Exact Borel subalgebras of path algebras of quivers of Dynkin type A

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ABSTRACT

Hereditary algebras are quasi-hereditary with respect to any adapted partial order on the indexing set of the isomorphism classes of their simple modules. For any adapted partial order on \(\{1, \ldots, n\}\), we compute the quiver and relations for the Ext-algebra of standard modules over the path algebra of a uniformly oriented linear quiver with \(n\) vertices. Such a path algebra always admits a regular exact Borel subalgebra in the sense of König and we show that there is always a regular exact Borel subalgebra containing the idempotents \(e_1, \ldots, e_n\) and find a minimal generating set for it. For a quiver \(Q\) and a deconcatenation \(Q = Q^1 \sqcup Q^2\) of \(Q\) at a sink or source \(s\), we describe the Ext-algebra of standard modules over \(KQ\) up to an isomorphism of associative algebras, in terms of that over \(KQ^1\) and \(KQ^2\). Moreover, we determine necessary and sufficient conditions for \(KQ\) to admit a regular exact Borel subalgebra, provided that \(KQ^1\) and \(KQ^2\) do. We use these results to obtain sufficient and necessary conditions for a path algebra of a linear quiver with arbitrary orientation to admit a regular exact Borel subalgebra.

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Contents

1. Introduction................................................................. 2
2. Notation and background .............................................. 3
   2.1. Gluing of subalgebras ............................................. 5
3. The path algebra of \(A_n\) ............................................. 5
   3.1. Ringel dual ..................................................... 13
   3.2. Regular exact Borel subalgebras of \(A_n\) ..................... 15
   3.3. \(A\infty\)-structure on \(\text{Ext}(\Delta, \Delta)\) .................... 22
4. Ext-algebras of standard modules over deconcatenations .......... 24
   4.1. Exact Borel subalgebras under deconcatenations .......... 34
   4.2. Regular exact Borel subalgebras for path algebras of linear quivers 39
   4.3. A nonlinear example ........................................... 41
Funding .................................................................. 42
Declaration of competing interest ........................................ 42
Acknowledgements ....................................................... 42
References ................................................................ 42

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1. Introduction

Highest weight categories were first introduced in [3] with the purpose of axiomatizing certain phenomena arising naturally in the representation theory of complex semisimple Lie algebras. This prompted the definition of a quasi-hereditary algebra (see [20]), which was shown by the authors to be precisely such a finite-dimensional algebra whose module category is equivalent to a highest weight category. Since their introduction, examples of quasi-hereditary algebras have been found to be abundant in representation theory. Classical examples include hereditary algebras, algebras of global dimension two, Schur algebras and blocks of BGG category $O$. The main protagonists of the representation theory of quasi-hereditary algebras are the standard modules. These are certain quotients of the indecomposable projective modules which depend on a chosen partial order on the indexing set of the isomorphism classes of simple modules. For blocks of BGG category $O$, this order is the Bruhat order. Associated to the standard modules is the category $\mathcal{F}(\Delta)$: the full subcategory of the module category consisting of those modules which admit a filtration whose subquotients are standard modules. Given a quasi-hereditary algebra, natural objectives of its representation theory are understanding the standard modules and the associated category $\mathcal{F}(\Delta)$.

In the endeavor of understanding $\mathcal{F}(\Delta)$ for a general quasi-hereditary algebra, a breakthrough was made by König, Külshammer and Ovsienko in [13]. They showed that any quasi-hereditary algebra $A$ is Morita equivalent to an algebra $\Lambda$, which admits a very particular subalgebra $B$, called a regular exact Borel subalgebra. This subalgebra has the surprising property that the image of its module category under the functor $\Lambda \otimes_B -$ is equivalent to $\mathcal{F}(\Delta)$.

In general, the problem of determining $\Lambda$ and $B$ for an arbitrary quasi-hereditary algebra $A$ may be very hard. One way to do this involves determining an $A_\infty$-structure on $\text{Ext}_A^*(\Delta, \Delta)$, the algebra of extensions between standard modules over $A$, which can be arduous. Some examples of where this has been done can be found in [12,21]. However, in the situations studied in this paper, the aforementioned $A_\infty$-structures can be easily computed. Moreover, we are able to use criteria from [4] to check when our algebras themselves contain regular exact Borel subalgebras.

A given associative algebra can have different quasi-hereditary structures, depending on the partial order mentioned above. In particular, two different orders may yield identical quasi-hereditary structures. This phenomenon motivates the definition of the essential order, an order with the property that two quasi-hereditary structures coincide if and only if they generate the same essential order. The algebras considered in this article are hereditary, and hereditary algebras are precisely those algebras which are quasi-hereditary with respect to any total order on an indexing set of the isomorphism classes of their simple modules. A natural question to ask is then how many different quasi-hereditary structures there are. When the algebra in question admits a duality on its module category which preserves simple modules, the structure is essentially unique, as shown by Coulembier in [6]. This contrasts the algebras studied in this article with the examples from [12,21], as the algebras appearing there admit a simple-preserving duality.

In a recent paper by Flores, Kimura and Rognerud, [9], the authors showed that when $A$ is the path algebra of a uniformly oriented linear quiver, the different quasi-hereditary structures are counted by the Catalan numbers. Moreover, they counted the quasi-hereditary structures of path algebras of more complicated quivers by means of a combinatorial process called “deconcatenation”. The main goal of the present article is to use the combinatorial techniques of Flores, Kimura and Rognerud to further study the quasi-hereditary structures of some of the algebras considered in [9]. In particular, we want to describe the Ext-algebras of their standard modules and their regular exact Borel subalgebras.

The following is a brief description of the main results of the present article. For a quasi-hereditary algebra $A$, denote by $\Delta$ the direct sum of standard modules, one from each isomorphism class, and denote by $\text{Ext}_A^*(\Delta, \Delta)$ the algebra of extensions between standard modules.
(A) Let $A_n$ be the path algebra of the following quiver:

\[ 1 \rightarrow 2 \rightarrow \ldots \rightarrow n - 1 \rightarrow n \]

For any partial order $\leq$ on $\{1, \ldots, n\}$ such that $(A_n, \leq)$ is quasi-hereditary, we construct a graded quiver $Q$ and an admissible ideal $I \subset KQ$, such that there is an isomorphism of graded associative algebras $\text{Ext}_{A_n}^*(\Delta, \Delta) \cong KQ/I$. According to results from [9], each quasi-hereditary structure corresponds to a unique binary tree, and we show that this tree encodes all necessary information about extensions between the standard modules. For all details, we refer to Section 3.

(B) Let $A_n$ be as in (A). For any partial order $\leq$ on $\{1, \ldots, n\}$ such that $(A_n, \leq)$ is quasi-hereditary, we check that $A_n$ admits a regular exact Borel subalgebra. We show that $A_n$ always admits a regular exact Borel subalgebra $B$ which contains the idempotents $e_1, \ldots, e_n$ and find a minimal generating set for $B$. For all details, we refer to Section 3.2.

(C) Let $Q = Q^1 \sqcup Q^2$ be a deconcatenation of the quiver $Q$ at a sink or source $v$. Put $A = KQ$ and $A^\ell = KQ^\ell$, for $\ell = 1, 2$. We describe $\text{Ext}_{A}^*(\Delta, \Delta)$ up to isomorphism in terms of the Ext-algebras of standard modules over $A^\ell$, via a certain “gluing” process.

(D) Given that $A^\ell$ admits a regular exact Borel subalgebra $B^\ell$, for $\ell = 1, 2$, we show the following:

(i) If $v$ is a source, $A$ admits a regular exact Borel subalgebra.

(ii) If $v$ is a sink, then $A$ admits a regular exact Borel subalgebra if and only if $v$ is minimal or maximal with respect to the essential order on $Q_0$.

In the cases where $A$ admits a regular exact Borel subalgebra, we construct a regular exact Borel subalgebra of $A$ from $B^1$ and $B^2$, using a similar “gluing” as in (C).

This article is organized in the following way: In Section 2 we give the necessary background on quasi-hereditary algebras and fix some notation. In Section 3, we recall the results of [9] on the quasi-hereditary structures of $A_n$ and expand upon them, proving (A). In Subsection 3.1, we compute the quiver and relations of the Ringel dual of $A_n$. Subsection 3.2 is devoted to the description of the regular exact Borel subalgebras of $A_n$. Subsection 3.3 briefly discusses $A_\infty$-structures on $\text{Ext}_{A_n}^*(\Delta, \Delta)$.

In Section 4, we give the background on deconcatenations from [9] and prove (C). Subsection 4.1 discusses how regular exact Borel subalgebras behave under deconcatenations and contains the proof of (D). In Subsection 4.2, we apply our results to the case where $Q$ is a linear quiver with arbitrary orientation.

2. Notation and background

Throughout, let $K$ be an algebraically closed field. For a quiver $Q = (Q_0, Q_1)$, denote by $KQ$ path algebra of $Q$. Let $I \subset KQ$ be an admissible ideal and let $A$ be the quotient $A = KQ/I$. We take $A$-module to mean finite-dimensional left $A$-module if nothing else is stated. For an arrow $\alpha \in Q_1$, denote by $s(\alpha)$ and $t(\alpha)$ the starting and terminal vertex of $\alpha$, respectively. We adopt the convention of writing composition of arrows from right to left. That is, for vertices and arrows arranged as

\[ x \xrightarrow{\alpha} y \xrightarrow{\beta} z, \]

we write the composition “first $\alpha$, then $\beta$” as $\beta \alpha$. We extend the notation of starting and terminal vertex to paths in $Q$, so if $p = \alpha_n \ldots \alpha_1$, then we put $s(p) = s(\alpha_1)$ and $t(p) = t(\alpha_n)$.

The isomorphism classes of the simple $A$-modules are indexed by the vertex set, $Q_0$, of $Q$. Denote by $L(i)$ the simple $A$-module associated to the vertex $i$. Denote by $P(i)$ and $I(i)$ the projective cover and injective envelope of $L(i)$, respectively. For an $A$-module $M$ and a simple module $L(i)$, denote by $[M : L(i)]$ the
Jordan-Hölder multiplicity of $L(i)$ in $M$. We denote the category of finite-dimensional left $A$-modules by $A$-mod.

1 Definition. [3] Let $A$ be a finite-dimensional algebra. Let $\{1, \ldots, n\}$ be an indexing set for the isomorphism classes of simple $A$-modules and let $\preceq$ be a partial order on $\{1, \ldots, n\}$. Let the standard module at $i$, denoted by $\Delta(i)$, be the largest quotient of $P(i)$, whose composition factors $L(j)$ are such that $j \preceq i$. The algebra $A$ is said to be quasi-hereditary with respect to $\preceq$ if the following hold.

(QH1) There is a surjection $P(i) \twoheadrightarrow \Delta(i)$ whose kernel admits a filtration with subquotients $\Delta(j)$, where $j \triangleright i$.

(QH2) There is a surjection $\Delta(i) \twoheadrightarrow L(i)$ whose kernel admits a filtration with subquotients $L(j)$, where $j \triangleleft i$.

Equivalently, let the costandard module at $i$, denoted by $\nabla(i)$, be the largest submodule of $I(i)$, whose composition factors are such that $j \preceq i$. The algebra $A$ is said to be quasi-hereditary with respect to $\preceq$ if the following hold.

(QH1)$'$ There is an injection $\nabla(i) \hookrightarrow I(i)$ whose cokernel admits a filtration with subquotients $\nabla(j)$, where $j \triangleright i$.

(QH2)$'$ There is an injection $L(i) \hookrightarrow \nabla(i)$ whose cokernel admits a filtration with subquotients $L(j)$, where $j \triangleleft i$.

For a quasi-hereditary algebra $A$, it is natural to consider two particular subcategories of its module category, $A$-mod. These are $\mathcal{F}(\Delta)$, the full subcategory of $A$-mod consisting of those modules which admit a filtration by standard modules, and $\mathcal{F}(\nabla)$, the full subcategory of $A$-mod consisting of those modules which admit a filtration by costandard modules.

For an $A$-module $M \in \mathcal{F}(\Delta)$ (or $\mathcal{F}(\nabla)$), the number of occurrences of a particular standard module $\Delta(i)$ (or costandard module $\nabla(i)$) as a subquotient of a filtration of $M$ is well-defined, and we denote this number by $(M : \Delta(i))$ (or $(M : \nabla(i))$).

In general, refining the partial order $\preceq$ may produce different standard and costandard modules. Traditionally, only so-called adapted orders on $\{1, \ldots, n\}$ are considered in order to avoid this.

2 Definition. [8] Let $A$ be a finite-dimensional algebra. Let $\{1, \ldots, n\}$ be an indexing set for the isomorphism classes of simple $A$-modules and let $\preceq$ be a partial order on $\{1, \ldots, n\}$. We say that $\preceq$ is adapted to $A$ if and only if for any $A$-module $M$ such that $\text{top } M \cong L(i)$ and $\text{soc } M \cong L(j)$,

where $i$ and $j$ are incomparable, there is $1 \leq k \leq n$ such that $i \lessdot k$, $j \lessdot k$ and $[M : L(k)] > 0$.

3 Lemma. [5] If $(A, \preceq)$ is a quasi-hereditary algebra, then $\preceq$ is adapted to $A$.

Two quasi-hereditary structures $(A, \preceq_1)$ and $(A, \preceq_2)$ are said to be equivalent if the sets of standard (and costandard) modules with respect to $\preceq_1$ and $\preceq_2$ coincide. We denote this relationship by $\preceq_1 \sim \preceq_2$. Then, more precisely:

$$\preceq_1 \sim \preceq_2 \iff \Delta_1(i) = \Delta_2(i) \wedge \nabla_1(i) = \nabla_2(i), \ \forall i \in Q_0.$$
Equivalence of different quasi-hereditary structures is also captured precisely by the essential order.

4 Definition. [6, Definition 1.2.5] Let $(A, \leq)$ be a quasi-hereditary algebra. Define the essential order $\leq^e$ of $\leq$ as the partial order transitively generated by the relations

\[ i \leq^e j \iff [\Delta(j) : L(i)] > 0 \quad \text{or} \quad (P(i) : \Delta(j)) > 0. \]

The essential order is related to equivalence of quasi-hereditary structures via

\[ \leq_1 \sim \leq_2 \iff \leq^e_1 = \leq^e_2. \]

2.1. Gluing of subalgebras

At various points in the present article, there will appear algebras which, intuitively, arise from gluing two subspaces of some ambient algebra at a shared dimension. Let $X$ and $Y$ be subspaces of some algebra $A$, which are closed under multiplication. Assume that $\dim X \cap Y = 1$ and choose complements $X'$ and $Y'$ of $X \cap Y$ in $X$ and $Y$, respectively. Assume moreover that $X' \cdot Y' = Y' \cdot X' = 0$ and that $1_A \in \text{span}(X, Y)$. Put

\[ C = X' \oplus Y' \oplus X \cap Y. \]

Then, $C$ is a subalgebra of $A$. We call $C$ a gluing of $X$ and $Y$ and write $C = X \diamond Y$. Note that when $X$ and $Y$ are graded, there is a natural grading on $X \diamond Y$ induced by the gradings on $X$ and $Y$.

3. The path algebra of $A_n$

In this section, we consider the path algebra of the uniformly oriented linear quiver

\[ A_n : 1 \overset{\alpha_1}{\longrightarrow} 2 \overset{\alpha_2}{\longrightarrow} \ldots \overset{\alpha_{n-2}}{\longrightarrow} n - 1 \overset{\alpha_{n-1}}{\longrightarrow} n. \]

Throughout the section, we put $A_n = K \mathcal{A}_n$. The simple modules over $A_n$ are indexed by the vertices $1, \ldots, n$. Moreover, the algebra $A_n$ is hereditary, and therefore, it is quasi-hereditary with respect to any adapted order $(\{1, \ldots, n\}, \leq)$ to $A_n$ [8]. Recall that the indecomposable $A_n$-modules, up to isomorphism, are given by the interval modules, which are modules having Loewy diagrams of the following form:

\[
\begin{array}{c}
\vdots \\
j - 1 \\
j \\
M(i, j) \\
i + 1 \\
i
\end{array}
\]

\[
\begin{array}{c}
\vdots \\
j - 1 \\
j \downarrow \\
j \downarrow \\
1 \overset{\alpha_1}{\longrightarrow} 2 \overset{\alpha_2}{\longrightarrow} \ldots \overset{\alpha_{n-2}}{\longrightarrow} n - 1 \overset{\alpha_{n-1}}{\longrightarrow} n.
\end{array}
\]
With this notation, we have

\[ L(i) = M(i, i), \quad P(i) = M(i, n) \quad \text{and} \quad I(i) = M(1, i), \quad \forall 1 \leq i \leq n. \]

Homomorphisms and extensions between interval modules are well understood, and we summarize this information in the following proposition. Note that [18] uses different conventions than the present article.

5 Proposition. [18,7,1] Let \( M(i_1, j_1) \) and \( M(i_2, j_2) \) be interval modules. Then, we have

(i) \[
\dim \text{Hom}_{A_n}(M(i_1, j_1), M(i_2, j_2)) = \begin{cases} 
1 & \text{if } i_2 \leq i_1 \leq j_2 \leq j_1; \\
0 & \text{otherwise}. 
\end{cases}
\]

(ii) \[
\dim \text{Ext}^1_{A_n}(M(i_1, j_1), M(i_2, j_2)) = \begin{cases} 
1 & \text{if } i_1 + 1 \leq i_2 \leq j_1 + 1 \leq j_2; \\
0 & \text{otherwise}. 
\end{cases}
\]

6 Definition. A binary tree \( T \) is either the empty set or a triple \( (s, L, R) \), where \( s \) is a singleton set, called the root of \( T \), and \( L \) and \( R \) are two binary trees, called the left and right subtrees of \( s \), respectively. The empty set has one leaf, and the set of leaves of \( T = (s, L, R) \) is the (disjoint) union of the sets of leaves of \( L \) and \( R \).

A binary search tree is a binary tree, whose vertices are labeled by integers, such that if a vertex \( x \) is labeled by \( k \), then the vertices of the left subtree of \( x \) are labeled by integers less than \( k \), and the vertices of the right subtree of \( x \) are labeled by integers greater than \( k \).

If \( T \) is a binary tree with \( n \) vertices, there exists a unique labeling of the vertices of \( T \) by the integers \( 1, \ldots, n \), turning \( T \) into a binary search tree. The procedure by which this labeling is obtained is known as the in-order algorithm, which recursively visits the left subtree, then the root, then the right subtree. The first vertex visited is labeled by 1, the second by 2, and so on.

7 Example. Consider the following binary tree with 6 vertices.

With the in-order algorithm, the vertices of the tree are labeled as follows, creating a binary search tree.
Any binary search tree $T$ with $n$ vertices induces a partial order on the set of its vertices $\{1, \ldots, n\}$. We denote this partial order by $\leq_T$. It is defined in the following way.

$$i \preceq_T j \iff i \text{ labels a vertex in the subtree of the vertex labeled by } j.$$

In the above example, the partial order $\leq_T$ would be given by:

$$1 \preceq_T 2 \preceq_T 3, \quad 5 \preceq_T 4 \preceq_T 6 \preceq_T 3.$$

At this point, it is natural to ask for which binary trees $T$ we obtain a quasi-hereditary algebra $(A_n, \preceq_T)$.

Recall that, since $A_n$ is hereditary, $A_n$ is quasi-hereditary with respect to any partial order which is adapted to $A_n$. It turns out that any partial order $\preceq$ with respect to which $A_n$ is quasi-hereditary, is equivalent (that is, produces the same set of standard and costandard modules) to a partial order produced by a binary tree. Conversely, for any binary tree $T$, the order $\preceq_T$ makes $A_n$ quasi-hereditary. More precisely, we have the following.

8 Proposition. [9, Proposition 4.4] Let $n$ be a natural number and let $\mathcal{T}$ be the set of binary trees with $n$ vertices. Denote by $\mathcal{A}$ the set of adapted orders to $A_n$. For any partial order $\preceq$ in $\mathcal{A}$, denote by $\overline{\preceq}$ the equivalence class of $\preceq$ with respect to the relation $\sim$. Then, the map from $\mathcal{T}$ to $\mathcal{A}/\sim$ defined by

$$T \mapsto \overline{\preceq}_T$$

is a bijection.

Now, given that any quasi-hereditary structure on $A_n$ may be associated to a binary tree, it is natural to describe this structure in terms of the associated binary tree. The following proposition is stated in the proof of [9, Proposition 4.4]. For the convenience of the reader, we give a proof.

9 Proposition. Let $T$ be a binary search tree and let $\preceq_T$ be the associated adapted order to $A_n$. Then, we have the following.

(i) The composition factors of the standard module $\Delta(i)$ are indexed by the labels of the vertices in the right subtree of the vertex labeled by $i$.

(ii) The composition factors of the costandard module $\nabla(i)$ are indexed by the labels of the vertices in the left subtree of the vertex labeled by $j$.

Proof. By definition, the standard module $\Delta(i)$ is a quotient of the indecomposable projective module $P(i)$. Since $P(i) = M(i, n)$, this implies that there exists an integer $s_i$, which satisfies $i \leq s_i \leq n$, such that
\(\Delta(i) = M(i, s_i)\). The composition factors of \(\Delta(i)\) are then \(L(i), \ldots, L(s_i)\), all occurring with Jordan-Hölder multiplicity 1. Denote by \(v_i\) the vertex labeled by \(i\).

The composition factors \(L(j)\) of \(\Delta(i)\) must satisfy \(v_j \leq_T v_i\). By definition of \(\leq_T\), the vertices \(v_j\) such that \(v_j \not< v_i\) are exactly the vertices of the left and right subtree of \(v_i\).

By construction of \(\leq_T\), the vertices in the left subtree of \(v_i\) are labeled by integers less than \(i\), which shows that the corresponding simple modules may not be composition factors of \(\Delta(i)\). Since \(\Delta(i)\) is the maximal quotient of \(P(i)\) such that its composition factors \(L(j)\) satisfy \(v_j \leq_T v_i\), we are done.

The argument for the form of the costandard modules is similar. \(\square\)

For a vertex \(v\) of \(T\), denote by \(\ell(v)\) and \(r(v)\) the vertices immediately down and to the left or down and to the right of \(v\), respectively. If such vertices do not exist, we write \(\ell(v) = 0\) or \(r(v) = 0\).

10 Proposition. Let \(T\) be a binary search tree and let \(\leq_T\) be the associated adapted order to \(A_n\).

(i) Suppose that \(i\) labels the vertex \(\ell(v)\) and that \(j\) labels the vertex \(v\). Then, we have
\[\dim \text{Ext}_{A_n}^1(\Delta(i), \Delta(j)) = 1.\]

(ii) Suppose that \(i\) labels the vertex \(r(v)\) and that \(j\) labels the vertex \(v\). Then, we have
\[\dim \text{Hom}_{A_n}(\Delta(i), \Delta(j)) = 1.\]

Proof. Put \(\Delta(i) = M(i, s_i)\) and \(\Delta(j) = M(j, s_j)\).

(i) Note that the first vertex visited by the in-order algorithm, after visiting the right subtree of \(\ell(v)\), is \(v\). Therefore, by Proposition 9, we have \(j = s_i + 1\). The condition of Proposition 5, part (ii), is then
\[i + 1 \leq s_i + 1 \leq s_i + 1 \leq s_j,\]
which is clearly satisfied.

(ii) With the in-order algorithm, the vertex \(r(v)\) is visited after \(v\). Then, using Proposition 9, we know that if \(\Delta(i) = M(i, s_i)\), then \(\Delta(j) = M(i + k, s_i)\), for some \(k \leq s_i - i\). The condition of Proposition 5, part (i), is then
\[i \leq i + k \leq s_i \leq s_i,\]
which is clearly satisfied. \(\square\)

11 Lemma.

(i) Let \(v_i\) and \(v_j\) be vertices labeled by \(i\) and \(j\), respectively. Assume that \(v_j\) is in the left subtree of \(v_i\). Then, we have
\[\text{Hom}_{A_n}(\Delta(i), \Delta(j)) = \text{Hom}_{A_n}(\Delta(j), \Delta(i)) = 0.\]

(ii) Let \(v_i\) and \(v_j\) be vertices labeled by \(i\) and \(j\), respectively. Assume that \(v_j\) is in the right subtree of \(v_i\). Then, we have
\[\text{Ext}_{A_n}^1(\Delta(i), \Delta(j)) = \text{Ext}_{A_n}^1(\Delta(j), \Delta(i)) = 0.\]
Let \( v_i \) and \( v_j \) be vertices, labeled by \( i \) and \( j \), respectively. Assume that \( v_i \) is not in the subtree of \( v_j \) and that \( v_j \) is not in the subtree of \( v_i \). Then, we have

\[
\text{Hom}_{A_n}(\Delta(i), \Delta(j)) = \text{Hom}_{A_n}(\Delta(j), \Delta(i)) = \text{Ext}^1_{A_n}(\Delta(i), \Delta(j)) = \text{Ext}^1_{A_n}(\Delta(j), \Delta(i)) = 0.
\]

**Proof.** By the proof of Proposition 9, we have \( \Delta(i) = M(i, s_i) \) and \( \Delta(j) = M(j, s_j) \), where the integers

\[
i + 1, i + 2, \ldots, s_i
\]

label the vertices in the right subtree of \( v_i \) and the integers

\[
j + 1, j + 2, \ldots, s_j
\]

label the vertices in the right subtree of \( v_j \).

(i) According to the in-order algorithm, the integers \( j + 1, \ldots, s_j \) are less than \( i \). We immediately have

\[
\text{Hom}_{A_n}(\Delta(i), \Delta(j)) = 0,
\]

since \( v_j \triangleright_T v_i \) and \( A_n \) is quasi-hereditary. Moreover by Proposition 5, part (ii), we have

\[
\text{Hom}_{A_n}(\Delta(j), \Delta(i)) = 0,
\]

since the condition \( i \leq j \leq s_i \leq s_j \) is not satisfied, because \( j < i \).

(ii) Since \( v_j \) is in the right subtree of \( v_i \), we have \( s_j \leq s_i \). We immediately have

\[
\text{Ext}^1_{A_n}(\Delta(i), \Delta(j)) = 0,
\]

since \( v_j \triangleright_T v_i \) and \( A_n \) is quasi-hereditary. Moreover, by Proposition 5, part (i), we have

\[
\text{Ext}^1_{A_n}(\Delta(j), \Delta(i)) = 0,
\]

since the condition \( i + 1 \leq j \leq s_i + 1 \leq s_j \) is not satisfied, because \( s_j \leq s_i \).

(iii) This is immediate, since \( A_n \) is quasi-hereditary and the vertices \( v_i \) and \( v_j \) are incomparable with respect to \( \triangleright_T \).

**12 Example.** Consider the following binary search tree, \( T \), with vertices labeled according to the in-order algorithm.

![Diagram of the binary search tree](image_url)

The partial order \( \leq_T \) is given by:

\[
\begin{align*}
4 & \leq_T 2 \\
4 & \leq_T 3 \\
5 & \leq_T 6 \\
1 & \leq_T 4 \\
2 & \leq_T 4 \\
3 & \leq_T 4 \\
5 & \leq_T 4
\end{align*}
\]
1 \triangleleft T 2 \triangleleft T 4, \quad 3 \triangleleft T 2 \triangleleft T 4, \quad \text{and} \quad 6 \triangleleft T 5 \triangleleft T 4.

For the standard and costandard modules, we have

\[ \Delta(1) \cong L(1), \quad \Delta(2) \cong M(2, 3), \quad \Delta(3) \cong L(3), \quad \Delta(4) \cong P(4), \quad \Delta(5) \cong P(6), \quad \Delta(6) \cong L(6), \]

\[ \nabla(1) \cong L(1), \quad \nabla(2) \cong M(1, 2), \quad \nabla(3) \cong L(3), \quad \nabla(4) \cong M(1, 4), \quad \nabla(5) \cong L(5), \quad \text{and} \quad \nabla(6) \cong L(6), \]

which we see by applying Proposition 9. Moreover, we have non-split short exact sequences

\[ \Delta(2) \hookrightarrow I(3) \twoheadrightarrow \Delta(1) \quad \text{and} \quad \Delta(4) \hookrightarrow P(2) \twoheadrightarrow \Delta(2), \]

giving us the extensions guaranteed by part (i) of Proposition 10.

Next, we see there are natural monomorphisms \( \Delta(3) \hookrightarrow \Delta(2) \) and \( \Delta(6) \hookrightarrow \Delta(5) \hookrightarrow \Delta(4) \), giving us the homomorphisms guaranteed by part (ii) of Proposition 10.

Lastly, we have

\[
\dim \text{Hom}_{A_n}(\Delta(1), \Delta(2)) = \dim \text{Hom}_{A_n}(\Delta(2), \Delta(1)) = \dim \text{Hom}_{A_n}(\Delta(1), \Delta(3)) = \dim \text{Hom}_{A_n}(\Delta(3), \Delta(1)) = 0;
\]

\[
\dim \text{Hom}_{A_n}(\Delta(2), \Delta(4)) = \dim \text{Hom}_{A_n}(\Delta(4), \Delta(2)) = \dim \text{Hom}_{A_n}(\Delta(1), \Delta(4)) = \dim \text{Hom}_{A_n}(\Delta(4), \Delta(1)) = 0;
\]

\[
\dim \text{Hom}_{A_n}(\Delta(2), \Delta(5)) = \dim \text{Hom}_{A_n}(\Delta(5), \Delta(2)) = \dim \text{Hom}_{A_n}(\Delta(2), \Delta(6)) = \dim \text{Hom}_{A_n}(\Delta(5), \Delta(6)) = \dim \text{Hom}_{A_n}(\Delta(6), \Delta(2)) = 0;
\]

\[
\dim \text{Hom}_{A_n}(\Delta(1), \Delta(5)) = \dim \text{Hom}_{A_n}(\Delta(5), \Delta(1)) = \dim \text{Hom}_{A_n}(\Delta(1), \Delta(6)) = \dim \text{Hom}_{A_n}(\Delta(6), \Delta(1)) = 0,
\]

as prescribed by Lemma 11.

13 Lemma. Let \( v, r(v) \) and \( \ell(r(v)) \) be vertices labeled by \( i, j \) and \( k \), respectively. Then, the multiplication map

\[
\text{Hom}_{A_n}(\Delta(j), \Delta(i)) \times \text{Ext}^1_{A_n}(\Delta(k), \Delta(j)) \to \text{Ext}^1_{A_n}(\Delta(k), \Delta(i))
\]

is the zero map.

Proof. Note that the vertices \( i, j \) and \( k \) are configured in the following way:

```
   i
   / \  \\  
   j  k
```

The multiplication map in question produces an extension in the space \( \text{Ext}^1_{A_n}(\Delta(k), \Delta(i)) \). But since \( \ell(r(v)) \) is a vertex in the right subtree of \( v \), this space is zero, by part (ii) of Lemma 11.

14 Lemma. Let \( v, \ell(v) \) and \( r(\ell(v)) \) be vertices labeled by \( i, j \) and \( k \), respectively. Then, the multiplication map

\[
\text{Ext}^1_{A_n}(\Delta(j), \Delta(i)) \times \text{Hom}_{A_n}(\Delta(k), \Delta(j)) \to \text{Ext}^1_{A_n}(\Delta(k), \Delta(i))
\]

is non-zero.
Proof. We start by observing that the vertices labeled by $i$, $j$ and $k$ are configured in the following way:

According to the in-order algorithm, we have $j < k < i$. Note that, since $A_n$ is hereditary, a minimal projective resolution of a standard module $\Delta(x)$ is of the form

$$\ker p_x \longrightarrow P(x) \overset{p_x}{\longrightarrow} \Delta(x).$$

Consider the following picture:

An extension from $\Delta(k)$ to $\Delta(i)$ is represented by a chain map, which has one (possibly) non-zero component, namely the map $g \circ f$. Now, assume that $\Delta(k) = M(k, s_k)$ and $\Delta(i) = M(i, s_i)$. Then, we have $\ker p_k = P(s_k + 1)$ and $\ker p_i = P(s_i + 1)$. Since $k < i$ and $s_k < s_i$, we have $\text{Hom}_{A_n}(P(k), P(i)) = \text{Hom}_{A_n}(\ker p_k, \ker p_i) = 0$. Therefore, the chain map, if it is non-zero, cannot be null-homotopic. By construction, $f$ is the unique (up to a scalar) non-zero map from $M(s_k + 1, n)$ to $M(s_j + 1, n)$. Next, note that in our configuration, the first vertex visited by the in-order algorithm, after visiting the entire right subtree of $\ell(v)$, is $v$. This implies that $s_j + 1 = i$, so that the map $g$ in the above picture is some scalar multiple of the identity homomorphism on $P(i)$. This shows that the composition $g \circ f$, and hence the chain map representing our extension, is non-zero.

For two interval modules $M(i_1, j_1)$ and $M(i_2, j_2)$, the space $\text{Hom}_{A_n}(M(i_1, j_1), M(i_2, j_2))$ is one-dimensional if it is non-zero. Fix a basis of this space, corresponding to the following homomorphism of representations:

$$K \leftarrow \ldots \leftarrow K \leftarrow \ldots \leftarrow K$$

Note that this choice of basis also gives an obvious choice of basis for the spaces $\text{Ext}_{A_n}^1(M(i_1, j_1), M(i_2, j_2))$. Suppose now that we have chosen such basis vectors

$$y \in \text{Hom}_{A_n}(\Delta(i), \Delta(j)); \quad x \in \text{Ext}_{A_n}^1(\Delta(j), \Delta(k)); \quad z \in \text{Ext}_{A_n}^1(\Delta(i), \Delta(k));$$

and that $i, j$ and $k$ are as in the statement of Lemma 14. Then, we have $xy = z$. 
15 Theorem. Let $T$ be a binary search tree with $n$ vertices, with the vertices labeled according to the in-order algorithm, and let $\preceq_T$ be the associated adapted order to $A_n$. Construct a quiver $Q$ in the following way:

(i) The vertices of $Q$ are $1, 2, \ldots, n$.
(ii) We draw an edge between $i$ and $j$ if and only if the integers $i$ and $j$ label either a set of vertices $\{v, \ell(v)\}$ or $\{v, r(v)\}$. If $i$ labels $\ell(v)$ and $j$ labels $v$, the orientation of the edge is from $\ell(v)$ to $v$, of degree 1, and is denoted by $e^1_{ij}$. If $i$ labels $r(v)$ and $j$ labels $v$, the orientation of the edge is from $r(v)$ to $v$, of degree 0, and is denoted by $f^1_{ij}$.

Let $I \subset KQ$ be the ideal generated by the following elements:

• $f^k_j e^l_i$, for all $i, j$ and $k$;
• $e^k_j f^l_i$, for all $i, j$ and $k$;

Then, there is an isomorphism of graded algebras $KQ/I \cong \text{Ext}_{A_n}^*(\Delta, \Delta)$.

16 Example. Consider the binary search tree

Then, the corresponding quiver $Q$ is

and the ideal $I = \langle e^2_2 e^2_1 \rangle$. The full arrows, $e^2_1$ and $e^4_1$, are of degree 1, while the dashed arrows, $f^2_3, f^4_3$ and $f^5_6$ are of degree 0.

Proof of Theorem 15. Fix basis vectors $\varphi^j_i \in \text{Hom}_{A_n}(\Delta(i), \Delta(j))$ and $e^j_i \in \text{Ext}_{A_n}^1(\Delta(i), \Delta(j))$ of all non-zero spaces of the form $\text{Hom}_{A_n}(\Delta(i), \Delta(j))$ and $\text{Ext}_{A_n}^1(\Delta(i), \Delta(j))$, as discussed prior to Theorem 15. Define a map $\Phi : KQ/I \to \text{Ext}_{A_n}^*(\Delta, \Delta)$ by

$$e^j_i \mapsto e^j_i, \quad f^j_i \mapsto \varphi^j_i, \quad e_i \mapsto 1_{\Delta(i)},$$

and extend by linearity. By applying Lemma 13 and Lemma 11, along with the fact that there are no extensions of degree greater than 1, since $A_n$ is hereditary, we see that $I \subset \ker \Phi$, so that $\Phi$ is well-defined.

To check that $\Phi$ is a homomorphism of algebras, it is enough to check compatibility of $\Phi$ with products of the form $e^k_j f^l_i$. We have:
\[ \Phi(e_i^j f_i^j) = \Phi(e_i^j) = e_i^k = e_j^k \epsilon_i^j = \Phi(e_i^j) \Phi(f_i^j). \]

As the image of \( \Phi \) contains bases of \( \text{Hom}_{A_n}(\Delta(i), \Delta(j)) \) and \( \text{Ext}^1_{A_n}(\Delta(i), \Delta(j)) \) for all \( i \) and \( j \), \( \Phi \) is surjective.

Next, we compare dimensions. Consider the degree 0 part of the space \( e_j^j KQ/te_i \). The dimension of this space is equal to the number of paths of degree 0 from \( i \) to \( j \) in \( Q \). Clearly, such a path must be a product of the form \( f_{k_s}^j f_{k_{s-1}}^j \ldots f_{k_1}^j t_i^j f_i^j \). Since vertices in \( Q \) have out-degree at most 1, it follows that such a path is unique. So the dimension of the degree 0 part \( e_j^j KQ/te_i \) is either 0 or 1. By construction of \( Q \), it is now clear that this dimension coincides with \( \dim \text{Hom}_{A_n}(\Delta(i), \Delta(j)) \). Next, consider the degree 1 part of the space \( e_j^j KQ/te_i \). Similarly, the dimension of this space is either 0 or 1, depending on the number of paths from \( i \) to \( j \) in \( Q \). Such a path is either of the form \( \epsilon_i^j \) or \( \epsilon_i^j f_i^j \). Again, by construction of \( Q \), we see that this dimension coincides with \( \text{Ext}^1_{A_n}(\Delta(i), \Delta(j)) \). Therefore, \( \Phi \) is a surjective linear map between vector spaces of the same dimension, hence an isomorphism.

\[ \square \]

3.1. Ringel dual

We recall that, for any quasi-hereditary algebra \( A \), there exists a module \( T \), called the characteristic tilting module, whose additive closure equals \( \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) \). The characteristic tilting module decomposes as

\[ T = \bigoplus_{i=1}^{n} T(i), \]

where each \( T(i) \) is indecomposable. The Ringel dual is then, by definition, \( R(A) = \text{End}(T)^{\text{op}} \). In this section, we compute quiver and relations of the Ringel dual of \((A_n, \leq_T)\) using the associated binary search tree. The following statement is part of the proof of Theorem 4.6 in \( [9] \). For the convenience of the reader, we give a proof.

\textbf{17 Proposition.} Let \( v \) be a vertex labeled by \( i \). The indecomposable summand \( T(i) \) of the characteristic tilting module is the interval module \( M(t_i, s_i) \), where \( t_i \) is the least integer labeling a vertex in the left subtree of \( v \) and \( s_i \) is the greatest integer labeling a vertex in the right subtree of \( v \).

\textbf{Proof.} To prove that \( T(i) = M(t_i, s_i) \), it suffices to show that \( \Delta(i) \) is isomorphic to a submodule of \( T(i) \), that the cokernel of this monomorphism admits a filtration by standard modules, and that \( T(i) \) is contained in \( \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) \) \( [19] \), Proposition 2 \]. We proceed by induction on the size of the subtree of \( v \). The basis of the induction is clear, since if \( \ell(v) = r(v) = \emptyset \), we have \( \Delta(i) = \nabla(i) = L(i) = T(i) \). Since \( t_i \leq i \), it is clear that \( \Delta(i) \) embeds into \( M(t_i, s_i) \). Clearly, the cokernel of this embedding is the module \( M(t_i, i - 1) \).

Suppose that \( j \) labels the vertex \( \ell(v) \). The composition factors of \( M(t_i, i - 1) \) are labeled by the vertices in the subtree of \( \ell(v) \), so by the induction hypothesis, \( M(t_i, i - 1) = T(j) \). In other words, we have a short exact sequence

\[ 0 \to \Delta(i) \hookrightarrow M(t_i, s_i) \to T(j) \to 0. \]

Since \( T(j) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) \), we have \( M(t_i, s_i) \in \mathcal{F}(\Delta) \). By a similar argument, we see that there is a short exact sequence

\[ 0 \to T(k) \hookrightarrow T(i) \to \nabla(i) \to 0, \]

where \( k \) labels the vertex \( r(v) \). Then, since \( T(k) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) \), we get that \( M(t_i, s_i) \in \mathcal{F}(\nabla) \). Now, \( M(t_i, s_i) \) is a module such that \( \Delta(i) \) embeds into it, the cokernel of this embedding is contained in \( \mathcal{F}(\Delta) \), and \( M(t_i, s_i) \in \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) \). This confirms that \( M(t_i, s_i) = T(i) \).

\[ \square \]
18 Corollary. Let the vertices $v$, $\ell(v)$ and $r(v)$ be labeled by $i$, $j$ and $k$, respectively. Then, for any $v$, there is an epimorphism $p^i_1: T(i) \rightarrow T(j)$ and a monomorphism $q^k_y: T(k) \hookrightarrow T(i)$.

19 Proposition. Let the vertices $\ell(\ell(v)), \ell(v), r(v)$ and $r(r(v))$ be labeled by $x, i, k$ and $y$, respectively. Let $p^i_1, p^x_j, q^k_y$ and $q^k_y$ be the homomorphisms from Corollary 18. Then, we have:

(i) $p^x_j p^i_1 \neq 0$ and $q^k_y q^k_y \neq 0$;
(ii) $p^i_1 q^k_y = 0$;
(iii) $\text{Hom}_{A_n}(T(j), T(i)) = 0$;
(iv) $\text{Hom}_{A_n}(T(i), T(k)) = 0$.

Proof. The vertices are configured in the following way:

![Diagram](image)

(i) This is immediate, as $p^x_j p^i_1$ is the composition of two epimorphisms and $q^k_y q^k_y$ is the composition of two monomorphisms.

(ii) Applying Proposition 17, we see that $T(k)$ and $T(j)$ have no common composition factors, which implies that $\text{Hom}_{A_n}(T(k), T(j)) = 0$, yielding the statement.

(iii) Put $T(j) = M(t_j, s_j)$ and $T(i) = M(t_i, s_i)$, where $t_j = t_i$ and $s_j < s_i$. Then, using Proposition 5, the condition for $\text{Hom}_{A_n}(T(j), T(i))$ being non-zero is

$$t_i \leq t_j \leq s_i \leq s_j,$$

which is not satisfied, since $s_j < s_i$.

(iv) Similar to (iii).

With these observations established, we are ready to describe the Ringel dual $R(A_n)$.

20 Theorem. Let $T$ be a binary search tree with $n$ vertices, with the vertices labeled according to the in-order algorithm, and let $\leq_T$ be the associated adapted order to $A_n$. Construct a quiver $Q$ in the following way:

(i) The vertices of $Q$ are $1, \ldots, n$.

(ii) We draw an edge between $i$ and $j$ if and only if the integers $i$ and $j$ label either a set of vertices $\{v, \ell(v)\}$ or $\{v, r(v)\}$. If $i$ labels $v$ and $j$ labels $\ell(v)$, the edge is denoted by $f^i_1$ and its orientation is from $i$ to $j$.

If $i$ labels $v$ and $j$ labels $r(v)$, the edge is denoted by $g^j_y$ and its orientation is from $j$ to $i$.

Let $I \subset KQ$ be the ideal generated by the elements $f^i_1 g^k_y$, for all $i, j$ and $k$. Then, there is an isomorphism of algebras $KQ/I \cong \text{End}(T)$, and consequently, $R(A_n) \cong KQ/I^{\text{op}}$.
21 Example. Consider the binary search tree

```
1
   / \
  2   3
 /     \
4       5
   \     /
    \  6
```

Then, the corresponding quiver $Q$ is

```
1   2   3
 / \ / \ /
4   f4 2 5  \ g4
   / \ / \ /
5   f1 2 3  \ g2
   / \ / \ /
6   g5 3 6
```

and the ideal $I \subset KQ$ is generated by the elements $f_2^2 g_5^4$ and $f_1^1 g_2^3$. 

**Proof of Theorem 20.** Recall the discussion prior to Theorem 15, which implies that for composable homomorphisms between indecomposable summands of $T$, we may choose basis vectors

$$x \in \text{Hom}_{A_n}(T(i), T(j)); \quad y \in \text{Hom}_{A_n}(T(j), T(k)); \quad z \in \text{Hom}_{A_n}(T(i), T(k))$$

such that $xy = z$. Define a map $\tilde{\Psi}: KQ \to \text{End}(T)$ by

$$f_i^j \mapsto p_i^j, \quad g_i^j \mapsto q_i^j, \quad e_i \mapsto 1_{T(i)}$$

and extend by linearity. Using Proposition 19, part (ii), we see that $I \subset \ker \tilde{\Psi}$, so that $\tilde{\Psi}$ factors uniquely through the quotient $KQ/I$. This gives us a well-defined homomorphism of algebras $\Psi: KQ/I \to \text{End}(T)$. Since the homomorphisms $p_i^j, q_i^j$ and $1_{T(i)}$ generate $\text{End}(T)$, $\Psi$ is surjective. It is now easy to see that $\dim KQ/I = \dim \text{End}(T)$, so that $\Psi$ is a surjective linear map between spaces of the same dimension, hence an isomorphism. 

3.2. Regular exact Borel subalgebras of $A_n$

In this section, we investigate when $(A_n, \preceq_T)$ admits a regular exact Borel subalgebra. Recall the following.

22 Definition. [14,2, Definition 3.4] Let $(\Lambda, \preceq)$ be a quasi-hereditary algebra with $n$ simple modules, up to isomorphism. Then, a subalgebra $B \subset \Lambda$ is called an **exact Borel subalgebra** provided that

(i) $B$ also has $n$ simple modules up to isomorphism and $(B, \preceq)$ is quasi-hereditary with simple standard modules,

(ii) the functor $\Lambda \otimes_B -$ is exact, and

(iii) there are isomorphisms $\Lambda \otimes_B L_B(i) \cong \Delta_A(i)$. 
If, in addition, the map $\text{Ext}^k_B(L_B(i), L_B(j)) \to \text{Ext}^k_A(\Lambda \otimes_B L_B(i), \Lambda \otimes_B L_B(j))$ induced by the functor $\Lambda \otimes_B -$ is an isomorphism for all $k \geq 1$ and $i, j \in \{1, \ldots, n\}$, $B \subset \Lambda$ is called a regular exact Borel subalgebra. If this map is an isomorphism for all $k \geq 2$ and an epimorphism for $k = 1$, $B \subset \Lambda$ is called a homological Borel exact Borel subalgebra.

We note that a quasi-hereditary algebra $(KQ/I, \preceq)$ being quasi-hereditary with simple standard modules is equivalent to said algebra being directed, meaning that for any arrow $i \xrightarrow{\alpha} j$, we have $i \prec j$.

**23 Theorem.** [4, Theorem D, part 5] Let $(A, \preceq)$ be a basic quasi-hereditary algebra whose isomorphism classes of simple modules are indexed by $\{1, \ldots, n\}$. Then $(A, \preceq)$ admits a regular exact Borel subalgebra if and only if $\text{rad } \Delta(i)$ is contained in $F(\nabla)$, for all $i \in \{1, \ldots, n\}$.

**24 Proposition.** $(A_n, \preceq_T)$ has a regular exact Borel subalgebra.

**Proof.** Let the vertex $v$ be labeled by $i$ and consider $\Delta(i) = M(i, s_i)$. Then, $\text{rad } \Delta(i) = M(i + 1, s_i)$. Let the vertex $r(v)$ be labeled by $k$. According to the in-order algorithm, the first vertex visited immediately after $v$ is the left-most vertex in the subtree of $r(v)$. This shows, in our earlier notation, that $t_k = i + 1$. This fact implies that

$$M(i + 1, s_i) = M(t_k, s_i) = M(t_k, s_k) = T(k),$$

according to the proof of Proposition 17. Since $T(k) \in F(\nabla)$, the statement follows from Theorem 23. □

Thanks to the observations made earlier in the current section, it is fairly easy to find the quiver, $Q_B$, of a regular exact Borel subalgebra $B \subset A_n$. Recall that, if $B \subset A_n$ is a regular exact Borel subalgebra, then there are isomorphisms of vector spaces

$$\text{Ext}^k_B(L(i), L(j)) \cong \text{Ext}^k_{A_n}(\Delta(i), \Delta(j)),$$

for all $k \geq 1$ and $i, j \in \{1, \ldots, n\}$. In particular, the spaces above have the same dimension. In our setup, these spaces are zero for $k > 1$. It is well-known that

$$\dim \text{Ext}^1_B(L(i), L(j)) = \text{number of arrows from } i \text{ to } j \text{ in } Q_B.$$

This means that we can determine the quiver $Q_B$ by computing extensions between standard modules over $A_n$. To this end, we apply Theorem 15, resulting in the following.

**25 Proposition.** Let $B \subset A_n$ be a regular exact Borel subalgebra. Then, there is an isomorphism of algebras $B \cong KQ_B$, where $Q_B$ is the following quiver.

(i) The vertex set of $Q_B$ is $\{1, \ldots, n\}$.
(ii) There is an arrow $i \to j$ if and only if $\dim \text{Ext}^1_{A_n}(\Delta(i), \Delta(j)) = 1$.

Next, we wish to determine how to realize $B$ as a subalgebra of $A_n$. In this endeavor, we are aided by the following results.

**26 Theorem** (Wedderburn-Mal’tsev). [22, 17] Let $A$ be a finite-dimensional algebra over an algebraically closed field. Then, there is a semi-simple subalgebra $S$ of $A$ such that
\[ A = \text{rad} A \oplus S \]

as a vector space. Moreover, if \( T \) is another semisimple subalgebra of \( A \) such that \( A = \text{rad} A \oplus T \), then there is an inner automorphism \( \sigma \) of \( A \) such that \( \sigma(S) = T \).

We remark that if \( S \) is a semi-simple subalgebra as in the theorem and \( T \) is another semi-simple subalgebra with \( \dim_K S = \dim_K T \), then necessarily \( A = \text{rad} A \oplus T \), as any semi-simple subalgebra intersects \( \text{rad} A \) trivially. Consequently, \( S \) and \( T \) are conjugate.

Let \( A = KQ/I \) be the path algebra of some quiver modulo an admissible ideal and let \( S \subset A \) be the subalgebra \( S = \langle e_1, \ldots, e_n \rangle \). Then, \( S \) is semi-simple and \( A = \text{rad} A \oplus S \). If \( \{d_1, \ldots, d_n\} \) is some other complete set of primitive orthogonal idempotents, then \( T = \langle d_1, \ldots, d_n \rangle \) is another semi-simple subalgebra of \( A \) with \( \dim_K S = \dim_K T \), so \( S \) and \( T \) must be conjugate. We conclude that for any two complete sets of primitive orthogonal idempotents in \( A \), there is an inner automorphism of \( A \) mapping one to the other.

**27 Theorem.** [23, Proposition 3.4] Let \( A \) be a finite-dimensional quasi-hereditary algebra and let \( B \subset A \) be an exact Borel subalgebra. Then, \( C = \sigma(B) \) is again an exact Borel subalgebra.

Let \( \{d_1, \ldots, d_n\} \) be a complete set of primitive orthogonal idempotents in \( B \). For any complete set of primitive orthogonal idempotents \( \{f_1, \ldots, f_n\} \) of \( A_n \), there is an inner automorphism \( \sigma \) of \( A_n \) mapping \( \{d_1, \ldots, d_n\} \) to \( \{f_1, \ldots, f_n\} \). But by Theorem 27, \( \sigma(B) \) is an exact Borel subalgebra of \( A_n \), and by construction it contains the idempotents \( \{f_1, \ldots, f_n\} \). We conclude that for any complete set of primitive orthogonal idempotents \( \{f_1, \ldots, f_n\} \) in \( A_n \), there is an exact Borel subalgebra \( C \) containing them. In particular, there is an exact Borel subalgebra containing the idempotents \( e_1, \ldots, e_n \).

**28 Example.** Consider \( A_2 \), the path algebra of \( 1 \xrightarrow{\alpha} 2 \) and let \( B \subset A_2 \) be a regular exact Borel subalgebra. Fix the order \( 1 \prec 2 \). Then we have

\[ \Delta(1) \cong L(1) \quad \text{and} \quad \Delta(2) \cong L(2). \]

Therefore, the quiver of \( B \) coincides with the quiver of \( A_2 \), and the (unique) regular exact Borel subalgebra is \( B = A_2 \). If we instead fix the order \( 2 \prec 1 \), we have

\[ \Delta(1) \cong P(1) \quad \text{and} \quad \Delta(2) \cong P(2). \]

Since there are no non-split extensions between projective modules, the quiver of \( B \) has no arrows. Thus, we have \( B \cong K \times K \). The obvious choice for \( B \) is then the subalgebra \( B = \langle e_1, e_2 \rangle \). Suppose \( xe_1 + ye_2 + z\alpha \in A_2 \) is an idempotent. Then we have:

\[ (xe_1 + ye_2 + z\alpha)^2 = x^2e_1 + y^2e_2 + z(x+y)\alpha = xe_1 + ye_2 + z\alpha \]

This implies that all possible complete sets of primitive orthogonal idempotents are \( \{e_1 + z\alpha, e_2 - z\alpha\} \), where \( z \in K \) is some scalar. Thus, we obtain exact Borel subalgebras \( B_2 = \langle e_1 + z\alpha, e_2 - z\alpha \rangle \). Since \( z\alpha \in \text{rad} A_2 \), the element \( 1 - z\alpha \) is invertible (with inverse \( 1 + z\alpha \)). We see that

\[ (1 + z\alpha)e_1(1 - z\alpha) = 1 + z\alpha \quad \text{and} \quad (1 + z\alpha)e_2(1 - z\alpha) = 1 - z\alpha, \]

so for all \( z \neq 0 \), the subalgebras \( B \) and \( B_2 \) are conjugate.
Next, we want to argue as in the discussion after Theorem 27 to conclude that there is always a regular exact Borel subalgebra containing the idempotents \( e_1, \ldots, e_n \). That argument crucially relies on inner automorphisms of quasi-hereditary algebras preserving exact Borel subalgebras, that is, on Theorem 27. Our next goal is to extend this statement to regular and homological exact Borel subalgebras.

For any automorphism \( \sigma \) of \( A \), there is an endofunctor \( \Phi_\sigma \) of \( A\)-mod defined by taking an \( A \)-module \( V \) to the module having the same underlying vector space, but with the action defined by \( a \cdot_\sigma v := \sigma(a) \cdot v \). For ease of notation, we denote \( \Phi_\sigma(V) \) by \( _\sigma V \), indicating that the action of \( A \) is twisted by the automorphism \( \sigma \). If \( B \subset A \) is a subalgebra, then so is \( C := \sigma(B) \). It is clear that \( \Phi_\sigma \) restricts, in a natural way, to a functor \( C\text{-mod} \rightarrow B\text{-mod} \).

Let \( b_1, \ldots, b_n \) be some complete set of primitive orthogonal idempotents for \( B \subset A \). Then, the set \( \{b_1, \ldots, b_n\} \) indexes the isomorphism classes of simple \( B \)-modules, and we denote by \( L_B(i) \), where \( 1 \leq i \leq n \), the simple module having as a basis the idempotent \( b_i \) and the action defined by letting \( b_i \) act as the identity and other basis elements as 0. Since \( \sigma : B \rightarrow C \) is an isomorphism, \( \sigma(b_1), \ldots, \sigma(b_n) \) is a complete set of primitive orthogonal idempotents for \( C \), and putting \( \sigma(b_i) = c_i \), for \( 1 \leq i \leq n \), we denote by \( L_C(i) \) the simple \( C \)-module having as a basis the idempotent \( c_i \) and the action defined by letting \( c_i \) act as the identity and other basis elements as 0. With this notation, consider the module \( _\sigma L_C(i) \) and fix the basis \( c_i \). On this basis, the idempotent acts as \( b_i \cdot_\sigma c_i := \sigma(b_i) \cdot c_i = c_i \cdot c_i = c_i \). Thus, with this notation, we have that \( _\sigma L_C(i) = L_B(i) \).

**29 Lemma.** Let \( A \) be an algebra and let \( B \subset A \) be a subalgebra. Let \( \sigma \) be an inner automorphism of \( A \), given by \( \sigma(x) = u^{-1}xu \) for some invertible element \( u \) and \( x \in A \). Put \( C = \sigma(B) \).

- (i) There is an isomorphism of \( A \)-\( A \)-bimodules \( A \cong _\sigma A \) given by left multiplication with \( u^{-1} \).
- (ii) For any \( C \)-module \( M \), there is an isomorphism of left \( A \)-modules \( _\sigma A \otimes_C M \rightarrow A \otimes_B _\sigma M \).
- (iii) For any \( C \)-module \( M \) there is an isomorphism of left \( A \)-modules \( A \otimes_C M \rightarrow A \otimes_B _\sigma M \).

**Proof.**

(i) That the map \( x \mapsto u^{-1}(x) \) is a linear bijection on \( A \) is clear, so we check that it is a homomorphism of bimodules. Indeed,

\[
u^{-1}(axb) = u^{-1}au^{-1}xb = \sigma(a)u^{-1}(x)b = a \cdot_\sigma u^{-1}(x) \cdot b.
\]

(ii) Define a map \( \varphi : _\sigma A \otimes_C M \rightarrow A \otimes_B _\sigma M \) on generators by \( a \otimes m \mapsto \sigma^{-1}(a) \otimes m \) and extend by linearity. Then, \( \varphi \) is well-defined:

\[
\varphi(ac \otimes C m) = \sigma^{-1}(ac) \otimes_B m = \sigma^{-1}(a)\sigma^{-1}(c) \otimes_B m = \sigma^{-1}(a) \otimes_B \sigma^{-1}(c) \cdot_\sigma m = \sigma^{-1}(a) \otimes_B c \cdot m = \varphi(a \otimes_C c \cdot m),
\]

for any \( c \in C \). Moreover, \( \varphi \) is a homomorphism of left \( A \)-modules:

\[
\varphi(x \cdot_\sigma (a \otimes_C m)) = \varphi(\sigma(x)a \otimes_C m) = \sigma^{-1}(\sigma(x)a) \otimes_B m = x\sigma^{-1}(a) \otimes_B m = x \cdot (\sigma^{-1}(a) \cdot_B m)
\]

for any \( x \in A \). Define a map \( \psi : A \otimes_B _\sigma M \rightarrow _\sigma A \otimes_C M \) on generators by \( a \otimes m \mapsto \sigma(a) \otimes m \) and extend by linearity. Then,

\[
\psi\varphi(a \otimes_C m) = \psi(\sigma^{-1}(a) \otimes_B m) = a \otimes_C m \quad \text{and} \quad \varphi\psi(a \otimes_B m) = \varphi(\sigma(a) \otimes_C m) = a \otimes_B m,
\]

so \( \varphi \) and \( \psi \) are mutually inverse bijections. We conclude that \( \varphi \) is an isomorphism.
(iii) Combine (i) and (ii).

\[\square\]

**30 Theorem.** Let \(A\) be a basic quasi-hereditary algebra and let \(B \subset A\) be a regular (respectively, homological) exact Borel subalgebra. Let \(\sigma\) be an inner automorphism of \(A\). Then, \(\sigma(B)\) is again a regular (respectively, homological) exact Borel subalgebra of \(A\).

**Proof.** Due to Theorem 27, we know that \(C\) is an exact Borel subalgebra. We interpret elements of the spaces \(\text{Ext}^k_B(L_B(i), L_B(j))\) and \(\text{Ext}^k_C(L_C(i), L_C(j))\) as exact sequences of length \(k + 2\), modulo equivalence. Since \(A \otimes_B -\) and \(A \otimes_C -\) are exact functors, applying them to exact sequences in \(\text{Ext}^k_B(L_B(i), L_B(j))\) and \(\text{Ext}^k_C(L_C(i), L_C(j))\) produces exact sequences in the spaces

\[\text{Ext}^k_A(A \otimes_B L_B(i), A \otimes_B L_B(j)) \quad \text{and} \quad \text{Ext}^k_A(A \otimes_C L_C(i), A \otimes_C L_C(j)).\]

Functoriality ensures that these maps are well-defined. Denote them by \(F_B\) and \(F_C\), respectively. We denote by \(\sigma_-\) the map \(\text{Ext}^k_C(L_C(i), L_C(j)) \to \text{Ext}^k_B(L_B(i), L_B(j))\) defined by taking an exact sequence

\[L_C(j) \to M_k \to \cdots \to M_1 \to L_C(i) \in \text{Ext}^k_C(L_C(i), L_C(j))\]

to the sequence

\[\sigma L_C(j) \to \sigma M_k \to \cdots \to \sigma M_1 \to \sigma L_C(i)\]

\[= \]

\[L_B(j) \to \sigma M_k \to \cdots \to \sigma M_1 \to L_B(i)\].

Consider the following diagram.

\[
\begin{array}{ccc}
\text{Ext}^k_B(L_B(i), L_B(j)) & \xleftarrow{\sigma_-} & \text{Ext}^k_C(L_C(i), L_C(j)) \\
F_B \downarrow & & \downarrow F_C \\
\text{Ext}^k_A(A \otimes_B L_B(i), A \otimes_B L_B(j)) & \xleftarrow{\sim} & \text{Ext}^k_A(A \otimes_C L_C(i), A \otimes_C L_C(j)) \\
\xrightarrow{\sim} & & \xrightarrow{\sim} \\
& \text{Ext}^k_A(\Delta(i), \Delta(j)) &
\end{array}
\]

Let \(L_C(j) \to M_k \to \cdots \to M_1 \to L_C(i)\) be an exact sequence, interpreted as an element of the space \(\text{Ext}^k_C(L_C(i), L_C(j))\). Applying first \(\sigma_-\) and then \(F_B\), we obtain

\[A \otimes_B L_B(j) \to A \otimes_B \sigma M_k \to \cdots \to A \otimes_B \sigma M_1 \to A \otimes_B L_B(i)\].

If we instead apply \(F_C\) to our original sequence \(L_C(j) \to M_k \to \cdots \to M_1 \to L_C(i)\), we get

\[A \otimes_C L_C(j) \to A \otimes_C M_k \to \cdots \to A \otimes_C M_1 \to A \otimes_C L_C(i)\].

Applying Lemma 29, we know there is an isomorphism of left \(A\)-modules, given by the composite

\[A \otimes_C M \to \sigma A \otimes_C M \to A \otimes_B \sigma M\].

This means that we may choose the dashed arrow in the diagram to be the map.
\[ A \otimes_C L_C(j) \to A \otimes_C M_k \to \cdots \to A \otimes_C M_1 \to A \otimes_C L_C(i) \]
\[ \implies \]
\[ A \otimes_B L_B(j) \to A \otimes_B M_k \to \cdots \to A \otimes_B M_1 \to A \otimes_B L_B(i), \]

which is an isomorphism and makes the top square commute, by construction. It follows that \( F_C \) is an isomorphism. For the second statement, the same argument as above ensures that the maps \( \text{Ext}^k_C(L_C(i), L_C(j)) \to \text{Ext}^k_A(A \otimes_C L_C(i), A \otimes_C L_C(j)) \) are isomorphisms for \( k \geq 2 \). For \( k = 1 \), the above diagram implies that we can solve for \( F_C \) to see that \( F_C \) is the composition of three epimorphisms, hence an epimorphism. \( \square \)

31 Proposition. Let \( B \subset A_n \) be a regular exact Borel subalgebra containing the idempotents \( e_1, \ldots, e_n \) and suppose that \( \Delta(i) = M(i, s_i) \).

(i) If \( i \leq s_i < n \), then \( \alpha_j \ldots \alpha_i \notin B \), for any \( i < j < s_i \), and \( \alpha_{s_i} \ldots \alpha_i \in B \).

(ii) If \( s_i = n \), that is, if \( \Delta(i) \) is projective, then \( \alpha_j \ldots \alpha_i \notin B \), for any \( i \leq j < n \).

Proof. (i) Assume towards a contradiction that \( \alpha_j \ldots \alpha_i \in B \) for some \( i < j < s_i \). Let \( p \in A_n \) be a path and consider the generator \( p \otimes e_i \) of \( A_n \otimes_B L(i) \). If \( s(p) \neq i \), then
\[ e_{j+1} \cdot (p \otimes e_i) = e_{j+1} \cdot (p \otimes e_{s(p)} e_i) = 0, \]

since \( e_{s(p)} \in B \). Similarly, if \( t(p) \neq j + 1 \), then
\[ e_{j+1} \cdot (p \otimes e_i) = e_{j+1} e_{t(p)} p \otimes e_i = 0. \]

If \( s(p) = i \) and \( t(p) = j + 1 \), then \( p = \alpha_j \ldots \alpha_i \), which we assume is contained in \( B \). We get
\[ \alpha_j \ldots \alpha_i \otimes e_i = e_j \otimes \alpha_j \ldots \alpha_i e_i = 0, \]

so that \( e_{j+1} \cdot (A_n \otimes_B L(i)) = 0 \). However, \( e_{j+1} \cdot \Delta(i) \neq 0 \), which is our desired contradiction.

For the second statement, assume towards a contradiction that \( \alpha_{s_i} \ldots \alpha_i \notin B \). Then, \( \alpha_{s_i} \ldots \alpha_i \otimes e_i \) is a non-zero element of \( A_n \otimes_B L(i) \). To see this, note that \( \alpha_j \ldots \alpha_i \notin B \) for any \( i < j < s_i \) by the first part.

It follows that also \( e_{s_i+1} \cdot (\alpha_{s_i} \ldots \alpha_i \otimes e_i) \neq 0 \). Now, we see that \( e_{s_i+1} \cdot \Delta(i) = 0 \), while \( e_{s_i+1} \cdot A_n \otimes_B L(i) \), leading to a similar contradiction as above.

(ii) Similar to the first part of (i). We remark that this is a separate case only because the first statement of part (ii) does not make sense when \( \Delta(i) \) is projective. \( \square \)

32 Proposition. Let \( B \subset A_n \) be a regular exact Borel subalgebra containing the idempotents \( e_1, \ldots, e_n \). Let \( S_B \) be the set of paths which are contained in \( B \) by Proposition 31 together with the idempotents \( e_1, \ldots, e_n \). Then, \( S_B \) is a minimal generating set for \( B \).

Proof. It is well known that any basic and connected finite-dimensional algebra over an algebraically closed field \( K \) is isomorphic to the quotient of the path algebra of some quiver by some admissible ideal. We recall the construction of this isomorphism. For details, we refer to [1, Theorem 3.7].

Let \( \Phi : KQ_B \to B \) be the isomorphism mentioned above. Then, \( \Phi \) is defined on the trivial paths in \( Q_B \) by \( \Phi(e_i) = e_i \), for \( 1 \leq i \leq n \). Note that this makes sense, as we may talk about the idempotents \( e_i \) as elements of \( KQ_B \) and of \( B \).

Consider the set of arrows \( i \to j \) in \( Q_B \). As discussed prior to Proposition 25, the cardinality of this set equals \( \dim \text{Ext}^1_{A_n}(\Delta(i), \Delta(j)) \), which may be 0 or 1, according to Theorem 15. Therefore, this set is either
empty or contains a single arrow, which we denote by \( x_i^j \). The arrow \( x_i^j \) should be mapped to an element \( y \in \text{rad} B \), so that the residue class \( y + \text{rad}^2 B \) forms a basis of \( e_j \frac{\text{rad} B}{\text{rad}^2 B} e_i \). This is achieved by defining \( \Phi(x_i^j) = \alpha_{j-1} \cdots \alpha_i \). Since we know that \( \Phi \) is an isomorphism, we know that the idempotents \( e_i \), together with paths of the form \( \Phi(x_i^j) \), constitute a minimal generating set for \( B \).

Recall that the arrow \( x_i^j \in Q_B \) exists precisely when \( \dim \text{Ext}^1_{A_n}(\Delta(i), \Delta(j)) = 1 \). Consider the following picture. The dashed edge between \( b \) and \( c \) signifies that \( c \) is a vertex in the right subtree of \( b \) (not necessarily the vertex \( r(b) \)), such that there exists a non-zero homomorphism from \( \Delta(c) \) to \( \Delta(b) \).

![Diagram of a tree with arrows](image)

We know, using Theorem 15, that possible non-zero extensions between standard modules appear as either \( \text{Ext}^1_{A_n}(\Delta(b), \Delta(a)) \) or as \( \text{Ext}^1_{A_n}(\Delta(c), \Delta(a)) \), where \( a, b \) and \( c \) are configured as above. Since \( b \) labels \( \ell(a) \), the vertex \( a \) is the first vertex visited immediately after visiting the entire right subtree of \( a \). Therefore, \( a = s_b + 1 \). Since \( c \) is in the right subtree of \( b \), we have \( s_b = s_c \), and therefore \( a = s_c + 1 \). Plugging this into the above, we see that the generators of \( B \) are of the form

\[
\Phi(x_i^{s_i+1}) = \alpha_{s_i} \cdots \alpha_i,
\]

whence it follows that \( S_B \) is a minimal generating set.

\[\square\]

**33 Remark.** Let \( B \) be the exact Borel subalgebra \( B \subset A_n \) containing the idempotents \( e_1, \ldots, e_n \). According to Proposition 32, we have a minimal generating set, \( S_B \), for \( B \). Then, using [23, Theorem 3.6], which states that exact Borel subalgebras of a basic quasi-hereditary algebra are unique up to application of an inner automorphism, it follows that the exact Borel subalgebras are precisely those given as \( C = \langle u^{-1} S_B u \rangle \), where \( u \in A_n \) is some invertible element. Moreover, since \( A_n \) always admits a regular exact Borel subalgebra, all exact Borel subalgebras of \( A_n \) are regular, according to Theorem 30.

In the particular case of \( A_n \), we have another proof of this fact. According to [16], any exact Borel subalgebra of a given quasi-hereditary algebra is unique up to a (not necessarily inner) automorphism. But, by [11], any automorphism of \( A_n \) is inner, and we argue as in the previous paragraph.

**34 Example.** We consider the binary search tree

![Binary search tree](image)

We saw, in the example following Theorem 15, that the Ext-algebra of standard modules over \( A_n \) is given by the quiver
modulo the relations \( I = \langle \varepsilon_2 \varepsilon_1^2 \rangle \). Here, there are three non-zero spaces of extensions, namely

\[
\dim \operatorname{Ext}^1_{A_n}(\Delta(1), \Delta(2)) = \dim \operatorname{Ext}^1_{A_n}(\Delta(2), \Delta(4)) = \dim \operatorname{Ext}^1_{A_n}(\Delta(3), \Delta(4)) = 1.
\]

Letting \( B \subset A_n \) be the regular exact Borel subalgebra containing \( e_1, \ldots, e_n \), we see that the quiver of \( B \) is

\[
1 \quad \xrightarrow{a} \quad 2 \quad \xrightarrow{b} \quad 3 \quad \xrightarrow{c} \quad 4 \quad \xrightarrow{d} \quad 5 \quad \xrightarrow{e} \quad 6.
\]

The arrows \( a, b \) and \( c \) correspond to the paths \( \alpha_1, \alpha_3 \alpha_2 \) and \( \alpha_3 \), respectively. A generating set \( S \) for \( B \) is then

\[
S = \{e_1, e_2, e_3, e_4, e_5, e_6, \alpha_1, \alpha_3, \alpha_3 \alpha_2\}.
\]

### 3.3. \( A_\infty \)-structure on \( \operatorname{Ext}^*_{A_n}(\Delta, \Delta) \)

The notion of an \( A_\infty \)-algebra is meant to capture the idea of an algebra which is not strictly associative, but associative only up to a system of “higher homotopies”. There is also the natural “multi-object” version, called an \( A_\infty \)-category, which we define below. In this section we are particularly interested in \( \operatorname{Ext}^*_{A_n}(\Delta, \Delta) \), which always carries the structure of an \( A_\infty \)-algebra. However, we will view \( \operatorname{Ext}^*_{A_n}(\Delta, \Delta) \) as an \( A_\infty \)-category with \( n \) objects, given by the standard modules. This will be useful since it enables us to enforce compatibility of the higher homotopies with the natural idempotents of \( \operatorname{Ext}^*_{A_n}(\Delta, \Delta) \): the identity homomorphisms \( 1_{\Delta(i)} \).

### 35 Definition. [10] Let \( K \) be a field. An \( A_\infty \)-category \( A \) consists of the following:

(i) a class of objects \( \operatorname{Ob}(A) \);

(ii) for each pair of objects \( A \) and \( B \), a \( \mathbb{Z} \)-graded vector space \( \operatorname{Hom}_A(A, B) \);

(iii) for all \( n \geq 1 \) and objects \( A_0, \ldots, A_n \), a homogeneous linear map

\[
m_n : \operatorname{Hom}_A(A_{n-1}, A_n) \otimes \operatorname{Hom}_A(A_{n-2}, A_{n-1}) \otimes \cdots \otimes \operatorname{Hom}_A(A_0, A_1) \to \operatorname{Hom}_A(A_0, A_n)
\]

of degree \( 2 - n \) such that

\[
\sum_{r+s+t=n} (-1)^{r+s+t} m_{r+1+t}(1^{\otimes r} \otimes m_s \otimes 1^{\otimes t}) = 0.
\]

Viewing \( \operatorname{Ext}^*_{A_n}(\Delta, \Delta) \) as an \( A_\infty \)-category as outlined above amounts to the following: if we have extensions \( \varepsilon_i \) arranged as

\[
\Delta(i_1) \xrightarrow{\varepsilon_1} \Delta(i_2) \xrightarrow{\varepsilon_2} \cdots \xrightarrow{\varepsilon_{\ell}} \Delta(i_{\ell}) \xrightarrow{\varepsilon_{\ell + 1}} \Delta(i_{\ell + 1})
\]
then $m_\ell(\varepsilon, \ldots, \varepsilon)$ is an extension from $\Delta(i_1)$ to $\Delta(i_{\ell+1})$. Writing out the $A_\infty$-relations for the first few $n$, we get:

- $n = 1$: $m_1 m_1 = 0$, meaning that $A$ is a complex.
- $n = 2$: $m_1 m_2 = m_2(m_1 \otimes 1 + 1 \otimes m_1)$, meaning that $m_1$ is a derivation with respect to $m_2$.
- $n = 3$: $m_2 (1 \otimes m_2 - m_2 \otimes 1) = m_1 m_3 + m_3 (m_1 \otimes 1 \otimes 1 + 1 \otimes m_1 \otimes 1 + 1 \otimes 1 \otimes m_1)$, meaning that $m_2$ is associative up to a homotopy given by $m_3$.

When evaluating at actual elements, even more signs appear according to the Koszul sign rule:

$$(f \otimes g)(x \otimes y) = (-1)^{|g||x|} f(x) \otimes g(y).$$

Here, $f$ and $g$ are homogeneous maps and $x$ and $y$ are homogeneous elements. The vertical bars denote degree. A graded category $A$ such that any $A_\infty$-structure on it satisfies $m_n = 0$ for $n \geq 3$ is called intrinsically formal.

36 Proposition. $\text{Ext}^*_{A_n}(\Delta, \Delta)$ is intrinsically formal.

Proof. Let $\varphi_1, \ldots, \varphi_\ell$ be homogeneous extensions in $\text{Ext}^*_{A_n}(\Delta, \Delta)$. Put $i = \sum_{j=1}^\ell \deg \varphi_j$. Since $m_\ell$ is of degree $2 - \ell$, the extension $m_\ell(\varphi, \ldots, \varphi_1)$ is of degree $2 - \ell + i$. Because $A_n$ is hereditary, the extension $m_\ell(\varphi, \ldots, \varphi_1)$ is of degree 0 or 1 if it is non-zero. This implies that there are the following cases:

$$2 - \ell + i = \begin{cases} 
0, & \text{if } i = \ell - 2 \\
1, & \text{if } i = \ell - 1. 
\end{cases}$$

Assume that $i = \ell - 1$ and that $m_\ell(\varphi, \ldots, \varphi)$ produces an extension in $\text{Ext}^1_{A_n}(\Delta(c), \Delta(a))$. We know that if such an extension is non-zero, the vertices $a$ and $c$ must be configured in the following way: here the dashed edge represents a path which is the concatenation of edges between a vertex $v$ and $r(v)$.

![Diagram](image)

Since $i = \ell - 1$, there is precisely one argument $\varphi_j$ of $m_\ell(\varphi, \ldots, \varphi_1)$ which is a homomorphism rather than a proper extension. Consider the following picture, where the dashed edge represents the homomorphism $\varphi_j : \Delta(x) \to \Delta(y)$ and the wavy arrows from $c$ to $x$ and from $y$ to $a$ represent the compositions $\varphi_{j-1}, \ldots, \varphi_1$ and $\varphi_\ell, \ldots, \varphi_{\ell+1}$, respectively.
Note that either of the wavy edges may represent a path of length > 1, rather than actual edges. Moreover, depending on j, the edge between c and x or the edge between y and a may be “empty”. We see that it is impossible to have the allowed configuration from the previous picture.

If i = ℓ – 2, the outcome of m_{ℓ}(φ_ℓ, . . . , φ_1) is an element of Hom_{A_n}(Δ(c), Δ(a)). Since ℓ ≥ 3, there is at least one argument φ_j of m_{ℓ}(φ_ℓ, . . . , φ_1) which is a proper extension. According to the allowed configuration above, this means that c is in the left subtree of a. Using Theorem 15, we get that Hom_{A_n}(Δ(c), Δ(a)) = 0. □

4. Ext-algebras of standard modules over deconcatenations

In the previous section, we studied the path algebra of A_n with linear orientation. To extend this study to arbitrary orientations, we apply the method of “deconcatenation” of a quiver Q at a sink or a source. In [9], the authors use this method to study the different quasi-hereditary structures of the path algebra KQ in terms of the quasi-hereditary structures on path algebras of certain subquivers of Q. In the present section, we extend this study to the Ext-algebra of standard modules.

37 Definition. [9, Definition 3.1] Let Q be a finite connected quiver and let v be a vertex of Q which is a sink or a source. A deconcatenation of Q at the vertex v is a union Q^1 ⊔ . . . ⊔ Q^ℓ of full subquivers Q^i of Q such that the following hold:

(i) Each Q^i is a connected full subquiver of Q with v ∈ Q^i_0.
(ii) For all i, j ∈ {1, . . . , ℓ}, such that i ≠ j, there holds Q_0 = (Q^i_0 \{v\}) ⊔ . . . ⊔ (Q^ℓ_0 \{v\}) ⊔ {v} and Q^i_0 ∩ Q^j_0 = \{v\}.
(iii) For x ∈ Q^i_0 \{v\} and y ∈ Q^j_0 \{v\}, where i, j ∈ {1, . . . , ℓ} are such that i ≠ j, there are no arrows between x and y in Q.

38 Example. Consider the quiver Q = 1 ← 2 ← 3 → 4 → 5. We see that Q has a deconcatenation at the source 3:

( 1 ← 2 ← 3 ) ⊔ ( 3 → 4 → 5 ).

We follow the notation of [9] and put

A^ℓ = A/⟨e_u | u ∈ Q_0 \ Q^ℓ_0⟩.

Then, A surjects onto A^ℓ, and consequently, there is a fully faithful and exact functor F^ℓ : A^ℓ-mod → A-mod. Regarding A^ℓ-mod as a full subcategory of A-mod via the functor F^ℓ, we remark that an A-module M is an A^ℓ-module if and only if e_uM = 0 for any u ∈ Q_0 \ Q^ℓ_0.
39 Lemma. Let $N$ be an $A^\ell$-module, let $M$ be an $A$-module and consider the quotient $\overline{M} = M / \sum e_u \cdot M$, where the sum ranges over all $u \in Q_0^\ell \setminus \{v\}$. Then, $\overline{M}$ is an $A^\ell$-module and there are isomorphisms of vector spaces

$$\text{Hom}_A(\overline{M}, N) \cong \text{Hom}_{A^\ell}(\overline{M}, N) \cong \text{Hom}_A(M, N).$$

Proof. The first isomorphism is immediate from the fact that $F^\ell$ is fully faithful. To see the second isomorphism, put $e^\ell = \sum e_u$ where the sum is as in the statement of the Lemma. By construction, $\overline{M}$ is the largest quotient of $M$ contained in $A^\ell$-mod, so that there is an isomorphism of $A$-modules

$$\overline{M} \cong (A/(e^\ell)) \otimes_A M.$$ 

Then, letting $G^\ell : A\text{-mod} \to A^\ell\text{-mod}$ denote the functor defined by $X \mapsto \overline{X}$, we have an adjoint pair $(G^\ell, F^\ell)$. Consequently, we have

$$\text{Hom}_A(M, N) \cong \text{Hom}_A(M, G^\ell(N)) \cong \text{Hom}_{A^\ell}(G^\ell(M), N) \cong \text{Hom}_{A^\ell}(\overline{M}, N),$$

finishing the proof.

40 Lemma. [9, Lemma 3.3] Let $Q^1 \sqcup Q^2$ be a deconcatenation of $Q$ at a sink or source $v$ and let $\ell = 1, 2$.

(i) For all $i \in Q^\ell_0$, we have $L(i) \cong L^\ell(i)$.
(ii) For all $i \in Q_0^\ell \setminus \{v\}$, we have $P(i) \cong P^\ell(i)$ and $I(i) \cong I^\ell(i)$.
(iii) If $v$ is a sink, we have $L(v) \cong P(v) \cong P^\ell(v)$, for $\ell = 1, 2$.
(iv) If $v$ is a source, we have $L(v) \cong I(v) \cong I^\ell(v)$, for $\ell = 1, 2$.
(v) For any non-zero $A$-module $M$, if both top $M$ and soc $M$ are simple, then we have either $M \in A^1$-mod or $M \in A^2$-mod.
(vi) Let $M$ be an $A^\ell$-module and let $i \in Q_0^\ell$ be a vertex. If $[M : L(i)] \neq 0$, then $i \in Q_0^\ell$.

Given a partial order $\preceq$ on $Q_0$ and a deconcatenation $Q_1 \sqcup Q_2$, of $Q$ at a sink or source $v$, we may form the restriction $\preceq|_{Q^\ell_0}$ to obtain a partial order $\preceq^\ell = \preceq|_{Q^\ell_0}$ on $Q^\ell_0$.

41 Lemma. [9, Lemma 3.4] Let $Q_1 \sqcup Q_2$ be a deconcatenation of $Q$ at a sink or source $v$ and let $\preceq$ be a partial order on $Q_0$. Denote by $\Delta$ and $\nabla$ the sets of standard and costandard $A$-modules associated to $\preceq$, respectively. Denote by $\Delta^\ell$ and $\nabla^\ell$ the sets of standard and costandard $A^\ell$-modules, respectively.

(i) For any $i \in Q^\ell_0 \setminus \{v\}$, there are isomorphisms of $A$-modules $\Delta(i) \cong \Delta^\ell(i)$ and $\nabla(i) \cong \nabla^\ell(i)$, for $\ell = 1, 2$.
(ii) If $v$ is a sink, there are isomorphisms of $A$-modules $L(v) \cong \Delta(v) \cong \Delta^\ell(v)$.
(iii) If $v$ is a source, there are isomorphisms of $A$-modules $L(v) \cong \nabla(v) \cong \nabla^\ell(v)$.
(iv) If $A$ is quasi-hereditary with respect to $\preceq$, then $A^\ell$ is quasi-hereditary with respect to $\preceq^\ell$, for $\ell = 1, 2$.

Let $Q = Q^1 \sqcup Q^2$ be a deconcatenation of $Q$ at a source or sink $v$ and let $\preceq^\ell$ be a partial order on $Q^\ell_0$ for $\ell = 1, 2$. Put $\Gamma = 2$ and $\Sigma = 1$. Define a partial order $\preceq = \preceq^{\ell_1} \sqcup \preceq^{\ell_2}$ on $Q_0$ as follows. For $i, j \in Q_0$, we say that $i \prec j$ if one of the following holds.

(i) We have $i, j \in Q^\ell_0$ and $i \prec^\ell j$, for some $\ell$.
(ii) We have $i \in Q^\ell_0$, $j \in Q^\Gamma_0$, $i \prec^\ell v$ and $v \prec^\ell j$. 


42 Lemma. [9, Lemma 3.5] Let $Q^1 \sqcup Q^2$ be a deconcatenation of $Q$ at a sink or source $v$. Let $\leq^\ell$ be a partial order on $Q_0^\ell$ and denote by $\Delta^\ell$ and $\nabla^\ell$ the sets of standard and costandard $A^\ell$-modules, associated to $\leq^\ell$, for $\ell = 1, 2$, respectively. Denote by $\Delta$ and $\nabla$ the sets of standard and costandard $A$-modules, respectively, associated to $\leq = \leq (\leq^1, \leq^2)$. Then, we have the following.

(i) For any $i \in Q_0^1 \setminus \{v\}$, there are isomorphisms of $A$-modules $\Delta(i) \cong \Delta^\ell(i)$ and $\nabla(i) \cong \nabla^\ell(i)$.
(ii) If $v$ is a sink, there are isomorphisms of $A$-modules $L(v) \cong \Delta(v) \cong \Delta^\ell(v)$.
(iii) If $v$ is a source, there are isomorphisms of $A$-modules $L(v) \cong \nabla(v) \cong \nabla^\ell(i)$.
(iv) If $\leq^\ell$ defines a quasi-hereditary structure on $A^\ell$ for $\ell = 1$ and $\ell = 2$, then $\leq = \leq (\leq^1, \leq^2)$ defines a quasi-hereditary structure on $A$.

We have the following observation about homomorphism spaces between indecomposable projective modules.

43 Lemma. Let $Q^1 \sqcup Q^2$ be a deconcatenation of $Q$ at the sink or source $v$. Then, for $i \in Q_0^1 \setminus \{v\}$ and $j \in Q_0^2 \setminus \{v\}$, there holds $\text{Hom}_A(P(i), P(j)) = \text{Hom}_A(P(j), P(i)) = 0$.

44 Lemma. Assume that $v$ is a source. Let $i \in Q_0^1 \setminus \{v\}$ and let $P^\bullet \to \Delta^\ell(i)$ be a minimal projective resolution of $\Delta^\ell(i)$ as an $A$-module. Then no term $P^k$ of the projective resolution $P^\bullet$ contains the projective module $P^\ell(v)$ as a direct summand.

Proof. We proceed by induction. The basis is clear, as $P^0 = P^\ell(i)$. Consider the following picture.

```
... \rightarrow P^{k+1} \xrightarrow{d_{k+1}} P^k \xrightarrow{d_k} \ldots \\
\text{ker } d_k
```

Assume that the statement holds for the module $P^k$ and assume towards a contradiction that the module $P^{k+1}$ contains a direct summand $P^\ell(v)$. By construction, $P^{k+1}$ is a projective cover of $\text{ker } d_k$, thus we conclude that the module top $\text{ker } d_k$ contains a direct summand $L^\ell(v)$. This fact, in turn, implies that there is a direct summand $P^\ell(t)$ of $P^k$, such that $[P^\ell(t) : L^\ell(v)] > 0$. Since a basis of $P^\ell(t)$ is given by paths in $A^\ell$ starting in $t$, this implies that there is a path in $A^\ell$ from $t$ to $v$. This is our desired contradiction, as $v$ was assumed to be a source.

45 Proposition. Let $Q^1 \sqcup Q^2$ be a deconcatenation of $Q$ at the sink or source $v$. Let $i \in Q_0^1 \setminus \{v\}$ and $j \in Q_0^2 \setminus \{v\}$.

(i) If $v$ is a source, there holds

$$\text{Ext}^k_A(\Delta(i), \Delta(j)) = \text{Ext}^k_A(\Delta(j), \Delta(i)) = 0,$$

for all $k \geq 0$.

(ii) If $v$ is a sink and maximal or minimal with respect to $\leq^c$, there holds

$$\text{Ext}^k_A(\Delta(i), \Delta(j)) = \text{Ext}^k_A(\Delta(j), \Delta(i)) = 0,$$

for all $k \geq 0$. 


Proof. (i) Let \( v \) be a source and let \( P^* \to \Delta(i) \) be a minimal projective resolution. Applying \( \text{Hom}_A(\_ , \Delta(j)) \) to \( P^* \), we consider the space \( \text{Hom}_A(P_k, \Delta(j)) \). This space decomposes into a direct sum with summands of the form \( \text{Hom}_A(P(\ell), \Delta(j)) \), where \( \ell \in Q_0 \setminus \{v\} \), according to Lemma 44. Then, it follows that
\[
\dim \text{Hom}_A(P(\ell), \Delta(j)) = \left[ \Delta(j) : L(\ell) \right] = 0.
\]
Consequently, there holds \( \dim \text{Hom}_A(P_k, \Delta(j)) = 0 \), which implies \( \text{Ext}_A^k(\Delta(i), \Delta(j)) = 0 \).

(ii) Let \( v \) be a sink and assume that \( v \) is maximal with respect to \( \leq^c \). Suppose there is an extension of degree \( k \) from \( \Delta(i) \) to \( \Delta(j) \). Then, there is a non-zero homomorphism in the space \( \text{Hom}_A(P_k, \Delta(j)) \). Note that the indecomposable direct summands of \( P_k \) are of the form \( P(\ell) \), with \( \ell \in Q_0 \). Since the \( \text{Hom} \)-functor is additive, consider the space \( \text{Hom}_A(P(\ell), \Delta(j)) \). For this space to have a non-zero element, we must have \( \left[ \Delta(j) : L(\ell) \right] > 0 \). The only possibility for such an \( \ell \) is \( \ell = v \). But then \( \left[ \Delta(j) : L(v) \right] > 0 \), implying \( v \leq^c j \), contradicting maximality of \( v \).

Assume instead that \( v \) is minimal with respect to \( \leq^c \). For \( k = 0 \), the claim of the proposition is
\[
\text{Hom}_A(\Delta(i), \Delta(j)) = 0,
\]
which is true because \( i \in Q_0 \setminus \{v\} \) and \( j \in Q_0 \setminus \{v\} \).

Next, we claim that \( \left[ P_k : L(v) \right] = 0 \) for all \( k \geq 1 \) and proceed by induction. Consider the following picture:

\[
\begin{array}{c}
\cdots \to P^1 \xrightarrow{\pi_i} P(i) \xrightarrow{\pi_i} \Delta(i) \\
\text{ker} \pi_i
\end{array}
\]

Since \( A \) is quasi-hereditary, the module \( \text{ker} \pi_i \) is contained in \( \mathcal{F}(\Delta(\leq^c i)) \), by which we mean that each standard module \( \Delta(a) \) occurring in its filtration satisfies \( i \leq^c a \). If
\[
\dim \text{Ext}_A^1(\Delta(i), \Delta(j)) > 0,
\]
then \( \dim \text{Hom}_A(P^1, \Delta(j)) > 0 \), which can only happen when \( P^1 \) has a direct summand isomorphic to \( L(v) \). This implies that \( L(v) \subset \text{top ker} \pi_i \), which means that some \( \Delta(b) \) with \( \left[ \Delta(b) : L(v) \right] > 0 \) occurs in the standard filtration of \( \text{ker} \pi_i \). Since \( L(v) \) is contained in \( \text{top ker} \pi_i \), the module \( \Delta(v) \) occurs in its standard filtration, since any other standard module \( \Delta(c) \) with a composition factor \( L(v) \) must satisfy \( L(v) \subset \text{rad} \Delta(c) \). Since \( \text{ker} \pi_i \in \mathcal{F}(\Delta(\leq^c i)) \), this implies \( v \leq^c i \), contradicting minimality of \( v \).

For the inductive step, assume that the module \( P^k \) is contained in \( \mathcal{F}(\Delta(\leq^c i)) \). Then, so is \( \text{ker} \pi_d(k) \), and we argue as above.

We note that the above arguments are symmetric with respect to swapping \( Q^1 \) and \( Q^2 \), so we are done. \( \square \)

46 Proposition. Let \( Q^1 \square Q^2 \) be a deconcatenation at a sink or source \( v \).

(i) If \( i, j \in Q_0 \setminus \{v\} \), there holds \( \text{Ext}_A^k(\Delta^e(i), \Delta^e(j)) \cong \text{Ext}_A^k(\Delta(i), \Delta(j)) \), for all \( k \geq 0 \).

(ii) If \( v \) is a sink, then \( \text{Ext}_A^k(\Delta^e(v), \Delta^e(j)) \cong \text{Ext}_A^k(\Delta(v), \Delta(j)) \), for all \( k \geq 0 \).

(iii) If \( v \) is a sink, then \( \text{Ext}_A^k(\Delta^e(i), \Delta^e(v)) \cong \text{Ext}_A^k(\Delta(i), \Delta(v)) \), for all \( k \geq 0 \).

Proof. Let \( P^* \to \Delta^e(i) \) and \( Q^* \to \Delta^e(j) \) be minimal projective resolutions. By Lemma 40, there is an isomorphism of \( A \)-modules \( P^e(x) \cong P(x) \), for all \( x \in Q_0 \setminus \{v\} \). Note that this isomorphism is the one induced by the functor \( F^e \), that is, \( F^e(P^e(x)) \cong P(x) \). Similarly, we have \( F^e(\Delta^e(i)) \cong \Delta(i) \) and \( F^e(\Delta^e(j)) \cong \Delta(j) \).
(i) The functor $F^\ell : A^\ell$-mod $\to A$-mod is fully faithful and exact. By Lemma 44, the terms of the projective resolution $P^\bullet \to \Delta^\ell(i)$ do not contain any direct summands isomorphic to $P^\ell(v)$. We conclude that $F^\ell(P^\bullet) \to F^\ell(\Delta^\ell(i)) \cong \Delta(i)$ is a minimal projective resolution. Similarly, we have that $F^\ell(Q^\bullet) \to F^\ell(\Delta^\ell(j)) \cong \Delta(j)$ is a minimal projective resolution. The statement follows.

(ii) If $v$ is a sink, we have $P^\ell(v) \cong L(v) \cong P(v)$ as $A$-modules, by Lemma 40, and $\Delta^\ell(v) \cong L(v) \cong \Delta(v)$ as $A$-modules, by Lemma 41. Thus, the statement follows by applying the functor $F^\ell$ to the minimal projective resolutions of $\Delta^\ell(v)$ and $\Delta^\ell(j)$.

(iii) Similar to (ii).

\[ \square \]

We recall the notation of [9], where $\overline{1} = 2$ and $\overline{2} = 1$.

47 Lemma. Let $Q_1 \sqcup Q_2$ be a deconcatenation of $Q$ at the source $v$ and let $i \in Q_0^I \setminus \{v\}$. Then, there are isomorphisms of vector spaces

\[ \Hom_A(P(i), P^\ell(v)) \cong \Hom_A(P(i), P(v)) \quad \text{and} \quad \Hom_A(P^\ell(v), P(i)) \cong \Hom_A(P(v), P(i)), \quad \text{for } \ell = 1, 2. \]

\[ \text{Proof.} \] The second statement is precisely Lemma 39 applied to the modules $P(i) \cong P^\ell(i)$ and $P(v)$. Note that in the notation of Lemma 39, we have $P^\ell(v) = P(v)$. For the first statement, let $p : P(v) \to P^\ell(v)$ denote the natural epimorphism. Then, there is a short exact sequence

\[ 0 \longrightarrow \ker p \longrightarrow P(v) \xrightarrow{p} P^\ell(v) \longrightarrow 0. \]

Note that $P^\ell(v)$ is an $A^\ell$-module while $\ker p$ is an $A^I$-module. Applying $\Hom_A(P(i), -)$, we obtain the long exact sequence:

\[ 0 \longrightarrow \Hom_A(P(i), \ker p) \longrightarrow \Hom_A(P(i), P(v)) \longrightarrow \Hom_A(P(i), P^\ell(v)) \longrightarrow \Ext_A(P(i), \ker p) \longrightarrow \ldots \]

By our previous observation, we have $\Hom_A(P(i), \ker p) = 0$. Moreover, since $P(i)$ is projective, we have $\Ext_A(P(i), \ker p) = 0$. We conclude that $\Hom_A(P(i), P(v)) \cong \Hom_A(P(i), P^\ell(v))$.

\[ \square \]

48 Lemma. Let $v$ be a source, let $P^\bullet \to \Delta^1(v)$ be a minimal projective resolution with terms $P^k$, $k \geq 0$ and let $Q^\bullet \to \Delta^2(v)$ be a minimal projective resolution with terms $Q^k$, $k \geq 0$. Then, there is a minimal projective resolution $T^\bullet \to \Delta(v)$, with terms $T^k = P^k \oplus Q^k$, for $k \geq 1$, and $T^0 = P(v)$.

\[ \text{Proof.} \] We proceed by induction. For the case $k = 1$, consider the projection $p : P(v) \to \Delta(v)$. The module $\ker p$ has a basis consisting of paths $q$ in $A$, such that $s(q) = v$ and $t(q) = j$ with $j \neq v$. Let $x_1, \ldots, x_n$ be those paths that satisfy $t(x_i) \in Q_0^I$ and let $y_1, \ldots, y_m$ be those paths that satisfy $t(y_j) \in Q_0^I$. Put $X = \text{span}\{x_1, \ldots, x_n\}$ and $Y = \text{span}\{y_1, \ldots, y_m\}$. Clearly, $\ker p = X \oplus Y$ as vector spaces. Since the action of $A$ on $X$ and $Y$ is given by left multiplication, we see that, in fact, $\ker p = X \oplus Y$ as $A$-modules.

It is clear that, as an $A^1$-module, $X$ is isomorphic to the kernel of the projection $p^1 : P^1(v) \to \Delta^1(v)$, so $P^1$ is a projective cover of $X$. Similarly, as an $A^2$-module, $Y$ is isomorphic to the kernel of the projection $p^2 : P^2(v) \to \Delta^2(v)$, so $Q^1$ is a projective cover of $Y$. When $k = 2$, we consider the picture:

\[ \begin{array}{ccc}
T^2 & \longrightarrow & P^1 \oplus Q^1 \\
\downarrow & & \downarrow (d^1, e^1) \\
\ker d^1 \oplus \ker e^1 & \longrightarrow & P(v) \\
& & \downarrow X \oplus Y
\end{array} \]
It is clear that we may view the map from $P^1 \oplus Q^1$ to $P(v)$ as $(d^1, e^1)$. It then follows that its kernel is the direct sum $\ker d^1 \oplus \ker e^1$, which has a projective cover $T^2 = P^2 \oplus Q^2$. This finishes the basis of the induction. Consider the following picture:

\[
\begin{array}{c}
T^{k+1} \xrightarrow{f^{k+1}} P^k \oplus Q^k \\
\downarrow \ker d^k \oplus \ker e^k \\
\end{array}
\]

\[
\begin{array}{c}
P^k \oplus Q^k \xrightarrow{(d^k 0 \ 0 e^k)} P^{k-1} \oplus Q^{k-1} \\
\end{array}
\]

That the right matrix above is diagonal follows from Lemma 43 and the induction hypothesis. Then, the kernel of the map \( \begin{pmatrix} d^k & 0 \\ 0 & e^k \end{pmatrix} \) is isomorphic to \( \ker d^k \oplus \ker e^k \), and a projective cover is therefore \( P^{k+1} \oplus Q^{k+1} \).

This shows that \( T^{k+1} = P^{k+1} \oplus Q^{k+1} \) and \( f^{k+1} = \begin{pmatrix} d^{k+1} & 0 \\ 0 & e^{k+1} \end{pmatrix} \). \( \square \)

49 Proposition. Let \( Q_1 \sqcup Q_2 \) be a deconcatenation of \( Q \) at the source \( v \) and let \( i \in Q_0 \setminus \{ v \} \). Then, there are isomorphisms of vector spaces

\[
\Ext^k_A(\Delta^\ell(i), \Delta^\ell(v)) \cong \Ext^k_A(\Delta(i), \Delta(v)) \quad \text{and} \quad \Ext^k_A(\Delta^\ell(v), \Delta^\ell(i)) \cong \Ext^k_A(\Delta(v), \Delta(i)),
\]

for all \( k \geq 0 \).

Proof. Let \( R^\bullet \to \Delta^\ell(i) \) and \( P^\bullet \to \Delta^\ell(v) \) be minimal projective resolutions. According to Lemma 48, there is a minimal projective resolution \( T^\bullet \to \Delta(v) \) with terms

\[
T^k = \begin{cases} 
P^k \oplus Q^k, & \text{if } k \geq 1 \\
P(v), & \text{if } k = 0, 
\end{cases}
\]

where the modules \( Q^k \) constitute a minimal projective resolution of \( \Delta^\ell(v) \). Let \( \alpha : T^\bullet \to P^\bullet \) be the natural chain map, where \( p : P(v) \to P^\ell(v) \) is the natural epimorphism:

\[
\begin{array}{c}
\ldots \to P^2 \oplus Q^2 \to P^1 \oplus Q^1 \to P(v) \\downarrow \begin{pmatrix} \text{id} & 0 \\ 0 & 0 \end{pmatrix} \downarrow \begin{pmatrix} \text{id} & 0 \\ 0 & 0 \end{pmatrix} \downarrow p \\
\ldots \to P^2 \to P^1 \to P^\ell(v) \\
\end{array}
\]

We claim that the map \( \Hom_{K(A)}(R^\bullet, T^\bullet[k]) \to \Hom_{K(A)}(R^\bullet, P^\bullet[k]) \) given by composition with \( \alpha \) is an isomorphism. Consider the following picture:

\[
\begin{array}{c}
\ldots \to R^{k+1} \rightarrow R^k \to \ldots \\
\downarrow \downarrow \downarrow \\
\ldots \to P^1 \oplus Q^1 \to P(v) \to \ldots \\
\downarrow \downarrow \downarrow \\
\ldots \to P^1 \to P^\ell(v) \rightarrow 0 \\
\end{array}
\]
For \( k \geq 1 \), we have \( \text{Hom}_{A}(R^{k+m}, P^m \oplus Q^m) \cong \text{Hom}_{A}(R^{k+m}, P^m) \), since \( R^{k+m} \in A^\ell \)-mod and \( Q^m \in A^\ell \)-mod. For \( k = 0 \), note that \( \text{Hom}_{A}(P(i), P(v)) \cong \text{Hom}_{A}(P(i), P^\ell(v)) \) according to Lemma 47, and that, in fact, an isomorphism is given by composition with \( p \).

When \( k = 0 \), the second statement follows directly from Lemma 39 applied to the modules \( \Delta(v) \) and \( \Delta(i) \). Let \( X \) be the kernel of the natural epimorphism \( P^\ell(v) \to \Delta^\ell(v) \) (that is, the first syzygy of \( \Delta^\ell(v) \)) and let \( \sigma(P^\bullet) \) be the complex:

\[
\cdots \longrightarrow P^2 \longrightarrow P^1 \longrightarrow X \longrightarrow 0
\]

Then, there is a chain map, \( \beta \), from \( \sigma(P^\bullet) \) to \( T \), given as follows:

\[
\begin{array}{ccc}
\cdots & \longrightarrow & P^2 \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & P^1 \longrightarrow X \\
\downarrow & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\cdots & \longrightarrow & P^2 \oplus Q^2 \longrightarrow P^1 \oplus Q^1 \longrightarrow P(v)
\end{array}
\]

Here, \( i \) is the natural embedding of \( X \) into \( P(v) \). Note that, by definition, \( X \subset \text{rad} P^\ell(v) \subset P(v) \). We claim that the map \( \text{Hom}_{K(A)}(T^\bullet, R^\bullet[k]) \to \text{Hom}_{K(A)}(\sigma(P^\bullet), R^\bullet[k]) \) given by precomposition with \( \beta \) is an isomorphism. Consider the following picture:

\[
\begin{array}{ccc}
\cdots & \longrightarrow & P^{k+1} \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & P^k \longrightarrow \cdots \longrightarrow X \\
\downarrow & & \downarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
\cdots & \longrightarrow & P^{k+1} \oplus Q^{k+1} \longrightarrow P^k \oplus Q^k \longrightarrow \cdots \longrightarrow P(v) \\
\downarrow & & \downarrow \\
\cdots & \longrightarrow & R^1 \longrightarrow P(i) \longrightarrow 0 \longrightarrow 0
\end{array}
\]

For \( k \geq 1 \), the claim follows immediately from the fact that \( \text{Hom}_{A}(P^{k+m}, R^m) \cong \text{Hom}_{A}(P^{k+m} \oplus Q^{k+m}, R^m) \), since \( R^m \in A^\ell \)-mod and \( Q^{k+m} \in A^\ell \)-mod. Similarly, there is a natural chain map \( \sigma(P^\bullet) \to P^\bullet \) given as follows:

\[
\begin{array}{ccc}
\cdots & \longrightarrow & P^2 \\
\downarrow & & \downarrow 1 \\
\cdots & \longrightarrow & P^1 \longrightarrow X \\
\downarrow & & \downarrow 1 \\
\cdots & \longrightarrow & P^2 \longrightarrow P^1 \longrightarrow P^\ell(v)
\end{array}
\]

Since we assume \( k \geq 1 \), it is clear that precomposition with the above chain map yields an isomorphism

\[
\text{Hom}_{K(A)}(P^\bullet, R^\bullet[k]) \cong \text{Hom}_{K(A)}(\sigma(P^\bullet), R^\bullet[k]).
\]

Combining these, we obtain

\[
\text{Hom}_{K(A)}(T^\bullet, R^\bullet[k]) \cong \text{Hom}_{K(A)}(\sigma(P^\bullet), R^\bullet[k]) \cong \text{Hom}_{K(A)}(P^\bullet, R^\bullet[k]),
\]

finishing the proof of the second statement.
Let $\text{Ext}^*_A(\Delta^\ell, \Delta^\ell)$ denote the Ext-algebra of standard modules over $A^\ell$, for $\ell = 1, 2$. Similarly, let $\text{Ext}^*_A(\Delta, \Delta)$ denote the Ext-algebra of standard modules over $A$. Fix the basis $1_{\Delta(v)}$ of the space $\text{End}_A(\Delta(v))$.

Recall the previously defined gluing process, which was discussed in Subsection 2.1.

**50 Theorem.** Let $Q^1 \sqcup Q^2$ be a deconcatenation of $Q$ at a sink or source $v$, such that if $v$ is a sink, then $v$ is minimal or maximal with respect to $\leq^c$. Then, there is an isomorphism of graded algebras $\text{Ext}^*_A(\Delta, \Delta) \cong \text{Ext}^*_A(\Delta^1, \Delta^1) \circ \text{Ext}^*_A(\Delta^2, \Delta^2)$.

**Proof.** We start by noting that, according to Propositions 46 and 49, there holds
\[
\text{Ext}^k_A(\Delta^\ell(i), \Delta^\ell(j)) \cong \text{Ext}^k_A(\Delta(i), \Delta(j)),
\]
for all $i, j \in Q^\ell_0$ and all $k \geq 0$. Put $C = \text{Ext}^*_A(\Delta^1, \Delta^1) \circ \text{Ext}^*_A(\Delta^2, \Delta^2)$. By definition of the multiplication on $C$, for any $x \in \text{Ext}^*_A(\Delta^1, \Delta^1)$ and $y \in \text{Ext}^*_A(\Delta^2, \Delta^2)$ such that $\deg x > 0$ and $\deg y > 0$, there holds $x \cdot y = y \cdot x = 0$. Similarly, according to Proposition 45, we have
\[
\text{Ext}^0_A(\Delta^1, \Delta^1) \cdot \text{Ext}^0_A(\Delta^2, \Delta^2) = \text{Ext}^0_A(\Delta^2, \Delta^2) \cdot \text{Ext}^0_A(\Delta^1, \Delta^1) = 0.
\]
Together, these facts imply that there is a homomorphism of graded algebras $\Phi : C \to \text{Ext}^*_A(\Delta, \Delta)$. Note that, because of Proposition 45, the subspaces $\Phi(\text{Ext}^*_A(\Delta^\ell, \Delta^\ell))$ generate $\text{Ext}^*_A(\Delta, \Delta)$, showing that $\Phi$ is surjective. Lastly, note that $\dim_K C = \dim_K \text{Ext}^*_A(\Delta, \Delta)$, so that $\Phi$ is a surjective linear map between vector spaces of the same dimension, hence a linear isomorphism. \hfill \Box

**51 Example.** We consider the deconcatenation $\{1 \leftarrow 2 \leftarrow 3\} \sqcup \{3 \to 4 \to 5\}$ of the quiver
\[
Q = 1 \leftarrow 2 \leftarrow 3 \to 4 \to 5.
\]
We draw the Loewy diagrams of the projective and standard modules over $A^1$.

\[
\begin{array}{c}
P^1(1) : 1, \\
\downarrow 2,
\end{array}
\begin{array}{c}
P^1(2) : 1, \\
\downarrow 2,
\end{array}
\begin{array}{c}
P^1(3) : 1, \\
\downarrow 2,
\end{array}
\begin{array}{c}
\Delta^1(1) \cong P^1(1), \\
\Delta^1(2) \cong P^1(2), \\
\Delta^1(3) \cong P^1(3).
\end{array}
\]

As the standard modules over $A^1$ are projective, there are no non-split extensions between them. We see that there are homomorphisms $f : \Delta^1(1) \to \Delta^1(2)$, $g : \Delta^1(2) \to \Delta^1(3)$ and $h : \Delta^1(1) \to \Delta^1(3)$, such that $h = g \circ f$. The quiver of the Ext-algebra $\text{Ext}^*_A(\Delta^1, \Delta^1)$ is then
\[
1 \xrightarrow{f} 2 \xrightarrow{g} 3,
\]
with $\deg f = \deg g = 0$, subject to no additional relations.

Next, we draw the Loewy diagrams of the projective and standard modules over $A^2$.

\[
\begin{array}{c}
P^2(3) : 4, \\
\downarrow 4,
\end{array}
\begin{array}{c}
P^2(4) : 5, \\
\downarrow 5,
\end{array}
\begin{array}{c}
P^2(5) : 5, \\
\end{array}
\begin{array}{c}
\Delta^2(3) : 3, \\
\Delta^2(4) : 4, \\
\Delta^2(5) : 5.
\end{array}
\]
As the standard modules over $A^2$ are simple, we see that we have non-split extensions $\alpha \in \text{Ext}_{A^2}^1(\Delta^2(3), \Delta^2(4))$ and $\beta \in \text{Ext}_{A^2}^1(\Delta^2(4), \Delta^2(5))$, such that $\beta \alpha = 0$. The quiver of the Ext-algebra $\text{Ext}_{A^2}^*(\Delta^2, \Delta^2)$ is then

$$3 \xrightarrow{\alpha} 4 \xrightarrow{\beta} 5,$$

subject to the relation $\beta \alpha = 0$. Finally, we draw the Loewy diagrams of the projective and standard modules over $A$.

$$\begin{align*}
P(1) : & 1, & P(2) : & 2, & P(3) : & 3, & P(4) : & 4, & P(5) : & 5, \\
& 1 & & & & & 1 & & & \end{align*}$$

$$\begin{align*}
\Delta(1) & \cong P(1), & \Delta(2) & \cong P(2), & \Delta(3) : & 2, & \Delta(4) & \cong L(4), & \Delta(5) & \cong P(5).
\end{align*}$$

We see that there are homomorphisms $f' : \Delta(1) \to \Delta(2)$, $g' : \Delta(2) \to \Delta(3)$ and $h' : \Delta(1) \to \Delta(3)$, such that $h' = g' \circ f'$. Additionally, we see that there are non-split extensions $\alpha' \in \text{Ext}_{A^1}^1(\Delta(3), \Delta(4))$ and $\beta' \in \text{Ext}_{A^1}^1(\Delta(4), \Delta(5))$, such that $\beta' \alpha' = 0$. The composition $\alpha' g'$ produces an extension in $\text{Ext}_{A^1}^1(\Delta(2), \Delta(4))$, and is therefore equal to zero, according to Lemma 46. Therefore, the Ext-algebra $\text{Ext}_{A^1}^*(\Delta, \Delta)$ is given by the quiver

$$1 \xrightarrow{f'} 2 \xrightarrow{g'} 3 \xrightarrow{\alpha'} 4 \xrightarrow{\beta'} 5 ,$$

subject to the relations $\beta' \alpha' = 0$ and $\alpha' g' = 0$. We observe that this algebra coincides with the algebra

$$C = \text{Ext}_{A^1}^*(\Delta^1, \Delta^1) \circ \text{Ext}_{A^2}^*(\Delta^2, \Delta^2).$$

We conclude this section with an example of what can happen when the condition that a sink $v$ is not minimal or maximal in the essential order.

52 Example. We consider the quiver $Q$ given by

$$\begin{align*}
1 & \xrightarrow{\alpha_1} 2 \xleftarrow{\beta} 3,
\end{align*}$$

where the vertical dots signify that there are $k$ arrows, $\alpha_1, \ldots, \alpha_k$, from 1 to 2. We fix the order $1 \triangleleft 2 \triangleleft 3$. Note that the quiver admits a deconcatenation at the sink 2. We have $\Delta(1) \cong L(1)$, $\Delta(2) \cong L(2)$ and $\Delta(3) \cong P(3)$. Let $M(i)$ be the module with the following Loewy diagram:

$$\begin{align*}
M(i) : & 1 \xrightarrow{\alpha_i} 2 \xleftarrow{\beta} 3,
\end{align*}$$
This means that a basis for $M(i)$ is given by $\{e_1, \alpha_i, e_3, \beta\}$ and that $\alpha_j \cdot M(i) = 0$, by definition, for $i \neq j$. Then, for all $1 \leq i \leq k$, there is a short exact sequence

$$0 \longrightarrow \Delta(3) \longrightarrow M(i) \longrightarrow \Delta(1) \longrightarrow 0.$$ 

We note that these sequences are inequivalent for different $i$ and that, indeed, we have $\dim \text{Ext}_A^1(\Delta(1), \Delta(3)) = k$.

Results from [13] guarantee that there exists a quasi-hereditary algebra $R$, which is Morita equivalent to $A = KQ$ and admits a regular exact Borel subalgebra $B \subset R$. For more details on how to obtain $B$ and $R$ from our data, we refer to the above and [15, Section 4.6].

We know that the quiver of $B$ should be given by the extensions of degree 1 between standard modules over $A$, that is, $\text{Ext}_A^1(\Delta, \Delta)$. The quiver of $B$ then looks as follows:

(i) Vertices are 1, 2 and 3.
(ii) There are $k$ arrows from 1 to 2, which we denote by $a_1, \ldots, a_k$. There are $k$ arrows from 1 to 3, which we denote by $b_1, \ldots, b_3$.

Then, the indecomposable projective modules over $B$ are as follows:

(i) We have:

$$[P_B(1) : L(1)] = 1, \quad [P_B(1) : L(2)] = k, \quad \text{and} \quad [P_B(1) : L(3)] = k.$$ 

(ii) We have $P_B(2) \cong L(2)$.
(iii) We have $P_B(3) \cong L(3)$.

Next, we want to compute $R \otimes_B P_B(i)$ for $1 \leq i \leq 3$. For each $i$, this will again be a projective module, so it will have a $\Delta$-filtration since $R$ is quasi-hereditary. Using that a composition series of $P_B(i)$ will correspond to a $\Delta$-filtration of $R \otimes_B P_B(i)$ under $R \otimes_B -$, we immediately conclude that

$$R \otimes_B P_B(2) \cong P_R(2), \quad \text{and} \quad R \otimes_B P_B(3) \cong P_R(3).$$

The module $R \otimes_B P_B(1) = T$ is a projective module such that

$$(T : \Delta_R(1)) = 1, \quad (T : \Delta_R(2)) = k, \quad \text{and} \quad (T : \Delta_R(3)) = k,$$

since the composition factors of $P_B(1)$ correspond to occurrences of standard modules in the $\Delta$-filtration of $T$. Since $A$ and $R$ are Morita equivalent in such a way that the quasi-hereditary structures are preserved, we may look at the standard modules over $A$, and see that the module $\hat{T} = P_A(1) \oplus P_A(3)^{\oplus k}$ satisfies

$$(\hat{T} : \Delta_B(1)) = 1, \quad (\hat{T} : \Delta_B(2)) = k, \quad \text{and} \quad (\hat{T} : \Delta_B(3)) = k.$$ 

We conclude that

$$R \otimes_B P_B(1) \cong P_R(1) \oplus P_R(3)^{\oplus k}.$$ 

Then, we have
\[ R \cong R \otimes_B B \cong R \otimes_B \bigoplus_{i=1}^3 P_B(i) \cong \bigoplus_{i=1}^3 R \otimes_B P_B(i) \cong P_R(1) \oplus P_R(2) \oplus P_R(3) \oplus P_R(3)^{\oplus k}. \]

It follows that

\[ R \cong \text{End}_A(A \oplus P_A(3)^{\oplus k})^\text{op}. \]

4.1. Exact Borel subalgebras under deconcatenations

In what follows, let \( A = KQ/I \) be a quasi-hereditary algebra and let \( Q = Q^1 \sqcup Q^2 \) be a deconcatenation at a sink or a source \( v \). Recall the notation

\[ A^\ell = A/(e_u \mid u \in Q_0 \setminus Q_0^\ell), \quad \ell = 1, 2. \]

Consider a deconcatenation \( Q^1 \sqcup Q^2 \) of \( Q \) at the sink or source \( v \). Let \( B^1 \subset A^1 \) be a subalgebra having a basis \( \{b_1, \ldots, b_n, e_v\} \) and let \( B^2 \subset A^2 \) be a subalgebra with basis \( \{c_1, \ldots, c_m, e_v\} \). Then \( B = B^1 \circ B^2 \) is a subalgebra of \( A \).

53 Proposition. Let \( Q = Q^1 \sqcup Q^2 \) be a deconcatenation at the source \( v \) and assume that we have regular exact Borel subalgebras \( B^1 \subset A^1 \) and \( B^2 \subset A^2 \). Then, \( A \) admits a regular exact Borel subalgebra.

Proof. Using Theorem 23, it suffices to check that \( \mathrm{rad} \Delta(i) \in \mathcal{F}(\nabla) \). For any \( i \in Q_0^1 \setminus \{v\} \), we have \( \Delta(i) \cong \Delta^\ell(i) \). Let \( 0 \subset M_n \subset \cdots \subset M_0 = \mathrm{rad} \Delta^\ell(i) \) be a filtration with costandard subquotients. We claim that this is also a filtration of \( \mathrm{rad} \Delta(i) \). Consider a subquotient of the filtration

\[ M_k/M_{k+1} \cong \nabla^\ell(j). \]

Then, \( \nabla^\ell(j) \cong \nabla(j) \) for all \( j \in Q_0^1 \). For \( j \neq v \), we appeal to Lemma 41. For \( j = v \), this is automatic as \( \nabla(v) \) is simple by virtue of \( v \) being a source. Left to check is that we have a \( \nabla \)-filtration of \( \mathrm{rad} \Delta(v) \). Since \( v \) is a source, we have a direct sum decomposition:

\[ \mathrm{rad} \Delta(v) = \mathrm{rad} \Delta^1(v) \oplus \mathrm{rad} \Delta^2(v). \]

Since \( A^1 \) has a regular exact Borel subalgebra, \( \mathrm{rad} \Delta^1(v) \) has a filtration whose subquotients are members of the set

\[ \{\nabla(i) \mid i \in Q_0^1 \setminus \{v\}\}. \]

To see that \( \nabla(v) \) does not occur as a composition factor, note that \( [\mathrm{rad} \Delta(v) : L(v)] = 0 \). Indeed, an occurrence of a composition factor \( L(v) \) would imply that \( [P(v) : L(v)] \geq 2 \), contradicting the fact that \( v \) is a source. Similarly, \( \mathrm{rad} \Delta^2(v) \) has a filtration whose subquotients are members of the set

\[ \{\nabla(i) \mid i \in Q_0^2 \setminus \{v\}\}. \]

We conclude that \( \mathrm{rad} \Delta(v) \in \mathcal{F}(\nabla) \), proving the statement.

54 Proposition. Let \( Q = Q^1 \sqcup Q^2 \) be a deconcatenation at the sink \( v \). Assume further that we have regular exact Borel subalgebras \( B^1 \subset A^1 \), \( B^2 \subset A^2 \) and that the vertex \( v \) is minimal with respect to the essential order on \( Q_0 \). Then, there exists a regular exact Borel subalgebra \( B \subset A \).
Proof. For any $i \in Q_0^1 \setminus \{v\}$, we have $\Delta(i) \cong \Delta^f(i)$. Let $0 \subset M_n \subset \cdots \subset M_0 = \text{rad} \Delta^f(i)$ be a filtration with costandard subquotients. We claim that this is also a filtration of $\text{rad} \Delta(i)$. Consider a subquotient of the filtration

$$M_k/M_{k+1} \cong \nabla^f(j).$$

Then $\nabla^f(j) \cong \nabla(j)$ for all $j \in Q_0^1 \setminus \{v\}$, according to Lemma 41. For $j = v$, we claim that when $v$ is a sink, $v$ is minimal in the essential order on $Q_0$ if and only if $\nabla(v)$ is simple. Indeed, if $v$ is minimal, then $\nabla(v)$ is clearly simple. Conversely, assume that $\nabla(v)$ is simple and that there exists another vertex $k$ such that $k \prec v$. Then, we have

$$[\Delta(v) : L(k)] > 0 \quad \text{or} \quad (P(k) : \Delta(v)) > 0.$$

The first condition cannot be satisfied, since $\Delta(v)$ is simple by virtue of $v$ being a sink. According to [8, Lemma 2.5], we have

$$(P(k) : \Delta(v)) = [\nabla(v) : L(k)],$$

so this condition may not be satisfied either, since $\nabla(v)$ was assumed to be simple. Therefore, for $i \in Q_0^1 \setminus \{v\}$, we have $\text{rad} \Delta(i) \in \mathcal{F}(\nabla)$, since $A^1$ and $A^2$ admit regular exact Borel subalgebras, by Theorem 23. Since $v$ is a sink, $\Delta(v) \cong L(v)$ which implies that $\text{rad} \Delta(v) = 0$, so there is nothing more to show. \qed

55 Lemma. Let $Q = Q^1 \sqcup Q^2$ be a deconcatenation at the sink $v$. Then, for all $i \in Q_0$, we have $[\nabla(i) : L(v)] > 0$ if and only if $i = v$.

Proof. By definition, $\nabla(i)$ is a submodule of $I(i)$, so $[\nabla(i) : L(v)] > 0$ if and only if there is a path from $v$ to $i$, not passing through a vertex greater than $i$ in $Q_0$. This is a contradiction, since $v$ is assumed to be a sink. \qed

From Lemma 55, it is also clear that for any module $M \in \mathcal{F}(\nabla)$, the costandard module $\nabla(v)$ appears as a subquotient in the filtration of $M$ if and only if $[M : L(v)] > 0$.

Now we are ready to provide a necessary and sufficient condition for $A$ to admit a regular exact Borel subalgebra $B \subset A$, where $A = KQ$ and $Q = Q^1 \sqcup Q^2$ is a deconcatenation at a sink $v$.

56 Proposition. Let $Q = Q^1 \sqcup Q^2$ be a deconcatenation at the sink $v$ and assume that $B^1 \subset A^1$ and $B^2 \subset A^2$ are regular exact Borel subalgebras. Moreover, assume that $v$ is not minimal with respect to the essential order on $Q_0$. Then, the following are equivalent.

(i) There exists a regular exact Borel subalgebra $C \subset A$.
(ii) $[\text{rad} \Delta(i) : L(v)] = 0$, for all $i \in Q_0$.
(iii) The vertex $v$ is maximal with respect to the essential order on $Q_0$.

Proof. We first show the equivalence (i) $\iff$ (ii). We assume that (ii) holds and claim that $\text{rad} \Delta(i) \in \mathcal{F}(\nabla)$, for all $i \in Q_0$. If $i = v$, we have $\Delta(v) \cong L(v)$ and consequently $\text{rad} \Delta(v) = 0$, so there is nothing to prove. Assume that $i \neq v$, where $i \in Q_0$. Then, $\Delta^f(i) \cong \Delta(i)$. According to the remark after Lemma 55, any $\nabla$-filtration of $\text{rad} \Delta^f(i)$ does not contain $\nabla^f(v)$ as a subquotient. Therefore, any $\nabla$-filtration of $\text{rad} \Delta^f(i)$ is automatically a $\nabla$-filtration of $\Delta(i)$, since all occurring subquotients in the filtration of $\text{rad} \Delta^f(i)$ are of the form $\nabla^f(j)$ with $j \neq v$. 
Conversely, if $C \subset A$ is a regular exact Borel subalgebra, then $\text{rad} \Delta(i) \in \mathcal{F}(\nabla)$ for $1 \leq i \leq n$. Let $i$ be such that $[\text{rad} \Delta(i) : L(v)] > 0$ and assume without loss of generality that $i \in Q_0^1 \setminus \{v\}$. Then, $\nabla(v)$ occurs in the $\nabla$-filtration of $\text{rad} \Delta(i)$, according to the remark after Lemma 55. Note that since $v$ is assumed not to be minimal in the essential order, $\nabla(v)$ is not simple. Since $Q^2$ is connected, $v$ has at least one neighbor, $j \in Q^2_0 \setminus \{v\}$. If $j \not\preceq v$, then $L(j)$ is a composition factor in $\nabla(v)$, which implies that $[\text{rad} \Delta(i) : L(j)] > 0$ since $\nabla(v)$ occurs in the $\nabla$-filtration of $\text{rad} \Delta(i)$. This is a contradiction, since $\text{rad} \Delta(i)$ is an $A^1$-module. If $v \preceq j$, then $[\text{rad} \Delta(j) : L(v)] > 0$, so that $\nabla(v)$ is a subquotient of $\text{rad} \Delta(j)$. This contradicts $\nabla(v)$ being a subquotient of $\text{rad} \Delta(i)$, unless $\nabla(v)$ is simple. However, we assumed that $v$ is not minimal in the essential order, guaranteeing it is not simple. If $v$ and $j$ are incomparable, we consider the module $M$, given by the following Loewy diagram:

$$
\begin{array}{c}
j \\
\downarrow \\
v
\end{array}
$$

Since $A^2$ is quasi-hereditary, $\preceq^e$ is adapted to $A^2$, according to Lemma 3. Then, by definition, there is a vertex $k$ such that $v \preceq k$, $j \preceq k$ and $[M : L(k)] \neq 0$. This is a contradiction, since $M$ has no other composition factors besides $L(v)$ and $L(j)$. This proves (i) $\iff$ (ii).

Next, we prove (ii) $\iff$ (iii). Assume (ii) and that $v$ is not maximal. Then there is another vertex $k$, such that $v \preceq^e k$. Then, by definition of the essential order, we have

$$
[\Delta(k) : L(v)] > 0 \quad \text{or} \quad (P(v) : \Delta(k)) > 0.
$$

The first condition can not be satisfied, as it contradicts (ii), and neither may the second, since $P(v)$ is simple by virtue of $v$ being a sink. Conversely, assume that $v$ is maximal and consider $P(i)$, for $i \in Q_0$. If $[P(i) : L(v)] = 0$, there is nothing to show, as $\text{rad} \Delta(i)$ is a submodule of a quotient of $P(i)$. Otherwise, if $[P(i) : L(v)] > 0$, there still holds $[\Delta(i) : L(v)] = 0$, since by definition, the composition factors $L(t)$ of $\Delta(i)$ satisfy $t \preceq i$.

57 Proposition. Let $Q = Q^1 \sqcup Q^2$ be a deconcatenation at a sink or a source $v$. Assume that there are regular exact Borel subalgebras $B^1 \subset A^1$ and $B^2 \subset A^2$ containing the sets of idempotents $\{e_i \mid i \in Q^1_0\}$ and $\{e_j \mid j \in Q^2_0\}$, respectively, and that $A$ admits a regular exact Borel subalgebra. Then $B^1 \circ B^2 \subset A$ is a regular exact Borel subalgebra of $A$.

Proof. Consider the regular exact Borel subalgebra $B^\ell$, for $\ell = 1, 2$. Note that the discussion prior to Proposition 25 implies that the Gabriel quiver of $B^\ell$ is the quiver $Q_{B^\ell}$:

(i) The set of vertices of $Q_{B^\ell}$ is $Q_{B^\ell, 0} = Q^\ell_0$.
(ii) There is an arrow $i \to j$ in $Q_{B^\ell}$ if and only if $\dim \text{Ext}_{A^\ell}(\Delta^\ell(i), \Delta^\ell(j)) = 1$.

To show that $B^1 \circ B^2$ is quasi-hereditary with simple standard modules, we check that it is directed. By construction, the quiver of $B^1 \circ B^2$ is a quiver $D$ such that there is a deconcatenation $D = Q_{B^1} \sqcup Q_{B^2}$ at $v$. Therefore, if there is an arrow $i \mathop{\rightarrow}^a j$ in $D$, then $i, j \in Q_{B^\ell}$. Therefore, since $B^\ell$ is a regular exact Borel subalgebra, $B^\ell$ is in particular directed, so we have $i \preceq j$.

Next, we check that the functor $A \otimes_{B^1 \circ B^2} -$ is exact. To this end, we claim that $A$ is projective as a right $B^1 \circ B^2$-module. Consider the decomposition of right $A^1$-modules

$$
A^1 \cong \bigoplus_{i \in Q^1_0} P^1(i)
$$
Since $A^1 \otimes_{B^1} -$ is an exact functor, $A^1$ is projective as a right $B^1$-module. Because $B^1 \subset A^1$ is a subalgebra, the decomposition above is also a decomposition of right $B^1$-modules. This, in turn, implies that the summands $P^1(i)$ above are projective as right $B^1$-modules. Because $B^1 \circ B^2$ surjects onto $B^1$, each $B^1$-module has the natural structure of a $B^1 \circ B^2$-module (see the remark prior to Lemma 40). Then, the above decomposition is also a decomposition of right $B^1 \circ B^2$-modules, since the action of $\text{rad} B^2$ on each summand is trivial. Now, consider the decomposition

$$A \cong \bigoplus_{i \in Q^1_0 \setminus \{e\}} P(i) \oplus \bigoplus_{j \in Q^0_0 \setminus \{v\}} P(j) \oplus P(v)$$

of right $A$-modules. By the above argument, the summands $P(i)$ in the first term are projective as right $B^1 \circ B^2$-modules (note that for $i \in Q^1_0 \setminus \{v\}$, we have $P(i) \cong P^1(i)$). Similarly, the summands of the second term are projective right $B^1 \circ B^2$-modules. Left to check is that $P(v)$ is a projective right $B^1 \circ B^2$-module.

Let $v$ be a sink. Then $P(v) \cong L(v)$ and there is nothing to show. Let $v$ be a source. As noted earlier, $P^f(v)$ is projective as a right $B^f$-module. Since $v$ is a source also in the quiver of $B^f$, the module $P^f_B(v)$ is the unique indecomposable projective $B^f$-module having $L(v)$ as a composition factor. This implies that there is an isomorphism $f^f$ of right $B^f$-modules:

$$f^f : P^f(v) \to P^f_B(v) \oplus M^f,$$

where $M^f$ is a direct sum of indecomposable projective $B^f$-modules, with some multiplicities. Note that $M^f$ does not contain $P^f_B(v)$ as a direct summand. We claim that there is an isomorphism of right $B^1 \circ B^2$-modules

$$f : P(v) \to P^f_B(v) \oplus M^1 \oplus M^2.$$

Define $f$ on paths in $P(v)$ by

$$p \mapsto \begin{cases} f^1(p) & \text{if } p \in A^1 \setminus \text{span}(e_v) \\ f^2(p) & \text{if } p \in A^2 \setminus \text{span}(e_v) \\ e_v & \text{if } p = e_v. \end{cases}$$

It is clear that $f$ is bijective and a homomorphism of $B^1 \circ B^2$-modules, hence an isomorphism.

Next, we check that $A \otimes_{B^1 \circ B^2} L(i) \cong \Delta(i)$, for all $i \in Q^0_0$. When $i \in Q^1_0 \setminus \{v\}$, we know that $\Delta^f(i) \cong \Delta(i)$. In particular, $\Delta(i)$ is an $A^f$-module. Let $\varphi : A^f \otimes_{B^f} L(i) \to \Delta(i)$ be an isomorphism. The module $A \otimes_{B^1 \circ B^2} L(i)$ is generated by elements of the form $p \otimes e_i$, where $p$ is a path in $A$. Note that, if $p$ is a path in $A^f$ not equal to $e_v$, then

$$p \otimes e_i = pe_s(p) \otimes e_i = p \otimes e_s(p)e_i = 0.$$

To see this, note that $p$ is a path in $A^f$ and $i \in Q^1_0 \setminus \{v\}$, so that $i \neq s(p)$. Then, we may define a homomorphism

$$\tilde{\varphi} : A \otimes_{B^1 \circ B^2} L(i) \to \Delta(i)$$

on the non-zero generators of $A \otimes_{B^1 \circ B^2} e_i$ by

$$p \otimes_{B^1 \circ B^2} e_i \mapsto \varphi(p \otimes_{B^f} e_i),$$
which is clearly an isomorphism. It remains to check that \( A \otimes_{B \circ B^2} L(v) \cong \Delta(v) \). When \( v \) is a sink, we have \( \Delta(v) \cong L(v) \), so the assertion is clear, given that \( A^\ell \otimes_{B^\ell} L(v) \cong \Delta(v) \). Assume that \( v \) is a source and let

\[
f_1 : A^1 \otimes_{B^1} L(v) \to \Delta^1(v), \quad \text{and} \quad f_2 : A^2 \otimes_{B^2} L(v) \to \Delta^2(v)
\]

be isomorphisms. The module \( A \otimes_{B \circ B^2} L(v) \) is generated by elements of the form \( p \otimes e_v \), where \( p \in A \) is some path. Note that with the exception of \( e_v \), every such \( p \) is contained in \( A^1 \) or in \( A^2 \). Define a map

\[
f : A \otimes_{B \circ B^2} L(v) \to \Delta(v)
\]
on generators by

\[
p \otimes e_v \mapsto \begin{cases} f_1(p \otimes e_v) & \text{if } p \in A^1 \setminus \text{span}(e_v), \\ f_2(p \otimes e_v) & \text{if } p \in A^2 \setminus \text{span}(e_v), \\ e_v \otimes e_v & \text{if } p = e_v. \end{cases}
\]

Note that, if \( p \in A^1 \) and \( q \in A^2 \), then \( p \otimes e_v \) and \( q \otimes e_v \) are either zero or linearly independent, except for when \( p = q = e_v \). Then, \( f \) is a well-defined homomorphism because \( f_1 \) and \( f_2 \) are. Similarly, \( f \) is injective because \( f_1 \) and \( f_2 \) are and the intersection of their images equals \( \text{span}(e_v) \). For surjectivity, note that as a vector space, we have \( \Delta(v) = \text{span}(e_v) \oplus \text{rad} \Delta^1(v) \oplus \text{rad} \Delta^2(v) \).

To finish the proof, we check that \( B^1 \circ B^2 \) is regular. Let \( i, j \in Q^I_0 \) and consider the following picture.

\[
\begin{array}{ccc}
\text{Ext}_A^k(\Delta(i), \Delta(j)) & \xrightarrow{\delta} & \text{Ext}_A^k(L(i), L(j)) \\
\text{Ext}_A^k(L(i), L(j)) & \xrightarrow{\gamma} & \text{Ext}_A^k(\Delta^1(i), \Delta^1(j)) \\
\end{array}
\]

Since \( B^1, B^2 \) and \( B^1 \circ B^2 \) enjoy the same relationship as \( A^1, A^2 \) and \( A = A^1 \circ A^2 \), Proposition 46 and Proposition 49 together imply the existence of the map \( \delta \) at the top of the diagram, once we note that \( B^1 \circ B^2 \) is quasi-hereditary with standard modules. In particular, it is an isomorphism. The map \( A^\ell \otimes_{B^\ell} - \) is an isomorphism because \( B^\ell \subset A^\ell \) is a regular exact Borel subalgebra. The map \( \gamma \) exists and is an isomorphism, according to Proposition 46 and Proposition 49.

To show that the diagram commutes, we chase a component of a chain map representing an extension in the space \( \text{Ext}_B^k(B^1 \circ B^2, L(i), L(v)) \), through the diagram. Such a component is a homomorphism between projective \( B^1 \circ B^2 \)-modules occurring in the resolutions of \( L(i) \) and \( L(j) \), respectively.

Let \( i, j \in Q^I_0 \setminus \{v\} \). In this case, the functors embedding \( B^\ell \)-mod into \( B^1 \circ B^2 \)-mod and \( A^\ell \)-mod into \( A \)-mod preserve minimal projective resolutions of standard modules (see the proof of Proposition 46, part (i)). Therefore, the maps \( \delta \) and \( \gamma \) may be thought of as identities in this case. Since the component of the chain map we are considering is a homomorphism between projective modules, it is given by a matrix whose entries are of the form \( \rho_P : P_{B \circ B^2}(x) \to P_{B \circ B^2}(y) \), that is, right multiplication by a linear combination of paths from \( y \) to \( x \) in \( B^1 \circ B^2 \). Under \( A \otimes_{B \circ B^2} - \), the homomorphism \( \rho_p \) is mapped to itself, but now viewed as multiplication by a linear combination of paths in \( A \) (this makes sense, since \( B^1 \circ B^2 \subset A \) is a subalgebra). The behavior of the map \( A^\ell \otimes_{B^\ell} - \) is similar. This shows that the diagram commutes. Since \( \delta, A^\ell \otimes_{B^\ell} - \) and \( \gamma \) are isomorphisms, so is \( A \otimes_{B \circ B^2} - \).

The same argument works when \( i = v \) or \( j = v \) and \( v \) is a sink, since then the statement about the projective resolutions still holds (see the proof of Proposition 46, part (ii)).
Left to consider is the case when \( i = v \) or \( j = v \) and \( v \) is a source. Assume that \( j = v \). We follow the notation of the proof of Proposition 49. The component we are considering is now a map \((f, 0) : R^{k+m} \rightarrow P^m \oplus Q^m\), where \( f \) is given by a matrix as in the previous case. When \( m > 0 \), the maps \( \delta \) and \( \gamma \) act by \((\rho_p, 0) \mapsto \rho_p\). Combining this with our previous observation about the vertical maps in the diagram, it commutes. When \( m = 0 \), we are considering \( \rho_p : P_{B^1 \circ B^2}(x) \rightarrow P_{B^1 \circ B^2}(v) \). From Lemma 44, it follows that \( x \neq v \). In this case, the maps \( \delta \) and \( \gamma \) act by postcomposing with the natural epimorphisms \( P_{B^1 \circ B^2}(v) \rightarrow P_{B^2}(v) \) and \( P(v) \rightarrow P^f(v) \), respectively (see the proof of Lemma 47). Since \( x \neq v \), the image of \( \delta \) is contained in \( \text{rad} P_{B^2}(v) \), on which the natural epimorphism acts as the identity. The situation is similar for \( \gamma \), resolving this case. The proof of the case \( i = v \) is similar.

We conclude this section with an easy observation about the behavior of (certain) \( A_\infty \)-structures under deconcatenations.

**58 Proposition.** Let \( Q = Q^1 \sqcup Q^2 \) be a deconcatenation at a sink or a source \( v \) such that if \( v \) is a sink, then \( v \) is minimal or maximal with respect to \( \leq \), and assume that \( \text{Ext}^n_A(\Delta, \Delta) \) is intrinsically formal, for \( \ell = 1, 2 \). Then, \( \text{Ext}^n_A(\Delta, \Delta) \) is intrinsically formal.

**Proof.** Let \( m_n \) denote the higher multiplications on \( \text{Ext}^n_A(\Delta, \Delta) \) and consider \( m_n(\varphi_n, \ldots, \varphi_1) \), where \( \varphi_i \in \text{Ext}^n_A(\Delta, \Delta) \) for \( 1 \leq i \leq n \). Suppose that \( m_n(\varphi_n, \ldots, \varphi_1) \in \text{Ext}^n_A(\Delta(a), \Delta(b)) \), for some \( a, b \in Q_0 \) and \( k \geq 0 \). If \( a, b \in Q_0 \), then \( m_n(\varphi_n, \ldots, \varphi_1) = 0 \), because \( \text{Ext}^n_A(\Delta, \Delta) \) is intrinsically formal. If \( a \in Q_0 \setminus \{v\} \) and \( b \in Q_0 \setminus \{v\} \), then \( m_n(\varphi_n, \ldots, \varphi_1) = 0 \), according to Proposition 45. If \( a = b = v \), then \( m_n(\varphi_n, \ldots, \varphi_1) = 0 \), since standard modules have no non-split self-extensions.

**4.2. Regular exact Borel subalgebras for path algebras of linear quivers**

Let \( Q \) be the linear quiver

\[
1 \longrightarrow 2 \longrightarrow \ldots \longrightarrow n-1 \longrightarrow n
\]

where the edges may be of either orientation, and put \( \Lambda = KQ \). Denote by \( A_m \) and \( B_\ell \) the special cases

\[
1 \longrightarrow \ldots \longrightarrow m \quad \text{and} \quad 1 \longleftarrow \ldots \longleftarrow \ell,
\]

respectively. By Proposition 24, \( KA_m \) admits a regular exact Borel subalgebra, regardless of the partial order on the vertices. We remark that all the statements from Section 2, describing the structure of \( KA_m \) in terms of its binary search tree, remain true for \( KB_\ell \), if we swap “left” and “right”. For instance, over \( KB_\ell \), the composition factors of a standard module \( \Delta_{KB_\ell}(i) \) are found in the left subtree of the vertex labeled by \( i \), rather than the right. From this, it follows that also \( KB_\ell \) admits a regular exact Borel subalgebra, regardless of the partial order on the vertices. Now, consider \( \Lambda = KQ \). The quiver \( Q \) admits an iterated deconcatenation

\[
Q = Q^1 \sqcup \cdots \sqcup Q^s,
\]

where, for each \( 1 \leq r \leq s \), the quiver \( Q^r \) is either \( A_m \) or \( B_\ell \). Recall that the partial order on the set of vertices of \( Q \) is constructed in the following way. When \( Q = Q^1 \sqcup Q^2 \), with \( Q^1_0 \cap Q^2_0 = \{v\} \) and \( i, j \in Q_0 \), we say that \( i \lessdot j \) if one of the following holds.

(i) We have \( i, j \in Q^1_0 \) and \( i \lessdot^f j \), for some \( \ell \).

(ii) We have \( i \in Q^1_0, j \in Q^2_0, i \lessdot v \) and \( v \lessdot^f j \).
It is clear that, in this setting, $v$ is maximal in $Q_0$ if and only if it is maximal in both $Q^7_0$ and $Q^7_0$. We conclude that we have a special case of Proposition 54 as follows.

59 Theorem. The algebra $\Lambda$ admits a regular exact Borel subalgebra $C \subset \Lambda$ if and only if each vertex $v \in Q_0$ which is a sink at which a deconcatenation occurs is minimal or maximal with respect to the essential order on $Q_0$.

60 Example. Consider the quiver

$$Q = 1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \leftarrow 5 \rightarrow 6 \rightarrow 7.$$ Deconcatenating $Q$ as far as possible we find that

$$Q = (1 \xrightarrow{x} 2 \xrightarrow{y} 3) \sqcup (3 \xleftarrow{a} 4 \xleftarrow{b} 5) \sqcup (5 \xrightarrow{u} 6 \xrightarrow{v} 7).$$

Suppose the order on $Q_0$ is the one given by the following orders on $Q^1_0, Q^2_0$ and $Q^3_0$.

$$1 \triangleleft_T 3, 2 \triangleleft_T 3, \quad 4 \triangleleft_T 5 \triangleleft_T 3, \quad 6 \triangleleft_T 5 \triangleleft_T 7,$$

corresponding to binary search trees

![Binary search trees](image)

We compute the standard modules.

$$\Delta(1) \cong L(1), \quad \Delta(2) \cong L(2), \quad \Delta(3) \cong L(3), \quad \Delta(4) \cong L(4), \quad \Delta(5) \cong M(4, 5), \quad \Delta(6) \cong L(6), \quad \Delta(7) \cong L(7)$$

The only standard module with non-trivial radical is $\Delta(5)$ with $\text{rad } \Delta(5) \cong L(4) \cong \nabla(4)$, so $\text{rad } \Delta(i) \in \mathcal{F}(\nabla)$ for all $1 \leq i \leq 7$. Then, $A^1, A^2$ and $A^3$ have regular exact Borel subalgebras given by

$$\begin{align*}
(1 \xrightarrow{x} 2 \xrightarrow{y} 3), \quad (3 \xleftarrow{a} 4 \xleftarrow{b} 5), \quad (5 \xrightarrow{u} 6 \xrightarrow{v} 7).
\end{align*}$$

We note that, since 3 is maximal in the first and second quivers, $A$ has a regular exact Borel subalgebra, by Proposition 57, given by the following quiver.

$$\begin{align*}
1 \xrightarrow{x} 2 \xrightarrow{3} 4 \xleftarrow{b} 5 \xrightarrow{6} 7.
\end{align*}$$

Similarly to the above, we now wish to extend the statement of Proposition 36 to the algebra $\Lambda$ discussed in this section.

61 Proposition. $\text{Ext}_\Lambda^*(\Delta, \Delta)$ is intrinsically formal.

Proof. This follows immediately from Proposition 36 and Proposition 58. □
In view of the situations considered in [9], it would be interesting to do as similar investigation with quivers of Dynkin type \( \mathbb{D} \) and \( \mathbb{E} \).

4.3. A nonlinear example

Consider the algebra \( A \), given by the quiver

\[
\begin{array}{ccc}
1 & \overset{a}{\rightarrow} & 2 \\
\downarrow b & & \downarrow d \\
3 & \overset{c}{\rightarrow} & 4
\end{array}
\]

subject to the relations \( ae = bd, \ ad = 0 \) and \( be = 0 \). With the usual order \( 1 < 2 < 3 < 4 \), the standard modules over \( A \) are

\[
\Delta(1) \cong L(1), \quad \Delta(2) \cong L(2), \quad \Delta(3) : 1 \overset{a}{\rightarrow} 2 \overset{b}{\rightarrow} 3, \quad \text{and} \quad \Delta(4) \cong L(4).
\]

The only standard module with non-trivial radical is \( \Delta(3) \), whose radical is isomorphic to the costandard module \( \nabla(2) \). Therefore, \( A \) admits a regular exact Borel subalgebra \( B \subset A \), which we find is given by the quiver

\[
\begin{array}{ccc}
1 & \overset{a}{\rightarrow} & 2 \\
\downarrow b & & \downarrow d \\
3 & \overset{c}{\rightarrow} & 4
\end{array}
\]

Consider now the algebra \( \Lambda \) given by the quiver

\[
\begin{array}{ccc}
1 & \overset{a}{\rightarrow} & 2 \\
\downarrow b & & \downarrow d \\
3 & \overset{c}{\rightarrow} & 4 \leftrightarrow 5 \\
\downarrow b' & & \downarrow d'
\end{array}
\]

subject to the relations

\[
ae = bd, \quad a'e' = b'd', \quad ad = 0, \quad a'd' = 0, \quad be = 0, \quad \text{and} \quad b'e' = 0,
\]

with the order on the vertices being \( 1 < 2 < 3 < 4, \quad 7 < 6 < 5 < 4 \). Then, \( 4 \) is maximal and Proposition 56 applies, and \( \Lambda \) has a regular exact Borel subalgebra \( C \subset \Lambda \), given by the quiver

\[
\begin{array}{ccc}
1 & \overset{a}{\rightarrow} & 2 \\
\downarrow b & & \downarrow \bar{d} \\
3 & \overset{c}{\rightarrow} & 4 \leftrightarrow 5 \\
\downarrow \bar{b}' & & \downarrow \bar{d}'
\end{array}
\]

according to Proposition 57.
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