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Abstract Logics and Lindström's Theorem

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Abstract

A definition of abstract logic is presented. This is used to explore and compare some abstract logics, such as logics with generalised quantifiers and infinitary logics, and their properties. Special focus is given to the properties of completeness, compactness, and the Löwenheim-Skolem property. A method of comparing different logics is presented and the concept of equivalent logics introduced. Lastly a proof is given for Lindström's theorem, which provides a characterization of elementary logic, also known as first-order logic, as the strongest logic for which both the compactness property and the Löwenheim-Skolem property, holds.

1 Introduction

During the 19^{th} and 20^{th} centuries, with the rise of formal logic as a field of study, there also arose a want for some canonical system of logic [1]. Today, much thanks to the completeness, compactness, and Löwenheim-Skolem theorems, the common consensus is that this canonical logic is, or at least should be, elementary logic, or as it is more commonly referred to as, first-order logic (for historical reasons though, this paper will use the term elementary logic). While, because of this, most modern day model theory is done in elementary logic, it has not resulted in every logician and model theorist completely abandoning every other logical system. In fact, while a lot of modern day mathematics can be formalised in elementary logic, in practice most of it is done informally, but in a way that often resembles second-order logic with statements like "there exists some function f" [1].

Furthermore, the study of different logics is central to the fields of finite model theory and descriptive complexity theory. Finite model theory concerns questions about what logics are needed to formally express certain properties of structures and is closely connected to descriptive complexity theory which tries to characterize complexity classes by what logics are needed to express what is captured by the complexity class. For example existential second-order logic (that is second order logic where every second-order quantifier is existential) characterizes the complexity class of NP on finite structures [2][3].

Thus there is still relevant research being done into logics that are not elementary logic. For example logics with expressive power that can be said to lie between that of elementary logic and that of second-order logic, some of which shall be explored here. We will look at some of the properties these logics have, and try to compare their expressive powers. We will conclude by providing a proof of Lindström's theorem, a very illuminating result when looking at different logics, which provides a classification of elementary logic as the strongest logic (in an expressive sense), that has the property of compactness, and for which there exists an analogue to the Löwenheim-Skolem theorem [1][4].

2 Model theory for Elementary Logic

Before going into abstract logics, which is the primary focus of this paper, we need to recall some basic model theory, its concepts and results.

2.1 Basic concepts

We begin with some basic concepts and terminology. Note that, as the reader is expected to already be somewhat familiar with model theory for elementary logic, this list is mostly provided as reference for the reader to be able look up concepts in, as they read the main part of the paper. All of these concepts and results can be found in [5] and [6].

An *ordinal number* is a set α which is well ordered by $\in$, where every member of α is also a subset of α. Finite ordinals are denoted by natural numbers, infinite ordinals are denoted as $\omega_\alpha + \beta$ where α and β are ordinals, with $\omega_{\alpha_0}+\beta_0 < \omega_{\alpha_1}+\beta_1$ if $\alpha_0 < \alpha_1$, $\omega_{\alpha+1}$ is the least ordinal, having ω_α as a member, for which there is no bijection between ω_α and $\omega\alpha + 1$, and $\omega_\alpha + \beta_0 < \omega_\alpha + \beta_1$ if $\beta_0 < \beta_1$.
A *cardinal number* is an ordinal α such that there does not exist a bijection between α and any $\beta \in \alpha$. ω_0 (or just ω) is the least infinite cardinal, and ω_1 the least uncountable cardinal. If there exists a bijection between a set A to a cardinal α, we say that A has *cardinality* α denoted $|A| = \alpha$.
A *constant/relation/function symbol* is, as the name suggests, a symbol representing a constant/relation/function. Each constant symbol, function symbol, and argument place of a function symbol or relation symbol is equipped with a *sort symbol* which is a symbol determining the sort of other symbols and thus what can be done with them, for example when looking at a vector field as a logical structure, one might want one sort symbol for vectors and another for scalars. In one sorted cases i.e. when all symbols are of the same sort, the sort symbol is dropped.
A *term* is any of the following: a variable, a constant symbol, or $f(t_0, ..., t_{n-1})$ where f is an n-ary function symbol and $t_0, ..., t_{n-1}$ are terms. A term can only be put in the argument place of a function or relation if the term and argument place are equipped with the same sort symbol.
A *Vocabulary* $\mathcal{V}$ is a set consisting of sort, constant, relation, and function symbols .
A $\mathcal{V}$*-structure* consists of two things, a (non-empty) set S together with an *interpretation* of $\mathcal{V}$ for said set, where an interpretation of $\mathcal{V}$ assigns each constant symbol in $\mathcal{V}$ to an element in S, each n-ary relation symbol in $\mathcal{V}$ to a subset of S^n, and each n-ary function symbol in $\mathcal{V}$ to a function from S^n to S.
The *size* of a structure $\mathfrak{M}$, denoted $|\mathfrak{M}|$, is the unique cardinal α for which there exists a bijection between α and the underlying set of $\mathfrak{M}$.
The *reduct* of a $\mathcal{V}$-structure $\mathfrak{M}$ with underlying set S, to a vocabulary $\mathcal{W} \subseteq \mathcal{V}$ is the structure $\mathfrak{M} \restriction \mathcal{W}$ with underlying set S and interpretation the interpretation of $\mathcal{V}$ in $\mathfrak{M}$ restricted to $\mathcal{W}$.
A ($\mathcal{V}$-)*sentence* is any of the following: $t_0 = t_1$, $R(t_0, ..., t_{n-1})$, $\neg\varphi$, $\varphi \wedge \psi$, or $\exists x\varphi$, where $t_0, ..., t_{n-1}$ are terms, R is a relation symbol, and φ and ψ are sentences.
A ($\mathcal{V}$-)*theory* is a set of sentences.
A *model* $\mathfrak{M}$ of a theory Γ is a $\mathcal{V}$-structure such that Γ is satisfied by $\mathfrak{M}$ i.e. every sentence in Γ is true in $\mathfrak{M}$. Furthermore, any $\mathcal{V}$-sentence φ is either true in $\mathfrak{M}$, denoted $\mathfrak{M} \models \varphi$, or φ is false in $\mathfrak{M}$, denoted $\mathfrak{M} \not\models \varphi$.
A *renaming* ϱ from a vocabulary $\mathcal{V}$ to vocabulary $\mathcal{W}$ is a bijective map that maps any symbol to a symbol of the same kind, i.e. sort symbols to sort symbols, relation symbols to relation symbols of the same arity, function symbols to function symbols of the same arity, and constant symbols to constant symbols, and such that the sort symbols that the constant symbols of $\mathcal{W}$ are equipped with correspond via ϱ.

The *structure renaming* of a $\mathcal{V}$-structure $\mathfrak{M}$ by a renaming $\varrho : \mathcal{V} \to \mathcal{W}$ is the map from $\mathfrak{M}$ to the structure $\varrho[\mathfrak{M}] = \mathfrak{N}$, with the same underlying set as $\mathfrak{M}$. This is given by ϱ such that elements in the underlying set of $\mathfrak{M}$ equipped with the sort symbol s are mapped to the same element but instead equipped with the sort symbol $\varrho(s)$, and for any symbol $x \in \mathcal{V}$, $\varrho(x^{\mathfrak{M}}) = \varrho(x)^{\mathfrak{N}}$.
Two structures $\mathfrak{M}$ and $\mathfrak{N}$ are *elementarily equivalent*, denoted $\mathfrak{M} \equiv \mathfrak{N}$, if they model exactly the same sentences.
An *isomorphism* from a $\mathcal{V}$-structure $\mathfrak{M}$ to a $\mathcal{V}$-structure $\mathfrak{N}$ is a function that is bijective and preserves every $\mathcal{V}$-formula. If such a function exists $\mathfrak{M}$ and $\mathfrak{N}$ are called *isomorphic*, denoted $\mathfrak{M} \simeq \mathfrak{N}$. If two structures are isomorphic, they are elementary equivalent, and finite elementary equivalent structures are isomorphic[6].
For an ordinal k, an *Ehrenfeucht–Fraïssé game* of length k between two $\mathcal{V}$-structures $\mathfrak{M}$ and $\mathfrak{N}$, denoted $G_k(\mathfrak{M}, \mathfrak{N})$ works as follows. There are two players, Spoiler and Duplicator. First Spoiler picks either some element from the underlying set of $\mathfrak{M}$, in which case we call this element m_0, or some element from the underlying set of $\mathfrak{N}$, in which case we will call this element n_0. If Spoiler picked m_0, then Duplicator picks an element from $\mathfrak{N}$ in which case this becomes the element we call n_0, and if Spoiler picked n_0, Duplicator instead picks an element from $\mathfrak{M}$ to be called m_0. This is then repeated with m_1 and n_1, m_2 and n_2... and so on until both Spoiler and Duplicator have picked k elements each (or until there are no more elements to pick) with the additional rule that for $i \neq j$, $m_i \neq m_j$ and $n_i \neq n_j$. When this is done we will have two sequences $\langle m_i \rangle_{i<k}$ and $\langle n_i \rangle_{i<k}$. We say that Duplicator wins the game if there is an isomorphism f from the substructure of $\mathfrak{M}$ generated by $\langle m_i \rangle_{i<k}$ to the substructure of $\mathfrak{N}$ generated by $\langle n_i \rangle_{i<k}$ such that $f(m_i) = n_i$ for all $i < k$ and we call f a *k-partial isomorphism* from $\mathfrak{M}$ to $\mathfrak{N}$. If there does not exist such an isomorphism we say that Spoiler wins. If there is a way for Duplicator to play to be guaranteed to win the game $G_\kappa(\mathfrak{M}, \mathfrak{N})$, no matter how Spoiler plays, we say that Duplicator *has a winning strategy* for the game $G_\kappa(\mathfrak{M}, \mathfrak{N})$.

2.2 Important results

We will now go through some of the most important results from model theory for elementary logic.

2.2.1 Completeness theorem

The completeness theorem draws a connection between syntactic and semantic consequence, i.e. between what can be proved using precisely defined proof rules from a theory and what is necessarily a consequence of that theory. It states that: For any theory Γ, and any sentence φ, there exists a formal proof showing φ is a consequence of Γ, denoted $\Gamma \vdash \varphi$, if and only if φ holds in every model of Γ denoted $\Gamma \models \varphi$ [6].

2.2.2 Compactness theorem

In essence the compactness theorem describes how the satisfiability of an infinite theory depends on its finite subtheories. More explicitly the compactness theorem of elementary logic states that: For any theory Γ, Γ is satisfiable if and only if every finite subtheory of Γ is satisfiable.
This might at first seem obvious, but when we consider the following finite theory $T = \{\varphi, \psi, \neg(\varphi \wedge \psi)\}$, where φ and ψ are sentences which can be true or false independently of each other, we see that clearly T is not satisfiable, however every proper subtheory of T is satisfiable. The compactness theorem states that for theories having infinite cardinality, a similar thing cannot happen [6].

2.2.3 Löwenheim-Skolem theorem

The Löwenheim-Skolem theorem is most often considered to be two separate theorems, namely the Upward Löwenheim-Skolem theorem and the Downward Löwenheim-Skolem theorem.
The Upward Löwenheim-Skolem (ULS) theorem states that: any theory having an infinitely large model, has models of arbitrarily large infinite cardinality.
The Downward Löwenheim-Skolem (DLS) theorem states that: Let $\mathcal{V}$ be a vocabulary and $\mathfrak{M}$ a $\mathcal{V}$-structure with $S_{\mathfrak{M}}$ as the underlying set. For any $X \subset S_{\mathfrak{M}}$, there exists a $\mathcal{V}$-structure $\mathfrak{N}$ with the underlying set $S_{\mathfrak{N}}$ such that the following holds:
(i) $\mathfrak{N}$ is an elementary substructure of $\mathfrak{M}$,
(ii) $X \subseteq S_{\mathfrak{N}}$
(iii) $|\mathfrak{N}| \leq \max(|X|, |\mathcal{V}|, \omega)$.
[6]

2.3 Properties not satisfied by Elementary Logic

2.3.1 Cardinality Quantification

For any $n \in \omega$ we can define a formula φ_n such that any model of a theory containing φ_n has cardinality n. For example $\varphi_2 = \exists x \exists y \forall z((x \neq y) \wedge ((z = x) \vee (z = y)))$. This kind of quantification however, is only possible for finite models since, if we try to construct a theory Γ such that all models of Γ have cardinality λ, where $\lambda \geq \omega$, the Upward Löwenheim-Skolem theorem states that there exists arbitrarily large (and hence strictly larger than λ) infinite models of Γ. While the ULS theorem often can be a useful tool, the lack of infinite cardinality quantification can lead to some problems, for example the lack of any categorical theories with infinite models which we will discuss further below.

2.3.2 Categoricity of theories with infinite models

A theory Γ is *categorical* if it has only one model up to isomorphism. In essence Γ being categorical means that Γ uniquely defines some structure, which clearly can be desirable when applying logic to other areas of mathematics. However as for cardinality quantification this is made impossible by the Upward Löwenheim-Skolem theorem when there exists an infinite model of Γ. Perhaps the most notable example of this is that there exist uncountable (non-standard) models of the first-order Peano axioms, which goes against our intuition that the natural numbers should be countable.

3 Abstract Logics

Up until this point in the paper we have worked in Elementary Logic (EL), or as it is more commonly referred to First-order Logic. The common name might give a clue that there are higher order logics, and indeed this is the case. Furthermore there exist logical systems apart from the ordered ones. We will here use the definition of (abstract) logic from [5] to look at some of these logical systems and their properties.

3.1 Defining a Logic

An *Abstract Logic* (*AL*), or sometimes just a Logic, is a pair $(\mathcal{L}, \models_{\mathcal{L}})$ where $\mathcal{L}$ is a mapping from vocabularies $\mathcal{V}$ to the class $\mathcal{L}[\mathcal{V}]$ of objects called $\mathcal{V}$-sentences on those vocabularies, and $\models_{\mathcal{L}}$ is a binary relation between $\mathcal{V}$-structures and $\mathcal{V}$-sentences such that:

(*i*) $(\mathcal{V} \subseteq \mathcal{W}) \Rightarrow (\mathcal{L}[\mathcal{V}] \subseteq \mathcal{L}[\mathcal{W}])$
This expresses that how $\mathcal{L}$ behaves when applied to a subvocabulary $\mathcal{V}$ of $\mathcal{W}$ is consistent with how $\mathcal{L}$ behaves when applied to all of $\mathcal{W}$.

(*ii*) For all $\mathcal{V}$-structures $\mathfrak{M}$, $(\mathfrak{M} \models_{\mathcal{L}} \varphi) \Rightarrow (\varphi \in \mathcal{L}[\mathcal{V}])$.
This expresses that anything that holds in $\mathfrak{M}$ according to $\models_{\mathcal{L}}$ must be expressible by $\mathcal{L}$ applied to $\mathcal{V}$ and thus lie in the class $\mathcal{L}[\mathcal{V}]$.

(*iii*) For all $\mathcal{V}$-structures $\mathfrak{M}$ and $\mathfrak{N}$, $(\mathfrak{M} \models_{\mathcal{L}} \varphi$ and $\mathfrak{M} \simeq \mathfrak{N}) \Rightarrow \mathfrak{N} \models_{\mathcal{L}} \varphi$.
This expresses that if $\mathfrak{M} \simeq \mathfrak{N}$ then anything that holds in $\mathfrak{M}$ must also hold in $\mathfrak{N}$.

(*iv*) For all $\mathcal{W}$-structures $\mathfrak{M}$, $(\varphi \in \mathcal{L}[\mathcal{V}]$ and $\mathcal{V} \subseteq \mathcal{W}) \Rightarrow (\mathfrak{M} \models_{\mathcal{L}} \varphi$ iff $\mathfrak{M} \upharpoonright \mathcal{V} \models_{\mathcal{L}} \varphi)$.
This expresses that the only sentences that hold in $\mathfrak{M}$ but not in a reduct of $\mathfrak{M}$ to a smaller vocabulary, are the sentences not expressible by the smaller vocabulary.

(v) For any renaming $\varrho : \mathcal{V} \to \mathcal{W}$, and any sentence $\varphi \in \mathcal{L}[\mathcal{V}]$, $\varrho(\varphi) \in \mathcal{L}[\mathcal{W}]$ is such that for any $\mathcal{V}$-structure $\mathfrak{M}$, $\mathfrak{M} \models_{\mathcal{L}} \varphi$ iff $\varrho[\mathfrak{M}] \models_{\mathcal{L}} \varrho(\varphi)$.
This expresses that a renaming applied to a structure does not affect logical truth.

(iii)-(v) are often called *the isomorphism property*, *the reduct property*, and the renaming property, respectively.

3.2 Properties of Abstract Logics

When working with *AL*s it is often easier to talk about properties rather than analogues of theorems existing, for example rather than saying that for an *AL* $\mathcal{L}$, there exists an analogue to the compactness theorem, we talk about $\mathcal{L}$ having the compactness property, similarly to how we talk about how EL does not have the cardinality quantification property. Listed below are a few important properties *AL*s can have and their definitions:

3.2.1 Completeness

An *AL* $\mathcal{L}$ is *complete* if there exists a set of non logical axioms and rules of deduction for formal proofs for $\mathcal{L}$ such that for any vocabulary $\mathcal{V}$, every theory $\Gamma \subseteq \mathcal{L}[\mathcal{V}]$, and any sentence φ: $\Gamma \vdash \varphi$ if and only if $\Gamma \models_{\mathcal{L}} \varphi$.

3.2.2 κ-compactness

For an infinite cardinal κ, an *AL* $\mathcal{L}$ is *κ-compact* if for any vocabulary $\mathcal{V}$ and all theories $\Gamma \subseteq \mathcal{L}[\mathcal{V}]$ such that $|\Gamma| \leq \kappa$: if every finite subtheory of Γ has a model, then Γ has a model.

3.2.3 Compactness

An *AL* $\mathcal{L}$ is *compact* if it is κ-compact for all infinite κ.

3.2.4 Löwenheim-Skolem property

An *AL* $\mathcal{L}$ has the *Löwenheim-Skolem property* if for any vocabulary $\mathcal{V}$ and every theory $\Gamma \subseteq \mathcal{L}[\mathcal{V}]$: if Γ has a model, then Γ has a model $\mathfrak{M}$ such that $|\mathfrak{M}| \leq \omega_0$.

3.2.5 Löwenheim-Skolem property down to κ

For an infinite cardinal κ, an *AL* $\mathcal{L}$ has the *Löwenheim-Skolem property down to* κ if for every theory $\Gamma \subseteq \mathcal{L}[\mathcal{V}]$: if Γ has a model, then Γ has a model $\mathfrak{M}$ such that $|\mathfrak{M}| \leq \kappa$.

3.2.6 Craig property

An AL $\mathcal{L}$ has the *Craig property* if for all vocabularies $\mathcal{V}$ and $\mathcal{W}$: if $\varphi_0 \in \mathcal{L}[\mathcal{V}]$, $\varphi_1 \in \mathcal{L}[\mathcal{W}]$, $\varphi_0 \models_{\mathcal{L}} \varphi_1$, and $\mathcal{V} \cap \mathcal{W}$ contains at least one sort symbol, then there exists a sentence, a so called *interpolant*, $\psi \in \mathcal{L}[\mathcal{V} \cap \mathcal{W}]$ such that $\varphi_0 \models_{\mathcal{L}} \psi$ and $\psi \models_{\mathcal{L}} \varphi_1$.

3.3 Examples of Abstract Logics

The number of abstract logics one can define from the rules in 3.1 is inexpressibly large so for the sake of simplicity we will only concern ourselves with those ALs that are extensions of EL, that is, logics $\mathcal{L}$ such that every EL-sentence is also an $\mathcal{L}$-sentence, and of those we will only look explicitly at a few.

3.3.1 Logics with cardinality quantifiers

Perhaps the simplest way to extend EL is to just add some new quantifiers, also known as *generalised quantifiers.* One such kind of generalised quantifier is the *cardinality quantifier* Q_α. For any ordinal α, $Q_\alpha x(\varphi(x))$ expresses that there are at least ω_α many different values that x can take such that $\varphi(x)$ holds. With the quantifier Q_1 for example, one can express whether or not something is uncountable, so by adding the sentence $\neg Q_1 x(x = x))$ to the first-order Peano axioms we get a theory which describes the natural numbers and does not allow for the uncountable non standard models that EL does, hence we get a theory for the natural numbers where every model is countable. Furthermore $\mathcal{L}(Q_1)$ is complete and has the Löwenheim-Skolem property down to ω_1. For completeness we would need to a set of axioms for formal proofs which can be found in [7] together with a proof of completeness for $\mathcal{L}(Q_1)$. Given a vocabulary with uncountably many constant symbols (and some arbitrary enumeration of these constant symbols), constructing the uncountable theory

$$\Gamma = \{\neg Q_1 x(x = x)\} \cup \{\neg c_\alpha = c_\beta | 0 \leq \alpha < \beta < \omega_1\}$$

gives us that $\mathcal{L}(Q_1)$ is not ω_1-compact since all finite subtheories of Γ has models but there exists no model of Γ itself, though it is ω-compact so for any countable theory compactness holds. Since the theory $\{Q_1(x = x)\}$ exists the Löwenheim-Skolem property does not hold for $\mathcal{L}(Q_1)$ and the Craig property is also not satisfied by $\mathcal{L}(Q_1)$.
One might now wonder why $\mathcal{L}(Q_1)$ was the first example given here instead of $\mathcal{L}(Q_0)$ since the distinction between infinite and finite intuitively might seem more basic than the distinction between uncountable and countable. The reason for this is that $\mathcal{L}(Q_0)$ is remarkably less well-behaved than $\mathcal{L}(Q_1)$, not being either complete, nor compact, and not satisfying the Craig property, so while it is possible to study its properties, the lack of completeness makes it much more of an endeavor [4] [5].

3.3.2 Infinitary logics

Instead of extending EL by adding generalised quantifiers we could modify the rules for constructing sentences and formulas to be a bit more liberal. The logic $\mathcal{L}_{\kappa,\lambda}$ where $\kappa, \lambda \geq \omega$, denotes EL with the alteration that we allow for all conjunctions and disjunctions with size strictly less than κ, and all homogeneous strings of quantifiers of length strictly less than λ, i.e. if $\alpha < \kappa$ and if for every $\beta \leq \alpha$, φ_β is an $\mathcal{L}_{\kappa,\lambda}$-formula, then $\bigwedge_{\beta\leq\alpha} \varphi_\beta$ is an $\mathcal{L}_{\kappa,\lambda}$-formula, and if $\gamma < \lambda$, and ψ is an $\mathcal{L}_{\kappa,\lambda}$-formula, then $Qx_0Qx_1...Qx_\gamma(\psi(x_0, x_1, ...x_\gamma))$ (where Q is a quantifier), is an $\mathcal{L}_{\kappa,\lambda}$-formula. Thus $\mathcal{L}_{\omega,\omega} = EL$. For any $\mathcal{L}_{\kappa,\lambda}$ with $\kappa \geq \omega_1$ compactness fails, though a version of the Löwenheim-Skolem property, which says that if T has a model then T has a model $\mathcal{M}$ such that $|\mathcal{M}| < \max(\kappa, \lambda)$, does hold for all $\mathcal{L}_{\kappa,\lambda}$. In particular $\mathcal{L}_{\omega_1,\omega}$ has the Löwenheim-Skolem property and is even complete and satisfies the Craig property. In $\mathcal{L}_{\omega_1,\omega}$ we can also construct the sentence

$$\bigwedge_{n<\omega} \exists x_0...x_n(\varphi(x_0) \wedge ... \wedge \varphi(x_n) \wedge x_0 \neq x_1 \wedge x_0 \neq x_2 ... \wedge x_{n-1} \neq x_n)$$

thus we have a logic where we can express the meaning of $Q_0x(\varphi(x))$ without said logic losing completeness [4][8].

3.3.3 Second-order logic

One very early extension of EL, earlier than both $\mathcal{L}(Q_\alpha)$ and $\mathcal{L}_{\kappa,\lambda}$, was $\mathcal{L}^2$ or *Second-order logic* which sprung up from statements like "there exists a function f (on the natural numbers) such that $f(x) > x$". Statements like this, referring to arbitrary functions, relations, or sets, cannot be expressed in any of the previous logics we have discussed. In $\mathcal{L}^2$ however, they can. This is achieved by allowing quantifiers, not just over elements in the domain of a structure, but also over subsets of the domain and over relations and functions on the domain. Now since we can quantify over functions, by introducing set theory we can say that for some set S, there exists a bijective function between S and some cardinal α and that for every $s \in S$ $\varphi(s)$ holds, thus we can express the meaning of any Q_α in $\mathcal{L}^2$. The expressive power of $\mathcal{L}^2$ is immensely powerful, allowing for concepts such as finiteness, countability, and well-ordering, and also giving us the tools needed to categorically define both the natural and the real numbers. To define formal proofs in $\mathcal{L}^2$ we need to add an axiom schema expressing that if there exists a specific relation between two elements, then there exists some relation between them. To use an example from the natural numbers: since $<(5, 7)$ (i.e. $5 < 7$) we can deduce that $\exists R(R(5, 7))$. What we must sacrifice for this expressive power however, are basically all of the useful properties we have for EL such as completeness, compactness, and the Löwenheim-Skolem property. Furthermore, in $\mathcal{L}^2$ the Craig property holds for only the one-sorted case [4] [5].

3.4 Comparing Logics

As we saw several times when looking at specific logics there is often an overlap of what can be expressed in different logics. In particular, all of the logics mentioned here are extensions of EL, and thus every EL-sentence is also a sentence in all of these logics. If we, instead of syntactic equality, regard semantic equivalence, we get the relation called the *strength* of logics. We say that for two logics $\mathcal{L}$ and $\mathcal{L}^*$, $\mathcal{L}^*$ is *at least as strong as* $\mathcal{L}$ if everything expressible in $\mathcal{L}$ is expressible in $\mathcal{L}^*$, that is, for every sentence $\varphi \in \mathcal{L}$ there exists a sentence $\psi \in \mathcal{L}^*$ such that φ and ψ have exactly the same models. $\mathcal{L}^*$ being at least as strong as $\mathcal{L}$ is denoted by $\mathcal{L} \leq \mathcal{L}^*$. If $\mathcal{L} \leq \mathcal{L}^*$ and $\mathcal{L}^* \leq \mathcal{L}$ we say that $\mathcal{L}$ and $\mathcal{L}^*$ are *equivalent* and denote this by $\mathcal{L} \equiv \mathcal{L}^*$[5][9]. Furthermore it follows that strength is a reflexive and transitive relation on logics, and that equivalence is a reflexive, transitive and symmetric relation. Thus the strength forms a partial order on the equivalence classes of equivalent logics.

While $\mathcal{L}^*$ being an extension of $\mathcal{L}$ trivially means that $\mathcal{L} \leq \mathcal{L}^*$, for $\mathcal{L}^*$ to be an extension of $\mathcal{L}$, every $\mathcal{L}$-sentence φ must also be a sentence in $\mathcal{L}^*$, so the converse, that if $\mathcal{L}^* \geq \mathcal{L}$ implies that $\mathcal{L}^*$ would be an extension of $\mathcal{L}$, is not necessarily true. The proof of this we have already showed since in $\mathcal{L}_{\omega_1,\omega}$ we can express the meaning of Q_0, so $\mathcal{L}(Q_0) \leq \mathcal{L}_{\omega_1,\omega}$, but Q_0 itself is not a part of $\mathcal{L}_{\omega_1,\omega}$, so the sentence $Q_0x(x = x)$ is an $\mathcal{L}(Q_0)$ sentence but not an $\mathcal{L}_{\omega_1,\omega}$-sentence. Thus $\mathcal{L}_{\omega_1,\omega}$ is not an extension of $\mathcal{L}(Q_0)$.

Of the logics discussed in this paper the weakest is EL as $\mathcal{L}(Q_\alpha)$, $\mathcal{L}_{\kappa,\lambda}$, and $\mathcal{L}^2$ are all extensions of EL. $\mathcal{L}^2$ meanwhile is stronger than all $\mathcal{L}(Q_\alpha)$ since any Q_α can be expressed by introducing some set theory and quantifying over α, $\forall(x \in \alpha)(\varphi(x))$. $\mathcal{L}(Q_1)$ is incomparable to both $\mathcal{L}(Q_0)$ and $\mathcal{L}_{\omega_1,\omega}$ and to show this we show that $\mathcal{L}(Q_1) \not\leq \mathcal{L}_{\omega_1,\omega}$ and $\mathcal{L}(Q_1) \not\geq \mathcal{L}(Q_0)$. To show the first of these statements, let $\mathcal{V} = \emptyset$. $Q_1x(x = x)$ is a sentence in $\mathcal{L}(Q_1)[\mathcal{V}]$, so if $\mathcal{L}(Q_1) \leq \mathcal{L}_{\omega_1,\omega}$ then there is some sentence $\psi \in \mathcal{L}_{\omega_1,\omega}$ with exactly the same models. Now since $\mathcal{L}_{\omega_1,\omega}$ has the Löwenheim-Skolem property this means that there exists a countable model of ψ, but $Q_1x(x = x)$ has no countable model, thus $\mathcal{L}(Q_1) \not\leq \mathcal{L}_{\omega_1,\omega}$. To show that $\mathcal{L}(Q_1) \not\geq \mathcal{L}(Q_0)$, let $\mathcal{V}$ be a vocabulary with infinitely many constant symbols (with some arbitrary enumeration of these constant symbols) and let $\Gamma \subset \mathcal{L}(Q_0)[\mathcal{V}]$ be the following theory

$$\Gamma = \{\neg Q_0x(x = x)\} \cup \{\neg c_\alpha = c_\beta | \alpha < \beta < \omega\}.$$

Assume $\mathcal{L}(Q_1) \geq \mathcal{L}(Q_0)$, then for every sentence in $\mathcal{L}(Q_0)$ there exists a sentence in $\mathcal{L}(Q_1)[\mathcal{V}]$ with exactly the same models. Let ζ be a mapping taking sentences in $\mathcal{L}(Q_0)[\mathcal{V}]$ to sentences in $\mathcal{L}(Q_1)[\mathcal{V}]$ with exactly the same models. Now construct the theory $\Gamma^* \subset \mathcal{L}(Q_1)[\mathcal{V}]$

$$\Gamma^* = \{\zeta(\neg Q_0x(x = x))\} \cup \{\zeta(\neg c_\alpha = c_\beta) | \alpha < \beta < \omega\}.$$

Every finite subtheory of Γ^* has a model, but Γ^* has no model, thus $\mathcal{L}(Q_1)$ is not ω-compact, but this contradicts what we stated previously, thus $\mathcal{L}(Q_1) \not\geq \mathcal{L}(Q_0)$. Lastly $\mathcal{L}_{\omega_1,\omega}$ and $\mathcal{L}^2$ are incomparable.

To show that $\mathcal{L}^2 \not\leq \mathcal{L}_{\omega_1,\omega}$ we simply need to consider that $\mathcal{L}(Q_1) \leq \mathcal{L}^2$, thus if $\mathcal{L}^2 \leq \mathcal{L}_{\omega_1,\omega}$, by transitivity of the strength of logics this would mean that $\mathcal{L}(Q_1) \leq \mathcal{L}_{\omega_1,\omega}$ thus contradicting our previous result that $\mathcal{L}(Q_1)$ and $\mathcal{L}_{\omega_1,\omega}$ are incomparable.
To show that $\mathcal{L}_{\omega_1,\omega} \not\leq \mathcal{L}^2$ is a little trickier. First let $\mathcal{V} = \{R\}$, where R is a binary relation symbol and let

$$\varphi = (\forall x, yR(x,y) \vee R(y,x) \vee (x = y)) \wedge (\forall x \neg(R(x,x)))$$

$$\wedge(\forall x, y, z(R(x,y) \wedge R(y,z) \to R(x,z))) \wedge (\forall x, y(R(x,y) \to \neg R(y,x))).$$

Now consider the following sentences in $\mathcal{L}_{\omega_1,\omega}[\mathcal{V}]$:

$$\psi_0 = \varphi \wedge (\bigwedge_{n<\omega} \exists x_0...x_n(R(x_0,x_1) \wedge R(x_0,x_2) \wedge ... \wedge R(x_1,x_2)$$

$$\wedge R(x_1,x_3) \wedge ... \wedge R(x_{n-1},x_n))),$$

$$\psi_1 = \varphi \wedge (\bigwedge_{n<\omega} \exists y_0, x_0...x_n(R(x_0,x_1) \wedge R(x_0,x_2) \wedge ... \wedge R(x_1,x_2)$$

$$\wedge R(x_1,x_3) \wedge ... \wedge R(x_{n-1},x_n) \wedge R(x_0,y_0) \wedge R(x_1,y_0) \wedge ... \wedge R(x_n,y_0))),$$

$$\psi_2 = \varphi \wedge (\bigwedge_{n<\omega} \exists y_0, y_1, x_0...x_n(R(x_0,x_1) \wedge R(x_0,x_2)...R(x_1,x_2)$$

$$\wedge R(x_1,x_3)...R(x_{n-1},x_n) \wedge R(x_0,y_0) \wedge R(x_1,y_0)... \wedge R(x_n,y_0) \wedge R(y_0,y_1))),$$

and so on. For any countable ordinal α, ψ_α describes an ordering of order type at least α (for more on order types see [10]). This means that for $\alpha \neq \beta$, ψ_α does not have the same models as ψ_β, since if we assume, without loss of generality, that $\alpha < \beta$, ψ_α has a model with an ordering of order type of exactly α which ψ_β does not. Thus for every countable ordinal we can associate a sentence in $\mathcal{L}_{\omega_1,\omega}[\mathcal{V}]$ which is unique with regard to models and since there are uncountably many countable ordinals, there are uncountably many distinct sentences in $\mathcal{L}_{\omega_1,\omega}[\mathcal{V}]$. This however is not the case for $\mathcal{L}^2[\mathcal{V}]$ since any sentence in $\mathcal{L}^2$ is finitely long so, since $\mathcal{V}$ is countable, there is only a countable number of sentences in $\mathcal{L}^2[\mathcal{V}]$.

Now we can illustrate the strength of all explicit logics discussed here with the following diagram:

$$\begin{tikzcd}
 & \mathcal{L}(Q_0) \arrow[r] \arrow[ddr] & \mathcal{L}_{\omega_1,\omega} \\
EL \arrow[ur] \arrow[dr] & & \\
 & \mathcal{L}(Q_1) \arrow[r] & \mathcal{L}^2
\end{tikzcd}$$

4 Lindström's Theorem

4.1 The theorem

Lindström's theorem provides a characterisation of elementary logic as the only logic with both the Löwenheim-Skolem property, the compactness property, that is at least as strong as EL.

Theorem (Lindström's theorem)**.** *For any abstract logic $\mathcal{L}$, if the following three properties hold:*
(i) *$EL \leq \mathcal{L}$,*
(ii) *the Löwenheim-Skolem property holds for $\mathcal{L}$,*
(iii) *$\mathcal{L}$ is ω-compact,*
then $\mathcal{L} \equiv EL$.

4.2 Proof

For the proof we will need the following proposition, the proof of which can be found (slightly reformulated) in [5] as theorem 4.3.1:

Proposition. *For countable structures $\mathfrak{M}$ and $\mathfrak{N}$, if Duplicator has a winning strategy for the Ehrenfeucht-Fraïssé game $G_\omega(\mathfrak{M}\mathfrak{N})$, then $\mathfrak{M} \simeq \mathfrak{N}$.*

With this result we are now ready to prove Lindström's theorem. The proof closely resembles the one in [11] with a few alterations and some alternative notation.

Proof. Assume $\mathcal{L}$ is an abstract logic such that (i), (ii), and (iii) holds but $\mathcal{L} \not\leq EL$. Let $\mathcal{V}$ be a vocabulary and π be a renaming from $\mathcal{V}$ to $\mathcal{V}'$ such that $\mathcal{V} \cap \mathcal{V}' = \emptyset$. Now let $\Gamma \subset \mathcal{L}[\mathcal{V} \cup \mathcal{V}']$ be the following set:

$$\{\forall x_0 ... x_{n-1}(R(x_0...x_{n-1}) \leftrightarrow \pi(R)(x_0...x_{n-1}))|1 \leq n < \omega, R \in \mathcal{V} \text{ is } n\text{:ary}\}$$

$$\cup\{\forall x_0...x_{n-1}(f(x_0...x_{n-1}) = \pi(f)(x_0...x_{n-1})|1 \leq n < \omega, f \in \mathcal{V} \text{ is n:ary}\}$$

$$\cup\{c = \pi(c)|c \in \mathcal{V}\}.$$

Furthermore, if ψ is a $\mathcal{L}[\mathcal{V}]$-sentence then $\Gamma \models_{\mathcal{L}} (\psi \leftrightarrow \pi(\psi))$. To show this, consider what π does to ψ within the context of Γ. Any constant symbol c is taken to the constant symbol $\pi(c)$, but $c = \pi(c)$ so the models of $\Gamma \cup \{\psi\}$ and $\Gamma \cup \{\pi_c(\psi)\}$ (where $\pi_c(\psi)$ denotes π applied only to the constant symbols in ψ) are exactly the same. For the rest of this proof, let $\overline{x}$ denote a tuple of variables, such that if f is an n-ary function then $f(\overline{x}) = f(x_0, ...x_{n-1})$. Any function f is taken to $\pi(f)$ but $f(\overline{x}) = \pi(f)(\overline{x})$, so again the models of $\Gamma \cup \{\psi\}$ and $\Gamma \cup \{\pi_f(\psi)\}$ (where $\pi_f(\psi)$ denotes π applied only to the function symbols in ψ) are exactly the same. Lastly π takes any relation R to $\pi(R)$, but $R \leftrightarrow \pi(R)(\overline{x})$ so the models of $\Gamma \cup \{\psi\}$ and $\Gamma \cup \{\pi_R(\psi)\}$ (where $\pi_R(\psi)$ denotes π applied only to the relation symbols in ψ) are exactly the same. Combining these three arguments we get that the models of $\Gamma \cup \{\psi\}$ and of $\Gamma \cup \{\pi_R \circ \pi_f \circ \pi_c(\psi)\}$ are exactly the

same, now note that (at least in the one sorted case which we here limit ourselves to) $\pi_R \circ \pi_f \circ \pi_c = \pi$, thus any model of $\Gamma \cup \{\psi\}$ and $\Gamma \cup \{\pi(\psi)\}$ are exactly the same and hence $\Gamma \models_{\mathcal{L}} (\psi \leftrightarrow \pi(\psi))$. Furthermore for every $\psi \in \mathcal{L}[\mathcal{V}]$ there exists a finite theory Γ_0 such that $\Gamma_0 \models_{\mathcal{L}} \psi \leftrightarrow \pi(\psi)$, since if it did not exist such a Γ_0, then for every finite $\Gamma_1 \subseteq \Gamma$, $\Gamma_1 \cup \{\neg(\psi \leftrightarrow \pi(\psi))\}$ would have a model, and thus, by *(iii)* $\Gamma \cup \{\neg(\psi \leftrightarrow \pi(\psi))\}$ would have a model which contradicts the result that that $\Gamma \models_{\mathcal{L}} (\psi \leftrightarrow \pi(\psi))$. Let $\mathcal{W}$ be the set of symbols of $\mathcal{V}$ occurring in Γ_0, then, since Γ_0 is finite and every sentence in Γ_0 contains finitely many symbols, $\mathcal{W}$ is finite, and for any $\mathcal{V}$-structures $\mathfrak{M}$ and $\mathfrak{N}$, if $(\mathfrak{M} \upharpoonright \mathcal{W}) \simeq (\mathfrak{N} \upharpoonright \mathcal{W})$ then $\mathfrak{M} \models_{\mathcal{L}} \psi$ iff $\mathfrak{N} \models_{\mathcal{L}} \psi$.
Thus we now have that

$$\begin{gathered}\text{given } \psi \in \mathcal{L}[\mathcal{V}]\text{, there exists } \mathcal{W} \subset \mathcal{V} \text{ such that } |\mathcal{W}| < \omega \text{ and}\\ \text{for any } \mathcal{V}\text{-structures } \mathfrak{M} \text{ and } \mathfrak{N}\\ ((\mathfrak{M} \upharpoonright \mathcal{W}) \simeq (\mathfrak{N} \upharpoonright \mathcal{W})) \implies ((\mathfrak{M} \models_{\mathcal{L}} \psi) \Leftrightarrow (\mathfrak{N} \models_{\mathcal{L}} \psi)).\end{gathered} \tag{1}$$

Now let $\psi \in \mathcal{L}[\mathcal{V}]$ be a sentence not equivalent to any $EL[\mathcal{V}]$-sentence, let $\mathcal{W} \subset \mathcal{V}$ be as in (1), and let $(\varphi_i)_{i\in\omega}$ be a sequence of all $EL[\mathcal{W}]$-sentences. For $n \in \omega$ we inductively define $\psi_0 = \varphi_0$ and for $n > 0$ $\psi_n = \varphi_n$ if $\varphi_n \wedge \bigwedge_{i<n} \psi_i$ is satisfiable and $\psi_n = \neg\varphi_n$ otherwise. Define $\Psi = \{\psi_n | n \in \omega\}$. It follows directly that seen as a theory $\Psi \subseteq EL[\mathcal{W}]$ is consistent and complete and as a theory $\Psi \subseteq \mathcal{L}[\mathcal{W}]$ is consistent (but incomplete). Since neither of the sentences ψ and $\neg\psi$ are equivalent to EL-sentences, there is nothing in Ψ which can contradict either ψ or $\neq \psi$, thus for any finite subset $\Phi \subseteq \Psi$ both $\{\psi\} \cup \Phi$ and $\{\neg\psi\} \cup \Phi$ have models so by (*iii*) $\Psi \cup \{\psi\}$ and $\Psi \cup \{\psi\}$ have models. Thus by (*ii*) there exist countable models $\mathfrak{M}$ and $\mathfrak{N}$ of $\Psi \cup \{\psi\}$ and $\Psi \cup \{\neg\psi\}$ respectively. From this we get that $\mathfrak{M} \upharpoonright \mathcal{W} \equiv \mathfrak{N} \upharpoonright \mathcal{W}$, and from (1) we get that $\mathfrak{M} \upharpoonright \mathcal{W} \not\simeq \mathfrak{N} \upharpoonright \mathcal{W}$. Since elementarily equivalent finite structures are isomorphic, and $\mathfrak{M}$ and $\mathfrak{N}$ are countable, we get that $|\mathfrak{M}| = |\mathfrak{N}| = \omega$. Thus without loss of generality we can assume that the underlying sets of $\mathfrak{M}$ and $\mathfrak{N}$ respectively are the same, we call this set S.
Thus in addition to (1) we now have that

$$\begin{gathered}\text{there exists } \mathcal{V}\text{-structures } \mathfrak{M} \text{ and } \mathfrak{N}\text{, with the same}\\ \text{underlying set } S\text{, such that}\\ \mathfrak{M} \models_{\mathcal{L}} \psi,\ \mathfrak{N} \models_{\mathcal{L}} \neg\psi,\ \text{and } \mathfrak{M} \upharpoonright \mathcal{W} \equiv \mathfrak{N} \upharpoonright \mathcal{W}\end{gathered} \tag{2}$$

We remind ourselves of the renaming π defined in the beginning of the proof and construct the vocabulary $\mathcal{V}^* = \mathcal{V} \cup \mathcal{V}' \cup \{f_n, g_n | n \in \omega\}$ where f_n and g_n are $(2n+1)$-ary function symbols. For each $n \in \omega$ fix an enumeration $\langle \chi_{n,i}(x_0, ..., x_{n-1}, x) | i \in \omega \rangle$ of all $EL[\mathcal{W}]$-formulas with free variables in $\{x_0, ..., x_{n-1}, x\}$. Now we construct the theory $\Delta \subseteq \mathcal{L}[\mathcal{V}^*]$

$$\Delta = \{\psi, \neg\pi(\psi)\} \cup \{\varphi \leftrightarrow \pi(\varphi) | \text{ for each } EL[\mathcal{W}]\text{-sentence } \varphi\}$$
$$\cup\{\forall\overline{x}\forall\overline{y}\forall x(\exists y(\bigwedge_{i=0}^{r}(\chi_{n,i}(\overline{x},x) \leftrightarrow \pi(\chi_{n,i})(\overline{y},y))) \rightarrow$$
$$\bigwedge_{i=0}^{r}(\chi_{n,i}(\overline{x},x) \leftrightarrow \pi(\chi_{n,i})(\overline{y},f_n(\overline{x},\overline{y},x)))),$$
$$\forall\overline{x}\forall\overline{y}\forall y(\exists x(\bigwedge_{i=0}^{r}(\chi_{n,i}(\overline{x},x) \leftrightarrow \pi(\chi_{n,i})(\overline{y},y))) \rightarrow$$
$$\bigwedge_{i=0}^{r}\chi_{n,i}(\overline{x},g_n(\overline{x},\overline{y},y)) \leftrightarrow \pi(\chi_{n,i})(\overline{y},y)))|n,r \in \omega\}.$$

We can see that the first row used here to construct Δ is satisfiable since ψ and $\neg\pi(\psi)$ are not EL-sentences. The middle and last row might seem very technical but looking closer at them one can see that all they do is dictate how f_n and g_n should work, in a way as to not interfere with anything else in Δ. Looking at the middle row for some r when $n = 0$ for example, we get the sentence

$$\forall x(\exists y(\bigwedge_{i=0}^{r}(\chi_{0,i}(x) \leftrightarrow \pi(\chi_{0,i})(y))) \rightarrow (\bigwedge_{i=0}^{r}(\chi_{0,i}(x) \leftrightarrow \pi(\chi_{0,i})(f_0(x))))).$$

For any $r \in \omega$ this is satisfiable thus any finite set consisting of this sentence for different values of r is satisfiable, furthermore we can see that for different values of n these sentences are completely independent so any finite set of sentences from the second row is satisfiable, the same argument can also be made for the last row. Now note that any finite $\Delta_0 \subseteq \Delta$ is satisfiable so by compactness Δ itself has a model, and by the Löwenheim-Skolem property there exists a countable model of Δ which we will call $\mathfrak{L}$. Now $\mathfrak{L}$, $\mathfrak{L} \upharpoonright \mathcal{V}$ and $\pi^-[\mathfrak{L} \upharpoonright \pi(\mathcal{V})]$, where π^- is the structure renaming given by the inverse of π, all have the underlying set S, furthermore $\mathfrak{L} \upharpoonright \mathcal{V} \models_{\mathcal{L}} \psi$, $\pi^-[\mathfrak{L} \upharpoonright \pi(\mathcal{V})] \models_{\mathcal{L}} \neg\psi$, and $(\mathfrak{L} \upharpoonright \mathcal{V}) \upharpoonright \mathcal{W} \equiv (\pi^-[\mathfrak{L} \upharpoonright \pi(\mathcal{V})]) \upharpoonright \mathcal{W}$ so $\mathfrak{L} \upharpoonright \mathcal{V}$ and $\pi^-[\mathfrak{L} \upharpoonright \pi(\mathcal{V})]$ satisfy all the assumptions we have made about $\mathfrak{M}$ and $\mathfrak{N}$ respectively so we can let $\mathfrak{M} = \mathfrak{L} \upharpoonright \mathcal{V}$ and $\mathfrak{N} = \pi^-[\mathfrak{L} \upharpoonright \pi(\mathcal{V})]$.

Now let $\langle s_i \rangle_{i\in\omega}$ be an enumeration of S and set up the Ehrenfeucht–Fraïssé game $G_\omega(\mathfrak{M} \upharpoonright \mathcal{W}, \mathfrak{N} \upharpoonright \mathcal{W})$. While the underlying sets of $\mathfrak{M} \upharpoonright \mathcal{W}$ and $\mathfrak{N} \upharpoonright \mathcal{W}$ are the same, the element $s_i \in S$ might not behave the same in both structures, however f_n and g_n at least guarantee that there exists some element $s_j \in S$ which does behave in $\mathfrak{N} \upharpoonright \mathcal{W}$ as s_i does in $\mathfrak{M} \upharpoonright \mathcal{W}$ or vice versa. By how f_n and g_n works according to Δ we can thus define two sequences $\langle \nu_i \rangle_{i\in\omega}$ and $\langle \mu_i \rangle_{i\in\omega}$ inductively, in the following way, $\nu_0 = f_0(s_0)$, $\nu_k = f_k(\langle s_i \rangle_{i<k}, \langle \mu_i \rangle_{i<k}, s_k)$, and $\mu_k = g_{k+1}(\langle s_i \rangle_{i\leq k}, \langle \nu_i \rangle_{i\leq k}, s_k)$. By Δ ν_i behaves in $\mathfrak{N} \upharpoonright \mathcal{W}$ as s_i does in $\mathfrak{M} \upharpoonright \mathcal{W}$, and μ_i behaves in $\mathfrak{M} \upharpoonright \mathcal{W}$ as s_i does in $\mathfrak{N} \upharpoonright \mathcal{W}$ for all i. Thus if Spoiler picks s_k from $\mathfrak{M} \upharpoonright \mathcal{W}$ then Duplicator picks ν_k, and if Spoiler picks s_k from $\mathfrak{N} \upharpoonright \mathcal{W}$

then Duplicator picks μ_k. Using this strategy, after t steps in the game, by mapping any s_k in the resulting substructure of $\mathfrak{M} \restriction \mathcal{W}$ to μ_k, and for the inverse mapping any s_k in the resulting substructure of $\mathfrak{N} \restriction \mathcal{W}$ to ν_k we get a t-partial isomorphism $f_t : \mathfrak{M} \to \mathfrak{N}$; the bijectivity is clear from the construction of the mapping, and preservation of $\mathcal{V}$-formulas follows from $\mathfrak{L}$ being a model of Δ. Now note that $f_t \subseteq f_{t+1}$ for all $t \in \omega$, so we get a countably infinite sequence $f_0 \subseteq f_1 \subseteq f_2 \subseteq ...$ of partial isomorphisms from $\mathfrak{M}$ to $\mathfrak{N}$. Now consider $\bigcup_{i \in \omega} f_i$, this is a ω-partial isomorphism from $\mathfrak{M}$ to $\mathfrak{N}$. Thus picking ν_k when Spoiler pick s_k from $\mathfrak{M} \restriction \mathcal{W}$ and picking μ_k when Spoiler pick s_k from $\mathfrak{N} \restriction \mathcal{W}$ is a winning strategy for Duplicator, so since $\mathfrak{M} \restriction \mathcal{W}$ and $\mathfrak{N} \restriction \mathcal{W}$ are countable our proposition gives us that $\mathfrak{M} \restriction \mathcal{W} \simeq \mathfrak{N} \restriction \mathcal{W}$. This however contradicts (1) since $\mathfrak{M} \models_{\mathcal{L}} \psi$ and $\mathfrak{N} \models_{\mathcal{L}} \neg\psi$ which proves the theorem. $\square$

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