Barbara Piechocinska

Physics from Wholeness

Dynamical Totality as a Conceptual Foundation for Physical Theories
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Abstract

Motivated by reductionism’s current inability to encompass the quantum theory we explore an indivisible and dynamical wholeness as an underlying foundation for physics. After reviewing the role of wholeness in the quantum theory we set a philosophical background aiming at introducing an ontology, based on a dynamical wholeness. Equipped with the philosophical background we then propose a mathematical realization by representing the dynamics with a non-trivial elementary embedding from the mathematical universe to itself. By letting the embedding interact with itself through application we obtain a left-distributive universal algebra that is isomorphic to special braids. Via the connection between braids and quantum and statistical physics we show that the mathematical structure obtained from wholeness yields known physics in a special case. In particular we point out the connections to algebras of observables, spin networks, and statistical mechanical models used in solid state physics, such as the Potts model. Furthermore we discuss the general case and there the possibility of interpreting the mathematical structure as a dynamics beyond unitary evolution, where entropy increase is involved.

Keywords: wholeness, elementary embedding, left-distributivity, braids, special braids, entropy, Temperley-Lieb algebra, von Neumann algebra, Potts model

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"Εν τῷ σοφῶν, ἑπίστασθαι γνώμην ἢ κυβερνᾶται πάντα διὰ πάντων."

Heracleitus [1]

“Wisdom is one thing—to know the thought whereby all things are steered through all things.”

Following Professor Paul G. Hewitt’s suggestion I am hereby taking a look at the elephant before measuring its tail...
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List of Symbols

· binary operation of application on elementary embeddings
  ○ binary operation of composition
  × associative binary operation
  ● action of a braid on a sequence of braid strand colors
  ∧ binary operation of conjugacy
  ∧ left-distributive binary operation, associated with coloring braid strands at a positive braid crossing, usually on LD-quasigroups
  ∨ left-distributive binary operation, associated with coloring braid strands at a negative braid crossing, usually on LD-quasigroups
  ∧ binary operation of braid exponentiation
  $<_{LD}$ left-distributive linear ordering
  $<_{Lex}$ Lexicographical ordering
  $|$ sign for “models”
  $\vec{a}$ sequence of braid strand colors or special braids (depending on context)
  $A$ set of colors, used to color braid strands or label used when encoding a crossing into the plane (depending on context)
  $Ad_a$ left translation on $A_j$ generated by $a \in A_j$
  $Ad$ mapping from $A_j$ to $Ad(A_j)$
  $Ad(A_j)$ adjoint monoid, submonoid of $A_j$ generated by left translations
  $A_j$ free universal left-distributive algebra generated by the elementary embedding $j$
  $\alpha$ ordinal number
  $\bar{b}$ closure of a braid
$B$ label used when encoding a crossing into the plane

$B_n$ braid group on $n$ braid stands

$B_\infty$ braid group on infinitely many braid strands

$B^p$ special braids

$\hat{\beta} = 1/k_BT$, where $k_B$ is Boltzmann's constant and $T$ the temperature, or ordinal number (depending on context)

$C$ field of complex numbers

$\text{crit}(A_j)$ the critical points of the elements in $A_j$

$\text{crit}(i)$ the critical point of the elementary embedding $i$

$d$ loop value

$\delta$ Kronecker delta

$e_i$ generator of von Neumann algebra

$E$ number of edges in a graph

$E(S)$ energy of a system in state $S$

$F$ mathematical field

$\phi$ formula in Set Theory

$f$ function

$g()$ two-form

$G$ graph, or group (depending on context)

$\text{gen}()$ two-form

$H$ Hamiltonian, or Hopf algebra (depending on context)

$h$ Planck's constant

$\bar{h}$ Planck's constant divided by $2\pi$

$i, k, l$ primarily used to denote elementary embeddings

$\text{inv}$ inverse map

$\text{inv}^*$ antipode

$x$
$I_L$ left ideal

$j$ non-trivial elementary embedding

$j_u$ isomorphism part of $j$, representing unfoldment

$j_e$ inclusion map part of $j$, representing enfoldment

$j''V$ the range of the elementary embedding $j : V \to V$

$K$ a link

$k_B$ Boltzmann’s constant

$\kappa$ critical point of elementary embedding

$L$ Laver sequence

$\lambda$ limit ordinal

$m$ mass, or multiplication (depending on context)

$m^*$ co-multiplication

$M$ matrix or time evolution operator (depending on context)

$n, p$ integers

$\nu$ $e^\beta - 1$

$\omega$ infinite countable cardinal, or two-form (depending on context)

$p$ momentum, or integer (depending on context)

$P()$ set of all subsets

$\psi$ quantum mechanical wave function

$Q$ quantum potential or number of states an element can be in (depending on context)

$ON$ class of ordinal numbers

$R$ amplitude of a quantum mechanical wave function

RI Reidemeister move one

RII Reidemeister move two

RIII Reidemeister move three
$S$ state of vertices in a graph, or phase of quantum mechanical wave function (depending on context)

$S_i$ state of vertex $i$ in a graph

$s$ state of a universe/link/braid when spliced

$|s|$ number of loops for a state $s$

$||s||$ number of loops -1 in Kauffman's bracket, or number of shaded regions for a state $s$

$(S, \wedge)$ a left-distributive system

$sh()$ shift endomorphism on $B_\infty$ mapping $\sigma_i$ to $\sigma_{i+1}$ for all $i$

$\sigma_i$ positive braid crossing of the $i$-th braid strand over the $i - 1$:st braid strand

$\sigma_i^{-1}$ negative braid crossing of the $i$-th braid strand under the $i - 1$:st braid strand

$t$ time parameter

$TL$ Temperley-Lieb algebra

$u$ unit map

$u^*$ co-unit

$U$ shade of a link, called universe

$U_c$ classical potential

$U_i$ generating element of a Temperley-Lieb algebra

$V$ mathematical universe, the cumulative hierarchy

$Vec$ vector space

$W$ Potts bracket

$WA$ Wholeness Axiom

$x$ position

$X, Y$ sets

$Z$ partition function

$ZFC$ Zermelo-Fraenkel set of axioms and the Axiom of Choice
Preface

This thesis is written in a way that illustrates the underlying philosophical considerations and intuitions of the author, that gave rise to the particular approach. Understanding the original ideas will hopefully motivate the mathematical choices made and facilitate the interpretational aspects. Perhaps the philosophical thoughts could even serve as an unconventional alternative for the construction of a new theory or in a further development of this particular approach. The more technically minded reader should have no trouble extracting the technical information, while keeping the interpretational and more philosophical aspects to a minimum.

The thesis is, to a large degree, based on work previously presented in the following articles:

- “A Physical Interpretation of a Monogenic Free Left-distributive Algebra” Manuscript in preparation.
1 Introduction

“Never solve a problem from its original perspective.”

(Charles Chic Thompson [2])

A particular perspective, a particular way of conceiving of our experiences, brings forth and emphasizes certain relationships. For instance, one perspective, proposed by Aristoteles, is to see the natural state of things as rest. This appears reasonable as it coincides with most of our everyday experiences, where whatever we push will eventually stop. With such a perspective one might be inclined to believe that the objects in the heavens follow a different set of rules than the objects on Earth. A contrasting perspective, proposed by Galilei, is to say that it is natural for a thing to continue to move at a constant speed, and attribute the slowing down to the medium in which the thing moves. This perspective opens up for new insights where new relationships come forth, such as Newton’s proposition that forces are to be related to the change in the velocity, instead of the change in the position. It also makes easier the unification of the laws of heavenly objects with the laws of the objects on Earth. And so we see that a particular perspective can disclose certain relationships that may be difficult to see from other perspectives. Thus an added perspective may improve our overall understanding of the phenomenon at hand.

In this thesis we propose a shift of perspective in the philosophy underlying physics and a consequent change in the mathematical foundation. The nature of the philosophical shift, motivated by the quantum theory’s indication of an underlying indivisible wholeness [3], [4], is to include the possibility of holistic aspects. The aim is twofold. Firstly, we wish to widen the current perspective so that we may advance our understanding and provide an ontology. This means that we would like our equations to describe Nature, as opposed to being limited to useful algorithms with predictive power, as is for instance the case of the Copenhagen interpretation of the quantum theory. Secondly, we are interested in exploring relationships, that the new perspective emphasizes and that may not be as obvious from the current, more reductionistic, perspectives. In particular we shall, by starting from wholeness, see how irreversibility comes in as a consequence of wholeness being whole and discuss the possibility of a description of an actual, non-trivial, dynamics beyond unitary evolution. We shall also see how the appearance of unitarity comes in,
even though it is only an approximative special case of the richer dynamics obtained from wholeness. Thus, connecting with the above perspectives on motion, one might say that the proposed shift in the perspective of the philosophy suggests two ideas. Firstly it suggests that motion itself is fundamental, not the things that move. Secondly it suggests a shift from seeing unitary motion as natural or fundamental to seeing irreversible motion as fundamental.

The set-up of the thesis is the following. We will start by giving an introduction to the background and motivation for this particular approach. Then, in chapter 2, we will introduce and discuss the underlying philosophy. Based on this philosophy we will, in chapter 3, proceed to introduce the mathematical background. Equipped with both the philosophy and the mathematics expressing the key ideas we will proceed, in chapter 4, by looking at physical interpretations of braids relevant to the quantum theory and models of interest to solid state physics. In the final chapter, chapter 5, we shall give examples of insights that an approach from wholeness can bring to already known physics and what potential it has for a further development of physics. In particular we shall discuss the possibility of describing entropy increase on a fundamental level in the dynamics.

1.1 Background and Motivation

A reductionistic approach to science involves an abstraction from wholeness and a focus on smaller and smaller parts, until encountering a part that appears manageable. Such a part is then investigated and defined. Manageable parts are then put together in hope of modelling the whole. This kind of an approach has indeed proven powerful in science. However, when the whole is simply abandoned there is a danger of excessive fragmentation and the scientific process does not proceed in an optimal way. An illustration of excessive fragmentation\(^1\) could be the smashing of a watch. If hit sufficiently hard, such a watch will fall into lots of small, random pieces. These pieces can then be investigated with the purpose of finding out the workings of a watch. However, these fragmentary pieces, such as pieces of shattered glass, are unlikely to be themselves central to how the watch works. An alternative way could be to look for the wholes and the meanings, and sub-meanings, in the context of the workings of the watch. This way one could end up with meaningful parts, instead of fragmented pieces. The crux is, of course, to find a way of doing this.

Excessive fragmentation, complicating and prolonging the reconstruction, is not the only possible hurdle that reductionism presents. There is a more serious one. Indeed, perhaps even an insurmountable one. The purely re-

\(^1\)Originally found in [5].
ductionist approach to physics rests on the assumption that the world can be modelled as being made up of separate identifiable parts that interact. In other words, it assumes that the whole can be seen as a sum of its parts and that the wholes do not directly affect the parts in an irreducible way. For centuries this seemed as a fair assumption in physics. It was not until the beginning of the twentieth century that experimental data demanded a drastically new theory. To accommodate the new results the quantum theory was invented. Despite the deeply rooted reductionist tradition, scientists discovered early on that the quantum theory was not purely reductionistic, and yet it was approached and formed from the classical side. It is for instance normal procedure to attempt to quantize classical systems, while it could be argued that the quantum theory, being more fundamental than classical theories, should be the starting point. There were, however, physicists who recognized the inappropriateness of fragmentarization. Werner Heisenberg said: “There is a fundamental error in separating the parts from the whole, the mistake of atomizing what should not be atomized. Unity and complementarity constitute reality.” [6].

The new aspects, not reducible to a purely reductionist perspective, were intimately related to what Bohr called wholeness. From the invention of the quantum theory Bohr has been clear to point out the key role that wholeness plays [7]. When two quantum systems interact, or have interacted, we can in general no longer view the system of interaction as being made up of two separately existent parts. Instead an inseparable totality is formed. The wave function of such an interacting system can, in general, not be expressed as a product of the wave functions of the two previously separated systems. And so a reductionist picture based on the interactions of static identifiable parts with definite properties cannot be maintained. The details of how a nonlocal, indivisible, and dynamical form of wholeness plays a central role are presented in the Ontological Interpretation of Quantum Theory [3] and a short summary of them will be given in the next section. The effects of a correlation related to the indivisible wholeness of quantum systems that transcends space and time (entanglement) have been experimentally observed [8]. Failing to conform to reductionism and being essential in the foundations of the quantum theory, it would be both intellectually satisfactory and of instrumental value in future research to have an understanding of what philosophical ideas wholeness entails.

Since the advent of the quantum theory, severe problems have arisen when attempting to form a consistent, unified description through physics. There appear to be incompatibilities between fundamental physical theories such as the quantum theory and the theories of relativity (especially the theory of general relativity). In attempting to find a common ground for both of them various physicists have suggested wholeness [4] (chapters 5-6), [3] (chapter 15), [9]. The prospect of being a common factor in both the quantum theory
and the theory of general relativity hints at the possibility of leading to a theory that transcends and includes the both of them.

In view of the present incompatibilities, the difficulty to fit contemporary physics into a reductionist framework, and the indications that an indivisible wholeness is essential in physics, it is suggestive that we attempt a scientific description that starts from wholeness. However, in order to engage in such an attempt we are in considerable need of new ways of thinking (fundamental concepts) as well as new mathematical tools, all due to the particularly elusive nature of wholeness. This thesis addresses that need and presents a proposition for new foundations of physics with the fundamental concepts and a mathematical theory capable of representing them.

1.1.1 Example of Wholeness in the Quantum Theory

When attempting to study physics in search of the underlying concepts we find ourselves to be quite fortunate because there are scientists who have already done a substantial amount of work in this direction. Of particular interest is the work of Bohm, Hiley, and collaborators. A large part of their work originates in intuitions deduced from Bohm’s interpretation of the quantum theory [10], [3], [11]. One of the great advantages of this interpretation is that it allows for an imaginative and intuitive understanding of quantum phenomena and thereby opens up the door to further insights that may become crucial for an extension of the quantum theory, possibly moving into a new theoretical framework. In particular, through the quantum potential, which as we shall see can be derived from the Schrödinger equation, this interpretation explicitly accentuates the essential role of a dynamical, unbroken wholeness. We shall now go into slightly greater detail in order to better see the holistic aspect of physics and how they are suggestive of Bohm’s implicate order. The starting point will be Bohm’s interpretation of the quantum theory.

Bohm’s Interpretation of the Quantum Theory

From being a statistical theory about the outcomes of measurements in which the actual phenomena involved cannot be analyzed [12](p.72), Bohm developed an essentially ontological interpretation of the quantum theory that provides intuitive understanding and further insight into the actuality of systems. In particular it provides a Hamilton-Jacobi like framework within which classical and quantum effects can be contrasted. Such comparisons offer a richer understanding of what Niels Bohr called the unanalyzable wholeness [7] and lead to the development of Bohm’s implicate order 2. It should also be men-

2Consequently, the implicate order can be seen as providing a kind of understanding of the wholeness found in the quantum theory that has the capability of going beyond the quantum theory [13].
tioned that aside from providing an ontology for, and an intuitive understand-

In Bohm’s view, the wave function is not regarded as a complete representa-

In the original interpretation of Bohm [10], a particle is assumed to exist

The treatment is similar to Hamilton-Jacobi theory, which can be seen as

where both $S$, the phase of $\psi$, and $R$, its amplitude, are real fields that are
dependent on the position, $\bar{x}$, and the time parameter $t$, and where $\hbar = h/(2\pi)$, $h$ being Planck’s constant. The next step is to derive the Quantum Hamilton-
Jacobi equation from the Schrödinger equation. Doing this for one quantum

where $m$ is the mass of the particle and $U_c = U_c(\bar{x}, t)$ is a (real) classical po-
tential. We now substitute (1.1) into (1.2), work out the Laplacian, rearrange
the terms and divide them up into two equations, one for the real part and the
other for the imaginary part. The real part then yields

where $U_c$ is still a classical potential and $Q$ is the quantum potential. More

For a discussion on formative cause, see last section in chapter 3
explicitly, we see that
\[ Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R}{R}. \] (1.4)

When \( Q \to 0 \) we move from the quantum domain towards the classical limit\(^4\) [11](p.225) and for \( Q = 0 \) equation (1.3) becomes the classical Hamilton-Jacobi equation. If we see the quantum potential as a type of potential energy we can view (1.3) as an extended version of the conservation of energy, valid in the quantum domain. In order to calculate the momentum of the quantum particle we use the guidance condition\(^5\)
\[ p = \nabla S \] (1.5)

The imaginary part of (1.2) when (1.1) is inserted into it is
\[ \frac{\partial (R^2)}{\partial t} = -\nabla (R^2 \frac{\nabla S}{m}). \] (1.6)

This can be seen as an equation for the conservation of probability.

The above approach grants us the unique opportunity of studying the quantum potential, which is responsible for the non-classical effects of the quantum theory. If we can understand its nature and physical relevance, we will come one step closer to intuitively understanding quantum phenomena and possibly going beyond them.

In order to pursue this we shall now stress how radically different the quantum potential is from classical potentials. Its dynamics goes beyond what can be seen as a mechanical interaction of external parts. To start with, we note that the quantum potential has no external source from which it can be seen to emanate, such as for instance the gravitational potential could be seen to emanate from a massive body in space. This is basically due to the fact that the quantum potential is constructed from the \( \psi \)-field, which itself depends on the whole system and lacks an external source. And so, the quantum potential’s dependence on the \( \psi \)-field introduces an irreducible dependence on the entire environment. When more particles are involved it becomes even more clear that the wave function depends on the whole system. Here we encounter a holistic aspect in which wholeness is more than just all the parts and their interactions.

Moreover we note from equation (1.4) that the quantum potential has a

\(^4\) Although for classical type of behavior it is sufficient that \( \nabla Q = 0 \).

\(^5\) It should be mentioned that an approach based on this condition was introduced by other authors as “Bohmian Mechanics” [15], [16]. The mathematics of it is in accordance with Bohm’s own approach. However, several parts, originally existing in Bohm’s interpretation and crucial to its philosophy (such as the quantum potential), have been omitted. For a short comparison between the two approaches, see [17].
different form than classical potentials. We find $R$ both in the numerator and denominator, which means that the quantum potential does not in general necessarily diminish with distance. Even though the $\psi$-field may go to zero as the distance increases, the quantum potential does not have to diminish. This means that remote features may have significant influence on particle movement. When more particles are involved, this can give rise to the phenomenon of non-locality. The quantum potential will in general depend on the positions of all $N$ particles,

$$Q = -\frac{\hbar^2}{2m} \frac{\nabla^2 R(x_1, x_2, ..., x_N)}{R(x_1, x_2, ..., x_N)},$$  \hspace{1cm} (1.7)

so that what can be interpreted as the quantum force $-\nabla_k Q$ on some particle $k$ depends on all particles.

Furthermore we note that it is not the intensity of the $\psi$-field that regulates the effect it has on the movement of a particle but its form. This again is due to the presence of $R$ in the denominator. An increase in the amplitude, by some factor $c$, has no bearing on the quantum potential, as $c$ divides out. This in turn suggests that the action of $\psi$ on the particle, through $Q$, is of a different kind than classical pushing or pulling (a mechanical transfer of momentum and energy), and is thoroughly discussed by Bohm and Hiley [3]. They suggest that the quantum field be treated as an information potential. The usage of the word information is different [17] from Shannon’s information [18], which refers to our ignorance or certainty about a system. Instead, Bohm and Hiley’s information “in-forms,” actively puts form into. The information in the quantum potential guides a particle along its way. It is information relevant to the objective movement of a particle, information that has meaning to the particle. When it guides a particle it is referred to as “active information.” Bohm and Hiley have suggested the metaphor of a boat being guided by the radar. The actual waves of the radar do not push or pull the boat, as do the waves of the sea. In a similar way the quantum potential guides through information that is meaningful to the particle, by putting form into the motion and not by pushing it.

**Dynamical Unfoldment and Enfoldment and the Implicate Order**

The dynamics suggested by the quantum potential clearly indicates a radical change from the classical framework. The quantum potential appears to suggest a dynamics where the irreducible totality of a system and its environment acts (from beyond any particular spatial source) and forms the explicit movement of some particle. In other words, the lack of an external local source for the quantum potential along with its irreducible dependence on the whole system and, in general the existence of non-locality, suggests that it operates from an irreducible order beyond space and time. Bohm called this the impli-
cate order. The implicate order is dominated by its holistic aspects and can be seen as a dynamical totality where things cannot be distinguished from each other and are instead inseparably intertwined. In other words, despite a lack of definite boarders and distinguishability (that we are used to seeing in space) all the relationships are intact and preserved.

Operating from the implicate order the quantum potential gives explicit form to the movement of a particle. Here we can talk about the explicate order. The explicate order is the order where things can be distinguished from each other, or seen to exist outside of each other, and approximated as independent. Physically, in terms of the quantum theory, a measurement outcome, technically an eigenvalue, can be seen as an explicate order.

An important fact here is that in order to form the movement of the particle the $\psi$-field needs information from the whole environment that is processed as a totality. We see how the implicate order, which is beyond the visible (explicate), plays an essential, formative role in the dynamics. And so, the movement of the part comes out of the totality, not the other way around. Since the totality is irreducible we cannot reduce it to parts. This is how wholeness is greater than the sum of its parts and it is reasonable that it should to be considered in any serious attempt to go beyond the quantum theory. This is the fundamental idea upon which we will continue to build and why we shall try to derive the movement of parts from the dynamics of wholeness instead of starting with parts and letting their sum describe a reductionistic wholeness.

To do this we shall need to discuss the nature of the process of going from the implicate order to the explicate order and the other way around. This will be done in terms of unfoldment and enfoldment. The dynamical concepts of unfoldment and enfoldment were introduced and developed by Bohm [4] in an effort to provide a common basis capable of accounting for both quantum and relativistic effects. The implicate order is the enfolded order while the explicate order is the unfolded order. To explain these aspects Bohm proposes an illustrative metaphor. One places a cylinder container with a smaller radius inside of a fixed one, with a larger radius. Then one pours glycerine between them. On top of the glycerine one then places a drop of ink. If, at this point, one starts turning the inner cylinder the ink-drop will smear out and become a line. After a while no ink will be visible. This is analogous to the enfoldment process. The ink-drop was first explicit and has now become enfolded into the order of the molecules and is implicit. Though the drop cannot be seen explicitly it is still there, implicitly in the order. If we now start turning the inner cylinder in the other direction the ink-drop will appear again. This is then analogous to the unfoldment process. The ink-drop becomes explicit again. This example descriptively expounds the ideas of unfoldment and enfoldment but should not be taken literally.

The idea that Bohm tried to convey was that every thing is in some sense
enfolded into the whole and that the whole is unfolded into every thing. Bohm called this dynamics the holomovement which is a holistic pulsation in which orders unfold and enfold. Emphasis should be put on the fact that this fundamental process is not a movement within space-time (like in the example of the ink-drop) but rather a process in which ultimately space-time and its contents are created. Recall that the quantum potential does not operate from any particular part in space but originates in an irreducible totality, and yet is locally accessible from any point within space where it puts explicit form into the movement. Likewise, the fundamental process originates in the implicate order which unfolds into an explicate order and forms it through the unfoldment. Because of the wording one might be inclined to think that there exist an implicate order and an explicate order and that they interact through unfoldment and enfoldment. This, however, is not what is being proposed here. Instead, focus should be put on the process, or movement. It is the nature of the unfoldment and enfoldment that is such that one can see it as giving rise to the explicate from the implicate. The explicate order does not have a permanent and independent existence. It is continually being created and dissolved. And the existence of the implicate order is of a subtle kind, it cannot be said to exist explicitly because it is implicit and beyond space and time. Therefore, it is suggested, movement is fundamental and can be expressed in terms of the enfoldment and unfoldment.

It has further been argued by Bohm and Hiley, in [3] (chapter 15.3), that essentially all the quantum mechanical laws of movement can already be seen as unfoldment and enfoldment. The value of a wave function at a particular time and point in space depends on the whole space at previous moments. Consequently a particular region can be said to enfold contributions from the whole space. Then, as it evolves, it unfolds into the whole space. The authors also show how Huygen’s principle and Feynman diagrams can be understood in terms of enfoldment and unfoldment. Huygen’s principle tells us that the propagation of a wavefront can be seen as each point on the wavefront acting as a wave-emitting source point. In other words, each point enfolds contributions from all other points on the wavefront and then unfolds, giving rise to a new wavefront.

**The generative order**

The implicate order was developed mainly in the context of the quantum theory. The unfoldment from an irreducible totality and the enfoldment back into it were related to the dynamics present in the quantum theory. That dynamics,
however, does not really generate anything new. It is a unitary dynamics, and as such it does not create any new contexts or introduce irreversibility. Unitarity is not unique to the quantum theory. Newton’s laws and Einstein’s equations have the same property.

In order to account for the observable entropy increase Bohm and Peat suggest that the holomovement should be creative and besides the implicate order there should be a generative order [5]. Time in a non-generative implicate order can be seen as an implication time, related to the parameter $t$ found in Schrödinger’s equation of motion. Time in a generative implicate order would include thermal time. Thus such an order would account for the irreversibility, entropy increase, and time evolution direction. This would also make the theory truly dynamical as it would open up for the possibility of a “now”. Present theories do not encompass such a concept. Take, for instance, Newton’s laws which provide us with a classical, deterministic trajectory of a moving body. Nowhere in the trajectory is there an indication of where “now” is or how it moves. In fact, it could be argued that the whole idea of mathematical analysis is rather suspicious conceptually when it comes to applying it to physics. Indeed, calculating, for instance, the velocity of something that exists right now involves taking something that does not exist yet, something from the future, and subtracting from it something else that does not exist anymore, something from the past. The generative order is an attempt at approaching dynamics in a different manner, where the “now” is real and dynamical, creating the explicate orders from the holomovement.

Summarizing this section we may say that the essential concept extracted from Bohm and Hiley’s work is the existence of a dynamical, unbroken wholeness. The dynamics seems to consist of unfoldments and enfoldments of explicate and implicate orders. It is furthermore desirable to have a generative component in the implicate order so that one will be able to account for observations like entropy increase.
2 The Underlying Philosophy

“In the beginning there was nothing. And the Lord said: ’Let there be light’ and there was still nothing, but now you could see it.”

(Terry Pratchett [20])

For a long time ontology had been an integral part of physics. Thus questions like “What is?” were seen as rudimentary. In the reductionist tradition, a tradition that has dominated physics over an extended period of time, one could answer the ontological question by saying that identifiable parts exist. Furthermore, one assumes that everything can be broken down into small identifiable parts. Thus when posing the question “What is fundamental?”, the smallest building blocks spring to mind. The idea is that if we can identify the smallest building blocks and understand how they interact, then we can describe everything that exists in terms of them.

However, in the physically measurable world, going towards the smallest part is no guarantee for fundamentality. Indeed, in the quantum theory we see, as described in the previous chapter, that the context, for instance the whole experimental set-up, irreducibly influences the behavior of a particular part. Given this, the idea of going to the smaller part no longer seems as appealing, or as fundamental. Simultaneously the context becomes more interesting. It would be interesting to find the order behind contexts because we lack this at the moment. A context can, in terms of physics, be seen as a physical set-up, an experimental set-up. Thus if we knew how contexts were related to each other we could go from one physical set-up to a completely different one. At present, however, time evolution is fundamentally described by equations that are unitary and work within contexts, not between contexts. As such they fail to provide an entropy increase.

Returning to ontology we note that as the quantum theory involves holistic aspects and a reductionistic ontology does not seem to fit in with it. Is there another ontology or do we have to abandon ontology all together? The dominating interpretation of the quantum theory, the Copenhagen interpretation, suggests that we abandon ontology. It is argued that due to the holistic aspects we cannot say what actually exists, only predict the outcomes of measurement. The equations of the quantum theory are no longer seen as descriptions of Nature but as abstract algorithms with predictive power. Physics is then no
longer anchored in an ontology that can be useful for physics. The philosophy behind the approach presented in this thesis aims at moving beyond a purely epistemological philosophy and also at providing an ontological ground in which the existence of each measurable thing has its being. The idea is that such an ontology can provide a basis for an understanding of the holistic aspects and may further guide us to discovering the order of contexts.

2.1 Old Philosophical Ideas

The fundamental ideas, upon which the approach presented here is based, are not new. In fact, one could argue that they originate in thousands of years old Eastern philosophical ideas. Or, looking at a slightly more recent history, the approach may be viewed as based on the teachings of two Greek philosophers that lived a few hundred years before Christ, namely Heraclitus of Ephesus and Parmenides of Elea. Their teachings have highly influenced western philosophy as well as western science.

Heraclitus saw the world as being in permanent motion. This leads to the notion that “being” is not stable because it changes at each moment and is therefore simultaneously “non-being”. Therefore existence is movement created when opposites fuse. Everything is dynamically unified and so reality is One, originating from one principle. Parmenides, on the other hand, is known for his teachings on the non-existence of movement. To him reality is an absolute, un-generated, unchanging One. Therefore the absolute “being” is fundamental. Though the views of Parmenides and Heraclitus appear to be directly contradictory they both suggest a certain fundamental Oneness. And it is through this Oneness that they can be united. The present philosophy aims at unifying both of these views by assuming wholeness and the wholeness preserving movement. From the “perspective” of wholeness there is no change. For wholeness to be wholeness it cannot change. This, however, still leaves the possibility of a wholeness preserving movement. It is this movement that gives rise to our physical reality. Therefore everything we perceive is indeed change.

2.2 The Underlying Philosophy

Faced with the idea of starting with wholeness we enquire whether it is possible to make philosophical sense of it. From the quantum theory we have seen that we have some indications as to what aspects of wholeness are interesting.
but it would be beneficial to have a clearer picture of what the philosophical background is. Understanding some of the philosophical assumptions can prove to be both motivating and aid in forming an understanding of how a physical theory from wholeness could be built. The philosophical considerations may give a first general idea of what is to be expected of the theory.

Effectively it can be said that the philosophical approach considered here is essentially ontological in its nature. Considerations regarding existence, reality, and their meaning are central as they lay at the background.

Before going into the philosophy of wholeness let us quickly summarize why, in principle and apart from the indications of the quantum theory, we are not satisfied with the view that the world is made up of things. Physics has for a long time been dealing with things as they are found, i.e. measured, in the explicate order. The problem is that things cannot be considered as fundamental because not only their properties but their very existence as the things they appear to be is intimately dependent on the surroundings [26](p.146). They are parts we approximate and abstract from a background, where the background is what actively makes the thing what it is. In other words, a thing can only be identified as itself in a particular context. If the surroundings change sufficiently, so will the thing. A liquid such as water is only a liquid in the right temperature interval. If the temperature is lowered sufficiently it will become a solid. This can be said to be true for all matter, even the particles that at times may have been considered to be fundamental in some sense. Whenever particles meet up with their antiparticles they annihilate and produce photons, which in their turn may be absorbed by matter. Thus things always appear to change and become other things, with other properties, even properties that can be seen as opposite. Therefore having one particular thing with its finite list of qualities and properties we cannot even hope to describe that thing for all times, let alone all things in terms of it. Instead it seems likely that we would require an infinite list of qualities and properties. Starting out from the assumption of infinity has the advantage of not denying any thing the possibility of it becoming something very different [26](p.154). And the assumption of infinity is precisely what the idea of starting from wholeness involves. Such an assumption tells us that a thing is primarily infinite and only secondarily finite. Finiteness comes about in a process of self-determination, or self-limitation, of the infinite.

2.2.1 The Thing-less Ground

“Nothing is real.” (John Lennon [27])

Dealing with things exclusively in the explicate order presents an ontological problem as well. Things are in general identified by their appearances
or outer representations as seen by others. A thing is an object as seen by a
subject. Remaining restricted to the level of external appearances, in this man-
ner, one is not capable of accounting for a thing’s internal existence as itself
[28](p.120-). The ontological existence becomes washed out when a thing is
“its” relation to other things.

One could still argue that there is room for an ontology, as one may take
relations to be the ontological entities. Then a thing is a collection of relations
just like other things are other, possibly intersecting, collections of relations.
The essence of the relations is then change, movement, or process. This is
because we cannot refer to things in our description or definition of a relation.
Consequently process becomes primary. At this point one may suggest that
we could have a process ontology where the ontological entities are different
kinds of processes. However, when examining what a process is we run into
the same kind of difficulty as with things. In trying to describe a process we
need some defining characteristics that let us differentiate between the dif-
ferent processes that exist. A process then reduces to “its” relations to other
processes. The essence of these relations is then a higher type of process.
These higher types of processes will then also suffer the same kind of difficul-
ties as the processes and things. We run into an infinite hierarchy. The higher
up on this hierarchy we come the more abstract the terminology becomes and
the more difficult it becomes to imagine what we are dealing with. Therefore
for the purpose of this particular discussion we can see the word “things” in a
broad manner as that which has defining characteristics.

The idea here is to equip things not only with external attributes found in
the explicate order such as position, shape, color, etc, but also with an internal
existence and uniqueness. Consequently we shall need to move beyond the
explicate order. The question is: How? What can be taken as the primal, or
fundamental, reality beyond the explicate order?

The suggestion here is that primal reality providing existence can only be
primal if it is thing-less. If the source, or constitutive element, of the thing we
consider primal were another thing, then what we consider to be primal would
fail to be so and we would not be able to account for internal existence. And,
as we saw earlier on, there appears to be no fundamental thing because things
change into each other and any thing is but an approximated abstraction and
therefore incapable of accounting for all things. If the source of a thing were
thing-less then the thing, although seen as a representative reality, would have
its existential and primal ground in thing-less-ness. And so thing-less-ness
can be seen as the primal reality beyond the explicate order.

Furthermore, we also note that since thing-less-ness is unique, and whole-

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2The word “thing” is used in a very broad manner. For something to be thing-less means that it
cannot be conceived of as some thing or some concept. It fails to have a representation and to
be an object.
ness is thing-less, we can also refer to wholeness as nothingness. A way of seeing the thing-less nature of wholeness or nothingness is to observe that they cannot be defined. Wholeness is “too full” while nothingness is “too empty”. They do not have defining characteristics or outer limits by which we could surround them and claim that they are that which is within those limits. The lack of thing-ness makes it impossible to differentiate between one nothingness and another. Just like the empty set in mathematics, nothingness is unique. It should be noted that we are referring to literal no-thing-ness and not just a hole, or lack of some particular thing. This thing-less-ness moves beyond the level of concepts. It moves beyond subjectivity and objectivity and representation. Inspired by Bohm [13] we shall refer to it as the implicate order. An important observation to make is that due to its thing-less nature it cannot be separated from the explicate order. In fact, since the explicate has its ground in the implicate it can be seen as a special case of it. Therefore the implicate is beyond the explicate while it also includes the explicate.

The implicate order is the thing-less mergence of the object and subject. It may be said to provide the ontological basis for every thing. The implicate order provides every thing with its 'internal' uniqueness. The explicate order provides the external form (objective representation) while its internal ontological existence is given by the implicate order. This implies a very special kind of order.

A Note on Paradox

“How wonderful, we have met with a paradox, now we have some hope of making progress.” (Niels Bohr [29])

Thing-less-ness introduces difficulties when it comes to expressing it in terms of language. Since it is beyond representation it is certainly beyond our language and repeatedly leads to paradox. Such paradoxes can, however, often be intuitively resolved, bringing about insight, if thing-less-ness is accepted. An example is the sentence “A thing is a thing if it is not a thing”. This is a statement summarizing the above section. It tells us that true ontological existence of a thing can only be based on thing-less-ness. The negation, “not”, does not remain on the level of subject and object but moves beyond into thing-less-ness.

As a curiosity it can be mentioned that mathematics itself can be said to contain paradox at its foundation. As most mathematics can be expressed in terms of Set Theory we can view Set Theory as rather fundamental. Set Theory itself is such that each set is defined in terms of the empty set. In other words, different sets are different ways of looking at the empty set. Furthermore, we note that the empty set itself can be seen as a bit of a paradox as it denotes the existence of nonexistence. In addition the extension of all para-
doxes (such as, for instance, the set of all square circles) is the empty set. And so, taking this perspective we can suggest that mathematics is founded on something containing all paradoxes.

2.2.2 The Order of the Implicate Order

The uniqueness of a definite thing in its explicate (measurable) existence has its ground in the absolute uniqueness of wholeness, which is its and every other thing’s source. In other words, for a thing to be can be said to mean for it to be unique through and through. On the explicate level uniqueness means that no other thing is like it (including position and time in the representation of the thing). On the implicate level for something to be unique means that it has its being in the thing-less ground that is One, truly unique, and inseparable from the thing\(^3\). However, every thing has its ground in the same wholeness. Therefore the ground of one particular thing is actively made up of all other things acting in a non-separable, or holistic way. Simultaneously, that particular thing lies at the ground of all other things as represented in the explicate order [28](p 147-). Another way of expressing this order is to say that wholeness is (in) each thing and each thing is (in) wholeness [4](chapter 7). This is the essence and the order of the implicate order. It is a holographic or self-similar type of order that is a consequence of no-thing-ness.

One might be tempted to separate existence and representation and see the thing-less ground as providing the existence but having nothing to do with the representation. One might even go one step further and say that since physics is concerned with things as we experience and measure them in the world of representation we can disregard the ground as it only provides the philosophical possibility of existence. Should such temptations make an appearance one should bear in mind that the thing-less ground cannot be separated from representation and this is the reason why holistic aspects play an active role in the evolution of quantum systems. Just like the famous fiction writer J.R.R. Tolkien says, “It does not do to leave a dragon out of your calculations, if you live near him”[30], and thing-less-ness is nearer to us than any thing could ever be.

The order of the implicate order is essential to grasp as it provides us with important insights. One such insight is that due to the fact that every thing penetrates into each thing and each thing penetrates into every thing we should expect there to be no particular preference as to points of view. In other words, we expect there to be relativity in the explicate order.

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\(^3\)One could object that apart from representation every thing is the same as every other thing and therefore no thing is unique. This is not true because the ground is thing-less and unique. Consequently there is no multiplicity and there are no other things to be the same as.
2.2.3 Movement

As we saw in the discussion of thing-less-ness it appears as though process is a more fundamental idea than that of static things. A more elegant way of expressing the primary role of thing-less-ness is to suggest that process, or movement, is fundamental. The fundamental movement does, in its totality, not contain any defining characteristics but gives rise to all kinds of processes. Static things are secondary and abstracted from the movement. This could at first appear confusing as we may well tend to ask what is moving, thereby attributing fundamentality to the thing. The what that is moving in this case is the holistic wholeness, which being thing-less lacks any defining characteristics and cannot be considered as a thing. Attempting to think of it as a thing easily leads to confusion and error. Therefore we maintain that movement is to be seen as fundamental. When dealing with the fundamental movement it should be remembered that due to thing-less-ness when seen as a totality, as mentioned above, the fundamental movement cannot be viewed as anything apart from the explicate order. Instead, it is the creative dynamical source of every thing. It is creative because it does not simply reform, or reshape, a thing into another but truly creates a thing out of no thing.

Taking movement to be more fundamental than our explicit concepts of objects and space is actually not as counterintuitive as it, at first, may seem.

From a scientific point of view we see that attempts have been made to describe physical reality in terms of fields. In [3](p.353-357) it is argued that these fields must be viewed as movement and all their properties as relations of that movement. What appears to be persistent is in fact continually being renewed.

From a more personal and psychological point of view the idea of motion being primary is not counterintuitive either. Although we are accustomed to operating with concepts of permanent or semi-permanent nature moving in space and time, it is not until relatively late in our development that we actually establish such concepts. Investigations into the development of human perception seem to imply that our own intuitive concepts of space and the objects around us are based on invariances in movement [31]. These findings suggest that as babies we start by experiencing a fluctuating totality from which we cannot differentiate any parts as thoughts, feelings, sensory input, etc. Then we start to recognize some features in the movement, we learn that we can affect the movement, and we learn to perform operations. The perception of the world in terms of permanent objects moving in space and time is an active skill that needs to be acquired. Once acquired the skill, we are no longer conscious of the process, instead we focus on the abstractions from the movement, the invariances. A motivating discussion on the subject of perception and physics with interesting references can be found in [32]. And so, our experience of the outside world in terms of perception is already based on
movement. Thus, movement as fundamental is in fact not a foreign idea for human beings.

A conceptual advantage regarding movement as fundamental is that we can abstract the concept of the explicate or implicate from the movement, instead of postulating their unorthodox existence and relationships. We have one fundamental movement with a nature such that it unfolds and enfolds wholeness. The unfoldment may be seen as giving rise to the explicate order from the implicate, while the enfoldment as going back into the implicate from the explicate. In this way the nature of the fundamental movement contains the order of the implicate order (as mentioned above). We can say that the implicate order is dynamic.

To summarize, we have the fundamental movement that may be referred to as a wholeness preserving movement unfolding and enfolding wholeness. The unfoldment gives rise to the level of the explicate order. This is the level of form, representation, and appearances. Through representation diversification and autonomy appear. A thing is represented as something. It is the level at which we have subjects and objects, neither of which are possible without the other as they are defined through each other. Beyond and including this level we have the implicate order that is arrived at through enfoldment. This is the thing-less ground of the explicate where representations have their inner existence. The existence of things in the explicate is only possible because the things are grounded in the implicate. An important reason why we should acknowledge the thing-less ground is because it provides us with the essential and active order upon which we can construct physical theories in addition to gaining insights regarding our present theories as well as the world and ourselves.

2.3 Meaning and Formative Causality Are in the Movement

This section is dedicated to conveying the importance of the movement and relating it to what has already been discussed in the introduction regarding the quantum theory. As already mentioned the idea is that all parts may be seen as arising from the dynamics of wholeness. This in itself indicates the fundamentality of movement. There is, however, another advantage of seeing the parts as arising from movement in physical processes. And that is that this may provide us with access to a new non-sequential, formative, causal structure where a concept like meaning is important. In other words, the main reason why the movement is so important is that the essence of the formative causal structure is not in the parts, though it may be encoded, or reflected, in the parts. Instead, the meaning of the formative causal structure becomes manifest only through the movement. The wholeness preserving movement
therefore has the potential of providing a platform in which physical theories may be extended beyond space and time, and the consecutive causality with which space and time often are associated. In particular, it may provide an insightful extension for the quantum theory, where, as is well known, the principle of local causality does not hold. Such an extension could consist in providing the order of the formative causality so that it would not only work within a certain context, a certain meaning, but be able to go between different contexts, or meanings. After reading the other sections of this thesis, and in particular the Discussion chapter, the reader will hopefully see that the discussion in this section is only a different way of approaching the idea of entropy increase and irreversibility.

The reason why we refer to the concept of meaning as important in the formative causal structure of the movement is the following. Meaning does not in general have the sequential connotation that causality tends to have, where one thing causes the next and a chain, or avalanche, of events is constructed. Instead it tends to be associated with contexts, totalities, and some all-embracing message. This is very much in line with the structure of the causality found through movement. Though it should be stressed that finding the new causal structure in which meaning plays an important role does not imply finding a well-defined, ultimate level of causality. It is not the missing complementary part that could resurrect determinism and bring it back into physics [33]. Such a finding would be highly unlikely [26] (for instance p.2-3) considering the fundamental ideas of wholeness conveyed here. This is because wholeness is not equivalent to a specification of outer limits within which the whole is enclosed. Instead, wholeness is infinite, of an implicate nature, and undefinable. Therefore, finding a well-defined, ultimate causal plane could be likened to finding the ultimate largest natural number in the infinite set of all natural numbers. Such a number does not exist. In fact, what we are suggesting here is that actual movement, originates in the undefinable4 and is possible because of the lack of defining characteristics5. We therefore find it likely that a well-defined, ultimate level of causality does not exist either. Instead, such a level would have to be undefinable and of an implicate nature, as is wholeness.

We will now follow with an attempt to give a better intuitive understanding of what is meant by a formative causal structure in the movement by looking at metaphorical analogies from our surroundings as well as at examples from physics.

One intuitive way of seeing how the meaning and formative causal structure are in the movement and not in the parts is by looking at language. Although it at times may appear as though language is reducible to letters and words

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4See the discussion on the generative order at the end of chapter 1.
5See the discussion on the critical point, 3.1.3 the next to last paragraph.
written down on a piece of paper, in reality, the true causal meaning of it, the meaning that forms it, can only be obtained through the act, or movement, of understanding. A word contains, encoded in it, a potential meaning, but the meaning is only extracted if it is understood. And it is only in the act of understanding that the meaningful structure, that may originally have been put into a word, may become manifest. An analysis of the letters and parts alone will not unveil the true formative causal structure. By analyzing a text written in some language, without seeing it as a language, one may become quite good at predicting the coming letters. At the same time one would totally miss the meaning along with a large part of its causal structure. The formative causal structure of the text is not found in the succession of one letter by the next. The author of the text had a certain understanding in mind, which determined the text at each stage. He may have tried to convey a general message and some sub-messages related to the general one, all of which gave rise to the text. They were part of the causal order that created, formed, it but cannot be extracted from the letters themselves. Instead we need the act of understanding to manifest the original causal order.

In a similar way music is in general perceived in themes that may be seen as constituting totalities and sub-totalities [34]. Its meaning and inspirational beauty are not in general found in the succession of distinct notes, but in the totality of a theme. It has been proposed that an essentially similar kind of causality to that found in the perception of music may be found in the quantum theory itself [34]. As is well known, the quantum theory does not exhibit a dynamical causality that determines successive quantum states in connection to measurement. The authors of [34] therefore suggest that “…the whole order and form of the development is the cause”.

A physical example of the structure of a movement being encoded into parts, where its meaning is not perceptible when examining the parts alone, is a hologram. A hologram has in itself encoded the structure of interfering light. And it is only when the appropriate frequency of light shines at the hologram that the original picture is reconstructed.

As we have seen in previous sections, the nature of the wholeness preserving movement is reflective. There is a mirroring action where wholeness mirrors itself. But it is not the mirror (the thing) but the action that is essential. It is through the action that understanding and meaning arise, not in the mirror. So the suggestion here is that by taking the wholeness preserving movement as fundamental we may gain some insight into a new formative causal structure where something similar to meaning plays a fundamental role. The meaning and the new causal structure are in the order of the dynamical wholeness preserving movement.
Why a New Causal Structure?

We will now try to specify exactly why we expect there to be a formative aspect to the causal structure beyond the sequential one. One way of stating the basic motivation is that due to complementarity neither a parts-based nor a totally holistic description is complete. Therefore, there must be an active dependence of what we see as parts, on the holistic properties. The sequentiality may be seen as the differentiated aspect of causality whereas the formative action as the undifferentiated aspect. Dependence on holistic properties may already be examined partially through conventional quantum theory without focusing on complementarity directly, but starting from the perspective of the violation of Bell’s theorem [35]. There we can see the existence of a dependence of the outcome of a particular measurement on non-local causes.

Since the wholeness preserving movement is the dynamics of wholeness and cannot consistently be separated from wholeness, its nature must be that of wholeness. Consequently, this tells us that although a reductionist view may be partially applicable it must be complemented by a holistic one. In other words, the source of physical reality requires more than just a reductionist approach. All classical theories have been mechanical in nature and focused on a reductionist description. The quantum theory, despite originating from reductionist ideas, as its very name “Quantum Mechanics” indicates, certainly contains aspects beyond reductionism. And so, it is not consistent with the kind of dynamical causality that is sequential in nature. Instead it offers probabilities. At this point one may wonder: Does the quantum theory offer probabilities because reality is truly probabilistic in its nature, or is it because the quantum theory has been constructed in a way that almost entirely focuses on reductionistic aspects? A probabilistic treatment is in general a reductionistic way of dealing with unknown causes. There certainly is room for yet undiscovered causal structures, the existence of which considerations from wholeness seem to indicate. In fact, such a formative causal structure can already be seen in the Ontological Interpretation of the Quantum Theory [3]. As discussed in chapter 1 section 1.1, instead of seeing the wave function as the square root of the probability of, say, finding a particle in some region, Bohm proposes to see it primarily as an actual field that provides a partial description of the formative cause, as well as a function from which actual probabilities can be derived [14]. Through the quantum potential we get a hint as to how the formative dynamics operates. Recall that the properties of the quantum potential differ radically from those of classical potentials. The potential does not have an external source, it does not necessarily decrease with distance, it depends in an irreducible way on the entire environment, and its interaction with a particle appears to be more in line with a formative cause than a kind of sequential cause/effect.
3 The Mathematical Framework

“Our knowledge about the Universe has an edge. ... what cannot be known is more revealing than what can.”

(J.D. Barrow [36])

Starting from wholeness demands a drastically different approach to physics, from the ones usually employed. The reason is that if we wish to stay true to the idea of wholeness we must recognize that it is not definable, and in that sense not knowable. Although we cannot define or fully describe wholeness we can attempt to build theories led by the intuition, or assumption, that there is an underlying wholeness. This is what we shall attempt. Thus we shall assume the existence of an undefinable dynamical wholeness, as an axiom, and see what the logical consequences are. Hopefully the assumption of the existence of something unknowable will smudge the edge of what is known and reveal new structures, providing new insights.

From the previous chapters we know that we are looking for way of representing the dynamics of wholeness. This movement or process should in its totality not be viewed as confined to physical space and time and is not reducible to a description of some parts that move in a sequential manner. Instead, it is the fundamental dynamics of wholeness, and that which through self-limitation gives rise to physical space and time along with parts which may be viewed as properties of, or invariances in, the movement. And so, the dynamics is such that any part, and the change of any part, may be traced back to the fundamental movement. Such a dynamics needs to be wholeness preserving. This is because wholeness may be said to be a totality implicitly containing all the parts that can be measured explicitly. This, in turn, appears to imply that for it to be truly whole it cannot change. If it were to change and become different, then this would imply that it was not truly whole in the first place because it did not contain itself after the change. At the same time we know that change, at least in our physical world, exists. Therefore, we are looking for a dynamical description that allows wholeness to remain essentially unchanged though moving, from the perspective of parts, and thereby allowing for change. We plan to capture this dynamical feature of wholeness, the wholeness preserving movement, mathematically in a fundamental way. This will be the fundamental concept. And so essentially, the proposition here
is to describe our physical reality as an expression of the dynamics of wholeness.

In this chapter we will start by providing a mathematical way of representing our fundamental concept, the wholeness preserving movement. Then we will see how, by interacting with itself, it creates different processes and thus generates a universal algebra. Finally, we will take a look at braids as they will be used in the physical interpretation of the process algebra.

3.1 Wholeness Axiom as the Dynamics of Wholeness

“Not everything that can be counted counts, and not everything that counts can be counted.” (Albert Einstein [37])

In order to provide a mathematical foundation that is capable of expressing features of wholeness at the very fundamental level, it is suggested that we start with Set Theory, since most of mathematics can be expressed in terms of it, and add the Wholeness Axiom introduced by Corazza [38] to its normal axioms. The normal axioms of Set Theory are the Zermelo-Fraenkel set of axioms and the Axiom of Choice (see Appendix A), abbreviated ZFC [39], [40], [41]. For approximatively eighty years it has been known that most of mathematics is derivable from them. These axioms refer to sets and are formulated in the formal language $\{\in\}$. This means that the only relation used in the formulation of the axioms is the membership relation, $\in$.

The mathematical theory within which we shall attempt to describe a theory from wholeness will contain the ZFC set of axioms together with the Wholeness Axiom and will be referred to as ZFC+WA. We will now proceed to see what the Wholeness Axiom is and what features of wholeness ZFC+WA is capable of expressing.

3.1.1 The Mathematical Universe $V$

An important concept for our purposes is the mathematical universe. The mathematical universe, $V$, is a proper class and can be expressed as:

$$V = V_0 \bigcup V_1 \bigcup V_2 \bigcup \ldots,$$

(3.1)

where the different stages are $V_0 = \emptyset$, $V_1 = P(V_0)$, $V_2 = P(V_1)$, ..., with $P(\cdot)$ denoting the power set (the set of all subsets). We say that $V$ is a proper class, this means that it is not a set. If $V$ were a set, it would be the set of all sets. However, as observed by Russell [42], such a statement leads to a self-referential paradox. The set of all sets would have to be an element of itself. This paradox is of the same type as the liar paradox: “This statement is a lie.”
If the statement is true then it is a lie, and if it is a lie then it is true\(^1\). This is why \(V\) is called a proper class.

It is interesting to note that there is a salient analogy between the mathematical universe \(V\) and wholeness. As previously mentioned, wholeness implicitly contains everything but is more than the sum of the parts, and is itself not an explicitly existent entity (a part). The nature of its existence is implicit. In a similar way \(V\) may be said to contain all the mathematically existent parts and yet be more than their sum and not itself an explicitly, mathematically existent entity. Saying that \(V\) is the union of \(V_0, V_1, V_2, \ldots\) is like saying that wholeness is, or contains, all the parts. The existence of each stage, \(V_x\) for any ordinal\(^2\) number \(x\) in \(V\), follows from the ZFC set of axioms. It is interesting to note, however, that the existence of \(V\) itself cannot be shown from the ZFC set of axioms, since \(V\) is not a set. In other words, \(V\) in its totality cannot be shown to be an explicitly existing mathematical object. The fact that all its stages can be shown to exist seems to imply that \(V\) should exist as well, however, its existence cannot be explicitly shown.

It should be stressed that our description of \(V\) is but a description of a feature of wholeness and far from a complete description of wholeness. Such a description cannot be expressed in explicit terms because the nature of wholeness is not reducible to the explicit. Our description in terms of \(V\) only refers to the explicit parts and states that it contains all parts but is more than that. This way of referring to wholeness is employed because the aim of this approach is to account for the existence and dynamics of the explicitly existent parts of reality.

3.1.2 The Wholeness Preserving Movement, \(j\)

As argued earlier, one of the fundamental concepts upon which we wish to base a theory from wholeness is the wholeness preserving movement. This means that we wish to postulate the existence of such a movement in order to enable a prediction of quantifiable characters of an experimentally verifiable entity in terms of the wholeness preserving movement. It turns out that the Wholeness Axiom can be seen as expressing the wholeness preserving movement. The Wholeness Axiom \([38]\) is the assumption that there is a non-trivial elementary embedding, \(j\), from the mathematical universe, \(V\), to itself:

\[
j: V \to V. \tag{3.2}\]

\(^1\)This type of reasoning, in particular statements like “This axiom is not provable”, is also what lead Gödel to his incompleteness theorem from 1931 \([43]\), which among numerous other applications is used to show that we cannot show the existence of large cardinals from ZFC.

\(^2\)Ordinal numbers denote the order of sets (Which one?) whereas cardinal numbers denote the size (What size? How many elements?).
For \( j \) to be an elementary embedding means that if some formula, \( \phi(x) \) holds true in the domain, which for \( j \) is \( V \), then \( \phi(j(x)) \) must hold true in the codomain, which is also \( V \) in this case. In other words, \( j \) is reflective and truth preserving. It can be said to be wholeness preserving because due to its non-triviality it actually changes parts while preserving all structures in \( V \). Non-triviality asserts that this embedding is not the identity and that there are sets that are taken by \( j \) to sets other than themselves.

Definition (3.2) can be axiomatized in a consistent manner\(^3\) so that ZFC+WA is obtained. This is done by adding \( j \) to the language \( \{\in\} \), so that we have \( \{\in, j\} \) and saying that the ZFC set of axioms is valid with the addition that all instances of Separation and no instance of Replacement\(^4\) are valid for \( j \), and the addition of axioms that express \( j \) as a non-trivial elementary embedding from \( V \) to itself (Non-triviality axiom and Elementarity axioms). The meaning of this in terms of wholeness will be discussed in the succeeding sections.

The Wholeness Axiom was developed to show the existence of all large cardinals, in mathematics. And so, elementary embeddings, such as \( j \), are closely connected with infinities. One of the aspects they have in common with infinities is that they disclose precisely the self-similar, or holographic, type of order we attribute to wholeness and find in the quantum theory. An example of self-similarity for the real numbers is that there are as many real numbers in, say, the interval between 0 and 1, as there are on the whole line. Elementary embeddings can be seen as displaying perhaps the ultimate form of self-similarity because one cannot distinguish a part from the whole with any definable formula.

### 3.1.3 The Critical Point \( \kappa \)

The introduction of the wholeness preserving movement, mathematically formulated as the Wholeness Axiom, brings about some interesting consequences. Of primary importance is the arising of a critical point. This happens because the axioms for \( j \), in particular non-triviality, ensure that there is a least ordinal moved by \( j \). This ordinal is denoted \( \kappa \) and is called the critical point of \( j \). In particular, we see that Non-triviality for \( j \) asserts that there has to be some set \( X \) for which \( j(X) \neq X \), and so, there has to be some smallest ordinal for which this condition holds true. The restriction of \( j \) to any set of a rank less than \( \kappa \) is the identity. In other words, for any set \( Y \), of rank less than \( \kappa \), \( j(Y) = Y \).

Being an elementary embedding, \( j \) is truth preserving. This means that properties, operations, and relations that hold true for some sets \( X_1, X_2, \ldots \in V \) also hold true for \( j(X_1), j(X_2), \ldots \in V \). This property of the elementary embedding

\(^3\)despite Kunen’s theorem [44]

\(^4\)The Separation and Replacement schema are axioms contained in the ZFC set of axioms and will be explained in a section below.
makes it possible for us to say more about $\kappa$. It turns out, for instance, that $j(\kappa) > \kappa$.

**Proof that $j(\kappa) > \kappa$**

Let $j : V \rightarrow V$ be a non-trivial elementary embedding from the mathematical universe to itself, with $\kappa$ as its critical point. Since $j$ is by definition non-trivial $j(\kappa) \neq \kappa$. Let us assume $j(\kappa) < \kappa$, and say that $j(\kappa) = y$. Then since $\kappa$ is the smallest ordinal moved by $j$ and $y < \kappa$ it must mean that $j(y) = y$. Because of elementarity ordinal order is preserved, meaning that if $y < \kappa$ then $j(y) < j(\kappa)$. But from the above we see that $j(y)$ cannot be lesser than $j(\kappa)$ because $j(y) = y = j(\kappa)$. And so we conclude that $j(\kappa) < \kappa$, is not consistent. Therefore it must be so that $j(\kappa) > \kappa$. ■

At this point one may wonder what $\kappa$ is. Let us start at the bottom of $V$. Could the empty set be taken to a set larger than itself? No. It could not, because to be the empty set, $\{\}$, is definable in ZFC, and $j$ has to preserve all definable properties. Could the set $\{\{\}\}$ be taken to some larger set? No, for the same reason as the empty set. Could the set of all integers be taken to some larger set? No, again. Climbing up the ordinals in $V$ one may start to wonder if any set will ever be taken to some larger set. When could that happen? That can happen at what we could intuitively call “the boarder of definability”. And so, the ability to move, here, comes from not being fixated by determined properties. The critical point is where the determining, defining, or fixating properties are loosened so that movement is made possible. Hence, the origin of movement is in the undefined, indetermined. Mathematically one can show that $\kappa$ is an infinite cardinal with all the large cardinal properties [38]. The idea that movement comes from the undefined is in line with the already discussed ideas of a creative generative order. The generative order explicitly creates something new from the undefinable holomovement.

Furthermore we note that because of $\kappa$ the range of $j$ will contain gaps. And so one could say $\kappa$ will decide which sets will be missing. Mathematically speaking it will be a nontransitive model because not every element of a set in $V$ is an element in the range of $j$. We can see that this happens because every set smaller than $\kappa$ will be taken to itself and $\kappa$ will be taken to some larger set. Since no set can be taken to something smaller than itself there is not set that will be taken to $\kappa$. Thus $\kappa$ will be missing in the range. In fact, all the ordinals from and including $\kappa$ up to $j(\kappa)$ will be missing. We suggest that the range be seen as an explicate order, while the domain and codomain as implicate orders. The codomain of $j$ is again a transitive model.
3.1.4 Separation and no Replacement for $j$

Recalling and recapitulating some features of wholeness we can say that wholeness is indescribable, it is more than the sum of parts, and yet it is locally accessible and that through which every part emerges. All of these features are represented in ZFC+WA. In order to see this, it is of interest to note that ZFC+WA does not restrict wholeness, or the wholeness preserving movement, to some explicitly existent mathematical object. All it does along those lines is to assume the existence of a wholeness preserving movement. The wholeness preserving movement is itself never defined by some specific formula. In fact, it cannot be defined within Set Theory. $j$ is neither a set nor a proper class. The fact that $j$ is not a set follows from $j$ being defined on all of $V$. To see why it is not a proper class either we need to take a look at the axioms of Separation and Replacement.

When axiomatizing the wholeness preserving movement, $j$, it is said that all instances of Separation and no instance of Replacement are valid for $j$. Given a set, $A \in V$, Separation allows us to talk of a subset of $A$ in which some property is true for all the elements of that subset. So, $j$ having all instances of Separation means that for all properties, $P$, depending on $j$, we can take any set, $A$, and look at the subset of $A$ containing all elements for which $P$ is true. That subset of $A$ is itself a set. Separation for $j$ makes $j$ interesting, powerful, and promising for a further development in terms of a physical theory, because it allows $j$ to act locally, meaning on any particular set. This allows us to use $j$ in local descriptions. What $j$ does on the entire mathematical universe, it also does locally on any set.

Replacement, on the other hand tells us that for any set $A$ and any rule that associates with each element $x$ of $A$ a set $Y_x$ there is a set $B$ that consists precisely of all $Y_x$, where $x \in A$. Replacement is useful for class functions. Class functions are functions that are not sets but proper classes defined on some proper class, such as for instance on $V$. Replacement guarantees that the range of a class function being restricted to some set is also a set. The range being a set makes its rank limited. So, we can say that Replacement makes sure that there can be no definable way of going through the top of $V$ in a certain amount of steps (indicated by the rank of the domain). No instances of Replacement for $j$ tells us that for all functions, $F$, depending on $j$, there may be no set consisting of all the $Y'_x$s associated with the elements $x$ of a given set. Indeed, Separation for $j$ can be used$^5$ to show that if $F$ is defined by letting $F(0) = \kappa, F(1) = j(\kappa), F(2) = j(j(\kappa)), \text{etc.}$, and letting $F(x) = \emptyset$ for any $x$ that is not a natural number, then the restriction of $F$ to the set of natural numbers has a range that goes all the way through $V$. More simply, the sequence $\kappa < j(\kappa) < j(j(\kappa)) < \ldots$ extends above every rank in the uni-

$^5$The proof makes essential use of Kunen’s theorem, see Proposition 3.6 in [38], and [44]
verse. One says that the 'critical sequence for \( j \) is cofinal in the mathematical universe'. Hence \( F' \)'s depending on \( j \) do not exist as sets or proper classes in \( V \) and may be capable of going through the top of \( V \). No Replacement for \( j \) assures us that \( j \) is not a proper class because it is not definable by a formula within Set Theory. If it were definable in Set Theory, then by ordinary Replacement in Set Theory, the \( F \) defined above would have the property that the range of \( F \) restricted to the natural numbers would be a set. As observed above, this is not the case.

Going back to our previously mentioned features of wholeness, \( j \) is indescribable and more than "the sum of the parts". At the same time it acts locally and is present in every part of \( V \).

3.1.5 Unfoldment and Enfoldment of \( j \)

The dynamical concepts of unfoldment and enfoldment were developed and introduced by Bohm [4] in an effort to provide a common basis capable of accounting for both quantum and relativistic effects. The holomovement, strongly reminiscent of our wholeness preserving movement, is by Bohm seen as a holistic pulsation in which orders unfold and enfold. The implicate order is the enfolded order while the explicate order is the unfolded order. The idea that Bohm tried to convey was that every thing is in some sense enfolded into the whole and that the whole is unfolded into every thing. Since the wholeness preserving movement, \( j \), can naturally be broken into two parts that highly resemble Bohm’s notions of unfoldment and enfoldment, these concepts will be adopted and used actively.

The wholeness preserving movement, \( j \), captures Bohm’s concepts in a nice way. We can factorize \( j \) into two parts,

\[
j = j_e \circ j_u. \tag{3.3}
\]

One part may be viewed as the unfoldment process and described as

\[
j_u : V \to j''(V), \tag{3.4}
\]

where \( j''(V) \) is the range of \( j \). The other part may be viewed as the enfoldment process and described by

\[
j_e : j''(V) \to V. \tag{3.5}
\]

The unfoldment process, \( j_u \), is an isomorphism, an elementary embedding in which the range and codomain coincide. The similarities between \( V \) and \( j''(V) \) are carried by the transformation. This is to say that if some formula \( \phi(x) \) holds true in \( V \) then \( \phi(j(x)) \) must hold true in \( j''(V) \). The unfoldment process is a structure-preserving movement that gives rise to \( j''(V) \). It may also be seen as a process that gives rise to separation and distinction. Through
it we obtain something different from $V$, namely $j''(V)$.

The enfoldment process, $j_e$, is an elementary embedding and an inclusion map. Of importance is that by being an inclusion map it may be seen as the unifying process. It is as if the process were that of the realization that $j''(V)$ is in fact not at all something different from $V$. Instead it is the same, the identity. Of further importance is to note that $j''(V)$ is an elementary submodel of $V$. In other words, $j''(V)$ and $V$ are similar without the similarities having to be carried by the movement. Therefore, if some formula $\phi(x)$ holds true in $j''(V)$ then it will also hold true in $V$.

An interesting property of elementary submodels is that they reflect all the properties of the models, even of the sets that cannot be found in the submodels. If some relation, $R$, between two elements of $V$, $a$ and $b$, that are not elements of $j''(V)$, holds, then the same relation must hold between some two elements of $j''(V)$. In other words, if $V |= aRb$ where $a, b \in V$ and $a, b$ are not elements of $j''(V)$, then there are such $c, d \in j''(V)$ that $j''(V) |= cRd$. Further we also note that since $j''(V)$ is a submodel of $V$, $cRd$ must also hold in $V$. Elementary submodels imply a kind of holographic packaging where all relations are preserved and contained within each submodel. And so, every elementary submodel is a sub-totality in the sense that it contains all the properties of the model.

Both the unfoldment and enfoldment are required in order to construct $j$. In other words, a differentiating process alone is not sufficient for a description of the dynamics of reality. It must be complemented by a unifying process. Neither process on its own is ever complete because they are very much related through $j$, which captures the creation and interplay between polarities.

In summary, $j$ is a reflective process that may be seen as a non-exhaustive\(^6\) complementary duality [45]. This dual process is necessary in order to provide the wholeness movement with its creative power.

### 3.1.6 A Preferred Direction in Creation

A significant point regarding $j$ is that it can be said to have a preferred direction in its creation. In order to see this we must first recognize that due to non-triviality $j$ must have a critical point, $\kappa$, which is the smallest ordinal that is not mapped onto itself. In other words $j(\kappa) \neq \kappa$. In fact, due to elementarity, $j(\kappa) > \kappa$. And so all sets smaller than $\kappa$ will be mapped onto themselves while all sets larger or equal to $\kappa$ will be moved to larger sets. This can be seen as directedness with a possible increase in size. But why should $j$ only work in one direction? Could one not also consider its inverse?

Despite the fact that $j$ is 1-1 it does not have a proper inverse. This is

---

\(^6\)If the complementarity were exhaustive in a definable way then the dynamics of wholeness would be reducible to a sum of parts.
due to the fact that not every element in the codomain is an image of an element in the domain. When specifying a function it is enough to specify its domain. A function’s codomain, a set that contains the range, is of little importance as a change in it does not affect the function itself. For an elementary embedding, on the other hand, the codomain plays a crucial role. This is because an elementary embedding has to satisfy the condition that its range is an elementary submodel\(^7\) of its codomain. Through its use of the codomain elementarity provides a holographic type of order that is essential to the embedding, where all relations are preserved, and yet beyond what can explicitly be reached (through the range). This is reminiscent of the implicate order which preserves all relations and is more and beyond the explicit parts.

Although we cannot talk about the inverse of \(j\) we could still talk about partial inverses. However, the partial inverse of \(j\), \(j^{-1} : j''(V) \to V\), where \(j''(V)\) it the range of \(j\), is not as interesting as \(j\) itself. It has been stripped of the essential information that \(j''(V)\) is an elementary submodel of \(V\), its domain is limited to \(j''(V)\), which means that it is no longer reflective, and it does not contain the same creative power as \(j\) (no large cardinal power). And so, we see that in large part due to its active dependence on the codomain, \(j\) in itself contains a preferred direction in creation.

As we shall see in the Discussion chapter the above will prove to be essential when it comes to physical time. The origin of the potential to evolve has to be found in the elementary embeddings if they are to be considered as the source of the physical universe along with its dynamics. The preferred direction of \(j\) could therefore account for a direction of time along which entropy increases, if this approach is developed further.

### 3.1.7 Laver Sequences–Emergence of All Sets in \(V\)

Taking the wholeness preserving movement to be fundamental it is desirable to have a way of seeing how the movement gives rise to every part. Is there a way of describing the emergence of all sets in the mathematical universe through the wholeness preserving movement? Although, strictly speaking, mathematics is in general not viewed as emerging from anything, it is possible to see all the sets in the mathematical universe as an expression of the dynamics of wholeness. In order to see how this can be done we can use Laver sequences [46]. The definition of a Laver sequence given here is a generalization due to Corazza. A Laver sequence is a set \(L\) of length \(\kappa\), \(\kappa\) being the critical point of \(j\), that has the property that for any set \(X\) in \(V\) there is an elementary embedding \(i : V_\alpha \to V_\beta\) such that the \(\kappa\):th term of \(i(L)\) is \(X\). In other words, every set in the mathematical universe can be located as the \(\kappa\):th term of an image of the Laver sequence \(L\) by some elementary embedding \(i\).

\(^7\)See the previous section for more on elementary submodels
Moreover, all such $i$’s are derived directly from $j$ itself. Therefore, every set exists as if in seed form in the single point $L$; its existence as a set becomes apparent when the appropriately derived embedding $i$ is applied to $L$. So, given $\kappa$ we can construct a sequence such that every set in the mathematical universe can be seen to emerge from it through movement. The existence of such sequences can be shown assuming the Wholeness Axiom. Another way of looking at Laver sequences is to see them as functions $f : \kappa \rightarrow V$, such that for any set $X \in V$ there is an elementary embedding, $i : V_\alpha \rightarrow V_\beta$, such that $i(f)(\kappa) = X$. As functions, Laver sequences live in the mathematical universe and may be said to be made out of parts. However, a detailed understanding of the parts does not facilitate the understanding of them. It is not until one “shines” an elementary embedding on them that they become “active” and one realizes their power to give rise to all sets. These global properties, not perceptible from the details but visible through movement, are what make them interesting. These properties may come to play a key role in finding new formative causal structures for physical theories from wholeness.

In short, the Wholeness Axiom provides wholeness preserving movement. As a consequence there is a $\kappa$ with which we can form a Laver sequence from which all sets in $V$ can be seen to emerge through the movement.

3.2 Process Algebra $A_j$

We see $j$ as a wholeness preserving movement. It would be interesting to see what kinds of processes it can give rise to. Therefore we shall have it “interact” with itself. This self-interaction will mathematically be given by application. As a result we will find that $j$ generates a universal algebra, meaning a set with the binary operation of application. A universal algebra is a set with some binary operations. Thus it is not necessarily a vector space. The elements of our process algebra will be different processes all unfolding from and enfolding to wholeness, or mathematically, different non-trivial elementary embeddings from the mathematical universe to itself. We will be especially interested in finding out how these processes relate to each other. This is because the processes can be seen as wholes, and interpreted as meaningful parts, and the structure of their interaction should tell us how to go from one meaningful part to another. When changing meaning we change the context and when changing contexts physically we can change the entropy. Thus the relation telling us how different contexts are related can provide us with an order that could potentially describe entropy increase.

Proofs of the here used mathematical results pertaining to universal algebras on elementary embeddings can, among other places, be found in [47](p.538-570). There one can also find a more substantial and general introduction to these algebras.
3.2.1 Application

For the elementary embeddings $j : V \rightarrow V$ application can be defined for any $j$-class $C$ as [48]:

$$j(C) = \bigcup_{\alpha \in ON} j(C \cap V_\alpha)$$

(3.6)

where $ON$ is the class of all ordinal numbers. Working with proper classes is technically difficult because we cannot quantify over them. Thus properties that can be derived for set versions of the proper classes may be difficult, or impossible, to prove for the proper classes without changing the axioms of Set Theory. Therefore we will work with a set version of $j$ and treat it as if it were an elementary embedding $j : V_\lambda \rightarrow V_\lambda$, where $\lambda$ is a limit ordinal. A limit ordinal is an ordinal number that does not have a predecessor. The existence of a non-trivial $j : V_\lambda \rightarrow V_\lambda$ is a large cardinal assertion called "$I_3$" and it is stronger than the Wholeness Axiom. Through its critical point one can show the consistency of all large cardinals below $V_\lambda$, whereas the Wholeness Axiom shows the existence of all large cardinals. In going to the set version we will keep the, for us, relevant mathematical structure but loose the philosophical proximity to wholeness. Limiting $j$ to a set objectifies it, giving the impression that it is a "thing" that can be looked at from the outside. The movement is no longer from the whole mathematical universe, $V$, to itself but from the set $V_\lambda$ to itself.

Seeing $j$ as the set version of the Wholeness Axiom we define the binary operation of application, $\cdot$, as:

$$j \cdot j = \bigcup_{\alpha < \lambda} j(j \cap V_\alpha).$$

(3.7)

We need this kind of definition in order for application to make sense. Normally we could not apply $j$ to itself because $j$ is not an element of the domain $V_\lambda$. However, seen as a function, in terms of a set of ordered pairs, we see that it is a subset of $V_\lambda$. Due to elementarity $j \cdot j : V_\lambda \rightarrow V_\lambda$ is also an elementary embedding. We shall some times omit explicitly writing the application sign - and use parenthesis to indicate the order in which the embeddings are applied to each other, so for instance $j \cdot (j \cdot j)$ will be written $j(jj)$. We note that application is different from composition, which it is also possible to to define for elementary embeddings. Exactly how it is different we will see in the next section.

With $j$ we can define $A_j$ to be the closure of $\{j\}$ under application, thus containing terms like $jj, j(jj), j(j(jj)), (jj)j$ etc. $A_j$ is now our universal process algebra. It has been shown [49] that $A_j$ is a monogenic free left-
distributive universal algebra. To be monogenic means to be generated by only one element, which in this case is $j$. What being free and left-distributive means we will see in the coming sections.

3.2.2 Composition

The composition of two elementary embeddings such as $i, k \in V_\lambda$ also gives rise to an elementary embedding on $V_\lambda$. For all $x$ in the domain of $i$ we define the composition of $k$ with $i$ as:

$$k \circ i(x) = k(i(x)).$$

(3.8)

And so a requirement for the above composition is that the range of $i$ be in the domain of $k$.

We note that composition is in general different from application. When we compose two mappings $i$ and $k$ and evaluate them at some $x$ so that $i \circ k(x) = i(k(x))$ we can see this as first evaluating $k$ at $x$ and then evaluating $i$ at $k(x)$. And so it is essential that the range of $k$ be in $i$. The domain of $k$, however, need not be in $i$. When dealing with application $i \cdot k(x) = i(k(x))$ we no longer evaluate the mappings one at a time. Instead we require that all of $k$ be in $i$. As an example of how application and composition differ on elementary embeddings say that the critical point of $j$ is $\kappa$. Then $j(j(\kappa)) = \kappa \neq j(j(\kappa))$ (or the same thing written differently $jj(\kappa) = \kappa \neq j(j(\kappa))$). In fact, composition can be seen as a kind of conjugation with respect to application in the following sense. For $x$ in the image of $i$ we have:

$$i(k)(x) = i \circ k \circ i^{-1}(x).$$

(3.9)

To see that this is so take $i$ and $k$ to be elementary embeddings from $V_\lambda$ to $V_\lambda$ and take $k(y) = z$, where $y$ and $z$ are sets in $V_\lambda$. Then $i$ being an elementary embedding means that $i(z) = i(k(i(y))) = i(k(y))$ (or written in a different form $i(z) = ik \circ i(y) = i \circ k(y)$). Putting $x = i(y)$ and thus assuming that $x$ is in the image of $i$ we can write $y = i^{-1}(x)$. Replacing $y$ with $i^{-1}(x)$ in the previous equation we get $ik(x) = i \circ k \circ i^{-1}(x)$, which is what we wanted to show.

Trivial Part of Application

It is interesting to note that in this limited version of application $ik$ goes from $i''V_\lambda$ to $i''V_\lambda$. Thus, when interpreting the range as the possible explicate order (as suggested in section 3.1.5 above) we have a unitary mapping from an explicate order to an explicate order. It is unitary because limiting application in this way $ik(x)$ has an inverse, $x = (ik)^{-1}(i(z))$. In fact all elements in $A_j$ will have inverses, if we limit application in this way because they all can be written as some embedding applied to some other embedding, except for $j$. We could choose to see $j$ as the identity from $j''V_\lambda$ to itself. However, if
we were to start with this limited version of application then we would never generate $A_j$. This is so because even if we assume that $j$ is not the identity we would have: $jj(x) = jojo^{-1}(x) = j(x)$. Thus application would be idempotent. Because it is unitary and fails to generate $A_j$ we can say that the limited version of application, as in equation (3.9), corresponds to the trivial part of application.

3.2.3 Critical Points

It is of interest to know what happens with the critical point of the elementary embeddings when we apply or compose them. For $i$ and $k$ elementary embeddings, in the case of application we have that

$$\text{crit}(i(k)) = i(\text{crit}(k)), \quad (3.10)$$

while in the case of composition we have

$$\text{crit}(i \circ k) = \text{inf}(\text{crit}(i), \text{crit}(k)). \quad (3.11)$$

So we see that iterative composition does not add new critical points to $A_j$.

3.2.4 Left-distributivity

For $A_j$ to be left-distributive means to have the elements satisfy the identity

$$a \cdot (b \cdot c) = (a \cdot b) \cdot (a \cdot c), \quad (3.12)$$

where $a, b, c \in A_j$ and $\cdot$ denotes the application operation.

Where does left-distributivity come from? In our case left-distributivity can be seen as a consequence of elementarity [50]. Being elementary means that all mathematically definable formulas are preserved. Let us take a function $f(x) = y$, where $x, y, f$ are sets in $V$, and the non-trivial elementary embedding $j$. To be a function is definable and so $j(f)$ must also be a function. To be the image of a function is also definable and since all properties are preserved $j(y)$ should be the image of $j(x)$ when $j(f)$ is applied to it, meaning that $j(y) = j(f(x)) = j(f)(j(x))$. As we have already seen it is possible to define application on elementary embeddings and so $x, y, f$ could themselves be elementary embeddings. But this implies left-distributivity as $j \cdot (f \cdot x) = (j \cdot f) \cdot (j \cdot x)$.

Left-distributivity tells us how the different processes relate to each other. It will come to be essential as it is behind the geometry of braids [47], and we will make the connection with physics through braids. Left-distributivity turns up in braids, as we soon shall see in detail, but also as conjugation in groups. In fact, we have already seen that the so called trivial part of appli-
cation reduces application to conjugation. We called it trivial because it is unitary and thus cannot generate new elements in $A_j$. (We have also seen that composition does not generate any new critical points.) This is to stress the difference between left-distributive operations such as conjugation and application. The non-trivial left-distributive operations, such as application, have only been studied since the end of the 20th century.

3.2.5 Useful Properties of $A_j$

**Left Cancellative**

An LD-system $(S, \wedge)$ is left cancellative if for $a, b, c \in S$, $a \wedge b = a \wedge c$ implies $b = c$.

$A_j$ is left cancellative because by definition elementary embeddings preserve all relations. In particular, for $i, j, k \in A_j$, if $i = k$ then $j(i) = j(k)$. Being left cancellative also means that left translations are injections.

**Not Right Cancellative**

An LD-system $(S, \wedge)$ is right cancellative if for $a, b, c \in S$, $b \wedge a = c \wedge a$ implies $b = c$.

$A_j$ is not right cancellative as $j(jj) = (jj)(jj)$ does not imply that $j = jj$. In fact, this shows that every right cancellative LD-system has to be an idempotent LD-system. $A_j$ is not idempotent. If application were idempotent then we would never generate $A_j$ through application.

**Not Left Divisible**

Given an LD-system $(S, \wedge)$, for all $a, c \in S$ there is at least one $b \in S$ such that $a \wedge b = c$.

$A_j$ is not left divisible as there is for instance no $i \in A_j$ such that $(jj)i = j$, where $j$ denotes the generator. However, whenever left division is defined it is unique. And left division is defined on the elements of $A_j$ composed of right iterated powers of $j$, that is $j, jj, (jj)j, ((jj)j)j, ...$. Left division corresponds to surjective left translations.

3.2.6 Left-distributive Monoid Structure in $A_j$

A way of relating $A_j$ to physics is through braids. This is what we will discuss in this thesis. However, an alternative way of relating $A_j$ to physics could possibly be through the left-distributive monoid structure that we will present in this section. One could for instance attempt to link it to Lie groups and Lie algebras. An example of a comparison between the structure of the adjoints presented here and adjoints in Lie algebras is given in Appendix C. But first we shall introduce the left-distributive monoid.
A left-distributive monoid is a set with an associative operation, $\times$, and a left-distributive operation, $\cdot$, that fulfills the following relations:

\[
\begin{align*}
(a \times b) \times c &= a \times (b \times c) \quad (3.13) \\
a \times b &= (a \cdot b) \times a \quad (3.14) \\
(a \times b) \cdot c &= a \cdot (b \cdot c) \quad (3.15) \\
a \cdot (b \times c) &= (a \cdot b) \times (a \cdot c) \quad (3.16) \\
id \cdot a &= a, a \cdot id = id \quad (3.17)
\end{align*}
\]

where $a, b, c$ are any elements of the monoid and $id$ is the identity element. It is possible to complete our LD-system, $A_j$, into an LD-monoid by carrying the LD-structure to the monoid of left-translations and then carrying the LD-monoid structure back to $A_j$. The details of how this can be done are described here below.

**Left Translation**

An action of an LD-system such as $A_j = (A_j, \cdot)$, where $\cdot$ denotes application, on itself is given by left translation, $Ad_a : A_j \to A_j$, which is a mapping defined by an element $a \in A_j$ such that it takes any $b \in A_j$ to $a \cdot b$ [47](p.492).

**Adjoint Monoid**

The left adjoint of $A_j$ is denoted by $Ad(A_j)$ and is the submonoid of endomorphisms in $A_j$ generated by left translations [47](p.501). And so $Ad(A_j)$ together with composition is a monoid containing terms like $Ad_a, Ad_b \circ Ad_c$, etc...

**Adjoint LD-monoid**

Whenever we have a monogenic LD-system, such as $A_j$ it is possible to complete the left adjoint monoid with a left distributive operation, making it into an LD-monoid, $(Ad(A_j), \cdot, \circ)$ [47](p.503). The idea is to use the mapping $Ad : A_j \to Ad(A_j)$ which maps some $a \in A_j$ to $Ad_a \in Ad(A_j)$. We let it carry the left-distributive operation from the LD-system to the monoid so that $Ad_a \cdot Ad_b = Ad_{a \cdot b}$. In general the left-distributive operation will be uniquely given by:

\[
(Ad_{i_1} \circ \ldots \circ Ad_{i_p}) \cdot (Ad_{k_1} \circ \ldots \circ Ad_{k_p}) = \\
Ad_{i_{1 \cdot \ldots \cdot k_1}} \circ \ldots \circ Ad_{i_{1 \cdot \ldots \cdot k_p}},
\]

(3.18)

As mentioned before we use the convention that when parenthesis are not explicitly shown they are all on the right, so $a \cdot b \cdot c$ is to be interpreted as $a \cdot (b \cdot c)$. 

37
We can now carry the LD-monoid structure back to $A_j$. The easiest way would be to use the mapping $Ad^{-1}$. $A_j$ is left cancellative and so $Ad$ is injective. However, $A_j$ is not left divisible, and therefore $Ad$ is not surjective. The image of $Ad$ is not closed under composition. There is, for instance, no element in $A_j$ that will give us $Ad_a \circ Ad_b$ as the image of $Ad$. As a result we cannot use $Ad^{-1}$ to carry the LD-monoid structure to $A_j$. We can, however, use the evaluation function from the adjoint LD-monoid to $A_j$ at $j$, so that $eval_j : Ad_{i_1} \circ \ldots \circ Ad_{i_n} \mapsto (i_1 \cdot (i_2 \cdot \ldots (i_n \cdot j)) \ldots)$. The evaluation function generates a left ideal, $I_L(A_j, j)$, on $A_j$. By defining the following two operations on the left ideal

$$\cdot_l : (i_1 \cdot \ldots \cdot i_n \cdot j) \cdot_l (k_1 \cdot \ldots \cdot k_p \cdot j) = (i_1 \cdot \ldots \cdot i_n \cdot k_1) \cdot \ldots \cdot (i_1 \cdot \ldots \cdot i_n \cdot k_p) \cdot j$$

$$\circ_l : (i \cdot j) \circ_l (k \cdot j) = i \cdot k \cdot j$$

we have that $(I_L(A_j, j), \cdot_l, \circ_l, j)$ is an LD-monoid, which is isomorphic to the left adjoint LD-monoid, and is called a $j$-coadjoint (47)(page 507). For a visual representation of $A_j$, the adjoint monoid and LD-monoid see figure 3.1.

![Figure 3.1: The LD-system $A_j$, its adjoint monoid $Ad(A_j)$ and the LD-monoid isomorphic to the left ideal $I_L(A_j, j)$.]
3.2.7 Freeness

For a system to be free can be seen to mean the following [47](p 178). Let us see \(S\) and \(S'\) as two systems satisfying the same set of algebraic identities. We could for instance see them as two LD-systems. Then some set \(X \in S\) is free with respect to \(S\) if for any \(S'\) and any mapping \(f : X \to S'\) there is a homomorphism from the subsystem of \(S\) generated by \(X\) to \(S'\). If \(X\), besides being free, generates \(S\), then it is a basis in \(S\). \(S\) is then free. This means that there is a unique way of extending \(f : X \to S'\) to the system generated by \(X\). Also, all free systems satisfying the same algebraic identities and based on sets of equal cardinality are isomorphic to each other.

3.2.8 Linear Orders on \(A_j\)

There are at least two linear orders that can be defined on \(A_j\). One is a lexicographical order and comes from the large cardinals, from the critical points of the elementary embeddings on \(A_j\). The other is a linear order that can be seen to come from the left-distributive structure. This order has been found to be of importance in applications such as braids, where it gives rise to the order on braids.

**Lexicographical Order on \(A_j\)**

A lexicographical order is the order that we find in the dictionaries where we compare words by looking at the first letters. If the letters are equal we go to the next ones and compare those and so on until we find two that do not match, in which case we order the words following the order of the alphabet. Mathematically a lexicographical order can be seen as the natural order of the cartesian product. So, given two ordered sets, \(A\) with \(a,a' \in A\) and \(B\) with \(b,b' \in B\), we have that \((a,b) \leq (a',b')\) if and only if \(a < a'\) or both \(a = a'\) and \(b \leq b'\). If the sets \(A\) and \(B\) are well-ordered then the well-ordering will be preserved by the Lexicographical order. A set is well-ordered if every one of its subsets has a least element. An example of a well-ordered set is any set of the ordinals.

Let \(crit(i)\) denote the critical point of the embedding \(i \in A_j\) and \(crit(A_j)\) denote the set of all the critical points in \(A_j\), that is

\[
\text{crit}(A_j) = \{\text{crit}(i) \mid i \in A_j\}.
\]

The Lexicographical order is based on the following two facts. Firstly, that if \(i,k \in A_j\) and \(i\) restricted to \(\text{crit}(A_j)\) is equal to \(k\) restricted to \(\text{crit}(A_j)\), then \(i = k\) [51]. And furthermore, \(\text{crit}(A_j)\) are ordinal numbers and therefore orderable. Let \(e : \omega \to \text{crit}(A_j)\) be the increasing enumeration of the critical points. Now we can define the order.
Definition (Lexicographical order) For \(i, k \in A_j\) we have that \(i <_{\text{Lex}} k\) if for the least \(n \in \omega\) such that \(i(e(n)) \neq k(e(n))\) \(i(e(n)) > k(e(n))\).

By defining it in this way we get that if \(\text{crit}(i) < \text{crit}(k)\) then \(i <_{\text{Lex}} k\). This is because if \(\text{crit}(i) < \text{crit}(k)\) then \(i\) will move its critical point while \(k\) will not, so \(i(\text{crit}(i)) > k(\text{crit}(i)) = \text{crit}(i)\). If the embeddings happen to have the same critical point then the embedding that moves \(e(n)\) the highest will be before the other embedding. As an example let us take \(j\) and \((jj)j\). We have that \(\text{crit}(j) = \kappa = \text{crit}((jj)j)\). We evaluate the embeddings at \(e(1)\), \(j(e(1)) = j(\kappa) > \kappa\) and \(((jj)j)(e(1)) = ((jj)j)(\kappa) = j(j(\kappa)) > j(\kappa) > \kappa\). And so \(((jj)j)(\kappa) > j(\kappa)\), hence \((jj)j <_{\text{Lex}} j\). An extreme case of this scenario occurs when \(i\) and \(k\) not only agree on the critical point but have the same critical sequences. Here is an example of how some elements would be ordered:

\[
(jj)j <_{\text{Lex}} j <_{\text{Lex}} jj <_{\text{Lex}} j(jj) <_{\text{Lex}} ((jj)j)(jj) <_{\text{Lex}} j(j(jj))
\]  

(3.22)

The Lexicographical ordering is order isomorphic to \(\omega\). This means that there is an order preserving mapping between the two, in particular a bijection \(f : A_j \rightarrow \omega\) such that for \(i, k \in A_j\) we have \(i \leq k\) if and only if \(f(i) \leq f(k)\).

Left-distributive Order on \(A_j\)

There is a linear order on \(A_j\), denoted by \(<_{LD}\) that can be seen to come from left-distributivity. To define it we need the notion of a left subterm. A term is an element in \(A_j\). Unless the term is \(j\) it is possible to divide it up into subterms. For instance, two subterms of \((jj)\) are \(j\), which is a left subterm, and \(jj\), which is a right subterm. An embedding like \((j(jj))(jj)\) has two left subterms. The first one is \(j(jj)\), while the second one is \(j\). When there are several left subterms we can call them iterated. So that for \((j(jj))(jj)\) we have that \(j(jj)\) is the first iterated left subterm while \(j\) is the second iterated left subterm. With iterated left subterms we now go to the linear order.

Definition (Left-distributive order) For \(i, k \in A_j\) we say that \(i <_{LD} k\) if there are some \(i' = i\) and \(k' = k\) such that \(i'\) is an iterated left subterm of \(k'\).

As an example we can take the elements \(j(jj)\) and \(jj\). Since \(j(jj) = (jj)(jj)\) we see that \(j\) is a left subterm of \((jj)(jj)\), and so \(jj <_{LD} j(jj)\). Note that \(<_{LD}\) is related to left divisibility. Saying that \(i'\) is an iterated left subterm of \(k'\) we mean that there exist a \(p \geq 1\) and \(l_1, l_2, ..., l_p \in A_j\) such that \(k' = (...)l_1 l_2 ... l_p\). And so \(<_{LD}\) can be seen as extending the order gotten from left divisibility. Here is an example of how some elements would be ordered:

\[
j <_{LD} jj <_{LD} (jj)j <_{LD} ((jj)j)(jj) <_{LD} j(jj) <_{LD} j(j(jj))
\]  

(3.23)

40
$X = \langle j, jj, j(jj), j(j(jj)), \ldots \rangle$ is an example of a subset of $A_j$ on which the $<_{Lex}$ and $<_{LD}$ coincide. We note that $X$ is well-ordered. This is so because there is a unique correspondence between it and the critical sequence, $\langle \kappa, j(\kappa), j(j(\kappa)), j(j(j(\kappa))), \ldots \rangle$. The critical sequence is ever increasing and well-ordered because it is a set of ordinals. For $X$, a critical point uniquely identifies the corresponding elementary embedding, and so we can say that we have a well-ordering on $X$ in $A_j$.

However, when extended to all of $A_j$ the two orders no longer coincide and $<_{LD}$ is itself not well-ordered. Looking at all of $A_j$, and so permitting terms like $(jj)j$, we see that we add new critical points that can be found between the critical points of the critical sequence. For instance, the critical point of $((jj)j)(jj)$ is strictly between $j(j(\kappa))$ and $j(j(j(\kappa)))$. We are also adding degeneracy, as, for instance, in $crit(j) = crit((jj)j) = \kappa$. And so for $A_j$ the critical point of an elementary embedding no longer uniquely identifies that embedding (as there may be degeneracy). At this point one could wonder if perhaps $<_{LD}$ is a way of splitting up the degeneracies and extending the ordering so that it will agree with the Lexicographical ordering. However, it is not. The two orders are different (as can be seen by comparing 3.22 with 3.23).

To see that $<_{LD}$ is not well-ordered we can construct an infinitely decreasing set. Such a set can be constructed starting with any element of the form $a(bc)$, where $a, b, c \in A_j$. Since, by left-distributivity, we have that

$$a(bc) = [((ab)(ab))][(ab)a]$$

we see that $[((ab)(ab))]$ is a left subterm of $a(bc)$ and so $[((ab)(ab))] <_{LD} a(bc)$. But $[((ab)(ab))]$ is of the same form as $a(bc)$ and can itself be expanded until we again reach a term that has a left subterm of the form $a(bc)$. That left subterm will then be smaller than $[((ab)(ab))]$. This process can be continued infinitely, meaning that there is no smallest element and that the set is therefore not well-ordered.

An interesting result pertaining to the left-distributive order [49] is that for $a, b \in A_j$ we have that $b <_{LD} a \cdot b$. This result could be useful in an attempt to connect application with time evolution, as suggested in the discussion.

**Discussion on the Orders**

Is there a relation between the two orders? To start with, we emphasize that it is not necessary to have elementary embeddings in order to have a structure that is left-distributive. We can choose to solely focus on the algebraic properties and, for instance, examine universal algebras that are isomorphic to $A_j$, but where the elements are not necessarily elementary embeddings. Similarly it is possible to have other sets that are order isomorphic to $\omega$ but not left-distributive. So the two orderings can certainly be studied independently.
However, it would be instructive to see how they can be related to each other. Here, we shall only mention that they can both be traced back to elementarity. From the section on left-distributivity we see that left-distributivity can be seen as a consequence of elementarity. How is elementarity related to the critical points of the embeddings, crit(\(A_j\))? A critical point, call it \(\kappa\), is the first ordinal that is beyond strictly definable properties. It contains something unknowable, implicate, and has therefore the ability of being moved by the embedding to something greater. Why is this so? Elementarity prescribes that all definable properties are to be preserved. And so the empty set will have to be preserved, and taken to the empty set. A first element will have to be preserved and taken to a first element. Continuing this way we see that any countable set will have to be taken to itself. Moving up higher in the mathematical universe we will reach the large cardinals. These can be defined by taking a certain definable property, for instance one that differentiates a finite set from an infinite one, and reinforcing it or using it at a higher level. So, for instance, an infinite set like the set of all natural numbers, \(\omega\), cannot be accessed from below, by taking the power set or union of its subsets. So \(\omega\) is an inaccessible cardinal. This property of inaccessibility can then be used on higher levels, resulting in inaccessible large cardinals. And so \(j\) also has to preserve all those properties, on the basis of which the different large cardinals are defined. Having to contain all the definable properties, yet being the least cardinal moved, it can be seen as the breaking point between the determined and undetermined. We note that it is only beyond the determined that movement can occur, thus meaning that some set \(\gamma\) can only be moved to something greater if \(\gamma \geq \kappa\).

The Importance of Orderability
It is already known [52] that \(<_{LD}\) has important consequences particularly when it comes to braids. The \(<_{LD}\)-order can be extended to the whole braid group (see section 3.3.5).

The \(<_{Lex}\)-order, on the other hand, can be used in \(A_j\) (with a corresponding alternative for braids) to define interesting structures such as for instance a metric.

Metric on \(A_j\)
Given the existence of linear orders on \(A_j\) it is possible to define a metric on it. Based on the Lexicographic order we can define a metric on \(A_j\) in the following way [53]. Let, as before, \(crit(A_j)\) be the set of all possible critical points generated by \(j\) through application. Then for each elementary embedding \(i \in A_j\) let there be an enumerating one to one function \(e(n) : \omega \rightarrow crit(A_j)\). Now we can define the distance between two embeddings \(i\) and \(k\),
written \( d(i,k) \) to be:

\[
d(i,k) = \begin{cases} 
0 & \text{if } i = k \\
\frac{1}{i + m} & \text{otherwise}
\end{cases}
\]  

(3.25)

where \( m \) is the smallest natural number for which \( i(e(m)) \neq k(e(m)) \). This distance function is a metric.

### 3.3 Braids

As it turns out, the process algebra \( A_j \) is isomorphic to so called special braids. It is through this connection with braids that we will physically interpret our process algebra. Therefore we start with an introduction to braids.

#### 3.3.1 Braid group, \( B_n \)

Braids are interweaved strands numbered and fastened at the ends. They can be seen as being made up of crossings. One usually denotes the crossing of braid strand \( i + 1 \) over braid strand \( i \) as \( \sigma_i \). The inverse crossing, that is of braid strand \( i \) over braid strand \( i + 1 \), is denoted \( \sigma_i^{-1} \) (see Figure 3.2).

\[
\begin{array}{c}
\sigma_i \\
\sigma_i^{-1}
\end{array}
\]

Out of these crossings it is now possible to build representations of braids by adding crossings below. Mathematically we introduce a product, so that, for instance, \( \sigma_i^2 \sigma_i^{-1} \) represents the braid diagram drawn in Figure 3.3. \( \sigma_i^p \sigma_i^{-p} \) is called a braid word. A braid \( b \in B_n \) is an equivalence class of all braid words representing it. A particular braid can thus be represented by a braid word. However, there is an infinite number of braid words that represent the same braid. As an example take the trivial braid, 1, in an \( n \)-strand system. It can, for instance, be represented by any one of the braid words \( \sigma_i^p \sigma_i^{-p} \), where \( i \in \{1,2,\ldots,n-1\} \) and \( p \) is any integer.
With the product on braid crossings one generates a group structure on the braids. The braid group, $B_n$, was introduced in 1925 by Artin [54]. It is generated by crossings, $\sigma_i$ and $\sigma_i^{-1}$, the unit braid 1, and the braid relations:

\[
\begin{align*}
\sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \\
\sigma_i\sigma_i^{-1} &= 1 \\
\sigma_i\sigma_j &= \sigma_j\sigma_i \quad \text{for} \ |i - j| > 1.
\end{align*}
\]

(3.26) (3.27) (3.28)

See Figure 3.4 for a picture of the braid relations. It is interesting to note that if we add the conditions $\sigma_i^2 = 1$ then the braid group turns into the symmetric group, describing the permutations.

3.3.2 Braids, Links, and Knots

Tying together the ends of a braid we get a link. In fact, every link can be gotten as the closure of a braid [56] (see Figure 3.5). Links are thus in

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Although the German mathematician Adolf Hurwitz is said to have used it in his work already in 1891 [55].
general made up of several loops that interweave. If there is only one loop have a knot, or an unknot. An unknot can be untangled into a simple loop without cutting the strand. In the study of knots the Reidemeister moves \([57]\) are essential. All moves on knots that do not involve cutting the strands can be formulated in terms of these three moves (see Figure 3.6).

\[
\begin{align*}
R_I & \equiv \begin{array}{c}
\text{IDEMPOTENCY}
\end{array} \\
R_{II} & \equiv \begin{array}{c}
\text{INVERSES}
\end{array} \\
R_{III} & \equiv \begin{array}{c}
\text{LEFT-DISTRIBUTIVITY}
\end{array}
\end{align*}
\]

**Figure 3.6:** The three Reidemeister moves. RI–idempotency, RII–inverses, and RIII–left-distributivity

### 3.3.3 Coloring of Strands

Braids and knots are often studied in terms of colorings. That is, one imagines that there is a certain set of colors available, call it \(A = \{a_1, a_2, \ldots, a_x\}\). These colors are used to color the strands of the braids. If the strands never change color, then by looking at the initial and final sequence of braid strand colors we only take into account the permutations of the strands. That is, we note that the strands have been permuted but we do not know how they were permuted.
From that information we cannot reconstruct the braid. In order to recover the braid as well we allow the colors to change. In particular, we introduce a binary operation $\wedge$ and say that if the initial colors of the braid strands number $i$ and $i+1$ are $a_i$ and $a_{i+1}$, then after the crossing $\sigma_i$ they will be $a_i \wedge a_{i+1}$ and $a_i$. Thus the braid strand crossing under another braid strand keeps its color while the one crossing over another strand changes the color to $a_i \wedge a_{i+1}$.

What kind of an operation is $\wedge$? From the first braid relation (or equivalently Reidemeister move III) we see that it has to be left-distributive

$$a \wedge (b \wedge c) = (a \wedge b)(a \wedge c)$$ (3.29)

(see Figure 3.7).

Figure 3.7: Left-distributivity from Reidemeister move III. $a, b, c$ represent the initial colors of the braid strands. We see that for the two braids to be equivalent we must have $a \wedge (b \wedge c) = (a \wedge b)(a \wedge c)$.

Thus the set of all colors $(A, \wedge)$ is a left-distributive system. It is possible to see positive braids as acting on sequences of colors taken from a left-distributive system. A positive braid is one that can be represented by a braid word that does not contain any inverse crossings, no $\sigma_i^{-1}$'s. For a sequence of colors $\bar{a} = (a_1, ..., a_n)$ with the colors taken from a system $(A, \wedge)$ Dehornoy defines the action of the positive braid word $\sigma_i b$, where $b$ is a product of positive braid crossings, on the sequence as $\bar{a} \cdot \sigma_i b = (a_1, ..., a_i \wedge a_{i+1}, a_i, a_{i+2}, ..., a_n) \cdot b$, and for $\varepsilon$, the empty braid word, $\bar{a} \cdot \varepsilon = \bar{a}$ [47](p 13). It is possible to extend the action of positive braids, $B_n^+$, on the $n$-th power of an LD-system to the action of all braids, $B_n$, on the $n$-th power of an LD-system. That, however, requires the LD-system to be left divisible. This is because in $B_n$ we also have inverse crossings and we have the second braid relation (or RII) that we have to satisfy. And so, as shown in figure 3.8, if $a$ and $b$ are any two colors at the top of the braid, then the inverse crossing will give us some color $c$. Followed by an ordinary crossing we end up with the colors $b \wedge c$ and $b$. Given invari-
ance under RII we see that \( a = b \land c \). And so there has to exist a unique \( c \) such that \( a = b \land c \). But this is left division. Thus the LD-system \((A, \land)\) has to be left divisible.

![Figure 3.8](a b c = a b)

*Figure 3.8:* Left division. For any \( a, b \in (A, \land) \) there has to be a unique \( c \) such that \( a = b \land c \).

When it comes to the inverse crossing one usually introduces another binary operation, \( \lor \). Now we say that if the initial colors of the braid strands number \( i \) and \( i + 1 \) are \( a_i \) and \( a_{i+1} \), then after the crossing \( \sigma_i^{-1} \) they will be \( a_{i+1} \) and \( a_i \lor a_{i+1} \lor a_i \). Thus, like in the case of \( \sigma_i \), the braid strand crossing under another braid strand keeps its color while the one crossing over another strand changes the color. This operation is also left-distributive (see Figure 3.9).

![Figure 3.9](a b c = a b)

*Figure 3.9:* Left-distributivity from Reidemeister move III. \( a, b, c \) represent the initial colors of the braid strands. We see that for the two braids to be equivalent we must have \( c \lor (b \lor a) = (c \lor b) \lor (c \lor a) \).

Furthermore, in the same fashion as shown above, invariance under the second of the braid relations (or Reidemeister move II), gives us the identities

\[
a \land (a \lor b) = a \lor (a \land b) = b,
\]

where \( a, b, c \) are colors. Finally, Reidemeister move I gives us idempotency

\[
a \land a = a \lor a = a.
\]

A system \((A, \land, \lor)\) that is invariant under the three Reidemeister moves
Table 3.1: Examples of special braids

<table>
<thead>
<tr>
<th>Expression</th>
<th>Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1^\wedge 1$</td>
<td>$\sigma_1$</td>
</tr>
<tr>
<td>$1^\wedge (1^\wedge 1)$</td>
<td>$\sigma_2 \sigma_1$</td>
</tr>
<tr>
<td>$1^\wedge (1^\wedge (1^\wedge 1))$</td>
<td>$\sigma_3 \sigma_2 \sigma_1$</td>
</tr>
<tr>
<td>$(1^\wedge 1)^\wedge 1$</td>
<td>$\sigma_1^2 \sigma_2^{-1}$</td>
</tr>
<tr>
<td>$(1^\wedge (1^\wedge 1))^\wedge 1$</td>
<td>$\sigma_2 \sigma_1 \sigma_2^{-1} \sigma_3^{-1}$</td>
</tr>
</tbody>
</table>

is called an idempotent LD-quasigroup or a quandle. A system $(A, \wedge, \vee)$ that is invariant only under Reidemeister moves II and III is called an LD-quasigroup. An LD-quasigroup is left cancellative and left divisible. Thus its left translations are bijections. And so it is possible to have an action of all braids, $B_n$, on the $n$-th power of an LD-quasigroup.

3.3.4 Special Braids, $B^{sp}$

Special braids [47](p.103) are of particular interest to us. Let us start with a definition of them. Let $B_\infty$ be the braid group with infinitely many braid strands. Let $sh$ be a shift endomorphism of $B_\infty$ that maps $\sigma_i$ to $\sigma_{i+1}$ for all $i$. We can now define $\wedge$ as the exponentiation on braids so that for braids $a, b \in B_\infty$ we have:

$$a^\wedge b = ash(b) \sigma_1 sh(a^{-1}).$$

(3.32)

See Figure 3.10. Special braids, $B^{sp}$, are then the closure of 1 under braid exponentiation, where 1 is the trivial braid. See the table 3.1 for examples of special braids.

![Figure 3.10: Picture of exponentiation on braids a and b where $a^\wedge b = ash(b) \sigma_1 sh(a^{-1})$.](image)

We have seen that braids can act on powers of LD-quasigroups, where we can interpret the elements of the LD-quasigroup as colors of braid strands. Special braids are left cancellative LD-systems. But they are not left divisible, as the LD-quasigroups are. However, when left division is defined it is unique.
This makes it possible to define a partial action of braids on left cancellative LD-systems, such as special braids. This means that we will have braids acting on sequences of (special) braids. For every braid \( b \) in \( B_n \) there will always be at least one sequence of special braids that the braid \( b \) will be able to act upon resulting in another sequence of special braids. Special braids are self-coloring braids. This means that if we let a special braid, \( b \), act on the sequence \((1, 1, 1, \ldots)\) we will get \((b, 1, 1, 1, \ldots)\) as the result (see Figure 3.11). All special braids are self-coloring and, in fact, all self-coloring braids are special [47].

Special braids can be written in terms of the \( \sigma_i \) and \( \sigma_i^{-1} \) operators. Even though they contain the inverse operators no special braid as a whole has an inverse, in special braids.

### 3.3.5 \( A_j \)’s Relation to Braids

We will now go into the reason why special braids are so interesting to us. This is because \( A_j \) is isomorphic to \( B^{sp} \), see Figure 3.12. Thus algebraically they could be seen as the same system. Thus application and braid exponentiation are operations of the same kind. We stress that they are different from operations like the left-distributive operations that we find in LD-quasigroups, or conjugation in groups. We have already seen that application is more than conjugation. In fact, conjugation can be seen as a trivial part of application.

At this point one may wonder what is lost in a transition from \( A_j \) to \( B^{sp} \).

We loose the large cardinal power. We only take into account the relational structure. We look at how processes relate to processes and disregard what they essentially are. We put aside the information that they are elementary embeddings with large cardinal power. We no longer see the elements as processes of the wholeness preserving movement whose innermost nature is wholeness, but only as processes in relation to other processes, or elements relating to other elements. In short, we loose the ontology.

### Order on Braids

We have already seen that there is a left-distributive linear ordering, \( \prec_{LD} \), on \( A_j \). Since \( A_j \) is isomorphic to \( B^{sp} \) this ordering also exists on special braids.
Figure 3.12: There is an isomorphism between the process algebra $A_j$ and special braids $B^{sp}$.

On special braids we have that for $a, b \in B^{sp}$, $a <_{LD} a^\times b$. We also have that $b <_{LD} a^\times b$.

Since it is possible to define a partial action of all braids, $B_n$, on special braids we can use this to extend the $<_{LD}$ to all braids. To start with we can extend $<_{LD}$ to a lexicographical version\(^9\) [47](p.108), call it $<_{seq}$, where we will compare sequences of special braids, $\vec{a}$, $\vec{a'}$. Thus $\vec{a} <_{seq} \vec{a'}$ if $a_i <_{LD} a'_i$ for the smallest $i$ when $a_i \neq a'_i$. Then for $b, b' \in B_n$ we will have that $b <_{LD} b'$ if $\vec{a} \bullet b <_{seq} \vec{a} \bullet b'$.

3.4 Kauffman’s Bracket

We have now seen what braids are and how they are related to the process algebra generated by the elementary embeddings. The next step will be to evince the connection between braids and physics. We shall do that by encoding the structure of braids into the plane and looking at planar algebras, whose structure will then be related to physics in the next chapter. A key concept will be a topological polynomial introduced by Kauffman [58] and called the bracket.

Let us start with Kauffman’s bracket for a link. First we need to find a way of encoding the structure of a three dimensional crossing into the plane. Following Kauffman we can imagine that when approaching a crossing on the

\(^9\)Not to be confused with the linear well-ordering $<_{let}$.
undermost strand the region on the left will be called $A$ while the region on the right will be called $B$ (see Figure 3.13). A crossing, $\sigma_1$, can now be spliced in two different ways, as $A\|$, uniting the two $A$-regions, or as $B\succ$, uniting the two $B$-regions (see Figure 3). Given any link we can say that a state, $s$, of the link is a particular choice of splicings at each crossing. Figure 3.14 shows a knot and a particular state of that knot. We will denote the number of loops in a link that has been encoded into the state $s$ by $\|s\|$. Thus in the example in Figure 3.14 $\|s\| = 2$. Finally, we will say that $<K|s>$ is the product of the labels $A,B$ when the link $K$ is in the state $s$. In Figure 3.14 $<K|s> = A^3$. Now the bracket can be defined as

$$<K> = \sum_s <K|s> d^{\|s\|}$$

where $d$ is the value of a loop. Usually one requires the bracket to be invariant under RIII and RII. This gives the conditions $B = A^{-1}$ and $d = -A^2 - A^{-2}$.

Two important properties of the bracket are that one is free to move the loops around. And so if $\bigcirc K$ denotes a knot with a loop next to it we have that $<\bigcirc K> = d <K>$. The other property is that if $\sigma_1$ denotes a crossing then $<\sigma_1> = A <\|> + B <\succ>$. As we shall see in the coming chapter, the bracket can be related to physics
in several ways. For instance, special cases of the bracket, that is assigning particular values to the $A, B,$ and $d,$ turn the bracket into a partition function for statistical physics models, such as the Potts model. Or, when $A = B = -1$ and $d = -2$ we get Penroses binor calculus, from which one can construct spin networks and attempt to derive the three dimensions of space.

**Kauffman’s Bracket for Braids**

The bracket can also be defined for braids. Since braids have free ends on top and bottom we need to close them in order to produce loops. This can be done by tying the first top end with the first bottom end, the second top end with the second bottom end and so on, see Figure 3.5. Then for any braid $b$ in $B_\infty$ (and so in particular also for any special braid) the bracket is given by:

$$\langle b \rangle = \sum_s \langle b|s \rangle d^{||s||}, \quad (3.34)$$

where $b$ is the closure of the braid $b$.

We shall now require that the bracket be invariant under RIII, as this corresponds to left-distributivity. This imposes the condition that $d = -A/B - B/A$ (see Appendix B for calculation). It is now interesting to note that invariance for the bracket under RIII automatically imposes invariance under RII up to a multiplication by $AB$. If $AB$ were equal to 1 we would have complete invariance under RII. This would imply $B = A^{-1}$ and that would reduce $d$ to $d = -A^2 - A^{-2}$. This is in fact what happens for the braid group, $B_n$, where besides RIII invariance we also have RII invariance.

Special braids can be written in terms of braid crossings $\sigma_i$’s and $\sigma_i^{-1}$’s. And so the braid crossings $\sigma_i$’s and $\sigma_i^{-1}$’s can be seen as the building blocks of special braids. For the building blocks we have RII invariance. However, when we look at the special braids themselves we see that no special braid has an inverse as a whole. We could still claim that we have RII invariance, just that we do not have the inverses to realize it when it comes to whole special braids. So we see that from the approach from wholeness it appears as though left-distributivity is more fundamental than unitarity. Unitarity only holds for the building blocks, not for whole processes.

### 3.4.1 Temperley-Lieb Algebra

When we splice a crossing we end up with elements like $||$ and $\times$. These types of elements can be seen as the generators of an algebra. For $n$ strands we will have the following generators $||...||_n = 1_n, \times ||...||_n = U_1, \times ||...||_n = U_2, \ldots, ||...||_n = U_{n-1}$. Multiplication can be seen as in braids, where we draw the element we multiply with at the bottom of the other element. From braids we will have the following relations

$$U_i U_{i \pm 1} U_i = U_i$$
\begin{align}
U_i^2 &= dU_i \\
U_i U_j &= U_j U_i \quad \text{for } |i - j| > 1.
\end{align}
(3.35)

See Figure 3.15 for a graphical display of these relations.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure3.15}
\caption{Illustration of generators of a Temperley-Lieb algebra and its relations.}
\end{figure}

The generators $U_1, U_2, \ldots U_{n-1}$ and the relations (3.35), where $d$ commutes with the generators, define a Temperley-Lieb algebra, $TL$ [59]. In our case $d$ is the loop value and thus $d = -A/B - B/A$. If we would have had invariance under RII as well, then we would have had a different loop value, namely $d = -A^2 - A^{-2}$. Since for some angle $\theta$ we have that $\cos(\theta) = (e^{i\theta} + e^{-i\theta})/2$ we could write $d = q + q^{-1}$ as $d = 2\cos(\pi/r) = e^{i\pi/r} + e^{-i\pi/r}$, where $r$ could be any complex number except for the rationals. Thus we could see $A/B$ as $-e^{i\pi/r}$.

Temperley-Lieb algebras are used in quantum and statistical physics. Their generators can for instance be used to build transfer matrices for different statistical mechanics models [60], [61]. They can also be seen as algebras of observables in quantum physics (see section 4.1.3).
4 The Physical Interpretation

“If we don’t change direction soon, we’ll end up where we’re going.”

(Irwin Corey [62])

The direction in which we have been moving so far has been from wholeness towards physics. Motivated primarily by the quantum theory we hypothesized that wholeness could be a fruitful foundation for physical theories. Consequently we expressed our ideas regarding wholeness in an ontology. Based on that ontology we then proceeded to formulate the dynamics of wholeness mathematically. The mathematical description lead us to a process algebra which turned out to be isomorphic with special braids. The key structure is the left-distributivity of the process algebra. This tells us how the processes relate to each other. Continuing, we saw that when encoded into the plane the special braids provide us with a Temperley-Lieb algebra.

Now, when faced with the task of physically interpreting the mathematical structures, we swiftly switch direction. Instead of going from wholeness towards physics we will move from physics towards the mathematical structures we have found to emanate from our conceptions of wholeness. In this section we will look at some examples of how braids and Temperley-Lieb algebras are used and interpreted in physics, focusing on statistical and quantum physics. As an example we will show how the Potts model can be reformulated in terms of Temperley-Lieb algebras. After those expositions the section will end by showing some of the more general structure that can be obtained from wholeness, leaving the interpretation open. A possible interpretation will be alluded to in the discussion chapter.

4.1 Braids and Physics

Braids turn up in physics in many places. They pop up not only in certain applications like the experimentally observed statistics of anyons in the Quantum Hall effect [63], or more theoretically as Yang-Baxter operators providing useful solutions for low dimensional quantum field theory [65] or lattice statistical physics [66], but also in the very fundamental structure of physics itself as in, for instance, the form of braided commutativity [67]. In fact, some mathematical physicists are studying braids and Hopf algebras are doing so in the
hope of unifying the quantum theory with general relativity [67], [68]. Other approaches to quantum gravity, such as for instance Loop Quantum Gravity [69], [70], also involve connections to braids.

We will first take a look at anyons, because this is the connection between braids and physics that most people are familiar with. That, however, will not be how we will choose to interpret our braids. Then, we will take a look at lattice statistical mechanics, von Neumann algebras, spin networks, quantum groups, and a general way in which braids are often interpreted.

4.1.1 Anyons

One of the perhaps best known examples of how braids can be used in physics is as the generalization of bosons and fermions to so called anyons [63], [64], which have been experimentally observed in the Quantum Hall effect. In three spatial dimensions it turns out that the only types of particles that can exist are either bosons, particles with integer spin such as the photon, or fermions, particles with half integer spin such as the electron. However, if we were to limit the motion of the particles from three spatial dimensions to only two, then it would be possible for the particles to have “any” spin, and thus one could call them anyons. Why is this so?

Imagine that we have a two dimensional world and two identical particles that we want to physically permute. There are two different ways in which this can be done (see Figure 4.1). The first possibility is to rotate particle 2 around particle 1 with the angle $\pi$ and then perform a translation. The second possibility is to rotate particle 2 around particle 1 with the angle $-\pi$ and then perform a translation. If we have a wave function describing these two particles given by $\Psi(1,2)$ then through the permutation the wave function will acquire an additional phase term. For possibility 1 $\Psi(1,2)$ will transform into $\Psi(1,2)e^{i\pi}$. For possibility 2 $\Psi(1,2)$ will transform into $\Psi(1,2)e^{-i\pi}$. The letter $n$ will give us the relevant statistics.

If we imagine that we are in more dimensions than two then we realize that the two different ways of exchanging the identical particles can no longer be distinguished. Looking from the dimension perpendicular to the plane of the particles we note that from below the plane possibility 1 looks like possibility 2 seen from the above, and vice versa. In other words, there is a continuous way of transforming possibility 1 into possibility 2 in three dimensions. This must mean that in three dimensions:

$$e^{i\pi} = e^{-i\pi}.$$ (4.1)

But this can only happen when $n = 0, 1 \pmod{2}$. The case where $n = 0$ corresponds to bosons while the case where $n = 1$ corresponds to fermions. And so we see that in three dimensions we only have bosons and fermions. The
relevant group when looking at the possible permutations is the symmetric permutation group. All we need to know is the initial positions and the final positions. We do not need to take into account the way in which the particles were permuted because they are all seen as equivalent.

Going back to two dimensions the equivalence no longer holds and we have to consider how the particles have been permuted. Thus the symmetric permutation group is no longer satisfactory. Having the initial and final positions is no longer sufficient. Looking at the $x$-$y$-plane from above we see two particles changing places. Looking from the side, along the $z$-axis, which represents the time direction, we see the world-lines of the two particles and these lines form braids as the particles rotate around each other. To describe this phenomenon we need the braid group. The braid group is a generalization of the symmetric group, as the addition of the condition $\sigma^2 = 1$ reduces it to the symmetric group.

Anyons are thus one way of seeing how the mathematical structure of braids can be used and interpreted in physics. Each braid strand corresponds to the world-line a particle in 2+1 dimensions. However, this is not the interpretation that we will use when interpreting our process algebra.

4.1.2 Lattice Statistical Physics

One particular connection between braids and statistical mechanics [71] is usually described in terms of knots, or more generally links. However, a link can always be written in terms of braids, as the closure of a braid. Links that
may be transformed into each other without the tearing of any strings, that is, using the Reidemeister moves [57], are said to be equivalent. It turns out that a link can be seen as a lattice on which we might, for instance, place spins either at the lattice sites (spin models) or at the edges (vertex models). The procedure is then to construct models such that all lattices corresponding to equivalent links give us the same partition function. This way the partition function is a link invariant. Let us take a closer look at a model that is widely used in solid state physics, namely the Potts model.

**The Potts Model**

The Potts model [72], [73] is primarily used in statistical mechanics to model the behavior of elements as they are allowed to interact with their neighbors. In the Potts model the state of an element can be assigned \(Q\) different values. A special case of the Potts model is the Ising model [74] when the states can only take two values, \(Q = 2\). This is, among other areas, useful in modelling systems of importance to solid state physics like spin models, binary alloys [75], lattice gas [76].\(^1\) One is usually interested in finding out information about phase transitions.

Let us have a closer look at the details of the Potts model and at how it is related to braids. Take a system represented by a graph, \(G\). One can imagine placing particles at the vertices and letting a particle at vertex \(i\) be in the state \(S_i\), whose value is chosen from the \(Q\) possible values. Let the edges, \(<i, j>\), represent the interactions between the particles. A particular state \(S\) of the system \(G\) is then a particular choice of \(S_i\)'s. Given \(n\) particles, or vertices in \(G\), there will be \(Q^n\) possible states.

The canonical partition function is given by \(Z_G = \sum_S e^{-E(S)\beta},\) where \(\beta = k_B T\), \(k_B\) Boltzmann’s constant, \(T\) the temperature; \(S\) the vertex state of the system; and \(E(S)\) the energy of the state. In the Potts model the energy \(E(S)\) is symmetric and given by \(E(S) = \sum_{<i, j>} \delta(S_i, S_j),\) with \(\delta\) being the Kronecker delta, and is thus equal to 1 if \(S_i = S_j\) and equal to 0 otherwise. The partition function is central for physical applications as all physically relevant quantities can be calculated from it. Given the partition function we can, for instance, calculate the probability of finding a system in a given state. The probability of finding the system related to \(G\) in the state \(S\) is \(\rho(S) = e^{-E(S)\beta}/Z_G\).

Thus the partition function is the key. The partition function can also be written as

\[
Z_G = \sum_S e^{-\sum_{<i, j>} \delta(S_i, S_j)\beta} = \sum_S \prod_{<i, j>} e^{\delta(S_i, S_j)\beta}.
\]  

\(^1\)However, the model has also found ample applications on the outskirts of physics as well as outside of physics, modelling things like the weather, neural networks, heart-cells, etc.
We can define \( \nu = e^\beta - 1 \). Substituting in \( \nu \) into 4.2 we get
\[
Z_G = \sum_S \prod_{<i,j>} (1 + \nu \delta(S_i, S_j)) = \sum_S \prod_{<i,j>} (1 + \nu \delta(S_i, S_j)).
\] (4.3)

The product over the edges can now be expanded into a sum, giving rise to \( 2^E \) terms, where \( E \) denotes the number of edges in the graph. And so for instance, if we have a graph with two vertices and two edges we have the product \((1 + \nu \delta_1)(1 + \nu \delta_2)\), which we expand into the following \( 2^2 \) terms: 
\[1 + 1\nu \delta_1 + 1\nu \delta_2 + \nu \delta_1 \nu \delta_2.\]
The terms in the expansion are now in a one-to-one correspondence with the possible configurations where each interaction bond has been assigned a 0 or a 1. We shall denote these possible configurations with a small \( s \). Note the difference between the small \( s \) and the large \( S \). The large \( S \) tells us what state the vertex elements are in. So, for instance, if we have a graph with two vertices, \( n = 2 \), two edges, \( E = 2 \), and the vertices can be in one of three possible states, \( Q = 3 \), then the number of possible vertex states is \( Q^n = 3^2 = 9 \). The number of possible edge configurations \( s \), however, is only \( 2^E = 2^2 = 4 \). Thus continuing with the example of a graph with two vertices and two edges we have that each term in the expansion of the product is made up of factors of 1 and of \( \nu \delta \)'s. 1 corresponds to an interaction bond that has been assigned the value 0, whereas \( \nu \delta \) corresponds to an interaction bond that has been assigned the value 1, see figure 4.2. Given a state \( s \) that

\[
\begin{align*}
1 & \quad 1\nu \delta_2 & \quad 1\nu \delta_1 & \quad \nu \delta_1 \nu \delta_2 \\
\text{Figure 4.2: Terms from the expansion of the product in a Potts partition function for a graph with two vertices and two edges, and schematic pictures of the corresponding configurations.}
\end{align*}
\]

has \( i(s) \) interaction bonds that are assigned the value 1 we will have a factor \( \nu^{i(s)} \). The delta functions will eliminate all the terms where the vertices are not aligned, not in the same state. Letting \( ||s|| \) denotes the number of clusters of vertices that are connected through interaction, for a certain state \( s \), we have that the number of contributing configurations will be \( Q^{||s||} \). And so we may rewrite the partition function as
\[
Z_G = \sum_s Q^{||s||} \nu^{i(s)},
\] (4.4)
where we are now summing over all the \( 2^E \) edge configuration states \( s \). This formulation of the partition function is also called a Whitney polynomial [77] or a dichromatic polynomial [78] and has the coloring [79] and percolation

59
problems as a special case. It is also possible to generalize this formulation so that models with different types of interactions may be included. A rectangular lattice might for instance have two different energies of interactions, $\varepsilon_v$ and $\varepsilon_h$, pertaining to vertical and horizontal bonds. This would give us $\nu_v = e^{\beta \varepsilon_v}$ and $\nu_h = e^{\beta \varepsilon_h}$ and consequently $Z_G = \sum_s Q^{i_v(s)} v_v^{i_v(s)} v_h^{i_h(s)}$, where $i_v(s)$ and $i_h(s)$ are the numbers of vertical and horizontal interaction bonds assigned value 1.

Let us now review a procedure of relating graphs to links. This procedure originates in [80] and [58](p.355). Starting with a graph, $G$, with vertices representing some elements that can be found in $Q$ states and edges representing interactions we can associate a so called universe, $U$, (or medial graph [81], or surrounding lattice [80]) which can be seen as the shadow cast by a link. This means that at each interaction, or edge in the graph $G$, we place the shadow of a crossing (see Figure 4.3). Then we draw a link in such a way that each vertex of $G$ becomes enclosed without sharing a link line with any other vertex of $G$. To make it visually more clear we shade each area of the universe $U$ that contains a vertex of $G$ in it (see Figure 4.3).

Each crossing in $U$

![Figure 4.3: Procedure of going from a graph $G$ to a universe $U$.](image)

now corresponds to an interaction in $G$. The existence of an interaction can be coded by the splicing of a crossing in the following way. If there is no interaction bond we denote it by a dotted line and separate the shaded areas. If there is an interaction bond we denote it by a whole line and join the shaded areas (see Figure 4.4).

![Figure 4.4: Two ways of splicing a shaded crossing.](image)

When all the crossings have been spliced we will end up with shaded areas such as the examples in Figure 4.5. This means that we can re-write the partition function for the universe $U$ instead of the graph $G$. Starting with a universe $U$ we shade it so that the outside, the background, is white and each edge is a boundary between a shaded and a white area. Then the partition function for $U$ becomes

$$Z_U = \sum_s Q^{i_v(s)} \nu_v^{i_v(s)} \nu_h^{i_h(s)}$$

(4.5)
where for a particular configuration, or spliced state $s$ we have that $|s|$ denotes the number of shaded regions and $i(s)$ the number of internal vertices of $U$, or equivalently the number of splicings where the shaded areas were joined. See Figure 4.6 for a comparison between starting from a graph $G$ and starting from a universe, $U$.

From equation (4.3) we see that the partition function has the form

$$Z\left(\begin{array}{c}
\end{array}\right) = Z\left(\begin{array}{c}
\end{array}\right) + \nu Z\left(\begin{array}{c}
\end{array}\right)$$

where the shaded ring denotes a shaded area and $K$ is a shaded link diagram. This is very reminiscent of Kauffman’s bracket for a link. If we choose to call
the shaded areas A and the white areas B (see Figure 3.13), then we can translate a shaded crossing into a braid crossing (see Figure 4.7). This way we can translate the shaded version of a link into an alternating link diagram. An alternating link diagram is one where a link strand alternates between passing a crossing above and below. In order to operate with states that are no longer shaded we will have to translate the number of shaded areas, \(|s|\), into the number of loops, or boundaries of the shaded areas, \(|s|\). If \(n\) is the number of vertices in the graph, or equivalently shaded areas in the universe before it is split, then we have that \(|s| = \frac{1}{2}(n - i(s) + |s|)\) [58](p.368). Replacing \(|s|\) with \(|s|\) the partition function for the Potts model now becomes

\[
Z_U = \sum_s Q^{\frac{1}{2}(n - i(s) + |s|)} = Q^{\frac{1}{2}n} \sum_s Q^{\frac{1}{2}|s|} / Q^{\frac{1}{2}} \nu^i(s).
\]

(4.6)

It can be written as \(Z_U = Q^{n/2} W_k\), where \(W_k\) is the Potts bracket [58](p.371) for the link \(K\) obtained from the universe \(U\). For the Potts bracket we have

\[
\begin{align*}
\mathbf{w}(\begin{array}{c}
\times \\
\end{array}) & = \mathbf{w}(\begin{array}{c}
\mid \\
\end{array}) + \nu^{1/2} \mathbf{w}(\begin{array}{c}
\bigcirc \\
\end{array}), \\
\mathbf{w}(\begin{array}{c}
\times \\
\end{array}) & = \nu^{1/2} \mathbf{w}(\begin{array}{c}
\mid \\
\end{array}) + \mathbf{w}(\begin{array}{c}
\bigcirc \\
\end{array}), \\
\mathbf{w}(\begin{array}{c}
\bigcirc \\
K
\end{array}) & = \nu^{-1/2} \mathbf{w}(K).
\end{align*}
\]

It is now possible to establish a direct connection with a Temperley-Lieb algebra. If we take the Potts bracket of a braid we will end up with states that are elements of a Temperley-Lieb algebra, where the Temperley-Lieb relations (see equations 3.35 or Figure 3.15) are respected. And so this means that given any graph \(G\) we can rewrite it in terms of a Temperley-Lieb algebra.

In the mathematics chapter, chapter 3, we saw how we can go from a dynamical wholeness to a Temperley-Lieb algebra. In particular we saw how the proposed dynamics of wholeness generates a universal left-distributive algebra that can be interpreted as special braids which when encoded into the
plane are a Temperley-Lieb algebra. In this section we saw how we can go from statistical physics to Temperley-Lieb algebras. In particular we saw how the Potts model can be expressed in terms of Temperley-Lieb algebras.

Transfer Matrices

The transfer matrix is a tool that is often used in calculating partition functions. The idea is to build the partition function from partition functions of the parts in a way that exploits the symmetries of the system, so that its dimension can be diminished. For instance, many two dimensional models can be seen as being made up of square matrices. A square matrix can be divided up into rows and columns. And so the partition function can be built in layers. An n-layer transfer matrix for the Potts models is given by

\[ T = \left( \prod_{i=1}^{n} (1 + vU_{2i-1}) \right) \left( \prod_{i=1}^{n-1} (v + U_{2i}) \right), \quad (4.7) \]

where the \( U_i \)’s are elements of a Temperley-Lieb algebra. Realizing this one can take advantage of the symmetries inherent in the algebraic system and use them to reduce the dimension of the problem.

4.1.3 Quantum Physics

The connection between braids and physics is not limited to statistical mechanics. Depending on features such as global properties and discreteness, the topology of braids can also be connected with quantum features. In fact, the connection between statistical mechanics and knot theory was first discovered by Jones [82] by noting the similarities between von Neumann algebras and the braid relations. A von Neumann algebra is a special type of \( C^* \)-algebra, used in algebraic approaches to the quantum theory such as [83]. Jones derived a link invariant, called the Jones polynomial, which through the connection with von Neumann algebras can be seen as providing expectation values. In this section we shall briefly talk about von Neumann algebras, spin networks, and quantum groups, as they are all related to braids.

von Neumann Algebras

von Neumann algebras are interesting from a physics point of view because they can be seen as algebras of observables in quantum physics [83]. Technically, a von Neumann algebra can be defined as the algebra of bounded linear operators on a Hilbert space that has an involution and is closed in the weak operator topology. It turns out that von Neumann algebras can be split into three types, or factors. Type I are those where the trace of the projection operators is an integer. The trace of a projection operator can be seen as giving the dimension of the space onto which the projection is done. Type I algebras
give us algebras with a trivial center, the center being all the commuting operators. In quantum field theories they are related to bosons [84](p.3). Type II algebras are those where the trace of the projection operators can take any value in $[0, \infty]$. They no longer have a trivial center. In fact, the center can be seen as being structurally related to classical physics, where all the operators commute. In quantum field theories type II algebras are related to fermions [84](p.75). The last type of von Neumann algebras, type III, are algebras where the trace is either 0 or infinity. These turn up in local quantum field theories [83] where one works with operators corresponding to transformations on a local region of space-time.

When it comes to relating von Neumann algebras to structures found from wholeness, the Temperley-Lieb algebras are closest at hand. It turns out that a Temperley-Lieb algebra can be seen as towers of von Neumann algebras [85]. Such a tower, which can in itself be seen as a von Neumann algebra, is generated by the elements $e_1, e_2, \ldots, e_{n-1}$ and the relations:

\begin{align*}
    e_i e_{i \pm 1} e_i &= \tau e_i \\
    e_i^2 &= e_i \\
    e_i e_j &= e_j e_i \quad \text{for} \quad |i - j| > 1. \quad (4.8)
\end{align*}

Where $\tau$ is a variable that commutes with all the elements. Putting $e_i = d^{-1} U_i$ we see that the von Neumann relations and Temperley-Lieb relations (3.35) become equivalent. And $\tau = d^{-2}$. And so, the Temperley-Lieb algebra is isomorphic to a tower algebra of von Neumann algebras. The Temperley-Lieb algebra seems to provide a way of ordering algebras of observables.

It is further interesting to note that one can decompose Kauffman’s bracket into two mappings. Using the bracket for special braids, one mapping is $\rho : B^{lp} \rightarrow TL$ and another called the trace $tr : TL \rightarrow Z[A, B]$, where $Z[A, B]$ is the ring of polynomials in $A$ and $B$. We define $\rho(\sigma_i) = A + BU_i$, and for any braid $b \in B^{lp}$ we have that

\begin{equation}
    \rho(b) = \Sigma_s < \overline{b}|s > U_s, \quad (4.9)
\end{equation}

where $\overline{b}$ is a closure of the braid $b$ and $< \overline{b}|s >$ is the product of $A$’s and $B$’s when the braid $b$ is in the state $s$. The trace is then

\begin{equation}
    < b > = tr(\rho(b)) = \Sigma_s < \overline{b}|s > < U_s >. \quad (4.10)
\end{equation}

As an example we have that

\begin{align*}
    < \sigma_1 > &= A < || > + B < \infty > = A d + B \\
    &= A(-\frac{A}{B} - \frac{B}{A}) + B = -\frac{A^2}{B}.
\end{align*}

And so we see that the bracket is structurally similar to the trace of a density
matrix in the quantum theory, which is of interest as it provides expectation values of observables.

Spin Networks

“You cannot have first space and then things to put into it, any more than you can have first a grin and then a Cheshire cat to fit on to it.” (Alfred North Whitehead [86])

In physics, space and time have traditionally played the role of an external reference frame into which things could be placed and within which they moved. Hence they provided an order, much like a coordinate system, within which we could describe the dynamics of objects. Newtonian physics, for example, postulated an absolute space and time in which mechanical bodies moved according to Newton’s laws. Since Einstein’s theories of relativity we have found space and time to play a more active role. It is no longer seen as absolute but as dependent on the observers’ frame of reference relative to the object of study. The quantum theory further complicated the issue of space and time by introducing quantization and suggesting that space and time may need to be quantized as well, in order to fit in with the theory of general relativity.

However, quantizing space and time we quantize the very order that is central to our descriptions. On what order should we base our physics descriptions instead? The suggestion in this thesis is, of course, to use the order given to us by wholeness. This would suggest the orders provided by the critical points or by left-distributivity (see section 3.2.8). By focusing on left-distributivity we are lead to Temperley-Lieb algebras, which are towers of von Neumann algebras, or ways of ordering algebras of observables in the quantum theories. However, in this section we shall look at another attempt of deriving the order of space and time and see that it is a special case of the approach from wholeness.

This approach was introduced by Penrose [87], [88] in the early 1970’s. The idea was to start with the combinatorial laws describing quantized angular momentum, given to us by the quantum theory, and see if they would produce something along the lines of three dimensional space. To do that one starts with entities having spin and builds spin networks from them. The approach turned out to be successful in recovering the topology of three dimensional space, but it is unclear how a natural metric structure should be recovered.

The entity with spin is called a spinor and is a complex two-vector. A spinor can live in the complex space \( \mathbb{C}^2 \). This is the appropriate space of a spin-1/2 particle. A vector in this space denotes the state of such a particle. This suggest the group \( SL(2, \mathbb{C}) \), the group of linear transformation on \( \mathbb{C}^2 \) with determinant 1, as the symmetry group. This is the group of \( 2 \times 2 \) matrices, \( M \),
that leave the matrix
\[ \epsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]
invariant. This means that \( M \epsilon M^T = \epsilon \), where \( M^T \) is the transpose of \( M \). \( SL(2, C) \) is the double cover of the Lorentz group, which is the group of symmetry operations on spacetime. When dealing with quantum theories it is convenient to use the double cover so that we can account for things like the electron requiring two turns to come back to its original state. This kind of a spinor is called a binor and the calculus of \( SL(2, C) \) invariant tensors is called binor calculus. Based on the binor one can proceed to built spin networks, which can be related to the topology of three dimensional space [89].

It turns out that binor calculus is a special case of Kauffman’s bracket. It is precisely the case when \( A = B = 1 \) and thus the loop value \( d = -2 \). We can write the binor identity symbolically as \( \times = |+\infty\). 

**Quantum Groups**

We have seen that the case of the bracket when \( A = 1 \) corresponds to the case when we have a set of operators that are invariant under \( SL(2, C) \), where \( C \) is the field of the complex numbers, but in general, from the braid perspective, it can be a commutative ring. One way of generalizing this is to drop the requirement of commutativity. This then leads to invariance under the so called quantum group \( SL(2)_q \) [90].

Quantum groups are generalizations of ordinary groups [91]. They are a special case of Hopf algebras, in particular they are deformed Hopf algebras (so that they are no longer commutative and depend on some constant, in physics often being Planck’s constant or the cosmological constant). To see what a Hopf algebra is one can start from a group \( G \) (usually a Lie group in physics applications). Being a group, \( G \) has multiplication, \( m \), an inverse, \( inv \), and a unit, \( u \):

\[
m : G \times G \to G \\
inv : G \to G \\
u : 1 \to G.
\]

Then one looks at all the functions \( F : G \to R \) (where \( R \) can in general be a commutative ring) and the algebra \( H \) that those functions form. The idea is then to induce homomorphisms by “pulling back” the functions, so that for instance, \( f : G \otimes G \to G \) becomes \( f* : F(G) \to F(G \otimes G) \). We get:

\[
m* : H \to H \otimes H \\
inv* : H \to H
\]
\[ u^* : H \rightarrow C, \quad (4.16) \]

where \( m^* \) is called co-multiplication, \( \text{inv}^* \) the antipode, and \( u^* \) the co-unit. With all of these \( H \) is now a Hopf algebra. In order to arrive at a quantum “group” we need deform it and make it depend on a constant. And so we deform it by saying that if \( p \) and \( q \) are the generators of \( H \), where \( pq = qp \), then we now decide that \( pq - qp = i\hbar \) (where \( i \) is the imaginary number). Thus we have a noncommutative structure. These kinds of noncommutative Hopf algebras are what one calls quantum groups, but strictly mathematically they are not groups.

Apart from generalizing groups, Hopf algebras, in general, possess another kind of symmetry, and that is self-duality. Mathematically self-duality in Hopf algebras means that they are both an algebra and a co-algebra. In other words that they include both the multiplication, \( m \), and the co-multiplication, \( m^* \), the inverse, \( \text{inv} \), and the antipode, \( \text{inv}^* \), the unit \( u \), and the co-unit \( u^* \). Self-duality is what some physicists [67] believe might be the key to unifying the quantum theory with the general theory of relativity. They see one side as providing the algebra and the other side as providing the geometry. The idea could perhaps be seen in the following way. The quantum theory is about algebras, and the “pulling back of functions”, in order to get the Hopf algebra, produces a connection between algebras and more geometrical concepts like spaces. The general theory of relativity tells us that (vacuum) energy is found in the geometrical curvature of spacetime. Vacuum energy is something that is usually treated by quantum field theories (in an algebraic manner) and the curvature of spacetime is certainly something geometric. So, one could hope that by imposing self-duality it would be possible to unify the two. In fact, making sure that this duality exists has in some examples, focusing on a quantum description of the motion of a particle, shown to impose restrictions on the metric that look somewhat similar to what appears in general relativity.

4.2 A Way of Interpreting Braids

In this section we shall take a look at a particular way of interpreting braids in physics. To start with we can imagine the braid strands as representing vector spaces, \( \text{Vec} \). Then several braid strands placed next to each other would be seen as tensor products of vector spaces (see Figure 4.8). A strand representing the vector space \( \text{Vec} \) may also be seen as an identity operator \( 1_{\text{Vec}} : \text{Vec} \rightarrow \text{Vec} \). This is because operators on vector spaces can be denoted as in Figure 4.9. There we have an operator \( T : \text{Vec}_1 \rightarrow \text{Vec}_2 \). The identity operator, taking a vector space to itself, is not depicted and a vector space can therefore also be seen as an identity operator.

We have seen that braids can be encoded into the plane. When doing that we
4.2.1 Symplectic Form

It is interesting to see what would be required of such a cup to be a symplectic form. To start with a symplectic form is antisymmetric, \( \omega(x, y) = -\omega(y, x) \) where \( x, y \) are vectors in a vector space represented by a strand. Diagrammatically this could be seen as in Figure 4.10. We have already seen that a braid, or here in particular a crossing, can be encoded into the plane. Thus we can write the crossing \( \sigma_1 \) as \( \sigma_1 = A \| + B \times \). The diagram in Figure 4.10 then gives us the equation: \( A \cup + Bd \cup = -\cup \), where \( d \) is the loop value. Or, rewriting it...
we get

\[(A + 1 + Bd) \cup = 0. \]  \hspace{1cm} (4.17)

**Case with RIII Invariance**

Given invariance under RIII only, as is the case of special braids, we know that \(d = -A/B - B/A\). Inserting this value into equation (4.17) gives us the condition

\[A = B^2.\]  \hspace{1cm} (4.18)

**Case with RIII and RII invariance**

Given invariance under RIII and RII, as is the case for the braid group, we have that \(d = -A^2 - A^{-2}\) and \(B = 1/A\). Inserting this \(d\) into equation (4.17) and replacing \(B\) with \(1/A\) gives us the condition

\[A^3 = 1.\]  \hspace{1cm} (4.19)

And so the three possible values for \(A\) are 1, \(-1/2 + i/2\sqrt{3}\), and \(-1/2 - i/2\sqrt{3}\).

Furthermore we note that a symplectic form does not keep track of the information regarding how the two vectors are being interchanged. In other words, an over crossing, \(\sigma_i\), is equivalent to an under crossing, \(\sigma_i^{-1}\), and can thus be seen as just a crossing, see Figure 4.11. Mathematically this gives

\[A \parallel + B \asymp A \asymp + B \parallel\]

which we can rewrite as

\[(\parallel - \asymp)(A - B) = 0.\]  \hspace{1cm} (4.20)

This gives us the condition that \(A = B\). This also corresponds to the non-degeneracy of the form. And so the two conditions taken together give us an antisymmetric and non-degenerate form.

**Case with RIII invariance**

In the case of RIII invariance the idea of not being able to distinguish between a crossing and its inverse, and thus the condition \(A = B\), gives us \(d = -A/A - A/A = -2\). The condition related to antisymmetry was \(A = B^2\).

Together we have \(A = B = B^2\). From invariance under RIII we know that \(A, B\)
cannot equal 0, and so we have that
\[ A = B = 1. \] (4.21)

**Case with RIII and RII invariance**

Indistinguishability between a crossing and its inverse gave us \( A = B \). For RII and RIII invariance we had that \( B = 1/A \). Thus \( A = B \) becomes \( A = 1/A \). This means that \( A = \pm 1 \). Thus the loop value becomes \( d = - (\pm 1)^2 - (\pm 1)^2 = -2 \). The condition related to antisymmetry gave us \( 1, -1/2 + i/2\sqrt{3}, \) and \( -1/2 - i/2\sqrt{3} \) as the three possible values for \( A \). For both of the conditions to hold we see that the only possibility is
\[ A = 1. \] (4.22)

### 4.2.2 Orthogonal Form

Let us investigate what the requirement of the cup is to be an orthogonal two-form, \( g : \text{Vec} \otimes \text{Vec} \rightarrow F \), where \( F \) is some field. To start with we want it to be symmetric. Thus to \( g(x, y) = g(y, x) \). In diagrammatic form this becomes Figure 4.12. Given \( \sigma_1 = A|| + B \approx \) this gives us the equation \( A \cup - \cup + Bd \cup = 0 \).

*Figure 4.12: An orthogonal form, \( g(x, y) \), of two vectors \( x, y \) is symmetric and thus equal to \( g(y, x) \).*

where \( d \) is the loop value. Rewritten we have
\[ (A - 1 + Bd) \cup = 0. \] (4.23)

**Case with RIII invariance**

Given invariance under RIII only we know that \( d = -A/B - B/A \). Inserting this value into equation (4.23) gives us the condition
\[ A = -B^2. \] (4.24)

As in the case of symplectic forms, there is no difference between over and under crossings. Thus we also have the condition \( A = B \). Taken together we have that \( A = B = -B^2 \). Since \( A, B \) cannot equal 0 we must have
\[ A = B = -1. \] (4.25)
Case with RIII and RII invariance

Given invariance under RIII and RII we have that \( d = -A^2 - A^{-2} \) and \( B = 1/A \). Inserting this \( d \) into equation (4.23) and replacing \( B \) with \( 1/A \) gives us the condition

\[
A^3 = -1. \tag{4.26}
\]

And so the three possible values for \( A \) are \(-1, 1/2 + i/2\sqrt{3}, \) and \( 1/2 - i/2\sqrt{3} \). As in the case of symplectic forms, there is no difference between over and under crossings. Thus we also have the condition \( A = \pm 1 \). Taken together the two conditions give us that

\[
A^3 = -1. \tag{4.27}
\]

To tie up with previous sections we mention that \( A = -1 \) is exactly the case of Penrose’s binor calculus where the binor identity was \( \times = || + \propto \).

4.2.3 General Two-forms

It is interesting to note that given invariance under RIII and RII we find that the non-degeneracy condition, or equivalently the condition that we do not differentiate between a crossing and its inverse, suggests that the only possibility for a two-form is for it to be symmetric or orthogonal. Thus \( A = \pm 1 \). In the case of RIII invariance, however, there is another possibility. Non-degeneracy only suggests that \( A = B \). Let us see what this implies for the two-form. We would like to know what happens when we switch the vectors in the form, \( \text{gen}(x,y) \). And so we let \( \text{gen}(x,y) = t \text{gen}(y,x) \), as shown in Figure 4.13. Given \( \times = A|| + A\propto \) we have from Figure (4.13) that \( \bigcup = t(A\bigcup + Ad\bigcup) \). With

\[
d = -2 \text{ we find that } t = -1/A. \text{ And so, a general non-degenerate }^2 \text{ two-form with only RIII-invariance can be written as }
\]

\[
\text{gen}(x,y) = -1/A \text{gen}(y,x). \tag{4.28}
\]

^2Had we also dropped the non-degeneracy conditions then we would have ended up with \( \text{gen}(x,y) = -A/B^2 \text{gen}(y,x) \) as described in the section below.
Continued Generalization of Two-forms

Let us continue with the idea of interpreting the cup, $\cup$, as a two-form. However, let us look at an even more general case. In the most general of situations related to special braids we have that $\sigma_1 = A|| + B \times$ and $d = -A/B - B/A$. This suggest that when we continue to see switching of the braid strands as a crossing, like in Figure 4.14 we have to multiply with a factor $-A/B^2$. Let

\[
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
\begin{bmatrix}
  x \\
  y
\end{bmatrix}
= -A/B^2
\]

Figure 4.14: A general two-form where switching the arguments introduces a factor $-A/B^2$.

us now represent the two-form, $\cup$, with matrices. Say that on the incoming strands we have are column vectors with entries $\begin{pmatrix} q' \\ p' \end{pmatrix}$ and $\begin{pmatrix} q \\ p \end{pmatrix}$. Then we can write the two-form as:

\[
\begin{pmatrix} q & p \end{pmatrix}
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\begin{pmatrix} q' \\ p' \end{pmatrix} = q(aq' + bp') + p(cq' + dp'),
\]

where $a, b, c, d$ are numbers that commute with each other and with the entries of the two-form. According to Figure 4.14 the above two-form should then be equal to:

\[
(-A/B^2)
\begin{pmatrix} q' & p' \end{pmatrix}
\begin{pmatrix} a & b \\ c & d \end{pmatrix}
\begin{pmatrix} q \\ p \end{pmatrix} = (-A/B^2)(aq + bp) + p'(cq + dp)).
\]

The two are equal if the following relations hold:

\[
q'q = -\frac{B^2a}{A}qq' \quad (4.29)
\]
\[
p'p = -\frac{B^2d}{A}pp' \quad (4.30)
\]
\[
p'q = -\frac{B^2b}{Ac}qp' \quad (4.31)
\]
\[
q'p = -\frac{B^2c}{Ab}pq' \quad (4.32)
\]

From these relations we see that $b, c \neq 0$. The fact that $A, B \neq 0$ we already knew from invariance under RIII. If we are particularly partial to exponential functions we can rewrite $-B^2/A$ as $Ae^{-2\pi/r}$. This is because with the loop value being $d = -A/B - B/A$ we can rewrite it as $d = 2\cos(\pi/r) = e^{i\pi/r} +$
$e^{-i\pi/r}$ and identify $-A/B = e^{i\pi/r}$.

What do these relations (4.29) mean? They are generalizations of structures that we find in physics. Let us relate these relations to the particular cases that we have looked at. We start by looking at the case where $a = d = 0$. This gives us the matrix

$$\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix}$$

and the relations $p'q = -\frac{B^2}{Ac}qp'$ and $q'p = -\frac{B^2}{Ac}pq'$. Let us assume that $p'$ and $q$ as well as $q'$ and $p$ commute. This gives us the conditions $1 = -\frac{B^2}{Ac}$ and $1 = -\frac{B^2}{Ac}$. Solving the first of these for $B^2$ and placing it into the second one we get $b = \pm c$. Solving the first for $c$ and placing it into the second one we get $A = \pm B^2$. This means that if $A = -B^2$ then $b = c$ and thus we get the matrix

$$b \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

If $b = 1$ we have the orthogonal case. If $A = B^2$ then $b = -c$ and thus we get the matrix

$$b \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$  

If $b = 1$ we have the symplectic case.

Let us, for a moment, relate what we have to Hamiltonian mechanics\footnote{Please see the appendix for a short review of Hamiltonian mechanics.}. For some function $u$ of $q$’s and $p$’s and $H$ the Hamiltonian, Hamilton’s equations can be written as

$$\frac{du}{dt} = \left( \frac{\partial u}{\partial q} \frac{\partial u}{\partial p} \right) \left( \begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) \left( \frac{\partial H}{\partial q} \frac{\partial H}{\partial p} \right) = \{u, H\}_p,$$

where $\{\}$$_p$ denotes the Poisson bracket. In the quantum theory these brackets are replaced with commutators and for some observable $\hat{O}$ and Hamiltonian operator $\hat{H}$ we have

$$i\hbar \frac{d\hat{O}}{dt} = \hat{H}\hat{O} - \hat{O}\hat{H} = [\hat{H}, \hat{O}] = i\hbar \{\hat{H}, \hat{O}\}_p.$$  \hfill (4.33)

And so we see that for the symplectic case with $b = 1$ and the entries chosen appropriately, we have the equations of motion of classical physics. If we choose to see $b$ as $i\hbar$ and choose the entries appropriately we have the equation of motion of quantum physics, Heisenberg’s equation.
If we are dealing with special braids then we realize that the entries along the braid strands are really special braids themselves.
5 Discussion

“If you wish to make an apple pie truly from scratch, you must first invent the universe.”

(Carl Sagan [92])

This thesis takes Sagans statement more seriously than perhaps most other publication in physics. The thesis is to a large degree about starting from scratch. The introduction put forward the notion that the main goal is to present a new perspective from which to view physics mainly for two reasons. The first reason being that a new perspective may add to the understanding of the physical world. In particular we hoped to find an ontology, where physics could be more than abstract algorithms producing predictions. The second reason is that a new perspective is likely to provide us with new orders, evincing relationships that may not be obvious from other perspectives.

In this chapter we shall discuss primarily these two aspects. We will start by seeing how the proposed ontology deepens our understanding. Thus we will talk about an area that is philosophically challenging, namely the quantum theory. There we shall give examples of insights that the conceptual background provides, such as for instance the understanding of the origin of Heisenberg’s indeterminacy principle. We will also discuss the potential of the approach, meaning what this approach has the potential of providing, if developed further. As we have seen, there is a general mathematical structure that should be investigated and interpreted further in order to see if it indeed is useful in physics. This is a project for the future. However, in a later part of this chapter we will use the philosophy and some mathematical indications to see what that general structure is suggestive of and might be capable of describing. The key suggestion is that this structure may have the potential of providing a dynamical evolution beyond unitarity, that includes entropy increase. In other words, we are suggesting that a natural way of describing entropy increase and time direction fundamentally, so to say from scratch, may be by starting from wholeness, or so to say holistic scratch. We shall end this chapter with a short series of frequently asked questions and the conclusions.
5.1 Discussion on the Quantum Theory

In this section we shall demonstrate the usefulness of the conceptual background that the perspective from wholeness provides. Let us start with an example of how the conceptual structure makes it possible to evolve our intuitive understanding of the theories we already have, such as the quantum theory. Let us in particular take a look at a concept that is at the heart of the quantum theory, namely Heisenberg’s principle of indeterminacy.

5.1.1 Insights into Heisenberg’s principle of indeterminacy

Here we shall argue that if an undivided wholeness is assumed, then the holistic view can provide insights into features of the quantum theory along with intuitive understanding. In order to better understand what a holistic view entails and adds, we shall, at this point, examine some fundamental differences in the concepts and requirements for reductionism and holism. A good description of these differences may be found in [93].

Reductionism presupposes the possibility of decomposition into identifiable parts. This means that we view the system that we study as essentially being made up of separate, identifiable parts, that interact with each other as described by laws. The description of the whole system is reducible to a description of its parts. In order to be able to have identifiable parts we need to have both strict causality and strict locality (or space-time continuity). And having both means that we can in principle always find identifiable parts. In other words, identifiable parts are equivalent to strict locality together with strict causality. Should either of these fail, then we will no longer be able to identify parts.

Let us now examine the holistic view. This involves using indivisible wholeness as most basic and fundamental. The holistic view assumes that there are no separate parts. Although there may be distinguishable modes on some level, on a deeper level we will see that separation and precise distinction are not possible. In other words, on that level we would not be able to find identifiable parts. As a consequence we could not have both strict causality and strict locality. We could have one of them, but that would imply that we would not be able to have the other. As we saw in the second paragraph, having both implies identifiable parts. Another alternative is that we could have neither strict causality nor strict locality. It is interesting to note that this is exactly what happens in the quantum theory [95], and can there be seen as an expression of Heisenberg’s principle of indeterminacy. One can intuitively think of position and time as expressing the local aspects, while energy and momentum can be considered to express the causal aspects [93]. Assuming wholeness, the appearance of a principle such as Heisenberg’s principle of indeterminacy can hereby be seen as a necessary consequence of the undivided
wholeness itself. We can see why strict causality and strict locality are not simultaneously possible.

5.1.2 Further Potential of the Approach From Wholeness

Seeing a principle like Heisenberg’s principle of indeterminacy as a necessary consequence of wholeness is one example of how an approach from wholeness can enrich our understanding of physics. This approach also has the potential of providing a philosophical foundation that is not dependent on the classical world. This would, for instance, mean that we would not require an external classical observer for the theory to make sense. Also, the theory could be constructed in a way that starts from wholeness and so does not depend on a classical world. The quantum theory has traditionally been constructed from classical considerations, often using the idea of quantization, where position and momentum variables are replaced by self adjoint operators satisfying commutation relations, which then fix a representation in Hilbert space. This is a non-unique procedure, depending on the choice of the classical system. However, it should in principle be possible to construct the quantum theory without starting with the classical deterministic picture [83](p.6). The approach from wholeness suggests a direction to follow.

Discussion on Time Evolution and Entropy

“Each act is virgin, even the repeated ones.” (René Char [94])

If we subscribe to the idea that physics should have equations that describe Nature, in other words that there should be an ontology, then the standard time evolution in the quantum theory must appear dissatisfactory. This is in part because the Schrödinger, or Heisenberg, equations of motion do not describe the actual dynamics. They are only a way of rewriting a process while parametrizing it with \( t \). This provides a unitary evolution, where the content of the process is the same. Nothing new is added as the time given by the parameter \( t \) changes. Given that there is no actual change we have no inherent direction in time. We can move forwards or backwards along \( t \). The actual change, where we find a particular outcome of a measurement, is not described by the above process. We can say that the Heisenberg equation is only “an abstraction of a fragment” [96], in the sense that it does not describe the actual happening. As such it cannot alter the entropy. Instead, entropy can be altered when a measurement is performed. Then there appears to be a “jump”. It would be desirable to, as Bell put it, “...have the jump in the equations and not just the talk--so that it would come about as a dynamical process in dynamically defined conditions.” [97].

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Thus it appears as though the missing part of the equations of motion has to do with entropy increase. Why is that part missing? Let us have a look at how physics works. In general we have the idea of a state and an evolution of the state. A state is similar to a static picture taken from a movie. It gives us an instantaneous picture of space as seen from our perspective. Then we have an evolution where both space and time are involved. At the end, we take another instantaneous picture. What problems could we expect to encounter when taking such instantaneous pictures of space? If we think of the movie analogy we see that by only looking at one frame we loose the context of the whole situation. We loose the meaning behind it. And, as argued in chapter 2 section 3, the meaning is in the movement. This situation is remedied in physics by taking a momentary picture only when we have equilibrium. Equilibrium essentially tells us that the state is time independent. This also provides us with a well defined context. However, since most situations cannot be seen as equilibrium, this limits the applicability of the method. Also, an externally put in context can be problematic when it comes to applying it to the whole universe. We see this in the quantum theory, which is extremely context dependent. In the quantum theory we need an external observer to add the proper context. A state in the quantum theory is a bit different than the suggested metaphor of a picture, as the positions and momenta are not simultaneously defined. Instead a state is given by a wave function or a density matrix and can be in superposition. However, the context is still being added from the outside, through the observer.

As we are discussing a missing part I would like to briefly mention Einstein’s suggestion that there is something missing in the quantum theory, that it is not complete. That criticism is of different origin than the one suggested here. It was made in regard to the existence of a non-local causality. If one is against non-locality then one can suggest that the quantum state does not contain all the relevant information pertaining to its nature. Instead, a quantum state contains the observers subjective and incomplete information about it. Taking that view suggestions have been made that one should attempt to eliminate all the subjectivity from the quantum theory. The idea was that one should “start afresh” and separate out the subjective traits, leaving an objective core which should provide a “clear window into nature.” Such a core should in some sense tell us about the nature of the quantum theory, as independent of us. It would be most interesting and revealing to see how much of the quantum theory can be salvaged after such a procedure. However, what if the nature of the quantum theory cannot be separated from us? This is indeed what the quantum theory seems to be telling us. Is there

\footnote{Although time is only involved in a direction-independent or unitary way.}
then a way of pursuing a fuller description without separating the subjective from the objective? This is, in fact, what is being proposed in this thesis. Let us start afresh by widening the perspective and going beyond the subjective and objective. This brings us to wholeness (see Section 2.2.1).

Thus, to sum up, Einstein was not in favor of a non-local causality, where Nature seemed to be connected in a totality. This made him suggest that the quantum theory is incomplete. I, on the other hand, am suggesting that there is a missing part in physics in general, not just the quantum theory, because we do not take the connection to totalities seriously enough, and thus do not take advantage of it in our physical descriptions.

A slightly different way of seeing the general structure of current physics is through variational principles [100]. Variational principals provide a single unifying principle from which the equations of motion can be derived. This principle is very general and works in fields like classical mechanics, quantum theory, relativity. However, variational principles all deal with equilibrium. In equilibrium there is a sense of time independence and therefore also time reversibility. And so they work within certain contexts, not between contexts.

A More General Approach

Is there another, a more general, way of doing physics? Could one include the web of contexts, telling us how to move from one situation with a certain context and a certain entropy, to another? Where could we find such a fundamental order of meanings?

A meaning has to do with a totality. The meaning in a situation can be seen as an all encompassing idea. Thus working with totalities, or wholes, has the potential of exposing a web of meanings. Starting from wholeness we start with the dynamics of a whole (the Wholeness Axiom, $j$). This then generates different processes (elements of $A_j$), where each process is a movement from wholeness to itself. Our web of meanings should tell us how the different wholes are related. This is, in our case, given to us by left-distributivity. Left-distributivity tells us how the totalities are related to each other.

Given this web of meanings, how are we to use it within physics? What we would like to happen is to see to it that the state has a meaning from the web of meanings, and not externally put in by observers. How will that change the notion of a state? Well, if the meaning is in the movement, as suggested in section 2.3, then perhaps we should aim for a process state instead of a static state, where a static state is an instantaneous picture of a process. Such process states would be of the same kind as the evolution that they undergo. They would no longer be instantaneous pictures with an externally put-in context. Instead, they would be meaningful processes themselves. This means that we no longer divide things up into things to be acted upon and actors (say a vector in a Hilbert space and an operator). Instead we see them both as processes.
This would suggest that we should view the processes in $A_j$ both as process states and as operators acting on these process states. But, does $A_j$ really have the potential of describing an actual dynamics beyond unitary evolution?

**Generalization of Conjugacy**

The unitary evolution of the quantum theory can be said to be described by conjugacy (see Appendix E). Conjugacy in the quantum theory can give us time evolution if we let $e_1$ and $e_2$ represent two observables (or explicate orders) and let $M = e^{iHt}$ be the time evolution operator with $\hbar$ set to 1, $e_1 = Me_2 M^{-1} = M^\wedge e_2$, where $\wedge$ denotes the operation of conjugacy. This time evolution is unitary and does, as argued above, not describe the actual happening. It is now interesting to note that application in $A_j$ can be seen as an extension of conjugacy. For application in $A_j$ we notice that for $a$ and $b$ elementary embeddings in $A_j$, for each $x$ in the image of $a$ we have that $a \cdot b(x) = aba^{-1}(x) = a^\wedge b(x)$. In other words, we see that application reduces to conjugacy for the sets that can be inverted. And, we see that this constitutes the trivial part of the application operation (see 3.2.2). The non-trivial part is not covered by conjugacy. It is, however, covered by application. If we were to interpret application as a way of describing physical time evolution, then we would see that unitary evolution, such as given by the Heisenberg equation, is only an abstraction of the trivial part of the dynamics.

**Special Braids as Process States and Operators**

What we are suggesting here is that the states should be process states and linked to the order of meanings. Do we see that when we interpret $A_j$ as special braids? As luck (?) should have it, this is exactly what we have with special braids. When it comes to physical interpretations of braids one often sees the colors of the braid strands as vector spaces. A vector in a vector space can be interpreted as a state. Thus we can interpret the colors of the incoming braid strands as the initial state. The braid itself can be seen as an operator and the outgoing braid strand colors as the final state. One way in which special braids are special is in being self-coloring (see section 3.3.4). This means that they can both be interpreted as the colors of the braid strands and as the braids that act on braid strands. And so, special braids certainly have the potential of describing what we are looking for.

A difference between special braids and braids in the braid group is that special braids do not have inverses as whole braids. This is what provides them with a sense of direction. However, special braids can be written in terms of braids from the braid group. This means that if we were to arbitrarily cut a special braid into pieces and then try to describe what parts it is made up of we would say that it is made up of the braids in the braid group, where all braids have inverses. And so, when taken out of context, the building blocks...
appear to be invertible. That invertibility, however, is not fundamental. It only appears when we allow fragmentation (see section 1.1). What is fundamental here, is left-distributivity.

**Maxwell’s Demon Resting In Peace?**

If left-distributivity turns out to be a fundamental order and time reversibility only a good approximation of arbitrarily cut out parts, then this will have implications for Maxwell’s demons. Maxwell’s demon is a hypothetical being that in some way manages to systematically and almost effortlessly decrease the entropy of a system and use this entropy decrease to do work. It was first conceived of by Maxwell to show the limitation of the second law of thermodynamics [101].

Since then many different versions have been summoned. The mechanical demons were joined by intelligent demons [102] and even intelligent quantum demons [103]. Perhaps the most commonly known version is the small intelligent classical demon that has a gas in two compartments (see Figure 5.1). Between the thermally isolating compartments there is a small door. When

![Figure 5.1: Picture of a classical Maxwell’s demon.](image)

the demon sees a rapidly moving molecule on the right approaching the door he opens it and lets it pass through to the left compartment. When he sees a slowly moving molecule on the left he similarly lets it pass to the right compartment. The argument is then that since the demon and the door are very tiny and the opening and closing of the door can be done with arbitrarily little work, the demon will have introduced a temperature difference between the two gases without doing any work. The temperature in the left compartment will have increased while the temperature of the gas in the right compartment will have decreased. This temperature difference can then be used to do work.

Why would one expect such demons to exist in the first place? Is there a general principle lurking backstage that would lead to such suspicions? Here is one way of seeing why. Equations of motion, such as Newton’s equations or even the Schrödinger equation in the quantum theory, are symmetrical with
respect to the direction of time. In other words, our fundamental description of dynamics is reversible and does not differentiate between the future and the past. However, when we come to systems that are made up of a large number of particles a direction seems to creep into the picture. The Second Law of Thermodynamics expresses this direction. And so, if we believe that reversibility is fundamental we would expect that there might be a way for us of exploiting it. This is exactly what Maxwell’s demons are designed to do. By systematically taking advantage of reversibility they violate the Second Law. A question that begs to be asked is: Is reversibility really fundamental? According to this approach from wholeness it is not. As we have seen, irreversibility comes in with the very assumption of wholeness being whole, and thus the dynamics being wholeness preserving. Thus, seen from the approach from wholeness, demons thriving on reversibility are not an option. Should irreversibility turn out to be fundamental, then there would be no incentive to further summon and exorcise these poor transcendental creatures, instead they could finally rest in peace.

The Now
When we no longer cut processes up into instantaneous states we have room for a “now”. The now is never fully defined. Due to Heisenberg’s indeterminacy relation we always have ambiguity in the now. There is always an unrealized potential that does not become realized until a later time, but at that time something else will be ambiguous. And so the now is always unknown, and can, seen from the approach from wholeness, be viewed as embedded in the holistic wholeness.

5.2 Frequently Asked Questions
This section is a short collection of some of the questions that I am asked most frequently.

5.2.1 The Appearance of Diversification
How does the diversification we perceive come about? If there is a dynamical implicate (holistic) wholeness that is fundamental, how can semi autonomous parts (explicate orders) appear? The key to answering this lies in the very nature of the wholeness preserving movement. The wholeness preserving movement is reflective. It mirrors wholeness in itself. The nature of this reflecting motion is that of polarization, opposition, or contradiction. This may be seen as projecting out a reflection of wholeness (unfoldment) and making it appear as though this reflection (range of \( j \)) is something very different from wholeness, and then realizing that it is not anything else but wholeness itself.
(enfoldment). This reflective nature contains the seed to opposition, differentiation, and therefore also to independence and autonomy, while at the same time preserving wholeness and disclosing its unity. Were it not for these opposing tendencies, then everything would be completely dependent upon everything else to the degree where distinguishability would not be possible. So opposition is a prerequisite for autonomy. But the autonomy of the explicate orders is not complete because the explicate orders are then enfolded into the whole.

And so, the wholeness preserving movement shows us not only how opposition comes from unity, but also how that which is in opposition is unified through wholeness.

5.2.2 Why Start from Set Theory?

But why do you start the mathematical approach with Set Theory? Is it not the pinnacle of reductionism? Are sets not motionless identifiable parts? Should one not try to start with something more along the lines of Category Theory, where process plays a more fundamental role?

The reason for starting from Set Theory is twofold. Firstly, we want the dynamics of wholeness to be mathematically represented in as fundamental a way as possible and we already know that Set Theory works as a foundation for mathematics, whereas the Category theoretic approach has not yet been developed that far. Secondly, even though Category Theory may be said to focus on process one does not get away from objects, which ultimately are motionless identifiable parts. And so, although the focus is on the process, the morphisms, the point of departure is still a static identifiable part. Processes are defined in terms of other definable processes which, in the end, are given by the static parts. Given the idea of starting from wholeness we are interested in accentuating process and basing it on the indefinable wholeness. The approach based on the Wholeness Axiom does accomplish something along those lines. The domain and codomain of the $j$ are the entire mathematical universe, which we can see as symbolizing wholeness (see 3.1.1). As a result $j$ itself is not explicitly describable with any set-theoretic formula, it is not a set or a proper class. And so we accomplish a sense of undefinability and process. Everything is process, the relevant structure comes from the relations between the processes, and the innermost nature of each process is wholeness.

5.2.3 Why Not Just Start With Special Braids?

Would it not be easier to start with special braids? This way we would not have to work with elementary embeddings.

As mentioned in the preface, this thesis is written in a way so that it will be clear from what philosophical ideas it originated. This is because the under-
lying ideas provide an understanding and may themselves be useful in further developments. In other words, an intuitive interpretation and understanding of what is being modelled along with the mathematical tools constitute a richer base in physics than just an abstract mathematical tool. This way we can use both our personal, experiential intuition and our mathematical intuition to further develop the model. The reader is invited to find any other, better suited interpretations, or free to abandon ontological interpretations all together. In the thesis there are perhaps especially two places where that could be done.

We start with the Wholeness Axiom, \( j : V \rightarrow V \), and say that we will interpret it as the dynamics of wholeness. Then, however, we say that as far as we can see none of the structural information relevant to this thesis is lost if we assume that \( j : V_\lambda \rightarrow V_\lambda \) where \( \lambda \) is a limit ordinal, and we construct the process algebra \( A_j \). What would we loose by starting with \( j : V_\lambda \rightarrow V_\lambda \) instead of with the Wholeness Axiom? Philosophically we would loose the proximity to wholeness. We no longer have a movement from the whole of \( V \) to itself but from the set \( V_\lambda \) to itself. Also, we reduce \( j \) to a set itself. Mathematically this changes the strength of \( j \) and has consequences such as the fact that we can no longer show the existence of all large cardinals. It might be so that this restriction has some consequences for the physical interpretation as well.

The next step we take is that we say that \( A_j \) is isomorphic to \( B^{\text{top}} \). Thus, when we are solely interested in the way in which the elements relate to each other we might as well work with \( B^{\text{top}} \) instead of with \( A_j \). At this point one might ask, why not start from \( B^{\text{top}} \)? In doing so, we loose contact with the underlying ideas of wholeness. We loose the ontology. The elements are no longer necessarily processes unfolding and enfolding wholeness. Mathematically we also loose the large cardinal strength and the well-ordered linear ordering, \( <_{\text{Lex}} \), that the large cardinals provide us with. If this ordering should turn out to be physically relevant, then we could not abandon the elementary embeddings in favor of special braids. However, if one is interested in creating a new understanding, a new philosophy, then starting with special braids could be an interesting mathematical starting point.

5.2.4 In Short, What Kind of a World Does This Model Describe?

This model describes the world as originating in an undefinable holistic wholeness which moves from itself to itself, creating a duality in the movement. Movement is all there is. Every thing is an abstraction from the movement, a temporary structure in the movement, that has its being in wholeness. The innermost nature of every thing, including the now, is wholeness. The movement is creative in nature, as it creates something out of no thing. By originating and culminating in a holistic, undefinable wholeness it allows for the
possibility of being non-trivial. The ambiguity creates the potential of a description of a real dynamics with a real now.

This approach needs to be developed further and checked. However, at the moment it appears as though evolution in time is about the changing of contexts, meanings. The essential evolution is in the evolution of the meanings, not the parts. The evolution of parts, as described by unitary evolution in the quantum theory is merely an abstraction of a fragment that this approach may have the potential of going beyond.

5.2.5 What Is Your Contribution to Physics?

In order to answer this, I first need to make clear what physics is. When looking it up in a dictionary, physics is often defined as something along the lines of “the study of matter and energy”. Although this may in some sense be true I believe that such a definition conceals one of the most crucial aspects of physics, namely imagination. Thus it, arguably falsely, portrays physics as something more static and objective than it really is. Let me expound.

In my view physics is about the agreement between our ideas and our experiences. Thus physics is not something that objectively exists “out there” for us to be discovered. Instead it originates in our imagination, and is then checked (through experiments) for factuality. Factuality is central in physics. Facts, or experimental outcomes, help us answer our scientific questions. However, despite the essential role of factuality, it is important to keep in mind that the facts, per se, do not provide us with the models. To construct models we need imagination, meaning the process that produces ideas and thus allows us to have knowledge about the experienced world. Factuality can help us in answering questions, but it does not pose them. What questions can we pose? Although we might like to answer that we can pose any questions, this is in fact not true. Our questions are always limited by our imagination. In particular, they are limited by the particular outlook, or perspective we have. And hence our physics is limited by our imagination, or by the particular perspective we have. By changing the perspective, we can change the questions we ask, and thus expand our understanding of some phenomenon and possibly even find new facts.

As a result, I would like to argue that a central part of the development of physics has to do with the ability of changing perspective and thus subsequently, putting forward new questions. New ways of conceiving of the facts that we are presented with are essential.

Applying new perspectives and interpretations is the main theme of my contribution. Let me now go into the details. The first contribution is a change of perspective. I suggest that we start from wholeness. The ideas of wholeness in physics and of it being fundamental are not new. What I believe is new
is to actually use it as a philosophical foundation on which to build physics. My suggestion is largely based on Bohms ideas on the implicate and generative orders. However, neither conceptually nor mathematically is this exactly Bohms idea of the implicate order. It is my impression that Bohm was looking for the order of the implicate order, but never quite found it. I also know that he believed that there should be a generative order, an order that would create something new and allow for entropy increase, but never fully explained what that should be or how it should work or be modelled mathematically. My contribution here is to suggest that the order of the implicate order is given by the generative order and in particular by the idea that wholeness does not change in totality. Hence my suggestion is that we should start with a wholeness preserving movement as the guiding principle for the mathematics modelling the generative order.

Furthermore, my contribution is to suggest a mathematical way of modelling the dynamics of wholeness. The mathematics itself, that I use, is not my original contribution. It is the way I choose to look at it that is original. Take for instance the Wholeness Axiom. I most certainly did not invent that mathematical axiom. My original contribution is to interpret it as the dynamics of wholeness from which we should be able to extract mathematical structures that we have already correlated with experiments as well as possible new structures to be checked for factuality. Continuing, the whole discussion on how mathematical properties, that I did not invent, relate to the ideas on wholeness, such as self-similarity, preservation of truths, unfoldment and enfoldment unfolded in a discussion with Corazza and cannot be found elsewhere.

Seeing the Wholeness Axiom as the dynamics of wholeness we now have to know how to continue, in order to eventually link it up with physics. My contribution here is to suggest that we focus on the universal algebra generated by it though application. Again, the mathematical structure of elementary embeddings acting on each other is not new, but the interpretation of it as a process algebra whose structure is of importance in physics is. Going on, the connection between a universal algebra, such as the process algebra that we arrive at, and special braids is not new. My contribution here is the suggestion that to interpret the process algebra we should go through braids because this can be connected to Kauffmans bracket and planar algebras, which in turn already have applications in physics. Finally, I also contribute to the discussion on entropy, suggesting that left-distributivity may be a fundamental order and time reversibility only comes about when a situation is taken out of its natural context and studied in a fragmentized manner.
5.2.6 Further Investigation

“The will is infinite and the execution confined, ... the desire is boundless and the act a slave to limit.” (William Shakespeare [104])

It would be very interesting to see what consequences the structure of special braids has in the different applications of planar algebras. It would, for instance, be interesting to further develop and interpret the ideas exposed in the section on the generalizations of two-forms (4.2.3). These should, among other ideas, provide generalized equations of motion which it would be interesting to interpret and see if they are of physical consequence.

It would furthermore be interesting to investigate the consequences of special braids in the areas where braids are used in general, such as statistical physics and quantum physics and perhaps especially quantum gravity. Besides applications there are several lines that could be worth while investigating.

One interesting issue to investigate is the lexicographical order on $A_j$ that comes from large cardinals, and allows us to construct a natural metric. One could investigate its relationship (see section 3.2.8) to the left-distributive order and, furthermore, its physical consequences.

As originally shown by Penrose, spin networks are related to three dimensionality. The connection to braids further discloses the links to topology in three dimensions. However, it is not clear how geometry should be tied to this. Perhaps there is a way of using the orderings on braids for that purpose. As we have seen it is possible to create a metric based on them. It would be interesting to see how that metric could be incorporated into networks.

Another possibly interesting issue is to investigate Laver sequences to see if or how they can be interpreted physically. It may also be interesting to focus on the study of Laver tables and see how the mathematical insights translate into physics. Furthermore, general mathematical work related to the structure of $A_j$, such as work on special braids, in particular the work of Dehornoy related to LD-monoids, parenthesized groups, ... could be further looked into to see how it can be interpreted in terms of physics.

If we do not make the assumption that a crossing and its inverse are equivalent, an assumption that is valid in three dimensional space, then, as we have seen, it turns out that there are more solutions to the two forms such as the cube root of $A$ (see section 4.2). Are they of physical relevance? Could they be related to particles that cannot exist individually in three dimensional space, such as quarks?
5.3 Conclusions

Based on deliberations on the quantum theory we proposed a shift in the philosophy underlying physics. This shift consisted of starting from a dynamical wholeness with holistic aspects. It lead us to propose a philosophy that can be seen as a wholeness-based process ontology, where a wholeness preserving movement is fundamental. Subsequently it was suggested that such a movement may be mathematically expressed in a fundamental way if the Wholeness Axiom is added to the currently established axioms of Set Theory. Such a mathematical foundation naturally allows for a description of the emergence of all parts of the mathematical universe. It also contains holistic features as indescribability, more than the sum of parts, local accessibility and the holographic type of order that we find in the quantum theory. By letting the wholeness preserving movement interact with itself it was found to generate a left-distributive universal algebra that is isomorphic to special braids. It was then shown how, through the connection to braids, this structure can in special cases be linked to quantum physics and statistical physics models used in solid state physics. These connections included the Potts model, spin networks, algebras of observables, and quantum groups. Finally, the general mathematical structure was discussed, alluding to a possibility of describing entropy increase, and showing how the conceptual background related to starting from wholeness facilitates the understanding of seemingly paradoxical features of existing theories.

In conclusion, it is suggested that a dynamical wholeness may be capable of providing a conceptual foundation for the development of physical theories. By starting from a broader perspective than current approaches it provides the guidelines for the creation of a theory that may be valid where other currently existing theories reach their limits. In particular, by focusing on wholes it suggests meaningful parts, as opposed to fragmented pieces, and with them an order that appears to have the potential of giving rise to a physical description beyond unitary evolution.

5.3.1 Final Note

The intention of this thesis was to point to an alternative way of conceiving of physics that could deepen our understanding of the physical world and open up for new descriptions that could reveal orders that current physics has not accessed. This approach, in particular, shows how wholeness is capable of providing a base as well as a guide in the development of a physical theory. It should be stressed that the approach displayed in this thesis is only one particular example of starting from wholeness. Specifically it chooses to use ZFC+WA as its mathematical basis and to focus on the algebraic structure derived from \( j \)'s interaction with itself through application which is then tied
to physics via braids. These choices, although motivated, are not the only possible choices that could follow from reflections on wholeness. Hopefully the shortcomings of this approach will serve to encourage people to develop better theories and not to abandon wholeness.

“The universe is full of magical things, patiently waiting for our wits to grow sharper.”

(Eden Phillpotts [106])
6 Summary in Swedish

6.1 Dynamisk helhet som grund för fysik

Genom århundraden har den reduktionistiska filosofin stolt och nästan en-väldigt regerat inom fysiken. Det vill säga att man har utgått från att man i princip kan beskriva världen genom att hitta de minsta byggnaderna och förstå hur de interagerar. Tiderna förändras, och det gör även fysiken. Ja, under det senaste århundradet har fysiken förändrats så dramatiskt att man kan fråga sig om det inte snart är dags att även ändra på den underliggande filosofin som ligger till grund för vår världsbild.

En av de mest omskakande aktörerna som gjorde intrång i fysiken i början av 1900-talet var kvantteorin. Kvantteorin kom fullt utrustad med minst sagt märkliga kvanteffekter. Här gavs materian både partikel och vågegenskaper, katter uppgavs kunna vara både levande och döda samtidigt och man påstod att om man kände till någontings hastighet med absolut precision så kunde man inte känna till dess läge, utan saken kunde befina sig precis var som helst i universum. Hur fysikerna än försökte att tämja denna teori och stänga in den i en reduktionistisk världsbild så lyckades alltid någon del att sticka ut. Det var så frustrerande att man nästan började fundera på om den reduktionistiska filosofin var ett bra utgångssätt till och börja med. En person som funderade i dessa banor var David Bohm och i hans matematiska formulering och tolkning av kvantteorin kan man spåra de märkliga kvanteffekterna till en bakomliggande odelbar (alltså holistisk) dynamisk helhet. Han kallade detta för den implicita ordningen (the Implicate Order). Utifrån denna holistiska

Figure 6.1: Kvantteorin passar inte in i ett reduktionistiskt ramverk.


Utrustade med denna kunskap om dynamiken är vi redo att formulera detta på ett matematiskt sätt. Lyckligtvis har en matematiker vid namn Paul Corazza nyligen formulerat ett sådant antagande och kallat det för helhetsaxiomet. Antar man att detta axiom är en matematisk beskrivning av helhetens dynamik så kan man börja titta på konsekvenserna.


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A The Axioms of Set Theory, ZFC

A formulation [105] of the Zermelo-Fraenkel set of axioms and the Axiom of Choice and of Foundation.

Empty Set: There is a set with no elements.

Infinity: There is an infinite set.

Paring: Whenever A and B are sets, there is another set containing just A and B, namely \( \{A, B\} \).

Power Set: For any set A, the collection \( P(A) \) of all subsets of A is also a set.

Union: If A is a set of sets, then \( \bigcup A \) is a set.

Extensionality: A and B are equal if and only if A and B have the same elements.

Choice: Every set can be well-ordered.

Foundation: There is no sequence of sets \( x_0, x_1, \ldots \) for which \( \ldots \in x_2 \in x_1 \in x_0 \).

Separation: For any formula \( \phi \{x_0, x_1, \ldots, x_n\} \), and any sets \( a_0, a_1, \ldots, a_n \), the collection of all sets \( y \) for which \( \phi \{y, a_1, a_2, \ldots, a_n\} \) is true forms another set.

Replacement: For any set A and any rule that associates with each element \( x \) of A a set \( Y_x \) there is a set B that consists precisely of all \( Y_x \), where \( x \in A \).
B Reidemeister Move III

Proof that left-distributivity as Reidemeister move III gives \( d = -A/B - B/A. \)

The third Reidemeister move tells us that:

\[ \langle \hat{\chi}_1 \rangle - \langle \hat{\chi}_2 \rangle. \]

Each of these braids can be spliced and encoded into the plane and thus decomposed into the following sums.

\[ \langle \hat{\chi}_1 \rangle = A^3\langle \mid | \rangle + 2A^2B\langle \hat{\gamma}_1 \rangle + A^2B\langle \hat{\chi}_1 \rangle + B^2A\langle \hat{\chi}_2 \rangle + B^2A\langle \hat{\gamma}_2 \rangle + B^3\langle \hat{\gamma}_1 \rangle \]

\[ \langle \hat{\chi}_2 \rangle = A^3\langle \mid | \rangle + 2A^2B\langle \hat{\chi}_2 \rangle + A^2B\langle \hat{\chi}_1 \rangle + B^2A\langle \hat{\gamma}_2 \rangle + B^2A\langle \hat{\gamma}_1 \rangle + B^3\langle \hat{\gamma}_1 \rangle \]

And so the difference

\[ \langle \hat{\chi}_1 \rangle - \langle \hat{\chi}_2 \rangle = 0 \]
gives us

\[(B^3 + A^2 B) \left( \langle \mathbf{H} \rangle - \langle \mathbf{X} \rangle \right) + B^2 A \left( \langle \mathbf{Y} \rangle - \langle \mathbf{O} \rangle \right) = 0.\]

If \(d = \langle o \rangle\) we get

\[(B^3 + A^2 B) \left( \langle \mathbf{H} \rangle - \langle \mathbf{X} \rangle \right) + B^2 A d \left( \langle \mathbf{Y} \rangle - \langle \mathbf{O} \rangle \right) = 0\]

which gives us

\[d = -\frac{(B^3 + A^2 B) \left( \langle \mathbf{H} \rangle - \langle \mathbf{X} \rangle \right)}{B^2 A \left( \langle \mathbf{Y} \rangle - \langle \mathbf{O} \rangle \right)} = -\frac{(A^2 + B^3)}{AB} = -\frac{A}{B} - \frac{B}{A}\]

and thereby shows that \(d = -A/B - B/A\) for RIII invariance.
C On Adjoints in Lie Algebras and LD-systems

This is a short comparative discussion on the concepts of adjoints in LD-systems and adjoints in Lie algebras.

Adjoints in Lie Algebras

A Lie algebra $L$ is a non-associative algebra with a binary operation called the Lie bracket, $[,] : L \times L \to L$, which is bilinear, antisymmetric, and satisfies the Jacobi identity ($[a,[b,c]] + [b,[a,c]] + [c,[a,b]] = 0$). A Lie group $G$, on the other hand, is a differential manifold with differential maps (product and inverse operations). The tangent vectors of the tangent space of the Lie group $G$ at identity $e$ can be seen as elements of a Lie algebra $TG_e$.

Left Translation

An action of a Lie group, $G$, on itself is given by diffeomorphisms called left and right translations, $L_g : G \to G$ and $R_g : G \to G$ respectively. Each element $g \in G$ defines an $L_g(h) = gh$ and a $R_g(h) = hg$ where $h \in G$. There are also induced maps on the tangent spaces, $L_g\ast : TG_h \to TG_{gh}$ and $R_g\ast : TG_h \to TG_h$.

Adjoint Representation

There is an inner automorphism of the Lie group $G$ which leaves the identity, $e$, fixed, namely: $A = R_g \ast L_g$. It’s derivative at identity is a linear map on the Lie algebra and is called the adjoint representation of $G$:

$$Ad_g = (A_g)_{se} = (R_g \ast L_g)_{se} : TG_e \to TG_e.$$  \hfill (C.1)

$Ad_g$ preserves the group operation , $Ad_g[x,y] = [Ad_g(x),Ad_g(y)]$ for $x,y \in TG_e$, and so it is a homomorphism.

Since it is an element $g$ of the group $G$ that defines the left and right translations which give the inner automorphism whose derivative at identity $Ad$ is, we can also see $Ad$ as a map from the group $G$ to the linear operators on the Lie algebra (see Figure C.1), so that

$$Ad(g) = Ad_g.$$  \hfill (C.2)
Figure C.1: The adjoint representation of a Lie group $G$.

Taking $Ad$’s derivative at the identity of the group we get:

$$ad = Ad_{e^0} : TG_e \rightarrow \text{End}(TG_e),$$

where $\text{End}(TG_e)$ denotes endomorphisms on the Lie algebra, which can be seen as the linear operations on $TG_e$. So $ad$ is very interesting because if we were working on complex numbers we could interpret it in a quantum way as giving us the unitary operators, the symmetries, of the observables (which are the vectors on $TG_e$). The image of $ad$ on some element, $x$, of the Lie algebra is

$$ad_x = \frac{d}{dt}igg|_{t=0}Ad_{e^t}.\tag{C.4}$$

Here $e^{tx}$ is an element of $G$ corresponding in the quantum case to a unitary operator denoting the one-parameter symmetry group with observable $x$. There is also a way of writing $ad$ just in terms of the Lie algebra. With $x, y \in TG_e$
we then have:

$$ad_t(y) = [x,y] \quad (C.5)$$

So the adjoint representation of two elements of a Lie algebra is given by the Lie bracket, as written here above. This $ad$ is what we are interested in finding an equivalent of in our algebras generated by $j$.

Adjoint Representation in LD-systems

**Left Translation**

An action of an LD-system such as $A_j = (A_j, \cdot)$, where $\cdot$ denotes application, on itself is given by left translation, $ad_a$, which is a mapping defined by an element $a \in A_j$ such that it takes any $b \in A_j$ to $a \cdot b$ ([47] p.492).

**Adjoint**

The adjoint of $A_j$ is denoted by $\text{Ad}(A_j)$ and is the submonoid of $\text{End}(A_j)$ generated by left translations, $ad$ ([47] p.501). And so $\text{Ad}(A_j)$ together with composition is a monoid containing terms like $ad_a, ad_b \circ ad_c$, etc...

**COMPARISON:** In the Lie group we did not use the left translations directly to define the adjoint. Instead we used automorphisms (homomorphisms that were also isomorphisms) from the group to itself. Using $A_j$ and following Dehornoy’s approach we use left translation directly to define the adjoint of $A_j$. But if we use the metric on $A_j$ proposed by Corazza (see equation (3.25)) we see that left translations are not isomorphisms.

Then, another difference is that in the Lie group we did not define the adjoint of the group from the group $G$ to the group of automorphisms of $G$, $\text{Aut}(G)$, but to a representation of $\text{Aut}(G)$, namely the induced map on the tangent space. So our $\text{Ad}(g)$ is the adjoint representation of $G$.

Similar to the Lie case we can see our adjoint as a mapping $\text{Ad}_a : A_j \rightarrow A_j$ or as $\text{Ad} : A_j \rightarrow \text{End}(A_j)$. To be more precise we have $\text{Ad} : A_j \rightarrow (\text{Ad}(A_j), \circ) \subset \text{End}(A_j)$ (see Figure C.2).

Now we are interested in finding something equivalent to the $ad$ of the Lie case where $ad : TG_e \rightarrow \text{End}(TG_e)$. We already have something similar-looking as $\text{Ad} : A_j \rightarrow (\text{Ad}(A_j), \circ) \subset \text{End}(A_j)$. However, $A_j$ and $(\text{Ad}(A_j), \circ)$ differ structurally in that $A_j$ does not have composition and $(\text{Ad}(A_j), \circ)$ does not have application. Consequently we choose to complete $(\text{Ad}(A_j), \circ, \cdot)$, by letting the adjoint carry the application operation so that $ad_a \cdot ad_b = ad_{a \cdot b}$. But having an LD-monoid means, according to Dehornoy ([47]p.492), that left translation $ad$ is the homomorphism:

$$\text{ad} : (\text{Ad}(A_j), \circ) \rightarrow \text{End}(\text{Ad}(A_j), \circ). \quad (C.6)$$
(See Figure C.2.) So we have found something similar to the Lie $ad$. Also,

![Diagram](image)

Figure C.2: The adjoints $\text{Ad}$ and $\text{ad}$ in our LD-system.

we note that for $a, b \in A_j$ we have $\text{ad}_a(b) = a \cdot b$, while in Lie algebras for $x, y \in TG_e$ we have $\text{ad}_x(y) = [x, y]$. 

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D Hamiltonian Mechanics

This is a quick review of some of the mathematical structures that are central in physics. We will mostly concentrate on the Hamiltonian mechanics. For a complete review see for instance [107].

Symplectic Geometry

One interesting way of describing the mathematical structure behind classical mechanics (as well as the quantum theory) is through symplectic geometry.

In classical mechanics the state of a system, such as say a particle moving in space and time, is described if its position $q = (q_1, q_2, q_3)$ and momenta $p = (p_1, p_2, p_3)$ are given. Given these we can calculate its trajectory in time by using Newton’s laws and therefore tell where the particle will be at any given time. In particular Newton’s second law gives us $\frac{d}{dt}p = F$ and $\frac{d}{dt}q = v$, where $F$ is the force, $v$ the velocity, and $t$ the time. One structured way of dealing with classical mechanics, based on Newtonian mechanics, is called “Hamiltonian mechanics”. It is based on Hamilton’s equations of motion:

$$\frac{dq}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial q},$$

where $H$ is the Hamiltonian function which can be seen as giving the energy of the system.

Let us now focus on the mathematical structure behind this theory. It turns out that Hamiltonian mechanics can be seen as geometry in phase space (the space of $p$’s and $q$’s). In order to see how that works we start with a symplectic manifold. A symplectic manifold is a pair $(M, \omega)$ where $M$ is a smooth manifold and $\omega$ is a symplectic differential 2-form from the tangent space of the manifold, $TM$, to the real numbers. Being symplectic means that $\omega$ is bilinear, antisymmetric, and non-degenerate. The non-degeneracy means that given any vector $u$ in the tangent space at some point if $\omega(u, v) = 0$ then the vector $v = 0$. This non-degeneracy of the symplectic form give us a ’natural’ isomorphism between the tangent spaces and their duals. In general, if a symplectic form is defined on some arbitrary vector space then the non-degeneracy

\[\text{There is also another structured way of dealing with classical mechanics which is called "Lagrangian mechanics".}\]
Figure D.1: Isomorphism between the tangent space and its dual.

Also tells us that the dimension of this vector space has to be even, $2n$, where in Hamiltonian mechanics the $2n$ are given by the $n q$’s and $n p$’s forming the phase space. Looking at a symplectic form geometrically, we can see it as a (skew-)scalar product, (instead of a usual scalar product).

**Isomorphism**

The isomorphism, $I : T^*M_x \rightarrow TM_x$, between the cotangent space $T^*M_x$ at some point $x \in M$ and the tangent space $TM_x$ at some $x \in M$ is given by the correspondence:

$$f_p(r) = \omega(r, p) \quad (D.2)$$

where $p, r \in TM_x$ and $f_p \in T^*M_x$ (see Figure D.1). And $f_p$ can be seen as the representative of $p$.

**Hamiltonian Vector Field**

Now, call $H : M \rightarrow \text{Real}$ the hamiltonian function and let it be a differentiable function on $M$, where $\text{Real}$ denotes the real numbers. Then $dH$ will be a 1-form in $T^*M_x$ and $IdH$ in $TM_x$ will give us a hamiltonian vector field. And so, rewriting the isomorphism in terms of $H$ we have $dH(r) = \omega(r, IdH)$ (see Figure D.2).
Hamiltonian Phase Flow

Evolution in time takes phase space points at some time $t$ to points at some other time in a way so that the energy (here represented by $H$) is conserved. Noether's theorem tells us that if there is a continuous symmetry then there is some quantity that is conserved. The symmetry corresponding to the conservation of energy will be given by the time evolution. Suppose that the vector field $IdH$, corresponding to the Hamiltonian function $H$, gives us a 1-parameter group of symplectomorphisms $g^t_H : M \to M$ so that

$$\frac{d}{dt} |_{t=0} g^t_H x = IdH(x). \quad (D.3)$$

A symplectomorphism is a diffeomorphism that preserves the symplectic form. Then we call $g^t_H$ the Hamiltonian phase flow. For $p = IdH$ conservation of energy is given by $dH(p) = \omega(p, p) = 0$. In general to every vector field $A(x) \in TM_x$ one can associate a 1-parameter group of symplectomorphisms, $g^t_A : M \to M$. The group containing all the symplectomorphisms is called the symplectic group and often denoted by $Sp(2n)$, where the $2n$ refers to the fact that the vectorspace is even dimensional. So this is the group of linear transformations that leave the skew-scalar product invariant.

Lie Algebras

Another useful way of expressing some of the structure of Hamiltonian mechanics (and quantum mechanics as we shall see later) is through Lie algebras.
A Lie algebra $L$ is a non-associative algebra with a binary operation called the Lie bracket, $[,] : L \times L \rightarrow L$, which is bilinear, antisymmetric, and satisfies the Jacobi identity ($[a, [b, c]] + [b, [a, c]] + [c, [a, b]] = 0$). A Lie group $G$, on the other hand, is a differential manifold with differential maps (product and inverse operations). The tangent vectors $A, B \in TG_e$ of the tangent space of the Lie group at identity $e$ can be seen as elements of a Lie algebra (see Figure D.3).

In our case we can see the symplectic group, $Sp(2n)$, as a Lie group and see the vector fields as elements of a Lie algebra. So, one particular subalgebra would be the Lie algebra of hamiltonian vector fields. But it is not only the vector fields that form a Lie algebra. Even the hamiltonian functions form a Lie algebra. Take $H : M \rightarrow \text{Real}$ to be a hamiltonian function and $F : M \rightarrow \text{Real}$ to be another. Then we can define the Lie bracket as the Poisson bracket $\{,\}_P$ as

$$\{F, H\}_P(x) = \frac{d}{dt}|_{t=0} F(g_t^e(x)). \quad (D.4)$$

And so in fact we get

$$\{F, H\}_P = dF(IdH) = \omega(IdH, IdF). \quad (D.5)$$

Hamilton's equations can be written in the following way. For some function $F$ of $q$'s and $p$'s in one dimension we have

$$\frac{dF}{dt} = \frac{\partial F}{\partial q} dq + \frac{\partial F}{\partial p} dp = \frac{\partial F}{\partial q} \frac{\partial H}{\partial p} - \frac{\partial F}{\partial p} \frac{\partial H}{\partial q} = \{F, H\}_P, \quad (D.6)$$
where $H$ is the Hamiltonian. Summarizing all of this into one picture we get Figure D.4.

$$\{F,H\}_P = \omega(I_H, I_{\mathrm{d}F})$$

In the quantum theory the Poisson bracket is replaced by the commutator. And so for some observable $O$ and the Hamiltonian operator $H$ the equation of motion is Heisenberg's equation:

$$i\hbar \frac{dO}{dt} = HO - OH = [H,O] = i\hbar \{H,O\}_P, \quad (D.7)$$

where $\hbar$ is Planck's constant divided by $2\pi$. 

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E Heisenberg’s Equation of Motion

In the Heisenberg picture the state does not evolve, instead the operators evolve. An operator $O_t$ at some time $t$ is given by:

$$O_t = M^{-1}(t)OM(t) \quad (E.1)$$

where $M(t)$ is a unitary time dependent transformation operator and $O$ is a fixed operator (like in the Schrödinger picture).

Here is how we can arrive at Heisenberg’s equation of motion. We can assume that $M(t) = e^{iHt}$ (actually it is $e^{\frac{it}{\hbar}}$ but since Planck’s constant depends on the units used we can always put $\hbar = 1$) and so slightly rewritten equation (E.1) becomes:

$$e^{iHt}O_t = Oe^{iHt} \quad (E.2)$$

Now we can take the derivative with respect to time. This will give us:

$$iHe^{iHt}O_t + e^{iHt} \frac{dO_t}{dt} = OiHe^{iHt} \quad (E.3)$$

Multiplication of this equation with $ie^{-iHt}$ results in:

$$-e^{-iHt}He^{iHt}O_t + i\frac{dO_t}{dt} = -e^{-iHt}OHe^{iHt} \quad (E.4)$$

Rearranging the terms we get:

$$i\frac{dO_t}{dt} = e^{-iHt}He^{iHt}O_t - e^{-iHt}OHe^{iHt} \quad (E.5)$$

Just like for the operator $O$ we have for the Hamiltonian operator $H$:

$$H_t = e^{-iHt}H e^{iHt} \quad (E.6)$$

And so equation (E.5) is actually:

$$i\frac{dO_t}{dt} = H_tO_t - O_tH_t = [H_t, O_t] \quad (E.7)$$

which is Heisenberg’s equation of motion. This equation is representation independent.
References


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