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Investigating grade-6 students' justifications during mathematical problem solving in small group interaction

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ABSTRACT

In this study, we focused on grade-6 students' justifications during mathematical problem solving in small group interaction. Video recordings from two classrooms were analyzed, using Harel and Sowder's (2007) taxonomy of proof schemes as a tool to categorize students' justifications, which showed that the frequency of various types of justifications corresponded to previous studies of adult students' demonstrations of proof schemes. Results also showed that both Non-referential Symbolic and Transformational justifications contributed to students' formulations of general arguments. In addition, agreements between proximately located justifications based on examples and calculations supported students' improvements of solutions. However, in cases of disagreements, justifications based on calculations often had most influence on the proving process even when the calculation was incorrect. Our results imply that teachers should emphasize the importance of agreement between calculations and empirical examples, as well as the significance of counterexamples, as a support to students' progression towards mathematical proving.

1. Introduction

Teaching and learning about proof is considered a both important and challenging part of mathematics education (e.g., Campbell, Boyle, & King, 2020; Hanna & Knipping, 2020). Similar to the teaching of algebra, which traditionally was not introduced until the lower-secondary-school level (Kaput, 2008), the notion of proof has not been included in the elementary-school curricula in most countries. Consequently, the majority of research studies on proving has been conducted at higher levels of education (Campbell et al., 2020). Within the research field of early algebra, studies have shown that it is beneficial to introduce algebraic ideas as early as the first school years, in order to facilitate the transition to formal algebra at the secondary-school level (Kieran, Pang, Schifter, & Ng, 2016). These findings have made their way to the educational system, where several countries have revised their mathematics curricula by including algebraic ideas from the earliest grades (Blanton et al., 2015). In a few countries, the elementary-school curricula have also started to emphasize the importance of activities related to proving, such as exploring and justifying mathematical arguments (Stylianides, Stylianides, & Weber, 2017), which may be a way to address the discontinuity that has characterized the teaching and learning of proof as students move to higher levels of education (Stylianides, 2007). However, in order to strengthen the role of proving in elementary-school mathematics curricula, more research on young students' ways of engaging in proving practices needs to be conducted.

In the study reported in this paper, we investigate grade-6 students' (i.e., 11—12-year-olds') justifications during mathematical

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problem solving in small groups. We do not refer to proofs and proving in a formal sense, such as in Bleiler-Baxter and Pair (2017) or Hemmi (2010). Instead, we consider proving as communicative practices of justification, constituted by students' explanations of how they reached a solution to a mathematical problem, and their arguments for supporting or refuting that solution.

Our analysis of communicative practices implies that individual students' understandings are not in focus (see Ingram, 2018; Knipping, 2008). Instead, we try to capture students' ways of engaging in proving practices, by analyzing actions that students use as justification and how these actions relate to one another. Therefore, we analyze not only the students' final solutions but also sequences of interaction during the ongoing process of solving the mathematical problem within which these solutions are constructed and negotiated.

The aim of this study is to increase the knowledge about grade-6 students' proving practices, in the shape of students' use of justifications, when engaging in solving a mathematical problem in small groups. The following questions guided our analysis:

RQ1 What types of justifications do grade-6 students use when solving mathematical problems in small groups?

RQ2 How are justifications related, with regard to local ordering and agreement, when they occur in immediate proximity to one another?

RQ3 What types of justifications are treated as most influential on the proving process?

2. Theoretical framework

In order to distinguish between different types of justifications, we use Harel and Sowder (1998, 2007) taxonomy of proof schemes as an analytical tool. During a series of teaching experiments, Harel and Sowder collected material in the form of classroom observations, interviews, and written tests, in order to analyze university students' understanding of mathematical proofs. They then identified certain recurrent features of the students' proving processes, which were arranged into the now well-known taxonomy of proof schemes.

In accordance with a sociocultural perspective on learning, Harel and Sowder (2007) stated that students' development of new knowledge is shaped by existing shared knowledge and by the social context. Moreover, they emphasized that students' proof schemes seldom include only formal mathematical proofs; thus, the taxonomy of proof schemes concerns a variety of factors that are involved when students treat an assertion as true during the proving process.

In this study, we use the titles of main classes and subcategories of the taxonomy that were introduced in Sowder and Harel (1998), as that notation turned out to be suitable when analyzing proving processes and justifications among grade-6 students. The only title from Harel and Sowder (2007) that we use is the subcategory of "Non-referential symbolic proof scheme," which emphasizes the absence of quantitative and/or spatial referents. Table 1 shows the titles of main classes and subcategories, including our abbreviations that will be used henceforth.

The *Externally based* proof schemes class represents students' references to factors outside themselves. This class includes the Authoritarian (**AU**), Ritual (**R**) and Non-referential Symbolic (**NRS**) subcategories. Students who justify solutions by referring to information given by some kind of authority, such as what the teacher or a more knowledgeable peer have told them, or what they have read in a textbook, demonstrate the AU proof scheme. Students who instead refer to the appearance of a solution (such as the two-column format of a geometrical proof) demonstrate the R proof scheme, whereas the NRS proof scheme is characterized by students' manipulations of mathematical symbols without consideration of what quantities the symbols refer to. One example of such a manipulation is the faulty treatment of the numerators and denominators when adding fractions: $a/b + c/\mathrm{d} = (a + c)/(b + d)$.

Referring to empirical examples implies a demonstration within the *Empirical* proof schemes class, which includes the Examples-based (**EB**) and Perceptual (**P**) subcategories. Students who refer to results of examinations of just a few examples (or only one example) demonstrate the EB proof scheme, whereas students who refer to their perception demonstrate the P proof scheme. The latter subcategory is mainly associated with the topic of geometry; students demonstrate the P proof scheme if they, for instance, refer to their visual perception when determining whether a triangle is equilateral or a parallelogram is a rectangle.

Students who follow abstract reasoning demonstrate the *Analytic* proof schemes class, which also has two subcategories: the Axiomatic (**AX**) and the Transformational (**T**) proof schemes. The AX proof scheme is characterized by students' references to mathematical axioms, while the T proof scheme is characterized by "generality, operational thought and logical inference" (Harel & Sowder, 2007:809). The criterion of generality implies that the student justifies a "for all" argument, i.e., "not isolated cases and no exception is accepted" (ibid.). The criterion of operational thought is instead related to students' formulation of goals and sub-goals, and the anticipation of their outcomes, while the third criterion implies that students understand and apply logical inference rules. Proving the divisibility rule of 9 may serve as an example. For instance, the number 837 is shown to be divisible by 9 as the sum of $(8 + 3 + 7)$ is divisible by 9 through the following representation:

$$(8 \cdot 100 + 3 \cdot 10 + 7) = (8 \cdot 99 + 3 \cdot 9) + (8 + 3 + 7) = 9(8 \cdot 11 + 3) + (8 + 3 + 7)$$

Students demonstrate the T proof scheme when they not only show that the specific number 837 is divisible by 9 but also indicate that the same process can be applied to any whole number (Harel & Sowder, 1998).

According to Harel and Sowder (1998), "proving" is a process that removes doubts about the truth of a conjecture. However, in the present study, we do not determine whether students are actually convinced that their (or their peers') conjectures are true or not. Instead, we analyze students' acts of justification during the ongoing process of solving a mathematical problem, using the categories of the taxonomy of proof schemes as a tool to distinguish different types, such as "EB justifications" or "T justifications".

In addition to investigating grade-6 students' justifications, the relations between different types of justifications are analyzed in

Table 1
The taxonomy of proof schemes, including our abbreviations of the titles of subcategories.

Externally based proof schemes			Empirical proof schemes		Analytic proof schemes	
Authoritarian (AU)	**Ritual (R)**	**Non-referential Symbolic (NRS)**	**Examples-based (EB)**	**Perceptual (P)**	**Axiomatic (AX)**	**Transformational (T)**

terms of their local ordering (i.e., how justifications follow one another), agreement (i.e., whether different types of justifications argue for the same solution), and influence on the proving process (i.e., how different justifications are treated during argumentative processes). This part of the analysis is based on a dialogical theory of communication (e.g., Linell, 2009) as students' acts of justification are analyzed with regard to their positions in the sequence of interaction and the meaning they make in interaction with preceding and following contributions. The theory emphasizes that communicative acts are locally achieved in the social interaction between participants; therefore, the argumentative relation between the act of justification and the preceding and following talk is taken into consideration when analyzing the justification's locally achieved meaning and its influence on the proving process.

3. Previous research

The following two subsections present previous studies on students' encounters with proving and proofs (3.1) and students' demonstrations of proof schemes (3.2). In subsection 3.1, the focus is on research regarding students who have not yet reached university level, while subsection 3.2 also considers studies on adult students' proof schemes.

3.1. Previous studies on young students' encounters with proving and proofs

During the past few decades, the research field of mathematics education has become increasingly influenced by perspectives that emphasize communication and participation as essential factors for the individual student's development of new knowledge (e.g., Hanna & Knipping, 2020). Despite the generally acknowledged "social turn" in mathematics education research (see Lerman, 2000), many studies on students' mathematical argumentation and use of proof have focused on individual cognitive processes (Blanton & Stylianou, 2014; Campbell et al., 2020).

The introduction of logical argumentation in the earliest school years has been emphasized as a potential support of students' mathematical development (e.g., Ball, Hoyles, Jahnke, & Movshovitz-Hadar, 2002; Ellis, 2007; Harel & Sowder, 2007; Stylianides, 2007). In addition, empirical studies have indicated that students at elementary-school level are able to use both inductive and deductive arguments when engaging in mathematical problem solving (e.g., Ayala-Altamirano & Molina, 2021; Flegas & Charalampos, 2013; Maher, 2009; Morris, 2009; Whitenack & Yackel, 2002). However, Flegas and Charalampos (2013) have also argued that educators need to know more about "the forms and elements of proof that young students have the capacity to use in the classroom" in order to be able to arrange a teaching that supports students' progression towards formal proving.

According to Healy and Hoyles (2000) and Lee (2016), secondary-school students are often more successful in assessing the validity of a completed proof than formulating their own proof. These findings, as well as Kanellos (2014) study of grade-9 students' solutions to mathematical problems, indicate that students who have not yet encountered teaching about formal mathematical proving may be familiar with the requirements of a proof, despite not being able to produce proofs of their own.

Examples often play an important role when students explain and justify solutions (Campbell et al., 2020; Ellis, Lockwood, Williams, Dogan, & Knuth, 2012; Healy & Hoyles, 2000; Stylianou, Chae, & Blanton, 2006), but it is also common that students consider proving to be constituted by a series of procedures that need to be followed in a mechanical way (Harel & Rabin, 2010; Liu & Manouchehri, 2013). Healy and Hoyles (2000) questionnaire among secondary-school students showed that although many students used examples when justifying solutions, they tended to treat references to examples as arguments with lower status than explanations and justifications that involved algebraic notation. To some extent, this can depend on the classroom culture. Based on the notion of sociomathematical norms (developed by Yackel & Cobb, 1996), Levenson, Tirosh, and Tsamir (2009) argued that students' choices of other types of explanations than those that the teacher encouraged them to use was due to the teacher's lack of evaluation of the students' arguments; the lack of feedback may have contributed to a sociomathematical norm that all explanations were equally acceptable.

Supporting students' ability to generalize has also been singled out as a particularly important part of students' progression towards mathematical proving (Campbell et al., 2020; Stylianides et al., 2017). Students' generalizations are enhanced by a classroom culture that encourages posing questions, sharing ideas, predicting outcomes of examples, clarifying solutions, and refining each other's suggestions (Ayala-Altamirano & Molina, 2021; Blanton & Stylianou, 2014; Ellis, 2007, 2011; Harel & Lesh, 2003). In addition, Ellis (2007) teaching experiment, which involved a small group of grade-7 students, indicated that there is a bidirectional relation between justification and generalization, in the sense that students' engagement in acts of justification can support their ability to generalize, and vice versa.

3.2. Previous studies on students' proof schemes

Harel and Rabin (2010) classroom observations showed that students' demonstrations of the AU proof scheme were related to teaching practices that were characterized by the teacher's unidirectional instruction on how to perform mathematical operations, whereas there was limited interaction among students. Within these practices, the teacher was established as the "sole arbiter of correctness" (Harel & Rabin, 2010:156), which contributed to students' focus on the teacher's instruction of how to use the correct procedures, instead of making their own attempts to solve the problem and raise questions about their peers' solutions. However, several studies using the taxonomy of proof schemes as an analytical tool (e.g., Ellis, Lockwood, Dogan, Williams, & Knuth, 2013; Erickson & Lockwood, 2021; Housman & Porter, 2003; Kanellos, 2014; Kanellos, Nardi, & Biza, 2018; Lee, 2016; Liu & Manouchehri, 2013; Sears, 2019; Sen & Guler, 2015) have not taken classroom interaction into account. Instead, they have analyzed individual students' proving, based on interviews and/or written material.

Previous studies have shown that the R and P proof schemes are common during work on geometry, whereas the NRS proof scheme primarily has been identified in students' reasoning about tasks where algebraic formulae are considered useful (Harel & Sowder, 1998; Kanellos, 2014; Liu & Manouchehri, 2013; Sen & Guler, 2015; Stylianou et al., 2006). Still, demonstrations of Empirical proof schemes are generally most common, regardless of topic (e.g., Harel & Sowder, 1998). Based on the analysis of interviews, Housman and Porter (2003) raised the explicit question whether instructing students to generate examples when solving mathematical problems may support students' progression from the Externally based to the Empirical and Analytic proof schemes. On the other hand, Stylianou et al. (2006) problematized the relation between students' problem-solving strategies and their demonstrations of proof schemes, as interviews showed that students who demonstrated Externally based proof schemes more often used examples to confirm solutions, rather than as tools for explorations that might support the formulation of general arguments.

Students' level of education influences what proof schemes they demonstrate. For instance, analyses of secondary-school students' proving have not identified any demonstrations of the AX proof scheme, as most students at this level of education have not yet encountered teaching about axioms (Kanellos, 2014; Sen & Guler, 2015). In addition, Blanton and Stylianou's (2014) teaching experiment in an undergraduate discrete mathematics course showed an increase of demonstrations of Analytic proof schemes after one year of teaching. When interviewed, students majoring in mathematics showed more demonstrations of Analytic proof schemes compared to students who had not completed mathematics courses at advanced levels (Housman & Porter, 2003; Sears, 2019), although Externally based proof schemes also have been identified in interviews with students taking advanced mathematics courses (Erickson & Lockwood, 2021).

When developing the taxonomy of proof schemes, Harel and Sowder (1998) not only identified various characteristics in students' arguments but also emphasized that students' proof schemes are neither hierarchical nor exclusive during the proving process. For instance, Harel and Lesh (2003) study of a small group of grade-8 students' work on a geometrical problem showed that students demonstrated both the P and the T proof scheme. In interviews with adult students about a set of conjectures, Housman and Porter (2003) and Sears (2019) in several cases also identified two or more proof schemes during the course of the interview. The majority of the students who were interviewed by Housman and Porter demonstrated Empirical proof schemes, while demonstrations of Analytic proof schemes were also quite common. Sears instead reported more co-occurrences of Externally based and Empirical proof schemes, as well as instances where students demonstrated all three proof scheme classes. When analyzing grade-9 students' written solutions to mathematical problems, Kanellos (2014) also identified several cases where students demonstrated Externally based and Analytic proof schemes but no apparent co-occurrences of all three classes of proof schemes. However, neither Housman and Porter, Sears, nor Kanellos explicitly reported how proof schemes that co-occurred were related during the proving process, for instance with regard to their internal ordering, or if they built on one another. Kanellos et al. (2018) further analyzed grade-9 students' written solutions to mathematical problems where more than one proof scheme was identified. In these cases, the proof schemes were noted as "combined, " but the ordering of the demonstrations was not taken into account in the analysis.

4. Method

The taxonomy of proof schemes was a result of the analysis of students' proving when working individually, in small groups, and in whole class settings (Harel & Sowder, 1998). However, previous studies that have used the taxonomy as a tool to analyze secondary-school students' proving (e.g., Kanellos, 2014; Kanellos et al., 2018; Lee, 2016; Liu & Manouchehri, 2013; Sen & Guler, 2015) have focused on written products in the form of answers to surveys and/or solutions to mathematical problems, which do not show the various steps taken during the proving process that resulted in the final product (Knipping, 2008). Our empirical material differs from the data of previous studies as we have analyzed video recordings of students' naturally occurring classroom interaction, which enabled the exploration of students' use of justifications during the interactional process of solving a mathematical problem. Thus, we analyzed not only what types of justifications students used but also how students' justifications related to one another argumentatively, as well as with regard to local ordering. In the following two subsections, we present the participants of the study, our empirical material, and our method of analysis.

4.1. Participants and empirical material

Our empirical material consisted of video recordings and students' notes on individual worksheets. The video recordings took place during mathematics lessons in two grade-6 classrooms in a medium-sized mainstream school, situated in a large municipality in Sweden. The material was collected by the first author as a sub-study of a research project focusing on various aspects of students'

There is a queue at the bus stop. In how many ways can:

a) 2 persons form a queue?
b) 3 persons form a queue?
c) 4 persons form a queue?

Fig. 1. The formulation of the mathematical problem "Queue at the bus stop".

collaboration (see Klang et al., 2020), which was approved by a Swedish Regional Ethics board (Dnr.2017/372). Informed consent was retrieved from the students' legal guardians.

The video recordings were conducted over two weeks. In total, 17 mathematics lessons were recorded (eight lessons in one classroom and nine lessons in the other). The teachers were asked to instruct their students in problem-solving activities, and to choose the topic of the problems. Each lesson lasted 55 min on average and was initiated by the teacher giving the students general information about the assigned problem; the teacher did not give specific instructions on how to solve the problem as one of the aims of the lessons was to encourage students to explore their own problem-solving strategies. The students were instructed to begin by working individually, thereafter to work in dyads, and then to combine dyads into groups of four. They were also instructed to not only share and explain their solutions but also to try to find a joint solution. At the end of the lesson, the teacher led a whole class discussion regarding the groups' various strategies of solving the mathematical problem.

During the observed lessons, four dyads in each class were recorded, and then followed when they formed two groups. Each group was recorded by two video cameras. During the whole lesson, the cameras were arranged in such a way that they captured not only students' verbal talk but also gaze, facial expressions, gestures, and use of artefacts. An additional camera captured the teacher's interaction with the students.

We have focused on dialogues and group discussions about a problem entitled "Queue at the bus stop." At the beginning of the lesson, the teacher handed out a worksheet where the problem was formulated as shown in Fig. 1.

On the back of the worksheet, there was an additional task (d) about the formulation of a "general rule" concerning how to calculate the number of queues for any number of persons.

The problem "Queue at the bus stop" is a combinatorial task, concerning permutations. According to Lockwood (2013), who developed a model for students' combinatorial thinking, the topic of combinatorics has the potential of offering a rich context for mathematical problem solving. Young students' strategies when solving combinatorial problems have also been thoroughly discussed by English (1991). However, our focus was on students' justifications during mathematical problem solving, and not on their strategies when solving a problem within a specific topic. Therefore, we will not analyze students' justifications from the point of view of investigating their problem-solving strategies in combinatorics.

In a pre-study (Fredriksdotter, Norén, & Bråting, 2021), we analyzed discussions within two groups of students, who presented identical solutions to task c, but oriented towards different social and sociomathematical norms. In the present study, we analyzed discussions regarding tasks a—d among all four observed groups, i.e., in total 16 students (the number of students present in the two classrooms was 22 and 23 respectively). We found that the students' work on the problem "Queue at the bus stop" was of particular interest as there was variety in their strategies of solving the problem, and in their way of justifying their solutions.

4.2. Method of analysis

The video recordings were transcribed by the first author and involved both spoken and embodied actions. The students' verbal talk was transcribed verbatim (including mistakes, repetitions and interruptions) along with pauses, overlapping talk, and information regarding prosody. Simultaneous use of spoken language and embodied actions such as depictive gestures (Streeck, 2009) and manipulation of objects was also noted.

In the transcribed dialogues and group discussions, we identified local interactional episodes in order to enable detailed analysis of relations between justifications that referred to the same solution. We based our analysis of episodes on how the students organized the talk topically, the type of activities they were engaged in (Korolija & Linell, 1996), as well as on the participation framework (Goodwin & Goodwin, 2004). Participation was viewed as an interactional achievement based on students' spoken and embodied orientation to one another, i.e., talk with consistent topic, activity type, and participants was coded as a coherent episode, while changes in any of these three aspects were coded as episode boundaries. Table 2, which represents the initial part of the dialogue between Alex and John that also involved their teacher, illustrates how episode boundaries were coded.

Within the episodes, students' justifications were identified and categorized according to the subcategories of Harel and Sowder's taxonomy of proof schemes. Thus, this part of the analysis was in line with the procedures of qualitative content analysis, where the development of categories for coding was concept-driven (Mayring, 2015). The focus of the analysis was not on individual students' understanding of the mathematical content of the task; instead, we focused on the students' actions (see Ingram, 2018; Knipping, 2008).

Early on, our analysis showed that students' justifications could be categorized according to four of the subcategories of Harel and

Table 2
Example illustrating episode boundaries.

Episode 1: Alex shares his solution to task a. John agrees with Alex' solution.	**Topic**: task a **Activities**: sharing and agreeing **Participants**: Alex and John
Episode 2: Alex asks Teacher if his solution is correct. Teacher points out that Alex' solution concerns the direction of the queue, rather than the arrangement of the persons forming the queue. Using the initials T and A, Teacher writes a table on Alex' worksheet, representing the two possible ways that the two of them could form a queue.	**Topic**: task a **Activities**: sharing and correcting **Participants**: Alex and Teacher
Episode 3: Alex resumes his dialogue with John, now discussing task b. Using the initials A, T, and J, Alex investigates the number of queues that he could form together with Teacher and John. Alex justifies his solution by referring to his table of initials.	**Topic**: task b **Activities**: problem solving and justification **Participants**: Alex and John

Sowder's taxonomy of proof schemes: AU, NRS, EB and T. Table 3 exemplifies our categorization in accordance with these four proof schemes.

The analysis of episode units in the group talk not only facilitated the identification of justifications; it was also useful during the categorization of justifications as "single" or "grouped." Justifications within the same episode were categorized as grouped if they followed each other in a sequence without being intermediated by other actions. For instance, Alex' justification of his solution (Table 2, Episode 3) was categorized as an EB justification, as he referred to a direct measurement of the number of queues. In addition, Alex' justification was categorized as "single" as it was not immediately followed by some other type of justification.

Groupings of justifications may instead be exemplified by the summary of an episode within Andy and Hugo's dialogue. When solving task c during individual work, Andy had written two tables showing six permutations of four digits, where the digit 1 was fixated in the first position in one of the tables, while the digit 2 was kept first in the other table. He then suggested the multiplication 6·4 as a solution to task c. When Andy presented 24 as his answer to task c, Hugo responded "how do you know that?" By referring to his table, Andy justified that four persons can form six queues when one person stays in the first position; thus, we categorized his argument as an EB justification. Thereafter, Andy justified his calculation 6·4, saying "if the same person is in the first position, then the others can stand in six different ways, and now there are four persons who can be first, so I just thought six times four." This part of Andy's justification demonstrated generality and operational thought; thus, we categorized it as a T justification. Andy's two justifications occurred in immediate proximity, without changes in topic, activity, or participants between them; therefore, we categorized them as a "group".

After discussing definitions of categories, "anchor samples" (such as the examples in Table 3) and coding rules (Mayring, 2015:377) all three authors analyzed the material individually, with regard to episode boundaries and types of justifications (including the analysis of "singles" and "groups"). The intercoder reliability was calculated through Miles and Huberman (1994) formula:

Reliability = [Number of agreements /(Total number of agreements + disagreements)]

According to Miles and Huberman (1994) the intercoder reliability is seldom better than 70% after the initial coding. In this study, there was an agreement on about 85% of the coding; thus, the intercoder reliability was high. Our disagreements primarily concerned how to code utterances where students suggested an incorrect solution. These cases of disagreement were discussed until consensus was reached.

The qualitative content analysis was followed by a quantitative analysis. In all, 141 justifications were identified, and categorized according to the taxonomy of proof schemes. The ratio of each type of justification was calculated, as well as the distribution between justifications that occurred as "singles" and "groups".

Thereafter, we analyzed sequences of turns (that is, we considered what preceded and what followed each justification) in order to investigate the local ordering and agreement between justifications that occurred together as groups. This part of the analysis also considered what justifications were treated as most influential on the proving process. In the example of Andy and Hugo's dialogue, the T justification followed the EB justification, which we represented as the local ordering EB → T. As the two types of justifications

Table 3
Examples of how justifications were categorized.

Proof schemes	Examples
Authoritarian (**AU**)	Referring to work done during a previous lesson was categorized as an AU justification as this action resembles referring to an example in a textbook.
Non-referential Symbolic (**NRS**)	Suggesting a numerical calculation that contradicted the empirical situation of forming a queue, or with results of examinations of examples, was categorized as an NRS justification. This action represents a symbol manipulation that lacks quantitative referents.
Examples-based (**EB**)	Students' use of depictive gestures representing persons' relative positions, and their talk about persons' positions in the queue, were categorized as EB justifications. These actions are equivalent to measuring a quantity by counting numbers of objects.
Transformational (**T**)	Suggesting a solution that implicitly argues that the number of queues for n persons can be calculated by multiplying n with the number of queues for $(n-1)$ persons, was categorized as a T justification; so was refuting a multiplication due to a counterexample. These actions demonstrate generality and operational thought.

Table 4
Symbols representing relations between different types of justifications.

Symbol	Use
→	Arrow indicates local ordering of justifications.
=	Equal sign indicates that justifications agreed.
≠	Not equal sign indicates that justifications disagreed.
T	Abbreviation written in bold italics indicates that the justification was treated as most influential.

agreed, in the sense that they argued for the same solution, we represented the argumentative relation as EB = T (whereas the notation EB ≠ T would have been used if the justifications had disagreed). In addition, we analyzed how different justifications influenced the proving process, where we highlighted the justification that was treated as most influential by writing its abbreviation in bold italics. For instance, after Andy's justification of his multiplication, Hugo repeated both the solution 24 and the multiplication 6·4, which indicated that the ***T*** justification was treated as most influential. Table 4 shows the symbols that we used during our analysis of relations between justifications.

Section 5 presents the results of our analysis, including excerpts from our transcripts, where the categorization of the students' justifications is marked in the rightmost margin of the excerpt. The excerpts within our report are translated into English. Appendix A contains information on the symbols used in the excerpts. Appendix B contains the excerpts in Swedish. All participants' names have been changed.

5. Results

In this section, we present our findings regarding students' justifications of solutions during mathematical problem solving in small groups. In subsection 5.1, we present the overall occurrence of various types of justifications in our material. In subsection 5.2, we present our findings of how different types of justifications related to one another in terms of local ordering and argumentative relation (i.e., agreement or disagreement). Thereafter, we present our findings of various justifications' influence on the proving process (subsection 5.3).

5.1. Overall occurrence of various types of justifications

We identified 141 utterances and actions that students used as justifications of solutions to the mathematical problem, which were categorized according to the subcategories of the taxonomy of proof schemes. Fig. 2 shows the relative frequency of the different types of justifications.

EB justifications were most common (Fig. 2) and primarily constituted by students' references to results of empirical examinations. Fig. 3 shows an example where a student has arranged three symbols in six different ways, which the student referred to when justifying the solution 6 to task b (about three persons).

The T and NRS justifications occurred to a comparable extent (Fig. 2) and were primarily constituted by suggestions of various multiplications. Two examples of justifications involving multiplications that we categorized as T justifications were 3·2 for task b, and 4·6 for task c, whereas the corresponding multiplications that we categorized as instances of NRS justifications were 3·3 and 4·4. In these particular examples, the multiplications that represent T justifications are correct whereas the corresponding multiplications that represent the NRS justifications are incorrect. However, whether a solution is correct or incorrect is not a criterion for categorizing it in

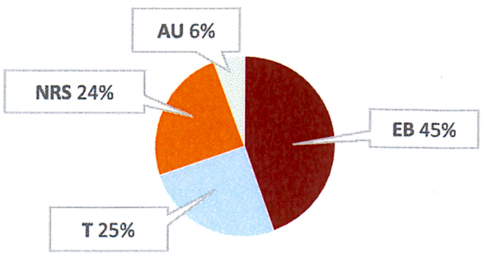

Fig. 2. Relative frequency of different types of justifications.

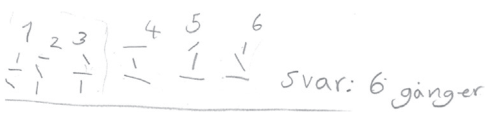

Fig. 3. Extract from Wanda's initial notes. *Note:* Next to her table, Wanda has written "Answer: 6 times" which clarifies that the ordering of the three symbols can be altered six times.

accordance with the taxonomy of proof schemes. For instance, when the student Andy (mentioned in subsection 4.2) was challenged by his peers to try to calculate the number of queues for five persons he set up the correct multiplication $24 \cdot 5$. However, he made an error when performing the calculation; Andy stated that five persons could form a queue in 95 ways instead of 120. Still, we categorized his calculation as a T justification, as it was based on the same type of generalization as his calculation $6 \cdot 4$ for four persons. Further details regarding our analysis of justifications of multiplications for tasks b and c are presented in subsection 5.2.

The category of AU justification was least common (Fig. 2). AU justifications primarily occurred in the form of students' references to a peer's solution while saying, for instance, "I solved it in the same way as she did it" or "he says that's a correct solution".

No instances of the R, P or AX justifications were identified. The absence of these types of justifications were most likely due to the topic of the task, and to the students' age. The R and P proof schemes are commonly demonstrated during work in geometry, whereas "Queue at the bus stop" (Fig. 1) is a combinatorial problem. Moreover, we cannot expect grade-6 students to demonstrate the AX proof scheme, as teaching about mathematical axioms is not part of the elementary-school curriculum in Sweden.

5.2. Relations between justifications

Fig. 4 shows the relative frequency of students' "single" and "grouped" justifications. Fig. 4 also highlights information regarding which types of justifications that occurred together as groups.

A majority of the 141 justifications occurred as single cases, immediately followed by other actions or activities, while the remaining cases formed groups of mainly two consecutive types of justifications (Fig. 4). The groups were primarily constituted by EB and T justifications, as well as by EB and NRS justifications (Fig. 4), and we therefore chose to focus on these groups. In subsection 5.2.1 and 5.2.2, we present our findings regarding relations between EB and T justifications, and between EB and NRS justifications.

5.2.1. Relations between Examples-based and Transformational justifications

When EB and T justifications occurred as a group, the local ordering was EB → T, i.e., the T justification followed the EB justification. Excerpt 1 shows how Maria, when justifying her solution to task c, at first referred to a set of six permutations of four digits and

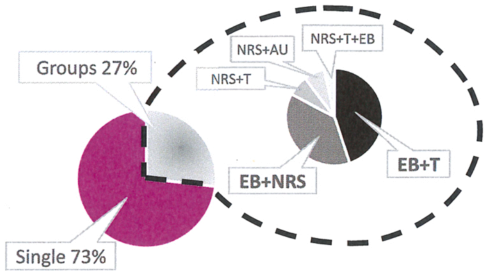

Fig. 4. Relative frequency of single and grouped justifications, highlighting information regarding groups of justifications.

Excerpt 1.

66.	Maria:	((points at Fig.5)) here are all the ways	EB
67.		all the possible ways with a one first right	
68.	Luiza:	mm	
69.	Maria:	there is no more with one (.)	
70.		because we've tried y'know ((points at Fig.5))	
71.		there (.) there (.) there well we've tried	
72.		I've tried all possible ways with a one first	
73.		and then I thought that there are	T
74.		exactly as many ways with two first	
75.		exactly as many ways with three first	
76.	Luiza:	but is it the same can't we do two three four one	
77.		(0.2)	
78.	Maria:	huh	
79.	Luiza:	tha- that's also a way you know two three	
80.	Maria:	exactly but that's my point	
	((lines 81-90 omitted: Maria repeats her justifications))		
91.	Maria:	so with four digits I just did (.)	T
92.		six times four and that's twenty four because I got	
93.		one two three four five six ways	
94.	Luiza:	but it c- there are more if you start	
95.	Maria:	twenty four (that's how many you can do)	
96.	Luiza:	oh yeah	

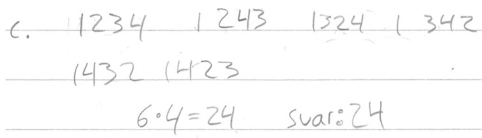

Fig. 5. Extract from Maria's notes. *Note:* Next to her multiplication, Maria has repeated that her "Answer" is 24.

then concluded that the number of queues for four persons can be calculated through the multiplication 6·4.

During Maria and Luiza's dialogue about task c, Maria at first referred to her list of six permutations (line 66—72), which we categorized as an EB justification. Maria then pointed out that if she instead had fixated 2 or 3 in the first position, there would still have been six permutations due to the possible arrangements of the three following digits (lines 73—75). Slightly later, she concluded that she could solve task c through the multiplication 6·4 (lines 91—93). In her justification of the multiplication, Maria demonstrated both generality and operational thought, and we therefore categorized it as a T justification.

The dialogue between Maria and Luiza in Excerpt 1 is a representative example of the local ordering and agreement between EB and T justifications. In cases where EB and T justifications occurred together, the T justification not only followed the EB justification but also agreed with it, in the sense that they both argued for the same solution. Thus, the local ordering EB → T and the argumentative relation EB = T were consistent.

In addition, Excerpt 1 is a representative example of how T justifications in our material built on the use of examples. Maria referred to her list of six permutations, which she explicitly pointed out to be the result of an empirical examination (i.e., an EB justification) in showing her notes and saying "I have tried all possible ways with a one first" (line 72). Maria then indicated that the examination of the example was followed by a generalization (i.e., a T justification) as she continued justifying her solution by saying "then I thought" (line 73), and later concluded that she "just did six times four" with four digits (line 91).

5.2.2. Relations between Examples-based and Non-referential Symbolic justifications

The relations between EB and NRS justifications were not as consistent as those in the groups of EB and T justifications. NRS justifications followed EB justifications and vice versa, i.e., we found examples of the local ordering NRS → EB as well as EB → NRS. In addition, the two categories of justifications were both in agreement and disagreement with one another.

The local ordering NRS → EB with the argumentative relation NRS = EB is illustrated by Excerpt 2, showing a discussion between Albert and Lisa, where Melvin also contributed with one objection. The excerpt shows how Albert and Lisa initially agreed that the general rule for calculating the number of queues of more than two persons was to square the number of persons forming the queue, where they referred to the operation of squaring as 'times itself.' Excerpt 2 also shows how Lisa applied the rule of 'times itself' to the task about three persons.

Excerpt 2.

14.	Albert:	you always do (.) like the number times its:el-	NRS
15.		but not for that one except for that one	
16.		((points at task a at his worksheet))	
17.	Lisa:	no I kno:w yeah I thought so too	
18.	Albert:	except for that one because it doesn't work	
19.	Lisa:	but why eh y'know why can't we do it for that one	
20.	Albert:	b- because i:t kind of doesn't work ((giggles))	
21.	Lisa:	((giggles)) but eh yeah yeah okay	
	((lines 22-32 omitted: Albert repeats his solution))		
	((**Episode boundary: talk about task a is initiated by Albert**))		
	((lines 33-74 omitted: justification of solution to task a))		
	((**Episode boundary: talk about task b is initiated by Lisa**))		
75.	Lisa:	but how eh why should we do three times three (.)	NRS
76.		because they can stand in three different places	EB
77.		↑nice	
78.	Albert:	eh y- yes yes	
79.	Lisa:	everyone can stand in three different places	
80.		and then it's three	
81.	Melvin:	but we weren't allowed to do it like that were we	
82.	Albert:	well I wrote that you do the number times	NRS
83.		itsel- itself	
84.	Lisa:	I wrote you take the number in the queue (.)	
85.		times the number in the queue	
86.	Albert:	yeah exactly exactly but that's what I meant	

Albert and Lisa agreed that the rule of ‘times itself’ would always apply, except when two persons are forming a queue (lines 14—18). Neither Albert nor Lisa made any attempt to check the validity of their rule, for instance through the examination of an example. We therefore categorized their explanation of ‘times itself’ as an NRS justification. Albert and Lisa could not explain why the rule of ‘times itself’ would not be applicable when two persons form a queue (lines 19—20); instead, they engaged in drawing a picture representing two persons forming two different queues (omitted lines 33—74), which confirmed that ‘times itself’ would not apply for two persons.

Quite abruptly, Lisa moved on to talk about task b by raising the rhetorical question about the multiplication “three times three” (line 75). Although Lisa did not repeat her and Albert’s general rule, the multiplication 3·3 could be associated to the rule of ‘times itself’ that the students initially agreed on. We therefore categorized Lisa’s utterance in line 75 as a resumption of the NRS justification in lines 14–18. Lisa did not examine an example of three persons, but her following utterance referred to the empirical situation of three persons forming a queue (line 76) and we therefore categorized her argument as an EB justification. Lisa’s use of the word “because” (line 76) when referring to the empirical situation shows that the example was used to confirm the multiplication 3·3, i.e., the argumentative relation of the justifications may be represented as NRS = EB, where the EB justification was used to support the NRS justification.

Excerpt 2 showed an example of the local ordering NRS → EB and the argumentative relation NRS = EB. The dialogue between Wanda and Leyla in Excerpt 3 instead shows an example where an NRS justification both followed and challenged an EB justification, i.e., Excerpt 3 shows the local ordering EB → NRS and the argumentative relation EB ≠ NRS. Prior to the dialogue in Excerpt 3, Wanda had shared her solution to task b about three persons, by showing a table where three symbols were arranged in six different ways (see Fig. 3 above). Wanda referred to an empirical examination, which we categorized as an EB justification. Leyla initially agreed with Wanda’s solution, and after agreeing on the number of queues for two persons as well, Leyla resumed their discussion about task b—and then suddenly proposed the multiplication 3·3. Excerpt 3 shows how Wanda again justified her solution by referring to her table of symbols, whereas Leyla maintained the suggestion of 3·3.

Excerpt 3.

42.	Wanda:	there i- I don’t know if there is any other way	
43.		or anything else that you can do	
	((lines 44-49 omitted: talk about Fig.3))		
50.	Wanda:	but I think like that ((points at table, Fig.3))	EB
51.	Leyla:	either six or nine	NRS
52.		(0.2)	
53.	Wanda:	mm ((nods, writes ”(9)” next to her table))	
54.	Leyla:	because nine then it’s three times three (.)	
55.		and then everyone gets to be: different	
56.	Wanda:	((looks at Leyla)) mm	

When arguing that she had found all possible permutations for three persons, Wanda referred to her table (line 50), which we categorized as an EB justification. Despite having studied Wanda’s table, Leyla suggested both six and nine as possible solutions to task b, and explicitly maintained the solution “three times three” (lines 54—55). As Leyla’s multiplication contradicted the result of Wanda’s empirical examination, we categorized it as an NRS justification. Leyla’s justification both followed and challenged the EB justification, i.e., the relations may be represented as EB → NRS and EB ≠ NRS.

5.3. The influence of various justifications on the proving process

When the subsequent justification challenged the preceding one, as in Excerpt 3, the argumentative process was apparent, as there was a local opposition between arguments. In Excerpt 3 the NRS justification was agreed upon, in the sense that both students accepted nine as the number of queues that three persons can form, despite contradicting the empirical example (Excerpt 3: lines 51—56). Excerpt 3 therefore represents an example where the ***NRS*** justification was treated as most influential on the proving process. In addition, Excerpt 3 represents a pattern within the argumentative relation between EB and NRS justifications as NRS justifications challenged EB justifications more often than vice versa. EB justifications were instead used to confirm NRS justifications (as in Lisa’s reference to an empirical situation in Excerpt 2) more often than the other way round.

Argumentative processes also occurred within episodes where justifications agreed. Excerpt 1 showed an example where a solution was objected to, despite the agreement of the EB and T justifications. Luiza questioned Maria’s generalization by referring to the example of putting the digit 2 in the first position (Excerpt 1: lines 75, 78), which prompted Maria to repeat her justification and her conclusion that task c could be solved through the multiplication 6·4 (Excerpt 1: omitted lines 80—90, lines 91—93). This was followed by Luiza’s agreement (Excerpt 1: line 96), which indicates that the ***T*** justification was treated as most influential. Excerpt 1 also represents a common trait of argumentative processes that followed T justifications. Just like Luiza, other students’ who listened to a peer’s justification of a general solution at first objected to it by referring to empirical examples—but then accepted the solution when the justification was repeated (sometimes more than once).

Excerpt 2 was another example where agreeing justifications (NRS = EB) were objected to. In Excerpt 2, Melvin questioned whether they were "allowed" to solve the problem the way Lisa suggested it (Excerpt 2: line 81). Albert and Lisa did not respond to Melvin's objection—and Melvin himself did not pursue the argumentation. Instead, Albert and Lisa repeated and again agreed upon their initial solution of 'times itself' (Excerpt 2: lines 82—85). Thus, Excerpt 2 showed an example where the ***NRS*** justification was treated as most influential on the proving process, without taking objections into account.

Excerpts 1—3 showed examples of argumentative processes that were initiated and completed within the same episode. However, some argumentative processes were extended across episode boundaries, which is illustrated by a discussion among Jason, Milton, Gloria and Emma in Excerpt 4. This excerpt shows how Jason initiated an episode of talk about task b, by suggesting the multiplication 3·2 as a way of finding the number of queues for three persons. It also shows how Jason's peers objected to his solution, which was followed by a change of topic before the discussion about task b was resumed.

Excerpt 4.

		((**Episode boundary: talk about task b is initiated by Jason**))	
13.	Jason:	for the three it's three times two	
14.	Milton:	is it	
15.	Jason:	yeah so it's six	
16.	Milton:	why is it three times two then	
17.	Gloria:	why is it three times two	
18.	Jason:	because eh if you (.)	
19.		I drew all the ways that are possible	**EB**
20.		((points at his picture of six permutations))	
21.		and there are six (0.2) it's not possible	
22.		to do it more ways (0.1) with three	
23.	Milton:	are you sure	
24.		(0.2)	
25.	Emma:	but it ought to be nine times ↑right (.)	**NRS**
26.		or three times three	
27.	Jason:	because if you (.)	
28.		((points at his picture)) because it's three right (.) look because	**EB**
29.		there are three different (.)	
30.		but the ones in the back (.) they can change once (.)	
31.		and then it's three times two if everyone changes	**T**
32.		because you can do two of each (0.2)	
33.		so that's why it's three times two you can do	
	((lines 34-41 omitted: comments about Jason's solution))		
42.	Milton:	↑I think it's three times three	**NRS**
43.	Emma:	me too	
		((**Episode boundary: talk about task a is initiated by Milton**))	
44.	Milton:	but the first one is two (.) everyone knows that right	
	((lines 45-55 omitted: talk about task a))		
		((**Episode boundary: talk about task b is resumed by Gloria**))	
56.	Gloria:	well b then I didn't manage to do c	
57.	Jason:	but b is six I think (.) because if (.)	**EB**
58.		try to draw more than six symbols	
59.		it's not possible (0.2) it's not possible	
	((lines 60-72 omitted: group members draw and comment on pictures))		
73.	Emma:	it's six ((looks up from her notes))	
74.	Gloria:	yes it's six	
75.	Jason:	yes	
76.	Milton:	exactly	

Milton and Gloria's questions (lines 16—17) prompted Jason to justify his solution, but before justifying the multiplication 3·2, Jason referred to his picture representing three persons forming six queues (lines 19—22). Thereafter, Milton challenged Jason's certainty (line 23), whereas Emma proposed the multiplication 3·3 (line 26). Jason continued justifying his solution, including the multiplication 3·2 (lines 28—33), but Milton and Emma still maintained the idea of squaring the number of persons (lines 42—43).

Within the first episode in Excerpt 4 (lines 13—43), different types of justifications were used, but the group did not agree upon a joint solution. However, instead of pursuing the argumentation, Milton changed the topic of talk (to the task about two persons) possibly as an effort to achieve consensus (line 44). The new topical episode was short (line 44, omitted lines 45—55), and the group soon resumed their discussion about task b (about three persons).

When resuming the discussion about task b, Gloria did not propose any solution (line 56), whereas Jason repeated that his answer was “six” (line 57). Although the multiplication 3·3 was not repeated, Jason explicitly encouraged his group mates to “try to draw more than six” permutations for three persons, adding “it’s not possible” (lines 58—59). Jason’s challenge prompted his peers to examine their own examples (omitted lines 60—72), which led them to agree that three persons can form six queues (lines 73—76). As the final answer was justified through examinations of examples, the ***EB*** justification was treated as most influential on the proving process.

The extended argumentation among Jason, Milton, Gloria and Emma indicates that students’ examinations of their own examples had influence on the proving process. However, the argumentation among Albert, Lisa, and Sofia, which also was extended across several episode boundaries, showed that examinations of examples did not make students refute all incorrect solutions. As shown in Excerpt 2, Albert and Lisa established the idea of ‘times itself’ during the initial part of their discussion (Excerpt 2: lines 14—17, 81—84). Applying the rule of ‘times itself’ to all numbers larger than two, Albert and Lisa’s solutions to tasks a, b, and c were 2, 9, and 16. Their peer Sofia instead proposed the solutions 2, 3, and 4, and the students thereafter spent most of their time discussing whether three persons can form 3 or 9 queues and whether four persons can form 4 or 16 queues.

About halfway into the lesson, Lisa started examining an empirical example of the number of queues that four persons can form (Fig. 6). When their teacher spontaneously joined the group in order to monitor their work, the students not only presented their answers but also continued to revise their solutions. During the discussion with their teacher, Lisa rejected the solutions 2, 3, and 4. She stated that these solutions were based on the assumption that everybody has to change places in order for the queue to be different, which she found to be invalid when examining the example. Excerpt 5 shows how Lisa referred to her empirical examination when again reconsidering her solution to task c.

Excerpt 5.

311.	Lisa:	these ones (.) here are different persons called	**EB**
312.		ELSA like this ((points at her notes, Fig.6))	
313.	Teacher:	Mm	
314.	Lisa	actually E could stay couldn’t it	
315.		and only S and L change	
316.	Teacher:	yeah exactly	
317.	Lisa:	and then it’s yeah it’s sixteen	

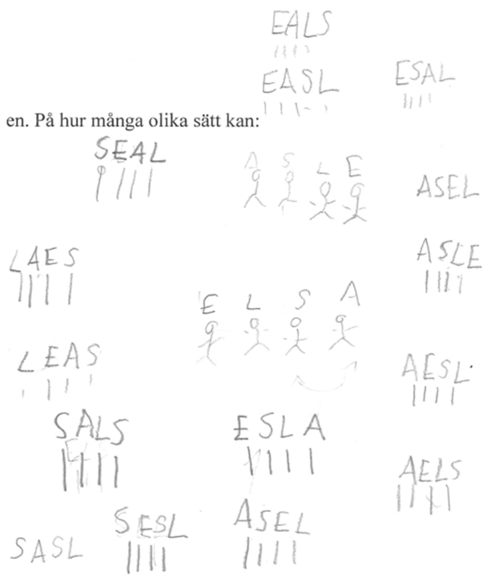

Fig. 6. Extract from Lisa’s notes *Note:* In all, Lisa represented 17 “queues” with the four letters ELSA, where ASLE and ASEL are repeated. There are also three “queues” that include a repetition of the letter S (SALS, SASL, SESL).

Using her empirical examination of the number of queues for four persons Lisa concluded, “E could stay” and “only S and L change” (line 314—315), as in the examples of EALS and EASL at the top of her notes (Fig. 6). We therefore categorized this part of Lisa's justification as an EB justification. Thereafter, Lisa stated that four persons could form sixteen queues (line 317). Although Lisa, in this episode, did not explicitly refer to the idea of ‘times itself,’ her conclusion was (again) associated to her and Albert's initial NRS justification of squaring the number of persons (Excerpt 2: lines 14—17, 82—85). Therefore, the ***NRS*** justification of ‘times itself’ was treated as most influential on the overall proving process.

5.4. Summary of results

Among the 141 analyzed justifications of solutions to the problem “Queue at the bus stop” we identified EB, T, NRS, and AU justifications, where EB justifications were most common. T and NRS justifications occurred to a comparable extent, while the frequency of the AU justifications was very low. A majority of the justifications occurred as “singles”. In cases where justifications occurred in immediate proximity to one another the groups most commonly consisted of EB and T justifications, and of EB and NRS justifications.

When EB and T justifications were grouped together, T followed EB and then always argued for the same solution. The relation between EB and NRS justifications was not as consistent. There was no apparent ordering of EB and NRS, and the justifications both agreed and disagreed with one another. However, NRS justifications more often challenged EB justifications than vice versa, whereas EB justifications more often were used to confirm an NRS justification than the other way round.

When a justification challenged the previous one, the argumentative process was apparent. However, objections also occurred within some episodes where justifications agreed and were then approached in different ways. In some cases, objections were followed by a repetition of a justification, whereas in other cases, further argumentation was avoided (for instance by ignoring the objection).

Several students chose to perform the operation of squaring the number of persons in order to find the number of queues. The idea of squaring was sometimes suggested and maintained even when an empirical example showing a correct number of permutations had been presented. Students who examined their own empirical example commonly refuted the operation of squaring. However, within one group, the solution of ‘times itself’ (i.e., the operation of squaring numbers) was maintained and jointly agreed upon, despite the students' examination of an empirical example.

6. Discussion

The aim of this study was to increase the knowledge of grade-6 students' proving processes, through the analysis of their use of justifications during mathematical problem solving in small groups. In subsection 6.1, we discuss our results concerning the frequency of the various types of justifications that we identified. Thereafter, we discuss relations between grouped justifications with regard to local ordering and in terms of agreement or disagreement. We also discuss what types of justifications were treated as most influential on the proving process, where we particularly focus on students' use of examples in relation to the practice of performing multiplications in order to solve the problem. Finally, we discuss our findings' possible implications for teaching (subsection 6.2).

6.1. Findings regarding grade-6 students' justifications

Previous studies have shown that the taxonomy of proof schemes is a useful tool when analyzing secondary-school students' proving (e.g., Ellis et al., 2013; Harel & Lesh, 2003; Kanellos, 2014; Lee, 2016; Liu & Manouchehri, 2013; Sen & Guler, 2015). The frequency of the various types of justifications that we identified in our material corresponds well to results of studies that have analyzed students' demonstrations of proof schemes at levels of education higher than grade 6. Our results therefore show that the taxonomy of proof schemes is useful also when analyzing grade-6 students' justifications of solutions to mathematical problems.

Studies that have identified students' demonstrations of more than one proof scheme have analyzed interviews with individual students and/or students' written solutions to mathematical problems (Erickson & Lockwood, 2021; Housman & Porter, 2003; Kanellos et al., 2018; Sears, 2019). Thus, previous investigations of cases where students demonstrated several proof schemes have been tied to individual students' results and not to a process of solving a problem in collaboration with peers. As we analyzed video recordings of students' interaction in small groups, we were able to analyze how different justifications related to one another during the ongoing process of solving a mathematical problem. Next, we discuss our findings regarding relations between different categories of justifications that occurred as groups.

We first consider relations between justifications that agreed, in the sense that they constituted arguments for the same solution. Our results include several cases where students' use of examples was immediately followed by a formulation of a general solution, according to the relations EB → T and EB = T. The consistency of these relations in our material supports results of previous studies which have indicated that students' use of examples may contribute to a development towards formulating mathematical proofs (e.g., Ellis et al., 2013; Housman & Porter, 2003; Stylianides, 2007; Stylianou et al., 2006). Moreover, our results show that grade-6 students have the ability to spontaneously transition from EB to T justifications, in the sense of shifting from performing empirical examinations to formulating general arguments without instructions from the teacher.

In some cases, the relations NRS → EB and NRS = EB occurred, where the reference to an example followed and confirmed an NRS justification. Both NRS and T justifications often included the use of multiplication, as well as arguments that a certain procedure was generally applicable; therefore, both NRS and T justifications could contribute to students' mathematical development in the sense of encouraging the ability to generalize. The crucial difference between NRS and T justifications was instead revealed by the relations

NRS = EB and EB = T. References to examples in some cases confirmed—and therefore agreed with—NRS justifications. However, the T justifications not only agreed with what was shown in a specific example but also explicitly built on results of previous empirical examinations. This emphasizes the significance of agreements between EB and T justifications. Although NRS justifications to some extent may support students' formulations of general arguments, our results show that the relation EB = T contributed even more to students' ability to generalize.

In addition to situations where justifications agreed, our results include several cases where justifications disagreed. This leads us to discuss our findings regarding argumentative processes that involved the relations EB ≠ NRS and NRS ≠ EB. Previous studies, based on interviews and written questionnaires (e.g., Erickson & Lockwood, 2021; Healy & Hoyles, 2000), have indicated that many students perceive references to examples as arguments with low status, although students often refer to examples when justifying solutions to mathematical problems. Again, our study differs from previous research as we analyzed naturally occurring classroom interaction. Our material therefore enabled the analysis of students' practices of using examples and calculations during the actual process of solving a mathematical problem and justifying their solutions. According to our results, the argumentative processes were extended over episode boundaries in cases where an EB justification challenged an incorrect multiplication (i.e., NRS ≠ EB). When, instead, an NRS justification containing an incorrect multiplication challenged an empirical example (i.e., EB ≠ NRS), the argumentative process was initiated and completed within a single episode. Thus, an incorrect multiplication more easily overruled results of empirical examinations than the other way round, which indicates that the practice of performing calculations had greater influence on the proving process than the practice of referring to examples.

The difference between examples and multiplications, in terms of influence on the proving process, was particularly apparent in cases where the NRS justification implied that students calculated the number of queues by squaring the numbers of persons. Therefore, we will now discuss our somewhat surprising finding that students were inclined to not only perform but also maintain the operation of squaring numbers, despite correct results of empirical examinations. In one group, the NRS justification of performing the operation of squaring was established during the initial part of their discussion, where it was agreed upon without taking objections into account (Excerpt 2, Section 5.2.2). Although the same students also conducted empirical examinations of examples, which offered possibilities to discover other solutions, the initially established justification of squaring numbers was treated as most influential on the overall proving process (Excerpt 5, Section 5.3). In addition, the practice of calculating the number of queues by squaring numbers was treated as more influential than other multiplications; in one group, the idea of squaring numbers was maintained although the correct multiplication had been justified (Excerpt 4, Section 5.3). Students' choices to perform multiplications may be interpreted as an orientation to a sociomathematical norm that solutions to mathematical problems have to include calculations (in line with discussions in Erickson & Lockwood, 2021; Levenson et al., 2009; Yackel & Cobb, 1996). However, neither our analysis of relations between various types of justifications nor the notion of sociomathematical norms, can explain the students' inclination towards using the specific operation of squaring numbers.

6.2. Implications for teaching

The importance of communication and participation in the mathematics classroom has been emphasized in mathematics education research during the past few decades (e.g., Hanna & Knipping, 2020; Lerman, 2000; Levenson et al., 2009). Compared to some of the previous studies that have used the taxonomy of proof schemes as an analytical tool (e.g., Lee, 2016; Sears, 2019; Sen & Guler, 2015) our results showed a relatively high frequency of T justifications that included general arguments. Possibly, these results were a consequence of students' interaction, as dialogues and group discussions offered opportunities to share and refine solutions.

Students' collaboration also enabled NRS justifications that included general arguments, which often were related to incorrect solutions and disagreement with EB justifications. In these cases, NRS justifications including incorrect calculations were often treated as most influential on the proving process. Our results therefore imply that teachers should not only encourage students to try to formulate general arguments but also emphasize the importance of agreement between various types of justifications. Emphasizing the significance of counterexamples, by pointing out that the identification of one single example that does not agree with a suggested calculation is a sufficient reason for rejecting that calculation, could possibly be another way to support students' improvement of their solutions. For instance, Excerpt 2 (Section 5.2.2) shows how students agreed to calculate the number of queues by squaring the number of persons, except when two persons form a queue. If the students had treated the number of queues for two persons as a counterexample (instead of an acceptable exception) to their rule of 'times itself' they might instead have rejected their incorrect solution.

Several studies (e.g. Campbell et al., 2020; Ellis et al., 2012; Healy & Hoyles, 2000; Stylianou et al., 2006) have discussed how students of various ages generate and use examples. For instance, Excerpt 1 (Section 5.2.1) shows how a student, when calculating the number of queues for four persons, pointed out that the procedure of fixating one person in the first position could be repeated (i.e., generalized) for all persons in the queue. Our findings regarding the relation EB = T therefore indicates that students should be encouraged to not only generate examples but also to identify general features of the specific example. Teachers' encouragement of T justifications during explorations of examples could contribute to students' ability to generalize since the empirical explorations then would support students in extending their reasoning beyond the specific example. In addition, the criterion of justifying "for all" arguments makes it possible to consider grade-6 students' practices involving T justifications as cases of "emergent proving," in the sense of being precursors to formal proofs. Therefore, teachers' encouragement of T justifications would also contribute to a teaching that supports students' progression towards mathematical proving.

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Declaration of Interest

None.

Appendix A. Transcription notation

Symbol	Use
word	Underlining indicates emphasis.
–	A dash indicates a cut-off utterance.
(word)	Parenthesis indicates that the transcriber had difficulties to hear the word/s.
((word))	Double parenthesis contain transcriber's descriptions.
(0.2)	Numbers in parenthesis indicate a pause in tenths of seconds.
(.)	A dot in parenthesis indicates a very short but hearable interval.
:	Colon indicates prolongation of the immediately prior sound.
↑	Arrow up indicates a pitch peak within an utterance.

Jefferson (2004:24ff)

Appendix B. Excerpts in Swedish

Excerpt 1.

66.	Maria:	((pekar på Fig. 5)) de här va dom sätten	EB
67.		alla dom sätten som gick me ett i början eller hur	
68.	Luiza:	mm	
69.	Maria:	de finns inget mer med ett (.)	
70.		för då har vi ju testat ((pekar på Fig.5))	
71.		där (.) där (.) där asså vi har testat	
72.		ja har testat alla sätt som går me ett i början	
73.		å sen tänkte ja att de finns	T
74.		exakt lika många sätt me två i början	
75.		exakt lika många sätt me tre i början	
76.	Luiza:	men blir de samma går det inte att ta två tre fyra ett	
77.		(0.2)	
78.	Maria:	va	
79.	Luiza:	de bl- de e ju också ett sätt två tre	
80.	Maria:	exakt men de är ju de ja menar	
	((rad 81-90 borttagna: Maria upprepar sina motiveringar))		
91.	Maria:	me fyra siffrer så då tog ja bara (.)	T
92.		sex gånger fyra å de e tjufyra för ja fick	
93.		en två tre fyr fem sex sätt	
94.	Luiza:	men de g- de finns fler om man börjar	
95.	Maria:	tjufyra (så många sätt kan man göra)	
96.	Luiza:	aha a	

Excerpt 2.

14.	Albert:	man gångrar allti (.) typ talen med s:ej själ-	NRS
15.		fast inte på den då förutom den	
16.		((pekar på deluppgift a på sitt arbetsblad))	
17.	Lisa:	nej ja ve:t a så tänkte jag också	
18.	Albert:	förutom den för den går inte	
19.	Lisa:	fast varför eh så här varför ska man inte göra på den	
20.	Albert:	f- för att de: liksom de går inte ((fnissar))	
21.	Lisa:	((fnissar)) men eh ah ah okej	
	((rad 22-32 borttagna: Albert upprepar sin lösning))		
	((**Episodgräns: samtal om uppgift a initierat av Albert**))		
	((rad 33-74 borttagna: lösning till uppgift a motiveras))		
	((**Episodgräns: samtal om uppgift b initierat av Lisa**))		
75.	Lisa:	men hur eh varför ska man köra tre gånger tre (.)	NRS
76.		förom kan stå på tre olika platser	EB
77.		↑bra	
78.	Albert:	eh j- ja ja	
79.	Lisa:	alla kan stå på tre olika platser	
80.		å då blire tre	
81.	Melvin:	men vi fick väl inte göra så	
82.	Albert:	asså ja skrev att man gångrar talet me	NRS
83.		sej själv självt	
84.	Lisa:	ja skrev man tar hur många som står i kön (.)	
85.		gånger hur många som står i kön	
86.	Albert:	a exakt exakt men de va de ja mena	

Excerpt 3.

42.	Wanda:	de finn- ja vet inte om de finns nån annan håll	
43.		eller nån annan som man kan göra	
	((rad 44-49 borttagna: samtal om Fig.3))		
50.	Wanda:	men ja tror de där ((pekar på Fig.3))	EB
51.	Leyla:	antingen sex eller nie	NRS
52.		(0.2)	
53.	Wanda:	mm ((nickar, skriver ”(9)” bredvid sin tabell))	
54.	Leyla:	för nie då blire tre gånger tre (.)	
55.		å då blire att alla får va: olika	
56.	Wanda:	((tittar på Leyla)) mm	

Excerpt 4.

		((**Episodgräns: samtal om uppgift b initierat av Jason**))	
13.	Jason:	på trean äre tre gånger två	
14.	Milton:	äre	
15.	Jason:	a såre blir sex	
16.	Milton:	varför ere tre gånger tvårå	
17.	Gloria:	varför ere tre gånger två	
18.	Jason:	för att eh om man (.)	
19.		ja ritade upp alla sorter som de går	EB
20.		((pekar på teckning som visar sex permutationer))	
21.		å de blev sex stycken (0.2) de gårnte å göra	
22.		flera olika sorter (0.1) me tre stycken	
23.	Milton:	eru säker	
24.		(0.2)	
25.	Emma:	men det borde ↑ju gå nie gånger (.)	NRS
26.		eller tre gånger tre	
27.	Jason:	för om man (.)	
28.		((pekar på teckning)) för de blir ju tre stycken (.)	EB
29.		kolla för de e tre olika (.)	
30.		men å dom här bak (.) dom kan byta plats en gång (.)	
31.		å då så blir de tre gånger två om alla byter plats	T
32.		för man kan göra två stycken av varje (0.2)	
33.		så därför blire tre gånger två man kan göra	
	((rad 34-41	borttagna: kommentarer om Jasons lösning))	
42.	Milton:	↑ja trore e tre gånger tre	NRS
43.	Emma:	ja me	
		((**Episodgräns: samtal om uppgift a initierat av Milton**))	
44.	Milton:	men på första är de ju två (.) de vet väl alla	
	((rad 45-55	borttagna: diskussion om deluppgift a))	
		((**Episodgräns: samtal om uppgift b återupptas av Gloria**))	
56.	Gloria:	ja och sen berå ja hann inte göra ce	
57.	Jason:	men be är sex tror ja (.) för att (.) om	EB
58.		testa å försöka rita upp fler än sex tecken	
59.		de går inte (0.2) de går inte	
	((rad 60–72	borttagna: gruppen ritar och kommenterar exempel))	
73.	Emma:	de e sex ((tittar upp från sina anteckningar))	
74.	Gloria:	jo de blir sex	
75.	Jason:	ja	
76.	Milton:	exakt	

Excerpt 5.

311.	Lisa:	de här (.) här e olika personer som heter	EB
312.		ELSA så här ((pekar på anteckningar, Fig.6))	
313.	Lärare:	mm	
314.	Lisa:	ejenklien så skulle ju E kunna va kvar	
315.		å ba att S å L byter	
316.	Lärare:	a precis	
317.	Lisa:	å då ere ju a de e sexton	

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