Computations of automorphic functions on Fuchsian groups

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Abstract

This thesis consists of four papers which all deal with computations of automorphic functions on cofinite Fuchsian groups.

In the first paper we develop an algorithm for numerical computation of the Eisenstein series. We focus in particular on the computation of the poles of the Eisenstein series. Using our numerical methods we study the spectrum of the Laplace-Beltrami operator as the surface is being deformed. Numerical evidence of the destruction of \( \Gamma_g(5) \)-cusp forms is presented.

In the second paper we use the algorithm described in the first paper. We present numerical investigations of the value distribution and distribution of Fourier coefficients of the Eisenstein series \( E(z,s) \) on arithmetic and non-arithmetic Fuchsian groups. Our numerics indicate a Gaussian limit value distribution for a real-valued rotation of \( E(z,s) \) as \( \text{Re } s = 1/2, \text{Im } s \to \infty \) and also, on non-arithmetic groups, a complex Gaussian limit distribution for \( E(z,s) \) when \( \text{Re } s > 1/2 \) near 1/2 and \( \text{Im } s \to \infty \), at least if we allow \( \text{Re } s \to 1/2 \) at some rate. Furthermore, on non-arithmetic groups and for fixed \( s \) with \( \text{Re } s \geq 1/2 \) near 1/2, our numerics indicate a Gaussian limit distribution for the appropriately normalized Fourier coefficients.

In the third paper we develop algorithms for computations of Green's function and its Fourier coefficients, \( F(z,s) \), on Fuchsian groups with one cusp. Also an analog of a Rankin-Selberg bound for \( F_s(z,s) \) is presented.

In the fourth paper we use the algorithms described in the third paper. We present some examples of numerical investigations of the value distribution of the Green's function and of its Fourier coefficients on PSL(2,Z). We also discuss the appearance of pseudo cusp forms in a numerical experiment by Hejhal.

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List of Papers

This thesis is based on the following papers, which are referred to in the text by their Roman numerals.


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1. Introduction

1.1 Quantum chaology

We consider the movement of a ball on a billiard table. Depending on the shape of the billiard, the movement of the ball can be either very predictable or completely unpredictable. Examples of these two scenarios are shown in figure 1.1.

Figure 1.1: The path of a billiard ball can be regular or chaotic depending on the shape of the billiard. Printed with kind permission from Carlo Beenakker, [Bee].

Already in 1898, Hadamard, [Had98], discovered that a billiard with negative curvature was favorable for studying chaotic movement. Such a billiard is not flat, instead it is curved as a saddle. He was able to demonstrate that the long-time behavior of the system is very sensitive to initial conditions, i.e., that the system is chaotic in modern terminology, cf., e.g., [Gut90, Chapter 19].

In the present work our basic setting will be a negatively curved billiard just like Hadamard’s, but with the ball replaced by a quantum mechanical particle. One essential difference is that quantum mechanics does not give information about the precise location of the particle, but only about the probability of encountering the particle in a certain region of the billiard. This is expressed in terms of the probability function, or wave function \( \Psi(x, t) \), which is a solution of the Schrödinger equation, [LL65, (17.6)]:

\[
i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi = E\Psi. \tag{1.1}
\]
Here \( m \) is the mass of the particle and \( E \) is its energy. The operator \( \Delta \) is the Laplace-Beltrami operator. In an isolated system the energy cannot take on arbitrary values. In fact, wave functions exist only for a discrete set of values of \( E \), in much the same way that a guitar string can only vibrate with certain frequencies. In this sense, the wave functions are never random or chaotic at all. But the processes which they describe are random. After all, we cannot know exactly \textit{when} or \textit{where} in quantum mechanics, we only know probabilities.

If we increase the energy of the particle it will eventually leave the regime of quantum mechanics and enter classical mechanics. If the corresponding classical motion is chaotic, then we expect that the original quantum system should exhibit some reflections of chaos. Quantum chaology is the study of such reflections of chaos in quantum mechanical systems.

For example, it is common to study statistical properties of \( \Psi \) and the spectrum in the \textit{semiclassical limit}, i.e., as Planck’s constant \( \hbar \to 0 \). If we declare \( \hbar = m = 1 \) then the semiclassical limit means effectively that we study large energies. In this direction, there is a result by Shnirelman, Colin de Verdière and Zelditch [Shn74, Col85, Zel87] which states that \textit{almost} all eigenfunctions of a quantum mechanical system whose classical dynamics is ergodic (we can think of this as meaning “chaotic”) should become \textit{equidistributed} in the semiclassical limit. In other words, the probability of finding the particle in some region, e.g., a box, on the billiard should be proportional to the area of that region. If all eigenfunctions without exception become equidistributed in the semiclassical limit, the phenomenon is called \textit{quantum unique ergodicity}. It is still not known precisely for what systems that happens.

After separating the time and space dependence as \( \Psi(\mathbf{x}, t) = \phi(\mathbf{x}) e^{-iEt/\hbar} \), equation (1.1) leads to

\[
\Delta \phi + \lambda \phi = 0 \tag{1.2}
\]

with \( \lambda = \frac{2mE}{\hbar^2} \). Typically, (1.2) has solutions for a discrete set of eigenvalues

\[
0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots,
\]

and if the billiard is unbounded, there may also be a continuous spectrum, e.g., some interval of the form \([\alpha, \infty)\).

Discrete eigenvalues correspond to discrete eigenfunctions, these are \textit{bound states}. An example of a bound state is a particle moving in a square potential well with infinite sides. Continuous eigenfunctions include the \textit{scattering states}. A finite height square well will allow for bound states inside the well and scattering states above the well. Scattering is often thought of as having three stages: at time \( t = -T \) (for large \( T \)) the particle is far away in some original state; at time \( t = 0 \) it interacts with
something, in the previous example a square potential; and at time $t = T$ it is again far away in a new state which was caused by the interaction. A so called scattering matrix pairs states before and after scattering. The poles of the scattering matrix correspond to nearly bound states, also called resonances. In the example with a potential well, a nearly bound state is a particle that becomes trapped inside the well with an associated lifetime (related to the resonance width).

1.2 Mathematical setting

A space with constant negative curvature is called a hyperbolic space. A good mathematical model for a hyperbolic space is the upper half-plane

$$\mathcal{H} = \{x + iy \in \mathbb{C}; \ y > 0\}$$

equipped with the hyperbolic metric $ds^2 = y^{-2}(dx^2 + dy^2)$. The Laplace-Beltrami operator in this metric is

$$\Delta = y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right).$$

Our usual sense of distances is not valid in hyperbolic space. This can be illustrated by the facts that the distance from any point in $\mathcal{H}$ to the real line is infinite; and that the shortest distance between two points could be along a circle, and not necessarily along a line as in flat space.

We will now explain how our billiard tables, which we often call surfaces, are constructed. Loosely speaking, we may start with some region $\mathcal{F}$ in $\mathcal{H}$ which is bounded by a finite number of sides, and we obtain a surface $\mathcal{S}$ by identifying the sides of $\mathcal{F}$ in pairs in a specified way. If a particle is moving in the region towards one side it can cross the border and immediately enter the region from another side. One can think of a sheet of paper that is folded so that two sides meet. We allow some sides to have infinite length, meaning that they extend to $i\infty$ or to the real line and meet another side there, where they form a so called cusp.

Mathematically the construction begins with the elements of $SL(2, \mathbb{R})$ acting as Möbius transformations of $\mathcal{H}$. More precisely, an element

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

of the group

$$SL(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}; \ a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\}$$
maps a point $z$ of $\mathcal{H}$ according to

$$Tz = \frac{az + b}{cz + d}.$$ 

The mappings $T \in SL(2, \mathbb{R})$ are isometries of $\mathcal{H}$. Now let $\Gamma$ be a Fuchsian group, i.e., a discrete subgroup of $PSL(2, \mathbb{R})$, where $PSL(2, \mathbb{R}) = SL(2, \mathbb{R})/\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. We take the surface $\mathcal{S}$ as the space $\Gamma\backslash \mathcal{H}$ of $\Gamma$-orbits in $\mathcal{H}$. A fundamental region for $\Gamma$ is a closed set $\mathcal{F} \subset \mathcal{H}$ such that

- $\mathcal{F}$ is the closure of its interior $\mathcal{F}^0$,
- no two points of $\mathcal{F}^0$ lie in the same $\Gamma$-orbit,
- the images of $\mathcal{F}$ under $\Gamma$ cover $\mathcal{H}$.

The region $\mathcal{F}$ will represent our billiard $\Gamma\backslash \mathcal{H}$.

We will always assume that $\Gamma\backslash \mathcal{H}$ is non-compact but of finite hyperbolic area. In this case it is known that $\mathcal{F}$ can be taken as a polygon with a finite number of sides and with at least one cusp. We say that two elements $z, w$ of $\mathcal{H}$ are equivalent under $\Gamma$ if there is an element $T$ in $\Gamma$ with $Tz = w$. The interior of $\mathcal{F}$ does not contain any two equivalent points.

The most simple example of this type of surface is the hyperbolic triangle $PSL(2, \mathbb{Z})\backslash \mathcal{H}$, shown in figure 1.2. This billiard was considered by Artin already in 1924.

Functions on the surface $\mathcal{S} = \Gamma\backslash \mathcal{H}$ can be thought of as functions on $\mathcal{H}$ that are automorphic with respect to $\Gamma$, i.e., satisfy

$$f(Tz) = f(z), \quad \text{for all } T \in \Gamma.$$ 

For example, this means that an automorphic function has the same value at any point $z \in \mathcal{F}$ as at a point outside of $\mathcal{F}$ that is equivalent to $z$.

Good references for the background theory of automorphic functions are [Apo76, Iwa97, Miy89].

Recall that we are interested in wave functions on $\mathcal{S}$. Thus we will mainly consider automorphic functions which are eigenfunctions of the hyperbolic Laplace-Beltrami operator $\Delta$, i.e., satisfy (1.2). The discrete eigenfunctions are called Maass waveforms. If, for example, the map $Sz = z + 1$ is an element of $\Gamma$ we may expand any eigenfunction in a Fourier series. It is well known that discrete eigenfunctions with eigenvalue $\lambda \geq 1/4$ have no constant term in their Fourier expansion, cf. [Hej83, pp. 26, 71]. These so called cusp forms thus have expansions like

$$\phi(x + iy) = \sum_{m=-\infty}^{\infty} c_m(s)y^{1/2}K_{s-1/2}(2\pi|m|y)e^{2\pi i mx}, \quad (1.3)$$

with some Fourier coefficients $c_m(s)$; and where $K_{\nu}(X)$ is the modified $K$-Bessel function.
Figure 1.2: The fundamental region $F$ for $PSL(2, \mathbb{Z}) \backslash \mathbb{H}$. The two sides extending to $i\infty$ are identified with each other, and so are the two parts of the circle $|z| = 1$ on either side of the vertical axis. We view this as a hyperbolic triangle because the two vertical sides meet at $i\infty$ where they form a cusp. The triangle is unbounded but of finite hyperbolic area.

The continuous spectrum is spanned by the Eisenstein series, $E(z; s)$, cf. [LL65, (5.15)]. The Eisenstein series is an automorphic function satisfying (1.2) but it is not in $L^2(\Gamma \backslash \mathbb{H})$. True wave functions are probability functions, and so they must be square integrable, in other words they belong to $L^2(\Gamma \backslash \mathbb{H})$. Therefore $E(z; s)$ itself is not a wave function. However, taking continuous superpositions of $E(z; 1/2 + ir)$ produces square integrable functions.

If there is only one cusp in $S$, the Fourier expansion of $E(z; s)$ is

$$E(z; s) = y^s + \varphi(s)y^{1-s} + \sum_{m=-\infty \atop m \neq 0}^{\infty} \varphi_m(s)y^{1/2}K_{s-1/2}(2\pi|m|y)e^{2\pi i m x} \quad (1.4)$$

and the eigenvalue is $\lambda = s(1 - s)$, [Hej92, pp. 65, 76]. It is well known that the continuous spectrum is the interval $[1/4, \infty)$, cf. [Sel89].

Computing Eisenstein series and cusp forms means finding the Fourier coefficients $\varphi_m(s)$ and $c_m(s)$ and the discrete eigenvalues $\lambda = s(1 - s)$ for cusp forms. Background material on spectral theory for Fuchsian groups
can be found in [Hej76b, Hej83, Iwa97]. Regarding the computational aspects see, e.g., [Hej92, HR92, HS01, BSV06, The05, Str05].

1.3 Connections to the Riemann zeta function

The Riemann zeta function is without doubt the most important function in analytic number theory. It is defined as a sum

\[ \zeta(s) = \sum_{n=1}^{\infty} n^{-s} \]

for \( \text{Re} \, s > 1 \), and continues analytically to all of \( \mathbb{C} \setminus \{1\} \). Its intimate relation to the primes lies in the fact that it can be expressed as a product over all primes \( p \) (for \( \text{Re} \, s > 1 \)):

\[ \zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} \]

The so called trivial zeros of \( \zeta(s) \) are the negative even integers. The Riemann hypothesis states that all the remaining zeros are on the critical line \( \text{Re} \, s = 1/2 \). To prove (or disprove!) the Riemann hypothesis is considered one of the greatest challenges in mathematics. In May 2000 the Clay Mathematics Institute announced that a million-dollar prize will be awarded for its proof. It has been shown numerically that billions of zeros lie on the critical line, [Odl01], and there are also other strong reasons to believe that the hypothesis is true. Hilbert said that if he would wake up after 500 years of sleep, the first thing he would ask is whether the Riemann hypothesis had been proved. Most experts would find it truly remarkable (and unsettling) if the hypothesis should turn out to be false.

It seems unlikely that nature is that perverse! (J. Brian Conrey, [Con03])

For a background on \( \zeta(s) \) we refer the reader to [Tit51].

The reason for exploring quantum chaos in the hyperbolic metric is not merely that it offers a successful mathematical model. In fact, there are deep connections between quantum chaology, hyperbolic metric and the research field of analytic number theory.

For example, on the modular surface shown in figure 1.2, the Eisenstein series (1.4) is related to the Riemann zeta function through the formula:

\[ \varphi(s) = \sqrt{\pi} \frac{\Gamma(s - 1/2)}{\Gamma(s)} \frac{\zeta(2s - 1)}{\zeta(2s)} \]

The function \( \varphi(s) \) is called the scattering matrix and its poles are the energy levels of resonances. The relation between these poles and the zeros of the zeta function via the factor \( \zeta(2s) \) is quite inspiring.
Another interesting feature is that spacings between zeros of the Riemann zeta function resemble the spacings between consecutive eigenvalues of certain random matrices called the Gaussian unitary ensemble, GUE, cf. figure 1.3. These random matrices are an important tool in quantum chaology, cf. [Odl01].

![Nearest neighbor spacings](image)

**Figure 1.3**: Probability density of the normalized spacings between zeros of $\zeta(s)$. Solid line: GUE prediction. Scatterplot: empirical data based on a billion zeros near zero $\approx 1.3 \cdot 10^{16}$. Printed with kind permission from Andrew M. Odlyzko, [Odl01].

Perhaps the most exciting connection is via the Hilbert-Pólya conjecture. Around 1910, Hilbert and Pólya independently suggested that the Riemann hypothesis could be proved by showing that the non-trivial zeros $s = 1/2 + i\lambda$ correspond to the eigenvalues $\lambda$ of a self-adjoint operator on a Hilbert space. From the theory of such operators it follows that all $\lambda$ are real and the proof would be complete. It seems that their hypothesis was based more on wishful thinking than on real evidence; and that neither Pólya or Hilbert had any specific space or operator in mind. Cf., e.g., the correspondence between Odlyzko and Pólya regarding the origin of the hypothesis, [Odl]. There has been hopes that the space in question might be of the type $\Gamma \backslash \mathcal{H}$ with $\Gamma$ a Fuchsian group, but it seems that it is then considerably more complicated than the surface in figure 1.2, cf., e.g., [Hej81, Hej76a]. The Hilbert-Pólya conjecture is still considered a promising approach for proving the Riemann hypothesis, in particular in view numerical evidence like that in figure 1.3.
1.4 Experimental mathematics

Mathematics has always been an experimental science. To try an idea we compute examples to see if the idea holds in some simple cases. The experiments might lead us to formulate a hypothesis and the final step is to give a rigorous proof of the hypothesis, which then transforms into a theorem. Traditionally it is the theorems and proofs that are published, but sometimes it can be more difficult to know what to prove than how to prove it.

If only I had the theorems! Then I should find the proofs easily enough.
(Bernhard Riemann)

Thus, there is certainly a great need for experiments in mathematics.

Along with the development of computers came the possibility of experimenting more and faster than had ever been possible with pencil and paper. The process of idea, experiment, hypothesis and proof remains the same, but the element of experiment has expanded to several weeks of super computer time. It has become important to share not only the final theorems and proofs but also the experimental results and methods.

The findings of Odlyzko (cf. figure 1.3) supporting the Riemann hypothesis is a fitting example. It is not unreasonable to suspect that without the abundant numerical evidence, the Riemann hypothesis would not have been as engaging as it is.

While it is obvious that the advances in technology have had tremendous impact, the importance of the development of algorithms has been somewhat more obscured. It has actually been the case in many areas of computational science that algorithmic advances have done much more to decrease computational time than have advances in technology. For example, the earliest computations of $\zeta(1/2 + it)$ for large $t$ required the order of $t$ operations, and a more recent method uses the order of $(\log t)^c$ operations for a constant $c$, [OS88]. Without this improvement it would perhaps have taken billions of years to compute the data in [Odl01] with a modern computer.

Quantum chaology is another area where computer experiments are of extreme importance. Sir Michael Berry, [Ber87], remarks that:

The phenomena of quantum chaology lie in the largely unexplored border country between quantum and classical mechanics; they are part of semiclassical mechanics. This is an area where rigorous mathematical development, as employed elsewhere in mechanics, is difficult. Most discoveries have been made by computer experiments with the quantum equations, guided by intuition and analogy. As the subject matures we can expect, on the one hand, more experiments on real physical systems, and, on the other, the precise formulation and proof of mathematical theorems.
2. Summary of papers

Sections 2.1–2.4 give an overview of the contents of papers I–IV with some explanation of the connection with the introduction presented in section 1.

2.1 Summary of paper I

Paper I deals with deformations of the surface $\Gamma \setminus \mathcal{H}$. We start with a surface $\Gamma_0 \setminus \mathcal{H}$ that is only slightly more complicated than the one in figure 1.2, this one has two cusps. The group we use is

$$\Gamma_0 = \Gamma_0(5) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2,\mathbb{Z}) ; \; c \equiv 0 \mod 5 \right\},$$

and the surface is shown in figure 2.1. We then deform the surface continuously and make sure that the deformed surfaces $\Gamma_{a,r} \setminus \mathcal{H}$ are of the same “type” as the original one. This is done by allowing the cusp at 0 to slide along the real axis to a new position $a$; and allowing the radius $r$ of one of the circles that constitutes the boundary to change. The rest of the geometry must then follow in a specified manner in order to maintain the correct “type”. The space of permitted surfaces is called Teichmüller space for $\Gamma_0$ and it is denoted $T(\Gamma_0)$. An example of a deformed surface is shown in figure 2.2.

The question has been raised whether there exist infinitely many cusp forms on a general surface like $\Gamma_{a,r} \setminus \mathcal{H}$, i.e., if there is an infinite discrete spectrum with eigenfunctions of the form

$$\phi(x + iy) = \sum_{\substack{m=-\infty \\to \\infty \\not= 0}} c_m(s)y^{1/2}K_{s-1/2}(2\pi|m|y)e^{2\pi imx}. \quad (2.1)$$

According to the Phillips-Sarnak hypothesis, the answer to this question is no, cf. [PS85].

We examine the Phillips-Sarnak hypothesis by computing discrete eigenvalues on $\Gamma_0 \setminus \mathcal{H}$, a surface which is known to have many cusp forms, and then deforming $\Gamma_0$ into $\Gamma_{a,r}$. We expect from [PS85] that the eigenvalues will not be preserved during continuous deformation. But eigenvalues cannot simply disappear, they must transform into something. It is known that if a cusp form is destroyed it creates a pole...
of the scattering matrix, cf. [Hej92, pp. 6, 95], [Hej83, pp. 231, 143–148]
and [PS92]. Physically this is quite natural: a bound state (cusp form)
which is destroyed during deformation of the surface should transform
into a \textit{nearly bound state} (resonance), and these are precisely the poles

---

Figure 2.1: The fundamental region for $\Gamma_0(5)$. The arrows indicate which sides
are identified with each other. The $\Box$ marks inequivalent elliptic fixpoints.

Figure 2.2: The fundamental region for a deformation of $\Gamma_0(5)$. The arrows
again indicate which sides are identified with each other, and the inequivalent
elliptic fixpoints are marked by $\Box$. Here we have used the parameter values
$a = 0.05$ and $r = 0.15$. 
of the scattering matrix, cf. section 1.1. Also recall that the scattering matrix is the function \( \varphi(s) \) in the Fourier expansion of the Eisenstein series (on one cusp groups),

\[
E(z; s) = y^s + \varphi(s)y^{1-s} + \sum_{m=-\infty}^{\infty} \varphi_m(s)y^{1/2}K_{s-1/2}(2\pi|m|y)e^{2\pi imz}. \tag{2.2}
\]

Numerical evidence of the destruction of cusp forms was gathered by computing the Eisenstein series on \( \Gamma_{a,r} \) and then localizing the poles of the scattering matrix. It was one of our main goals to track how these poles move as the group is deformed.

We know that these poles vary real analytically with the parameters \( a \) and \( r \), [PS92]. Due to symmetry reasons, a pole \( \xi = 1/2 - \eta + i\gamma \) must satisfy (near \( a = 0, r = 1/5 \), and \( s_0 = 1/2 + iR_0 \)):

\[
\eta = A_1a^2 + A_2(r - \frac{1}{5})^2 + A_3a^2(r - \frac{1}{5}) + A_4(r - \frac{1}{5})^3 \\
+ A_5a^4 + A_6a^2(r - \frac{1}{5})^2 + A_7(r - \frac{1}{5})^4 + A_8a^4(r - \frac{1}{5})^3 + A_9a^2(r - \frac{1}{5})^5 + A_{10}(r - \frac{1}{5})^6 + A_{11}a^6 + \ldots \tag{2.3}
\]

\[
\gamma = R_0 + B_0(r - \frac{1}{5}) + B_1a^2 + B_2(r - \frac{1}{5})^2 \\
+ B_3a^2(r - \frac{1}{5}) + B_4(r - \frac{1}{5})^3 + B_5a^4 + \ldots \tag{2.4}
\]

with \( A_1 \geq 0, A_2 \geq 0 \). To track the pole means precisely to find the Taylor coefficients \( A_1, A_2, \ldots \) and \( B_0, B_1, \ldots \). In figure 2.3 we show computed values of \( \frac{1}{2} + \eta \) with \( a \) varied and \( r = 1/5 \) kept fixed. On the left hand side there is a second order curve fit. In the example on the right hand side, the points do not seem to lie on the second order curve but on a fourth order curve. It seems that in this case we have \( A_1 = 0 \) and \( A_5 \) as the lowest order Taylor coefficient.

This is explained by the following lemma which is proved in the paper:

**Lemma 2.1.1** With notation as in (2.3) we have

\[
A_1A_2 = 0.
\]

This lemma also explains an interesting property noted in the experiments by Farmer and Lemurell in [FL05]. Phillips and Sarnak conjecture that cusp forms are destroyed by almost all deformations in Teichmüller space. However, they do not exclude the possibility for cusp forms to be stable under deformation along certain paths in \( T(\Gamma_0) \). In fact, in [FL05], Farmer and Lemurell are able to numerically track what seems to be stable cusp forms as the group is deformed. They conjecture that for every cusp form on a non-compact Riemann surface with non-trivial Teichmüller space there is a continuous family of Teichmüller deformations on which the
Figure 2.3: The stars correspond to poles of $\varphi(s)$ found near the cusp forms having $R_0 = 3.028$ and $R_0 = 5.436$. $\text{Re}(\rho) = 1/2 + \eta$ is plotted as a function of the parameter $a$. The curve in the first plot is $1/2 + 3.4389a^2$. The curves in the second plot are $1/2 + 0.0144a^2$ and $1/2 + 1744.6a^4$. We see that the stars appear to lie on the fourth order curve.

cusp form lives. A consequence of Lemma 2.1.1 is that if $A_1 = 0$ then the curve $(a(t), r(t))$ in the Farmer-Lemurell hypothesis is parallel to the $a$-axis at $t = 0$, which is precisely what is found experimentally in [FL05].

The Taylor expansion (2.3) may also be used to decide whether a cusp form is destroyed for all deformations in $T(\Gamma_0)$. In paper I we prove a lemma that states a first sufficient condition for total destruction:

**Lemma 2.1.2** Let $\eta(a, r)$ be as in equation (2.3) and let

$$P_4(a, r) = A_1a^2 + A_2(r - \frac{1}{5})^2 + A_3a^2(r - \frac{1}{5}) + A_4(r - \frac{1}{5})^3$$

$$+ A_5a^4 + A_6a^2(r - \frac{1}{5})^2 + A_7(r - \frac{1}{5})^4$$

be the fourth order Taylor approximation of $\eta(a, r)$. Suppose $A_1 = 0$ and $A_2 > 0$. Then we necessarily have

$$4A_2A_5 - A_3^2 \geq 0.$$

Furthermore, if $4A_2A_5 - A_3^2 > 0$ then both $P_4(a, r) > 0$ and $\eta(a, r) > 0$ hold in some punctured neighborhood of $(a, r) = (0, 1/5)$.

Thus the Farmer-Lemurell hypothesis suggests that we should always have $4A_2A_5 - A_3^2 = 0$.

A number of algorithms are needed to conduct these experiments and a substantial part of the work has been to develop these algorithms. Dennis Hejhal has developed an algorithm for computing eigenvalues and Fourier coefficients of cusp forms on groups with one cusp. See [Hej92] and [Hej99]. This algorithm is not directly applicable to $\Gamma_0(5)$ because...
this group has two cusps. However, eigenvalues on $\Gamma_0(5)$ are related to eigenvalues on the one cusp group

$$\tilde{\Gamma}_0(5) = \langle \Gamma_0(5), W_0 \rangle,$$

(2.5)

where $W_0$ is the Fricke involution $z \mapsto -\frac{1}{5z}$, and on $\tilde{\Gamma}_0(5)$ we may use Hejhal’s algorithm. The analogous construction is applied to $\Gamma_{a,r}$. The idea of working with $\tilde{\Gamma}_0(5)$ and $\tilde{\Gamma}_{a,r}$ was shared with us at an early stage by Stefan Lemurell and David Farmer, cf. [FL05] for their related work.

The algorithms that we have developed are:

- An algorithm for computing $E(z; s)$ on one cusp groups (based on Hejhal’s algorithm).
- An algorithm for computing the $K$-Bessel function $K_\nu(X)$ with complex $\nu$ (based on an algorithm for $K_\nu(X)$ with purely imaginary $\nu$ developed by Hejhal, cf. [Hej92]).
- A pull-back algorithm for finding images of points $z \in \mathcal{H}$ in the fundamental region of a Fuchsian group (based on ideas in [Str00]).
- A method for tracking poles of $\varphi(s)$ as the group is deformed.

Several weeks of computer time were spent on gathering data for the results in paper I.

2.2 Summary of paper II

Paper II makes further use of the algorithm for $E(z; s)$ developed in paper I. The purpose is to explore some questions in quantum chaology by studying statistical properties of the continuous spectrum.

More precisely, the values of $E(z; s)$ are studied in relation to the question of equidistribution in the semiclassical limit, i.e., as $\text{Re } s = 1/2$ and $\text{Im } s$ becomes large. Specifically, we compare results on the arithmetic group

$$\Gamma_0 = \Gamma_0(5) = \left\{ \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \in PSL(2,\mathbb{Z}) ; \hspace{1em} c \equiv 0 \mod 5 \right\},$$

which has some special symmetry properties implying an infinite discrete spectrum, cf. section 2.1, with results on “general” groups $\Gamma_{a,r}$, where no such symmetries are present and we expect there to be no discrete eigenfunctions (by the Phillips-Sarnak hypothesis). On $\Gamma_{a,r}$, the pursuit of quantum chaos must therefore depend entirely on the Eisenstein series.

Taking the concept of equidistribution a little further, it is believed that the relative frequency measures

$$\frac{\mu\{z \in F; \sigma^{-1}\left| E\left(z; \frac{1}{2} + iR\right) \right| \in [a, b]\}}{\mu(F)}$$

over rectangles $F \subseteq \Gamma \setminus \mathcal{H}$, should approach a Gaussian value distribution as $R \to \infty$ for a general surface $\Gamma \setminus \mathcal{H}$, cf. [Ber77] and [HR92, §§6, 7(item
Here $\mu$ is the hyperbolic area measure and $\sigma$ is the standard deviation, which can be thought of as a scale factor.

Several numerical experiments have previously been carried out which strongly support Gaussian distribution for cusp forms on $\Gamma \backslash \mathcal{H}$ with, e.g., $\Gamma = PSL(2, \mathbb{Z})$ (as in figure 1.2) and similar triangle groups, cf. [HR92, Hej99]. Regarding the values of the Eisenstein series on the arithmetic surface $PSL(2, \mathbb{Z}) \backslash \mathcal{H}$, promising experimental results along with some heuristics are presented in [HR92].

Our results in paper II suggest that quantum chaos is “closer” on the general, non-arithmetic groups $\Gamma_{a,r}$ than on $\Gamma_0$. This is shown in figures 2.4 ($\Gamma_0$) and 2.5 ($\Gamma_{a,r}$ with $a = 0.1, r = 0.19$). The solid lines are the conjectured Gaussians and the dots are the computed $E(z; s)$-statistics. The series of three plots with Im $s$ increasing from left to right may give us a hint about what happens in the semiclassical limit. It is clear that the fit improves with increasing eigenvalue and that the fit is better for the non-arithmetic group than for the arithmetic $\Gamma_0$.

In addition to the statistical investigations of $E(z; s)$-values, we have performed numerical studies of several aspects of the Fourier coefficients of $E(z; s)$ and we have devoted some computer time to producing beau-

![Figure 2.4](image1.png)  
*Figure 2.4: Histograms of $E(z; s)$ (scaled) for the arithmetic $\tilde{\Gamma}_0(5)$. The solid lines are the conjectured Gaussians.*

![Figure 2.5](image2.png)  
*Figure 2.5: Histograms of $E(z; s)$ (scaled) for the non-arithmetic $\Gamma_{a,r}$ with $a = 0.1, r = 0.19$. The solid lines are the conjectured Gaussians.*
Figure 2.6: Pictures of the Eisenstein series on the arithmetic group $\tilde{\Gamma}_0(5)$. Minimum and maximum values are given in parenthesis.

Beautiful pictures of the Eisenstein series. We include two examples here in figure 2.6. Values at $500 \times 500$ points of $E(x + iy; s)$ for $-0.75 < x < 0.75$, $0.15 < y < 0.65$ were computed and given colors ranging through blue, green, yellow and red as the values pass from their minimum to their maximum.

2.3 Summary of paper III

Paper III has its origin in the connection between spectral theory and the Riemann zeta function, cf. section 1.3. Recall that the hypothesis of Hilbert and Pólya states that the non trivial zeros, $s = 1/2 + i\lambda$, of the Riemann zeta function, $\zeta(s)$, correspond to the eigenvalues $\lambda$ of a self-adjoint operator on a Hilbert space. If this hypothesis is true then the Riemann hypothesis will follow because such eigenvalues are necessarily real and thus $1/2 + i\lambda$ would be along the line $\text{Re } s = 1/2$.

In 1977, an experiment conducted by Hartmut Haas seemed to produce numerical evidence that the Hilbert-Pólya operator had been discovered. Indeed, the first few zeros of the Riemann zeta function seemed to appear in Haas’ attempt to compute eigenvalues of the Laplacian on $PSL(2, \mathbb{Z})\backslash \mathcal{H}$.

However, the true circumstances were revealed when Dennis Hejhal investigated the matter closer, cf. [Hej81]. He found that Haas had forgotten to take necessary precautions to rule out the possibility of singularities.
The eigenfunctions corresponding to the remarkable eigenvalues all had logarithmic singularities at \( 1/2 + i \sqrt{3}/2 \).

These events triggered Hejhal’s interest for cusp forms with logarithmic singularities, which he named *pseudo cusp forms*. Elaborating on Haas’ somewhat accidental method of computing them, Hejhal was able to find many more eigenvalues of pseudo cusp forms. However, several open questions still remain to be explored, cf. [Hej81, Hej92].

A pseudo cusp form is a special case of a Green’s function. In paper III we develop a method for numerical computations of Green’s functions on Fuchsian groups. Our method is designed for eigenfunctions with logarithmic singularities. The primary computational challenges are that multiple layers of singularities have to be dealt with, and that the Fourier coefficients increase very rapidly in magnitude.

As with cusp forms and Eisenstein series the key to computations lies in the fact that we may expand an eigenfunction in a Fourier series. For the Green’s function it takes the form, for \( z = x + iy \) with \( y > \text{Im} T w \) for all \( T \in \Gamma \),

\[
G_s(z; w) = \frac{E(w; s)y^{1-s}}{1 - 2s} - \sum_{n \neq 0} F_{-n}(w; s)y^{1/2}K_{s-1/2}(2\pi|n|y)e^{2\piinx},
\]

(2.6)

cf. [Hej83, p. 42]. Recall that \( K_\nu(X) \) denotes the \( K \)-Bessel function. Those \( G_s(z; w) \) with \( E(w; s) = 0 \) are pseudo cusp forms, cf. [Hej81, §4].

The above Fourier series is divergent unless \( y > \text{Im} T w \) and this is certainly a problem if one would like to use the ideas from Hejhal’s algorithm to compute \( G_s(z; w) \). Actually, Hejhal’s algorithm relies on the fact that the same Fourier expansion is valid below the fundamental region as well as inside the fundamental region. Therefore, it becomes necessary to work with a modified function \( G^r_s(z; w) \):

\[
G^r_s(z; w) = G_s(z; w) - \sum \{ \text{all singularities above level } y = v_{r+1} \},
\]

(2.7)

where \( \text{Im} T w = v_{r+1} \) holds for some \( T \in \Gamma \). More precisely, \( G^r_s(z; w) \) is defined in such a way that it is an eigenfunction of the Laplacian with the same eigenvalue as \( G_s(z; w) \) and it has a Fourier expansion valid for \( y > v_{r+1} \). The price we pay is that \( G^r_s(z; w) \) is not automorphic.

It is well known that the Fourier coefficients \( F_n(z; s) \) themselves can be written as Fourier expansions and that, in some sense, \( F_n(z; s) \) is a kind of generalized Eisenstein series:

\[
F_n(z; s) = \begin{cases} y^{1/2}I_{s-1/2}(2\pi|n|y)e^{2\piinx} : & n = 1, 2, 3, \ldots \\ y^s : & n = 0 \\ + \varphi_0^n(s) \frac{y^{1-s}}{2s-1} + \sum_{m \neq 0} \varphi_m^n(s)y^{1/2}K_{s-1/2}(2\pi|m|y)e^{2\piinx}. \end{cases}
\]

(2.8)
Here $I_\nu(X)$ is the $I$-Bessel function and $F_0(z; s) = E(z; s)$. Cf. [Hej83, §6.8]. In paper III we show how the algorithm for computing Eisenstein series described in paper I may be adapted to the functions $F_n(z; s)$.

For this purpose it is useful to know that the Fourier coefficients of $F_n(z; s)$ do not grow too fast. We present an analog of a Rankin-Selberg bound for $F_n(z; s)$, cf., e.g., [Iwa02, Theorem 3.2], [Iwa97, Theorem 5.1]:

**Theorem 2.3.1** Let $\Gamma$ be a cofinite Fuchsian group with one normalized cusp at $i\infty$, let $s \in \mathbb{C}$, $s$ not a pole of $F_n(z; s)$, and suppose $n \neq 0$. Then we have

$$\sum_{1 \leq |m| \leq N} |\varphi_m^n(s)|^2 = O\left(Ne^{8\pi c_\Gamma^{-1}\sqrt{|n|N}}\right)$$

as $N \to \infty$. The implied constant depends on $\Gamma$, $n$ and $s$.

As a consequence we have

$$\varphi_m^n(s) = O\left(\sqrt{|m|}e^{4\pi c_\Gamma^{-1}\sqrt{|nm|}}\right) \quad \text{as} \quad |m| \to \infty. \quad (2.9)$$

with

$$c_\Gamma = \min\{|c| : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \setminus \Gamma_\infty\}. \quad (2.10)$$

We also prove that the exponent in Theorem 2.3.1 is the best possible:

**Theorem 2.3.2** Let $\Gamma$ be a cofinite Fuchsian group with one normalized cusp at $i\infty$. Fix any $n \neq 0$ and $s \in \mathbb{C}$ not a pole of $F_n(z; s)$. Then, for any constants $0 < \delta < 4\pi c_\Gamma^{-1}\sqrt{|n|}$ and $C > 0$ there exist infinitely many numbers $m \in \mathbb{Z}$ such that

$$|\varphi_m^n(s)| > Ce^{\delta\sqrt{|m|}}.$$

During the course of this work we have developed methods for computing the modified Bessel functions of the third kind, i.e., the $K$- and $I$-Bessel functions, with power series expansions and asymptotic expansions. It was only after a significant amount of work was done in this direction that I became aware of [GST02, GST03, GST04], where similar methods were discussed.

To summarize, the algorithms developed in Paper III are the following:
- An algorithm for computing $G_s(z; w)$ on one cusp groups.
- An algorithm for computing $F_n(z; s)$ on one cusp groups.
- Algorithms for computing the $K$- and $I$-Bessel functions.

After explaining these algorithms we discuss some computational details and present thorough testings of the results. The tests are built up in
Figure 2.7: Pictures of $G_s(z; 1/2 + i\sqrt{3}/2)$. The plots show pseudo cusp forms with $s = 1/2 + 14.13\ldots$ and $s = 1/2 + 98.83\ldots$, on the left and right hand side, respectively. Minimum and maximum values are given in parenthesis.

several stages and use a number of theoretical and algorithmic relations. For example, values of $F_n(z; s)$ computed with the $F_n(z; s)$-algorithm are compared to $F_n(z; s)$-values as Taylor coefficients computed with the $G_s(z; w)$-algorithm.

Finally in paper III, we provide some pictures of $G_s(z; w)$. Here we include as an example the real valued $G_s(x + iy; w)$ on $PSL(2, \mathbb{Z})$ with $-0.75 < x < 0.75, 0.75 < y < 2.25$ and $w = 1/2 + i\sqrt{3}/2$. We have computed values at $500 \times 500$ points and given them colors ranging through blue, green, yellow and red as the values pass from their minimum to their maximum, except for points very near the singularities which have been given the value 0 and therefore are green, cf. figure 2.7.

2.4 Summary of paper IV

The algorithms developed in paper III can be used for a variety of numerical investigations of the properties of $G_s(z; w)$ and $F_n(z; s)$. In paper IV we present some preliminary results of such investigations.

In particular, we use the $G_s(z; w)$-algorithm to begin to explore questions in quantum chaology. Recall that $G_s(z; w)$ is an eigenfunction of the Laplacian with a logarithmic singularity at the point $w$. Physically, eigenfunctions with singularities correspond to wave functions on a surface with a point scatterer, i.e., a point-like obstacle situated in the billiard, scattering the wave functions. See for example, [SC97, Shi94] where
eigenfunctions in terms of the Green’s function are used; and [dVvCL98] for a discussion of some physical situations successfully modeled by a point scatterer.

While quantum waves are inevitably influenced by a point scatterer, a classical particle will not even notice the obstacle unless it directly strikes it. This has the consequence that a surface that is classically non-chaotic, like a square, may show chaotic behavior on the quantum level if we introduce a point scatterer. Cf., e.g., the computations in [Š90] on the Sinai billiard with a circular obstacle of radius \( r = 0 \). Circumstances like these make eigenfunctions with singularities very appealing for statistical studies like those done in paper II.

In paper IV we present data that might give us a hint about the value distribution of \( G_s(z; w) \) in the semiclassical limit. Our data seems to indicate that the value distribution of \( G_s(z; w) \) on compact regions approaches a Gaussian distribution as the eigenvalue tends to infinity.

The Fourier coefficients \( F_n(z; s) \) of the Green’s function may be viewed as generalizations of the Eisenstein series, cf. (2.8). It is therefore natural to perform similar tests with \( F_n(z; s) \) as was done with \( E(z; s) \) in paper II. Paper IV presents some results of such tests regarding the value distribution of \( F_n(z; s) \) in the semiclassical limit for \( n = 1, \ldots, 5 \). Although the improvement of the fit to a Gaussian curve is not as strong as with \( E(z; s) \), our results for \( F_n(z; s) \) may suggest that the values of \( F_n(z; s) \) have a Gaussian limit distribution on compact regions.

In [Hej92] Hejhal used a version of his cusp form algorithm to search for eigenfunction with singularities on \( PSL(2, \mathbb{Z}) \). At first sight, the functions that were picked up all seemed to be of the form

\[
G_s(z; \rho) \quad \text{with} \quad E(\rho; s) = 0,
\]

where \( E(\rho; s) \) is the Eisenstein series, and \( \rho \) is the lower left corner of the fundamental region for \( PSL(2, \mathbb{Z}) \), cf. figure 1.2. For example, Hejhal was able to detect \( G_s(z; \rho) \) with \( s \) being the first zero of the Riemann zeta function, i.e., \( s = 1/2 + i14.134725 \ldots \). The fact that no other \( G_s(z; w) \) appears is somewhat puzzling considering that for each \( w \) sufficiently near \( \rho \) there exists some \( s \)-value near \( 1/2 + i14.13 \ldots \) such that \( E(w; s) = 0 \) holds. One might expect that the corresponding \( G_s(z; w) \) also would be visible. Note that because Hejhal’s algorithm is designed for cusp forms it requires a zero first order Fourier coefficient, cf. (1.3), therefore it would not pick up a \( G_s(z; w) \) for \( w \) with \( E(w; s) \neq 0 \).

In paper IV we present some heuristics and numerics produced with our \( G_s(z; w) \)-algorithm with the purpose of discussing why the functions \( G_s(z; \rho) \) with \( E(\rho; s) = 0 \) seem to stand out in Hejhal’s experiments. Somewhat simplified, the main ideas are as following. We study a special type of functions related to the Fourier coefficients of the computed eigenfunctions. The geometry of \( PSL(2, \mathbb{Z}) \) forces the nodal curves through
\( \rho \) of these functions to stay close together and close to the nodal curve through \( \rho \) for the Eisenstein series. This may cause sign changes which are detected by the computer program. The corresponding nodal curves through other points on \( \partial \mathcal{F} \) near \( \rho \) seem to be too far apart to cause similar sign changes.
Beräkningar av automorfa funktioner på Fuchsiska grupper

Kvantkaologi


Om vi låter partikeln få större och större energi kommer den mer och mer att bete sig som den vanliga biljardbollen. Om ytan är formad så att biljardbollens bana är kaotisk så borde den kvantmekaniska partikelmns rörelse visa spår av kaos på något sätt. Kvantkaologi handlar om att studera dessa spår av klassiskt kaos i kvantmekaniska system.

Egenvärden och vågfunktioner
Kvantmekaniska partiklars rörelse beskrivs av vågfunktioner. Ofta existerar bara vågfunktioner med vissa bestämda energier. Man kan tänka på energinivåerna i en atom, eller att en gitarrsträng bara kan svänga med bestämda frekvenser, gitarrsträngens rörelse är ju också en vågrörelse. Dessa förutbeständiga energier är diskreta egenvärden. Vi ordnar
dem i en växande följd:

\[ 0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots \]  

(2.12)

De motsvarande vågfunktionerna är *bundna tillstånd*.

Om biljardbordet är obegränsat, alltså om det fortsätter oändligt långt åt något håll, så kan det också finnas *obundna tillstånd* med tillåtna energier i ett sammanhängande intervall, som till exempel \([\alpha, \infty)\). Dessa energier är också egenvärden. Kvantkaologi handlar om att söka spår av kaos i vågfunktioner och egenvärden och därför är det viktigt att ha bra metoder för att beräkna dessa.

**Algoritmer och experiment**

I min avhandling har jag utvecklat algoritmer för numeriska beräkningar av vågfunktioner och egenvärden. Det betyder att jag i avhandlingen beskriver hur man steg för steg kan utföra dessa beräkningar med hjälp av datorer. Jag förklarar också de matematiska teorier som jag har byggt algoritmerna på och redovisar tester som jag använt för att förvissa mig om att algoritmerna och datorprogrammen där jag implementerat algoritmerna är korrekta.

Även om en stor del av arbetet är just att utveckla beräkningsmetoder så är huvudmålet att använda de färdiga algoritmerna till att skaffa ny kunskap. I min avhandling redovisar jag resultatet av de experiment som jag utfört för att utforska det tidigare okända.

**Något om mina resultat**

diskret egenvärde så har jag numeriska bevis för att det egenvärdet har försvunnit under deformationen.

Med hjälp av mina algoritmer har jag beräknat egenvärden ur följden (2.12) på den ursprungliga symmetriska ytan och nollställen till spridningsmatrisen på deformerade ytor och därmed numeriskt bekräftat Phillips och Sarnaks hypotes.


Mycket riktigt visar mina resultat att spåren av kaos blir tydligare då egenvärdet ökar. Det illustreras i figur 2.4 på sidan 22 där mina testpunkter följer kurvan bättre och bättre från vänster till höger.


Den fjärde artikeln visar några exempel på experiment som är möjliga att utföra med algoritmerna från den tredje artikeln. Till exempel görs undersökningar relaterade till kvantkaologi för vågfunktioner på ytor med hinder.

I bilden av vågfunktionerna i figur 2.7 på sidan 26 finns hinder i de små cirklarna strax över markörerna för 0.5 och −0.5.
First of all I would like to express my deep gratitude towards my two advisors Dennis Hejhal and Andreas Strömbergsson. This thesis would not exist without your help and encouragement.

Dennis, thank you for your enthusiasm. I have never met anyone with such an ability to bring out the fun in mathematics. Sometimes I came to our meetings feeling down about my research, but I always left enlightened, inspired and convinced that my work was important. I also want to thank you for introducing me to this exciting field of research, for listening to what I wanted to do, for teaching me how to write mathematics, and for suggesting excellent research problems.

Andreas, we first met in Salongsorkestern were we both played the violin, but it was when you were my fellow graduate student that we got to know each other well. I was in my beginning year and you had the room across the hall from me in Bastun. You invited me to come and ask you any questions I might have, and even though it was a lot of questions, you became a very good friend. When you came back to Uppsala after your post doctoral studies the invitation for mathematical questions was still open. This was a hospitality I used frequently, and so you became my advisor. I want to thank you for always taking the time to answer my questions and always making me feel that my questions are interesting! When you became my advisor you never forgot that you were my friend first, and I know we will stay friends for a long time after my thesis defense.

There are two more people without whose help I could not have written this thesis. I want to thank Carl Edström and Christian Nygaard for keeping our computer systems in such excellent condition. It is truly amazing how you have helped me these years with software installation, data storage and raising my disk quota (over and over again); and always with a friendly smile!

The final work with this introduction went smoothly only thanks to Anders Frisk and Erik Melin who let me in on all the secrets of the \LaTeX\ thesis templates. Also I want to thank Anders Källström for making the nice book cover for my thesis.

I have had helpful discussions about my research with a number of people. Here Fredrik Strömberg certainly stands out, you are a great friend! Thank you also for making my trip to Clausthal so nice, even
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I want to thank all of my colleagues and friends who have made my everyday work at the department enjoyable.

Especially, I want to thank Anders Pelander for sharing my office with me in room 7110, I really enjoyed our conversations. (Do you remember the ants that came after the sandwiches I kept in my backpack when I was pregnant?)

Lunch time has been a welcome break in my work day and this is thanks to all the nice people around our lunch table. During my first years when I had lunch with the statisticians in house 4 on Polacksbacken I got to know some of the best people of our department. Special thanks to Allan Gut who cared when times were difficult. After the move to Ångström me and the statisticians joined in with mathematicians and administrative personnel around one big lunch table. I really appreciate the friendly atmosphere around our table.

I never had time to take coffee breaks, at least not in real life. But some days would have been difficult to get through without the friendship and support from Jenny, Åsa M and Åsa L in our virtual coffee breaks. The three of you are the best friends I could ask for!

Anna, Kajsa, Martina, Ingrid, Salla, Johanna, Cecilia, Bodil, Susanna, Malin and Alice, you know that Lilla $\mu$ has meant a lot to me and that is thanks to all of you. I look forward to having dinners and board game evenings with you also in the future!

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