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On Axioms and Images in the History of Mathematics

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Abstract
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This dissertation deals with aspects of axiomatization, intuition and visualization in the history of mathematics. Particular focus is put on the end of the 19th century, before David Hilbert's (1862–1943) work on the axiomatization of Euclidean geometry. The thesis consists of three papers. In the first paper the Swedish mathematician Torsten Brodén (1857–1931) and his work on the foundations of Euclidean geometry from 1890 and 1912, is studied. A thorough analysis of his foundational work is made as well as an investigation into his general view on science and mathematics. Furthermore, his thoughts on geometry and its nature and what consequences his view has for how he proceeds in developing the axiomatic system, is studied. In the second paper different aspects of visualizations in mathematics are investigated. In particular, it is argued that the meaning of a visualization is not revealed by the visualization and that a visualization can be problematic to a person if this person, due to a limited knowledge or limited experience, has a simplified view of what the picture represents. A historical study considers the discussion on the role of intuition in mathematics which followed in the wake of Karl Weierstrass' (1815–1897) construction of a nowhere differentiable function in 1872. In the third paper certain aspects of the thinking of the two scientists Felix Klein (1849–1925) and Heinrich Hertz (1857–1894) are studied. It is investigated how Klein and Hertz related to the idea of naïve images and visual thinking shortly before the development of modern axiomatics. Klein in several of his writings emphasized his belief that intuition plays an important part in mathematics. Hertz argued that we form images in our mind when we experience the world, but these images may contain elements that do not exist in nature.

Keywords: History of mathematics, axiomatization, intuition, visualization, images, Euclidean geometry

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List of Papers

This thesis is based on the following papers:

I Pejlare, J. (2007). Torsten Brodén and the foundations of Euclidean geometry, *Historia Mathematica* **34**, 402–427.

II Bråting, K., Pejlare, J. Visualizations in mathematics. To appear in *Erkenntnis.*

III Pejlare, J. The role of intuition and images in mathematics: The cases of Felix Klein and Heinrich Hertz. Submitted.

Contents

1. Introduction

Modern axiomatics, as we know it today, was developed by David Hilbert (1862–1943) during the beginning of the 20th century. His first edition of the *Grundlagen der Geometrie*, which provided an axiomatization of Euclidean geometry, was published in 1899, but was revised several times. He built up Euclidean geometry from the undefined concepts "point", "line" and "plane" and from a few undefined relations between them. The properties of the undefined concepts and relations are specified by the axioms as expressing certain related facts fundamental to our intuition.

Hilbert's work was the result of a long tradition of research into the foundations of geometry. The historically most important event in the development of geometry was Euclid's systematic treatment of the subject in the form of a uniform axiomatic-deductive system. His work entitled *Elements*,[1] written about 300 BC, still maintains its importance as one of the most valuable scientific books of all time. Influenced by the work of Aristotle, Euclid set himself the task of presenting geometry in the form of a logical system based on a number of definitions, postulates and common notions. It was believed that, in establishing this system, he was creating a sufficient foundation for the construction of geometry.

However, Euclid's *Elements* received a lot of criticism. One of the main issues concerned logical gaps in the proofs, where at some points assumptions that were not stated were used. This happened already in the proof of the first proposition, where an equilateral triangle is constructed. To do this two circles are drawn through each others' centers. The corners of the triangle will now be in the centers of the two circles and in one of the points of intersection of the two circles. However, it does not follow from the postulates and common notions that such a point of intersection actually exists, even if it seems to be the case from the visual point of view. If we, for example, consider the rational plane $\mathbf{Q}^2$, instead of the real plane $\mathbf{R}^2$, there are no points of intersection in this case. Thus we could say that Euclid in the *Elements* assumed, without saying so explicitly, the continuity of the two circles. In a similar way, continuity of the straight line is assumed.

These defects are subtle ones, since we are not assuming something contrary to our experience; the tacit assumptions are so evident that there do not appear to be any assumptions. These gaps in Euclid's *Elements* were probably

[1] For a complete treatment of the *Elements*, see Heath (1956). An overview of the history of geometry can be found in Eves (1990) and Kline (1972).

not considered to be of a very serious kind, since intuition could fill them in. Of particular interest was instead the problem whether or not Euclid's fifth postulate, also called the parallel axiom, is necessary for the construction of geometry, that is, whether or not the parallel axiom is independent of the other postulates and common notions. The parallel axiom is formulated in the following way:[2]

> That, if a straight line falling on two straight lines make the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles.

In the efforts to eliminate the doubts about the parallel axiom two approaches were followed. One was to replace it with a more self-evident statement. The other was to prove that it is a logical consequence of the remaining postulates, and that it therefore may be omitted without loss to the theory.

In spite of considerable efforts by several mathematicians for about two millenia, no one was able to do this. This is no wonder, since, as was eventually found out, the parallel axiom is independent of, and thus cannot be derived from, the other postulates and common notions, and also cannot be omitted in Euclidean geometry. This observation was probably first made by Carl Friedrich Gauss (1777–1855), who claimed that he already in 1792, at the age of 15, had grasped the idea that there could be a logical geometry in which the parallel axiom did not hold, that is, a non-Euclidean geometry.[3] However, he never published anything of his work on the parallel axiom and non-Euclidean geometry.

Generally credited with the creation of non-Euclidean geometry are Nikolai Ivanovich Lobachevsky (1793–1856) and János Bolyai (1802–1860). Lobachevsky published his first article on non-Euclidean geometry in 1829–1830 in the *Kasan Bulletin*. Bolyai's article on non-Euclidean geometry was published in 1832.[4] Lobachevsky and Bolyai independently arrived at their systems of geometry, which are essentially the same. They both took all the explicit and implicit assumptions of Euclid's *Elements*, except the parallel axiom, for granted. Instead of the parallel axiom they included an axiom contradicting it, with the consequence that all parallel lines in a given direction converge asymptotically.

[2](Heath, 1956, p. 202). An equivalent formulation of the parallel axiom is Playfair's axiom: "Through a given point only one parallel can be drawn to a given straight line." (Heath, 1956, p. 220).

[3]Gauss made this claim in letters to friends and colleagues, for example in a letter to Taurinus of November 8, 1824, and in a letter to Schumacher of November 28, 1846. For details, see (Gauss, 1973).

[4]The article was published as an abstract to his father Wolfgang Bolyai's book *Tentamen*. A translation into German can be found in (J. Bolyai and W. Bolyai, 1913).

The realization that the parallel axiom could not be deduced from the other assumptions, and thus could be exchanged with an axiom contradicting it, implied that Euclidean geometry was no longer the only possible geometry. Therefore Euclidean geometry is not necessarily the geometry of physical space. Immanuel Kant (1724–1804) had regarded geometry as synthetic a priori, that is, geometrical knowledge is based on an immediate awareness of space and this awareness accompanies all our perceptions of spatial things without being determined by them (Torretti, 1978, p. 164). But with several possible geometries references may have to be made to experience to decide which one describes the world. For example, Hermann von Helmholtz (1821–1894) criticized Kant and instead emphasized the empirical origin of geometry and insisted that only experience can decide between the different geometries.

The discovery of non-Euclidean geometry also made mathematicians realize that the deficiencies in Euclid's *Elements* was a serious problem, and a reconstruction of the foundations of Euclidean geometry had to be made. However, the development of non-Euclidean geometry remained unknown to mathematicians in general until the 1860s (Kline, 1972, p. 879). Instead, because of its beauty and simplicity, projective geometry, which may be regarded as a non-metric geometry, since it ignores distances and sizes, received more attention (Torretti, 1978, p. 110). In 1873 Felix Klein (1849–1925) proved that projective geometry is independent of the parallel axiom, and hence is valid in both Euclidean and non-Euclidean geometries. Therefore projective geometry can be considered to be more fundamental than these. Klein is also well-known for his use of intuitive models for "seeing" things in new perspectives (Glas, 2000, p. 80). For example, he constructed Euclidean models of non-Euclidean geometries to be able to study less visualizable geometries in a more intuitive manner. He expressed his view on geometric intuition and its role in science in several of his writings.

In 1882 Moritz Pasch (1843–1930) managed to develop a complete axiomatic system for projective geometry.[5] He explicitly formulated all primitive notions and axioms, and he understood the importance of a logical deduction of all the geometrical theorems from them. Furthermore, he rejected pictures as irrelevant to geometrical foundations; he insisted that every conclusion which occurs in a proof must be confirmed by a picture, but it is not justified by the picture (Pasch, 1882, p. 43).

Contro argues for two lines of development for research into the foundations of geometry after Pasch, one in Italy and another in Germany that was completed with the work of Hilbert (Contro, 1976, p. 291).

The most complete of the Italian geometers is probably Mario Pieri (1860–1913), who focused on metamathematical issues while characterizing the nature of an axiomatic theory (Marchisotto, 1993, p. 288). For Pieri the subject

[5]The work can also be found in (Pasch, 1976) together with an appendix by Max Dehn. The axiomatic system is investigated in detail by (Contro, 1976).

of geometry is not the intuitive notion of space, but space envisioned as subject to all interpretations that fulfill certain conditions. But his work was only one result of an Italian school that had been active for decades. Other important Italian mathematicians who contributed to the field of geometry were Federigo Enriques (1871–1946), Gino Fano (1871–1952), Giuseppe Peano (1858–1932) and Giuseppe Veronese (1854–1917). In Italy the formal and logical point of view regarding an axiomatic theory was emphasized (Contro, 1976, p. 292). Apparently a complete and rigorous organization of the foundations of geometry was achieved in Italy already in the 1890s. The question of foundations had a direct connection to issues arising from teaching (Avellone, Brigaglia and Zappulla, 2002). However, their work did not receive the attention abroad which it deserved, and became overshadowed by the work of Hilbert.

Not only in geometry, but also in analysis, the role of geometric intuition was discredited during the second half of the 19th century since it can be deceptive. For example, it was for a long time not uncommon to believe that every continuous function must be everywhere differentiable, except at isolated points (Volkert, 1987). But in 1872 Karl Weierstrass (1815–1897) constructed a function that is continuous but nowhere differentiable. The result was proved analytically, leaving obscure what the geometrical nature of the function may be, and was used to discredit the role of visual representations in analysis (Mancosu, 2005, p. 16). Klein, on the other hand, wanted to preserve visual elements in mathematics, insisting that mathematics cannot be built from the axioms alone. He argued that the axioms are exact idealizations originating in inexact naïve intuition and that mathematics would become lifeless if intuition was suppressed.

In this thesis I consider aspects of axiomatization, intuition and visualization in the history of mathematics. In particular, I consider the period at the end of the 19th century, before Hilbert's work on the axiomatization of Euclidean geometry. In the first paper I study the Swedish mathematician Torsten Brodén (1857–1931) and his work on the foundations of Euclidean geometry from 1890 and 1912. I make a thorough analysis of his foundational work and investigate his general view on science and mathematics. Furthermore, I investigate his thoughts on geometry and its nature and what consequences his view has for the way in which he proceeds in developing the axiomatic system.

The second paper is a joint work with Kajsa Bråting. We study different aspects of visualizations in mathematics. In particular, we argue that the meaning of a visualization is not revealed by the visualization and that a visualization can be problematic to a person if, due to limited knowledge or limited experience, this person has a simplified view of what the picture represents. In a historical study we consider, among other things, the discussion on the role of intuition in mathematics which followed in the wake of Weierstrass' construction of a continuous but nowhere differentiable function in 1872.

In the third paper I study certain aspect of the thinking of the two scientists Felix Klein and Heinrich Hertz (1857–1894). Hertz is well-known for his work on electrodynamics and for his contributions to the foundations of mechanics, and considerably influenced Hilbert in his work on the foundations of physics (Corry, 2004). I investigate how Klein and Hertz related to the idea of naïve images and visual thinking shortly before the development of modern axiomatics. Klein emphasized in several of his writings his belief that intuition plays an important part in mathematics. Hertz argued that we form images in our mind when we experience the world, but these images may contain elements that do not exist in nature.

2. Overview of the Thesis

2.1 Summary of Paper I

A summary of this paper has been presented at the conferences “History and Pedagogy of Mathematics” in Uppsala in 2004 and “Research in Progress” in Oxford in 2005.

In this paper I study the Swedish mathematician Torsten Brodén’s work on the foundations of Euclidean geometry from 1890 and 1912. In the 1890 article he tried to give a philosophical justification for his axiomatization. On the one hand, he appealed to Helmholtz and wanted to obtain a theoretical basis for the fact that the external reality as described by Euclidean geometry corresponds to experience. But, on the other hand, he considered geometry to be a priori.

The aim of Brodén’s 1890 article seems to be to take part in a contemporary pedagogical debate on the problems in Swedish schools. He wanted to decide if it is true that the value of geometry as a school subject lies in the possibility for it to be treated in a strictly “scientific” way. His axiomatic system is the result of his detailed investigation into what a scientific geometry should look like. He argued that a scientific system should be built up from a number of undefined “basic notions” and a number of unproven “axioms”, satisfying certain criteria. Of particular interest are the criteria regarding the sufficiency of the axioms for arranging geometry under certain logical forms and the independence of the axioms.

I consider Brodén’s axiomatic system for Euclidean geometry from 1890 in detail and compare it with his later work on the foundations of geometry. He insisted that geometry reduces all phenomena to motion, which can be characterized by a collection of objects and a collection of relations between them. From this he concluded that the two basic notions “point” and “immediate equality of distance” are enough. Original in Brodén’s axiomatic system is his use of symmetries, the symmetric correspondence in the line and the symmetric equivalence in the plane. This seems to be an unusual approach at this time. For example, he rotated a line around a fixed point by performing a composition of reflections about two lines through a fixed point.

In 1890 Brodén gave two continuity axioms from which a bijection between the points of the line and the real numbers follows. In doing this he transfered George Cantor’s (1845–1918) idea to construct real numbers from Cauchy sequences of rational numbers to the straight line. I argue that these two axioms implies the two continuity axioms of Hilbert from 1903, the Archimedean axiom and the completeness axiom. Brodén in 1912 claimed that he anticipated Hilbert when he in his 1890 axiomatization of Euclidean geometry gave a formulation of a completeness axiom. I argue, however, that Brodén in 1912 exchanges this axioms into a weaker one, and Hilbert's Archimedean axiom does no longer follow.

Furthermore, Brodén gave an explicit proof for the sufficiency of the axioms for establishing Euclidean geometry. In the proof Brodén constructed a coordinate system and deduced the distance formula for calculating the distance between two arbitrary points. In this formula, he claimed, the entire Euclidean geometry lies embedded since "everything" can be derived from it. I argue that Brodén's demand of sufficiency could possibly be interpreted as some kind of consistency proof. However, I do not believe that Brodén had a general concept of consistency, as was later developed by Hilbert.

2.2 Summary of Paper II

This paper is a joint work with Kajsa Bråting. We have presented our results at the conference "Towards a New Epistemology of Mathematics" in Berlin in 2006.

In this paper we study visualizations in mathematics from a historical and a didactical perspective. We criticize some different views on mathematical visualizations that focus too much on pictures as being independent of the observer. For example, during the latter half of the 19th century visual thinking fell into disrepute since it can be deceptive. One reason could have been Weierstrass' construction of a continuous but nowhere differentiable function. Before this discovery it had not been an uncommon belief among mathematicians that a continuous function must be differentiable, except at isolated points. As a reaction to Weierstrass' function Klein wanted to discuss the limitations of our intuition of space. He indicated the need for informal thinking in mathematics and had a problem with mathematics, such as Weierstrass' function, that he could not verify through naïve intuition. Furthermore, the Swedish mathematician Helge von Koch (1870–1924) found it difficult to understand mathematics without "seeing" the mathematical results. Referring to Klein's naïve intuition, von Koch constructed a continuous but nowhere differentiable function such that it from the visual representation would be possible to see this result. However, with support from an empirical study of university students' solutions of a mathematical problem, we argue that for a person not familiar with the existence of such functions, this result may not be so easy to "see". A visualization can be problematic to a person if this person, due to limited knowledge or limited experience, has a simplified view of what the picture represents. Furthermore, we argue that a person with enough mathematical experience and familiarity with the theory can read what is unsaid in the picture "between the lines". Thus, we need to know some mathematics to be able to know what to look for in a visualization and let the unsaid become meaningful.

Moreover, we argue that the meaning of the visualization is not revealed by the visualization; there must be an interaction between the visualization and the person interpreting it. Removed from its mathematical context, the visualization loses its meaning. A historical example we consider in connection to this is the angle of contact. In the 17th century there was a debate between Thomas Hobbes (1588–1679) and John Wallis (1616–1703) whether there exist an angle between a circle and its tangent, and, if such an angle exist, what quantity it has. It seems that they sometimes did not base their arguments on mathematical definitions, instead relying too much on the visualization and trying to "see" the correct answer. Furthermore, we argue that a visualization may be interpreted in different ways depending on context and on what question should be answered. For example, depending on what definition of an angle we use, the angle of contact may be zero or it may not exist.

2.3 Summary of Paper III

Parts of this paper has been presented at the conferences "Filosofidagarna 2005" in Uppsala and "Towards a New Epistemology of Mathematics" in Berlin in 2006.

In this paper I study certain aspects of the thinking of the two scientists Klein and Hertz at the end of the 19th century, before the development of modern axiomatics by Hilbert. In particular, I discuss their philosophical views of mathematics and mechanics and how they related to the idea of naïve images and visual thinking in science.

Klein insisted that intuition is the origin of geometry and also important to its practice, and he objected that the axioms are arbitrary statements which we set up as we please. He rejected Pasch's demand that the full intuitive content of geometry could be expressed in the axioms and also objected to Weierstrass who wanted to suppress intuition from mathematics and rely only on arithmetical proofs. Klein insisted that the axioms are idealizations of inexact naïve intuition of space and claimed that mathematics will become lifeless if intuition is suppressed. He furthermore insisted that naïve intuition always precedes refined intuition, being the result of logical deduction from the exact axioms. For Klein it was not the formal arguments and final results that were of greatest importance, but the road to discovery. Moreover, he emphasized the importance of a dynamical interaction at different levels between visual naïve thinking and refined axiomatization. Thus, he tried to save naïve intuition as an essential part of mathematics and its origin using visual and intuitive arguments to get new perspectives and a deeper understanding of mathematics.

Hertz, on the other hand, had a very different philosophy compared to Klein. According to him we, in order to build up a scientific theory describing reality, form images in our mind. If the image of reality is sufficiently good, we can predict events that will occur after a certain time in the external world. He insisted that it is possible to form different images of the same object and introduced three criteria on the basis of which the images may be compared with each other such that the most appropriate one can be chosen. Furthermore, Hertz argued that images may contain elements that do not exist in nature. For example, in his image of mechanics he permitted concealed masses that do not have any connection to our sensory system. Thus, he wanted to clear out the concrete visual elements as a foundation of the concepts of modern mathematics, showing a similarity to modern axiomatics later developed by Hilbert.

3. Summary in Swedish

I den här avhandlingen diskuteras aspekter av axiomatisering, åskådning och visualisering i matematikens historia. Framför allt studeras utvecklingen under slutet av 1800-talet, det vill säga perioden som föregick David Hilberts utveckling av den moderna axiomatiseringen. Hilbert publicerade år 1899 den första upplagan av *Grundlagen der Geometrie*, i vilken han presenterade en axiomatisering av den Euklidiska geometrin. Detta arbete var resultatet av en lång tradition av forskning om geometrins grundvalar, som tog sin början i Euklides *Elementa* från 300-talet före Kristus.

Under den första hälften av 1800-talet utvecklades den icke-Euklidiska geometrin av Carl Friedrich Gauss, Nikolai Ivanovich Lobachevsky och János Bolyai. Insikten att Euklides parallellaxiom kunde ersättas med ett axiom som motsäger detta och att det existerade många möjliga geometrier medförde att den Euklidiska geometrin inte nödvändigtvis var den geometri som beskriver det fysiska rummet. Immanuel Kant hade ansett att geometrin var syntetisk a priori, men med många möjliga geometrier kan det vara nödvändigt att referera till erfarenheten för att avgöra vilken geometri som beskriver rummet. Bland andra Hermann von Helmholtz kritiserade Kant och menade att geometrin har sitt ursprung i empirin.

Under slutet av 1800-talet bedrevs mycket forskning i Italien och Tyskland om geometrins grundvalar. I Italien betonades speciellt axiomatiseringens formella och logiska sida, framför allt av Mario Pieri. Utvecklingen i Tyskland fullbordades med Hilberts arbete. Ett viktigt bidrag gavs även av Moritz Pasch, som 1882 konstruerade ett fullständigt axiomatiskt system för den projektiva geometrin. Pasch tillbakavisade bilder som relevanta i geometrins grundvalar. Han menade att slutsatser som dras i ett bevis kan bekräftas med bilder, men enbart bilder kan inte utgöra bevis.

I avhandlingens första artikel studeras den svenska matematikern Torsten Brodéns arbete om geometrins grundvalar från 1890 och 1912. Syftet med Brodéns artikel från 1890 var att bidra till en pedagogisk debatt om problem i den svenska skolan. I ett försök att avgöra huruvida värdet för geometrin som ett skolämne ligger i dess möjlighet att behandlas på ett strikt vetenskapligt sätt, gjorde han en detaljerad undersökning av hur en vetenskaplig geometri måste se ut. I avhandlingen undersöks det i detalj hur Brodén bygger upp sitt axiomatiska system med utgångspunkt från hans filosofiska syn på geometrins natur.

Inte bara i geometrin, utan även i analysen, misskrediterades åskådningens roll under den andra hälften av 1800-talet. År 1872 konstruerade Karl Weierstrass en funktion som var kontinuerlig men ingenstans deriverbar. Innan dess var det en inte ovanlig föreställning bland matematiker att kontinuerliga funktioner var deriverbara överallt förutom i isolerade punkter. Inspirerad av Weierstrass resultat diskuterade Felix Klein begränsningar av vår åskådning av rummet. Klein ansåg att det finns ett behov av informellt tänkande i matematiken och menade att det var problematisk med exempel som Weierstrass funktion som han inte kunde verifiera med hjälp av naiv åskådning. Även Helge von Koch hade problem med att förstå matematiska resultet som han inte kunde “se”. För att förstå existensen av kontinuerliga men ingenstans deriverbara funktioner konstruerade han en funktion som är en variant av hans numera välkända “snöflinga”. Han menade att det, utifrån den visuella representationen av denna funktion, skulle vara möjligt att “se”, och därmed förstå, existensen av kontinuerliga men ingenstans deriverbara funktioner. Med utgångspunkt från bland annat detta historiska exempel diskuteras i avhandlingens andra artikel visualiseringar i matematik. I artikeln kritiseras synen på matematiska visualiseringar som fokuserar för mycket på bilder som varande oberoende av betraktaren. Det argumenteras för att en visualisering kan vara problematisk för en person som på grund av begränsad erfarenhet eller kunskap har en förenklad syn på vad bilden representerar. Med stöd av en empirisk undersökning av universitetsstudenters lösning av ett matematiskt problem argumenteras det vidare för att en person som inte är väl förtrogen med till exempel kontinuerliga men ingenstans deriverbara funktioner, är detta resultat inte så lätt att “se” utifrån en visualisering.

I avhandlingens tredje artikel studeras Felix Kleins och Heinrich Hertz filosofiska syn på matematiken och mekaniken och hur de relaterade till idén om naiva bilder och visuellt tänkande i vetenskap. Klein menade att den naiva åskådningen är en viktig del av geometrin och dess ursprung och var kritisk gentemot Weierstrass som ville bannlysa åskådningen från matematiken och enbart förlita sig på aritmetiska bevis. Klein menade att om åskådningen blir bannlyst så blir matematiken livlös. Vidare argumenterade han för att axiomen är exakta idealiseringar som har sitt ursprung i inexakt åskådning. Klein försökte bevara åskådningen som en väsentlig del av matematiken och dess ursprung genom att använda visuella och åskådliga argument för att få nya perspektiv och en djupare förståelse för matematiken. Hertz menade att vi gör oss bilder av världen när vi upplever den, och eftersom vi aldrig kan uppleva världen exakt så kommer bilderna enbart att vara bilder och kan innehålla element som inte existerar i naturen. När han konstruerade en ny axiomatisering av mekaniken ville han rensa ut konkreta visuella element som grund för begreppen. Detta innebär att hans bilder är formella och inte kopplade till något visuellt, vilket visar en likhet med Hilberts senare axiomatik.

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