily a consistently complete algebraic cpo even if D is consistently complete.

There is a large class of algebraic cpos that is closed both under function spaces and Plotkin power domain construction, namely the bifinite domains. We here give a standard definition. In Paper I and Paper II we consider a non-standard definition.

Definition 1.2.14. Let $(D; \sqsubseteq, \perp)$ be an algebraic cpo. Given $A \in \wp_f(D_c)$, let $\mathrm{mub}(A)$ denote the set of minimal upper bounds of A. Define $\mathrm{mub}^n(A)$ for $n \in \mathbb{N}$ by

1. $\mathrm{mub}^0(A) := A$.
2. $\mathrm{mub}^{n+1}(A) := \bigcup\{\mathrm{mub}(B) : B \in \wp_f(\mathrm{mub}^n(A))\}$.

Definition 1.2.15. Let $(D; \sqsubseteq, \perp)$ be an algebraic cpo. We say that D is a *bifinite domain* if for all $A \in \wp_f(D_c)$ it holds that

1. for all upper bounds x of A there is $b \in \mathrm{mub}(A)$ such that $b \sqsubseteq x$.
2. $\mathrm{mc}(A) := \bigcup_{n \in \omega} \mathrm{mub}^n(A)$ is finite.

Note that if D is a consistently complete algebraic cpo and $A \subseteq_f D_c$ then $\mathrm{mc}(A) = \mathrm{mub}^1(A)$.

Theorem 1.2.16. *If D is a bifinite domain then $P_P(D)$ is a bifinite domain.*

This was first proved in [22]. The bifinite domains form a maximal full cartesian closed subcategory of the algebraic cpos [15].

1.2.4 Effectivity

In this section we give some basic definitions and results concerning computability for domains. We base our computability theory on the Mal'cev-Ershov-Rabin theory of numberings in order to extend computability from the natural numbers to domains. This computability concept is *concrete*, in the sense that computations may in principle be coded and executed on a digital computer.

We use the following fundamental concepts of recursion theory. We choose a primitive recursive pairing function $\langle\cdot,\cdot\rangle\colon \omega\times\omega\to\omega$ along with its primitive recursive projections π_1 and π_2.

Let A be a set. A *numbering* of A is a surjective function $\alpha\colon\omega\to A$. It should be thought of as a coding of A by natural numbers. A subset $S\subseteq A$ is α-*semidecidable* if $\alpha^{-1}(S)$ is recursively enumerable (r.e.) and S is α-*decidable* if $\alpha^{-1}(S)$ is recursive.

Suppose $\alpha\colon\omega\to A$ is a numbering of a set A. Then let $\alpha^*\colon\omega\to\mathscr{P}_f(A)$ be the numbering defined by $\alpha^*(e)=\alpha[K_e]$, where $K_e\subseteq\omega$ is the finite subset with canonical index e. If β is a numbering of B then $\alpha\times\beta\colon\omega\to A\times B$ is the numbering defined by

$$\alpha\times\beta(n)=(\alpha(\pi_1(n)),\beta(\pi_2(n))).$$

Definition 1.2.17. A continuous cpo $D=(D;\sqsubseteq,\perp)$ is *weakly effective* if D has a base B for which there is a surjective function

$$\alpha\colon\omega\to B$$

such that the relation $\alpha(n)\ll\alpha(m)$ is a recursively enumerable relation on ω.

We denote a continuous cpo weakly effective under a numbering α by (D,α). Implicit in this notation is a fixed base $B=\alpha[\omega]$. We will use the notation B for such a base. Thus we let $\mathrm{approx}_\alpha(x)=\{a\in B\colon a\ll x\}$.

Computable elements are those that can be effectively approximated. A function is said to be effective if it can be effectively approximated.

Definition 1.2.18. Let (D,α) and (E,β) be weakly effective domains.

1. An element $x\in D$ is α-*computable* if the set

   $$\{n\in\omega:\alpha(n)\ll x\}=\alpha^{-1}(\mathrm{approx}_\alpha(x))$$

   is r.e. An r.e. index for the set $\alpha^{-1}(\mathrm{approx}_\alpha(x))$ is an *index* for x. The set of α-computable elements of D is denoted by $D_{k,\alpha}$.
2. A continuous function $f\colon D\to E$ is (α,β)-*effective* if the relation

   $$\beta(m)\ll f(\alpha(n))$$

   is r.e. An r.e. index for the set $\{(m,n)\colon\beta(m)\ll f(\alpha(n))\}$ is an *index* for f.

For the work in this thesis we need a stronger notion than that of a weakly effective domain, since a goal is to construct cartesian closed categories of effective domains, and the function space of two weakly effective domains is not necessarily a weakly effective domain. We here present a standard definition of effectivity for consistently complete algebraic cpos, from which one obtains a cartesian closed category.

Definition 1.2.19. A consistently complete algebraic cpo $D = (D; \sqsubseteq, \perp)$ is *effective* if there is a numbering $\alpha : \omega \to D_c$ such that the following relations are recursive:

1. $\alpha(m) \sqsubseteq \alpha(n)$;
2. $\exists k(\alpha(m), \alpha(n) \sqsubseteq \alpha(k))$; and
3. $\alpha(m) \sqcup \alpha(n) = \alpha(k)$.

Let (D, α) and (E, β) be effective consistently complete algebraic cpos. By Theorem 1.2.12, $[D \to E]_c$ is the set

$$\{\bigsqcup_{i=1}^{n} \langle a_i; b_i \rangle : a_i \in D_c,\ b_i \in E_c,\ \{\langle a_1; b_1 \rangle, \ldots, \langle a_n; b_n \rangle\} \text{ consistent}\}.$$

Furthermore,

$$\bigsqcup_{i=1}^{n} \langle a_i, b_i \rangle \sqsubseteq \bigsqcup_{j=1}^{m} \langle c_j, d_j \rangle \iff (\forall i)(\langle a_i, b_i \rangle \sqsubseteq \bigsqcup_{j=1}^{m} \langle c_j, d_j \rangle)$$

and

$$(\bigsqcup_{j=1}^{m} \langle c_j, d_j \rangle)(x) = \bigsqcup \{d_j : c_j \sqsubseteq x\}.$$

The characterisation in Proposition 1.2.11 shows that $D_c \times E_c$ is $(\alpha \times \beta)^*$-decidable. Thus we obtain a numbering γ of $[D \to E]_c$ such that the relations in Definition 1.2.19 are recursive.

Theorem 1.2.20. *Let (D, α) and (E, β) be effective consistently complete algebraic cpos. Then $[D \to E]$ is an effective consistently complete algebraic cpo with a numbering obtained uniformly from α and β.*

1.3 Nets and convergence spaces

In this section we present the underlying definitions for nets and convergence spaces. General references for this material are [16] and [5].

1.3.1 Basic definitions

Let $\Sigma := (\Sigma, \leq_\Sigma)$ and $\Sigma' := (\Sigma', \leq_{\Sigma'})$ be two directed sets. The *product* $\Sigma \times \Sigma'$ of Σ and Σ' is the directed set $(\Sigma \times \Sigma', \leq)$ where $\leq$ is the *product ordering* defined by $(a, a') \leq (b, b')$ if and only if $a \leq_\Sigma b$ and $a' \leq_{\Sigma'} b'$.

Let X be a set. A *net* on X is a function $S : \Sigma \to X$, where $\Sigma = (\Sigma, \leq)$ is a directed partial order. (Note that the definition of a net in [16] is for Σ being a directed preorder.) Σ is called the *index set* of the net. We sometimes write

$\{x_\sigma : \sigma \in \Sigma\}$ or $(x_\sigma)_{\sigma\in\Sigma}$ for the net S, where $x_\sigma = S(\sigma)$. If $f : X \to Y$ is a function, then we define the net $f \circ S := (f(x_\sigma))_{\sigma\in\Sigma}$.

A net $S' : (\Sigma', \leq') \to X$ is a *subnet* of a net $S : (\Sigma, \leq) \to X$ if there is a function $f : \Sigma' \to \Sigma$ such that for all $\sigma' \in \Sigma'$ we have $S(f(\sigma')) = S'(\sigma')$ and such that the following condition holds:

$$(\forall \sigma \in \Sigma)(\exists \sigma'_0 \in \Sigma')(\forall \sigma' \geq' \sigma'_0)(f(\sigma') \geq \sigma).$$

For each $\sigma \in \Sigma$ we define the special subnet $S_{\geq\sigma} := (x_{\sigma'})_{\sigma'\geq\sigma}$. We call $S_{\geq\sigma}$ the *tail* of S from σ.

The net S on X is *eventually in* a set $A \subseteq X$ if there is $\sigma_0 \in \Sigma$ such that $x_\sigma \in A$ for all $\sigma \geq \sigma_0$. In other words, S is eventually in A from σ_0 if and only if $S_{\geq\sigma_0} = \{x_\sigma : \sigma \geq \sigma_0\} \subseteq A$. Similarly, S is *frequently in* A if for all $\sigma \in \Sigma$ there is $\sigma' \in \Sigma$ such that $\sigma' \geq \sigma$ and $x_{\sigma'} \in A$.

Let (X, τ_X) and (Y, τ_Y) be topological spaces and suppose that $f : X \to Y$ is a continuous function. If S is a net on X such that S converges with respect to τ_X to $x \in X$ then $f \circ S$ converges with respect to τ_Y to $f(x)$.

1.3.2 κ-convergence spaces

In this section we present our cardinality restricted version of *convergence spaces* or $\mathscr{L}^*$-spaces [5].

Let X be a set and let $\to_X$ be a relation between nets S on X and elements x in X. We then call $\to_X$ a *convergence relation* on X and the pair $(X, \to_X)$ a *set with convergence relation*.

Let κ be an infinite cardinal. A *κ-net* on X is a net S on X such that the index set of S is of cardinality less than or equal to κ. A subnet S' of a κ-net S is a *κ-subnet* if S' is a κ-net. A convergence relation $\to_X$ on X is a *κ-convergence relation* (or *κ-limit relation*) if the convergence relation is a relation between κ-nets S on X and elements $x \in X$. For each $x \in X$ we let $(x)_\kappa$ denote any κ-net with constant value x.

Definition 1.3.1. Let X be a set and let $\to$ be a κ-limit relation on X. Then $(X, \to)$ is a *κ-convergence space* if $\to$ satisfies the following properties, for each κ-net S on X and each $x \in X$:

1. $(x)_\alpha \to x$ for each infinite cardinal $\alpha \leq \kappa$;
2. if $S \to x$ and S' is a κ-subnet of S then $S' \to x$; and
3. if $S \not\to x$ then there is a κ-subnet S' of S such that for all κ-subnets S'' of S' we have $S'' \not\to x$.

A convergence relation $\to_X$ on X satisfying axioms 1 and 2 above induces a reasonable topology on X. A set $U \subseteq X$ is *open* if whenever $x \in U$ and S is a net such that if $S \to_X x$ then S is eventually in U. The set of open sets in X is denoted $\tau_{\to_X}$ and is called the *induced topology* on $(X, \to_X)$.

A topological space (X,τ) induces a convergence relation $\to_\tau$ on X, defined by $S \to_\tau x$ if and only if S is eventually in each $U \in \tau$ such that $x \in U$. We call a convergence relation obtained in this way *topological*. Note that we have that $(X,\to_\tau)$ is a convergence space.

The topology $\tau_{\to_X}$ induced by a κ-convergence relation $\to_X$ on a set X is κ-*sequential* in the sense of [20], which means that $\tau_{\to_X}$ can be described by nets indexed by sets of at most cardinality κ. Thus $(X,\tau_{\to_X})$ is a κ*-net space* in the terminology of Paper III. Conversely, a κ-net space (X,τ) can be considered as a κ-convergence space $(X,\to_X)$ under a κ-convergence relation induced by the topology. More precisely, we let $\to_X$ be the κ-limit relation obtained by restricting the induced topological convergence relation $\to_\tau$ to κ-nets.

1.3.3 Continuous functions

In this section we consider functions between sets with convergence relations.

Definition 1.3.2. Let $(X,\to_X)$ and $(Y,\to_Y)$ be two sets with convergence relations and let $f : X \to Y$ be a function.

1. f is κ*-continuous* (with respect to $(X,\to_X)$ and $(Y,\to_Y)$) if $S \to_X x$ implies $f \circ S \to_Y f(x)$, for all κ-nets S on X and $x \in X$.
2. f is *continuous* (with respect to $(X,\to_X)$ and $(Y,\to_Y)$) if the condition in 1 holds for each cardinal κ.

We also define the κ-continuous and continuous functions when $X = (X,\tau)$ is a topological space. We let f be κ*-continuous* if f is continuous with respect to $(X,\to_X)$ and $(Y,\to_Y)$, where $\to_X$ is the κ-limit relation induced by τ. Then f is continuous if f is κ-continuous for each cardinal κ.

We let $[X \xrightarrow{\kappa} Y]$ ($[X \to Y]$) be the set of κ-continuous (continuous) functions from X to Y. There is a natural convergence relation on $[X \xrightarrow{\kappa} Y]$, related to the notion of continuous convergence, first defined for ω-sequences by H. Hahn in [14]. Let $T : \Gamma \to [X \xrightarrow{\kappa} Y]$ and $S : \Sigma \to X$ be nets. Define the net $T[S] : \Gamma \times \Sigma \to Y$ by $T[S](\gamma,\sigma) := T(\gamma)(S(\sigma))$.

Definition 1.3.3. Let $T : \Gamma \to [X \xrightarrow{\kappa} Y]$ be a net and let $f \in [X \xrightarrow{\kappa} Y]$.

1. T *converges continuously* to f if for each net S on X and $x \in X$ such that $S \to_X x$ we have $T[S] \to_Y f(x)$.
2. Suppose that T is a κ-net. Then T *converges* κ*-continuously* to f if the condition in 1 holds for each κ-net S.

1.3.4 Weak κ-convergence spaces

In this section we weaken the axioms for a κ-convergence space and obtain a new and larger class of spaces containing the κ-convergence spaces. This is inspired by [25]. We need the following convergence relation. Let $x \in X$ and let $S : (\Sigma, \leq) \to X$ be a κ-net on a set with convergence relation $(X, \to_X)$ such that $S \to_X x \in X$. We define the directed set $\bar{\Sigma} := \Sigma \cup \{t_\Sigma\}$, where the ordering on $\bar{\Sigma}$ is $\leq$ extended with $\sigma < t_\Sigma$ for each $\sigma \in \Sigma$. Define the convergence relation $\to_{\bar{\Sigma}}$ on $\bar{\Sigma}$ in the following way. Let $R : (\Gamma, \leq') \to \bar{\Sigma}$ be a net. Then $R \to_{\bar{\Sigma}} t_\Sigma$ if for all $\sigma \in \Sigma$ there is $\gamma_0 \in \Gamma$ such that for all $\gamma \in \Gamma$ such that $\gamma \geq' \gamma_0$ we have $R(\gamma) \geq \sigma$. Furthermore, if R is eventually equal to σ then $R \to_{\bar{\Sigma}} \sigma$.

Definition 1.3.4. Let X be a set and let $\to$ be a κ-limit relation on X. Then $(X, \to)$ is a *weak κ-convergence space* if $\to$ satisfies the following properties for each κ-net $S : (\Sigma, \leq) \to X$ and $x \in X$:

1. $(x)_\alpha \to x$ for each infinite cardinal $\alpha \leq \kappa$;
2. if $S \to x$ then the function $g_S : (\bar{\Sigma}, \to_{\bar{\Sigma}}) \to (X, \to)$, defined by $g_S(\sigma) = x_\sigma$ for $\sigma \in \Sigma$ and $g_S(t_\Sigma) = x$, is κ-continuous; and
3. if $S_{\geq \sigma} \to x$ for some $\sigma \in \Sigma$ then $S \to x$.

If $(X, \to_X)$ is a weak κ-convergence space then we call $\to_X$ a *weak κ-limit relation*. The function g_S is called the function *induced by S*.

The natural extension of the $\mathscr{L}^*$-spaces is the following.

Definition 1.3.5. Let X be a set and let $\to_X$ be a limit relation on X. Then $(X, \to_X)$ is a *weak convergence space* if $\to_X$ satisfies the three conditions of Definition 1.3.4, for each cardinal κ.

1.4 Admissible domain representations

In this section we describe domains with totality and domain representations. In particular we present the notion of an admissible domain representation, an important tool for the study of κ-net spaces and weak κ-convergence spaces carried out in Paper III and Paper IV. Reference material for this section can be found in [3, 4, 26].

1.4.1 Basic definitions

Let D be a domain and let $D^R \subseteq D$. We call the pair (D, D^R) a *domain with totality*. We will speak of D^R as the *set of representing elements* of D. Let (D, D^R) be a domain with totality, let X be a set and let $\varphi : D^R \to X$ be a surjective function. Then φ is called a *representing function* from D^R to X.

We now review different notions of a domain representation. Let λ and κ be infinite cardinals. A *λ-continuous domain representation* of a set with convergence relation $(X, \to_X)$ is a triple $D = (D, D^R, \varphi)$, where (D, D^R) is a domain with totality and $\varphi : (D^R, \to_D) \to (X, \to_X)$ is a λ-continuous representing function, $\to_D$ being the convergence relation obtained from the Scott topology on D. We call D a *domain representation* of X if D is a λ-continuous domain representation of X, for each cardinal λ.

The domain representation $D = (D, D^R, \varphi)$ is *consistently complete* if D is a consistently complete algebraic cpo. It is *κ-based* if $|D_c| \leq \kappa$ and *locally κ-based* if $|\text{approx}(x)| \leq \kappa$ for each $x \in D$. It is *dense* if D^R is topologically dense in D, i.e. if D^R intersects every Scott-open set in D. We then call (D, D^R) a *domain with dense totality*.

Let (X, τ) be a topological space. Then $D = (D, D^R, \varphi)$ is a domain representation of X if D is a domain representation of the associated convergence space $(X, \to_\tau)$. Note that this is equivalent to $\varphi : D^R \to X$ being continuous in standard topological sense. This notion of domain representation is used in Paper III.

1.4.2 Admissible domain representations

In this section we present the notion of an admissible domain representation.

Definition 1.4.1. Let λ and κ be infinite cardinals and let $D = (D, D^R, \varphi)$ be a λ-continuous domain representation of a set with convergence relation $(X, \to_X)$.

1. D is a *κ-admissible λ-continuous domain representation* of $(X, \to_X)$ if for each κ-based domain with dense totality (E, E^R) and each λ-continuous function $\phi : E^R \to X$ there is a continuous function $\bar{\phi} : E \to D$ such that $\phi(x) = \varphi \circ \bar{\phi}(x)$ holds, for each $x \in E^R$.
2. D is an *admissible λ-continuous domain representation* of $(X, \to_X)$ if D is a κ-admissible λ-continuous domain representation of X, for each cardinal κ.
3. We say that D is a *locally κ-admissible λ-continuous domain representation* of $(X, \to_X)$ if the condition in 1 holds for each locally κ-based domain with dense totality (E, E^R).

If $(X, \to_X)$ has a κ-admissible domain representation then it follows that there is a κ-admissible domain representation D of X such that the domain D is consistently complete.

Proposition 1.4.2. *Let $D = (D, D^R, \varphi)$ be a κ-admissible, λ-continuous and α-based domain representation of a set with convergence relation $(X, \to_X)$. Then there is a κ-admissible, λ-continuous and α-based consistently complete domain representation of X.*

We will sometimes exclusively consider domains with dense totalities.

Proposition 1.4.3. *Let (D, D^R) be a domain with totality. There is a subdomain D' of D such that D^R is dense in D' and such that the relative topologies on*

D^R induced by D and D' are identical. Furthermore, if D is a consistently complete algebraic cpo then D' is a consistently complete algebraic cpo.

It follows that if $D = (D, D^R, \varphi)$ is a κ-admissible, λ-continuous and α-based consistently complete domain representation of a set with convergence relation $(X, \to_X)$ then there is a dense, κ-admissible, λ-continuous and α-based consistently complete domain representation $D' = (D', D^R, \varphi)$ of X.

1.4.3 Representing functions

In this section we consider the representability of functions between sets with convergence relations that have a κ-admissible domain representation. The basic definition is the following.

Definition 1.4.4. Let $(X, \to_X)$ and $(Y, \to_Y)$ be sets with convergence relations and let $D = (D, D^R, \varphi)$ and $E = (E, E^R, \psi)$ be κ-continuous domain representations of X and Y respectively. A function $f : X \to Y$ is *representable with respect to D and E* if there exists a continuous $\bar{f} : D \to E$ such that $(\forall x \in D^R)(f(\varphi(x)) = \psi(\bar{f}(x)))$.

We then say that $\bar{f}$ *represents* f.

The striking fact is that if we have suitable κ-admissible and κ-continuous domain representations of two weak κ-convergence spaces X and Y then we have representability of exactly the κ-continuous functions from X to Y.

Theorem 1.4.5. *Let $(X, \to_X)$ and $(Y, \to_Y)$ be weak κ-convergence spaces and suppose that $D = (D, D^R, \varphi)$ and $E = (E, E^R, \psi)$ are κ-admissible and κ-continuous domain representations of X and Y and that D is a dense and κ-based domain representation. Then $f : X \to Y$ is representable if and only if f is κ-continuous.*

In Paper III we prove the corresponding theorem for topological spaces.

Theorem 1.4.6. *Let X and Y be topological spaces and suppose that $D = (D, D^R, \varphi)$ is a dense, κ-admissible and κ-based domain representation of X and that $E = (E, E^R, \psi)$ is a κ-admissible domain representation of Y. Then $f : X \to Y$ is representable if and only if f is κ-continuous.*

We obtain a natural surjection onto $[X \xrightarrow{\kappa} Y]$ as a result of these theorems. Let $(X, \to_X)$ and $(Y, \to_Y)$ be weak κ-convergence spaces and let $D = (D, D^R, \varphi)$ and $E = (E, E^R, \psi)$ be dense, κ-admissible, κ-continuous and κ-based consistently complete domain representations of X and Y respectively. Note that we may by Propositions 1.4.2 and 1.4.3 assume that the domain representations are dense and consistently complete. Then $[D \to E]$ is a κ-based consistently complete algebraic cpo. Define the set of representing elements $[D \to E]^R$ as follows. Let $g \in [D \to E]^R$ if and only if $g[D^R] \subseteq E^R$ and

$$(\forall x, y \in D^R)(\varphi(x) = \varphi(y) \Rightarrow \psi \circ g(x) = \psi \circ g(y)).$$

By Theorem 1.4.5 there is a surjective function $\chi : [D \to E]^R \to [X \xrightarrow{\kappa} Y]$, defined by $\chi(g) = f$ if g represents f. Correspondingly, if X and Y are topological spaces then $\chi : [D \to E]^R \to [X \xrightarrow{\kappa} Y]$ is a surjection by Theorem 1.4.6.

We close this background chapter by noting that a fundamental step in the proof of cartesian closure for the categories considered in Paper III and Paper IV is the following theorem (and its analogous formulation for topological spaces), showing in particular that the function χ is κ-continuous.

Theorem 1.4.7. *Let $(X, \to_X)$ and $(Y, \to_Y)$ be weak κ-convergence spaces and let $D = (D, D^R, \varphi)$ and $E = (E, E^R, \psi)$ be dense, κ-admissible, κ-based and κ-continuous consistently complete domain representations of X and Y. Then $([D \to E], [D \to E]^R, \chi)$ is a κ-admissible, κ-based and κ-continuous consistently complete domain representation of $([X \xrightarrow{\kappa} Y], \to_{[X \xrightarrow{\kappa} Y]})$.*

2 Overview of thesis

In this chapter we summarise the main results. The thesis consists of four papers, which can be viewed as two pairs. The first pair is two papers on the problem of constructing large cartesian closed categories of effective continuous cpos, while the second pair deals with the concept of an admissible domain representation.

2.1 Summary of Paper I

Paper I is joint work with Viggo Stoltenberg-Hansen. It develops two different notions of effectivity on continuous cpos. We consider a notion of an effective bifinite domain, which is a generalisation of the notion of an effective consistently complete algebraic cpo. We also consider effectivity on continuous cpos induced by effectivity on algebraic cpos via projection pairs.

2.1.1 Effective bifinite domains

We use the following, slightly original, definition of a bifinite domain. First our definition of a *complete set* and *complete cover*. An inspiration for this definition can be found in [13].

Definition 2.1.1. Let $(P; \sqsubseteq, \bot)$ be a partial order with a least element.

1. $B \subseteq P$ is a *complete set* (in P) if
$$(\forall C \subseteq B)(\forall x \sqsupseteq C)(\exists b \in B)(C \sqsubseteq b \sqsubseteq x).$$
2. A family $\mathscr{F} = \{B_i : i \in I\}$ of finite subsets of P is a *complete cover* of P if each B_i is complete and for each $A \subseteq_f P$ there is $i \in I$ such that $A \subseteq B_i$.

What we require of a bifinite domain is that each finite subset of compact elements be covered by a finite complete set of compact elements.

Definition 2.1.2. D is a *bifinite domain* if D is an algebraic cpo and D_c has a complete cover.

We note that, according the standard definition of bifinite domains, we have that $\{\mathrm{mc}(A) : A \in \wp_f^*(D_c)\}$ is a complete cover of D_c. Thus it is easy to see that our definition is equivalent with the standard ones.

The compact elements of the function space of two bifinite domains can be obtained from finite sets of step functions $\{<a_i;b_i> : i \in I\}$, characterised mainly by the requirement that the first coordinates $\{a_i : i \in I\}$ form a complete set. Hence we have a natural definition of an effective bifinite domain.

Definition 2.1.3. A bifinite domain D is an *effective bifinite domain* if there is a numbering $\alpha\colon \omega \to D_c$ such that

1. the relation $\alpha(n) \sqsubseteq \alpha(m)$ is recursive, i.e. $\sqsubseteq$ is α-decidable; and
2. there is a complete cover $\mathscr{F}$ of D_c such that $\mathscr{F}$ is α^*-decidable.

The major result we prove is that the function space $[D \to E]$ of two effective bifinite domains (D,α) and (E,β) is again an effective bifinite domain. This follows in an elegant way, since it suffices to consider step functions obtained from an α^*-decidable cover of D_c.

Theorem 2.1.4. *Let (D,α) and (E,β) be effective bifinite domains. Then $[D \to E]$ is an effective bifinite domain with a numbering obtained uniformly from α and β.*

It follows that the category of effective bifinite domains with effective continuous functions is cartesian closed.

2.1.2 Smyth effective domains

The theory of effectivity on algebraic cpos induces a theory of effectivity on continuous cpos via projection pairs. We show that if we start with a cartesian closed category of effective algebraic cpos then we obtain in this way a cartesian closed category of effective continuous cpos. This was first done by Smyth in [29], where effective consistently complete algebraic cpos were considered. Using Theorem 2.1.4 we extend Smyth's result and obtain a cartesian closed category of projections of effective bifinite domains.

We here need to review some of the terminology used in the paper. Let E be an algebraic cpo, let D be a cpo, and let (e,p) be a projection pair from D to E. Recall that one element in a projection pair (e,p) from D to E determines the other. Thus we let (E,p,D) denote that $p\colon E \to D$ is a projection onto D and we then denote the corresponding embedding by e. We say that (E,p,D) is an *AP-domain* if E is an algebraic cpo.

Definition 2.1.5. Let (e_i,p_i) be a projection pair from D_i to E_i for $i = 1,2$. Define $\mathscr{E}\colon [D_1 \to D_2] \to [E_1 \to E_2]$ and $\mathscr{P}\colon [E_1 \to E_2] \to [D_1 \to D_2]$ by

$$\mathscr{E}(g) = e_2 \circ g \circ p_1 \text{ and } \mathscr{P}(f) = p_2 \circ f \circ e_1.$$

Then $(\mathscr{E},\mathscr{P})$ is a projection pair from $[D_1 \to D_2]$ to $[E_1 \to E_2]$. Now we define a relation $\prec$ on $E_c \times E$ by

$$a \prec x \Longleftrightarrow a \sqsubseteq ep(x).$$

It is via this relation that we define effectivity on the AP-domains.

Definition 2.1.6. 1. Let (E,p,D) be an AP-domain. Then $((E,p,D),\alpha)$ is *Smyth effective* if $\alpha\colon\omega\to E_c$ is a numbering such that the relation $\prec$ on E_c is α-semidecidable.
2. Let $((E,p,D),\alpha)$ be Smyth effective. Then $x\in D$ is *α-Smyth computable* if the relation $a\prec e(x)$ is α-semidecidable.
3. Let $((E_1,p_1,D_1),\alpha)$ and $((E_2,p_2,D_2),\beta)$ be Smyth effective. Then a continuous function $f\colon D_1\to D_2$ is *(α,β)-Smyth effective* if the relation $b\prec\mathscr{E}(f)(a)$ is (α,β)-semidecidable.

Here is the main theorem. As a result, we can build Smyth effective type structures over continuous cpos as long as they are projections of bifinite domains.

Theorem 2.1.7. *Let $((E_1,p_1,D_1),\alpha)$ and $((E_2,p_2,D_2),\beta)$ be Smyth effective AP-domains and suppose that (E_1,α) and (E_2,β) are effective bifinite domains. Then there is a numbering γ of $[E_1\to E_2]$, obtained uniformly from α and β, such that $(([E_1\to E_2],\mathscr{P},[D_1\to D_2]),\gamma)$ is a Smyth effective AP-domain and $([E_1\to E_2],\gamma)$ is an effective bifinite domain.*

2.2 Summary of Paper II

This paper presents two categories of effective continuous cpos. We define a new criterion on the basis of a cpo as to make the resulting category of consistently complete continuous cpos cartesian closed. We also generalise the definition of a complete set, used as a definition of effective bifinite domains in Paper I, and investigate what closure results that can be obtained.

2.2.1 Cartesian closure for almost algebraic cpos

It seems necessary to impose extra requirements on a basis for a continuous cpo in order to obtain a characterisation of the basic relations on the function space in terms of the relations on the base domains. We therefore make the following definition.

Definition 2.2.1. Let $D=(D;\sqsubseteq,\perp)$ be a continuous cpo. A basis B of D is called *almost algebraic* if the following hold for all $a,b\in B$:
1. There is a sequence $(a_n)_{n\in\omega}\subseteq B$ with $a_0\gg a_1\gg\cdots\gg a$.
   If $b\gg a$ then there exists $n\in\omega$ such that $b\gg a_n$.
2. $\Uparrow a\subseteq\Uparrow b\Rightarrow b\sqsubseteq a$.

The assumption of a base being almost algebraic is sufficient to prove the following crucial lemma characterising the way-below relation between step functions and continuous functions.

Lemma 2.2.2. *Let D and E be consistently complete continuous cpos with bases B_D and B_E and suppose that B_D is almost algebraic and that B_E is countable. Let $f \in [D \to E]$, $a \in B_D$ and $b \in B_E$. Then $<a;b> \ll f \Leftrightarrow b \ll f(a)$.*

Lemma 2.2.2 is important in the proof of the following theorem. We consider a notion of effectivity for consistently complete continuous cpos, which is a natural adaption of Definition 1.2.19 to continuous cpos by requiring $\ll$ to be recursive on a basis B. Note that B then have to be *closed* in the sense that for all $b, c \in B$ we have $b \sqcup c \in B$.

Theorem 2.2.3. *The category of effective consistently complete continuous cpos with closed and almost algebraic bases and effective continuous functions as morphisms is cartesian closed.*

2.2.2 Effective C-bifinite domains

We generalise the definition of an effective bifinite domain in Paper I. First, the straightforward generalisation of the definition of a complete set and complete cover.

Definition 2.2.4. Let $(D; \sqsubseteq, \perp)$ be a cpo.

1. $B \subseteq D$ is a *wa-complete* set if
$$\forall C \subseteq B \forall x \gg C \exists b \in B\,(x \gg b \sqsupseteq C).$$
2. A family $\mathscr{F} = \{B_i : i \in I\}$ of finite subsets of B is a *way-above-complete cover* of B if each B_i is wa-complete and for each $A \subseteq_f B$ there is $i \in I$ such that $A \subseteq B_i$.

Then this is our generalisation of a bifinite domain to continuous cpos.

Definition 2.2.5. Let $(D; \sqsubseteq, \perp)$ be a continuous cpo. We say that D is a *c-bifinite domain* if there is a basis B such that B has a wa-complete cover. B is then called a *c-bifinite basis* for D.

One can show that any algebraic cpo with c-bifinite basis is bifinite, and that any consistently complete continuous cpo is c-bifinite. Thus we have a natural extension of both categories. We have been unable to prove the existence of a non-algebraic c-bifinite domain which is not consistently complete.

We also prove the analogue of Lemma 2.2.2 for c-bifinite domains. As in the case for consistently complete continuous cpos, this lemma plays a fundamental role in the proof of the next theorem.

Theorem 2.2.6. *Let D and E be c-bifinite domains with c-bifinite bases B_D and B_E, respectively. Suppose that B_D is almost algebraic and that B_E is countable. Then $[D \to E]$ is a c-bifinite domain.*

The open problem is whether it is possible to obtain an almost algebraic basis $B_{[D \to E]}$ for $[D \to E]$ from almost algebraic and c-bifinite bases B_D and B_E. If this is possible, then we will have a cartesian closed subcategory of the c-bifinite domains.

One pleasing aspect of the c-bifinite domains is that they are closed under the Plotkin power domain construction.

Theorem 2.2.7. *Let D be a c-bifinite domain with a c-bifinite basis B. Then $P_P(D)$ is a c-bifinite domain with c-bifinite basis $B_{P_P(D)} := \{[A] : A \in \wp_f^*(B)\}$.*

We also show that if B is an almost algebraic basis for D then $B_{P_P(D)}$ is an almost algebraic basis for $P_P(D)$.

We finish by giving the following natural notion of effectivity for c-bifinite domains, generalising Definition 2.1.3.

Definition 2.2.8. Let $(D; \sqsubseteq, \perp)$ be a continuous cpo and let B be a c-bifinite basis for D. We say that B is *effective* if there is a numbering $\alpha : \omega \to B$ such that

1. $\sqsubseteq$ on B is α-decidable;
2. $\ll$ on B is α-decidable; and
3. there is an α^*-computable wa-complete cover $\mathscr{F}$ of B, i.e. the relation $\alpha(m) \in \alpha^*(n)$ is recursive.

We call (D, α) an *effective c-bifinite domain* if D has an effective c-bifinite basis. With this definition we prove effective versions of Theorems 2.2.6 and 2.2.7.

2.3 Summary of Paper III

This paper considers admissible domain representations of topological spaces. We show two major results. The first is a characterisation theorem of when a topological space has an admissible representation, while the second presents a cartesian closed category of topological spaces with a dense, countably based and countably admissible domain representation.

2.3.1 The characterisation theorem

In this section we present the characterisation theorem. As a tool in the proof we define the notion of a κ-net base. An origin of the concept of a κ-net base can be found in [2].

Definition 2.3.1. Let (X, τ) be a topological space. A *κ-net base* $\mathscr{B} \subseteq 2^X$ is a family such that for all $O \in \tau$, for all $x \in O$ and for all κ-nets $S \to x$, there is $B \in \mathscr{B}$ such that $x \in B \subseteq O$ and such that S is eventually in B.

Let κ and λ be two infinite cardinals such that $\kappa \leq \lambda$.

Theorem 2.3.2. *A topological space (X,τ) has a κ-based and locally λ-admissible domain representation if and only if (X,τ) is a T_0-space and has a λ-net base of cardinality less than or equal to κ.*

The proof in one direction is straightforward. Define the κ-net base for X as $\mathscr{B}_D := \{\varphi[\uparrow a \cap D^R] : a \in D_c\}$, if $D = (D, D^R, \varphi)$ is a κ-based and locally λ-admissible domain representation of X. The T_0-property is obtained as a consequence of the relatively small actual number of continuous functions between two domains E and D, compared with the potential number of continuous functions from E into a non-T_0-space X.

The proof in the other direction is carried out by constructing a domain representation of X from a κ-net base $\mathscr{B}$ for X as $D = \mathrm{Idl}(\mathscr{B})$. An elegant step is showing that D is locally λ-admissible. Let (E, E^R) be a locally λ-based domain with dense totality and let $\phi : E^R \to X$ be a continuous function. Suppose that D is not locally λ-admissible. Then it is possible to construct a net $S : \Sigma \to E^R$ with index set $\Sigma = E_c \times \mathscr{B}$, for which the application of ϕ on S leads to a contradiction, showing that D is locally λ-admissible.

2.3.2 Cartesian closure

In this section we present a cartesian closed category of spaces with a dense, countably admissible and countably based domain representation.

Definition 2.3.3. Let $\omega\mathbf{ADM}$ be the category with objects (X,D) where $X = (X,\tau)$ is a topological space and $D = (D, D^R, \varphi)$ is a countably based, ω-admissible and consistently complete dense domain representation of X. Let (X,D) and (Y,E) be objects in $\omega\mathbf{ADM}$. The morphisms $[\bar{f}] : (X,D) \to (Y,E)$ are equivalence classes of functions $\bar{f} \in [D \to E]^R$, where two functions $\bar{f}$ and $\bar{g}$ are equivalent if and only if $\chi(\bar{f}) = \chi(\bar{g})$.

The topology $\tau_{[X \xrightarrow{\kappa} Y]}$ on $[X \xrightarrow{\kappa} Y]$ we consider is obtained from the subbase

$$\mathscr{B}^{\kappa}_{X \to Y} := \{M(S \cup \{x\}, U) : S \to x \wedge U \in \tau_Y\}.$$

Here S varies over κ-nets on X and $M(A,B) := \{f \in [X \xrightarrow{\kappa} Y] : f[A] \subseteq B\}$. We show that convergence with respect to $\tau_{[X \xrightarrow{\kappa} Y]}$ and κ-continuous convergence are equivalent on $[X \xrightarrow{\kappa} Y]$ when $\kappa = \omega$, which is necessary in order to show that $\omega\mathbf{ADM}$ is cartesian closed. Then $([X \xrightarrow{\omega} Y], [D \to E]')$ is the exponential in $\omega\mathbf{ADM}$, where the domain representation $[D \to E]'$ is obtained via Proposition 1.4.3 from the natural domain representation $[D \to E]$ of $[X \xrightarrow{\omega} Y]$ obtained in Theorem 1.4.7.

Theorem 2.3.4. *ωADM is cartesian closed.*

It is straightforward to show that the function $\chi : [D \to E]^R \to [X \xrightarrow{\omega} Y]$ presented in Section 1.4.3 is continuous. This is because $[D \to E]$ is a countably based domain, which means that it suffices to show that χ is sequentially continuous. To show ω-admissibility, let (F, F^R) be a countably based domain with dense totality and let $\phi : F^R \to X$ be continuous. We then define a continuous function $\upsilon : F^R \times D^R \to Y$ by $\upsilon(z,w) = \phi(z)(\varphi(w))$. Thus we obtain a continuous $\bar{\upsilon} : F \times D \to E$, by the κ-admissibility of E. Then the continuous function $\mathrm{curry}(\bar{\upsilon}) : F \to [D \to E]$ witnesses that $[D \to E]$ is κ-admissible.

2.4 Summary of Paper IV

This paper considers admissible domain representations of sets with convergence relations. We present two major results. The first is that the categories of weak κ-convergence spaces and weak convergence spaces are cartesian closed.

Theorem 2.4.1. *The category $w\mathscr{L}_\kappa^*$ of weak κ-convergence spaces with continuous functions as morphisms is cartesian closed.*

The proof is similar to the proof in [5] of that the category of $\mathscr{L}^*$-spaces is cartesian closed. The critical ingredient is showing that if $(X, \to_X)$ and $(Y, \to_Y)$ are weak κ-convergence spaces then $([X \xrightarrow{\kappa} Y], \to_{[X \xrightarrow{\kappa} Y]})$ is a weak κ-convergence space.

The second result is Theorem 2.4.3 below, which characterises some cartesian closed categories of weak κ-convergence spaces with an admissible domain representation. First the precise definition.

Definition 2.4.2. Let $\lambda ADM\alpha w\mathscr{L}_\kappa^*$ be the category with objects (X, D), where X is a weak κ-convergence space and D is a dense, λ-admissible, α-based and κ-continuous consistently complete domain representation of X. The morphisms between two objects (X, D) and (Y, E) of $\lambda ADM\alpha w\mathscr{L}_\kappa^*$ are the continuous functions $f : X \to Y$.

Theorem 2.4.3. *Let α, λ and κ be infinite cardinals such that $\alpha \leq \lambda \geq \kappa$. Then $\lambda ADM\alpha w\mathscr{L}_\kappa^*$ is a cartesian closed category.*

The proof of Theorem 2.4.3 is similar in style to the proof of Theorem 2.3.4, noting that the notion of continuity on weak κ-convergence spaces can be viewed as tailor-made to make the proof work for each cardinality. As a corollary of Theorem 2.4.3 we obtain analogous results for the associated categories of κ-convergence spaces and weak convergence spaces.

3 Summary in Swedish

3.1 Effektiva domäner och admissibla domänrepresentationer [1]

Denna avhandling är inom området domänteori. En domän är en partiellt ordnad mängd med ett minsta element, där mängden innehåller den minsta övre gränsen till alla riktade delmängder. Avhandlingen kan delas in i två delar.

Vi studerar i avhandlingens första två artiklar effektiva domäner, det vill säga vi använder Mal'cev-Ershov-Rabins teori för numreringar för att ge en uppräkning av mängden baselement i en domän och relationer på den. Detta ger ett beräkningsbarhetsbegrepp för ouppräkneliga strukturer. Vi definierar i den första artikeln en kartesiskt sluten kategori av effektiva bifinita domäner och använder den för att inducera effektivitet på kontinuerliga domäner via projektionspar. I den andra artikeln definierar vi två kategorier av effektiva kontinuerliga domäner och studerar vilka slutenhetsegenskaper de har.

I avhandlingens andra del studerar vi domänrepresentation av olika klasser av matematiska strukturer. En gemensam nämnare för många av de typer av strukturer och domänrepresentationer som undersöks är att deras utseende är kopplade till ett eller flera oändliga kardinaltal. Speciellt undersöker vi domänrepresentation av de strukturer X, vilkas utseende beskrivs av en familj av nät på X där indexmängden har begränsad kardinalitet. Den tredje artikeln behandlar fallet när X är ett topologiskt rum. Det viktigaste resultatet vi visar är att det finns en naturlig kartesiskt sluten kategori i vilken objekten är par (X,D), där X är ett T_0-rum och D är en uppräkneligt baserad och uppräkneligt *admissibel domänrepresentation* av X. Att en domänrepresentation D av X är uppräkneligt admissibel betyder väsentligen att varje annan uppräkneligt baserad domänrepresentation E av X kan reduceras till D via en kontinuerlig funktion från E till D. Vi visar också en karakterisering av de topologiska rum vilka har λ-admissibla och κ-baserade domänrepresentationer, där κ och λ är

[1] Instruktionerna för den obligatoriska svenskspråkiga sammanfattningen talar om att syftet är att sprida ny kunskap ut i det svenska samhället, samtidigt som nivån på sammanfattningen ska ligga på ungefär samma nivå som själva avhandlingen. Dessa två målsättningar står till viss del i konflikt med varandra, då det inte finns någon svenskspråkig person med de nödvändiga matematiska förkunskaperna som saknar de nödvändiga elementära kunskaperna i engelska vilka krävs för att förstå avhandlingstexten. Därför delas denna sammanfattning upp i två delar, varav en populärvetenskaplig.

två oändliga kardinaltal.

Den fjärde artikeln behandlar fallet när X är en mängd med en tvåställig konvergensrelation mellan nät och element på X. Vi introducerar och behandlar främst den kartesiskt slutna kategorin av vad vi kallar för *svaga κ-konvergensrum*, vilka kan ses som en generalisering av en försvagning av de tre axiomen för Kuratowskis limitrum för ett godtyckligt oändligt kardinaltal κ. Vårt huvudresultat i denna artikel är att klassen av svaga κ-konvergensrum som har en λ-admissibel, κ-kontinuerlig och α-baserad domänrepresentation är kartesiskt sluten om $\alpha \leq \lambda \geq \kappa$. Som en naturlig följd av dessa satser fås analoga resultat för de relaterade kartesiskt slutna kategorierna av κ-konvergensrum och svaga konvergensrum. Inte heller dessa kategorier verkar ha behandlats explicit i litteraturen tidigare.

3.2 Populärvetenskaplig sammanfattning

Denna avhandling handlar om hur man kan studera matematiska strukturer genom att approximera eller representera dem med *domäner*.

Bland de första tal man brukar lära sig finns de *naturliga talen* $0,1,2,3,\ldots$ De naturliga talen är små och lätthanterliga i den meningen att varje tal kan skrivas med ett ändligt antal siffror. Dessutom är $\mathbb{N}$, *mängden av alla naturliga tal*, en liten mängd i betydelsen att den är *uppräknelig*. Med det menas att vi kan räkna upp de naturliga talen i en följd ett efter ett, och säga "det här är det första, andra, tredje naturliga talet" och så vidare, och vi kommer för varje naturligt tal n inom en ändlig tidsrymd ha räknat upp n, även om det totalt sett finns oändligt många naturliga tal. Därmed kan de naturliga talen i princip beskrivas av en dator.

Många matematiska strukturer innehåller element som är svårhanterliga i den meningen att de innehåller "många" element som är "stora". Ett klassiskt exempel är $\mathbb{R}$, de *reella talen* (i vardagligt tal även kallad *tallinjen*). Det finns reella tal vilka det behövs oändligt många siffror för att skriva. Ett vanligt exempel är $\pi \approx 3,14159\ldots$ Dessutom finns det så många reella tal att $\mathbb{R}$ inte är en uppräknelig mängd. Den är alltså så stor att vi inte direkt kan använda de reella talen till att räkna med i våra datorer. Vi måste istället använda *representationer* eller approximationer av reella tal när vi till exempel ska räkna ut multiplikationen $\pi * 2$. Det finns dock ett sätt att beskriva ett reellt tal som en uppräknelig följd av tal från en uppräknelig delmängd av $\mathbb{R}$, nämligen mängden $\mathbb{Q}$, *de rationella talen* (eller *bråktalen* som de också har kallats). Denna observation ligger till grund för konstruktionen av en domän som representerar $\mathbb{R}$.

Låt $[a,b]$ beteckna intervallet av alla reella tal från och med a till och med b. Att ett intervall $[a,b]$ innehåller ett reellt tal r uttrycks av formeln $r \in [a,b]$, och vi säger då att r är ett element i $[a,b]$. Ett sätt att representera ett reellt tal r är som en uppräknelig följd av minskande intervall $[a_i,b_i]$ vilka alla innehåller r. Det betyder att vi räknar upp intervall $[a_0,b_0],[a_1,b_1],\ldots[a_i,b_i],[a_{i+1},b_{i+1}],\ldots$ så att för alla $i \in \mathbb{N}$ gäller det att $[a_{i+1},b_{i+1}]$ är inkluderat i eller en delmängd av $[a_i,b_i]$. Här är $[a_{i+1},b_{i+1}]$ en delmängd av $[a_i,b_i]$ om det för alla x gäller att om $x \in [a_{i+1},b_{i+1}]$ så $x \in [a_i,b_i]$. För varje steg i uppräkningen får vi då en allt bättre approximation av var på tallinjen det reella talet befinner sig. Om vi dessutom väljer ändpunkterna a_i och b_i i intervallen som rationella tal sådana att $a_i < a_{i+1} \leq r$ och $b_i > b_{i+1} \geq r$ så kan vi göra intervallet $[a_i,b_i]$ godtyckligt litet och ändå vara säkra på att r finns i $[a_i,b_i]$.

Vi betraktar nu mängden $P = \{[a,b] : a \leq b \text{ och } a,b \in \mathbb{Q}\}$ av alla intervall med rationella ändpunkter. Vi tänker oss att vi placerar intervallen som punkter på ett papper framför oss, så att ett intervall I ordnas ovanför ett annat intervall J på papperet om och endast om I är en delmängd av J. Vi får en tvådimensionell struktur, där vi säger att en punkt I är en bättre approximation än en annan punkt J om I ordnas ovanför J på papperet.

Vi lägger till en punkt längst ner på papperet. Denna punkt är tänkt att motsvara det intervall som utgörs av hela tallinjen (vilken saknar ändpunkter och därför inte finns med i P). Vi lägger också till överst på papperet en punkt för varje reellt tal r. Dessa punkter placeras så att ett reellt tal r ordnas ovanför precis de intervall $I \in P$ sådana att $r \in I$. Framför oss på papperet har vi då den så kallade *intervalldomänen* för de reella talen. För den gäller att ett reellt tal r approximeras av ett intervall I om $r \in I$.

Vi vill dessutom kunna representera en beräkning på de reella talen, det vill säga en reell funktion som tar ett reellt tal r och enligt någon matematisk beskrivning f ger som resultat ett (eventuellt annat) reellt tal $f(r)$. En funktion ger alltså alltid samma resultat $f(r)$ varenda gång man tillämpar den på r. Med hjälp av intervalldomänen är det möjligt att ge en domän som representerar en stor mängd $\mathbb{F}$ av ofta använda reella funktioner.

Om nu D är intervalldomänen för $\mathbb{R}$, så kan vi konstruera en *domän av funktioner* $[D \to D]$ vilken representerar $\mathbb{F}$, likt D representerar $\mathbb{R}$. Och eftersom D kan beskrivas genom att ge en uppräkning av elementen i P och den ovan beskrivna inklusionsordningen mellan dessa så kan vi i princip beskriva D genom ett datorprogram. Domänen D är i detta fall exempel på en *effektiv domän*. Vi kan alltså "koda" D på vår dator och utföra beräkningar på $\mathbb{R}$ genom att generera bättre och bättre approximativa svar med hjälp av intervalldomänen. En bonus är att denna metod kan användas för att lösa vissa problem där den normalt använda metoden för datorberäkningar av reella funktioner inte fungerar.

Studiet av intervalldomänen är en viktig ingrediens inom *domänteorin*, det område av den matematiska logiken som denna avhandling behandlar. Vi behandlar speciellt problemet med att konstruera *kartesiskt slutna kategorier* av olika specialtyper av domäner. En kartesiskt sluten kategori K av domäner har bland annat den önskvärda egenskapen att givet två domäner D och E i K så gäller att $[D \to E]$ är i K. Kartesiskt slutna kategorier av domäner har bland annat även använts för att studera vissa matematiska aspekter av programmeringsspråk, så kallad *denotationssemantik*.

De specifika resultaten i avhandlingen kan delas in i två delar. Den första behandlar problemet att hitta stora kartesiskt slutna kategorier av effektiva domäner. Den andra gäller problemet att beskriva vissa matematiska strukturer vi kan kalla för *konvergensrum*. Ett konvergensrum X är en mängd punkter tillsammans med ett förhållande eller en relation R mellan delmängder S av X och element $x \in X$. Vi kallar S för *nät*, eftersom vi kan tänka oss relationen R som beskrivande vilka punkter $x \in X$ som "fångas in" av nätet S. Vi visar att om vi ställer vissa krav på utseendet av konvergensrummen, bland annat genom att begränsa storleken på näten, så kan vi hitta ett antal olika klasser av konvergensrum vilka på ett naturligt sätt kan representeras med hjälp av kartesiskt slutna kategorier av domäner.

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“Gibt es einen, der nicht trivial ist?”[2]

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[2] Last words of an imaginary countess.

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Göran Hamrin

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