Open Quantum Systems

Effects in Interferometry, Quantum Computation, and Adiabatic Evolution

JOHAN ÅBERG
Dissertation presented at Uppsala University to be publicly examined in Häggsalen, Ångströmlaboratoriet, Lägerhyddsvägen 1, Uppsala, Thursday, September 22, 2005 at 10:15 for the degree of Doctor of Philosophy. The examination will be conducted in English.

Abstract
Uppsala. ISBN 91-554-6307-X.

The effects of open system evolution on single particle interferometry, quantum computation, and the adiabatic approximation are investigated.

Single particle interferometry: Three concepts concerning completely positive maps (CPMs) and trace preserving CPMs (channels), named subspace preserving (SP) CPMs, subspace local channels, and gluing of CPMs, are introduced. SP channels preserve probability weights on given orthogonal sum decompositions of the Hilbert space of a quantum system. Subspace locality determines what channels act locally with respect to such decompositions. Gluings are the possible total channels obtainable if two evolution devices, characterized by channels, act jointly on a superposition of a particle in their inputs. It is shown that gluings are not uniquely determined by the two channels. We determine all possible interference patterns in single particle interferometry for given channels acting in the interferometer paths. It is shown that the standard interferometric setup cannot distinguish all gluings, but a generalized setup can.

Quantum computing: The robustness of local and global adiabatic quantum search subject to decoherence in the instantaneous eigenbasis of the search Hamiltonian, is examined. In both the global and local search case the asymptotic time-complexity of the ideal closed case is preserved, as long as the Hamiltonian dynamics is present. In the case of pure decoherence, where the environment monitors the search Hamiltonian, it is shown that the local adiabatic quantum search performs as the classical search with scaling N, and that the global search scales like N\(^{1/2}\), where N is the list length. We consider success probabilities p<1 and prove bounds on the run-time with the same scaling as in the conditions for the p → 1 limit.

Adiabatic evolution: We generalize the adiabatic approximation to the case of open quantum systems in the joint limit of slow change and weak open system disturbances.

Keywords: Open Systems, Completely Positive Maps, Channels, Decoherence, Quantum Information, Quantum Computing, Adiabatic Approximation, Quantum Search, Time-Complexity

Johan Åberg, Department of Physical Chemistry, Quantum Chemistry, Box 518, Uppsala University, SE-75120 Uppsala, Sweden

© Johan Åberg 2005

ISSN 1651-6214
ISBN 91-554-6307-X
urn:nbn:se:uu:diva-5893 (http://urn.kb.se/resolve?urn=urn:nbn:se:uu:diva-5893)
Till min familj
List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.

I  Subspace preservation, subspace locality, and gluing of completely positive maps
   J. Åberg

II Operations and single-particle interferometry
   J. Åberg

III Robustness of the adiabatic quantum search
   J. Åberg, D. Kult, and E. Sjöqvist

IV Quantum adiabatic search with decoherence in the instantaneous energy eigenbasis
   J. Åberg, D. Kult, and E. Sjöqvist
   *preprint*: quant-ph/0507010

V Adiabatic approximation for weakly open systems
   P. Thunström, J. Åberg, and E. Sjöqvist
   *preprint*: quant-ph/0504046
   Accepted for publication in *Physical Review A.*
## Contents

1 Introduction ................................................................. 1
2 Background ................................................................. 5
   2.1 Density operators ...................................................... 5
   2.2 Reduced density operators ........................................... 6
   2.3 State evolution of closed and open systems ..................... 7
   2.4 Completely positive maps .......................................... 7
   2.5 The two-path interferometer ....................................... 10
   2.6 Master equations .................................................... 12
   2.7 Decoherence ........................................................... 13
   2.8 The adiabatic approximation ...................................... 14
   2.9 Quantum computing ................................................. 17
   2.10 Adiabatic quantum computing ..................................... 18
3 Summary of the papers ................................................... 21
   3.1 Interferometry: Papers I and II .................................. 21
      3.1.1 Subspace preservation ....................................... 21
      3.1.2 Subspace locality ............................................. 22
      3.1.3 Gluing of completely positive maps ......................... 25
      3.1.4 On measuring gluings with interferometers ............... 30
   3.2 Quantum computation: Papers III and IV ....................... 31
   3.3 Adiabatic evolution: Paper V ....................................... 36
4 Conclusions ................................................................. 41
5 Summary in Swedish: Öppna kvantsystem. Inverkan på
   interferometri, kvantdatorer och adiabatisk tidsutveckling ....... 43
References ................................................................. 49
1. Introduction

Quantum systems that interact with a quantum mechanical environment are sometimes called open quantum systems, as to differ them from their closed counterparts that evolve in isolation from the rest of the world. Open systems have been studied for quite a while. In spite of this, it is only in more recent years that open systems have become more generally known within the physics community. There are, I believe, two main reasons for this increasing awareness. First, open system effects play an important role to understand the emergence of a classical world from quantum mechanics. Second, open system effects constitute a prime obstacle to the realization of a functioning quantum computer.

It seems likely that open system effects will become more important in other parts of physics as well, for example in molecular physics, chemical physics, and condensed matter theory, especially concerning the dynamics of quantum systems. As the theoretical methods and the computational modeling of these systems become more accurate, and are being compared with more sophisticated experiments, open system effects should reasonably become more noticeable.

One very pronounced difference between classical and quantum mechanics is the ability of single quantum mechanical particles to show interference effects. This is a reflection of the superposition principle, which allows a quantum system to be in states that in some sense are combinations of classical configurations. Based upon the single-particle interference effects, one may construct single-particle interferometers, which are virtually like playgrounds for displaying the many intriguing aspects of quantum mechanics.

Open system effects may have a strong influence on superposition. Decoherence, for example, may reduce superposition, and thereby make systems behave more classically. It thus seems reasonable to expect that open system evolution may have a pronounced influence on single-particle interferometry. In the first project of this thesis we focus on the question of how to understand and model interferometry in the presence of open system effects. We consider particles with internal degrees of freedom, like for example intrinsic spin. The particles are subject to interactions with other quantum systems, which causes state changes, or “operations”, on the internal state of the particles. These operations can be imagined as evolution devices, which are inserted into paths
of the two-path interferometer. We focus on how to understand the effect of these operations on the interferometer. Given the knowledge of how operations affect the interferometer we may turn the question around and ask how the interferometer can be used as a tool to detect open system effects. This would make it possible to use single particle interferometers to reveal what type of open system evolution is present in the system.

Quantum computing is presently a very flourishing field of research, spanning from purely theoretical to experimental investigations. Apart from the many insights into physics that these studies have provided, much of the enthusiasm stems from the observation that a functioning quantum computer would be a very valuable tool, as its computational power for some classes of problems appears to go far beyond ordinary computers.

Decoherence is one of the main problems to overcome in the construction of a functioning quantum computer. Due to this, much effort is spent on designing ways to protect quantum computers from decoherence, and inventing computational schemes that are intrinsically resilient to certain types of errors. In the second project of this thesis we address the issue of decoherence in quantum computing. We focus on a special type of quantum computer called the adiabatic quantum computer. This is a promising candidate for a robust quantum computer, since it appears to have some level of intrinsic robustness against disturbances. In order to investigate this question in detail we consider the effect of a specific type of decoherence on adiabatic quantum computing schemes to solve a problem called “search in a disordered list”. With respect to this choice of decoherence model we analyze how the efficiency of the adiabatic search scheme is affected by decoherence. In other words, we ask whether the benefit of having a quantum computer to solve this computational problem vanishes as decoherence sets in.

In general it is not an easy task to determine the evolution of quantum systems, neither theoretically nor numerically. This is especially true if the system depends on time-dependent parameters. One example is time-dependent (classical) electromagnetic fields influencing a molecule. In the case of such time-dependent systems, analytical solutions are very rare. One may, however, apply various kinds of approximation methods. One method is the adiabatic approximation. This approximation technique can be used when the parameter changes are slow. In general the adiabatic approximation becomes better the slower the change of the external parameters. Apart from being very useful as an approximation tool, the study of adiabatic evolution have also lead to the concept of geometric phases. This concept provides insights into the nature of quantum mechanics, but can also be used to implement parts of a quantum computer, in terms of the so called geometric quantum gates.

In the third project of this thesis we generalize the adiabatic theorem to quantum systems influenced by weak open system effects. This generalized
approximation may be used in situations where we wish to investigate how
an ideal adiabatic evolution for a closed system is affected by open system
effects. One example is the above mentioned geometric quantum gates, which
in reality are bound to be affected by decoherence. With the present approx-
imation technique one may study how decoherence influence the functioning
of these gates.
2. Background

In this chapter we briefly describe some basic concepts that play a central role in this thesis.

2.1 Density operators

In quantum mechanics a Hilbert space is associated to each quantum system. Such a Hilbert space $\mathcal{H}$ is a complex linear space and represents all possible pure states $^1$ of the system. However, there are cases when pure states do not provide a satisfactory description. In contrast to the pure states, the mixed states can be used to describe more general situations, and can be represented using density operators.

When an observable $A$ is measured over an ensemble corresponding to the state vector $|\psi\rangle$ the expectation value is $\langle \psi | A | \psi \rangle$. One can think of this in terms of a machine that prepares quantum systems according to some well defined procedure. We let the machine repeatedly produce systems using the same preparation procedure each time. On each system the observable $A$ is measured and the average of the outcomes tends to the expectation value $\langle \psi | A | \psi \rangle$. Now, suppose there is some instability in the machinery of the preparation device. The machine acts as a random generator producing a state $|\psi_k\rangle$ with probability $p_k$. $^2$ When the observable $A$ is measured, the average of the outcomes tends to $\sum_k p_k \langle \psi_k | A | \psi_k \rangle$, which is the weighted average of the expectation values of all the output states. The expectation value can be written

$$\sum_k p_k \langle \psi_k | A | \psi_k \rangle = \text{Tr}(\rho A),$$

(2.1)

where the operator $\rho$ is

$$\rho = \sum_k p_k |\psi_k\rangle \langle \psi_k|.$$

(2.2)

The linear operator $\rho$ is called the density operator. In this example $\rho$ characterizes the average output of the preparation device.

$^1$A state is pure when it always gives a definite outcome for some maximal test [1].

$^2$We assume that the preparations are independent and identically distributed.
Density operators are positive, i.e., $\langle \phi | \rho | \phi \rangle \geq 0$ for all $|\phi\rangle \in \mathcal{H}$. Moreover, they fulfill $\text{Tr}(\rho) = 1$. Both these properties are reflections of the fact that quantum mechanics assigns probabilities to outcomes of measurements.

2.2 Reduced density operators

Density operators do not necessarily describe the output of random preparation devices. They are also suitable to use if we wish to describe the statistical properties of a subsystem of a larger quantum system. The state of the total system is given by the density operator of the total system. The properties of a subsystem, however, is given by the reduced density operator.

Consider a total system consisting of subsystems 1 and 2, with Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. The total system has the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$, where $\otimes$ denotes the tensor product. One example is two distinguishable particles, where each particle is a subsystem. Another example is one single particle with an intrinsic spin, where the spin degree of freedom is one subsystem and the spatial degree of freedom is the other subsystem.

Suppose we measure an observable $A_1$ on subsystem 1, but that we are unable to do any measurements on subsystem 2. To measure observable $A_1$ on system 1 and nothing on system 2 corresponds to the operator $A_1 \otimes \hat{1}_2$, where $\hat{1}_2$ is the identity operator on $\mathcal{H}_2$. As we have seen, the expectation value of the observable is given by $\text{Tr} (A_1 \otimes \hat{1}_2 \rho)$, where $\rho$ is the density operator of the total system. Let \{|$1_k\rangle$\}$_k$ be an orthonormal basis (ON-basis) of $\mathcal{H}_1$ and let \{|$2_l\rangle$\}$_l$ be an ON-basis of $\mathcal{H}_2$. Then

$$\text{Tr} (A_1 \otimes \hat{1}_2 \rho) = \sum_{kl} \langle 2_l | \langle 1_k | A_1 \otimes \hat{1}_2 \rho | 1_k \rangle | 2_l \rangle = \text{Tr}(A_1 \rho_1), \quad (2.3)$$

where

$$\rho_1 = \sum_l \langle 2_l | \rho | 2_l \rangle \quad (2.4)$$

is the reduced density operator. The operation that takes the density operator $\rho$ to the reduced density operator $\rho_1$ is called partial trace, and can be defined as

$$\text{Tr}_2(\rho) = \sum_l \langle 2_l | \rho | 2_l \rangle, \quad (2.5)$$

where the subscript indicates the subsystem to be “traced out”. One can show that the operation $\text{Tr}_2$ is independent of the choice of ON-basis \{|$2_l\rangle$\}$_l$. 

6
2.3 State evolution of closed and open systems

The basic equation governing time evolution in quantum mechanics is the Schrödinger equation (units chosen such that $\hbar = 1$ from now on)

$$i \frac{d}{dt} |\psi\rangle = H(t) |\psi\rangle,$$  \hspace{1cm} (2.6)

where $H(t)$ is a possibly time-dependent Hamiltonian. The state change from an initial time 0 to some fixed time $t$, can be described by the linear map $U(t)$ as

$$|\psi(t)\rangle = U(t) |\psi(0)\rangle, \hspace{1cm} U(t) = \mathcal{T} e^{-i \int_{0}^{t} H(s) ds},$$  \hspace{1cm} (2.7)

where $\mathcal{T}$ is the time-ordering operation [2]. For every fixed time $t$ the evolution operator $U(t)$ is unitary. If the initial state is mixed, rather than pure, the state change can be expressed as

$$\rho(t) = U(t) \rho(0) U(t)^\dagger.$$  \hspace{1cm} (2.8)

The type of evolution described by (2.8) we call a unitary evolution, and is associated with closed quantum systems. Loosely speaking, closed quantum systems are those that do not interact with other quantum systems. Any quantum system evolving according to the Schrödinger equation is a closed system. One example is an electron moving in a time-dependent electromagnetic potential. As a contrast, if there is another quantum system with which the system of interest interacts, then the latter does in general not evolve like a closed system. The two systems considered as a whole may be a closed system and the evolution be unitary, but the evolution of the reduced density operator of the system of interest can in general not be described by equation (2.8). We say that the evolution is non-unitary and that the system of interest is an open system.

2.4 Completely positive maps

How should we mathematically model the evolution of open systems? Since the density operators correspond to physical states it seems reasonable to look for maps that take density operators to density operators. Among these we wish to find those that are realizable in principle. We assume these maps to be linear, since time evolution in ordinary quantum mechanics is linear. As mentioned before, density operators are positive and have trace 1. We let $Q \geq$
0 denote that the operator $Q$ is positive. Now, consider linear maps $\Phi$ that satisfy $\Phi(Q) \geq 0$ for all $Q \geq 0$, and $\text{Tr}(\Phi(Q)) = \text{Tr}(Q)$ for all operators $Q$. We call such maps positive and trace preserving. Since these maps are linear and do map density operators to density operators one may at first sight be tempted to regard these as corresponding to physical operations. However, the situation is slightly more subtle.

We will now construct a counterexample; a linear map that is positive and trace preserving, but which nevertheless “behaves badly”. Let $\{|\psi_k\rangle\}_k$ be an orthonormal basis of the Hilbert space $\mathcal{H}$, and consider the following map

$$T(\rho) = \sum_{kk'} |\psi_k\rangle \langle \psi_{k'}| \rho |\psi_{k'}\rangle.$$  \hspace{1cm} (2.9)

The mapping $T$ performs the transposition of the operator $\rho$, with respect to the orthonormal basis $\{|\psi_k\rangle\}_k$. One can check that $T$ is linear and trace preserving. To show that it is positive it is sufficient to show that $T(|\phi\rangle \langle \phi|) \geq 0$ for every $|\phi\rangle \in \mathcal{H}$.\footnote{If $A$ and $B$ are Hermitian operators on $\mathcal{H}$, then $A \succeq B$ means that $\langle \psi| A|\psi\rangle \geq \langle \psi| B|\psi\rangle$ for all $|\psi\rangle \in \mathcal{H}$.}

Now, suppose that we have two systems and that $T$ acts on one of them, but leaves the other unaffected. On the total system this corresponds to the mapping $T \otimes I$, where $I$ is an identity map. It is reasonable to require that $T \otimes I$ should be a positive map if $T$ is a physical operation, since we otherwise could obtain unphysical states by operating on a subsystem.

Consider two spin-half particles, each with spin up $|\uparrow_k\rangle$ and spin down $|\downarrow_k\rangle$ states. Suppose this two-spin system is in the entangled state $\psi = \frac{1}{\sqrt{2}} (|\uparrow_1\rangle |\downarrow_2\rangle - |\downarrow_1\rangle |\uparrow_2\rangle)$. If $T$ is defined with respect to the $\{|\uparrow_1\rangle, |\downarrow_1\rangle\}$ basis, one finds that the operator $[T \otimes I_2] (|\psi\rangle \langle \psi|)$ has the eigenvalues $\frac{1}{2}$, $\frac{1}{2}$, $\frac{1}{2}$, and $-\frac{1}{2}$. Hence, $[T \otimes I_2] (|\psi\rangle \langle \psi|)$ is not a positive operator. This means that if $T$ operates on one subsystem of an entangled pair we may obtain non-physical states. We can conclude that, although $T$ is positive and trace preserving, it is not acceptable as a physical operation.

We have now seen that linearity, positivity, and trace preservation are not sufficient to get a good characterization of physical operations. How to proceed? There is a hint hidden in the counterexample. The counterexample was constructed by letting $T$ act on a subsystem of a larger system. What if we demand all such extensions to be positive maps?

---

\textsuperscript{3}It is sufficient to let $T$ act on pure states since $T$ is linear and since every density operator can be written as $\rho = \sum_k \lambda_k |\phi_k\rangle \langle \phi_k|$, where each $|\phi_k\rangle \in \mathcal{H}$. In the finite-dimensional case this follows from the positivity of the density operator.

\textsuperscript{4}A state $\rho$ (on a system with two subsystems) is entangled if and only if it cannot be written $\rho = \sum_k \lambda_k \rho_{1,k} \otimes \rho_{2,k}$, where $\rho_{1,k}$ and $\rho_{2,k}$ are density operators on subsystems 1 and 2, respectively, and where the numbers $\lambda_k$ satisfy $\lambda_k \geq 0$ and $\sum \lambda_k = 1$. 

---

8
Figure 2.1: Visualization of $\Phi(\rho) = \text{Tr}_a(U\rho \otimes |a\rangle\langle a|U^\dagger)$. Channels can be realized using a joint unitary evolution on a system and an ancillary system.

A linear map $\Phi$ is called a completely positive map (CPM) if $[\Phi \otimes I_n](\rho) \geq 0$ for all positive operators $\rho$ on $\mathcal{H} \otimes \mathcal{H}_n$, and all $n$. We use the words channel and operation as synonymous to trace preserving CPMs. We can conclude that channels take density operators to density operators, even if acting on a subsystem of a larger system.

The next question is whether channels are physically realizable. In other words, can there be a quantum system that evolves as prescribed by such a map? Consider a quantum system interacting with another system. This other system is usually referred to as the ancilla to differ it from the system of interest. The system and ancilla are initially in a product state $\rho \otimes |a\rangle\langle a|$. They evolve jointly as a closed system under some fixed elapse of time $t$. After the evolution has taken place, the reduced density operator of the system is

$$\Phi(\rho) = \text{Tr}_a(U\rho \otimes |a\rangle\langle a|U^\dagger).$$

(2.10)

For any ancillary Hilbert space $\mathcal{H}_a$, any normalized vector $|a\rangle$ in $\mathcal{H}_a$, and any unitary operator $U$ on $\mathcal{H} \otimes \mathcal{H}_a$, the mapping $\Phi$ is a trace preserving completely positive map [3] (see figure 2.1). Moreover, it can be shown [3] that for any channel there exists some ancillary system with Hilbert space $\mathcal{H}_a$, some element $|a\rangle$ in $\mathcal{H}_a$, and some unitary operator $U$ on $\mathcal{H} \otimes \mathcal{H}_a$, such that (2.10) holds for all density operators $\rho$ on $\mathcal{H}$. In other words, every channel can be realized using a joint unitary evolution on the system and an ancillary system.

---

Here the author lies a bit. This assumes that the channel is a mapping from $L(\mathcal{H})$ to $L(\mathcal{H})$, where $L(\mathcal{H})$ denotes the set of linear operators on $\mathcal{H}$. (See paper I.)
A useful way to represent CPMs are *Kraus representations* [3]. Given a CPM $\Phi$ there always exists a set operators $\{V_k\}_k$ such that

$$\Phi(\rho) = \sum_k V_k \rho V_k^\dagger.$$  \hfill (2.11)

If $\Phi$ is trace preserving, the set of operators $\{V_k\}_k$ satisfies

$$\sum_k V_k^\dagger V_k = \hat{1}.$$  \hfill (2.12)

Conversely, a set of operators $\{V_k\}_k$ that satisfies condition (2.12) defines a channel via equation (2.11).

Before ending this section it should be pointed out that channels are not always appropriate to use as models of open system dynamics. Channels can be viewed as describing controlled experiments, where we use preparation procedures generating systems that have no correlation with other relevant degrees of freedom. Moreover, it assumes an evolution device, or an environment, that always can be “reset” in a fixed initial state. If these prerequisites are not satisfied, then channels usually cannot be expected to provide good models. One example when the channel concept may be inappropriate is a system that continuously interacts with some environment not under our control. Suppose we do not have the possibility to prepare the system, and are only able to make limited observations on it. It is reasonable to expect that the system and its environment is more or less always in a changing entangled state. Since there are many states on the combined system that correspond to the same reduced density operator, a full knowledge of the reduced density operator at some moment does not in general provide sufficient information to predict the future state of the system. Thus, in such cases one cannot even expect a proper mapping describing the state change.

2.5 The two-path interferometer

The single particle interference effect is a manifest illustration of the difference between classical and quantum mechanics. The standard textbook example is the double-slit experiment where particles are sent, one by one, onto a double-slit [4]. Under suitable conditions the distribution of hits behind the

---

7In the case $\{V_k\}_k$ has an infinite number of elements, a convergence criteria has to be satisfied. If the Hilbert space is separable and the index set $I$ is countably infinite, then $\{V_k\}_{k \in I}$ has to be such that $\sum_{k \in I} V_k^\dagger V_k \leq \hat{1}$, for all finite $I_0 \subset I$ (see [3]).

8To “reset” the environment might be to let it relax to thermal equilibrium before the next run of the experiment.

9There are exceptions to this, as will be discussed in section 2.6.
Figure 2.2: The two-path interferometer gives the probability of detecting the particle in path 1, as a function of the variable phase shift $\phi$.

double-slit shows interference effects reminiscent of the interference patterns obtained with classical waves.

The two-path interferometer is very simple, yet it can be used to model many of the intriguing effects in single particle interferometry. To describe the idea it is convenient to think of a particle such as a photon. A central part is the beam-splitter. If a laser beam shines on a beam-splitter at a suitable angle, then half of the intensity is transmitted, while the other half is reflected. This can also be done with single photons. However, in this latter case there is no splitting of intensity. The effect of the beam-splitter is rather to cause a superposition of the photon being transmitted or reflected. In other words, the effect of the beam-splitter is to create a superposition of one single photon propagating in two directions.

We model the path-states of the photon as a two-dimensional Hilbert space with an orthonormal basis $|1\rangle, |2\rangle$. The element $|1\rangle$ corresponds to a photon localized in path 1, and $|2\rangle$ corresponds to the photon localized in path 2. (See figure 2.2.) The effect of the beam-splitter can be modeled by the unitary operator

$$U_{BS} = \frac{1}{\sqrt{2}} (|1\rangle + |2\rangle) \langle 1| + \frac{1}{\sqrt{2}} (|1\rangle - |2\rangle) \langle 2|. \quad (2.13)$$

The next step is to insert a variable phase shifter into one of the paths. The effect of the phase shifter is to add a relative phase between the two paths, and can be modeled by the unitary operator $U_{ps} = |1\rangle \langle 1| + e^{i\phi} |2\rangle \langle 2|$. With mirrors
the two paths can be made to recombine. At the focus a second beam-splitter is inserted, which makes the interference possible. After the second beam-splitter a detector is placed in path 1, say. This detector measures whether the photon is present or not. The probability to find the photon in path 1 is $p_1 = \frac{1}{2}(1 + \cos(\phi - \phi_0))$, where the constant $\phi_0$ depends on details in the interferometer setup. Hence, the detection probability exhibits an oscillatory dependence on the variable phase $\phi$.

The two-path interferometer model can be made richer if we assume that the particle has an internal degree of freedom. One example of this can be found in [5]. In the case of a particle with an internal degree of freedom we may determine the visibility and phase shift of the interference patterns for each possible input internal state $\rho_I$. In total the result can be presented as a complex valued function of $\rho_I$, where the absolute value and argument correspond to the visibility and phase shift.

2.6 Master equations

In section 2.4 we discussed mappings that describe the state change between two fixed moments in time. However, sometimes one wishes to describe the entire evolution, rather than one single map. A master equation is a differential equation for the density operator of a system; one of the simplest examples being the Schrödinger equation on Master equation form,

$$\frac{d}{dt}\rho(t) = -i[H(t), \rho(t)]. \quad (2.14)$$

Master equations can be used to model non-unitary evolution of open systems. There are many master equations in the literature (for some examples, see [6, 7, 8, 9]) but there is a special class of master equations with especially appealing properties. These can be written on the Lindblad form

$$\frac{d}{dt}\rho = -i[H, \rho] + \sum_n V_n \rho V_n^\dagger - \frac{1}{2} \sum_n V_n^\dagger V_n \rho - \frac{1}{2} \rho \sum_n V_n^\dagger V_n, \quad (2.15)$$

where $H$ is a time-independent Hermitian operator and $V_n$ are arbitrary time-independent operators.

A master equation that can be written on the Lindblad form induces a one-parameter family of linear maps $\Phi_x$ such that $\rho(t_2) = \Phi_{t_2-t_1}\rho(t_1)$, for $t_2 \geq t_1$. Each $\Phi_x$ is trace preserving and completely positive [10, 11]. Given the knowledge of the state at time $t_1$, these maps allow us to calculate the state at all times $t_2 \geq t_1$. In other words, we only need the present state of the system in order to know the future. This type of evolution is sometimes referred to as Markovian.
In the end of section 2.4 it was mentioned that the total state of the system and environment has to be known in order to determine the future of the system. This is quite in contrast with Markovian evolution. Generally we can therefore not expect systems to be governed by Markovian master equations. However, Markovian evolution may in some cases be taken as an approximation, valid on a time scale much larger than the time scale of the fluctuations in the environment.\textsuperscript{10} Justifications of Markovian master equations from an underlying model of a system and environment, are often obtained in some weak interaction limit [14, 15].

2.7 Decoherence

The word “decoherence” is a bit difficult to define precisely, partially because it is used in many not entirely equivalent senses. One way to put it would be to say that decoherence is a process, caused by the interaction between a system and its environment, that have a tendency to reduce entanglement with respect to certain subsystem decompositions, and to diminish superposition with respect to certain orthonormal bases.\textsuperscript{11} Which subsystem decomposition, and which basis, depends on the details of the system-environment interaction. In this thesis, we only use the word decoherence to signify reduction of superposition with respect to a given orthonormal basis. If the density operator is represented as a matrix in this basis, then the reduction of superposition corresponds to a diminishing of the off-diagonal elements in this matrix. As mentioned above, this kind of reduction may occur due to interactions with an environment. One of the simplest possible models of this is provided by the Hamiltonian $H = W \otimes Q$, where $W$ and $Q$ are time-independent Hermitian operators acting on the system and the environment, respectively. We assume that $W$ has a complete orthonormal eigenbasis $|w_k\rangle$ and eigenvalues $w_k$. If the system and environment start in the product state $\rho \otimes |a\rangle\langle a|$, and if we represent the reduced density operator of the system as a matrix $\rho_{kl} = \langle w_k|\rho|w_l\rangle$,

\textsuperscript{10}A state change of the system may induce a change in the environment, which in turn, may “kick back” on the system after a while. In this way the environment has a memory of the earlier states of the system that may influence the evolution. If we assume the evolution to be Markovian we disregard this feedback. For a discussion on time-scales and Markovian master equations, see [12]. For discussions on the problems with Markovian master equations in relation to thermodynamics, see [13].

\textsuperscript{11}One may consider more general types of decoherence that are not associated with ON-bases, but rather with more general collections of states, called pointer states. These pointer states are, within a certain time window, comparably stable against the system-environment interaction [16]. The stability may be measured in how fast the von Neumann entropy $S(\rho) = -\text{Tr}(\rho \ln \rho)$ of the reduced density operator of the system increases. This can be turned into a procedure, called the predictability sieve, which can be used to find the pointer states [17, 18].
\[ \rho_{kl}(t) = \rho_{kl}(0) \langle \psi_l(t) | \psi_k(t) \rangle, \quad | \psi_k(t) \rangle = e^{-itw_k Q} | \alpha \rangle. \quad (2.16) \]

The degree by which the off-diagonal element \( \rho_{kl} \) becomes suppressed is determined by the extent to which the vectors \( | \psi_k(t) \rangle \) and \( | \psi_l(t) \rangle \) become orthogonal. One way to put this is to say that the superposition gets reduced to the extent that the system leaves traces in the environment.

Using master equations it is quite easy to construct models for decoherence. As before, we assume an Hermitian operator \( W \) on the system of interest, with the same eigenbasis and eigenvalues as above. Consider the following master equation

\[ \frac{d}{dt} \rho = -B [W, [W, \rho]], \quad (2.17) \]

where \( B \geq 0 \) is a constant. This can be rewritten as \( \frac{d}{dt} \rho_{kl} = -B (w_k - w_l)^2 \rho_{kl} \), with solution

\[ \rho_{kl}(t) = \rho_{kl}(0) e^{-it(w_k - w_l)^2}. \quad (2.18) \]

Equation (2.17) is an example of a wide-open system [19], where the evolution is solely governed by decoherence. In most cases a master equation also contains a Hamiltonian term, in which case the evolution becomes more complicated. Qualitatively, however, the right-hand side of equation (2.17) always tends to diminish superposition with respect to the \( | w_k \rangle \) basis, although other processes may contribute to make the evolution more complicated.

### 2.8 The adiabatic approximation

The adiabatic theorem [20] concerns quantum systems whose evolution depends on some external parameters, e.g., a molecule exposed to electric or magnetic fields. The adiabatic theorem essentially states that if the system starts in an eigenstate of the Hamiltonian, and the external parameters are changed sufficiently slowly, then the state remains close to the instantaneous eigenstate of the Hamiltonian. Related to the adiabatic theorem is the adiabatic approximation, which describes the evolution in the limit of very slow changes. Apart from being a valuable approximation tool, it has lead to insights into the nature of quantum mechanics, such as geometric phases [21] and more generally into quantum holonomies [22].

The adiabatic approximation is usually formulated in terms of the Schrödinger equation and pure states. The approximation decouples the total evolution in such a way that each eigenspace evolves on its own, thus yielding a collection of smaller and possibly much simpler equations. In anticipation of what to come we will here make a perhaps unusual formulation of the
adiabatic approximation, in terms of master equations.

Our starting point is the Schrödinger equation on master equation form
\[
\frac{d}{dt} \rho(t) = -i [H(t/T), \rho(t)], \tag{2.19}
\]
where \( t \in [0, T] \) and \( T \) is the run-time. Note that the family of Hamiltonians stays fixed for all choices of \( T \). The choice of \( T \) affects only the speed by which the path of Hamiltonians is traversed. It is convenient to change variables to \( s = t/T \), which results in
\[
\frac{d}{ds} \rho(s) = -iT [H(s), \rho(s)]. \tag{2.20}
\]
We let \( P_k(s) \) denote the projectors onto the eigenspaces of \( H(s) \) such that
\[
H(s) = \sum_{k=1}^{K} E_k(s) P_k(s), \tag{2.21}
\]
where \( E_k(s) \) are the corresponding eigenvalues. The adiabatic approximation can be stated in terms of the following equation
\[
\frac{d}{ds} \rho(s) = -i [TH(s) + Q(s), \rho(s)], \tag{2.22}
\]
where
\[
Q(s) = i \sum_k \dot{P}_k(s) P_k(s). \tag{2.23}
\]
Formulated in this way it is not at all obvious that there is any simplifying decoupling in the approximate equation. However, we shall rewrite equation (2.22) in a way that makes the decoupling evident.

Since the Hamiltonian is time-dependent (or \( s \)-dependent) the eigenspaces rotate in the Hilbert space as time passes. It shows to be convenient to consider unitary operators \( U(s) \) that map each eigenspace at \( s \) back to the corresponding eigenspace at \( s = 0 \). In other words
\[
U(s)P_k(s)U^\dagger(s) = P_k(0). \tag{2.24}
\]
Note that these unitary operators are not unique, there are many possible choices and we may use any sufficiently smooth family. We also define a new density operator \( \tilde{\rho}(s) \) by
\[
\tilde{\rho}(s) = U(s)\rho(s)U^\dagger(s). \tag{2.25}
\]
In terms of $\tilde{\rho}(s)$, equation (2.22) takes the form

$$\dot{\tilde{\rho}} = -iT[\tilde{H}(s), \tilde{\rho}(s)] - i \left[ \sum_l Z_l(s), \tilde{\rho}(s) \right], \tag{2.26}$$

where

$$\tilde{H}(s) = U(s)H(s)U^+(s) = \sum_k E_k(s)P_k(0),$$

$$Z_l(s) = i P_l(0)U(s)U^+(s)P_l(0). \tag{2.27}$$

We decompose the density operator $\tilde{\rho}(s)$ as

$$\tilde{\rho} = \sum_{kl} \tilde{\rho}^{(kl)}, \quad \tilde{\rho}^{(kl)} = P_k(0)\tilde{\rho}P_l(0). \tag{2.28}$$

We refer to $\tilde{\rho}^{(ll)}$ as the diagonal terms, while we refer to $\tilde{\rho}^{(kl)}$, with $k \neq l$, as the off-diagonal terms. With (2.28) into (2.26) the approximate equation can be written as

$$\frac{d}{ds} \tilde{\rho}^{(kl)} = -iT(E_k(s) - E_l(s))\tilde{\rho}^{(kl)}(s) - iZ_k(s)\tilde{\rho}^{(kl)}(s) + i\tilde{\rho}^{(kl)}(s)Z_l(s). \tag{2.29}$$

Clearly, all the diagonal and off-diagonal terms are decoupled from each other.

The very point with the adiabatic approximation is that it becomes more accurate the slower the parameter of the Hamiltonian is changed. This corresponds to making $T$ larger. If we compare the original equation (2.20) with the adiabatic approximation (2.22) one can see that the only difference is the operator $Q(s)$. Note also that $Q(s)$ is not multiplied with $T$. Intuitively, the term $TH$ dominates over $Q$ as $T$ grows, and $Q$ affects the evolution less. Thus, it seems rather reasonable that the adiabatic approximation should be accurate when $T$ is large.

How large should $T$ be in order for the adiabatic approximation to be accurate? A very often used condition for adiabaticity [20, 23, 24, 25] is

$$T \gg \frac{\delta}{g^2}, \tag{2.30}$$

where

$$g = \min_{s \in [0,1]} (E_1(s) - E_0(s)),$$

$$\delta = \max_{s \in [0,1]} \left| \langle E_1(s) | \frac{dH}{ds} | E_0(s) \rangle \right|. \tag{2.31}$$

Here $E_0$ and $E_1$ are the energies of the ground state and the first excited states,
respectively. If the system starts in the ground state, then equation (2.30) gives the condition for remaining close to it. Equation (2.30) does not provide us with an explicit number that tells us what a sufficiently large $T$ is, but it provides us with the scale in which “large” is to be understood. Moreover, we see from equation (2.30) that the smaller the energy gap, the slower we have to change the parameter in order to stay close to the ground state.

2.9 Quantum computing

One of the key questions in quantum computing is: Can computations be performed more efficiently, compared to ordinary computers, if computational devices are based on the laws of quantum mechanics? In order to make this a precise question we need to have a clear idea of what “efficiency” is supposed to mean. Imagine a computational problem, e.g., to find a specific element in a list of several elements. The time it takes to solve this problem depends on the hardware, whether it is solved by hand or by a supercomputer. To obtain a measure of efficiency that does not depend on the hardware, one can determine how the run-time, or the number of elementary operations, grows with the problem size. The problem size is usually measured in the number of bits needed to describe the problem, such as, e.g., the number of bits needed to describe an integer that we wish to factorize. By using this efficiency measure we can also define a measure of how difficult a computational problem is. The complexity of a computational problem can be defined as the efficiency of the most efficient algorithm that solves it. Needless to say, it is generally quite difficult to determine the computational complexity of a problem. A method with a time-complexity that is polynomial is called efficient or tractable, while a method with a time-complexity that grows faster than any polynomial is called hard, inefficient, or intractable.

Suppose we have a list of objects. In this list there is a specific element that we wish to find. Moreover, suppose the list is totally random. It is not difficult to realize that, on average, one has to look through half of the list before the desired element is found. In other words, the effort grows linearly with the list length in the classical setting. This linear dependence can thus be seen as the classical time-complexity of this problem. It has been shown that a quantum computer may perform this task with an effort that increases like $\sqrt{N}$, using the so-called Grover search [26]. This is not a very dramatic increase of efficiency, although it may become a pronounced advantage when the list becomes very long. More important, though, is the fact that this efficiency is better than what is achieved classically. In other words, there appears to be a

\[\text{There are similar expressions if one starts in an excited state, but then one must take into account both the next excited state and the state below.}\]
fundamental difference between classical and quantum computation.

Another example is the factorization of integers into prime numbers. For this problem there exists a quantum algorithm with a polynomial time-complexity in the size of the integer (measured in bits) [27, 28, 29]. At present the best classical algorithm is the number field sieve, which does not have a polynomial efficiency [30]. Note though, that it has not yet been proved that the factorization problem is an intractable problem in the classical setting.\(^\text{13}\)

Several different “flavors” of quantum computers have been introduced. The oldest is the circuit model [33], which resembles the ordinary computer in that it builds up all operations as sequences of single qubit and two-qubit operations [33], in the same way as the ordinary computer does with bits. Another example is the measurement based quantum computer, where the computation is initiated in a fixed highly entangled state and the whole computation is driven by measurements [34, 35]. The present investigation is mainly focused on a third approach that goes under the name “adiabatic quantum computing” [24, 36].

2.10 Adiabatic quantum computing

Suppose it would be possible to encode computational problems into Hamiltonians in such a way that the ground state gives the solution to the problem. Below we discuss how this might be done. Suppose there is another Hamiltonian with a known ground state that is easily prepared. Suppose, moreover, that there is some external parameter that interpolates between the systems governed by the simple initial Hamiltonian and the complicated problem Hamiltonian. According to the adiabatic theorem we can start the system in the ground state of the initial Hamiltonian, change the external parameter sufficiently slowly, and end up close to the ground state of the problem Hamiltonian. As the ground state will be found with a high probability, we can measure the energy and read off the answer to our computational problem. In order to find out whether this can be done efficiently we have to determine what is a “sufficiently slow” change of the parameter. In other words, we have to determine the run-time as a function of the problem size.

How can computational problems be encoded into Hamiltonians in such a way that the ground state gives the solution to the computational problem? Does one not need to know the answer in order to construct such a system? Hopefully, the following example will clarify that this is not the case. Consider

\(^{13}\)This is partially related to the long standing \(P \neq NP\) conjecture in classical computational theory [31, 32], where \(P\) denotes those problems that can be solved in polynomial time and \(NP\) those that can be solved nondeterministically in polynomial time.
Figure 2.3: An electron is free to move on a line above boxes with a positively charged object hidden in one of them. The electron feels a potential with a minimum above the box with the object. Likewise the ground state wave function of the electron is peaked around the same box. Thus, the Hamiltonian of the electron has a ground state that indicates the box in which the object is hidden. In order to construct this system we do not need know in which box the object is.

a collection of closed boxes. An object is hidden in one of these boxes and we wish to find out in which. Now, suppose the object is positively charged and that we can place an electron on a line above the boxes. Clearly, the electron feels a potential with a minimum right above the box with the object. It follows that the ground state wave-function will be peaked around the wanted box. (See figure 2.3.) Note that we have achieved what we wanted: An Hamiltonian (for the electron) with a ground state that indicates the solution of our search problem. Moreover, we do not need to know in advance where the object is in order to construct this system. We only need to put an electron on a line above the boxes. This way to solve the search in a disordered list seems somewhat difficult to achieve in practice. Moreover, it seems difficult to treat it theoretically. In order to analyze the quantum search we use another construction.

To the $N$-element search problem we associate an $N$-dimensional Hilbert space, in which we choose some orthonormal basis $\{|k\rangle\}_{k=1}^N$. To the position $k$ in the list we associate the basis element $|k\rangle$. The desired element with the unknown position is denoted $|\mu\rangle$. The problem Hamiltonian can be written

$$H_1 = -|\mu\rangle\langle\mu|.$$  \hspace{1cm} (2.32)

This Hamiltonian has the properties which we demand from a problem Hamiltonian, since the ground state $|\mu\rangle$ of $H_1$ indicates the position of the desired element. The ground state $|\mu\rangle$ has energy $-1$, while all other states correspond to the degenerate eigenvalue $0$. As the initial Hamiltonian we choose

$$H_0 = -|\psi\rangle\langle\psi|,$$  \hspace{1cm} (2.33)

One might note that the object can be classical to its nature. In other words, the list does not have to be stored in a quantum memory. It is the search engine (the electron in the example above), so to speak, that has to be quantum mechanical.
Figure 2.4: The eigenenergies of the search Hamiltonian $H(s)$ defined in equation (2.35), for two list lengths, $N = 10$ (solid lines) and $N = 100$ (dashed lines).

where

$$|\psi\rangle = \frac{1}{\sqrt{N}} \sum_{k=1}^{N} |k\rangle.$$  \hspace{1cm} (2.34)

In words, the ground state of the initial Hamiltonian is a uniform superposition of all the basis elements. We further choose the interpolation

$$H(s) = (1-s)H_0 + sH_1 = -(1-s)|\psi\rangle\langle\psi| - s|\mu\rangle\langle\mu|.$$ \hspace{1cm} (2.35)

The eigenvalues of $H(s)$ as a function of $s$ is shown in figure 2.4. We let $s = t/T$, where $T$ is the run-time. With this arrangement we obtain what we refer to as the global search scheme. It can be shown that with global search one obtains the efficiency $T \sim N$ [24], i.e., no improvement over the classical search. To use of a uniform speed of change of the parameter $s$ is not a particularly good idea. As seen in figure 2.4 the energy difference between the ground state and the first excited state is smallest in the vicinity of $s = 0.5$. As mentioned in section 2.8, the smaller the energy gap between ground state and the first excited state, the slower the parameter has to be changed in order for the evolution to not kick the system out of the ground state. In the global search scheme we use the run-time in a rather wasteful way, as the same speed of change is used no matter whether the gap is large or small. In view of this it seems a better strategy to adjust the speed to the to size of the energy gap. If this is done in a suitable way one obtains the so called local search [37, 38], which has the efficiency of the Grover search $T \sim \sqrt{N}$ [26].
3. Summary of the papers

This chapter presents a brief summary of the conceptually most important aspects and findings of the papers in this thesis.

3.1 Interferometry: Papers I and II

Paper I introduces and analyzes three concepts related to CPMs, and in paper II these concepts are applied to interferometers. The investigation presented in paper I were initiated due to a wish to answer some basic questions concerning the effects of open system evolution in interferometers. It soon turned out that these questions could be generalized from the ordinary two-path interferometer into a more abstract setting based on orthogonal decompositions of Hilbert spaces. Although formulated in this more general way in paper I\(^1\) we will here formulate the results in terms of the two-path interferometer. In paper II the focus is on how interference patterns are effected by open system evolution, and how interferometry can be used as a tool to measure open system effects.

3.1.1 Subspace preservation

Channels can describe many types of operations, also operations that one does not naturally associate with interferometers. Due to this one might restrict the set of all possible channels on the system. For a macroscopic interferometer it seems a rather natural assumption that the particle does not suddenly jump from one path to the other. Hence, there is no transport of particles between the two paths. What are the operations on the particle that do not make it “jump” between the paths, but is else arbitrary? The Hilbert space of the particle in the two paths of the interferometer is an orthogonal sum of two subspaces, one \(\mathcal{H}_1\) representing the pure states localized in path 1, and the other \(\mathcal{H}_2\) representing pure states localized in path 2. We wish to find those channels that do not “move” the state from one subspace to the other. To be able to answer this question we have to be a bit more precise about what we mean by “no transport of the particle”.

\(^1\)In this summary we assume that the “evolution devices” have the same input system as output system. Moreover, we assume that the orthogonal decomposition is the same for the input as for the output. None of this is assumed in paper I.
Let $P_1$ and $P_2$ be the projection operators onto the subspace $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. A channel $\Phi$ is subspace preserving (SP) on $(\mathcal{H}_1, \mathcal{H}_2)$ if
\[
\text{Tr}(P_1 \Phi(\rho)) = \text{Tr}(P_1 \rho),
\]
for all possible input density operators $\rho$ on the total Hilbert space. In words, this definition entails that it is the same probability to find the system in subspace $\mathcal{H}_1$ after the operation has been performed, as it was before. Note that the definition equivalently could be stated in terms of $P_2$. Note also the related study [39], which investigates the possible state changes caused by the measurement of an observable. Propositions 10 and 14 in paper I list all SP CPMs.

3.1.2 Subspace locality

The subspace preserving channels constitutes the class of operations that do not transport the particle between the two paths of the interferometer. This class includes many channels that, in some sense, coordinate their actions in the two paths. What if we would like to model operations that act locally in the two paths? Subspace locality is designed to take care of this question.

Locality in the standard sense

Suppose we have two separate locations and wish to discuss operations on these two locations. Quantum cryptography [40, 41] and teleportation [42] are examples of such situations. The standard method to model two locations is to associate a subsystem to each location.

There are two aspects of locality and non-locality. First is the question of states, secondly the question of operations. Although related, these should not be confused. The question of states concerns the correlation properties of states on the two systems, like whether a state is entangled or not. The other question, which is in focus here, is the locality or non-locality of operations, like whether an operation can be performed without any communication between the two systems. An operation is regarded to be local if it can be implemented by separate and independent operations on each location. Since the two locations are associated to subsystems, one with Hilbert space $\mathcal{H}_A$ and the other with Hilbert space $\mathcal{H}_B$, the total system has Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$. Channels are local with respect to these subsystems if and only if they are product channels $\Phi_A \otimes \Phi_B$, where channel $\Phi_A$ operates on density operators on $\mathcal{H}_A$, and $\Phi_B$ operates on density operators on $\mathcal{H}_B$. These types of operations can be performed without any type communication or sharing between the two locations. Each location minds its own business, so to speak.
Subspace local quantum channels

There are situations where the separation into locations is not immediately associated with a tensor product decomposition, but rather a decomposition into an orthogonal sum. One example is a single particle in a two-path interferometer, where each of the paths correspond to a subspace. The question is how to formalize the intuitive idea of an operation that acts locally in two subspaces. One way would be to generate operations in an intuitively “subspace local way”. As stated in section 2.4, every channel can be realized using a unitary channel on the system and an ancilla, via the equation

$$\Phi(\rho) = \text{Tr}_a(U\rho \otimes |a\rangle \langle a|U^\dagger).$$

(3.2)

Suppose we have two ancillary systems $a1$ and $a2$, with Hilbert spaces $\mathcal{H}_{a1}$ and $\mathcal{H}_{a2}$, respectively. We let $\mathcal{H}_I$ denote the internal state space, and $\mathcal{H}_S = \text{Sp}\{|1\rangle, |2\rangle\}$ the spatial state space, where $\text{Sp}$ denotes the linear span. We put one ancilla in each path and consider the following unitary operator

$$U_{tot} = |1\rangle \langle 1| \otimes U_1 \otimes \hat{1}_{a2} + |2\rangle \langle 2| \otimes \hat{1}_{a1} \otimes U_2,$$

(3.3)

where $U_1$ is a unitary operator acting on $\mathcal{H}_I \otimes \mathcal{H}_{a1}$ and $U_2$ is a unitary operator acting on $\mathcal{H}_I \otimes \mathcal{H}_{a2}$ (see figure 3.1). Note that $\rho$ is the total state of the system of interest, i.e., $\rho$ is a density operator on $\mathcal{H}_I \otimes \mathcal{H}_S$. In words the action of $U_{tot}$ can be described as follows. If the particle passes path 1, then the internal degree of freedom of the particle interacts with ancilla 1, but ancilla 2 is not affected. If the particle passes path 2, then it interacts with ancilla 2, but ancilla 1 is unaffected. If we imagine ancilla 1 to be located at path 1, and ancilla 2 at path 2, it seems reasonable to regard the channel

$$\Phi_{tot}(\rho) = \text{Tr}_{a1,a2}(U_{tot}\rho \otimes |a1\rangle \langle a1| \otimes |a2\rangle \langle a2|U_{tot}^\dagger)$$

(3.4)

as subspace local. But are there channels that we intuitively regard as subspace local, but that cannot be generated using this approach? What is the relation to the definition of locality with respect to subsystems? Is there any danger of some kind of inconsistency between this definition of subspace locality and the ordinary definition of locality?

We now use another approach to define subspace locality. To describe the basic ideas behind this definition of subspace local channels we have to shift perspective. When discussing the two-path system we have so far let the Hilbert space describe the state of the particle, i.e., how the particle is distributed over the two paths. Another approach is to ask how the two paths are occupied by the particle. We focus on the “occupation states” of paths 1 and 2,

2True for channels that are mappings from $\mathcal{L}(\mathcal{H})$ to $\mathcal{L}(\mathcal{H})$, where $\mathcal{L}(\mathcal{H})$ denotes the set of linear operators on $\mathcal{H}$.

3 In the special case of two-path systems, the definition above and the one below coincide.
Figure 3.1: Visualization of equations (3.3) and (3.4). One possible definition of sub-space local channels would be those that can be realized with locally acting unitary operations with local ancillary systems.

rather than the state of the particle. From this point of view, the two paths can be regarded as two subsystems. On these subsystems the standard definition of local channels can be used.

In order to describe the occupation states of the paths one can make a second quantization of the original Hilbert space. The original Hilbert space $\mathcal{H}$ is referred to as the first-quantized space. By the second quantization we allow a variable particle number, including a vacuum state. If $\mathcal{H}_1$ and $\mathcal{H}_2$ are the first-quantized subspaces, we let $F(\mathcal{H}_1)$ and $F(\mathcal{H}_2)$ be the second quantized spaces. A key observation is the following

$$F(\mathcal{H}_1 \oplus \mathcal{H}_2) \simeq F(\mathcal{H}_1) \otimes F(\mathcal{H}_2).$$

(3.5)

In words this says that a second quantization of an orthogonal sum of two subspaces is equivalent to a tensor product of second quantizations of the two subspaces. The second quantization allows us to regard the two locations as two subsystems. With respect to this separation into subsystems the ordinary definition of local channels as product channels can be used.

Much of the efforts in paper I are invested in dealing with the complications introduced by the second quantization, and in translating the definition of subspace locality back to the first-quantized spaces. We will here give a brief overview of the set of subspace local (SL) channels. A more detailed account is provided by Proposition 34 in paper I.

The set of SL channels is partitioned into four sub-families. These four families can be characterized in terms of how they handle the probability weights on the subspaces (see figure 3.2). The first family preserves the probability

\[\text{24}\]
weights on the locations.

\[ \text{Tr}(P_1 \Phi(\rho)) = \text{Tr}(P_1 \rho). \]  \hspace{1cm} (3.6)

This family we call local subspace preserving (LSP). The reason for this name is that the set of LSP channels is the intersection between the set of SP channels and the set of SL channels. Hence, the LSP channels are those subspace preserving channels that simultaneously are subspace local. The second family swaps the probability weights on the two locations,

\[ \text{Tr}(P_2 \Phi(\rho)) = \text{Tr}(P_1 \rho). \]  \hspace{1cm} (3.7)

The remaining two channels concentrate the probability weight into one of the two locations,

\[ \text{Tr}(P_k \Phi(\rho)) = \text{Tr}(\rho) = 1, \quad k = 1, 2. \]  \hspace{1cm} (3.8)

At first sight it may seem strange that channels with the power to redistribute the probability weights should be regarded as local operations. However, seen from the point of view of second quantization this is rather natural. Suppose we have two devices, one on each location. One of these has the function to annihilate (absorb, deflect) particles inserted into the input. No matter the input, the machine always returns a vacuum state. The other device is slightly more interesting. If it receives a particle in the input, it evolves the state of the particle and returns it. If it receives no particle, it returns a particle in some fixed internal state. If this pair of machines acts on single particle input states, the output state is always concentrated in the output of the second device. Although the two machines act locally, there is a redistribution of the probability weights. One may note that this argument depends on the assumption that there exist identical particles. When removing a particle at one location there must be an identical particle to insert at the other location, otherwise the particle has to be transported from one location to the other, which is clearly not a local operation. This indicates that one may need to take into account possible extra constraints when investigating what are the realizable subspace local channels.

3.1.3 Gluing of completely positive maps

Imagine two evolution devices. Each of these devices has the function to evolve the internal state of a particle. The internal state evolution caused by each device is described by channels \( \Phi_1 \) and \( \Phi_2 \), respectively. Imagine we have one single particle. What happens to the particle if it is put in a superposition between the inputs of the machines? We know \( \Phi_1 \) and \( \Phi_2 \) and wish to deduce the evolution these two machines cause when acting jointly on
the superposition. This situation may arise in a two-path interferometer. The first beam-splitter causes the particle to be in a superposition between the two paths. When inserting the two devices, one into each path, the question is what operation these generate in the interferometer? It may be tempting to assume that the global evolution caused by the two machines is uniquely determined by the two channels $\Phi_1$ and $\Phi_2$. This is, however, not the case, as is shown in paper I. The following example may provide some intuitive understanding.

Consider a phase-shifter; the device used in the two-path interferometer. The effect of the phase shifter is to add a relative phase to the state of a particle passing through. If a particle passes the phase-shifter and the internal state is measured on the other side, we cannot notice any effect. In other words, the internal state channel of the phase-shifter is the identity CPM. However, when inserted into the interferometer the phase-shifter causes a measurable phase shift in the interference pattern. (See figure 3.3.) Thus, the internal state channel of the phase-shifter is not sufficient to determine the action of the phase-shifter, when it acts in an interferometer. In other words, there is some additional information needed to characterize this device.

In terms of the two-path system, gluings can be described in the following way. If a channel $\Phi_{tot}$ on the two-path system is a gluing of $\Phi_1$ and $\Phi_2$, then an input state $|1\rangle\langle 1| \otimes \rho_I$ gives the output state $|1\rangle\langle 1| \otimes \Phi_1(\rho_I)$, while if the input state is $|2\rangle\langle 2| \otimes \rho_I$ then the output is $|2\rangle\langle 2| \otimes \Phi_2(\rho_I)$. (See figure 3.4.)

Since the gluings are not unique, a natural question is what is the set of

---

**Figure 3.2:** The set of all subspace local channels decomposes into four subfamilies. One of these consists of the LSP channels, which is the intersection between the SL channels and the SP channels.
Figure 3.3: A constant phase-shifter $\chi$ acts like the identity channel on the internal state $\rho_I$ of a particle, but causes a measurable phase shift when inserted into an interferometer. The interference pattern gives the probability of detecting the particle, as a function of the variable phase shift $\phi$.

Figure 3.4: Suppose $\Phi_1$ and $\Phi_2$ are two channels acting on the internal state of the particle. $\Phi_{\text{tot}}$ is such that if the particle is localized in path 1 in the internal state $\rho_I$, then the output is localized in path 1 and in the new internal state $\Phi_1(\rho_I)$. If the input is localized in path 2, then the output is localized in path 2 and in the internal state $\Phi_2(\rho_I)$. Under these conditions $\Phi_{\text{tot}}$ is a trace preserving gluing of the channels $\Phi_1$ and $\Phi_2$. 
gluings of two given channels? Consider two channels $\Phi_1$ and $\Phi_2$ with linearly independent Kraus representations $\{V_k\}_k$ and $\{W_l\}_l$, respectively. The set of trace preserving gluings of $\Phi_1$ and $\Phi_2$ can be described via a matrix $C$

$$
\Phi_{\text{tot}}(\rho) = |1\rangle \langle 1| \otimes \sum_{n=1}^{N} V_n (1|\rho|1)V_n^\dagger \\
+ |2\rangle \langle 2| \otimes \sum_{m=1}^{M} W_m (2|\rho|2)W_m^\dagger \\
+ |1\rangle \langle 1| \otimes \sum_{n,m} C_{n,m} V_n (1|\rho|2)W_m^\dagger \\
+ |2\rangle \langle 1| \otimes \sum_{n,m} C_{n,m}^* W_m (2|\rho|1)V_n^\dagger,
$$

(3.9)

for all density operators $\rho$ on $\mathcal{H}_S \otimes \mathcal{H}_I$, where the matrix $C$ satisfies the condition

$$
I \geq CC^\dagger,
$$

(3.10)

where $I$ is an identity matrix, and where $\mathcal{H}_S$ and $\mathcal{H}_I$ are the spatial and internal state spaces. Thus, equation (3.9) provides us with all the trace preserving gluings of two given channels.

Another question is the cause of the non-uniqueness. Every trace preserving gluing of two channels is necessary an SP channel. It follows that trace preserving gluings are rather “allowing” in the sense that the two devices may interact or share correlated resources. Since $\Phi_1$ and $\Phi_2$ give only the action of each machine alone, it seems reasonable that if the machines can interact, then $\Phi_1$ and $\Phi_2$ do not provide information of how the machines act jointly. If we require the gluing to be SL, however, we prevent the machines from interacting or sharing correlated resources. Is the non-uniqueness removed by this constraint? If so, it would mean that the non-uniqueness of the gluing is merely due to possible interaction of the devices, and the question would be settled. However, gluings are not unique, even if we assume them to be SL.

Instead of discussing general LSP gluings of two arbitrary channels we here concentrate on gluings of a channel and an identity channel. Such gluings correspond to a device in one path, but nothing in the other. Like in the case of the phase-shifter, the non-uniqueness of gluings can be seen as an effect of an insufficient description the device. The reason is that the device operates on more general states when inside an interferometer. Because of the first

---

5The linearly independent Kraus representations are not unique. Any choice can be used in order to express the set of gluings.
6This is a special case of Proposition 19 in paper I.
7This may seem as restricting the generality of this discussion, but this is not the case. In paper I it is shown that every LSP gluing $\Phi$ can be decomposed as $\Phi = \Phi_2 \circ \Phi_1$, where $\Phi_2$ is a gluing of a channel on path 1 and an identity channel on path 2, and vice versa for channel $\Phi_1$. 

28
beam-splitter, the particle is in a superposition of propagating in path 1 and 2. Hence, what the machine gets is not just a particle in some specific internal state, but some combination of receiving a particle and not receiving a particle. The internal state channel $\Phi_1$ does not tell us what the machine do with such states. Viewed from the perspective of the occupation number representation, as described in the previous section, this can be understood. Fortunately, we only have to describe single-particle and vacuum states on each path. The occupation state space of path 1 can be taken as $\mathcal{H}_1 = \mathcal{H} \oplus \text{Sp}\{|0_1\rangle\}$. In words this means that on path 1 we may have a particle with various internal states, but also no particle, as well as various linear combinations of the particle and the vacuum state. The same construction is made for path 2, which results in the total state space $\mathcal{H}_1 \otimes \mathcal{H}_2$. The combination of a device in path 1 and nothing in path 2 can be described as a product channel $\tilde{\Phi}_1 \otimes I_2$, where $I_2$ is the identity channel on path 2. Moreover, if the internal state channel is $\Phi_1$, with a linearly independent Kraus representation $\{V_k\}_k$, it can be shown that $\tilde{\Phi}_1$ has to be on the form

$$\tilde{\Phi}_1(\tilde{\rho}) = \Phi_1(\tilde{\rho}) + V \tilde{\rho} |0_1\rangle\langle 0_1| + |0_1\rangle\langle 0_1| \tilde{\rho} V^\dagger + |0_1\rangle\langle 0_1| \tilde{\rho} |0_1\rangle\langle 0_1|, \quad (3.11)$$

where the operator $V$ is

$$V = \sum_k c_k V_k, \quad (3.12)$$

and where the complex numbers $c_k$ satisfy

$$\sum_k |c_k|^2 \leq 1. \quad (3.13)$$

In view of equation (3.11) one can see what is missing from the description of the device. The channel $\Phi_1$ describes only what happens when there is a particle “fully present” in path 1. The missing part is provided by the operator $V$ that describes what happens to superpositions of single particle states and the vacuum state. In other words, to give a complete description of a device, in the context of the two-path system, it is not sufficient to describe what the device does to a particle fully present. One also has to describe what it does to linear combinations of single particle-states and the vacuum state.\(^8\)

---

\(^8\)One may put it this way, although the device in the interferometer acts on a part of an entangled state rather than on such a superposition. The superposition of the particle propagating in the upper or the lower path $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$, in the occupation number representation corresponds to $\frac{1}{\sqrt{2}}(|1_1\rangle|0_2\rangle + |0_1\rangle|1_2\rangle)$, where $|0_k\rangle$, $|1_k\rangle$ are the vacuum and the single occupation states of path $k$. Hence, the device operates on a part of an entangled system.
3.1.4 On measuring gluings with interferometers

In paper II the focus is on gluings in two-path interferometers. The basic question is in what way given operations in the two paths of an interferometer affects the interference pattern. In view of the previous sections it should not come as a surprise that the interference pattern is not uniquely determined by the channels alone. Note the closely related [45], which considers how interferometry can be used to characterize certain aspects of quantum processes and introduces measures of coherence related to this.

In paper II it is determined what the possible interference effects are, given the two channels. It turns out to be the gluing that determines the interference effects, not the channels per se. Can the interference effects be used to determine the gluing? With the setup described in section 2.5, the answer is partially yes and partially no. It depends on the channels that are glued. In the ordinary two-path interferometer the gluing matrix $C$, in equation (3.9), can be uniquely determined if $\{W_k^\dagger V_l\}_{kl}$ is a linearly independent set. The set $\{W_k^\dagger V_l\}_{kl}$ is generally not linearly independent, but there is an important exception. If we consider gluings of a channel and an identity channel, then the gluing matrix $C$ can be determined uniquely using the standard interferometer. If one wishes to determine all types of gluings, the interferometer has to be generalized.

In the construction of the two-path interferometer in section 2.5 we used a variable phase-shifter in order to obtain an interference pattern. To generalize the interferometer we replace this variable phase-shifter with a variable unitary transformation on the internal state space. Although not needed in principle we keep the variable phase-shifter due to convenience. (See figure 3.5.) In the ordinary interferometer we described the interference effects as a complex valued function of the input internal state of the particle, where the absolute value and argument of the complex number correspond to the visibility and phase of the interference pattern. In the generalized interferometer the complex valued function depends on two variables, the input internal state and the unitary transformation on the internal state space. The reason why it is practical to keep the phase-shifter is that we then may perform ordinary interference experiments for fixed values of $\rho_I$ and $U$. By varying $\rho_I$ and $U$ we may then map out this generalized interference function. In paper II it is shown that this generalized interferometer has the capacity to uniquely determine all trace preserving gluings of two arbitrary channels.

Since the ordinary interferometric setup has the power to determine trace preserving gluings of a channel and an identity channel, the interferometer has the capacity to reveal more about the global evolution than direct measurements, as the latter only reveal the internal state evolution channel. However, the description in terms of $\tilde{\Phi}_1$ in equation (3.11) suggests that another strategy would be possible. If the evolution device is subject to a process tomography
Figure 3.5: The generalized interferometer has the capacity to uniquely determine arbitrary gluings of arbitrary known channels $\Phi_1$ and $\Phi_2$. The probability is recorded as a function of the internal input state $\rho_I$ and the variable unitary operator $U$ acting on the internal state space. The variable phase shift $\phi$ is not necessary, but is added due to convenience.

[46, 47, 48] on the extended Hilbert space $\tilde{H}_1$, then $\tilde{\Phi}_1$ could be determined and provide the same information as the interferometer would. To achieve this one would need to prepare superpositions between the single-particle and vacuum state. Furthermore, the measurements on the output has to be sufficiently general on $\tilde{H}_1$. Whether this is possible is likely to depend on what kind of systems or particles considered.

3.2 Quantum computation: Papers III and IV

Papers III and IV concern adiabatic quantum computing. More specifically these papers consider the question of the resilience of adiabatic quantum computing against decoherence. The adiabatic quantum computer is a promising candidate for actual realizations of quantum computers, as it appears to have some level of intrinsic resistance against disturbances of various kinds. For example, since the adiabatic schemes operate close to the energy ground state it seems reasonable to expect it to be robust against relaxation [36]. The resistance to noise has been examined in [49] and it has been argued that adiabatic quantum computers should be resilient to decoherence [36, 50]. Unitary control errors and resistance to decoherence have been numerically investigated in Ref. [51]. Another study of the effect of decoherence on adiabatic quantum computation is found in [52]. The motivation for the present study was to obtain analytical results concerning the effect of decoherence on the adiabatic quantum computer, in terms of how the efficiency is affected. We study adia-
adiabatic quantum search in a disordered list, subject to decoherence with respect to the instantaneous energy eigenbasis. This problem and decoherence model are sufficiently simple to allow an entirely analytical treatment. Although certainly not the most general form of decoherence, it seems a reasonable starting point, since the eigenstates has such a prominent role in adiabatic computing. Furthermore there is evidence [53] that decoherence with respect to the eigenbasis is obtained in certain regimes of system-environment interaction.

We consider master equations of the following form

\[
\frac{d}{ds} \rho(s) = -iA[H(s), \rho(s)] - BT[W(s), [W(s), \rho(s)]], \tag{3.14}
\]

where \(H(s)\) is as defined in equation (2.35) and where we assume that the Hermitian operator \(W(s)\) satisfies \([H(s), W(s)] = 0\) in order to obtain decoherence with respect to the instantaneous energy eigenbasis. The constants \(A\) and \(B \geq 0\) give the strength of the Hamiltonian evolution compared to the decohering evolution. We wish to find out how the run-time depends on the list length for given strengths of the open system effect. We let

\[
A = \cos(\omega \pi/2), \quad B = \sin(\omega \pi/2). \tag{3.15}
\]

In this way we obtain a parameter \(\omega\) that takes us from the ideal closed case \((\omega = 0)\), to the wide-open case \((\omega = 1)\) where the evolution is entirely determined by the decoherence term. The question is: What happens with the efficiency as a function of the degree of openness \(\omega\)? The efficiency we measure in how fast the run-time grows with the list-length \(N\). As was described in section 2.10 the functioning of the adiabatic quantum computer depends on the possibility to find the ground state of the problem Hamiltonian. The adiabatic approximation is, in general, truly valid only in the limit of infinite run-times. In order to have a finite run-time we must accept a probability \(p < 1\) to find the system in the ground-state. This probability \(p\), we refer to as the success probability, since this is the probability to obtain an answer to our computational problem in one single run. In the present case of the local and global search the run time grows like \(T \sim N^a\). We refer to the exponent \(a\) as the scaling of the run-time.

One way to answer the question of how the efficiency depends on the degree of openness \(\omega\) is to make numerical simulations. In figure 3.6, the logarithm of the run-time \(T\) of the local search scheme is plotted as a function of the logarithm of the list-length \(N\). More specifically, \(T\) is the run-time needed to obtain precisely the success-probability 0.5, where we have an environment that monitors the instantaneous energy of the system, i.e., \(W(s) = H(s)\). Each of the curves corresponds to a given degree of openness \(\omega\) ranging from the closed case \(\omega = 0\) to the wide-open case \(\omega = 1\). From figure 3.6 it appears as if all curves tend to the slope 1/2, except the wide-open case for which the slope
Figure 3.6: The upper plot shows $\log_2 T$ vs $\log_2 N$, where $T$ is the run-time needed to obtain the success probability $p = 0.5$ in the local search case and for the list-length $N$. Each curve corresponds to a degree of openness $\omega$. Counted from below the curves correspond to $\omega = 0, 0.1, \ldots, 0.9, 1$. The lower plot shows a numerical approximation of the derivatives of the curves in the upper plot. As seen it appears as if the slopes in the semi-open cases ($\omega < 1$) tends to 0.5, while for the wide open case ($\omega = 1$) the slope approaches 1.
seems to tend to 1. This is perhaps even more clearly seen in the lower plot of figure 3.6, where a numerical approximation of the derivative of the curves in the upper part is shown. From figure 3.6 it appears as if the asymptotic slope is independent of the degree of openness, as long as $\omega < 1$. But what about the success probability $p$? In figure 3.6, we used a fixed success probability $p = 0.5$. How does the scaling depend on this choice? The independence of the scaling with respect to the success-probability can be seen quite clearly in the upper plot in figure 3.7. This plot is the result of numerical simulations of the local search, for three values of $\omega$, and for a range of $p$ values. In the lower plot of figure 3.7 are the numerically calculated derivatives of the graphs in the upper plot.

The purpose of papers III and IV is more or less to prove that what we see in figures 3.6 and 3.7 essentially reflect the true behavior of the present model. To summarize the findings of papers III and IV we may consider how the sufficient run-time $T$ to obtain at least the success probability $p$, scales with the list-length $N$, and how it depends on the degree of openness $\omega$. In the global search case we obtain

$$T \sim C_1(p, \omega)N, \quad \omega < 1, \quad (3.16)$$

$$T \sim D_1(p)N^{3/2}, \quad \omega = 1, \quad (3.17)$$

and in the local search case

$$T \sim C_2(p, \omega)N^{1/2}, \quad \omega < 1, \quad (3.18)$$

$$T \sim D_2(p)N, \quad \omega = 1. \quad (3.19)$$

It is to be noted that in the wide-open cases $\omega = 1$ we have chosen to use $W(s) = H(s)$. In the wide-open case this choice matters, while within certain limits, this does not matter in case $\omega < 1$. In words, equations (3.16)-(3.19) mean that the scaling of the run-time is independent of the success probability $p$, as well as the degree of openness $\omega$, as long as $\omega < 1$. The scaling is affected only by the transition from semi-open to wide-open search, where it in both the global and local case adds $1/2$ to the scaling. These results indicate that the presence of the Hamiltonian evolution in some sense protects the adiabatic search from decoherence with respect to the energy eigenbasis.

---

9 We note that in the wide-open cases $\omega = 1$ we have chosen to use $W(s) = H(s)$. In the wide-open case this choice matters, while within certain limits, this does not matter in case $\omega < 1$. In words, equations (3.16)-(3.19) mean that the scaling of the run-time is independent of the success probability $p$, as well as the degree of openness $\omega$, as long as $\omega < 1$. The scaling is affected only by the transition from semi-open to wide-open search, where it in both the global and local case adds $1/2$ to the scaling. These results indicate that the presence of the Hamiltonian evolution in some sense protects the adiabatic search from decoherence with respect to the energy eigenbasis.

---

9 $W$ is not allowed to fluctuate too violently with respect to the parameter $s$. This is expressed by the condition in equation (33) in paper IV.
Figure 3.7: The upper plot shows the \( \log_2 T \) vs \( \log_2 N \) plots, where \( T \) is the runtime needed to obtain a given success probability in the local search case and for the list length \( N \). As seen there are three “bundles” of curves. Each bundle corresponds to a given degree of openness. The uppermost bundle corresponds to the wide-open case \( \omega = 1 \), the middle corresponds to \( \omega = 0.9 \), and the lowest bundle corresponds to \( \omega = 0.5 \). Within each bundle each curve corresponds to a fixed success probability \( p = 0.4, 0.5, 0.6, 0.7, 0.8, 0.9 \) counted from below. The lower plot shows numerical approximations to the derivatives of the curves in the upper plot.
3.3 Adiabatic evolution: Paper V

In the previous section we have seen that the adiabatic theorem plays an important role in the construction of the adiabatic quantum computer. The adiabatic theorem and the adiabatic approximation are widely used to obtain simplified equations of motion of quantum systems. The adiabatic theorem is also important for holonomic implementations of quantum circuits [54]. Like all other quantum devices, holonomic circuits are affected by decoherence. In paper V we present a generalization of the adiabatic theorem to systems that are weakly disturbed by open system effects.

As we have seen in section 2.8 the ordinary adiabatic approximation can be implemented by a decoupling of the instantaneous eigenspaces of the Hamiltonian. When open system effects are added to the evolution, this approach to adiabaticity has to be reconsidered.

One type of generalization is introduced in [55]. In this generalization the adiabatic approximation is characterized in terms of Jordan block decompositions of the right-hand side of equation (3.20) regarded as a super-operator.\footnote{The word super-operator is sometimes used to signify a linear map acting on a linear space of linear operators.}

We do not consider this Jordan decomposition approach but focus on the eigenspaces of the time-dependent Hamiltonian. Our approach generalizes [56, 57], as it removes the restriction of non-degeneracy of the Hamiltonian.

The starting point of the approximation is a master equation of the form

\[
\frac{d}{dt} \rho(t) = -i[H(t/T), \rho(t)] + \Gamma_D(t/T) \rho(t),
\]

where \( H \) is a time dependent Hamiltonian and \( D_{t/T} \) is a linear super-operator. For the equation to give a solution that is a proper density operator, the super-operator \( D_{t/T} \) cannot be arbitrary. However, for the formulation of the approximation we do not need to consider this. We make the change of variables \( s = t/T \) and obtain

\[
\frac{d}{ds} \rho(s) = -i[H(s), \rho(s)] + \Gamma_D(s) \rho(s).
\]

The generalized adiabatic approximation can be written in the following way

\[
\frac{d}{ds} \rho = -i[TH(s) + Q(s), \rho] + \Gamma_T \sum_{k l k' l'} g_{k lk'} P_k(s) D_{s} \left( P_{l'}(s) \rho P_{l'}(s) \right) P_l(s),
\]

where

\[
Q(s) = i \sum_k \hat{P}_k(s) P_k(s).
\]

(3.22)
The first part of equation (3.22) we recognize as the ordinary adiabatic approximation in the form of equation (2.22). The second part involves the quantities \( g_{kk'kl'l'} \), which have the value zero or one depending on the behavior of the energy levels as functions of \( s \).

Analogously to the procedure in section 2.8, we can rewrite (3.22) as

\[
\frac{d}{ds} \tilde{\rho}(kl) = -iT \left( E_k(s) - E_l(s) \right) \tilde{\rho}(kl)(s) - iZ_k(s) \tilde{\rho}(kl)(s) + i\tilde{\rho}(kl)(s)Z_l(s) + i\tilde{\rho}(kl)(s)Z_l(s) + \Gamma T \sum_{k'l'} g_{kk'kl'l'} P_k(0) \tilde{D}_s(\tilde{\rho}(k'l')) P_l(0),
\]

(3.24)

where \( Z_l(s) \) is as defined in equation (2.27) and where

\[
\tilde{D}_s(\tilde{\rho}) = U(s) D_s(U^\dagger(s) \tilde{\rho}(s) U(s)) U^\dagger(s).
\]

(3.25)

Note that when \( \Gamma = 0 \) we obtain (2.22) and the standard adiabatic approximation.

The properties of \( g_{kk'kl'l'} \) imply that the diagonal terms \( \tilde{\rho}(ll) \) always evolve according to

\[
\frac{d}{ds} \tilde{\rho}(ll) = -i[Z_l(s), \tilde{\rho}(ll)(s)] + \Gamma T \sum_k P_l(0) \tilde{D}_s(\tilde{\rho}(kk)) P_l(0).
\]

(3.26)

The second term in this equation introduces a coupling between the diagonal terms of the density operator. Equation (3.26) implies that the diagonal terms of the approximate solution always evolve independently of the off-diagonal terms. In the simplest case where \( g_{kk'kl'l'} = \delta_{kk'} \delta_{ll'} \), for \( k \neq l \), the off-diagonal terms evolve independently of each other and of the diagonal terms.

When is this a good approximation? Before we discuss this question we should remind ourselves of the conditions for the ordinary adiabatic approximation. The adiabatic approximation gets better the larger the run-time \( T \) is. As we saw in section 2.8 there exists a condition (2.30) that gives the scale of the necessary run-time. In the present case the situation is somewhat more complicated as we have two parameters, the run-time \( T \) and the strength parameter \( \Gamma \). Since we are interested in a generalization of the adiabatic approximation we naturally consider the \( T \to \infty \) limit. In the general case it also seems quite intuitively reasonable to expect that the limit \( \Gamma \to 0 \) would be needed, since we consider very long run-times during which the destructive open system effects can accumulate.

To investigate the quality of the approximation as a function of \( T \) and \( \Gamma \) we apply the generalized adiabatic approximation to a 4-level system whose
evolution is governed by the master equation

$$\dot{\rho} = -iT[H(s), \rho] - \Gamma T[A, [A, \rho]],$$  (3.27)

where $H(s)$ is a time-dependent Hamiltonian of the form

$$H(s) = e^{-iZ}H_0e^{iz},$$  (3.28)

and $A$, $Z$, and $H_0$ are Hermitian. These operators, as well as a pure initial state, have been generated randomly. Figure 3.8 shows the maximum error in the Hilbert-Schmidt norm\(^{11}\) $\max_{s \in [0,1]} ||\rho(s) - \rho^a(s)||$ for various choices of $T$ and $\Gamma$. Here, $\rho(s)$ is the solution of equation (3.27) and $\rho^a(s)$ the solution of the corresponding approximate equation (3.22). As seen in Fig. 3.8 the approximation indeed seems to behave as expected. In the ideal case, $\Gamma = 0$, the error appears to go to zero as $T$ increases, while for non-vanishing $\Gamma$ there seems to be a minimum error.

In paper V we are able to prove that the solution of the approximate equations converges to the solution of the original equation in the joint limit $T \to \infty$ and $\Gamma \to 0$, under the condition that $\Gamma T$ is bounded. A physical interpretation of this condition might be to assume that the strength parameter $\Gamma$ depends on the run-time $T$. If $\Gamma = \alpha/T$, with $\alpha \geq 0$ a constant independent of $T$, then the error would go to zero when $T \to \infty$. However, this is not a very realistic picture. In practice, the open-system effects are often residual uncontrollable errors and the strength $\Gamma$ is given, and we cannot decrease $\Gamma$ as $T$ increases. On the other hand, it is quite clear that the approximation is good if the run-time $T$ is sufficiently large, the characteristic time-scale of the open-system effects $\Gamma^{-1}$ is sufficiently large, and the run-time $T$ is in the order of or smaller than $\Gamma^{-1}$. Unlike the standard adiabatic approximation, where the error can be made arbitrarily small by increasing the run-time, the present approximation appears to be limited, since for a given open-system strength $\Gamma$, the error cannot be made arbitrarily small as the run-time has to be at the same order or smaller than the characteristic time-scale of the open-system effects.

Having said all this, it has to be emphasized that these conditions are sufficient for the approximate solution to approach the true solution. They may, however, not always be necessary. In paper V we numerically investigate a specific example of a holonomic implementation of a single qubit gate. The numerical results indicate that for this system there is no need for a small $\Gamma$. Within the range of $\Gamma$ considered, the only condition for a good approximation appears to be that $T$ should be large.

\(^{11}\)The Hilbert-Schmidt norm is $||A|| = \sqrt{\text{Tr}(A^\dagger A)}$, which can be shown to be a norm on spaces of linear operators [58].
Figure 3.8: The figure shows the maximum error $\max_{s \in [0,1]} ||\rho(s) - \rho^a(s)||$ between the solution $\rho(s)$ of the original equation (3.27) and the solution $\rho^a(s)$ of the approximate equation. The error is plotted as function of the run-time $T$ and the decoherence strength parameter $\Gamma$. The norm $|| \cdot ||$ is the Hilbert-Schmidt norm.
4. Conclusions

In this thesis we discuss effects of open system evolution in quantum mechanics. Under this general heading three different projects have been pursued, namely single-particle interferometry, quantum computing, and adiabatic evolution.

Within the context of single particle interferometry three concepts are introduced and investigated. The subspace preserving channels are those that do not redistribute probability with respect to given orthogonal decompositions of the Hilbert space of the quantum system. In a two-path interferometer this would correspond to operations that do not transport the particle between the two paths in the interferometer. Expressions to generate all such channels for a given decomposition are proved. In order to model operations that act locally in the two paths of the interferometer we introduce the concept of subspace local channels. Given two operations acting in the two paths of the interferometer, one might ask what total operation acts on the interferometer. It is shown that the total operation is not uniquely determined by these two channels. The set of gluings of the two channels provides all the possible total channels compatible with the two given channels. We deduce expressions by which all possible gluings can be generated. The non-uniqueness of the gluing is partially due to the possibility of communication or sharing of correlated resources. By assuming that the gluing is subspace local this interdependence can be cut off. It is shown that the gluing is still not unique in spite of the assumption of subspace locality. However, this non-uniqueness can be understood in terms of an occupation number description of the two paths. The non-uniqueness of gluings can be translated into a non-uniqueness of the interference patterns obtained in an interferometer. We investigate how single particle interferometry can be used to determine open system effects. A step toward the application of these results would be to investigate how absorption affects the gluing phenomenon. Does particle loss induce more effects than just reduction of visibility in interference experiments? One might also consider to classify SP channels with respect to the resources needed to implement them. Assuming the definition of subspace locality is a reasonable one, we know that some SP channels are not subspace local. But, how “violently” non-local are they? Is the use of shared correlation enough as a resource to generate them, or is communication necessary?
In the project on quantum computing we study the effect of decoherence on the adiabatic quantum computer. More precisely, we investigate the problem of “search in a disordered list”, using the global and local adiabatic search schemes under the influence of decoherence in the instantaneous energy eigenbasis of the system. We are able to show that, as long as the Hamiltonian dynamics is present, the run-time scales like $T \sim N$ and $T \sim N^{1/2}$ in the global and local search, respectively. In other words, the efficiency of the ideal global and local search is unaffected by this type of decoherence. This changes abruptly when the Hamiltonian dynamics vanishes. In the special case when the dynamics is solely governed by an environment that monitors the instantaneous energy of the system, the run-time scales like $T \sim N^{3/2}$ and $T \sim N$ in the global and local case, respectively. To conclude, the adiabatic quantum search appears to have an intrinsic resilience to this type of decoherence, which supports the adiabatic quantum computer as a promising candidate for quantum computing. One obvious generalization of these investigations would be to consider adiabatic evolution for arbitrary families of Hamiltonians under the influence of eigenbasis decoherence. How does the condition (2.30) for the adiabatic approximation change under the influence of this type of decoherence? One might also consider a wide-open adiabatic theorem and a wide-open adiabatic approximation. What would the analogue of the adiabatic condition be? The analytical investigation presented here might be difficult to generalize to 3-SAT and other NP-complete problems. One might consider purely numerical simulations. If the results of such simulations would give qualitatively similar results as those found for the adiabatic quantum search, they might provide further indications to the resilience of the adiabatic scheme.

In the third project we introduce a generalization of the adiabatic approximation to quantum systems that are weakly open. We are able to show that this generalized adiabatic approximation can be obtained in the limit of long run-times and weak open system effects, under the condition that the run-time is in the order of or smaller than the characteristic time scale of the open system effect. However, a numerical solution of a specific model problem does indicate that the approximation may have a larger range of applicability in some cases. To clarify when and why this is the case could be a possible direction for future studies. One immediate application of the generalized adiabatic approximation would be to study various holonomic implementations of quantum gates and their resilience to noise and decoherence.
5. Summary in Swedish: Öppna kvantsystem. Inverkan på interferometri, kvantdatorer och adiabatisk tidsutveckling


De forskningsprojekt vilka ligger till grund för denna avhandling har syftat till att teoretiskt undersöka olika aspekter av öppna system i kvantmekaniken. Under detta överblickande tema har tre delprojekt undersökts. Det första handlar om interferometri, det andra om inverkan av dekoherens på kvantdatorer och det tredje om så kallad adiabatisk tidsutveckling i öppna system. Nedan ges en kort beskrivning av var och ett av dessa delprojekt.

Interferometri:
I detta delprojekt har jag undersökt en speciell sorts interferometer som kan kallas för en "enpartikel tvåvägsinterferometer". Den bygger på den ovan nämnda superpositionsprincipen. Lite grovt kan man beskriva den på följande sätt: En partikel, till exempel en foton (ljuspartikel), skickas mot en halvt speg-

I detta projekt har jag undersökt vad som händer om partikeln i tvåvägsinterferometern påverkas på sin väg mellan de två halvspeglarna, där denna påverkan kan bero på att man har ett öppet system. Man kan se det som att man i de två vägarna har apparater som interagerar med partikeln. Jag har teoretiskt studerat en rad olika frågor i samband med detta. Till exempel har denna undersökning utmynnat i en klassificering av alla tänkbara sätt att påverka partikeln som inte leder till att partikeln hoppar mellan de två vägarna. En annan frågeställning har varit: Om man känner till vad dessa två apparater var för sig gör med en partikeln, kan man då räkna ut interferensmönstret? Svaret på den frågan visar sig vara ja, men under förutsättning att man ger en tillräckligt omfattande beskrivning av apparaterna. Vidare har jag undersökt hur man skall modellera idén om två apparater som inte påverkar varandra eller samagerar när de verkar i de två banorna.

Kvantdatorer:


Jag hävdade tidigare att detta sätt att klassificera beräkningsproblem inte beror på hårdvaran. Det var faktiskt inte riktigt sant. I den klassiska beräkn-

Det finns dock en baksida av allt detta, och det är att det är mycket svårt att konstruera en kvantdator. Ett av de svåraste hindren utgörs av dekoherensen. En viktig del i arbetet med att försöka bygga en kvantdator ligger därför i att bekämpa dekoherensen och att försöka hitta konstruktioner som inte är så känsliga mot den. En lovande variant kallas för den adiabatiska kvantdatorn. (Vad "adiabatisk" betyder förklaras nedan.)

Två av delarbetena i denna avhandling har handlat om att undersöka den adiabatiska kvantdatorns förmåga att motstå dekoherens. Mer specifikt har vi undersökt hur bra den adiabatiska kvantdatorn löser problemet "sökning i oordnad lista" under påverkan av en viss klass av dekoherens. Det visar sig att man fortfarande kan få $\sqrt{N}$-effektivitet, trots närvaron av denna typ av dekoherens. Detta ger stöd åt tanken att den adiabatiska kvantdatorn har en viss naturlig motståndskraft mot dekoherens.

Adiabatisk tidsutveckling:


I detta projekt har vi tagit fram en generalisering av den adiabatiska approx-
imationen som är lämplig att använda när man har en önskad ideal adiabatisk tidsutveckling som är störd av effekter som uppkommer på grund av att man har ett öppet system. Genom denna approximation kan man få information om hur effekterna av det öppna systemet påverkar den tänkta ideala tidsutvecklingen. Vi visar att denna generaliserade approximationsmetod fungerar bra i en viss gräns av långsamma förändringar och små störningar. Genom numeriska undersökningar visar det sig dock att det finns tillfällen när effekterna av det öppna systemet inte är små, men då approximationen tycks fungera bra i alla fall. Detta antyder att det finns möjligheter att vidareutveckla denna approximationsmetod.
Acknowledgments

Jag fasar inför att skriva detta. Traditionen verkar påbjuda orgier i långa namnlistor, och jag är rädd för att jag skall råka glömma bort någon som rätteligen borde stå här.

Hur som helst börjar jag med att tacka mina huvudhandledare som har varit Osvaldo Goscinski och därefter Erik Sjöqvist, samt min biträdesvänhandledare Sten Lunell. Med Erik har jag, förutom oräkneliga fysikdiskussioner, haft ändlösa överläggningar om språk och formuleringskonst, exempelvis i samband med denna avhandling. Dessutom har vi pratat en hel del strunt, och i ljuset av detta kan jag bara säga en sak: Güdd!


Dessutom tänkte jag passa på att tacka alla personer som jag har gått runt och stört när jag inte har haft lust att jobba. Jag tror nog ni vet vilka ni är så jag tänker inte räkna upp er.

Nu när jag ändå är i farten så det väl lika bra att jag dessutom tackar alla mina vänner och min familj. Tack allihop!

Johan Åberg
Uppsala, Augusti 2005
References


Acta Universitatis Upsaliensis

Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology 77

Editor: The Dean of the Faculty of Science and Technology

A doctoral dissertation from the Faculty of Science and Technology, Uppsala University, is usually a summary of a number of papers. A few copies of the complete dissertation are kept at major Swedish research libraries, while the summary alone is distributed internationally through the series Digital Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology. (Prior to January, 2005, the series was published under the title “Comprehensive Summaries of Uppsala Dissertations from the Faculty of Science and Technology”.)

Distribution: publications.uuse
urn:nbn:se:uu:diva-5893