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Simple transitive 2-representations of 2-categories associated to self-injective cores

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ABSTRACT

Given a finite dimensional algebra A , we consider certain sets of idempotents of A , called self-injective cores, to which we associate 2-subcategories of the 2-category \mathcal{C}_A of projective bimodules over A . We classify the simple transitive 2-representations of such 2-subcategories, vastly generalizing a previous result of Zimmermann.

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1. Introduction

Motivated by the extensive use of categorical actions in various areas of mathematics, notably in famous results such as the introduction of Khovanov homology in [6] and the proof of Broué's abelian defect group conjecture for symmetric groups in [2], Mazorchuk and Miemietz initiated a systematic study of 2-representations of the so-called finitary 2-categories in the series of papers [8–13].

One of the main developments established in the above listed papers is the introduction of the notion of a simple transitive 2-representation, which is a 2-analogue of the classical notion of a simple module. Similar to the Jordan-Hölder theorem for simple modules, simple transitive 2-representations admit a weak Jordan-Hölder theory. This immediately leads to the very natural problem of classifying such 2-representations, up to (weak) isomorphism.

One of the first such classification results was given in [12], stating that if A is a self-injective finite dimensional algebra, then the simple transitive 2-representations of the 2-category \mathcal{C}_A of projective bimodules over A are exhausted by the so-called cell 2-representations. Cell 2-representations are obtained from the cell structure of \mathcal{C}_A , which is defined similarly to the cell structure on a semigroup resulting from Green's relations and can be defined for any finitary 2-category. This classification result was later significantly improved in [17], where it was shown that one does not need to assume A to be self-injective.

The 2-category \mathcal{C}_A is biequivalent to the delooping of the \mathbb{k} -linear monoidal category $\mathcal{C}_A = (\text{add}\{A, A \otimes_{\mathbb{k}} A\}, \otimes_A)$. As such, it has a unique object, its 1-morphisms correspond to A -

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bimodules in $\text{add}\{A, A \otimes_{\mathbb{k}} A\}$, with composition corresponding to the tensor product over A , and 2-morphisms correspond to bimodule homomorphisms. Given a fixed complete set of pairwise orthogonal, primitive idempotents $1 = e_0 + \cdots + e_k$ for A and subsets U, V of $\{e_0, \dots, e_k\}$, it is easy to verify that $\text{add}\{A, Ae \otimes fA \mid e \in U, f \in V\}$ gives a monoidal subcategory of \mathcal{C}_A , and hence gives rise to a respective 2-subcategory $\mathcal{D}_{U \times V}$ of \mathcal{C}_A . If $V = \{e_0, \dots, e_k\}$, then $\mathcal{D}_{U \times V}$ is given by a union of right cells of \mathcal{C}_A , and if $U = \{e_0, \dots, e_k\}$, then $\mathcal{D}_{U \times V}$ is given by a union of left cells.

The cell structure of $\mathcal{D}_{U \times V}$ can be understood in terms of that of \mathcal{C}_A . However, whenever U and V do not coincide, the crucial symmetry between the left cells and right cells of \mathcal{C}_A breaks upon restriction to $\mathcal{D}_{U \times V}$. In particular, if $U \neq V$, then $\mathcal{D}_{U \times V}$ is not weakly fiat, so if one is to study its simple transitive 2-representations, the rich theory of 2-representations of weakly fiat 2-categories developed in, among others, [14] and [15], cannot be applied directly.

When studying simple transitive 2-representations of $\mathcal{D}_{U \times V}$, one should expect some of the techniques of [12] (and subsequent papers, such as [18, 19]) to work also in the setting of the 2-subcategories. At the same time, the lack of cell symmetry may result in the existence of non-cell simple transitive 2-representations. If that is the case, one may hope that it would be relatively easy to construct non-cell 2-representations of such 2-subcategories, and that the constructions used to that end would then generalize to other, more difficult settings.

A special case of the above-described problem was studied in [21]. Let A be the zigzag algebra on a star-shaped graph Γ on $k + 1$ vertices $\{0, 1, \dots, k\}$, with the unique internal node labeled by 0. Let $U = \{e_0, e_1, \dots, e_k\}$ and $V = \{e_0\}$. In this case, the existence of non-cell simple transitive 2-representations of $\mathcal{D}_{U \times V}$ was conjectured in [21], and later positively verified in [20]. In contrast to this, [21, Theorem 6.2] shows that if we instead choose $U = \{e_0\}$ and $V = \{e_0, \dots, e_k\}$, then any simple transitive 2-representation is equivalent to a cell 2-representation.

In order to generalize [21, Theorem 6.2], we avoid the explicit calculations used in the proof given in [21] and give a clearer connection between that proof and the fact that the zigzag algebra is self-injective. To that end, we observe that we only need to use the self-injective property when speaking of the idempotents in U , which motivates the main definition of the document:

Definition (Definition 3.3). A subset $U \subseteq \{e_0, \dots, e_k\}$ is a *self-injective core* if, for every $e \in U$, there is $f \in U$ such that $Ae \simeq (fA)^*$.

Our main result is the following:

Theorem (Theorem 4.5). *Let $U \subseteq V \subseteq \{e_0, e_1, \dots, e_k\}$, with U a self-injective core for A . Then any simple transitive 2-representation of $\mathcal{D}_{U, V}$ is equivalent to a cell 2-representation.*

In particular, the above theorem classifies simple transitive 2-representations whenever $V = \{e_0, e_1, \dots, e_k\}$ and A is self-injective, which implies [21, Theorem 6.2]. The generalization is vast: from a fixed choice of A together with fixed choices of U and V , we arrive at a general condition for A and U , with an abundance of instances going beyond the setting of [21].

The article is structured as follows. Section 2 contains the necessary preliminaries for 2-representation theory. In Section 3, we give a more detailed account of the 2-categories of the form $\mathcal{D}_{U \times V}$, including their cell structure and the structure of their cell 2-representations. Section 4 consists of the proof of the main theorem. In Section 5, we describe an example illustrating how 2-categories of the form $\mathcal{D}_{U \times V}$ appear naturally in the context of general fiat 2-categories.

2. Preliminaries

Throughout, we let \mathbb{k} be an algebraically closed field. By $(-)^*$ we denote the duality functor $\text{Hom}_{\mathbb{k}}(-, \mathbb{k})$. Given integers m, n with $m < n$, we let $[[m, n]]$ denote the set $\{m, m + 1, \dots, n\}$. Further, let $[[n]] := [[1, n]]$.

2.1. 2-Categories and their 2-representations

We say that a category \mathcal{C} is *finitary over \mathbb{k}* if it is additive, idempotent split, \mathbb{k} -linear, and has finitely many isomorphism classes of indecomposable objects.

Definition 2.1. A 2-category \mathcal{C} is *finitary over \mathbb{k}* if

- it has finitely many objects;
- for every $i, j \in \mathcal{C}$, the category $\mathcal{C}(i, j)$ is finitary;
- horizontal composition in \mathcal{C} is \mathbb{k} -bilinear;
- for any object $i \in \mathcal{C}$, the identity 1-morphism 1_i is an indecomposable object of $\mathcal{C}(i, i)$.

For the remainder of this section, let \mathcal{C} be a 2-category which is finitary over \mathbb{k} . Following [9], we consider the following 2-categories:

- $\mathfrak{A}_{\mathbb{k}}$ -the 2-category whose objects are small, finitary (over \mathbb{k}) categories, 1-morphisms are additive \mathbb{k} -linear functors between such categories and whose 2-morphisms are all natural transformations between such functors;
- $\mathfrak{R}_{\mathbb{k}}$ -the 2-category whose objects are small abelian \mathbb{k} -linear categories, 1-morphisms are right exact \mathbb{k} -linear functors between such categories and whose 2-morphisms are all natural transformations between such functors.

Definition 2.2. A *finitary 2-representation* of \mathcal{C} is a 2-functor \mathbf{M} from \mathcal{C} to $\mathfrak{A}_{\mathbb{k}}$, such that $\mathbf{M}_{i, j}$ is \mathbb{k} -linear, for all $i, j \in \text{Ob}\mathcal{C}$. An *abelian 2-representation* of \mathcal{C} is such a 2-functor whose codomain is $\mathfrak{R}_{\mathbb{k}}$.

Together with strong transformations and modifications, the collection of finitary 2-representations of \mathcal{C} forms a 2-category $\mathcal{C} - \text{afmod}$. Similarly, we obtain the 2-category $\mathcal{C} - \text{mod}$ of abelian 2-representations of \mathcal{C} , see [10, Section 2.3] for details. We say that two 2-representations \mathbf{M}, \mathbf{N} are *equivalent* if there exists an invertible strong transformation $\mathbf{M} \rightarrow \mathbf{N}$. In particular, a strong transformation $\Phi \in \mathcal{C} - \text{afmod}(\mathbf{M}, \mathbf{N})$ such that all of its components are equivalences of categories, is invertible, as shown for instance in [10, Proposition 2].

A \mathcal{C} -stable ideal \mathbf{I} is a tuple $(\mathbf{I}(i))_{i \in \text{Ob}\mathcal{C}}$ of ideals $\mathbf{I}(i)$ of $\mathbf{M}(i)$, such that, for any morphism $X \xrightarrow{f} Y$ in $\mathbf{I}(i)$ and any 1-morphism $F \in \text{Ob}\mathcal{C}(i, j)$, the morphism $\mathbf{M}F(f)$ lies in $\mathbf{I}(j)$.

Definition 2.3. A 2-representation is said to be *transitive* if, for any indecomposable $X \in \text{Ob}\mathbf{M}(j)$ and $Y \in \text{Ob}\mathbf{M}(k)$, there is a 1-morphism G in $\mathcal{C}(j, k)$ such that Y is isomorphic to a direct summand of $\mathbf{M}(G)X$. A 2-representation is *simple transitive* if it admits no non-trivial \mathcal{C} -stable ideals.

As shown in [12, Lemma 4], every transitive 2-representation admits a unique simple transitive quotient. Further, from [12, Section 4.2] it follows that a simple transitive 2-representation is transitive.

If \mathcal{C} has a unique object i , we define the *rank* of a finitary 2-representation \mathbf{M} of \mathcal{C} as the number of isoclasses of indecomposable objects of $\mathbf{M}(i)$. If \mathcal{C} has multiple objects, one can define the rank as a tuple of positive integers.

We say that \mathcal{C} is *weakly fiat* if every 1-morphism of \mathcal{C} admits both left and right adjoints, giving rise to a weak antiautomorphism $(-)^*$ of finite order, see [13, Section 2.5]. If $(-)^*$ is involutive, we say that \mathcal{C} is *fiat*.

2.2. Abelianization

Given a finitary 2-representation \mathbf{M} of \mathcal{C} , let $\overline{\mathbf{M}}$ denote the *projective abelianization* of \mathbf{M} , as initially defined in [9], using the projective abelianization of finitary categories given in [4].

The abelianization is an abelian 2-representation of \mathcal{C} associated to \mathbf{M} , and, more generally, we obtain a 2-functor $\overline{} : \mathcal{C} - \text{afmod} \rightarrow \mathcal{C} - \text{mod}$. The finitary 2-representation \mathbf{M} can be recovered from $\overline{\mathbf{M}}$ by restriction to certain subcategories of $\overline{\mathbf{M}}(\mathfrak{i})$ equivalent to $\overline{\mathbf{M}}(\mathfrak{i}) - \text{proj}$. If \mathcal{C} is weakly fiat, then, since $\overline{\mathbf{M}}$ is abelian, the functor $\overline{\mathbf{M}}F$ is exact, for any 1-morphism F of \mathcal{C} . An improved construction was given in [14, Section 3]. For our purposes, either of the two constructions can be used.

As observed in [8, Section 3.1], there are finite dimensional algebras $A_{\mathfrak{i}}$ such that we have $\mathbf{M}(\mathfrak{i}) \simeq A_{\mathfrak{i}} - \text{proj}$ and $\overline{\mathbf{M}}(\mathfrak{i}) \simeq A_{\mathfrak{i}} - \text{mod}$, for all $\mathfrak{i} \in \text{Ob}\mathcal{C}$. Since, for a 1-morphism F of $\mathcal{C}(\mathfrak{i}, \mathfrak{j})$, the functor $\overline{\mathbf{M}}F$ is right-exact, it follows by the Eilenberg-Watts theorem that there is a finitely generated $A_{\mathfrak{j}} - A_{\mathfrak{i}}$ -bimodule F such that $\mathbf{M}F$ corresponds to $F \otimes_{A_{\mathfrak{i}}} -$. Similarly, for a 2-morphism $\alpha : F \rightarrow G$, the natural transformation $\mathbf{M}\alpha$ can be identified with a bimodule homomorphism.

2.3. Cells and cell 2-representations

Let $\mathcal{S}(\mathcal{C})$ denote the set of isomorphism classes of indecomposable 1-morphisms of \mathcal{C} . Given an object X of a category \mathcal{C} , we denote its isomorphism class by $[X]$. If \mathfrak{K} is a set of isomorphism classes of objects in a category \mathcal{C} , and X is an object of \mathcal{C} , we will sometimes abuse notation and write $X \in \mathfrak{K}$ for $[X] \in \mathfrak{K}$. In particular, if F is an indecomposable 1-morphism of \mathcal{C} , we may write $F \in \mathcal{S}(\mathcal{C})$.

Following [9], given $F, G \in \mathcal{S}(\mathcal{C})$, we write $F \geq_L G$ if there is a 1-morphism H such that F is isomorphic to a direct summand of $H \circ G$. This gives the *left preorder* L on $\mathcal{S}(\mathcal{C})$. The *right preorder* R and the *two-sided preorder* J are defined similarly. The equivalence classes of the induced equivalence relations are called the left, right and two-sided *cells* respectively. (Alternatively, L -cells, R -cells and J -cells.)

Let \mathcal{L} be a left cell of \mathcal{C} . We briefly summarize the construction of the *cell 2-representation* $\mathbf{C}_{\mathcal{L}}$, following [12, Section 3.3]. There is a unique $\mathfrak{i} \in \text{Ob}\mathcal{C}$ such that $F \in \mathcal{L}$ implies that \mathfrak{i} is the domain of F . Consider the 2-functor $\mathbf{P}_{\mathfrak{i}} := \mathcal{C}(\mathfrak{i}, -)$. This 2-functor gives a finitary 2-representation of \mathcal{C} , and the additive closure of the set of 1-morphisms F satisfying $F \geq_L \mathcal{L}$ gives a 2-subrepresentation $\mathbf{K}_{\mathcal{L}}$ of $\mathbf{P}_{\mathfrak{i}}$. Taking the quotient of the latter by the ideal generated by $\{\text{id}_F \mid F \geq_L \mathcal{L}\}$ gives a transitive 2-representation $\mathbf{N}_{\mathcal{L}}$. The cell 2-representation $\mathbf{C}_{\mathcal{L}}$ is the unique simple transitive quotient of $\mathbf{N}_{\mathcal{L}}$.

A two-sided cell \mathcal{J} of \mathcal{C} is said to be *idempotent* if there are non-zero 1-morphisms $F, G, H \in \mathcal{J}$ such that H is isomorphic to a direct summand of $G \circ F$. As was shown in [1, Lemma 3], the set of J -cells of \mathcal{C} not annihilating a fixed transitive 2-representation \mathbf{M} admits a J -greatest element, called the *apex of \mathbf{M}* . The apex of a transitive 2-representation must be idempotent, and it coincides with the apex of its unique simple transitive quotient.

We now formulate and give a proof of a statement often implicitly used in the literature, for instance in [12, Proposition 9], [18, 21].

Assume that \mathcal{C} has a single object \mathfrak{i} , and that \mathcal{C} admits an idempotent J -cell \mathcal{J} . Let \mathbf{M} be a simple transitive 2-representation with apex \mathcal{J} . Fix a complete, irredundant set $X = \{X_1, \dots, X_n\}$ of representatives of isomorphism classes of indecomposable objects of $\mathbf{M}(\mathfrak{i})$. By construction of projective abelianization, $\bigoplus_{k=1}^n X_k$ is a projective generator for $\overline{\mathbf{M}}(\mathfrak{i}) \simeq A_{\mathfrak{i}} - \text{mod}$. Let $Q := A_{\mathfrak{i}}$. We conclude that $\mathbf{M}(\mathfrak{i}) \simeq Q - \text{proj}$, and $Q = \text{End}(\bigoplus_{k=1}^n X_k)^{\text{op}}$. Further, the set $\{f_k\}_{k=1}^n := \{\text{id}_{X_k}\}_{k=1}^n$ is a complete set of pairwise orthogonal, primitive idempotents for Q . Choose a left cell $\mathcal{L} = \{F_1, \dots, F_r\}$ in \mathcal{J} . Finally, recall that the *Cartan matrix of $\mathbf{M}(\mathfrak{i})$ with respect to X* (and similarly for any finitary category with a fixed list of indecomposables) is the $n \times n$ matrix whose (i, j) th entry is given by $\dim \text{Hom}_{\mathbf{M}(\mathfrak{i})}(X_i, X_j)$.

Lemma 2.4. *If there is an ordering of X such that*

- The Cartan matrix of $\mathbf{M}(\mathfrak{i})$ is equal to that of $\text{add}(\mathcal{L})$;
- There is an index $l \in [[n]]$ such that, for $k \in [[n]]$, the functor \mathbf{MF}_k is naturally isomorphic to the functor $Qf_k \otimes_{\mathbb{k}f_l} Q \otimes_Q -$.

then \mathbf{M} is equivalent to the cell 2-representation $\mathbf{C}_{\mathcal{L}}$.

Proof. As described in [12, Section 5.2], the embedding $\mathbf{M}(\mathfrak{i}) \rightarrow \overline{\mathbf{M}}(\mathfrak{i})$, sending X to the projective object $0 \rightarrow X$, gives an equivalence $\mathbf{M}(\mathfrak{i}) \simeq \overline{\mathbf{M}}(\mathfrak{i}) - \text{proj}$. From the latter of our assumptions, we conclude that for any 1-morphism F of \mathcal{C} and any $X \in \text{Ob} \overline{\mathbf{M}}(\mathfrak{i})$, the object $\mathbf{MF}X$ is projective.

In particular, $\overline{\mathbf{M}}(\mathfrak{i}) - \text{proj}$ becomes a finitary 2-representation of \mathcal{C} , equivalent to \mathbf{M} , which we denote by $\overline{\mathbf{M}} - \text{proj}$. Given $k \in [[n]]$, we let L_k be the simple top of the object $0 \rightarrow X_k$ of $\overline{\mathbf{M}}(\mathfrak{i})$.

The Yoneda lemma for $\mathbf{P}_{\mathfrak{i}}$ given in [9, Lemma 9] gives a unique 2-transformation from $\mathbf{P}_{\mathfrak{i}}$ to $\overline{\mathbf{M}} - \text{proj}$, induced by the assignment $\mathbb{1}_{\mathfrak{i}} \mapsto L_l$. This 2-transformation sends F to $\overline{\mathbf{M}}FL_l$. We may restrict this to a 2-transformation from $\mathbf{K}_{\mathcal{L}}$. Since \mathcal{J} is idempotent, [7, Corollary 19] implies that the left cells it contains are pairwise L -incomparable, and so $F >_L \mathcal{L}$ implies $F >_J \mathcal{J}$. Since \mathcal{J} is the apex of \mathbf{M} , a 1-morphism F satisfying $F >_L \mathcal{L}$ is sent to zero by the above described 2-transformation. Thus, the above 2-transformation from $\mathbf{K}_{\mathcal{L}}$ factors through an induced 2-transformation $\sigma : \mathbf{N}_{\mathcal{L}} \rightarrow \overline{\mathbf{M}} - \text{proj}$, from the transitive quotient $\mathbf{N}_{\mathcal{L}}$ of $\mathbf{K}_{\mathcal{L}}$.

The image of F_k under σ is

$$\overline{\mathbf{M}}F_k L_l \simeq Qf_k \otimes_{\mathbb{k}f_l} Q \otimes_Q L_l \simeq Qf_k,$$

which is indecomposable. This shows that all the isomorphism classes of indecomposable objects of $Q - \text{proj}$ are in the essential image of σ , and so σ is essentially surjective.

The kernel of σ is an ideal of $\mathbf{N}_{\mathcal{L}}$, which does not contain any identity 2-morphisms of \mathcal{D} , since $F_k L_l \neq 0$, for all k . Thus it is contained in the maximal ideal \mathbf{I} of $\mathbf{N}_{\mathcal{L}}$, which defines the cell 2-representation $\mathbf{C}_{\mathcal{L}}$, via $\mathbf{C}_{\mathcal{L}} = \mathbf{N}_{\mathcal{L}}/\mathbf{I}$.

We claim that $\text{Ker} \sigma = \mathbf{I}$. It suffices to show that $\mathbf{I} \subseteq \text{Ker} \sigma$. The 2-transformation

$$\tilde{\sigma} : \mathbf{N}_{\mathcal{L}}/\text{Ker}(\sigma) \rightarrow \overline{\mathbf{M}} - \text{proj}$$

is, by the definition of $\text{Ker} \sigma$, given by a faithful functor, so that, for all $s, t \in [[n]]$, the linear map

$$\tilde{\sigma}_{st} : \text{Hom}_{\mathbf{N}_{\mathcal{L}}/\text{Ker}(\sigma)}(F_s, F_t) \rightarrow \text{Hom}_{Q-\text{proj}}(Qe_s, Qe_t)$$

is injective. Hence

$$\dim \text{Hom}_{\mathbf{N}_{\mathcal{L}}/\text{Ker}(\sigma)}(F_s, F_t) \leq \dim \text{Hom}_{Q-\text{proj}}(Qe_s, Qe_t).$$

Since $\text{Ker}(\sigma) \subseteq \mathbf{I}$, we have

$$\dim \text{Hom}_{\mathbf{C}_{\mathcal{L}}}(F_s, F_t) \leq \dim \text{Hom}_{\mathbf{N}_{\mathcal{L}}/\text{Ker}(\sigma)}(F_s, F_t).$$

The equality of Cartan matrices for $\mathbf{C}_{\mathcal{L}}$ and \mathbf{M} implies that the lower and the upper bounds for $\dim \text{Hom}_{\mathbf{N}_{\mathcal{L}}/\text{Ker}(\sigma)}(F_s, F_t)$ coincide, so

$$\dim \mathbf{I}(F_s, F_t) = \dim \text{Ker}(\sigma)(F_s, F_t), \text{ and } \text{Ker}(\sigma) = \mathbf{I}.$$

We thus have a 2-transformation $\tilde{\sigma} : \mathbf{C}_{\mathcal{L}} \rightarrow \overline{\mathbf{M}} - \text{proj}$ such that $\tilde{\sigma}_{st}$ is an injective linear map between equidimensional spaces, and thus is an isomorphism, for all s, t . This shows that $\tilde{\sigma}$ is essentially surjective, full and faithful, and so it gives an equivalence of 2-representations. Thus, $\mathbf{C}_{\mathcal{L}}$ is equivalent to $\overline{\mathbf{M}} - \text{proj}$, and hence also to \mathbf{M} . □

2.4. Decategorification and action matrices

Following [9, Section 2.4], we define the *decategorification* of \mathcal{C} as the preadditive category $[\mathcal{C}]$ given by

- $\text{Ob}[\mathcal{C}] := \text{Ob}\mathcal{C}$;
- Given $i, j \in \text{Ob}\mathcal{C}$, we let $[\mathcal{C}](i, j) := [\mathcal{C}(i, j)]_{\oplus}$, the split Grothendieck group of $\mathcal{C}(i, j)$, with composition induced by composition in \mathcal{C} .

Similarly, for a finitary 2-representation \mathbf{M} , its decategorification is a \mathbb{Z} -bilinear functor from $[\mathcal{C}]$ to \mathbf{Ab} .

Given $i, j \in \text{Ob}\mathcal{C}$, choose complete, irredundant sets of representatives of isoclasses of indecomposable objects of the categories $\mathbf{M}(i)$ and $\mathbf{M}(j)$ and denote them by $\{X_1, \dots, X_n\}$ and $\{Y_1, \dots, Y_m\}$, respectively. Let $F \in \text{Ob}\mathcal{C}(i, j)$. With respect to the induced bases X, Y in the respective split Grothendieck groups, the *action matrix* $[F]^{X, Y}$ is the $m \times n$ matrix such that the entry $[F]_{kl}^{X, Y}$ is the multiplicity of Y_k as a direct summand of $\mathbf{M}F X_l$. Under a fixed choice of bases, we will denote $[F]^{X, Y}$ simply by $[F]$. If there is possible ambiguity regarding the 2-representation with respect to which we denote the action matrix, we will add it as a subscript in our notation, for instance $[F]_{\mathbf{M}}$ in the case above.

2.5. Discrete extensions of 2-representations

The notion of a discrete extension of 2-representations was introduced and studied in [1]. We only give a brief summary of the notions we will use in this text and refer to [1] for the details.

A *short exact sequence*

$$0 \rightarrow \mathbf{K} \rightarrow \mathbf{M} \rightarrow \mathbf{N} \rightarrow 0 \quad (1)$$

of 2-representations of \mathcal{C} consists of

- a finitary 2-representation \mathbf{M} ;
- a finitary 2-subrepresentation \mathbf{K} of \mathbf{M} ;
- the 2-representation \mathbf{N} given by the quotients of the categories $\mathbf{M}(i)$ by the ideals generated by the identity morphisms of all objects of $\mathbf{K}(i)$;
- the inclusion 2-transformation $\mathbf{K} \rightarrow \mathbf{M}$;
- the projection 2-transformation $\mathbf{M} \rightarrow \mathbf{N}$.

Given $i \in \text{Ob}\mathcal{C}$, a short exact sequence such as the one above induces a partition of the set of indecomposable objects of $\mathbf{M}(i)$ into two subsets, one consisting of the objects of $\mathbf{K}(i)$ and the other consisting of the indecomposable objects not sent to zero under the projection $\mathbf{M} \rightarrow \mathbf{N}$. Let $\{Y_1, \dots, Y_s\}$ be the former set of indecomposable objects and let $\{X_1, \dots, X_r\}$ be the latter.

We say that the short exact sequence (1) is *trivial* if, for any 1-morphism F of \mathcal{C} , there are no $l \in [[s]], k \in [[r]]$, such that Y_l is a direct summand of $\mathbf{M}F X_k$.

Let \mathbf{K}, \mathbf{N} be transitive, finitary 2-representations of \mathcal{C} . If every short exact sequence

$$0 \rightarrow \mathbf{K}' \rightarrow \mathbf{M} \rightarrow \mathbf{N}' \rightarrow 0$$

with $\mathbf{K} \simeq \mathbf{K}'$ and $\mathbf{N} \simeq \mathbf{N}'$ is trivial, we have

$$\text{Dext}(\mathbf{N}, \mathbf{K}) = \emptyset.$$

We will not use the general definition of $\text{Dext}(\mathbf{N}, \mathbf{K})$, and refer the interested reader to [1].

2.6. The 2-category \mathcal{C}_A

Let A be a finite dimensional, basic, connected algebra. Fix a small category \mathcal{A} equivalent to $A - \text{mod}$. The 2-category \mathcal{C}_A consists of

- a single object i ;
- endofunctors of \mathcal{A} isomorphic to tensoring with A - A -bimodules in the category $\text{add}\{A \otimes_{\mathbb{k}} A, A\}$ as 1-morphisms (in other words, so-called projective functors of \mathcal{A});
- all natural transformations between such functors as 2-morphisms.

In particular, \mathcal{C}_A is finitary. Fix a complete set of pairwise orthogonal, primitive idempotents $\{e_1, \dots, e_m\}$ and an equivalence $A - \text{mod} \xrightarrow{\sim} \mathcal{A}$. Under these choices, the isomorphism classes of indecomposable 1-morphisms correspond bijectively to the set consisting of the regular bimodule ${}_A A_A$ and the projective bimodules $\{Ae_i \otimes_{\mathbb{k}} e_j A \mid i, j \in [[m]]\}$. We then further choose a unique representative of every isomorphism class corresponding to a bimodule of the form $Ae_i \otimes_{\mathbb{k}} e_j A$, and denote this representative by F_{ij} . On the level of isomorphism classes of 1-morphisms, the composition $F_{ij} \circ F_{kl}$ corresponds to the tensor product

$$(Ae_i \otimes_{\mathbb{k}} e_j A) \otimes_A (Ae_k \otimes_{\mathbb{k}} e_l A) \simeq (Ae_i \otimes_{\mathbb{k}} e_l A)^{\oplus \dim e_j A e_k},$$

and so we may write

$$F_{ij} \circ F_{kl} \simeq F_{il}^{\oplus \dim e_j A e_k}. \tag{2}$$

Using our notation for 1-morphisms, given a subset $W \subseteq [[m]] \times [[m]]$, we denote the set $\{[F_{ij}] \mid (i, j) \in W\}$ by $\mathcal{S}(W)$. A 1-morphism F belongs to $\mathcal{S}([[m]] \times [[m]])$ if and only if F is an indecomposable 1-morphism not isomorphic to the identity 1-morphism.

Further, to W as above we associate the multiplicity-free 1-morphism $\bigoplus_{(i,j) \in W} F_{ij}$, which we denote by F_W . In particular, $F_{ij} = F_{\{(i,j)\}}$.

Unless otherwise stated, we implicitly assume a fixed complete set of pairwise orthogonal, primitive idempotents $\{e_1, \dots, e_m\}$ for A , with respect to which we use the above introduced notation.

A 2-category of the form \mathcal{C}_A is weakly fiat if and only if A is self-injective. This is an immediate consequence of the following lemma:

Lemma 2.5. [8, Lemma 45] *Let f, e be primitive, mutually orthogonal idempotents of A . Then*

$$((Ae \otimes_{\mathbb{k}} fA) \otimes_A -, ((fA)^* \otimes_{\mathbb{k}} eA) \otimes_A -)$$

is an adjoint pair of endofunctors of $A - \text{mod}$.

Proof. This is an immediate consequence of the proof of [8, Lemma 45], since the assumption about A being weakly symmetric is used there only to show that $(fA)^*$ is projective. \square

2.6.1. Cell structure and simple transitive 2-representations of \mathcal{C}_A

Using the composition rule given in (2), one may determine the cell structure of \mathcal{C}_A . The following is its representation using a so-called eggbox diagram, commonly used in the theory of semigroups:

$$\boxed{\mathbb{1}_i}$$

F_{11}	F_{12}	\cdots	F_{1m}
F_{21}	F_{22}	\cdots	F_{2m}
\vdots	\vdots	\ddots	\vdots
F_{m1}	F_{m2}	\cdots	F_{mm}

In the above diagram, the rectangular arrays are the J -cells of \mathcal{C}_A , the rows within the arrays right cells and the columns give left cells. The diagram thus illustrates that \mathcal{C}_A has two J -cells $\mathcal{J}_0, \mathcal{J}_1$, satisfying $\mathcal{J}_1 >_J \mathcal{J}_0$, where \mathcal{J}_1 is partitioned into m left cells and m right cells, with each intersection of a left cell with a right cell given by a singleton. By construction we then have $m + 1$ cell 2-representations: the apex of a cell 2-representation $\mathbf{C}_{\mathcal{L}}$ here is the J -cell containing \mathcal{L} .

The simple transitive 2-representations of \mathcal{C}_A have been completely classified:

Theorem 2.6. [17, Theorem 9]

- Every simple transitive 2-representation of \mathcal{C}_A is equivalent to a cell 2-representation.
- All cell 2-representations associated to left cells contained in \mathcal{J}_1 are mutually equivalent.
- Thus, up to equivalence, the 2-category \mathcal{C}_A admits exactly two simple transitive 2-representations.

The main aim of the next two sections is to obtain a similar result for certain 2-subcategories of \mathcal{C}_A .

3. Combinatorial 2-subcategories of \mathcal{C}_A

A *multisemigroup* is a set S together with a function μ from $S \times S$ to the power set 2^S , satisfying the *associativity condition*

$$\bigcup_{x \in \mu(r,s)} \mu(x,t) = \bigcup_{y \in \mu(s,t)} \mu(r,y).$$

A *multisubsemigroup* of S is a subset R of S such that, for any $x, y \in R$, we have $\mu(x, y) \subseteq R$. The restriction of μ to $R \times R$ endows R with the structure of a multisemigroup.

Let \mathcal{C} be a finitary 2-category. Denote by $\mathcal{S}(\mathcal{C})^0$ the set $\mathcal{S}(\mathcal{C}) \setminus \{0\}$. Recall from [9, 3.3] that the *multisemigroup* of \mathcal{C} is defined as the set $\mathcal{S}(\mathcal{C})^0$ together with the function μ defined by

$$\mu([F], [G]) = \begin{cases} \{0\} & \text{if } F \circ G \text{ is undefined;} \\ \{0\} & \text{if } F \circ G = 0 \\ \{[H] \in \mathcal{S}(\mathcal{C}) \mid H \text{ is a direct summand of } F \circ G\} & \text{otherwise.} \end{cases}$$

together with $\mu([F], 0) = \mu(0, [F]) = \mu(0, 0) = \{0\}$.

A subcategory \mathcal{D} of a category \mathcal{C} is called *replete* if, given $X \in \text{Ob}\mathcal{D}$ and an isomorphism $f : X \xrightarrow{\sim} Y$ in \mathcal{C} , the object Y lies in $\text{Ob}\mathcal{D}$, and f is a morphism in \mathcal{D} . Further, \mathcal{D} is called *wide* if $\text{Ob}\mathcal{D} = \text{Ob}\mathcal{C}$.

If a 2-subcategory \mathcal{D} of a 2-category \mathcal{C} is such that, for all $i, j \in \text{Ob}\mathcal{D}$, the subcategory $\mathcal{D}(i, j)$ of $\mathcal{C}(i, j)$ is replete, we say that \mathcal{D} is *2-replete*. Similarly, we say that a 2-subcategory \mathcal{D} of a 2-category \mathcal{C} is *wide* if $\text{Ob}\mathcal{D} = \text{Ob}\mathcal{C}$.

In the remainder of this text, we will only consider finitary, wide, 2-replete, 2-full 2-subcategories of finitary 2-categories. Hence, we introduce the following terminology:

Definition 3.1. Let \mathcal{C} be a finitary 2-category. We say that a 2-subcategory \mathcal{D} of \mathcal{C} is *combinatorial* if it is finitary, wide, 2-replete and 2-full.

Proposition 3.2. *The map*

$$\left\{ \begin{array}{l} \text{Combinatorial} \\ \text{2-subcategories of } \mathcal{C} \end{array} \right\} \rightarrow \left\{ \begin{array}{l} \text{Multisubsemigroups of } \mathcal{S}(\mathcal{C})^0 \\ \text{containing } 0 \text{ and } [1_i] \text{ for every } i \in \text{Ob}\mathcal{C} \end{array} \right\}$$

$$\mathcal{D} \mapsto \mathcal{S}(\mathcal{D})^0$$

is a bijection.

Proof. By definition, $\mathcal{S}(\mathcal{D})^0$ contains 0, and, since \mathcal{D} is wide, $\mathcal{S}(\mathcal{D})^0$ necessarily contains $[1_i]$, for every $i \in \text{Ob}\mathcal{C}$. If F is a 1-morphism in $\mathcal{D}(i, j)$ and G a summand of F in $\mathcal{C}(i, j)$, then $\mathcal{D}(i, j)$, being idempotent split, contains a 1-morphism isomorphic to G , and thus, since \mathcal{D} is 2-replete, it also contains G itself. Hence the map in the proposition is well-defined.

The set $\mathcal{S}(\mathcal{D})^0$ uniquely determines the collection of 1-morphisms of \mathcal{D} - the latter is given by all 1-morphisms isomorphic to finite direct sums of 1-morphisms in the isomorphism classes of $\mathcal{S}(\mathcal{D})$. Further, 2-fullness implies that the collection of 2-morphisms in \mathcal{D} is determined by that of 1-morphisms in \mathcal{D} . Finally, being wide implies $\text{Ob}\mathcal{D} = \text{Ob}\mathcal{C}$. It follows that $\mathcal{S}(\mathcal{D})^0$ uniquely determines \mathcal{D} , so the map is injective.

Finally, we show that the map is surjective, since given a multisubsemigroup S of $\mathcal{S}(\mathcal{C})^0$, the set $\text{Ob}\mathcal{C}$, together with the collection of all 1-morphisms isomorphic to finite direct sums of 1-morphisms in the classes of S and the collection of all 2-morphisms of \mathcal{C} between such 1-morphisms, gives a wide, 2-replete, 2-full, finitary 2-subcategory \mathcal{D}_S of \mathcal{C} . In particular, it is closed under composition of 1-morphisms: given indecomposable 1-morphisms F, G of \mathcal{D}_S , all the isomorphism classes of indecomposable direct summands of $G \circ F$ again lie in S and so $G \circ F$ is a 1-morphism of \mathcal{D}_S . □

From the description of the 1-morphisms of \mathcal{C}_A in Section 2.6, one sees that the multisemigroup of \mathcal{C}_A is actually a monoid with zero, with multiplication μ defined by

$$\mu([F_{ij}], [F_{kl}]) = \begin{cases} \{[F_{il}]\} & \text{if } e_j A e_k \neq 0 \\ \{0\} & \text{otherwise.} \end{cases}$$

We may relabel the element $[1_i]$ as 1, and the elements $[F_{ij}]$ of this monoid as (i, j) and identify the monoid with that obtained by adjoining the unit 1 to the resulting semigroup structure on $([[m]] \times [[m]]) \cup \{0\}$. Denote this monoid by $\mathcal{N}(A)$. Given a submonoid U of $\mathcal{N}(A)$, let \mathcal{D}_U denote the combinatorial 2-subcategory of \mathcal{C}_A corresponding to it under the bijection of Proposition 3.2.

Definition 3.3. Let A be a finite dimensional, basic, connected algebra and choose a complete set of pairwise orthogonal, primitive idempotents $\{e_1, \dots, e_m\}$ for A . A non-empty subset $U \subseteq [[m]]$ is a *self-injective core* for A if, for any $i \in U$, there is $j \in U$ such that

$$(e_i A)^* \simeq A e_j,$$

Let \mathcal{D} be a combinatorial subcategory of \mathcal{C}_A , where we have fixed a complete set of pairwise orthogonal, primitive idempotents $\{e_1, \dots, e_n\}$ for A . Define the sets $N_L(\mathcal{D})$ and $N_R(\mathcal{D})$ as

$$N_L(\mathcal{D}) := \{i \in [[m]] \mid \text{there is } F_{kl} \in \text{Ob}\mathcal{D}(i, i) \text{ such that } k = i\};$$

$$N_R(\mathcal{D}) := \{i \in [[m]] \mid \text{there is } F_{kl} \in \text{Ob}\mathcal{D}(i, i) \text{ such that } l = i\}.$$

Lemma 3.4. *If a combinatorial subcategory \mathcal{D} of \mathcal{C}_A is weakly fiat, then*

$$\mathbb{N}_L(\mathcal{D}) = \mathbb{N}_R(\mathcal{D}),$$

and this set is a self-injective core for A .

Proof. From Lemma 2.5 we conclude that the indecomposable 1-morphism F_{ij} of \mathcal{D} has a right adjoint in \mathcal{D} if and only if there is $k \in [[m]]$ such that there is an isomorphism of bimodules

$$(e_j A)^* \otimes_{\mathbb{k}} e_i A \simeq A e_k \otimes_{\mathbb{k}} e_i A$$

and $F_{ki} \in \text{Ob}\mathcal{D}(\dot{i}, \dot{i})$. Said bimodules are isomorphic if and only if $(e_j A)^* \simeq A e_k$. In particular, we have shown that $F_{ij} \in \text{Ob}\mathcal{D}(\dot{i}, \dot{i})$ implies $F_{ki} \in \text{Ob}\mathcal{D}(\dot{i}, \dot{i})$, so \mathbb{N}_L is a subset of \mathbb{N}_R .

As remarked in Section 2.1, taking right adjoints gives a weak antiautomorphism of finite order, and so every indecomposable 1-morphism F_{ij} of \mathcal{D} itself is a right adjoint. Thus, by the argument above, there is l such that

$$(e_l A)^* \otimes_{\mathbb{k}} e_j A \simeq A e_i \otimes_{\mathbb{k}} e_j A$$

and $F_{kl} \in \text{Ob}\mathcal{D}(\dot{i}, \dot{i})$. In particular, $(e_l A)^* \simeq A e_i$. Similarly to the first part of the proof, this shows that \mathbb{N}_R is a subset of \mathbb{N}_L . Hence $\mathbb{N}_L = \mathbb{N}_R$.

Given $j \in \mathbb{N}_R$, choose $i \in [[m]]$ such that $F_{ij} \in \text{Ob}\mathcal{D}(\dot{i}, \dot{i})$. We have shown that, in that case, there is $k \in \mathbb{N}_L$ such that $(e_j A)^* \simeq A e_k$. Since $\mathbb{N}_R = \mathbb{N}_L$, this shows that \mathbb{N}_R is a self-injective core. \square

Corollary 3.5. *A 2-subcategory of \mathcal{C}_A of the form $\mathcal{D}_{U_1 \times U_2}$ is weakly fiat if and only if $U_1 = U_2$ and U_1 is a self-injective core.*

Proof. Assume that \mathcal{D} is weakly fiat. From the definition we have $U_1 = \mathbb{N}_L$ and $U_2 = \mathbb{N}_R$. From Lemma 3.4 it follows that $U_1 = U_2$ and that U_1 is a self-injective core.

Assume that U_1 is a self-injective core and let F_{ij} be an indecomposable 1-morphism of $\mathcal{D}_{U \times U}$. Let $k \in U_1$ be such that $(e_j A)^* \simeq A e_k$. Then, from Lemma 2.5 it follows that F_{ki} is right adjoint to F_{ij} . Hence, $\mathcal{D}_{U_1 \times U_1}$ is weakly fiat. \square

Proposition 3.6. *Let U be a self-injective core for A and let $e = \sum_{i \in U} e_i$. The centralizer subalgebra eAe is self-injective.*

Proof. The set $\{e_i \mid i \in U\}$ is a complete set of pairwise orthogonal, primitive idempotents for eAe . Given $i \in U$, the functor $\text{Hom}_{A\text{-mod}}(Ae, -) : A\text{-mod} \rightarrow eAe\text{-mod}$ sends the indecomposable projective Ae_i to the indecomposable projective eAe_i and the indecomposable injective $(e_i A)^*$ to the indecomposable injective $(e_i Ae)^*$. Since U is a self-injective core, we may choose $j \in U$ such that $Ae_i \simeq (e_j A)^*$. Then

$$(e_j Ae)^* \simeq \text{Hom}_{A\text{-mod}}(Ae, (e_j A)^*) \simeq \text{Hom}_{A\text{-mod}}(Ae, Ae_i) \simeq eAe_i,$$

which shows that eAe indeed is self-injective. \square

It is easy to verify that, given two subsets U_1, U_2 of $[[m]]$, the set $(U_1 \times U_2) \cup \{0, 1\}$ gives a submonoid of $\mathcal{N}(A)$.

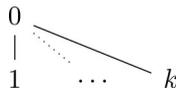
Definition 3.7. Let U, V be subsets of $[[m]]$. We say that a combinatorial 2-subcategory \mathcal{D} of \mathcal{C}_A is *U-superdiagonal* if $U \subseteq V$ and $\mathcal{D} = \mathcal{D}_{U \times V}$. We say that \mathcal{D} is *U-subdiagonal* if $V \subseteq U$ and $\mathcal{D} = \mathcal{D}_{U \times V}$. If \mathcal{D} is both *U-superdiagonal* and *U-subdiagonal*, then $\mathcal{D} = \mathcal{D}_{U \times U}$ and we say that \mathcal{D} is *U-diagonal*.

In this document, we will focus on the U -superdiagonal case. The remaining results of this section also hold in the U -subdiagonal case, although Proposition 3.13 must be modified as described in its formulation. Further, in the U -superdiagonal case, the vacuous J -cells also constitute left cells, whereas in the U -subdiagonal case these constitute right cells. The crucial difference, which is the reason for restricting our attention to the U -superdiagonal case, is that Proposition 4.2 is not true in the U -subdiagonal case. Neither is the main result of this document, Theorem 4.5: a counterexample is given by [21, Theorem 5.10].

Proposition 3.8. *A combinatorial 2-subcategory \mathcal{D} of \mathcal{C}_A is U -superdiagonal if and only if it contains $\mathcal{D}_{U \times U}$ and $N_L(\mathcal{D}) = U$.*

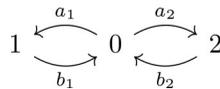
Proof. Clearly, \mathcal{D} being U -superdiagonal implies both that \mathcal{D} contains $\mathcal{D}_{U \times U}$ and that $N_L(\mathcal{D}) = U$. Assume that \mathcal{D} contains $\mathcal{D}_{U \times U}$ and $N_L = U$. Let $j \in N_R$ and let $i \in U$ be such that $F_{ij} \in \text{Ob}\mathcal{D}(i, i)$. Given $k \in U$, from the assumption we know that F_{ki} is a 1-morphism of \mathcal{D} . Since $e_i A e_i \neq 0$, the 1-morphism F_{kj} is a direct summand of $F_{ki} \circ F_{ij}$. Since \mathcal{D} is finitary, it follows that F_{kj} is a 1-morphism of \mathcal{D} . This shows that $\mathcal{D} = \mathcal{D}_{U \times N_R(\mathcal{D})}$. \square

Given a graph Γ , we use the notation of [3] and denote the zigzag algebra on Γ by $Z_{\rightleftharpoons}(\Gamma)$. As defined in [5], the algebra $Z_{\rightleftharpoons}(\Gamma)$ is the quotient of the path algebra of the double quiver on Γ by the ideal generated by paths $i \rightarrow j \rightarrow k$ for $i \neq k$, together with elements of the form $\alpha - \beta$, where α, β are different 2-cycles at the same vertex of Γ . From [5, Proposition 1], we know that a zigzag algebra $Z_{\rightleftharpoons}(\Gamma)$ is weakly symmetric. Let S_k be the star graph on $k + 1$ vertices, labeled as follows:

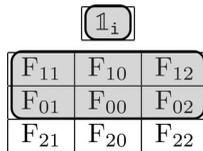


Following [21], we call a zigzag algebra of the form $Z_{\rightleftharpoons}(S_k)$ a *star algebra*. The above labeling of the vertices of S_k induces a complete set of pairwise orthogonal, primitive idempotents $\{e_0, e_1, \dots, e_k\}$ for $Z_{\rightleftharpoons}(S_k)$.

Example 3.9. Consider the star algebra $A := Z_{\rightleftharpoons}(S_2)$, by definition given as the quotient of the path algebra of



by the ideal generated by $\{a_2 b_1, a_1 b_2, b_2 a_2 - b_1 a_1\}$. The set $\{1, 0\}$ is a self-injective core for A , and the shaded part of the eggbox diagram of \mathcal{C}_A below corresponds to the $\{0, 1\}$ -superdiagonal 2-subcategory $\mathcal{D}_{\{1,0\} \times \{1,0,2\}}$.



Example 3.10. A weakly fiat, combinatorial 2-subcategory of \mathcal{C}_A is not necessarily of the form $\mathcal{D}_{U \times U}$. Consider again the star algebra $Z_{\rightleftharpoons}(S_2)$ of Example 3.9. We have

$$F_{11} \circ F_{11} \simeq F_{11}, \quad F_{22} \circ F_{22} \simeq F_{22} \quad \text{and} \quad F_{11} \circ F_{22} = F_{22} \circ F_{11} = 0.$$

Hence, we may consider the combinatorial 2-subcategory $\mathcal{D}_{\{(1,1), (2,2)\}}$. Being a zigzag algebra, $Z_{\rightleftharpoons}(\mathcal{S}_2)$ is weakly symmetric, and so from Lemma 2.5 it follows that both F_{11} and F_{22} are self-adjoint. Hence $\mathcal{D}_{\{(1,1), (2,2)\}}$ is fiat, and in particular also weakly fiat. But the set $\{(1,1), (2,2)\}$ is not a product of subsets of $\{1, 0, 2\}$.

Remark 3.11. A self-injective core is not unique for an algebra A with a fixed complete set of pairwise orthogonal, primitive idempotents $\{e_1, \dots, e_m\}$. If A is weakly symmetric, then any non-empty subset of $[[m]]$ gives a self-injective core, yielding $2^m - 1$ different cores for A . More generally, if A is self-injective with Nakayama permutation ν of $[[m]]$, then the self-injective cores of A are given by the unions of orbits of ν .

Moreover, not every algebra A admits a self-injective core: if there are no non-zero projective-injective modules over A , then A cannot have a self-injective core. The existence of such a module does not guarantee the existence of a self-injective core, either. A family of counterexamples is given by hereditary algebras of type A . Let us label the uniformly oriented quiver for A_n in the standard way:

$$1 \longrightarrow 2 \longrightarrow \cdots \longrightarrow n$$

Then the unique non-zero projective-injective module is given by $A_n e_1 \simeq (e_n A_n)^*$. A self-injective core containing 1 would thus have to contain n . However, $A_n e_n$ is not injective.

In view of the above-described non-uniqueness, our choice of terminology may seem peculiar. It is motivated by the essential role of the self-injective core U and its associated U -diagonal 2-subcategory of \mathcal{C}_A in the classification of simple transitive 2-representations of any U -superdiagonal 2-subcategory of \mathcal{C}_A .

Example 3.12. Let $\{e_1, \dots, e_m\}$ be a complete set of pairwise orthogonal, primitive idempotents for A . Choose $U \subseteq [[m]]$ and let $e = \sum_{i \in U} e_i$. Assume that eAe is self-injective. This is not sufficient to conclude that U is a self-injective core for A . Consider the algebra $A = \mathbb{k}Q/I$, where Q is the quiver

$$\begin{array}{ccc} & \alpha & \\ & \curvearrowright & \\ 1 & & 2 \\ & \curvearrowleft & \\ & \beta & \end{array}$$

and I is the ideal $\langle \alpha\beta \rangle$. The algebra $e_2 A e_2$ is isomorphic to \mathbb{k} , and hence in particular it is self-injective. But $\{2\}$ is not a self-injective core, since the module Ae_2 is not injective.

3.1. Cells of U -superdiagonal 2-subcategories of \mathcal{C}_A

For the remainder of this document, we let $U \subseteq [[m]]$ be a self-injective core for A and we let \mathcal{D} be a U -superdiagonal 2-subcategory of \mathcal{C}_A .

It is rather clear that the left, right and two-sided preorders of a combinatorial 2-subcategory \mathcal{D} of \mathcal{C}_A are coarser than the restrictions of the respective preorders for \mathcal{C}_A to the set $\mathcal{S}(\mathcal{D})$. In many cases they are not strictly coarser:

Proposition 3.13.

1. The left preorder of \mathcal{D} coincides with the restriction of the left preorder of \mathcal{C}_A to $\mathcal{S}(\mathcal{D})$.
2. Dually, if \mathcal{B} is a U -subdiagonal 2-subcategory of \mathcal{C}_A , then the right preorder of \mathcal{B} coincides with the restriction of the right preorder of \mathcal{C}_A to $\mathcal{S}(\mathcal{B})$.

3. Hence, the left, right and two-sided preorders of $\mathcal{D}_{U \times U}$ all coincide with the respective restrictions.

Proof. We prove the first statement. The second one is dual, and the third one is an immediate consequence of the first two.

Let \leq_L^A denote the left preorder of \mathcal{C}_A and let $\leq_L^{\mathcal{D}}$ denote the left preorder of \mathcal{D} . In view of the observation preceding the proposition, it suffices to show that, for indecomposable 1-morphisms F, G of \mathcal{D} , the condition $F \leq_L^A G$ implies $F \leq_L^{\mathcal{D}} G$. It is clear that $[\mathbb{1}_i]$ remains the unique minimal element the cell structure of a combinatorial 2-subcategory of \mathcal{C}_A , so we may write $F = F_{ij}$ and $G = F_{kl}$ with $i, k \in U$. $F_{ij} \leq_L^A F_{kl}$ is equivalent to $j = l$. We thus need to show that, for $i, k \in U$ and j such that $F_{ij}, F_{kj} \in \text{Ob}\mathcal{D}(i, i)$, there is a 1-morphism H of \mathcal{D} such that F_{kj} is a direct summand of $H \circ F_{ij}$. Let $H = F_{ki}$. Since $i, k \in U$ and \mathcal{D} contains $\mathcal{D}_{U \times U}$, we know that F_{ki} is a 1-morphism of \mathcal{D} . And since $e_i A e_i \neq 0$, we see that F_{kj} indeed is a direct summand of $F_{ki} \circ F_{ij} \simeq F_{kj}^{\oplus \dim e_i A e_i}$. The result follows. \square

Proposition 3.14. *Let $F \in \mathcal{S}(\mathcal{D}) \setminus \{[\mathbb{1}_i]\}$. Let $i \in U$ and $j \in [[m]]$ be such that $F = F_{ij}$.*

- a. *The following are equivalent:*
 - i. *There is a 1-morphism $G \in \mathcal{S}(\mathcal{D}_{U \times U})$ such that F is J -equivalent to G inside \mathcal{D} .*
 - ii. *There is $h \in U$ such that $e_j A e_h \neq 0$.*
- b. *If there is no h as above, then F lies in a maximal J -cell of \mathcal{D} , which is not idempotent.*

Proof. Since G is not isomorphic to $\mathbb{1}_i$, without loss of generality we may assume $G = F_{kl}$, for some $k, l \in U$.

Suppose that there is $h \in U$ such that $e_j A e_h \neq 0$. We then have

$$F_{kl} \in \text{Ob add}\{F_{ki} \circ F_{ij} \circ F_{hl}\},$$

where $h, i, k, l \in U$, so that $F_{ki}, F_{hl} \in \text{Ob}\mathcal{D}(i, i)$. This shows that $F_{kl} \geq_J F_{ij}$. Further, we have $F_{ij} \geq_J F_{kl}$, since

$$F_{ij} \in \text{Ob add}\{F_{ik} \circ F_{kl} \circ F_{lj}\}.$$

Note that we did not use the assumption about h for this latter condition.

Now assume that G is as specified in 1a. Again let $G = F_{kl}$, with $k, l \in U$. By definition, there are H, H' such that $F_{kl} \simeq H \circ F_{ij} \circ H'$. Due to the biadditivity of composition of 1-morphisms, we may assume H, H' to be indecomposable. If we write $H' = F_{xy}$, then, clearly, it is necessary that $e_j A e_x \neq 0$. Further, we have $x \in N_L(\mathcal{D}) = U$. We may now let $h := x$.

If F_{ij} is not J -equivalent to the 1-morphisms of $\mathcal{D}_{U \times U}$, then $e_j A e_h = 0$, for all $h \in U$. But \mathcal{D} is U -superdiagonal, hence in particular $N_L = U$. This implies that, for any $H \in \mathcal{S}(\mathcal{D}) \setminus \{\mathbb{1}_i\}$, we have $F_{ij} \circ H = 0$. Thus, for $H' \in \mathcal{S}(\mathcal{D})$, the statements $H' \geq_L^{\mathcal{D}} F_{ij}$ and $H' \geq_J^{\mathcal{D}} F_{ij}$ are equivalent. From Proposition 3.13 we infer that the left cell $\mathcal{L}_j = \{[F_{yj}] \mid y \text{ such that } F_{yj} \in \text{Ob}\mathcal{D}(i, i)\}$ is a maximal left cell in \mathcal{D} , and so

$$\mathcal{L}_j = \{H' \in \mathcal{S}(\mathcal{D}) \mid H' \geq_J^{\mathcal{D}} F_{ij}\}.$$

It follows that \mathcal{L}_j is a maximal J -cell of \mathcal{D} . All of its elements are annihilated by right composition with nonidentity indecomposable 1-morphisms, and so composition of any two elements of \mathcal{L}_j is zero, which proves that \mathcal{L}_j is not idempotent. \square

As a consequence, \mathcal{D} has exactly two idempotent J -cells. The J -minimal among the two is given by $\mathbb{1}_i$, the other is the J -cell containing $F_{ij} \in \mathcal{S}(U \times U)$. We denote the former by $\mathcal{J}_0^{\mathcal{D}}$ and the latter by $\mathcal{J}_1^{\mathcal{D}}$. If there is no risk of ambiguity, we may omit the superscript \mathcal{D} and write $\mathcal{J}_0, \mathcal{J}_1$.

The left and right cell structures of \mathcal{D} restricted to the union of the idempotent J -cells $\mathcal{J}_0, \mathcal{J}_1$ is given by the restriction of the respective cell structures of \mathcal{C}_A . Additionally, \mathcal{D} admits a (possibly empty) set of mutually incomparable, J -maximal, non-idempotent J -cells, each strictly J -greater than the idempotent J -cells. Such a J -cell is also a left cell, and the right cells inside it are singletons. We refer to such J -cells as *vacuous cells*.

The terminology is motivated by the fact that vacuous cells can be ignored when considering our problem of classification of simple transitive 2-representations. A vacuous cell \mathcal{J} is J -maximal and non-idempotent, and hence, as a consequence of [1, Proposition 3], it is annihilated by every simple transitive 2-representation, so we may replace \mathcal{D} by its quotient by the 2-ideal generated by $\{\text{id}_F \mid F \in \mathcal{J}\}$.

Proposition 3.15. *Let $\mathcal{L}, \mathcal{L}'$ be two left cells of \mathcal{D} . The cell 2-representations $\mathbf{C}_{\mathcal{L}}, \mathbf{C}_{\mathcal{L}'}$ are equivalent if and only if they have the same apex.*

Proof. Equivalent 2-representations have the same apex, so we only need to prove that cell 2-representations with the same apex are equivalent. From Propositions 3.13, 3.14 we conclude that the apex of a cell 2-representation not associated to the minimal left cell $\{[\mathbb{1}_{\mathbb{1}}]\}$ is \mathcal{J}_1 . Indeed, if $F_{kl} \in \mathcal{L}$, then $F_{kk} \circ F_{kl} \neq 0$ shows that the apex of $\mathbf{C}_{\mathcal{L}}$ is \mathcal{J}_1 . This shows that \mathcal{J}_0 is the unique left cell whose cell 2-representation has \mathcal{J}_0 as apex. It thus suffices to prove the claim for $\mathcal{L}, \mathcal{L}' \neq \mathcal{J}_0$. In that case, there are $j, j' \in \mathbb{N}_R(\mathcal{D})$ such that $\mathcal{L} = \mathcal{S}(U \times \{j\})$ and $\mathcal{L}' = \mathcal{S}(U \times \{j'\})$.

Recall that $\mathbf{C}_{\mathcal{L}} = \mathbf{N}_{\mathcal{L}}/\mathbf{I}_{\mathcal{L}}$, where the target category of $\mathbf{N}_{\mathcal{L}}$ is $\text{add } \mathcal{S}(U \times \{j\})$. Consider the left cell $\overline{\mathcal{L}} = \mathcal{S}([m] \times \{j\})$ of \mathcal{C}_A . Similarly, we have $\mathbf{C}_{\overline{\mathcal{L}}} = \mathbf{N}_{\overline{\mathcal{L}}}/\mathbf{I}_{\overline{\mathcal{L}}}$, with $\mathbf{N}_{\overline{\mathcal{L}}}(\mathbb{1}) = \text{add } \mathcal{S}([m] \times \{j\})$. Let $e_U = \sum_{i \in U} e_i$. By the definition of $F_{U \times \{j\}}$, there is a canonical algebra isomorphism $\text{End}_{\mathcal{D}(\mathbb{1}, \mathbb{1})}(F_{U \times \{j\}}) \simeq e_U A e_U \otimes_{\mathbb{k}} e_j A e_j$. One may easily verify that $\mathbf{I}_{\mathcal{L}}$ is determined on the level of indecomposable objects by the ideal $e_U A e_U \otimes_{\mathbb{k}} e_j (\text{Rad} A) e_j$. Similarly, $\text{End}_{\mathcal{C}_A(\mathbb{1}, \mathbb{1})}(F_{[m] \times \{j\}}) \simeq A \otimes_{\mathbb{k}} e_j A e_j$ and $\mathbf{I}_{\overline{\mathcal{L}}}$ is determined by the ideal $A \otimes_{\mathbb{k}} e_j (\text{Rad} A) e_j$. This observation is used in [12, Proposition 9].

The inclusion of $F_{U \times \{j\}}$ in $F_{[m] \times \{j\}}$ corresponds to the canonical inclusion of $e_U A e_U \otimes_{\mathbb{k}} e_j A e_j$ in $A \otimes_{\mathbb{k}} e_j A e_j$. It follows that $\mathbf{C}_{\mathcal{L}}$ is a 2-subrepresentation of the restriction of $\mathbf{C}_{\overline{\mathcal{L}}}$ to a 2-representation of \mathcal{D} .

From [12, Proposition 9], we know that the functor $\mathbf{C}_{\overline{\mathcal{L}}}(\mathbb{1}) \rightarrow \mathbf{C}_{\overline{\mathcal{L}}}(\mathbb{1})$, given by sending F_{ij} to $F_{ij'}$ on the level of objects, and on the level of morphisms corresponding to the map

$$(A \otimes_{\mathbb{k}} e_j A e_j) / (A \otimes_{\mathbb{k}} e_j (\text{Rad} A) e_j) \simeq A \xrightarrow{\text{id}_A} A \simeq (A \otimes_{\mathbb{k}} e_{j'} A e_{j'}) / (A \otimes_{\mathbb{k}} e_{j'} (\text{Rad} A) e_{j'}),$$

gives an equivalence $\mathbf{C}_{\mathcal{L}} \xrightarrow{\sim} \mathbf{C}_{\overline{\mathcal{L}'}}$. Using the commutativity of

$$\begin{array}{ccc} (A \otimes_{\mathbb{k}} e_j A e_j) / (A \otimes_{\mathbb{k}} e_j (\text{Rad} A) e_j) & \xrightarrow{\sim} & A \\ \uparrow & & \uparrow \\ (e_U A e_U \otimes_{\mathbb{k}} e_j A e_j) / (e_U A e_U \otimes_{\mathbb{k}} e_j (\text{Rad} A) e_j) & \xrightarrow{\sim} & e_U A e_U \end{array},$$

we conclude that the equivalence $\mathbf{C}_{\overline{\mathcal{L}}} \xrightarrow{\sim} \mathbf{C}_{\overline{\mathcal{L}'}}$ restricts to an equivalence $\mathbf{C}_{\mathcal{L}} \simeq \mathbf{C}_{\mathcal{L}'}$, from which the result follows. □

3.2. 2-Representations of fiat U-diagonal 2-subcategories of \mathcal{C}_A

Recall that, by Proposition 3.5, the 2-category $\mathcal{D}_{U \times U}$ is weakly fiat. Let \mathcal{D} be a combinatorial 2-subcategory of \mathcal{C}_A containing $\mathcal{D}_{U \times U}$. Let \mathbf{M} be a simple transitive 2-representation of \mathcal{D} . The following is an immediate consequence of [21, Theorem 3.1]:

Lemma 3.16. *Given a 1-morphism $F \in \text{Ob } \mathcal{D}_{U \times U}(\mathfrak{i}, \mathfrak{i})$, the functor $\overline{\mathbf{M}}F$ is a projective functor.*

Recall from [8, Section 4.8] that a J -cell of a finitary 2-category is called *strongly regular* if

- any two left (respectively right) cells in \mathcal{J} are not comparable with respect to the left (respectively right) order;
- the intersection between any right and any left cell in \mathcal{J} is a singleton.

From the results of the preceding section, it is clear that all the J -cells of \mathcal{D} are strongly regular. Combining that with Proposition 3.5, we find that $\mathcal{D}_{U \times U}$ is weakly fiat with strongly regular J -cells. The following is then an immediate consequence of [13, Theorem 33]:

Proposition 3.17. *Any simple transitive 2-representation of $\mathcal{D}_{U \times U}$ is equivalent to a cell 2-representation.*

For the remainder of this document, let \mathcal{L}_0 be the left cell of $\mathcal{D}_{U \times U}$ consisting of $[1_{\mathfrak{i}}]$ and let \mathcal{L}_1 be a left cell contained in $\mathcal{J}_1^{\mathcal{D}_{U \times U}}$. Since any left cell of $\mathcal{D}_{U \times U}$ is also a left cell of \mathcal{D} , both \mathcal{L}_0 and \mathcal{L}_1 give left cells of both \mathcal{D} and $\mathcal{D}_{U \times U}$. We denote the associated 2-representations of \mathcal{D} by $\mathbf{C}_{\mathcal{L}_0}, \mathbf{C}_{\mathcal{L}_1}$, and the associated 2-representations of $\mathcal{D}_{U \times U}$ by $\mathbf{C}_{\mathcal{L}_0}^U, \mathbf{C}_{\mathcal{L}_1}^U$.

Proposition 3.18. *We have*

1. $\text{Dext}(\mathbf{C}_{\mathcal{L}_0}^U, \mathbf{C}_{\mathcal{L}_0}^U) = \emptyset$
2. $\text{Dext}(\mathbf{C}_{\mathcal{L}_1}^U, \mathbf{C}_{\mathcal{L}_1}^U) = \emptyset$
3. $\text{Dext}(\mathbf{C}_{\mathcal{L}_1}^U, \mathbf{C}_{\mathcal{L}_0}^U) = \emptyset$.

Proof. The above statement is a minor modification of [1, Theorem 6.22], and the proof therein requires only two changes for our case. First, we restrict the index set, from $[[n]]$ in [1], to U in our case. Second, rather than assume that A is self-injective, we only assume that U is a self-injective core. Both the assumptions imply the properties which are used in the proof given in [1]. □

Proposition 3.18 immediately implies the following statement:

Corollary 3.19. *Let \mathbf{M} be a finitary 2-representation of $\mathcal{D}_{U \times U}$. There is a labeling of the indecomposable objects of $\mathbf{M}(\mathfrak{i})$, with respect to which we have*

$$[\mathbf{F}_{U \times U}]_{\mathbf{M}} = \left(\left(\begin{array}{c|c|c} [\mathbf{F}_{U \times U}]_{\mathbf{C}_{\mathcal{L}_1}^U} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & [\mathbf{F}_{U \times U}]_{\mathbf{C}_{\mathcal{L}_0}^U} \end{array} \right) \middle| * \right) \\ \hline \begin{array}{c|c} & 0 \\ \hline & 0 \end{array}$$

Each diagonal block of the form $[\mathbf{F}_{U \times U}]_{\mathbf{C}_{\mathcal{L}_i}^U}$ corresponds to a simple transitive subquotient equivalent to $\mathbf{C}_{\mathcal{L}_i}^U$ in the weak Jordan-Hölder series of \mathbf{M} , and the bottom right diagonal block 0 corresponds to the subquotients equivalent to $\mathbf{C}_{\mathcal{L}_0}^U$.

4. The main result and its generalizations

4.1. The main result

Let $Z_{\rightleftharpoons}(S_k)$ be the star algebra defined in Example 3.9. The set $\{0\}$ defines a self-injective core for $Z_{\rightleftharpoons}(S_k)$. Consider the $\{0\}$ -superdiagonal 2-subcategory of $\mathcal{C}_{Z_{\rightleftharpoons}(S_k)}$ given by $\mathcal{D}_{\{0\} \times [[0, k]]}$. The main goal of this section is to generalize the following result:

Theorem 4.1. [21, Theorem 6.2] Any simple transitive 2-representation of the 2-category $\mathcal{D}_{\{0\} \times [[0, k]]}$ is equivalent to a cell 2-representation.

Given a finitary 2-subcategory \mathcal{B} of a finitary 2-category \mathcal{C} , denote by

$$\text{Res}_{\mathcal{B}}^{\mathcal{C}}(-) : \mathcal{C} - \text{afmod} \rightarrow \mathcal{B} - \text{afmod}$$

the restriction 2-functor given by precomposition with the inclusion 2-functor from \mathcal{B} to \mathcal{C} . Let \mathbf{M} be a finitary 2-representation of \mathcal{C} . Following [12, Theorem 8], we may choose a transitive subquotient \mathbf{N} of $\text{Res}_{\mathcal{B}}^{\mathcal{C}}(\mathbf{M})$. For any $\mathfrak{i} \in \text{Ob}\mathcal{C}$, we may label a complete, irredundant list of representatives of isomorphism classes of indecomposable objects of $\mathbf{M}(\mathfrak{i})$ as X_1, \dots, X_l , so that there is $k \leq l$ such that the list X_1, \dots, X_k of representatives is complete and irredundant for $\mathbf{N}(\mathfrak{i})$. Under this labeling, the Cartan matrix $C^{\mathbf{N}(\mathfrak{i})}$ is the diagonal block submatrix of $C^{\mathbf{M}(\mathfrak{i})}$ corresponding to the indices $1, \dots, k$. By [12, Lemma 4], the 2-representation \mathbf{N} admits a unique maximal \mathcal{B} -stable ideal \mathbf{I} . The quotient \mathbf{N}/\mathbf{I} is simple transitive. The projection 2-transformation $\mathbf{N} \rightarrow \mathbf{N}/\mathbf{I}$ does not map any of the indecomposable objects to 0, and so the Cartan matrices are of the same size, and, for $i, j \in [[k]]$, we have the entry-wise inequality

$$C_{ij}^{\mathbf{M}(\mathfrak{i})} = C_{ij}^{\mathbf{N}(\mathfrak{i})} \geq C_{ij}^{\mathbf{N}(\mathfrak{i})/\mathbf{I}(\mathfrak{i})} = C_{ij}^{\mathbf{N}/\mathbf{I}(\mathfrak{i})}. \quad (3)$$

For the remainder of this section, we will let \mathbf{M} be a transitive 2-representation of the U -superdiagonal 2-subcategory \mathcal{D} of \mathcal{C}_A , fixed in Section 3. Similarly to [21, Theorem 6.2], if the apex of \mathbf{M} is \mathcal{J}_0 , our remaining claims, including the main result, follow immediately. We thus assume that the apex of \mathbf{M} is \mathcal{J}_1 .

Proposition 4.2. *The 2-representation $\text{Res}_{\mathcal{D}_{U \times U}}^{\mathcal{D}}(\mathbf{M})$ is transitive.*

Proof. From the description of the cell structure of \mathcal{D} given in Section 3.1, we know that the set $\mathcal{S}(U \times U)$ of indecomposable 1-morphisms of $\mathcal{D}_{U \times U}$ is a union of left cells of \mathcal{D} . Hence, for any $G \in \mathcal{S}(\mathcal{D})$, we have $G \circ F_{U \times U} \in \text{add } \mathcal{S}(U \times U)$.

This shows that the additive closure of the essential images of the functors in $\{\mathbf{M}F \mid F \in \mathcal{S}(U \times U)\}$ is stable under the functorial action of \mathcal{C} given by \mathbf{M} . Using action notation, we may write

$$\text{add}(\mathcal{D} \cdot (F_{U \times U} \cdot \mathbf{M}(\mathfrak{i}))) = \text{add}((\mathcal{D} \circ F_{U \times U}) \cdot \mathbf{M}(\mathfrak{i})) = \text{add}(F_{U \times U} \cdot \mathbf{M}(\mathfrak{i})).$$

Since the apex of \mathbf{M} is \mathcal{J}_1 , we have $\text{add}\{F_{U \times U} \cdot \mathbf{M}(\mathfrak{i})\} \neq 0$. Further, by assumption, \mathbf{M} is transitive, which yields $\text{add}\{F_{U \times U} \cdot \mathbf{M}(\mathfrak{i})\} = \mathbf{M}(\mathfrak{i})$.

This shows that all the rows of the action matrix $[F_{U \times U}]_{\mathbf{M}}$ are non-zero, and, in view of Corollary 3.19, we find that all simple transitive weak subquotients of $\text{Res}_{\mathcal{D}_{U \times U}}^{\mathcal{D}}(\mathbf{M})$ are equivalent to $C_{\mathcal{L}_1}^U$. Thus, if we let k be the length of the weak Jordan-Hölder series of $\text{Res}_{\mathcal{D}_{U \times U}}^{\mathcal{D}} \mathbf{M}$, Corollary 3.19 implies that the matrix $[F_{U \times U}]_{\mathbf{M}} = [F_{U \times U}] \text{Res}_{\mathcal{D}_{U \times U}}^{\mathcal{D}}(\mathbf{M})$ is the $k \times k$ block diagonal matrix

$$\left(\begin{array}{c|c|c} [F_{U \times U}]_{\mathbf{C}_{\mathcal{L}_1}} & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & [F_{U \times U}]_{\mathbf{C}_{\mathcal{L}_1}} \end{array} \right).$$

Observe that all the entries of $[F_{U \times U}]_{\mathbf{C}_{\mathcal{L}_1}}$ are positive integers. Due to the additivity of action matrices, we have $[F_{U \times U}]_{\mathbf{M}} = \sum_{(i,j) \in U \times U} [F_{ij}]_{\mathbf{M}}$. From our earlier observations it follows that $F_{\mathcal{S}(\mathcal{Q})} \circ F_{U \times U}$ has a direct sum decomposition with summands in $\mathcal{S}(U \times U)$. As a consequence, there are non-negative integers $\{a_{ij} \mid (i,j) \in U \times U\}$ such that

$$[F_{\mathcal{S}(\mathcal{Q})} \circ F_{U \times U}]_{\mathbf{M}} = \sum_{(i,j) \in U \times U} a_{ij} [F_{ij}]_{\mathbf{M}}.$$

Since $\mathcal{S}(U \times U)$ is a subset of $\mathcal{S}(\mathcal{Q})$, the integers a_{ij} are positive, for all $i, j \in U$. Using what we know about $[F_{U \times U}]_{\mathbf{M}}$, we can write

$$[F_{\mathcal{S}(\mathcal{Q})} \circ F_{U \times U}]_{\mathbf{M}} = \left(\begin{array}{c|c|c} \mathbf{C}_1 & 0 & 0 \\ \hline 0 & \ddots & 0 \\ \hline 0 & 0 & \mathbf{C}_k \end{array} \right),$$

where the entries of the block submatrices $\mathbf{C}_1, \dots, \mathbf{C}_k$ are positive integers. From the biadditivity of composition of 1-morphisms we have

$$[F_{\mathcal{S}(\mathcal{Q})} \circ F_{U \times U}]_{\mathbf{M}} = [F_{\mathcal{S}(\mathcal{Q})}]_{\mathbf{M}} [F_{U \times U}]_{\mathbf{M}}.$$

Since \mathbf{M} is transitive, the entries of the left factor in this product are positive integers, and we have shown that the entries of the right factor are non-negative integers with positive diagonal entries. This shows that all the entries of the matrix $[F_{\mathcal{S}(\mathcal{Q})} \circ F_{U \times U}]_{\mathbf{M}}$ are positive. Thus $k = 1$, and so $\text{Res}_{\mathcal{Q}_{U \times U}}^{\mathcal{Q}}(\mathbf{M})$ is transitive. □

Corollary 4.3. *The rank of \mathbf{M} coincides with the rank of $\mathbf{C}_{\mathcal{L}_1}$, which by definition equals $|U|$. Further, the action matrices of 1-morphisms of $\mathcal{Q}_{U \times U}$ for \mathbf{M} coincide with those for $\mathbf{C}_{\mathcal{L}_1}$.*

Proof. Combining Proposition 4.2 with Proposition 3.17, we conclude that the unique simple transitive quotient of $\text{Res}_{\mathcal{Q}_{U \times U}}^{\mathcal{Q}}(\mathbf{M})$ is equivalent to said cell 2-representation. The claim follows. □

Assume that \mathbf{M} is simple transitive. Let $r := |U|$. Without loss of generality, assume that $U = [[r]]$. Let Q be the basic algebra introduced in Section 2.3, satisfying $\mathbf{M}(\dot{i}) \simeq Q - \text{proj}$, together with the complete set of pairwise orthogonal, primitive idempotents described therein. Given $s, t \in [[r]]$, choose an endofunctor of $\overline{\mathbf{M}}(\dot{i}) \simeq Q - \text{mod}$ naturally isomorphic to the indecomposable projective functor $Qf_s \otimes_{\mathbb{k}f_i} Q \otimes_Q -$, and denote it by G_{st} . This is consistent with the notation $\{F_{ij} \mid i, j \in [[m]]\}$ for indecomposable 1-morphisms of \mathcal{C}_A , since such 1-morphisms correspond to the indecomposable projective endofunctors of $A - \text{mod}$. Using this notation, Lemma 3.16 implies $\overline{\mathbf{M}}\mathbf{F} \in \text{Ob add}\{G_{st} \mid s, t \in [[r]]\}$.

We recall a notation convention introduced in [19]. Given $i, j \in U$, we write

$$X_{ij} = \left\{ \begin{array}{l} s \in [[r]] \mid \text{there is } t \in [[r]] \text{ such that } G_{st} \text{ is} \\ \text{isomorphic to a direct summand of } \overline{\mathbf{M}}\mathbf{F}_{ij} \end{array} \right\},$$

$$Y_{ij} = \left\{ \begin{array}{l} t \in [[r]] \mid \text{there is } s \in [[r]] \text{ such that } G_{st} \text{ is} \\ \text{isomorphic to a direct summand of } \overline{\mathbf{M}}\mathbf{F}_{ij} \end{array} \right\}.$$

Clearly, the essential image of G_{st} in $\overline{\mathbf{M}}(\mathfrak{i})$ is given by $\text{add}\{Qe_s\}$. Hence, X_{ij} coincides with the set of non-zero rows of $[F_{ij}]_{\mathbf{M}}$. From Corollary 4.3 we know that only the i th row of $[F_{ij}]_{\mathbf{M}}$ is non-zero. Hence $X_{ij} = \{i\}$ for all $i, j \in U$. The following statement, as well as its proof, is completely analogous to [19, Lemma 20] and [19, Lemma 22]:

Lemma 4.4. *For any $i, j \in U$ we have $Y_{ij} = \{j\}$. Thus, $\mathbf{M}F_{ij} \simeq G_j^{\oplus m_{ij}}$ for some positive integers m_{ij} .*

Theorem 4.5. *Let \mathcal{D} be a U -superdiagonal 2-subcategory of \mathcal{C}_A . Any simple transitive 2-representation of \mathcal{D} is equivalent to a cell 2-representation.*

Proof. In view of our prior observations regarding 2-representations with apex \mathcal{J}_0 , it suffices to show the claim for the above-described simple transitive 2-representation \mathbf{M} . Let C^Q denote the Cartan matrix of $Q - \text{proj}$ and let C^{eAe} denote the Cartan matrix of $eAe - \text{proj}$, where e is the idempotent $e_U = \sum_{i \in U} e_i$ of A . By definition we have $([F_{ij}]_{C_{\mathcal{L}_1}})_{il} = ([F_{ij}]_{C_{\mathcal{L}_1}^U})_{il} = C_{jl}^{eAe}$, for $i, j, l \in U$.

Corollary 4.3 yields $[F_{ij}]_{\mathbf{M}} = [F_{ij}]_{C_{\mathcal{L}_1}}$. By Lemma 4.4, we have $([F_{ij}]_{\mathbf{M}})_{il} = m_{ij}C_{jl}^Q$. To summarize, for $i, j, l \in U$, we have

$$C_{jl}^{eAe} = ([F_{ij}]_{C_{\mathcal{L}_1}})_{il} = ([F_{ij}]_{\mathbf{M}})_{il} = m_{ij}C_{jl}^Q. \quad (4)$$

Proposition 4.2 shows that the unique simple transitive quotient of $\text{Res}_{\mathcal{D}_{U \times U}}^{\mathcal{D}}(\mathbf{M})$ is equivalent to $C_{\mathcal{L}_1}^U$. Hence the inequality (3) implies that

$$C_{ij}^Q \geq C_{ij}^{eAe}, \text{ for } i, j \in U. \quad (5)$$

Combining (4) and (5) we see that $m_{ij} = 1$ and $C_{ij}^Q = C_{ij}^{eAe}$, for $i, j, l \in U$. The result now follows from Lemma 2.4. \square

4.2. Generalization to $\mathcal{C}_{A, X}$

Let Z be the algebra of 2-endomorphisms of $\mathbb{1}_i \in \text{Ob}\mathcal{C}_A(i, i)$ which factor through any 1-morphism in $\text{add}\mathcal{S}([m] \times [m])$. Given any subalgebra X of $\text{End}_{\mathcal{C}_A(i, i)}(\mathbb{1}_i)$ containing Z , the 2-category $\mathcal{C}_{A, X}$, initially introduced in [11, Section 4.5], is defined by having the same collection of 1-morphisms as \mathcal{C}_A and the same spaces of 2-morphisms as \mathcal{C}_A except $\text{End}_{\mathcal{C}_{A, X}(i, i)}(\mathbb{1}_i) = X$.

Importantly, $\mathcal{C}_{A, X}$ contains all adjunction 2-morphisms of \mathcal{C}_A , and thus the inclusion 2-functor $\mathcal{C}_{A, X} \hookrightarrow \mathcal{C}_A$ yields a bijection between adjunctions in the former and the latter. Observe that beyond our use of adjunctions, our arguments do not involve $\text{End}_{\mathcal{C}_A(i, i)}(\mathbb{1}_i)$, hence our results generalize verbatim from \mathcal{C}_A to $\mathcal{C}_{A, X}$, for any X .

5. Self-injective cores and Duflo involutions

One of the central tools used in the study of finitary 2-representations of weakly fiat 2-categories is the use of *Duflo involutions*. Importantly, these were used for the first definition of a cell 2-representation in [8, Section 4.5]. Other important applications and connections with other 2-representation theoretic concepts can be found in [16, Section 4] and [14, Section 6.3].

Let \mathcal{C} be a weakly fiat 2-category. Let $\mathfrak{i} \in \text{Ob}\mathcal{C}$, and let \mathcal{L} be a left cell of \mathcal{C} whose elements have \mathfrak{i} as their domain. Consider the abelianized principal 2-representation $\overline{\mathbf{P}}_{\mathfrak{i}} = \overline{\mathcal{C}}(\mathfrak{i}, -)$. Let $\hat{P}_{\mathbb{1}_{\mathfrak{i}}}$ be the image of $\mathbb{1}_{\mathfrak{i}}$ under the canonical embedding $\mathbf{P}_{\mathfrak{i}}(i) \hookrightarrow \overline{\mathbf{P}}_{\mathfrak{i}}(i)$. By [13, Proposition 27], there is a unique submodule K of $\hat{P}_{\mathbb{1}_{\mathfrak{i}}}$ such that every simple subquotient $\hat{P}_{\mathbb{1}_{\mathfrak{i}}}/K$ is annihilated by any $F \in \mathcal{L}$, and such that K has a simple top L , with $\overline{\mathbf{P}}_{\mathfrak{i}}F(L) \neq 0$, for any $F \in \mathcal{L}$. The projective cover $G_{\mathcal{L}}$ of K lies in $\overline{\mathbf{P}}_{\mathfrak{i}}(\mathfrak{i}) - \text{proj} \simeq \mathbf{P}_{\mathfrak{i}}(\mathfrak{i}) \simeq \mathcal{C}(\mathfrak{i}, \mathfrak{i})$. Abusing notation, we denote by $G_{\mathcal{L}}$ the

1-morphism of \mathcal{C} corresponding to $G_{\mathcal{L}}$ under this equivalence. The *Duflo involution in \mathcal{L}* is the 1-morphism $G_{\mathcal{L}}$.

Let \mathcal{C} be a weakly fiat 2-category with a unique object and let \mathcal{J} be a strongly regular idempotent J -cell of \mathcal{C} . Using [16, Theorem 4.28], rather than study the simple transitive 2-representations of \mathcal{C} with apex \mathcal{J} , one may equivalently study the simple transitive 2-representations of an associated 2-category $\mathcal{C}_{\mathcal{J}}$ with apex \mathcal{J} . Indeed, the respective 2-categories of simple transitive 2-representations are biequivalent. Further, this biequivalence sends cell 2-representations to cell 2-representations.

By definition, $\mathcal{C}_{\mathcal{J}}$ has only two J -cells – one containing $1_{\dot{i}}$ and one given by \mathcal{J} . Further, it is \mathcal{J} -simple (see [9, Section 6.2]). It follows from [13, Theorem 32] that $\mathcal{C}_{\mathcal{J}}$ is biequivalent to $\mathcal{C}_{A,X}$, for a weakly symmetric A and a suitable algebra $X \subseteq \text{End}_{\mathcal{C}_A(\dot{i},\dot{i})}(1_{\dot{i}})$. For our purposes we may thus restrict ourselves to the case $\mathcal{C} := \mathcal{C}_{A,X}$.

Let \mathcal{L} be a left cell of \mathcal{J} . Let $G_{\mathcal{L}}$ be the Duflo involution in \mathcal{L} , and let $L_{G_{\mathcal{L}}}$ be the simple top of the image of G in $\overline{\mathbf{P}}_{\dot{i}}$, which is used to define the Duflo involution. Following [9, Proposition 22], the finitary 2-subrepresentation of $\overline{\mathbf{P}}_{\dot{i}}$ given by $\text{add}\{\overline{\mathbf{P}}_{\dot{i}}(\text{FG}_{\mathcal{L}})L_{G_{\mathcal{L}}} \mid F \in \mathcal{C}(\dot{i},\dot{i})\}$ is equivalent to $\mathbf{C}_{\mathcal{L}}$. We conclude that $\overline{\mathbf{P}}_{\dot{i}}(G_{\mathcal{L}})L_{G_{\mathcal{L}}}$ is a generator for $\mathbf{C}_{\mathcal{L}}$ in the sense of [15, Definition 2.19].

Given a 1-morphism F of \mathcal{C} , let $*F$ denote its left adjoint. By [13, Proposition 28], the right cell $*\mathcal{L} = \{*F \mid F \in \mathcal{L}\}$ contains $G_{\mathcal{L}}$. For $*F \in *\mathcal{L}$, we have $(*F)G_{\mathcal{L}} \in \text{add}\{G_{\mathcal{L}}\}$. Further, if $F \in \mathcal{S}(\mathcal{C}) \setminus (\{1_{\dot{i}}\} \cup *\mathcal{L})$, then an immediate consequence of [8, Proposition 17(b)] is that $\overline{\mathbf{P}}_{\dot{i}}(\text{FG}_{\mathcal{L}})L_{G_{\mathcal{L}}}$ has no summand in $\text{add}\{G_{\mathcal{L}}L_{G_{\mathcal{L}}}\}$.

As a consequence, the superdiagonal 2-subcategory $\mathcal{D}_{*\mathcal{L}}$ of \mathcal{C} given by the right cell $*\mathcal{L}$ is precisely the 2-subcategory of \mathcal{C} stabilizing the subcategory $\text{add}\{G_{\mathcal{L}}L_{G_{\mathcal{L}}}\}$, the additive closure of the generator of $\mathbf{C}_{\mathcal{L}}$. If \mathcal{C} is fiat, then $\mathcal{D}_{*\mathcal{L}}$ is additionally a self-injective core, and so, using the generalization of Theorem 4.5 from Section 4.2, we conclude that $\text{add}\{G_{\mathcal{L}}L_{G_{\mathcal{L}}}\}$ is uniquely characterized as the target for the unique (up to equivalence) simple transitive 2-representation of $\mathcal{D}_{*\mathcal{L}}$ which does not annihilate the Duflo involution $G_{\mathcal{L}}$.

More generally, given a self-injective core $U \subseteq [[m]]$ such that $G_{\mathcal{L}}$ lies in $\mathcal{D}_{U \times U}$, the 2-subcategory $\mathcal{D}_{U \times [[m]]}$ is precisely the 2-subcategory stabilizing

$$\text{add}\{\overline{\mathbf{P}}_{\dot{i}}(\text{FG}_{\mathcal{L}})L_{G_{\mathcal{L}}} \mid F \in \mathcal{R}_j, \text{ for some } j \in U\},$$

and, again as a consequence of Theorem 4.5, this category is characterized as the target for the unique simple transitive 2-representation of $\mathcal{D}_{U \times [[m]]}$ not annihilating $G_{\mathcal{L}}$.

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