An Overview Of The Rado Graph

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Abstract. This paper examines the Rado graph, the unique, countably infinite, universal graph. Many of the central properties are covered in detail, and various constructions are provided, using results from a variety of fields of mathematics. A variant of the Rado graph was initially constructed by Ackermann. The actual Rado graph was studied later, by Erdős and Rényi, before Rado rediscovered it from a different perspective. A multitude of other authors have since then contributed to the subject.

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1. Introduction

Let’s say we draw some vertices on a paper, and then, for each possible edge, we toss a coin. For heads, we draw the edge, for tails, we do not. What kind of graph would we attain? Obviously, this depends on the outcomes of every part of the random process, as well as how many vertices we have. If there are \( n \) vertices, we can have anything from 0 to \( \binom{n(n-1)}{2} \) number of edges, but what if we have a countable set of vertices? If we could form some sort of limit graph of this process, can we say anything about it? One might be tempted to think that since we are still working with a random process, we may attain all sorts of graph, some highly different from others. However, this is not the case. What may be surprising is that we can always expect to attain the same graph. Of course, one graph will perhaps have an edge between the first two vertices, provided some enumeration, while another could have a non-edge, however, we can always expect two such graphs to be isomorphic to each other. In other words, almost all graphs attained through this coin-tossing procedure are isomorphic. For this reason, we will (when formality is not of necessity) treat them as the same graph. This graph is called the Rado graph, sometimes the random graph, and is often denoted by \( R \).

The first question one might have is: how does this graph look? Or, more formally: can this graph be described in a more specific way? One might be surprised at the answer, which is that this graph is relatively easy to describe. Aside from being countably infinite, we only need one additional property to define it uniquely, namely, the extension property. One might find some difficulty to first grasp it, but the intuition for the property is rather straightforward. To name one of the many interesting implications this property have, it is what allows us to find any graph, that is at most countably infinite, as an induced subgraph of \( R \).

The extension property will be discussed in the first major part of the paper, together with some of its implications. Other implications are discussed in the final chapter, after we have seen some constructions.

If we ponder the implications of the statement in the initial example, that this coin-tossing procedure almost always gives the same graph, we may draw the conclusion that almost all countable graphs are the same, in the sense that they are all isomorphic to \( R \).

The next question one might thus consider is: how do we do this formally? How can we show that we can expect to always attain the same graph? This is somewhat troublesome to do, since we are dealing with an uncountable probability space. In particular, if we fix a countable set of vertices, the number of graphs on said vertex set is uncountable. To solve this, we need to introduce a probability measure.

The example presented in the beginning is one method of constructing the Rado graph, and it is related to the probability measure mentioned above. We will see multiple other constructions later on.
2. Notation and terminology

The reader is assumed to have some background knowledge of graph theory and logic, however, since notation and terminology may vary, some concepts and background is given in this chapter. For a more detailed reference, see the book *A First Course in Logic* by Shawn Hedman [Hed04].

Throughout this paper, graph refers to simple graph, meaning an undirected graph where no vertex is adjacent to itself. A graph $H$ is a subgraph of $G$ if the vertex set $V(H)$ is a subset of $V(G)$ and the only edges in the edge set $E(H)$ are between vertices in $V(H)$ and $E(H) \subseteq E(G)$. An induced subgraph $H$ of $G$ is a subgraph such that for every edge $e$ in $E(G)$ between vertices in $V(H)$, we have $e \in E(H)$. If we choose some subset $S$ of the vertex set $V(G)$, then the induced subgraph $H$ with vertex set $S$ is said to be induced by $S$. This is written $H = G[S]$.

For cardinality of sets, a denumerable set will in this paper refer to a set with the same cardinality as the natural numbers. A countable set refers to a set that is finite or denumerable. Also, for this paper, we include 0 in the natural numbers.

As a reminder of some notation and terminology of First-Order logic, we use a vocabulary of symbols together with variables and specific logic symbols to form formulas. In this paper we will only use the standard vocabulary $V = \{\sim\}$ for graphs, where $\sim$ is a binary relation symbol representing that two vertices are adjacent.

A formula with no free variables is called a sentence. A set of sentences $T$ with respect to some vocabulary is called a theory if it is satisfiable. That is, it is satisfied by a model, also called a structure. This is true if and only if $T$ is consistent, which means that it cannot prove a contradiction. Usually, calligraphic letters such as $\mathcal{M}$ are used for structures, and their underlying sets, often called universes, are often written using a standard font, such as $M$. Note that vocabularies are also often written using calligraphic letters.

For graphs, substructure means the same as induced subgraph, and since we only work with a finite, relational vocabulary, finitely generated is the same as finite. To be more precise, if we choose some subset $S$ of the vertices and wish to generate a substructure (induced subgraph) $G$ as small as possible (with respect to inclusion) containing $S$, we simply end up with the subgraph induced by $S$. If $S$ is finite, then this $G$ is also finite.

For a vocabulary $\mathcal{V}$, a $\mathcal{V}$-theory $T$ is said to be complete if for every $\mathcal{V}$-sentence $\varphi$, either $\varphi \in T$ or $\neg \varphi \in T$. The set of sentences that can be proven from a theory $T$ is called the deductive closure of $T$, and is usually written $\bar{T}$. For a cardinal $\kappa$, a theory $T$ is said to be $\kappa$-categorical if all its models of cardinality $\kappa$ are isomorphic, and such a model exists. Recall Vaught’s Theorem, which states that if a theory $T$ is $\kappa$-categorical for some infinite $\kappa$, then $\bar{T}$ is complete.

If one wishes to read more about the Rado graph, there are multiple additional results covered in *The random graph* by Peter J. Cameron [Cam97]. Many of the results in this article are taken from there. Some of the more central parts have been done more formally in this paper, while Cameron’s survey covers a broader area of the topic, such as the automorphism group on the graph. The book *Model theory* by Wilfrid Hodges [Hod93] is also a good source, mainly for the model theoretic perspective of $R$. The final chapter of this paper, where $R$ is constructed as a Fraïssé limit, uses the results provided in Hodges’ work. For the curious reader, the papers by Ackermann [Ack37], Erdős and Rényi [ER63], and Rado [Rad64] mentioned in the abstract are included in the bibliography.
3. Core properties

The most central property of the Rado graph is the extension property. An intuition for it, and a (possible) explanation for the name, is the idea that whenever one chooses a finite (induced) subgraph of the Rado graph, it is always possible to extend this subgraph by adding another vertex, in any way one wants. That is, one may choose exactly which vertices in the subgraph the new vertex should be adjacent to, and which vertices it should not.

This might seem like a relatively small piece to the puzzle, but together with denumerability, this is, as mentioned in the introduction, the only thing we really need. We will see that any two denumerable graphs satisfying the extension property are isomorphic, meaning that for any denumerable graph, if this property can be verified, then it is isomorphic to $\mathbb{R}$. Since this property is so important, we will dedicate an entire chapter to it, and we will examine how we may write it as a theory of First-Order Logic. The reason this is of interest has to do with the same thing, namely that since all denumerable models this theory describes are isomorphic, the theory is complete, and we thus only need this theory to determine if any sentence in First-Order Logic is true or false in the Rado graph. In a later chapter, we will see how this connects to logical sentences in finite graphs.

**Definition 3.1.** Let $G = (V, E)$ be a graph. $G$ is said to have the extension property if for any finite, disjoint $U, W \subseteq V$, there is a $z \in V \setminus (U \cup W)$ such that $\{z, u\} \in E$ for all $u \in U$ and $\{z, w\} \notin E$ for all $w \in W$.

For a given pair of finite, disjoint $U, W \subseteq V$, we say that a $z$ that abides by the above requirement is correctly joined. We should also note that if some graph $G$ has the extension property, then it must be infinite. Otherwise, we could let $U$ be the entire vertex set, and then there would be no other vertex that could satisfy this requirement, since $V \setminus (U \cup W)$ would be empty. Moreover, a vertex cannot be adjacent to itself, so even if we were allowed to choose any vertex in $V(G)$, none could satisfy the requirement in the definition.

**Theorem 3.2.** All denumerable graphs satisfying the extension property are isomorphic.

**Proof.** This is proved using a so called back and forth argument. Let $G$ and $H$ be denumerable graphs satisfying the extension property. Let $V(G) = \{v_0, v_1, \ldots\}$ and $V(H) = \{w_0, w_1, \ldots\}$. We now form an initial isomorphism $f_0$ as follows

$$G_0 := G[\{v_0\}] \cong H[\{w_0\}] =: H_0$$

Now assume we have constructed the isomorphism $f_n : G_n \rightarrow H_n$, where the isomorphic subgraphs are as follows

$$G_n = G[\{a_0, a_1, \ldots, a_n\}] \cong H[\{b_0, b_1, \ldots, b_n\}] = H_n$$

where $a_j \mapsto b_j$.

Note that the vertices $\{a_i : i \leq n\}$ and $\{b_i : i \leq n\}$ may have another indexing compared to the initial assumption. That is, we might not have $a_i = v_i$ or $b_i = w_i$ for some $i$.

We now construct the next isomorphism $f_{n+1}$. If $n$ is even, let $a_{n+1}$ be the first vertex in $V(G)$ not already chosen, according to the enumeration $\{v_0, v_1 \ldots\}$. Let
\[ G_{n+1} = G[\{a_0, a_1, \ldots, a_{n+1}\}] \]

Since \( H \) satisfies the extension property, there is a vertex \( b_{n+1} \in V(H) \setminus \{b_0, \ldots, b_n\} \) such that

\[ G_{n+1} \cong H[\{b_0, b_1, \ldots, b_{n+1}\}] = H_{n+1} \]

Specifically, we extend \( f_n \) to \( f_{n+1} \) with

\[ f_{n+1}(a_{n+1}) = b_{n+1} \]

If \( n \) is odd, let \( b_{n+1} \) be the first vertex in \( V(H) \) not already chosen, according to the enumeration \( \{w_0, w_1, \ldots\} \). Let

\[ H_{n+1} = H[\{b_0, b_1, \ldots, b_{n+1}\}] \]

As above, we find \( a_{n+1} \in V(G) \setminus \{a_0, \ldots, a_{n}\} \) such that

\[ H_{n+1} \cong G[\{a_0, a_1, \ldots, a_{n+1}\}] = G_{n+1} \]

with \( f_n \) extended to \( f_{n+1} \) and

\[ f_{n+1}(a_{n+1}) = b_{n+1} \]

We have now constructed \( f_{n+1} \). If we view all the \( f_i \) from a set theoretic view, we have \( f_i \subseteq f_{i+1} \), which can even be reformulated as \( f_i \subseteq f_j \) when \( i < j \). Thus, we may define

\[ f = \bigcup_{i \in \mathbb{N}} f_i \]

This \( f \) is an isomorphism between the entire graphs \( G \) and \( H \), which gives us

\[ G \cong H. \]

Thus it is reasonable to speak of all denumerable graphs satisfying the extension property as a single graph. This graph is called the Rado graph, sometimes the random graph since it can be constructed using the earlier mentioned coin-tossing method.

Note that the above result, Theorem 3.2, also shows that it is enough to verify the extension property for a denumerable graph to see that it is isomorphic to the Rado graph.

With the vocabulary \( \mathcal{V} = \{\sim\} \) where \( \sim \) is a binary relation symbol, we can write the extension property as a First-Order theory in the following way. That it is a theory, i.e. satisfiable, will be shown by constructing a structure that models it, i.e. by constructing the Rado graph. Let

\[ \varphi_n = \forall u_1, \ldots, u_n, w_1, \ldots, w_n \left( \bigwedge_{i, j \leq n} u_i \neq w_j \rightarrow \exists z \left( \bigwedge_{i \leq n} (u_i \sim z) \land \bigwedge_{i \leq n} \neg (w_i \sim z) \right) \right) \]

Then an "extension theory" may simply be formed as the union of these sentences

\[ (3.1) \quad T_e = \{ \varphi_n : n \in \mathbb{N} \} \]

If we add the axioms for simple graphs, we attain the following theory
(3.2) \[ T = T_e \cup \{ \forall x (x \sim x) \} \cup \{ \forall x, y ((x \sim y) \rightarrow (y \sim x)) \} \]

Since all simple, countable graphs obeying the extension property are isomorphic, \( T \) is \( \aleph_0 \)-categorical. This also implies that the deductive closure of the theory is complete, by Vaught’s Theorem. All of this simply restates what we have already seen in a different way, that we only need the sentences in \( T \) (and denumerability of the model) in order to describe the Rado graph. In particular, if we wish to evaluate whether or not a First-Order sentence is true in \( R \), we only need the axioms in \( T \) to prove or disprove the sentence.
4. Constructions

To show that the Rado graph actually exists, we construct it. In this part we will examine six ways of constructing the graph, with a seventh construction given later after some additional properties have been verified.

Two of these constructions are probabilistic, while another relies on the assumed consistency of ZF. These, especially the one relying on ZF, may therefore be more akin to existence proofs rather than constructions. However, since they are all constructions in some sense, they are presented in this chapter.

We will begin with a measure-theoretic existence proof, which we will transform into a construction. This will take up a large portion of the chapter, since we need to present a lot of measure theory to begin with.

4.1. Measure Theoretic construction. We mentioned earlier that "almost all" graphs constructed using the coin-tossing procedure, as described in the introduction, are isomorphic to the Rado graph. In order to show this formally, a measure is needed. We are specifically looking for a probability measure, since we wish to determine the probability of attaining the Rado graph. This can be thought of as the ratio between the graphs on \( \mathbb{N} \) isomorphic to the Rado graph, and all possible graphs on \( \mathbb{N} \).

Since we are dealing with uncountable sets, we cannot simply divide and find a ratio, we need to come up with a different way of measuring. The most fundamental aspect of this measure is the idea that if we would fix one edge to either be in the edge set or not, then we are in essence choosing to keep "one half" of the graphs and discarding "the other half".

The measure of the entire space will be 1, since we are constructing a probability measure. This gives us the intuition that the measure of the set of all graphs where this edge is fixed to be (or not be) in the edge set would be \( \frac{1}{2} \). Likewise, when additional edges are fixed, the measure of the remaining set is cut in half for every edge added.

One issue that is often encountered in measure theory is that it is many times impossible to find a function that measures every subset in a reasonable way. Some sets must be discarded as non-measurable in order for us to be able to correctly measure other, so called measurable, sets. The class of measurable sets form what is called a \( \sigma \)-algebra, and defining it is the first step in this section.

The measure-theoretic results in this part are gathered from the first chapter in Foundations of modern analysis by Avner Friedman [Fri82].

**Definition 4.1.** Let \( X \) be a set. Let \( \Sigma \) be a set of subsets of \( X \). If the following hold, then \( \Sigma \) is called a \( \sigma \)-algebra.

- \( \emptyset \in \Sigma \)
- \( X \in \Sigma \)
- \( A, B \in \Sigma \implies A \setminus B \in \Sigma \)
- \( \Sigma \) is closed under countable unions

The following two lemmas are good to take notice of, as they will be useful later on.

**Lemma 4.2.** \( \Sigma \) is closed under complement.

**Proof.** For the property

\[ A, B \in \Sigma \implies A \setminus B \in \Sigma \]

let \( A = X \), then for any \( B \), we have \( B^c \in \Sigma \). \( \square \)

**Lemma 4.3.** \( \Sigma \) is closed under countable intersections.

**Proof.** Using De Morgan's law and the previous lemma, we find
\[
\bigcap_{n \in \mathbb{N}} E_n = \left( \bigcup_{n \in \mathbb{N}} (E_n)^c \right)^c 
\]

Now that we know the initial properties of a class of measurable sets, we are ready to define what it means for a function to be a measure.

**Definition 4.4.** Let \( X \) be a set and \( \Sigma \) a \( \sigma \)-algebra over \( X \). \( \mu : \Sigma \to [0, \infty] \) is called a measure if it abides by the following

- \( \mu(\emptyset) = 0 \)
- **Countable additivity (or complete additivity),** for disjoint \( E_i \):
  \[
  \mu \left( \bigcup_{n \in \mathbb{N}} E_n \right) = \sum_{n \in \mathbb{N}} \mu(E_n)
  \]

As one might guess, it is the final property, countable additivity, that forces us to discard some sets as non-measurable. The following theorem might not seem like much, but it will be useful later on, when proving another theorem that relies on the measure we will construct. The idea of splitting a set in two, which is the main step in the proof, is also used again later on.

**Theorem 4.5.** If \( \mu : \Sigma \to [0, \infty] \) is a measure and \( E, F \in \Sigma \) with \( E \subseteq F \), then \( \mu(E) \leq \mu(F) \)

**Proof.** Write \( F \) as a disjoint union \( F = E \cup (F \setminus E) \). Then, by additivity,
\[
\mu(F) = \mu(E) + \mu(F \setminus E)
\]

where \( \mu(F \setminus E) \geq 0 \). Thus \( \mu(E) \leq \mu(F) \). \( \square \)

That \( \mu(F \setminus E) \) is defined is guaranteed by the definition of a \( \sigma \)-algebra.

Now that we have defined what a measure is, and found our first important property of measures, we would like to know how to construct such a function. In many cases, defining a function for uncountable sets such that the function also abides by various properties can be troublesome and tedious. We have also not yet found any easy way of determining what subsets are measurable, and if we were to construct a measure function, we would need to show that the sets that are expected to be measurable do indeed form a \( \sigma \)-algebra. To get around this, we will examine two theorems that allow us to solve these troubles in a quick way. We will begin this with another definition.

**Definition 4.6.** For a set \( X \), the function \( \mu^* : \mathcal{P}(X) \to [0, \infty] \) is an outer measure if it abides by the following

- \( \mu^*(\emptyset) = 0 \)
- \( E, F \in \mathcal{P}(X), \ E \subseteq F \implies \mu^*(E) \leq \mu^*(F) \)
- **Countable sub-additivity:**
  \[
  \mu^* \left( \bigcup_{n \in \mathbb{N}} E_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(E_n)
  \]

In total, this is a weaker definition of a measure function. However, it is defined on all subsets (of \( X \)), not just some. The reason this works is because of the weaker condition of sub-additivity that is chosen instead of complete additivity. Before we construct a measure on the set of graphs with vertices \( \mathbb{N} \), we will construct an outer measure. This allows us to construct the measure function without worrying about the part about the class of measurable sets being a \( \sigma \)-algebra, which we may solve afterwards.
Once we have an outer measure, we would like to translate this into an actual measure, which we may do, loosely speaking, by restricting the outer measure to the measurable sets of the measure function. To do this, we first need to know what a measurable set actually is.

**Definition 4.7.** For an outer measure \( \mu^* \), a set \( E \) is called \( \mu^* \)-measurable if, for any subset \( A \subseteq X \),

\[
\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)
\]

Now to the first of the two theorems in this step. This theorem is what allows us to later on ignore the struggle of verifying the properties of \( \sigma \)-algebras.

**Theorem 4.8.** Let \( \mu^* \) be an outer measure and let \( \Sigma \) be the class of all \( \mu^* \)-measurable sets. Then \( \Sigma \) is a \( \sigma \)-algebra and the restriction of \( \mu^* \) to \( \Sigma \) is a measure.

**Proof.** See Friedman [Fri82]. □

So far we have seen what a measure is, and how to turn an outer measure into a measure. The question that remains to answer for this part is how to construct an outer measure in the first place. The idea described in the introduction to this section is a solid one to begin with, but there are many subsets that we would need to measure whose measure is significantly harder to define. To get around this issue, we will use what is called a sequential covering class. This lets us define the measure function on a select few of the subsets, and then more easily define it on the rest of the subsets.

**Definition 4.9.** A sequential covering class \( K \) is a class of subsets of a set \( X \) such that \( \emptyset \in K \) and for every \( A \subseteq X \) there is a sequence of \( E_i \in K \) such that \( A \subseteq \bigcup_{i \in \mathbb{N}} E_i \).  

**Theorem 4.10.** Let \( K \) be a sequential covering class of \( X \) and let \( \lambda : K \to [0, \infty] \) with \( \lambda(\emptyset) = 0 \). For each \( A \subseteq X \), let

\[
\mu^*(A) = \inf \left\{ \sum_{n \in \mathbb{N}} \lambda(E_n) : E_n \in K, A \subseteq \bigcup_{n \in \mathbb{N}} E_n \right\}
\]

Then \( \mu^* \) is an outer measure.

**Proof.** See Friedman [Fri82]. □

This is the theorem we will use to define the outer measure. As described in the introduction to this chapter, an important property of the measure will, loosely speaking, be that if we wish to measure the set of all graphs with one edge fixed to be (or not be) in the edge set, then the measure should be \( \frac{1}{2} \). If two edges are fixed, then it should be \( \frac{1}{4} \). If all edges between \( n \) vertices are fixed, then the measure should be \( \left( \frac{1}{2} \right)^{\binom{n}{2}} \). There are thus a few subsets which we now have some understanding of how to define the (outer) measure on. These will form the sequential covering class, and the class of all subsets that can be formed using this covering class in combination with the various properties of a \( \sigma \)-algebra will be contained in the class of measurable sets of this outer measure. For instance, a countable union of the sets described above will be in the class of measurable sets. Thus, so long as we can show that a set is constructable using the sets in this sequential covering class together with said properties, we know that it is measurable.
Definition 4.11. Let $\Omega$ be the set of all graphs with vertex set $\mathbb{N}$. For a finite graph $G$ with $V(G) \subseteq \mathbb{N}$, let $[G]$ be the set of all $\mathcal{M} \in \Omega$ such that $\mathcal{M}$ restricted to $V(G)$ is exactly $G$. Let

$$\mathcal{R} = \{\emptyset\} \cup \{[G] : V(G) \subseteq \mathbb{N}, |V(G)| < \omega\}$$

Now $\mathcal{R}$ is a sequential covering class of $\Omega$. To see this, let $V(G) = \emptyset$, which gives $[G] = \Omega$. We may define an outer measure $\mu^*$ on $\Omega$ in the following way. Let $G$ be a finite graph with $V(G) \subseteq \mathbb{N}$, let $A \subseteq \Omega$ and let $\lambda: 2^\Omega \to [0,1]$. Note that we use 1 rather than $\infty$ as the upper limit, since we are constructing a probability measure. Let

$$\lambda(\emptyset) = 0$$

$$\lambda([G]) = \left(\frac{1}{2}\right)^{|V(G)|}$$

$$\mu^*(A) = \inf\left\{ \sum_{n \in \mathbb{N}} \lambda(E_n) : E_n \in \mathcal{R}, A \subseteq \bigcup_{n \in \mathbb{N}} E_n \right\}$$

By theorem 4.10, $\mu^*$ is an outer measure. Thus it abides by countable sub-additivity. In this particular case, it means that for a sequence of sets of graphs $(E_n)_{n \in \mathbb{N}}$ with $E_n \subseteq \Omega$ for all $n \in \mathbb{N}$ we have

$$\mu^* \left( \bigcup_{n \in \mathbb{N}} E_n \right) \leq \sum_{n \in \mathbb{N}} \mu^*(E_n)$$

We may now form a measure $\mu$ by restricting $\mu^*$ to the measurable sets in $2^\Omega$. As defined above, the measurable sets are precisely the sets $E$ such that for any set $A$, the following holds.

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \setminus E)$$

Since $\mu$ is a measure, it has countable additivity. Let $\Sigma$ denote the set of $\mu^*$-measurable sets. Then for any sequence of pairwise disjoint sets $(E_n)_{n \in \mathbb{N}}$ with $E_n \in \Sigma$ for all $n \in \mathbb{N}$,

$$\mu \left( \bigcup_{n \in \mathbb{N}} E_n \right) = \sum_{n \in \mathbb{N}} \mu(E_n)$$

Should the sets not be pairwise disjoint, then countable sub-additivity holds instead of countable additivity.

The next step for us to tackle is to understand what sets we are actually interested in measuring. The sets we used for the sequential covering class were the least complicated ones to define the measure on, which is why they were used, but they are generally not the sets we will be interested in later on. Instead, we will examine sets that may be constructed from those sets, and by showing that they can be constructed, we will both gain some understanding of their measure, as well as verifying that they are indeed measurable.

Definition 4.12. For finite, pairwise disjoint $U, W, A \subseteq \mathbb{N}$, let $E(U, W, A)$ be the set of all $\mathcal{M} \in \Omega$ such that there is no $a \in A$ with the property $\forall u \in U (a \sim u) \land \forall w \in W (a \sim w)$.

Let $E(U, W)$ be the set of all $\mathcal{M} \in \Omega$ such that there is no $a \in \mathbb{N} \setminus (U \cup W)$ with $\forall u \in U (a \sim u) \land \forall w \in W (a \sim w)$. 
We may first see that for all finite, disjoint \( U, W, A \subset \mathbb{N} \),
\[
E(U, W) \subseteq E(U, W, A)
\]

Now we will show that the sets \( E(U, W) \) and \( E(U, W, A) \) are measurable.

Let \( D \) be the set of all \( G \) such that \( V(G) = U \cup W \cup A \) and such that there is no \( a \in A \) with the property \( \forall u \in U \ (a \sim u) \land \forall w \in W \ (a \sim w) \). There are finitely many edge configurations between the vertices in \( U \cup W \cup A \), and thus when we restrict us to graphs where there is no \( a \) such that the above holds, we see that \( D \) must be finite. If we then form \( D' = \bigcup_{G \in D} [G] \), we see that \( D' \) is a finite union of sets on the form \( [G] \).

Since a set \( [G] \) is measurable for a finite \( G \) we find that \( D' \) must be measurable. We now note that \( D' = E(U, W, A) \).

For the next part, we may simply form a sequence of \( D'_i \) for \( i \in \mathbb{N} \) by constructing \( D'_i \) in the above way, using \( U, W, A_i \), where the sequence of \( A_i \) is such that
\[
\bigcup_{i \in \mathbb{N}} A_i = \mathbb{N} \setminus (U \cup W)
\]

Then we may see that
\[
\bigcap_{i \in \mathbb{N}} D'_i = E(U, W)
\]

Since the measurable sets are closed under countable intersection by lemma (4.3), \( E(U, W) \) is measurable.

Thus, since all \( E \) defined this way are \( \mu^* \)-measurable, we have
\[
\mu(E(U, W)) \leq \mu(E(U, W, A))
\]

We may now show that we can bound \( \mu(E(U, W)) \) by taking larger and larger \( A \), where \( \mu(E(U, W, A)) \) will tend to zero. Firstly, the probability that any one \( a \) does not meet the criteria \( \forall u \in U \ (a \sim u) \land \forall w \in W \ (a \sim w) \) is
\[
1 - \left( \frac{1}{2} \right)^{|U \cup W|}
\]

Thus, if we examine an arbitrarily large, finite set \( A \) of vertices we find
\[
\mu(E(U, W, A)) = \left( 1 - \left( \frac{1}{2} \right)^{|U \cup W|} \right)^{|A|}
\]

Note the hidden independence argument in this part. If we fix the edges and non-edges between some \( a \) and vertices in \( U \cup W \), the set of these edges will be disjoint from a similar set of edges formed with some other \( a \). Therefore, the event of some \( a \) meeting or not meeting the criteria is not dependent on other \( a \) meeting or not meeting the criteria. Thus, we may find the measure for the entire \( A \) by simply taking the product of all the measures for the individual \( a \).

If we inspect the sequence \( \mu(E(U, W, A_i)) \) where the cardinality of the sets \( A_i \) in the sequence of finite sets \( (A_i)_{i \in \mathbb{N}} \) is unbounded, we only need to note that \( 0 < 1 - \left( \frac{1}{2} \right)^{|U \cup W|} < 1 \). Using this we arrive at the conclusion that \( \mu(E(U, W, A_i)) \) can take a value arbitrarily close to zero, provided that \( A_i \) is large enough. Since
\[
\mu(E(U, W)) \leq \mu(E(U, W, A_i))
\]

for all \( A_i \), we find that we can bound \( \mu(E(U, W)) \) with values arbitrarily close to zero, and since \( \mu \) is non-negative, we get \( \mu(E(U, W)) = 0 \).
Since all $U, W$ examined are finite and their elements are chosen from a countable set, the set of all possible disjoint combinations of such $U, W$ is countable. We may thus use countable sub-additivity, since the sets on the form $E(U, W)$ might not be disjoint, and we find

$$\mu \left( \bigcup_{U \cap W = \emptyset \atop |U \cup W| < \aleph_0} E(U, W) \right) \leq \sum_{U \cap W = \emptyset \atop |U \cup W| < \aleph_0} \mu(E(U, W)) = \sum_{i \in \mathbb{N}} 0 = 0$$

Thus

$$\mu \left( \bigcup_{U \cap W = \emptyset \atop |U \cup W| < \aleph_0} E(U, W) \right) = 0$$

Since the measurable sets are closed under complement and countable intersection, we arrive at

$$\mu \left( \left( \bigcup_{U \cap W = \emptyset \atop |U \cup W| < \aleph_0} E(U, W) \right)^c \right) = \mu \left( \bigcap_{U \cap W = \emptyset \atop |U \cup W| < \aleph_0} E(U, W)^c \right) = 1$$

How should we interpret this? The last equation shows that the set of all graphs on $\mathbb{N}$, such that for any finite, disjoint $U, W \subseteq \mathbb{N}$ there is a vertex adjacent to all $u \in U$ but no $w \in W$, has measure 1. Also note that since the graphs satisfying the extension property 3.1 has measure 1, the Rado graph does indeed exist.

We may now transform this existence proof into an actual construction, which will be the same as the one presented in the introduction. Let $G$ be a graph with $V(G) = \mathbb{N}$. Let any edge between two vertices be in $E(G)$ with probability $\frac{1}{2}$, independently of each other.

**Theorem 4.13.** $G$, as constructed above, satisfies the extension property 3.1 with probability 1.

**Proof.** Since every graph on $\mathbb{N}$ is equally likely to be obtained, and since the measure of graphs on $\mathbb{N}$ that satisfies the extension property 3.1 has measure 1, from the previous section, we will, with probability 1, attain the Rado graph.

□

Since the Rado graph does exist, the theory $T$ from equation (3.2) is satisfiable. This construction is also the reason the Rado graph is sometimes called the random graph. Though it is not shown here, the above theorem holds for any probability of edges strictly between zero and one. That is, for any edge, we can let it be in $E(G)$ with probability $p$, where $0 < p < 1$, as long as this selection is independent for all edges, and we use the same $p$ for all edges.

### 4.2. Coin-tossing for multiple edges at once

For a rather similar construction, we form a graph on $\mathbb{N}$ in the following way. Let $S \subseteq \mathbb{N} \setminus \{0\}$ be chosen randomly, such that $P(n \in S) = \frac{1}{2}$ for any $n \in \mathbb{N} \setminus \{0\}$, independently of each other. Let $x \sim y \iff |x - y| \in S$. This graph satisfies the extension property 3.1 with probability 1.

To be more specific regarding the random selection, we may use a similar argument in creating a probability measure as for the previous construction.
Since every graph attainable can be represented by a subset of natural numbers, not including zero, we wish to define this measure on sets of subsets of natural numbers. That is, the outer measure function will be defined on subsets of \( P(\mathbb{N} \setminus \{0\}) \). If any argument seems unclear, please check the reasoning for the previous construction, since many parts are reused.

For finite, disjoint sets of natural numbers \( X \) and \( Y \), define

\[
[X,Y]^* = \{ S \subseteq \mathbb{N} : X \subseteq S, S \cap Y = \emptyset \}
\]

Then we may use the set

\[
\mathcal{L} = \{\emptyset\} \cup \{ [X,Y]^* : X,Y \subseteq \mathbb{N}, X \cap Y = \emptyset, |X \cup Y| < \aleph_0 \}
\]

as our sequential covering class.

We now define an outer measure \( \mu^* \) as follows. Let \( X,Y \subseteq \mathbb{N} \) be finite and disjoint. Let \( A \subseteq P(\mathbb{N} \setminus \{0\}) \). Let \( \lambda : P(P(\mathbb{N} \setminus \{0\})) \to [0,1] \). Let

\[
\lambda(\emptyset) = 0
\]

\[
\lambda([X,Y]^*) = \left( \frac{1}{2} \right)^{|X|+|Y|}
\]

\[
\mu^*(A) = \inf \left\{ \sum_{n \in \mathbb{N}} \lambda(E_n) : E_n \in \mathcal{L}, A \subseteq \bigcup_{n \in \mathbb{N}} E_n \right\}
\]

Now let \( \mu \) be the restriction of \( \mu^* \) to the set of measurable sets.

Moving forward, we will now describe a way of coding the graphs introduced in the beginning of the section dedicated to this construction. Since we only need to know which numbers are in \( S \), every possible graph we may attain can be described using an infinite sequence \( s^* \) of 0’s and 1’s, where we let the \( i \)th place in the sequence be a 1 if \( i \in S \), and 0 otherwise. We now have an isomorphism between the set of graphs we may attain, and denumerable sequences of 0’s and 1’s.

How can we translate this back to our measure? If we construct a corresponding measure \( \tilde{\mu} \) from the above mentioned isomorphism, we find that if we fix a total of \( n \) places with 0’s or 1’s in a sequence and measure the set of all sequences abiding by this restriction, the measure of this set will be \( \left( \frac{1}{2} \right)^n \).

For the next part, let us assume that we have constructed a graph \( G \) with sequence \( s^* \) in the above way, and have been given some finite, disjoint \( U,W \subseteq \mathbb{N} \). In order to find a way of verifying whether there is some \( z \) correctly joined to \( U,W \), we wish to describe this situation in a different way.

Let’s say that we choose some \( z \in \mathbb{N} \setminus (U \cup W) \) and examine it. In order for \( z \) to be correctly joined, we must have that \( |z - u| \in S \) for all \( u \in U \) and \( |z - w| \notin S \) for all \( w \in W \). This is the same as saying that \( s^* \) has a 1 in place \( |z - u| \), for all \( u \in U \), and a 0 in place \( |z - w| \), for every \( w \in W \). If we repeat this examination for other vertices, the question of finding some vertex that is correctly joined simply comes down to finding a finite sequence of 1’s and 0’s (albeit with some potential holes) in the infinite sequence \( s^* \).

Remember that the vertices are natural numbers. If we let \( l = 1 + \max(U \cup W) \), and examine vertices \( z_1, z_2, \ldots \) where \( z_i = i \cdot l \), we have guaranteed that the difference between these \( z_i \) is large enough for any two examinations to be independent. That is, the event
of \( z_i \) being correctly joined and the event of \( z_j \) being correctly joined are independent for \( i \neq j \).

Now let \( U \) and \( W \) be as above and let \( n \in \mathbb{N} \). Using the above way to define the set \( \{ z_i : i \in \mathbb{N} \} \), let \( E(U, W, n) \) be the set of sequences where no \( z_i \) for \( i \leq n \) is correctly joined to these \( U, W \). This set can be formed as a countable intersection of sets \( B_i \), where each \( B_i \) is the set of sequences with a 1 in place \( z_i - u \) for all \( u \in U \) and a 0 in place \( z_i - w \) for all \( w \in W \). For this particular situation, note that all \( z_i \) are larger than all \( u \in U \) and \( w \in W \).

Let \( E(U, W) \) be the set of sequences where no \( z_i \) is correctly joined for all \( i \in \mathbb{N} \). We can see that

\[
E(U, W) = \bigcap_{n \in \mathbb{N}} E(U, W, n)
\]

which tells us that \( E(U, W) \) is measurable, and

\[
E(U, W) \subseteq E(U, W, n)
\]

for all \( n \in \mathbb{N} \).

Using our definition of \( \tilde{\mu} \), we may find

\[
\tilde{\mu}(E(U, W, n)) = \left( 1 - \left( \frac{1}{2} \right)^{|U \cup W|} \right)^n.
\]

This expression tends to 0 as \( n \) tends to infinity, and since we also have

\[
\tilde{\mu}(E(U, W)) \leq \tilde{\mu}(E(U, W, n))
\]

for all \( n \in \mathbb{N} \), we find

\[
\tilde{\mu}(E(U, W)) = 0
\]

which tells us that for every pair of finite, disjoint \( U, W \subseteq \mathbb{N} \), the measure of the set of all graphs attainable in the above way, such that every \( z_i \) fails to be correctly joined is 0. By countable sub-additivity, the measure of the set of attainable graphs such that there is some finite, disjoint \( U, W \subseteq \mathbb{N} \) where all \( z_i \) fails is 0. Since the measurable sets are closed under complement and countable intersections, the set of attainable graphs where every finite, disjoint \( U, W \subseteq \mathbb{N} \) has at least one \( z_i \) correctly joined is 1. Thus the measure of attainable graphs satisfying the extension property 3.1 is 1. We may thus say that the extension property is satisfied with probability 1.

4.3. Set Theoretic construction. For a set theoretic construction, we may examine a countable model of set theory. The axioms of set theory can be found in anything from books to articles and so on. I have decided to use Combinatorial Set Theory by Lorenz J. Halbeisen [Hal12] as a reference for these. This is mainly since the names of the axioms can vary depending on the author, and inconsistency would make the paper unreadable.

Let \( \mathcal{M} \) be a countable model of set theory, with universe \( M \), which existence is guaranteed by the Downward-Löwenheim-Skolem Theorem, see Hedman [Hed01]. The vocabulary consists of a single relation symbol, \( \{ E \} \), which denotes membership. We may say that \( aEb \) represents \( a \in b \).

When we say that \( \mathcal{M} \) is a model for set theory, we mean that it satisfies the set of First Order sentences that describe the axioms of ZF. We use the equality symbol ‘=’, and note that for this construction, we do not need the Axiom of Choice. We do, however, need to assume that ZF is consistent. Otherwise, such a model \( \mathcal{M} \) cannot exist.

For elements \( x, y \in M \), form the graph \( G_M \) with vertex set \( M \) and let \( x \sim y \iff xEy \lor yEx \).
In this construction, we must take care to not confuse elements of $\mathcal{M}$ with sets. $\mathcal{M}$ is a model for set theory, so the elements in $\mathcal{M}$ will represent sets in a set theoretic universe. However, they are not necessarily sets themselves, and we may therefore not treat them as such. When we say "$a$ represents $A$", we mean that $bEa \iff B \in A$, where $b$ represents $B$ and $\emptyset$ is represented by the element $e \in M$ such that for all $b \in M$, $(b \notin e)$. This distinction between sets and their representatives is important, because in different models of set theory, the axioms of set theory will appear to behave differently. For instance, from the Downward-Löwenheim-Skolem Theorem we know that there is a countable model for set theory.

The Axioms of Empty Set, Pairing, Separation, Union and Infinity guarantees that such a model contains an element $a$ with $bEa$ for infinitely many $b$. This element, $a$, seems to contradict the Axiom of Power Set, since the power set of an infinite set is uncountable, and one might think that all the subsets that are elements of the power set should have representations in $\mathcal{M}$.

However, this is actually not true. The Axiom of Power set guarantees that if an element $b$ is a representation of a subset of $A$, then it will be related to $a$ by $bEa$. This assumes that such a $b$ is actually a member of $\mathcal{M}$ in the first place, but if a subset of $A$ cannot be properly defined by a First-Order formula, we cannot be sure that said subset even has a representation in $\mathcal{M}$ at all. In that case, the Axiom of Power set does not say anything at all about said subset, since the axiom only concerns elements of $\mathcal{M}$.

If we were to instead inspect an uncountable model for set theory, the Axiom of Power set applied to an element representing an infinite set could yield an element representing an uncountable set, with uncountably many elements related to it. This is, however, not what we will be deepening our understanding of today. All in all, the existence of a countable model for set theory is not paradoxical, however, we must take care to distinguish between sets and their representatives, since we might otherwise be confused by the behaviour of the axioms.

To get to something more important, we will see that all finite sets of elements in $\mathcal{M}$ have representations in $\mathcal{M}$. For this, we use the following argument. For any finite $S \subseteq M$ we may use the Axiom of Pairing and Axiom of Separation to see that for every $b \in S$, there is $b^*$ representing $\{B\}$, where $b$ represents $B$. If we apply the Axioms of Pairing and Separation again, we may see that for $b_1, b_2 \in S$ representing $B_1, B_2$, we have $c \in M$ representing $C = \{B_1, B_2\}$. Using the Axiom of Union shows that $\{B_1, B_2\}$ has a representative in $\mathcal{M}$. Similarly, we may see that $\{B_3\}$ has a representative, which may be used to show that $\{\{B_1, B_2\}, \{B_3\}\}$ has a representative. Again, the Axiom of Union shows that $\{B_1, B_2, B_3\}$ has a representative. Using this argument inductively shows that every finite $A \subseteq M$ has a representative $a \in M$.

Now to the part of actually verifying the extension property. As explained earlier, we work with the graph $G_M$ where $x \sim y \iff xEy \lor yEx$.

Let $U, W \subseteq M$ be finite and disjoint. Since all finite sets have representations, we will be able to find elements in $\mathcal{M}$ that represents $U$ and $W$, which will be denoted by $z_1$ and $z_2$, respectively. That is,

$$x Ez_1 \iff x \in U$$
$$x Ez_2 \iff x \in W.$$  

Using the Axiom of Union, Pairing, and Separation yields the existence of $z \in M$ such that

$$xEz \iff xEz_1 \lor x = z_2$$

Informally we may write
This $z$ is the vertex we are looking for, namely the one which is adjacent to all $u \in U$ but no $w \in W$. Now to verifying it. We have $uEz$ for all $u \in U$. Thus, for any $u \in U$, we have $u \sim z$. The next part is just a tiny bit more tricky. We will show that for any $w \in W$, $w \not\sim z$. Let $w \in W$. If $wEz$, then there are two cases to examine. The first possibility is that $w = u$ for some $u \in U$, which contradicts our assumption that $U$ and $W$ are disjoint. The other option is that $w = z_2$, but then $w \in w$, contradicting the Axiom of Foundation that guarantees that no set can be a member of itself. If instead $zEw$, then $wEzEw$, which would imply that there is a set $Z$ such that $w \in W \in Z \in w$. This would give rise to an infinite descending sequence of sets, namely $\ldots w \in Z \in w \in W \in Z \in w$, which again contradicts the Axiom of Foundation. This yields $w \in W \implies \neg(wEz) \land \neg(zEw)$. Thus $z \sim u$ for all $u \in U$ and $z \not\sim w$ for all $w \in W$. This shows that the graph $G_M$ satisfies the extension property \[3.1\].

4.4. **Binary representation.** Another way of constructing the Rado graph is by examining the binary representation of natural numbers. Let the vertex set be the natural numbers and let $i < j$ be adjacent if and only if $a_i = 1$ where $a_k \in \{0, 1\}$ for all $k \in \mathbb{N}$ when we write $j$ as a sum of powers of 2 with coefficients $a_k$:

$$j = \sum_{k \in \mathbb{N}} a_k 2^k$$

To verify the extension property we examine $U, W \subseteq \mathbb{N}$, finite, disjoint

$$U = \{u_1, \ldots, u_m\}$$

$$W = \{w_1, \ldots, w_n\}$$

Let

$$l = 1 + \max(U \cup W)$$

and

$$r = 2^l + \sum_{i=1}^m 2^{u_i}$$

Now $r$ is adjacent to all $u_i \in U$ and no $w_i \in W$. Thus, this graph satisfies the extension property \[3.1\].

4.5. **Quadratic reciprocity.** This next construction uses quadratic reciprocity on a subset of the primes. The book *An Introduction to the Theory of Numbers* by Ivan Niven, Herbert Zuckerman and Hugh Montgomery [NZM91] is a good source for anyone seeking a deeper understanding of number theory and quadratic reciprocity. If nothing else is stated, number theoretic results used in this part can be found there.

For odd prime numbers $p, q$, we examine the equation

$$x^2 \equiv p \mod q$$

If this equation has a solution, then we say that $p$ is a quadratic residue mod $q$. This is usually written as $\left( \frac{p}{q} \right) = 1$. The theorem for quadratic reciprocity is as follows

\[ \left( \frac{p}{q} \right) \left( \frac{q}{p} \right) = (-1)^{\frac{p-1}{2} \cdot \frac{q-1}{2}} \]

The proof of this theorem is beyond the scope of this paper. For primes congruent to 1 (mod 4), the theorem gives

\[ \left( \frac{p}{q} \right) = 1 \iff \left( \frac{q}{p} \right) = 1 \]

Let \( \mathbb{P} \) be the primes congruent to 1 (mod 4). We now construct a graph \( G \) with nodes \( \mathbb{P} \) and let \( p \sim q \) if and only if \( \left( \frac{p}{q} \right) = 1 \).

To show that \( G \) satisfies the extension property 3.1 we let \( U = \{ u_1, \ldots, u_m \} \) and \( W = \{ w_1, \ldots, w_n \} \), and for every \( u_i \) we fix a quadratic residue \( a_i \), and for every \( w_i \), we fix a quadratic non-residue \( b_i \). Since 1 is always a quadratic residue, it is easy to see that these \( a_i \) do exist. That the \( b_i \) also exist follows from the fact that there are an equal number of quadratic residues and quadratic non-residues for primes, excluding zero. We will also not choose any \( a_i \) or \( b_i \) to be zero. Since all \( u_i \) and \( w_i \) are primes, we may use the Chinese Remainder Theorem with these, and since they are all odd, we may add another equation with modulus 4. The Chinese Remainder Theorem then states that there is a solution to the following equation.

\[ x \equiv 1 \mod 4, \quad x \equiv a_i \mod u_i, \quad x \equiv b_i \mod w_i \]

This solution, \( x \), will be coprime with both 4 and all \( u_i \) and \( w_i \). It will be unique modulo the product of all these, call it \( y \).

For the next step, we need Dirichlet’s Theorem for prime numbers, which reads as follows.

Theorem 4.15. For coprime integers \( a \) and \( d \), there are infinitely many primes on the form \( a + nd \)

The proof for this theorem is also beyond the scope of this paper. If one wishes to read more about this topic in particular, there is a chapter in the book *Introduction to Analytic Number Theory* by Tom Apostol [Apo76] dedicated to proving the theorem. Since \( x \) and \( y \) are coprime, we may find a prime \( z \) on the form \( x + ny \), which is precisely the vertex we are looking for. This shows that the desired vertex \( z \) actually exists, and the extension property 3.1 holds.

4.6. First-Order Logic. Now since \( R \) is attained with probability 1 by the random construction in Theorem 4.13, we have a rather straight-forward intuition for the zero-one laws for graphs.

The previous constructions that we have seen are existence proofs as much as they are constructions. The following construction, as it is done here, does rely upon the assumption that \( R \) actually exists, and therefore only works as another way of construction, and not as a proof of existence. It could probably be modified to be a proof of existence as well, but that is not done in this paper.

We say that a First-Order sentence \( \theta \) is true in almost all finite graphs if the probability of \( \theta \) being true tends to one as the number of vertices go to infinity. More formally, we have the following definition.

**Definition 4.16.** Let \( \theta \) be a First-Order sentence. Let \( B_n \) be the set of graphs with vertex set \( \{ 1, \ldots, n \} \) and let \( B_n^\ast \) be the set of such graphs where also \( \theta \) is true. We say that \( \theta \) is true in almost all finite graphs if the following hold
Theorem 4.17. Let $\Phi$ be the set of First-Order sentences that are true in almost all finite graphs. Then $\Phi$ is a theory and $R$ models $\Phi$.

Proof. This can be shown by verifying that a sentence $\theta$ is true in almost all finite graphs if and only if it is true in $R$.

Recall the extension axioms $T_e = \{ \phi_n : n \in \mathbb{N} \}$ as described in equation (3.1), and recall that these were the only ones we needed to define $R$ using FO-logic, provided that we work with simple graphs. (If one wishes to be extra clear, one can add the axioms for simple graphs and use $T$ as in (3.2).

When we talk about the probability of a sentence being true in a finite graph of some size, we examine all graphs with a specific vertex set of that size, for instance the vertex set $\{1, \ldots, n\}$ for size $n$. Then the probability of the sentence being true is simply the probability of choosing a graph where the sentence is true, and the probability of choosing any one graph is the uniform probability. That is, for graphs of size $n$ with a given vertex set, the probability of choosing any one such graph $G$ is

$$P(G) = \frac{1}{|B_n|} = \frac{1}{2^{\binom{n}{2}}} = \left(\frac{1}{2}\right)^{\binom{n}{2}}.$$

We shall first see that any of these $\varphi_n$ is true in almost all finite graphs. The probability that it fails in a graph with $N$ vertices can be bounded by $N^{2n}(1 - \frac{1}{2^n})^{N-2n}$. This is because there is not more than $N^{2n}$ ways of choosing $2n$ distinct points, and the second part of the expression is a bound for the probability that no other point is correctly joined. To see that we can use the same $n$ for both $U$ and $W$ while still maintaining this bound on the probability, we may simply see that if we have a set $U$ of $n$ distinct points and another set $W$, also of $n$ distinct points, we may remove some of the points from one of the sets (for instance by letting $u_i = u_j$ for some $i \neq j$), and the probability that there is no vertex correctly joined will certainly not increase.

Since $N^{2n}(1 - \frac{1}{2^n})^{N-2n}$ tends to $0$ as $N \to \infty$, we see that $\varphi_n$ is true in almost all finite graphs.

If $\theta$ can be logically deduced from $T_e$, then a finite subset $\Sigma$ of $T_e$ is sufficient to prove $\theta$, since FO proofs are finite. Since $\Sigma$ is finite and all its sentences are true in almost all finite graphs, $\theta$ must also be true in almost all finite graphs.

For the other way around, we may simply see that if $\theta$ cannot be deduced from $T_e$, then $\theta$ cannot be true in $R$. Then $\neg \theta$ is true in $R$ and can be deduced from $T_e$, and using the above argument, we find that $\neg \theta$ holds in almost all finite graphs. Specifically, $\theta$ does not hold in almost all finite graphs. This shows that $\Phi$ is precisely the sentences true in $R$, and thus, $\Phi$ is a theory and $R$ models $\Phi$.

Note that the above proof shows that $\Phi$ is complete. Since both $\Phi$ and $T$ are modeled by $R$, we find that $\bar{T} = \Phi$.

Something else that is interesting about the above argument is that it shows that there is a zero-one law for finite graphs. Even though it is not of direct importance to this paper, I will still state it for its general significance.

Theorem 4.18. Let $\theta$ be a First-Order sentence of the language of graph theory. Then either $\theta$ is true in almost all finite graphs, or false in almost all finite graphs.
5. Additional properties

This section will focus on other properties of the Rado graph, rather than the most fundamental one. For instance, due to the extension property, we can always choose an induced subgraph of $R$, perhaps only one vertex, and extend it with additional vertices from $R$ in any way we want. Since this can go on indefinitely, we may find any graph of countable cardinality as an induced subgraph of $R$. We may describe this as saying that the Rado graph contains all countable graphs.

The other major part of this chapter is to state enough definitions and show the required results for a final construction of $R$, which is as the Fraïssé Limit of all finite graphs. What that is will be explained in more detail as we get to that part, but it is closely related to the fact that $R$ contains all finite graphs as induced subgraphs.

**Theorem 5.1.** Let $G$ be a simple graph of cardinality at most $\aleph_0$. Then there is an induced subgraph $H$ of $R$ such that

$$G \cong H$$

**Proof.** We may use a similar proof as for Theorem 3.2

Let $G$ be a graph with $V(G) = \{v_0, v_1, \ldots\}$. Let

$$G_0 = G[\{v_0\}]$$

and embed $G_0$ into $R$ such that we for some vertex $w_0 \in V(R)$ have that

$$v_0 \mapsto w_0$$

is an isomorphism, namely $f_0 : G[\{v_0\}] \to R[\{w_0\}]$. This yields

$$G_0 \cong H_0 := R[\{w_0\}]$$

Assume $G_k$ and $H_k$ have been constructed such that

$$G_k = G[\{v_0, \ldots, v_k\}] \cong H_k := R[\{w_0, \ldots, w_k\}]$$

such that

$$f_k : G_k \to H_k$$

is an isomorphism. Then, by the extension property, there is $w_{k+1} \in V(R) \setminus \{w_0, \ldots, w_k\}$ such that we may extend the isomorphism $f_k$ to $f_{k+1} : G_{k+1} \to H_{k+1}$ where

$$G_{k+1} := G[\{v_0, \ldots, v_k, v_{k+1}\}] \cong H_{k+1} := R[\{w_0, \ldots, w_{k+1}\}]$$

Thus we may define

$$f = \bigcup_{i \in \mathbb{N}} f_i$$

which will be an isomorphism defined on the entire graph $G$ with an induced subgraph $H$ of $R$ as its image. Specifically, we find

$$G \cong H$$

where $H$ is an induced subgraph of $R$. □

**Definition 5.2.** The age of a structure is the class of all its finitely generated substructures.
Remember that for graphs, finitely generated substructure is the same as finite, induced subgraph. Thus it is not even necessary to examine any infinite case to see that the finitely generated substructures of \( R \) are precisely all finite graphs, since \( R \) contains all finite graphs as induced subgraphs. Thus, the age of \( R \) is simply the class of all finite graphs.

**Definition 5.3.** A structure \( D \) is ultrahomogeneous if every isomorphism between finitely generated substructures of \( D \) extends to an automorphism on \( D \).

**Theorem 5.4.** The Rado graph is ultrahomogeneous.

*Proof.* In the proof for Theorem 3.2, where we showed that two denumerable graphs satisfying the extension property are isomorphic, we began with one vertex from each of the two graphs and the (only) isomorphism between them. However, the argument is not dependent on the number of vertices present in the initial isomorphism, the only requirement is that there is an isomorphism. Thus, if we begin with an isomorphism between two finite, induced subgraphs of \( R \), we can simply use the back-and-forth argument from there to construct an automorphism. \( \square \)

For the rest of the chapter, let \( K \) be a non-empty finite or countable class of finitely generated structures of some vocabulary \( L \). We will now state some more definitions, these are used mainly for the purpose of setting up requirements for a class so that we may be sure that it is an age of some structure.

**Definition 5.5.** \( K \) has the Hereditary Property (HP) if the following holds. If \( A \in K \) and \( B \) is a finitely generated substructure of \( A \) then \( B \) is isomorphic to a structure in \( K \).

**Definition 5.6.** \( K \) has the Joint Embedding Property (JEP) if the following holds. If \( A, B \in K \) then there is \( C \in K \) such that both \( A \) and \( B \) are embeddable in \( C \).

**Definition 5.7.** \( K \) has the Amalgamation Property (AP) if the following holds. If \( A, B, C \in K \) and \( e : A \rightarrow B, f : A \rightarrow C \) are embeddings, then there are \( D \in K \) and embeddings \( g : B \rightarrow D \) and \( h : C \rightarrow D \) such that \( ge = hf \).

We will now state Fraïssé’s Theorem, see Hodges [Hod93], and use it for a final construction of the Rado graph.

**Theorem 5.8.** Let \( L \) be a countable vocabulary and let \( K \) have HP, JEP, AP. Then there is an \( L \)-structure \( D \), unique up to isomorphism, satisfying the following

1. \( D \) has cardinality \( \leq \aleph_0 \)
2. \( K \) is the age of \( D \)
3. \( D \) is ultrahomogeneous

The structure \( D \) guaranteed by Fraïssé’s Theorem is usually called the Fraïssé limit of \( K \).

We will now show that the class of all finite graphs, from now on referred to as \( K \), satisfies HP, JEP, and AP.

HP is rather straightforward to show. Since every finitely generated substructure of a graph itself is a finite graph, it does not matter which finite graph \( A \) we examine. Every \( B \) that is a finitely generated substructure of \( A \) will be in \( K \), since \( K \) contains all finite graphs.

To show JEP, we may, for any \( A, B \in K \), simply form a new graph \( C \) with vertex set \( V(A) \cup V(B) \), where \( C \) restricted to \( V(A) \) is equivalent (or at least isomorphic) to \( A \) and \( C \) restricted to \( V(B) \) is equivalent (or at least isomorphic) to \( B \). Then it is clear that both \( A \) and \( B \) are embeddable in \( C \) and \( C \) is a finite graph, so \( C \in K \).

We finally verify AP. Let \( A, B, C \in K \) and let \( A \) be embeddable into \( B \) by embedding \( e \) and let \( A \) also be embeddable into \( C \) by embedding \( f \). Form a new graph \( D \) with
vertex set $V(B) \cup S$ where $|S| = |V(C)| - |V(A)|$ with $S$ and $V(B)$ disjoint. Now, let $D$ be equivalent to $B$ on $V(B)$ and let $D$ be isomorphic to $C$ on $e(A) \cup S$. This is possible since $A$ could be embedded in both $B$ and $C$, and since $V(B)$ and $S$ are disjoint, we may choose the edges between vertices in $S$ and edges between $S$ and $e(A)$ arbitrarily. In particular, we can make sure that $D$ restricted to $e(A) \cup S$ is isomorphic to $C$.

We have now concluded that the class of all finite graphs, $K$, satisfies $HP$, $JEP$, and $AP$.

Since $R$ is countable, ultrahomogeneous, and has the class of all finite graphs as its age, we may now conclude that another, and for this paper final, way of constructing the Rado graph is as the Fraïssé limit of all finite graphs.

To the specific terminology and properties we may also add that the Rado graph is symmetric, meaning that there is an automorphism that takes any ordered pair of adjacent vertices to any other pair of adjacent vertices. To see that this is true, we may simply use Theorem 5.4, that $R$ is ultrahomogeneous.

One final property we might want to mention is that even though every countable graph can be embedded in $R$, it is not true that distances are preserved through such an embedding. For any two vertices in $R$, there is always a vertex adjacent to both of them. Thus the diameter of $R$ is 2. Since the diameter of a finite or countable graph can be arbitrarily large, even infinite, the distances between any two vertices cannot always be preserved when such a graph is embedded in $R$. We say that $R$ is not universal for isometric embeddings. However, $R$ is universal for isometric embeddings of diameter 2.
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