Monotone Lagrangians in cotangent bundles of spheres

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We study the compact monotone Fukaya category of T^*S^n, for $n \geq 2$, and show that it is split-generated by two classes of objects: the zero-section S^n (equipped with suitable bounding cochains) and a 1-parameter family of monotone Lagrangian tori $(S^1 \times S^{n-1})_\tau$, with monotonicity constants $\tau > 0$ (equipped with rank 1 unitary local systems). As a consequence, any closed orientable spin monotone Lagrangian (possibly equipped with auxiliary data) with non-trivial Floer cohomology is non-displaceable from either S^n or one of the $(S^1 \times S^{n-1})_\tau$. In the case of T^*S^3, the monotone Lagrangians $(S^1 \times S^2)_\tau$ can be replaced by a family of monotone tori T^3_τ.
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Contents

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1. Introduction

An embedded Lagrangian L in a cotangent bundle $(T^*Q, d(pdq))$, is *exact* if $pdq|_L = df$ for some function $f : L \to \mathbb{R}$. Arnold's nearby Lagrangian conjecture predicts that if Q and L are closed, then L is Hamiltonian-isotopic to the zero-section $Q \subset T^*Q$. This result is currently known to hold only for a limited list of examples, including $Q = S^2$ [20] and T^2 [12]. The work of many authors has also led to a proof that the composition $L \to T^*Q \to Q$ (where the first map is the embedding and the second is projection to the zero-section) is a simple homotopy equivalence [4].

Very little is known if one drops the requirement of L being exact. We will consider the case of L *monotone*, by which we mean that there is a constant $\tau \geq 0$ such that, for every smooth map $u : (D^2, \partial D^2) \to (T^*Q, L)$,

$$
\int_{D^2} u^* \omega = \tau \cdot \mu(u)
$$

where $\mu(u)$ is the Maslov index of u. Note that we allow the case $\tau = 0$, which happens, for instance when L is exact. For some results about monotone Lagrangians in cotangent bundles, see for instance [17].

Remark 1.1. Under suitable conditions, $\tau = 0$ implies that L is exact. This is the case, for example, if the following conditions are both satisfied:

(1) the map $h^* \colon \operatorname{Hom}(H_2(T^*Q, L), \mathbb{R}) \to \operatorname{Hom}(\pi_2(T^*Q, L), \mathbb{R})$ is injective (h^* is the dual of the relative Hurewicz homomorphism, see for example [11, Section 6.17]);
(2) the map $H^1(T^*Q; \mathbb{R}) \to H^1(L; \mathbb{R})$ induced by inclusion is trivial.

Recall that there is an isomorphism $U \colon H^2(T^*Q, L; \mathbb{R}) \to \operatorname{Hom}(H_2(T^*Q, L), \mathbb{R})$ given by the universal coefficients theorem. Note that if (1) holds, then $h^* \circ U \colon H^2(T^*Q, L; \mathbb{R}) \to \operatorname{Hom}(\pi_2(T^*Q, L), \mathbb{R})$ is also injective. This is useful when showing that L is exact.

Condition (2) above holds if $Q = S^n$ with $n \geq 2$. The fact that $\mathbb{R}$ is an injective $\mathbb{Z}$-module implies that condition (1) holds if $Q = S^n$ with $n \geq 1$, using the five-lemma (see for example [19, page 129]). Hence, if $Q = S^n$ with $n \geq 2$, a monotone Lagrangian $L \subset T^*S^n$ is exact iff $\tau = 0$.

The focus of this paper is on closed monotone Lagrangians in cotangent bundles of spheres S^n (with the standard smooth structure), from the point of view of Floer theory, more specifically using *wrapped Floer cohomology*. Unless otherwise specified, we will always assume that $n \geq 2$. Given closed Lagrangians L, L' (possibly equipped with additional data like bounding cochains or local systems) in a symplectic manifold, one can sometimes define their Floer cohomology $HF^*(L, L')$, which is invariant under Hamiltonian perturbations of either L or L'. If $HF^*(L, L') \neq 0$, then L is not Hamiltonian-displaceable from L' (which means that $\varphi(L) \cap L' \neq \emptyset$ for every Hamiltonian diffeomorphism φ) [13]. Unless we say otherwise, we will take Floer cohomology with coefficients in the Novikov field over $\mathbb{C}$, which is denoted by $\mathbb{K}$ and defined in Section 3.1.

There is a 1-parameter family of disjoint monotone Lagrangians $(S^1 \times S^{n-1})_\tau \subset T^*S^n$, of different monotonicity constants $\tau > 0$, whose construction will be reviewed below. These Lagrangians can be equipped with local systems such that their Floer cohomologies are non-trivial. In T^*S^3, the same holds for a 1-parameter family of disjoint monotone Lagrangian tori T^3_τ, see [10]. We will review the construction of these tori below as well. We will prove the following result.

Theorem 1.2. *Take $n \geq 2$ and let $L \subset T^*S^n$ be a closed orientable spin monotone Lagrangian with a unitary local system of rank 1 for which $HF^*(L, L; \mathbb{K}) \neq 0$. Then, either $HF^*(L, S^n; \mathbb{K}) \neq 0$ (where the zero-section S^n is equipped with a suitable bounding cochain) or there is a $\tau > 0$ for which $HF^*(L, (S^1 \times S^{n-1})_\tau; \mathbb{K}) \neq 0$ (where $(S^1 \times S^{n-1})_\tau$ is equipped with a suitable unitary local system of rank 1). In particular, L is not Hamiltonian-displaceable from either S^n or from $(S^1 \times S^{n-1})_\tau$, for some $\tau > 0$.*

Furthermore, in T^*S^3 we can replace the $(S^1 \times S^2)_\tau$ with the tori T^3_τ.

Our work towards the proof of Theorem 1.2 will also imply the following.

Theorem 1.3. *Let $\tau, \tau' > 0$. Then $\tau = \tau'$ iff the Lagrangians $(S^1 \times S^2)_\tau$ and $T^3_{\tau'}$ can be equipped with unitary local systems of rank 1 with respect to which $HF^*((S^1 \times S^2)_\tau, T^3_{\tau'}; \mathbb{K}) \neq 0$. In particular, $(S^1 \times S^2)_\tau$ is not Hamiltonian-displaceable from T^3_τ.*

We now describe the structure of the proof of Theorem 1.2. The Lagrangians L in the statement give objects in a monotone wrapped Fukaya category of T^*S^n, which

also includes a cotangent fiber $F = T^*_q S^n$ (for some $q \in S^n$). This is an A_∞-category (with only a $\mathbb{Z}/2\mathbb{Z}$-grading, since we allow monotone Lagrangians), which we denote temporarily by $\mathcal{W}$ (and will refer to it as $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(Y; \mathbb{K})$ in Section 3.4).

The category $\mathcal{W}$ is split-generated by the cotangent fiber F, as can be proven by adapting a result in [2] (which in its original form was for the wrapped Fukaya category of exact Lagrangians). Let us consider some algebraic consequences of this split-generation result. Let $A_\mathbb{K} := HW^*(F, F; \mathbb{K})$ be the wrapped Floer cohomology algebra of F. The graded algebra $A_\mathbb{K}$ is isomorphic to $H_{-*}(\Omega_q S^n; \mathbb{K})$, where $q \in S^n$ is a basepoint and Ω_q denotes the based loop space, see [3]. Hence, since $n \geq 2$, $A_\mathbb{K}$ is isomorphic to a polynomial algebra $\mathbb{K}[u]$, where $\deg(u) = 1 - n$. There is a *Yoneda functor*

$$Y : \mathcal{W} \to \mathrm{mod}(A_\mathbb{K})$$
$$L \mapsto HF^*(F, L; \mathbb{K})$$

where $\mathrm{mod}(A_\mathbb{K})$ is the category of $\mathbb{Z}/2\mathbb{Z}$-graded right $A_\mathbb{K}$-modules., with the morphism space between two objects M, M' in $\mathrm{mod}(A_\mathbb{K})$ being $\mathrm{Ext}^*_{A_\mathbb{K}}(M, M')$.

The split-generation result mentioned above, together with formality results for A_∞-modules over $A_\mathbb{K}$ that we prove in Section 5, imply that Y is a cohomologically full and faithful functor, which means that it induces isomorphisms on cohomology

$$HW^*(L, L'; \mathbb{K}) \cong \mathrm{Ext}^*_{A_\mathbb{K}}(Y(L), Y(L'))$$

for any pair of objects L, L'. See Proposition 3.8 below for more details. Take the sub-category $\mathcal{F} \subset \mathcal{W}$ (denoted as $\mathcal{F}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(Y; \mathbb{K})$ in Section 3.4) that does not include the object F, but only compact monotone Lagrangians. Given an object L of this subcategory (where we suppress the additional data of local systems or bounding cochains), $HW^*(F, L; \mathbb{K}) = HF^*(F, L; \mathbb{K})$ is a finite dimensional $\mathbb{K}$-vector space, so Y restricts to a cohomologically full and faithful embedding

$$Y_c : \mathcal{F} \to \mathrm{mod}_{pr}(A_\mathbb{K}) \tag{1.1}$$

into the subcategory $\mathrm{mod}_{pr}(A_\mathbb{K}) \subset \mathrm{mod}(A_\mathbb{K})$ of proper $A_\mathbb{K}$-modules M (those which are finite dimensional over $A_\mathbb{K}$). The approach of this paper will be to study the category $\mathcal{F}$ by analyzing the algebraic category $\mathrm{mod}_{pr}(A_\mathbb{K})$.

Corollaries 6.5 and 6.8 below give generators for $\mathrm{mod}_{pr}(A_\mathbb{K})$, and imply the following result (which will appear below as Corollaries 4.16 and 4.20).

Theorem 1.4. *Given $n \geq 2$, the functor Y_c in (1.1), when extended to the split-closure of $\mathcal{F}$ (which is the monotone Fukaya category of T^*S^n), is a quasi-equivalence of categories. The category $\mathcal{F}$ is split-generated by the uncountable collection of objects consisting of S^n (equipped with suitable bounding cochains) and the $(S^1 \times S^{n-1})_\tau$ (equipped with unitary local systems of rank 1). In the case of T^*S^3, we can replace the $(S^1 \times S^2)_\tau$ with the tori T^3_τ.*

Proof of Theorem 1.2. Given L such that $HF^*(L, L; \mathbb{K}) \neq 0$, L is a non-trivial object in $\mathcal{F}$. Theorem 1.4 then implies that $HF^*(L, L'; \mathbb{K}) \neq 0$, where L' is one of the split-generators. $\square$

Remark 1.5 *(Relation to mirror symmetry).* As mentioned, the tori T_τ^3 were studied in [10]. They are fibers of an SYZ fibration in the complement of an anticanonical divisor H in T^*S^3 (H is anticanonical in the sense that the Lagrangian tori in the SYZ fibration have vanishing Maslov class in the complement of H). In this setting, the authors compute the disk potentials associated to SYZ-fibers by studying wall-crossing for pseudoholomorphic disks. This information is used to construct a Landau–Ginzburg model that is mirror to T^*S^3. The critical locus of the Landau–Ginzburg potential is an affine line. If the mirror is constructed over the Novikov field, then the points in this critical line with negative valuation correspond to (split summands of) the monotone Lagrangians T_τ^3, equipped with suitable unitary local systems of rank 1. The points with non-negative valuation correspond to bounding cochains on the zero section S^3.

Remark 1.6 *(Relation to abstract flux).*

The monotone Lagrangians $(S^1 \times S^{n-1})_\tau$ can be obtained geometrically as follows. Let $f : S^n \to \mathbb{R}$ be a Morse function with exactly two critical points. The graph of df intersects the zero section of T^*S^n transversely in the two critical points, and one can perform surgery on this transverse intersection to produce the family $(S^1 \times S^{n-1})_\tau$. Similarly, the tori T_τ^3 can be obtained by taking a Morse–Bott function $g : S^3 \to \mathbb{R}$ whose critical locus is a Hopf link, and performing surgery in T^*S^3 on the clean intersection of the zero section and the graph of dg.

Recall that given a compact manifold Q and a class $\alpha \in H^1(Q; \mathbb{R})$, one can take the *flux deformation* of the zero-section of T^*Q in the direction of α, by flowing Q along a symplectic vector field X such that $[\omega(., X)] = i^*\alpha$ (where $i : Q \to T^*Q$ is the inclusion). Using the Weinstein tubular neighborhood theorem, one can similarly deform a compact Lagrangian L in a symplectic manifold (M, ω) along a class $\alpha \in H^1(L; \mathbb{R})$. Motivated by [33], one can think of the family of Lagrangians $(S^1 \times S^{n-1})_\tau$ (respectively, T_τ^3) as an *abstract flux deformation* of two copies of the zero section S^n (respectively, S^3) in the direction of a class $\beta \in H^n(S^n; \mathbb{R})$ (respectively, $H^3(S^3; \mathbb{R})$), if n is odd. The case of n even is more subtle, as we will see.

This paper is organized as follows. In Section 2, we present the construction of the monotone Lagrangians $(S^1 \times S^{n-1})_\tau$ in T^*S^n and T_τ^3 in T^*S^3. In Section 3, we recall the definitions of several versions of Fukaya categories of T^*S^n, including a monotone wrapped Fukaya category where Lagrangians are allowed to intersect cleanly. In Section 4, we perform several Floer cohomology computations, with a view towards proving Theorem 1.4. The remaining sections have a more algebraic nature, and are about A_∞-algebras and A_∞-modules. In Section 5, we establish formality results for a category of

modules associated to a cotangent fiber in T^*S^n. In Section 6, we obtain generators for that category of modules.

Acknowledgments

2. Monotone Lagrangians in T^*S^n

2.1. Lagrangians in T^*S^n

Recall that T^*S^n is symplectomorphic to the complex affine quadric

$$X_n = \{(z_0, \ldots, z_n) \in \mathbb{C}^{n+1} \mid z_0^2 + \ldots + z_n^2 = 1\},$$

equipped with the Kähler form ω obtained from the restriction of $\frac{i}{2} \sum_{j=0}^{n} dz_j \wedge d\overline{z_j}$ on $\mathbb{C}^{n+1}$ [25, Exercise 6.20]. Assume that $n \geq 2$. The projection to the first coordinate defines a Lefschetz fibration

$$\pi_n \colon X_n \to \mathbb{C}$$

$$(z_0, \ldots, z_n) \mapsto z_0$$

with critical values ± 1. For every regular value $p \neq \pm 1$, the fiber $\pi_n^{-1}(p)$ is symplectomorphic to T^*S^{n-1}, and contains the Lagrangian sphere

$$V_p := \{(p, \sqrt{1-p^2}\, x_1, \ldots, \sqrt{1-p^2}\, x_n) \in X_n \mid (x_1, \ldots, x_n) \in S^{n-1}\},$$

where $S^{n-1} \subset \mathbb{R}^n$ is the unit sphere and $\sqrt{1-p^2}$ is one of the two square roots of $1-p^2$. Write also $V_{\pm 1} = \{(\pm 1, 0, \ldots, 0)\}$.

We will be interested in the following types of Lagrangians that project to curves under π_n. See Fig. 1 for relevant examples of such curves.

Definition 2.1. Given a curve $C \subset \mathbb{C} \setminus \{-1, 1\}$ that is the image of an embedding of S^1, let

$$L_C := \bigcup_{z \in C} V_z.$$

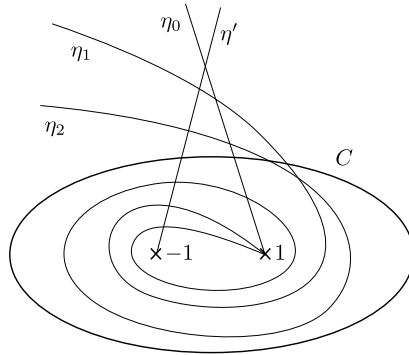Given an embedding $\eta : [0, \infty) \to \mathbb{C}$ such that

Fig. 1. Some curves in $\mathbb{C} \setminus \{\pm 1\}$.

- $\eta(0) \in \{-1, 1\}$,
- $\eta((0, \infty)) \subset \mathbb{C} \setminus \{-1, 1\}$ and
- $\eta(t) = at + b$ for some $a \in \mathbb{C}^*$, $b \in \mathbb{C}$ and t large enough, let

$$F_\eta := \bigcup_{t \geq 0} V_{\eta(t)}.$$

Lemma 2.2. *The subsets L_C and F_η of X_n in Definition 2.1 are Lagrangian submanifolds. If C encloses both points ± 1, then L_C is diffeomorphic to $S^1 \times S^{n-1}$, while F_η is Hamiltonian-isotopic to a cotangent fiber in $T^* S^n$.*

Proof. The L_C and F_η are Lagrangians because parallel transport with respect to the connection induced by the symplectic fibration π_n preserves the spheres V_p (they are vanishing cycles for arbitrary vanishing paths in the base), see [32, Lemma 16.3].

Since there are only two types of S^{n-1}-bundles over S^1, and the closed curve C encircles two critical values which have the same monodromy (a Dehn twist), it follows that L_C is the trivial bundle.

We now consider the Lagrangians F_η. Take $\eta_\pm$ such that $\eta_\pm(t) = \pm(t + 1)$ for all $t \geq 0$. Then, $F_{\eta_\pm}$ is mapped to $T_{\pm 1}^* S^n$ by the symplectomorphism $X_n \to T^* S^n$ in [25, Exercise 6.20]. For any other η, one can construct an isotopy to one of the $\eta_\pm$ that lifts to a Hamiltonian isotopy by applying Moser's trick. $\square$

Remark 2.3. The Floer cohomology of the Lagrangian submanifolds $L_C \cong S^1 \times S^{n-1}$ in $T^* S^n$ in the previous lemma was studied in [7]. These are a particular case of the *generalized Polterovich Lagrangians* in [26].

Remark 2.4. In this Lefschetz fibration description $\pi_n : X_n \to \mathbb{C}$ of $T^* S^n$, the zero section S^n is the Lagrangian lift of the interval $[-1, 1] \subset \mathbb{C}$.

Let us continue with our study of the Lagrangians L_C, where C encloses $\{\pm 1\}$. Much of what follows in this section is an adaptation of results in [24, Section 2.2]. The homology long exact sequence of the pair (T^*S^n, L_C) implies that

$$H_2(T^*S^n, L; \mathbb{Z}) \cong H_2(T^*S^n; \mathbb{Z}) \oplus H_1(L_C; \mathbb{Z})$$

if $n \geq 2$. The group $H_2(T^*S^n; \mathbb{Z})$ vanishes unless $n = 2$, in which case both ω and $c_1(T^*S^2)$ vanish on $H_2(T^*S^2; \mathbb{Z})$. For $n \geq 3$, the group $H_1(L_C; \mathbb{Z})$ has rank 1 and $H_2(T^*S^n, L_C; \mathbb{Z}) \cong \mathbb{Z}$ is generated by a class β such that $\pi_n \circ \beta$ covers C once. For $n = 2$, $H_1(L_C; \mathbb{Z})$ has rank 2 and we can pick $\alpha, \beta \in H_2(T^*S^2, L_C; \mathbb{Z})$ such that their boundaries give a basis for $H_1(L_C; \mathbb{Z}) \cong \mathbb{Z}^2$, with the following properties: α is a Lefschetz thimble for some vanishing cycle V_p (hence it has vanishing Maslov index and symplectic area), while the boundary of $\pi_2 \circ \beta$ covers C once. We will now study the ω-area and Maslov index of the disks β.

We need some auxiliary notation. Denote by $\sigma_{std} := \frac{i}{2} dz \wedge d\overline{z} = r dr \wedge d\theta$ the standard area form in $\mathbb{C}$. Define, on the set $\mathbb{C} \setminus \{\pm 1\}$ of regular values of the Lefschetz fibration π_n, the 2-form

$$\sigma := \frac{i}{2} dz_0 \wedge d\overline{z_0} + f^* \sigma_{std},$$

where $f : \mathbb{C} \setminus \{\pm 1\} \to \mathbb{C} \setminus \{0\}$ is given by $f(z) = \frac{1 - z^2}{\sqrt{2|1 - z^2|}}$. The function f can be thought of as the composition of the two maps

$$\mathbb{C} \setminus \{\pm 1\} \to \mathbb{C} \setminus \{0\} \qquad \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$$

$$z \mapsto 1 - z^2 \qquad\qquad re^{i\theta} \mapsto \sqrt{\frac{r}{2}} e^{i\theta}$$

The first map is holomorphic and the second is smooth and orientation-preserving, so σ defines a positive measure on $\mathbb{C} \setminus \{\pm 1\}$. It extends to all of $\mathbb{C}$, as a measure that is absolutely continuous with respect to the Lebesgue measure.

Lemma 2.5. *Given a disk* $\beta : (D^2, \partial D^2) \to (X_n, L_C)$ *such that* $\pi_n \circ \beta$ *covers* C *once, we have*

$$\int_\beta \omega = \int_{\pi_n(\beta)} \sigma.$$

Proof. Take β as in the lemma. We can assume the boundary of β to be given by $c(t) = \left(\gamma(t), \sqrt{1 - \gamma(t)^2}\, s(t) \right)$, where $\gamma : [0, 1] \to \mathbb{C} \setminus \{\pm 1\}$ is a degree 1 parametrization of C and $s(t) = (s_1(t), \ldots, s_{n+1}(t)) \in S^{n-1}$. Here, $\sqrt{\cdot}$ is the analytic continuation of a branch of the square root along the path $1 - \gamma^2$. Write $g(t) := \sqrt{1 - \gamma(t)^2}$. We have

$$\int_\beta \omega - \frac{i}{2} dz_0 \wedge d\overline{z_0} = \int_c \sum_{j=1}^n \frac{i}{4}(z_j \, d\overline{z_j} - \overline{z_j} \, dz_j) = \frac{i}{4}\int_0^1 g\overline{g}' - \overline{g}g' dt, \qquad (2.1)$$

using on the first identity Stokes' theorem and the fact that $\frac{i}{4}(z_j \, d\overline{z_j} - \overline{z_j} \, dz_j)$ is a primitive of $\frac{i}{2} dz_j \wedge d\overline{z_j}$. A calculation shows that the right side of (2.1) can be written as

$$\int_0^1 g^* \left(\frac{1}{2}r^2 d\theta\right) = \int_C f^* \left(\frac{1}{2}r^2 d\theta\right),$$

where f is the function defined before the lemma. Identifying C with the boundary of $\pi_n(\beta)$ and using Stokes' theorem, the integral on the right equals

$$\int_{\pi_n(\beta)} f^* \sigma_{std},$$

which finishes the proof. □

Remark 2.6. The previous argument also goes through if C is a piecewise smooth curve. This will be helpful in Section 4, when computing operations μ^k involving several Lagrangians that fiber over paths in $\mathbb{C}$.

Corollary 2.7. *Suppose that the simple curves C and C' in $\mathbb{C} \setminus \{-1, 1\}$ both enclose $\{-1, 1\}$. Then, they bound the same σ-area if and only if L_C and $L_{C'}$ are Hamiltonian-isotopic.*

Proof. The proof is similar to that of [24, Corollary 2.5]. □

Lemma 2.8. *The Maslov index of an oriented disk in X_n with boundary in L_C, whose boundary projects to a degree 1 cover of C, is $2(n-1)$. The Lagrangians L_C are monotone with monotonicity constant $\tau_C = \frac{\int_{\Omega_C} \sigma}{2(n-1)}$, where $\Omega_C \subset \mathbb{C}$ is the region bounded by C in the plane.*

Proof. We begin by considering the Lagrangian lift L_0 of the unit circle in the model Lefschetz fibration $\pi : \mathbb{C}^n \to \mathbb{C}$, where $\pi(z_1, \ldots, z_n) = z_1^2 + \ldots + z_n^2$. The vanishing cycle over $p \in \mathbb{C} \setminus \{0\}$ of a vanishing path through p is

$$V'_p := \{\sqrt{p}(x_1, \ldots, x_n) \,|\, (x_1, \ldots, x_n) \in S^{n-1}\},$$

see [32, Example 16.5]. We can use the holomorphic volume form

$$\Omega = dz_1 \wedge \ldots \wedge dz_n \qquad (2.2)$$

on $\mathbb{C}^n$ to compute the Maslov index of a disk with boundary in L_0. Let u be such a disk, of positive symplectic area and with boundary projecting to a simple cover of the unit circle. Let $\gamma : S^1 \to L$ be a parametrization of this boundary loop such that $\pi(\gamma(t)) = e^{it}$. The imaginary part of $\left(e^{-i(nt+\pi)/2}\,\Omega\right)|_{L_0}$ vanishes, hence the Maslov index of u is n (see [35] for similar computations).

To compute the Maslov class of L_C in the statement of the lemma, we observe that C is Lagrangian-isotopic to a connected sum $C_{-1}\#C_1$, where $C_{\pm 1}$ is a small simple loop around ± 1 (this argument is inspired by [31]). By picking a local trivialization of the Lefschetz fibration π_n near ± 1, we see that the Maslov class of $L_{C_{\pm 1}}$ can be identified with that of L_0 above. This implies that one can think of a disk in X_n with positive symplectic area, and with boundary in L_C projecting to a simple cover of C, as a connected sum of two disks as in the previous paragraph. Hence, the Maslov index of the disk with boundary in L_C is $2n - 2$, as wanted.

The monotonicity of L_C and the value of τ_C now follow from Lemma 2.5. $\square$

Recall that, given a monotone Lagrangian L in a symplectic manifold (M, ω) and a choice of basis $h_1, \ldots, h_m$ for the free part of $H_1(L; \mathbb{Z})$, we can define the *disk potential* $W_L : (\mathbb{C}^*)^m \to \mathbb{C}$ as

$$W_L(x_1, \ldots, x_m) = \sum_{u \in \mathcal{M}} \pm x^{\partial u}, \tag{2.3}$$

where $\mathcal{M}$ is the moduli space of J-holomorphic maps $u : (D^2, \partial D^2) \to (M, L)$ of Maslov index 2, such that $u(1) = p$, for a generic choice of point $p \in L$ and compatible almost complex structure J on (M, ω). The sign associated to u depends on the spin structure of L. If we write $\langle \partial u, h_i \rangle$ for the h_i-component of $[\partial u]$ in the free part of $H_1(L; \mathbb{Z})$, then $x^{\partial u}$ stands for the product $x_1^{\langle \partial u, h_1 \rangle} \ldots x_m^{\langle \partial u, h_m \rangle}$. The disk potential does not depend on the choices of generic p and J.

Lemma 2.9. *For $n = 2$, the disk potential of L_C is $W_{L_C} = x_1(1 + x_2)^2$, in a basis $h_1, h_2 \in H_1(L_C; \mathbb{Z}) \cong \mathbb{Z}^2$ where h_1 is a loop projecting to C in degree 1 and h_2 is a fiber of the projection $\pi_2|_{L_C}$. The disk potential is zero if $n > 2$.*

Proof. For $n = 2$, the disk potential is computed in [24, Lemma 2.19], using a degeneration argument from [31]. In the proof of [8, Corollary 5.13], the relevant Maslov index 2 disks are also computed explicitly, using the integrable complex structure in the target. The case $n > 2$ follows from Lemma 2.8. $\square$

Fix $\tau > 0$ and a smooth embedded loop $C_\tau \subset \mathbb{C} \setminus \{-1, 1\}$ that winds once around -1 and 1 and bounds σ-area $2(n-1)\tau$. Denote by L_τ, or $(S^1 \times S^{n-1})_\tau$, the corresponding Lagrangian L_{C_τ}. By Lemma 2.8, L_τ is monotone with monotonicity constant τ. Observe that we can exhaust $\mathbb{C} \setminus [-1, 1]$ by a collection of disjoint simple curves C, such that the corresponding monotonicity constants τ_C cover $\mathbb{R}_{>0}$ without repetitions. The matching

sphere over the interval $[-1, 1] \subset \mathbb{C}$ is the zero section $S^n \subset T^*S^n$. Assume that C_τ is the curve C in Fig. 1, and denote by F_i the lifts of the paths η_i in the same figure. Similarly, denote by F' the lift of the path η'.

Recall that two Lagrangian submanifolds $L, L' \subset (X, \omega)$ *intersect cleanly* if $K := L \cap L'$ is a manifold and for every $x \in K$ we have $T_x K = T_x L \cap T_x L' \subset T_x X$.

Lemma 2.10. *For every $i \geq 0$, F_i and L_τ intersect cleanly. For every $i, j \geq 0$, F_i and F_j intersect cleanly. Also, all these Lagrangians intersect F' cleanly.*

Proof. This follows from the fact that the Lagrangians project under the map $\pi_n : X_n \to \mathbb{C}$ to curves that intersect transversely. $\quad\square$

2.2. More Lagrangians in T^*S^3

It will be useful to also consider an alternative description of the complex affine quadric 3-fold, which is symplectomorphic to T^*S^3. We borrow some notation from [10]. Write

$$X = \{(z, u_1, v_1, u_2, v_2) \in \mathbb{C}^5 \,|\, u_1 v_1 = z + 1, u_2 v_2 = z - 1\}.$$

Consider the Lefschetz fibrations

$$\pi^i : \mathbb{C}^2 \to \mathbb{C}$$

$$(u_i, v_i) \mapsto u_i v_i + (-1)^i,$$

where $i \in \{1, 2\}$. The map π^i has a unique critical value at $(-1)^i$ and, given $p \in \mathbb{C} \setminus \{(-1)^i\}$, the vanishing circle in $(\pi^i)^{-1}(p)$ of a vanishing path through p is

$$V_{i,p} := \{(u_i, v_i) \in \mathbb{C}^2 \,|\, \pi^i(u_i, v_i) = p, |u_i| = |v_i|\}.$$

Write also $V_{i,(-1)^i} = \{(0, 0)\}$. For more details, see [10] and [32, Example 16.5]. The affine quadric X is the fiber product of these two fibrations:
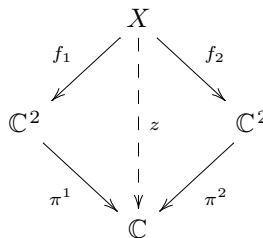
The map $z : X \to \mathbb{C}$ is not a Lefschetz fibration, but it can be thought of as a Morse–Bott analogue, with critical values ± 1 and such that the critical locus over ± 1 is a copy of $\mathbb{C}^*$. We will consider the following analogues of the Lagrangians L_C and F_η from the previous section. It will again be useful to have Fig. 1 in mind.

Definition 2.11. Given a curve $C \subset \mathbb{C} \setminus \{\pm 1\}$ that is the image of an embedding of S^1, let

$$T_C := \bigcup_{z \in C} V_{1,z} \times V_{2,z}.$$

Given an embedding $\eta : [0, \infty) \to \mathbb{C}$ such that

- $\eta(0) = 1$,
- $\eta\big((0, \infty)\big) \subset \mathbb{C} \setminus \{\pm 1\}$ and
- $\eta(t) = at + b$ for some $a \in \mathbb{C}^*$, $b \in \mathbb{C}$ and t large enough,

let

$$N_\eta := \bigcup_{t \geq 0} V_{1,\eta(t)} \times V_{2,\eta(t)}.$$

Several arguments in the previous section can be adapted to this setting. This time, if C encloses $\{-1, 1\}$, then T_C is diffeomorphic to a torus T^3 and we have

$$H_2(T^* S^3, T_C; \mathbb{Z}) \cong H_1(T_C; \mathbb{Z}) \cong \mathbb{Z}^3.$$

We can pick a basis α_1, α_2, β for this relative homology group, such that α_1 is a fiber product of a Lefschetz thimble for π^1 by a point, and α_2 is a fiber product of a point by a Lefschetz thimble for π^2. We choose β so that its boundary projects to a degree 1 cover of C. The fact that the α_i come from Lefschetz thimbles for the π^i implies that they have vanishing area and Maslov index. We are left with determining the area and index of β.

As before, there is a positive measure σ' on $\mathbb{C}$, absolutely continuous with respect to the Lebesgue measure and smooth in $\mathbb{C} \setminus \{\pm 1\}$, such that the following result holds.

Lemma 2.12. *T_C and N_η are Lagrangian submanifolds of X. The Lagrangian T_C is diffeomorphic to T^3. Given β as above, its ω-area is $\int_{\Omega_C} \sigma'$, where $\Omega_C \subset \mathbb{C}$ is the region bounded by C, and its Maslov index is 2. Therefore, T_C is monotone with monotonicity constant $\tau_C = \int_{\Omega_C} \sigma'/2$. The N_η are Hamiltonian-isotopic to the conormal Lagrangian of the unknot in S^3. In particular, they are diffeomorphic to $S^1 \times \mathbb{R}^2$ and are exact.*

Proof. The proof uses arguments similar to the ones in the previous section, so we omit them. See [10] for the proofs of some of these statements. □

We can also write the disk potential of T_C.

Lemma 2.13. *The disk potential of T_C is*

$$W_{T_C} = x_1(1 + x_2)(1 + x_3),$$

in a basis $h_1, h_2, h_3 \in H_1(T_C; \mathbb{Z}) \cong \mathbb{Z}^3$ such that h_1 is a loop projecting to C in degree 1, $h_2 = V_{1,z} \times \{p_2\}$ for some $z \in C$ and $p_2 \in V_{2,z}$, and $h_3 = \{p_1\} \times V_{2,z}$ for some $z \in C$ and $p_1 \in V_{1,z}$.

Proof. This is computed in [10]. $\quad\square$

We can again exhaust $\mathbb{C} \setminus [-1, 1]$ by disjoint simple closed curves C, such that the collection of monotonicity constants τ_C of the T_C covers $\mathbb{R}_{>0}$ injectively. Fix $\tau > 0$ and denote by T_τ^3 the Lagrangian torus with monotonicity τ in this family. Assume that T_τ^3 is the lift of the curve C in Fig. 1. Denote also by N_i, resp. N' the lifts of the paths η_i, resp. η', in Fig. 1.

Lemma 2.14. *For every $i \geq 0$, N_i and T_τ^3 intersect cleanly. For every $i, j \geq 0$, N_i and N_j intersect cleanly. All these Lagrangians intersect N' cleanly.*

Proof. As in Lemma 2.10, this result follows from the fact that the Lagrangians project to curves in the plane that intersect transversely. $\quad\square$

3. Wrapped Fukaya categories

The wrapped Fukaya category of a Liouville domain M was introduced in [5]. In the original definition, its objects are exact Lagrangians in the completed Liouville manifold $\widehat{M}$. The Lagrangians are either compact or agree outside of a compact set with the product of $\mathbb{R}$ with a Legendrian submanifold of the contact manifold ∂M. We will consider various versions of the wrapped Fukaya category, possibly allowing for closed monotone Lagrangians, as in [29]. For Lagrangians intersecting cleanly, we will use a Morse–Bott formalism similar to that of [33] to compute the associated A_∞-maps μ^k.

Remark 3.1. The monotone wrapped Fukaya category is defined in [29], for a non-compact monotone symplectic manifold (E, ω) that is convex at infinity. In that reference, the monotonicity constant of E is assumed to be strictly positive. Since the first Chern class of T^*S^n vanishes, we are interested in the case of vanishing monotonicity constant. The main results in [29] still hold in this case, in particular Theorem 1.1 and equation (9.1), which we will make use of in this article.

3.1. Coefficients

Some of the Floer cohomology groups we will study are defined over $\mathbb{Z}$, and some over a *Novikov field*. Given a commutative ring R, which for us will always be either $\mathbb{Z}$ or $\mathbb{C}$, write

$$\mathbb{K}_R := \left\{ \sum_{i=0}^\infty a_i T^{\lambda_i} \mid a_i \in R, \lambda_i \in \mathbb{R}, \lambda_i < \lambda_{i+1}, \lim_{i \to \infty} \lambda_i = \infty \right\}.$$

We will be mostly interested in $\mathbb{K}_{\mathbb{C}}$, which will be denoted simply by $\mathbb{K}$. We could replace $\mathbb{C}$ with any algebraically closed field of characteristic zero, so that the Novikov field is algebraically closed. See Section 6 for more on this point.

There is a *valuation* map

$$\mathrm{val}\colon \mathbb{K}_R \to (-\infty, \infty]$$

$$\sum_{i=0}^{\infty} a_i T^{\lambda_i} \mapsto \min\{\lambda_i \mid a_i \neq 0\}$$

where $\mathrm{val}(0) = \infty$. Say that $\alpha \in \mathbb{K}_R$ is *unitary* if $\mathrm{val}(\alpha) = 0$. Denote by $U^*_{\mathbb{K}_R} := \{\alpha \in \mathbb{K}_R \mid \mathrm{val}(\alpha) = 0$ and α is invertible$\}$ the group of invertible unitary elements in $\mathbb{K}_R$ (if R is a field, $\mathrm{val}(\alpha) = 0$ implies that α is invertible), by $\mathbb{K}_{R,0} := \{\alpha \in \mathbb{K}_R \mid \mathrm{val}(\alpha) \geq 0\}$ the *Novikov ring* and by $\mathbb{K}_{R,+} := \{\alpha \in \mathbb{K}_R \mid \mathrm{val}(\alpha) > 0\}$ the maximal ideal in $\mathbb{K}_{R,0}$.

3.2. Morse–Bott Floer cohomology for clean intersections

We will use a Morse–Bott version of the Fukaya category, where Lagrangians are allowed to intersect cleanly, as in [33, Section 3.2]. For more details on a construction of the Fukaya category with a Morse–Bott definition of the A_∞-algebra of endomorphisms of a Lagrangian submanifold, see [36, Section 4]. These references assume that the Lagrangians are exact, which precludes disk bubbling. Lemma 3.6 below guarantees that if (L, ξ) is a Lagrangian with a rank 1 unitary local system giving a non-trivial object in the Fukaya category, then ξ corresponds to a zero of the disk potential of L. This is useful when considering 2-parameter families of pearly trees of holomorphic disks (to prove the A_∞-relations, for instance), since the vanishing of the disk potentials implies the cancellation of configurations with disk bubbles. Therefore, we can assume for many purposes that the relevant Lagrangians bound no holomorphic disks. For more on the Floer cohomology of cleanly intersecting Lagrangians, see for example [28], [14], [15], [9], and [30].

Let us briefly define the relevant Floer complexes. Let L_0, L_1 be two Lagrangians such that each L_i is equipped with:

- an orientation and a spin structure;
- a local system ξ on a trivial $\mathbb{K}_R$-bundle $E_i = \oplus_k E_{i,k}$, where the direct sum is finite and the summand $E_{i,k}$ of grading k is a finite rank trivial vector bundle over L_i, in degree k. If the rank of E_i is 1, then the local system can be unitary (with holonomy taking values in $U^*_{\mathbb{K}_R}$). If the rank of E_i is bigger than 1, then the local system has trivial holonomy.

Remark 3.2. In this article, the zero section in T^*S^n, with $n > 1$ even, is the only class of Lagrangians that we will equip with local systems of rank greater than 1.

Remark 3.3. The choices of spin structures on the Lagrangians are necessary to orient moduli spaces of holomorphic curves. Nevertheless, in our computations we will not be very careful in specifying spin structures on the Lagrangians. This is because the effect of changing the spin structure on a Lagrangian is a change in signs associated to holomorphic curves, and this change can be compensated by the choice of a different local system ξ on the Lagrangian.

The categories of exact Lagrangians we will consider are $\mathbb{Z}$-graded, so we will need additional choices of gradings for their objects (as in [32, Section 12a]). The categories of monotone Lagrangians will only be $\mathbb{Z}/2\mathbb{Z}$-graded, so we will not need to choose gradings in that case.

Let L_0, L_1 be Lagrangians as above, with local systems ξ_i on $\mathbb{K}_R$-bundles E_i over L_i. Write $\mathcal{L}_i := (L_i, \xi_i)$. Assume that the Lagrangians intersect cleanly and let $f : L_0 \cap L_1 \to \mathbb{R}$ be a Morse function. In accordance with [33, Equation (3.9)], we define the cochain complex

$$CF^*(\mathcal{L}_0, \mathcal{L}_1) := \bigoplus_{C \subset L_0 \cap L_1} \quad \bigoplus_{p \in \mathrm{Crit}(f|_C)} \left(\mathrm{Hom}_{\mathbb{K}_R}\left((E_0)_p, (E_1)_p\right) \otimes_{\mathbb{Z}} \mathfrak{o}_C \right)[-\deg(p)]$$

where the $C \subset L_0 \cap L_1$ are connected components of the intersection, $\mathfrak{o}_C$ is the orientation line of C (a rank 1 local system over $\mathbb{Z}$ depending on the spin structures of the L_i), and $\mathrm{Hom}_{\mathbb{K}_R}$ denotes $\mathbb{K}_R$-linear maps. The Floer degree associated to the critical point p is $\deg(p) = \dim(C) - \mathrm{ind}(p) + \deg(C)$, where $\mathrm{ind}(p)$ is the Morse index of p as a critical point of $f|_C$ and $\deg(C)$ is an absolute Maslov index, which depends on the gradings of the L_i.[1] Recall that according to the cohomological convention that we use, $[-k]$ adds k to the degree.

The operations μ^k are defined on tensor products of these chain complexes, via counts of *pearly trees*. We give a very brief description of these, referring the reader to [33, Section 3.2] for more details. Given a collection $\mathcal{L}_0, \ldots, \mathcal{L}_k$ of Lagrangians with local systems, a pearly tree contributing to

$$\mu^k : CF^*(\mathcal{L}_{k-1}, \mathcal{L}_k) \otimes \ldots \otimes CF^*(\mathcal{L}_0, \mathcal{L}_1) \to CF^*(\mathcal{L}_0, \mathcal{L}_k)$$

is a collection of perturbed pseudoholomorphic disks (with respect to auxiliary almost complex structures and perturbing 1-forms) with boundary punctures and Lagrangian boundary conditions, connected by gradient flow lines of auxiliary Morse functions and metrics on the clean intersections of the L_i. This collection of disks and flow lines can be concatenated into a continuous map from a disk with $n+1$ boundary punctures to the symplectic manifold, with boundary components of the disk mapping to the Lagrangians $L_0, \ldots, L_k$, see Fig. 2. The contribution of a rigid configuration of disks and flow lines

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[1] With this choice of grading, the graded vector space $CF^*(\mathcal{L}_0, \mathcal{L}_1)$ can be identified with $CF^*(\varphi(\mathcal{L}_0), \mathcal{L}_1)$, for a Hamiltonian isotopy φ supported near $L_0 \cap L_1$ and constructed from f.
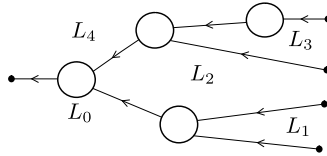
Fig. 2. A pearly tree contributing to μ^4.

to μ^k is determined by the areas of the pseudoholomorphic disks (which are encoded in the exponents of the variable T in the Novikov field), by signs specified by the spin structures on the L_i, and by parallel transport with respect to the local systems ξ_i on the E_i along the boundary components of the concatenated disk (with the input elements of $\mathrm{Hom}_{\mathbb{K}_R}(E_i, E_{i+1})$ applied at the boundary punctures). The μ^k satisfy the A_∞-relations, which can be written in abbreviated form as $\mu \circ \mu = 0$.

We will also consider Fukaya categories containing additional objects. A *bounding cochain* on an object $\mathcal{L}$ in a $\mathbb{Z}/2\mathbb{Z}$-graded Fukaya category is $b \in CF^{\mathrm{odd}}(\mathcal{L}, \mathcal{L})$ satisfying the *Maurer–Cartan equation*

$$\sum_{k=1}^{\infty} \mu^k(b, \ldots, b) = 0, \tag{3.1}$$

see [15] (to ensure that all the A_∞-operations converge, we assume that b has strictly positive valuation). We can enlarge our category by allowing objects of the form $(\mathcal{L}, b)$, for such b. The object $\mathcal{L}$ can be identified with $(\mathcal{L}, 0)$. Given objects $\hat{\mathcal{L}}_0 = (\mathcal{L}_0, b_0), \ldots, \hat{\mathcal{L}}_k = (\mathcal{L}_k, b_k)$ in the enlarged category, the A_∞-maps

$$\hat{\mu}^k : CF^*(\hat{\mathcal{L}}_{k-1}, \hat{\mathcal{L}}_k) \otimes \ldots \otimes CF^*(\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_1) \to CF^*(\hat{\mathcal{L}}_0, \hat{\mathcal{L}}_k)$$

are given by

$$\hat{\mu}^k(x_k, \ldots, x_1) := \sum_{l_0, \ldots, l_k \geq 0} \mu^{(k + \sum_i l_i)}(\underbrace{b_k, \ldots, b_k}_{l_k}, x_k, b_{k-1}, \ldots, b_1, x_1, \underbrace{b_0, \ldots, b_0}_{l_1}).$$

The fact that the b_i satisfy the Maurer–Cartan equation (3.1) implies the A_∞-equations $\hat{\mu} \circ \hat{\mu} = 0$. Since $\hat{\mu}^k$ agrees with μ^k when all $b_i = 0$, we will continue to write μ^k instead of $\hat{\mu}^k$.

3.3. Wrapped Floer cohomology

We will use a model for wrapped Floer cohomology from [6], presented in [18] and [38]. Let L_0 be a non-compact Lagrangian, which in this paper will be either a cotangent fiber F or a conormal Lagrangian of the unknot $N \subset T^*S^3$. We pick a family L_i of Lagrangians that are lifts of paths η_i in the base of the Lefschetz fibration π_n from Section 2.1 (in the case of F), or in the base of the fiber product of Lefschetz fibrations

π^i from Section 2.2 (in the case of N), where the path η_i wraps i times around the two critical values, see Fig. 1. Then, given another Lagrangian L', we have

$$HW^*(L_0, L') := \lim_{i \to \infty} HF^*(L_i, L'),$$

with the limit taken with respect to the continuation maps relating L_i and L_{i+1}. For the equivalence of this model with the usual definitions involving fast growing Hamiltonians, see [18, Lemma 3.37] and [38, Proposition 2.6]. In these references, the wrapped Fukaya category is defined by localizing the Fukaya category on the continuation maps that were just mentioned. We will combine this approach to wrapped Floer cohomology with the definition of Morse–Bott Floer cohomology above, where the Lagrangians intersect cleanly and are possibly equipped with local systems and bounding cochains.

3.4. Wrapped Fukaya categories

We will consider several versions of the Fukaya A_∞-category of T^*S^n. Recall that R is either $\mathbb{Z}$ or $\mathbb{C}$.

- $\mathcal{W}^{\mathbb{Z}}(T^*S^n; \mathbb{Z})$ is a category whose objects are either the F_η from Definition 2.1 or compact oriented exact Lagrangians. When $n = 3$, we also include the objects N_η from Definition 2.11. Objects are equipped with $\mathbb{Z}$-gradings and spin structures. Morphism spaces are wrapped Floer cochain complexes with coefficients in $\mathbb{Z}$. The differential and higher A_∞-operations count rigid pearly trees, without keeping track of areas (which can be thought of as setting $T = 1$ in the Novikov field $\mathbb{K}_{\mathbb{Z}}$).
- $\mathcal{W}^{\mathbb{Z}}(T^*S^n; \mathbb{K}_R)$ has the same objects as $\mathcal{W}^{\mathbb{Z}}(T^*S^n; \mathbb{Z})$. The difference is that the morphism spaces are now wrapped Floer cochain complexes *with coefficients in $\mathbb{K}_R$*, to keep track of the symplectic areas of the disks in the pearly trees that contribute to the A_∞ operations.
- $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K}_R)$ is obtained from $\mathcal{W}^{\mathbb{Z}}(T^*S^n; \mathbb{K}_R)$ by collapsing the $\mathbb{Z}$-gradings to $\mathbb{Z}/2\mathbb{Z}$-gradings. If n is odd, allow also objects of the form (S^n, b_α), where S^n is the zero section and $b_\alpha = \alpha[pt]$ is a bounding cochain with $\alpha \in \mathbb{K}_{R,0}$ and $[pt] \in H^n(S^n; \mathbb{K}_R)$. See Remark 3.4 below for why we impose $\alpha \in \mathbb{K}_{R,0}$. We have implicitly chosen a perfect Morse function on S^n, and $[pt]$ is given by the minimum of that function (the maximum yields the unit in the A_∞-algebra of S^n). Since S^n bounds no disks, it is clear that all the summands in (3.1) vanish for $b = b_\alpha$. If n is even, we want to allow instead objects corresponding to bounding cochains in $S^n \oplus S^n[1]$ (summing and shifting objects are allowed in the additive enlargement of the Fukaya category). We implement this by equipping S^n with the trivial graded $\mathbb{K}_R$-bundle $E := \mathbb{K}_R \oplus \mathbb{K}_R[1]$, and bounding cochains $b_{\alpha,\beta} \in H^{odd}(S^n; \text{End}(E))$ of the form

$$b_{\alpha,\beta} := \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}_{[pt]},$$

where $\alpha, \beta \in \mathbb{K}_{R,0}$ and the matrix represents an endomorphism of the fiber of E at the minimum of the auxiliary perfect Morse function on S^n.

- $\mathcal{W}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^* S^n; \mathbb{K}_R)$ is an extension of $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}(T^* S^n; \mathbb{K}_R)$, including closed monotone Lagrangians. The objects are equipped with orientations and spin structures, and are $\mathbb{Z}/2\mathbb{Z}$-graded. We also equip monotone Lagrangians with unitary rank 1 local systems over $\mathbb{K}_R$. The construction of the monotone wrapped Fukaya category is given in [29]. See also [37] for a definition of the monotone Fukaya category in a closed setting.
- $\mathcal{F}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^* S^n; \mathbb{K}_R)$ is the full subcategory of $\mathcal{W}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^* S^n; \mathbb{K}_R)$ containing only those objects whose underlying Lagrangians are closed.

It is an important fact that in all these categories the isomorphism class of objects is preserved by Hamiltonian isotopies; in the presence of bounding cochains, this means that if b is a bounding cochain on L, and L' is Hamiltonian-isotopic to L, then there is a bounding cochain b' on L' so that the two corresponding objects of the Fukaya category are isomorphic.

Remark 3.4. If we equip Lagrangians with bounding cochains valued in the maximal ideal $\mathbb{K}_{R,+}$ of $\mathbb{K}_{R,0}$, then we are guaranteed convergence of all the A_∞-operations deformed by such bounding cochains. In our case, since the degree of $[pt] \in H^*(S^n; \mathbb{Z})$ is $n > 1$, we could in fact allow bounding cochains $\alpha[pt]$ for arbitrary $\alpha \in \mathbb{K}_R$ in the category of exact Lagrangians $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}(T^* S^n; \mathbb{K}_R)$. Nevertheless, for bounding cochains $\alpha[pt]$ with $\mathrm{val}(\alpha) < 0$, we could run into convergence issues when taking morphisms with monotone Lagrangians in $\mathcal{W}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^* S^n; \mathbb{K}_R)$, which is why we restrict to bounding cochains with coefficients in $\mathbb{K}_{R,0}$. With minor modifications to our arguments, we could also have equipped all objects in $\mathcal{W}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^* S^n; \mathbb{K}_R)$ with finite rank unitary local systems and suitable bounding cochains.

Observe that we can define several functors between these categories:

- $\mathcal{G}_1 \colon \mathcal{W}^{\mathbb{Z}}(T^* S^n; \mathbb{Z}) \to \mathcal{W}^{\mathbb{Z}}(T^* S^n; \mathbb{K}_R)$ is the identity on objects. Fix a primitive f_L for every exact Lagrangian L. Given exact Lagrangians L_0, L_1 in $\mathcal{W}^{\mathbb{Z}}(T^* S^n; \mathbb{Z})$, map $x \in CW^*(L_0, L_1; \mathbb{Z})$ to $T^{f_1(x) - f_0(x)} x \in CW^*(L_0, L_1; \mathbb{K}_R)$, where $f_i := f_{L_i}$. If u is a pearly tree contributing to μ^k in $\mathcal{W}^{\mathbb{Z}}(T^* S^n; \mathbb{Z})$, then the contribution of u to μ^k in $\mathcal{W}^{\mathbb{Z}}(T^* S^n; \mathbb{K}_R)$ is weighted by the factor $T^{\int_{D^2} u^* \omega}$, where the integral is over all the holomorphic disks in the pearly tree. The functor $\mathcal{G}_1$ depends on the choices of primitives f_L, but different choices yield isomorphic functors (we could eliminate this choice by incorporating the primitives in the definition of objects of the exact Fukaya category).
- $\mathcal{G}_2 \colon \mathcal{W}^{\mathbb{Z}}(T^* S^n; \mathbb{K}_R) \to \mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}(T^* S^n; \mathbb{K}_R)$ is given by collapsing the $\mathbb{Z}$-grading to a $\mathbb{Z}/2\mathbb{Z}$-grading, followed by inclusion of objects.

- $\mathcal{G}_3 \colon \mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K}_R) \;\to\; \mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^n; \mathbb{K}_R)$ is given by inclusion of objects, as are $\mathcal{G}_4 \colon \mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^n; \mathbb{K}_{\mathbb{Z}}) \;\to\; \mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^n; \mathbb{K})$ (recall that $\mathbb{K} = \mathbb{K}_{\mathbb{C}}$) and $\mathcal{G}_5 \colon \mathcal{F}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^n; \mathbb{K}_{\mathbb{Z}}) \to \mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^n; \mathbb{K})$.

Remark 3.5. Let L be a monotone Lagrangian. A unitary rank 1 local system ξ on the trivial $\mathbb{K}_R$-bundle over L can be specified by a homomorphism (the holonomy)

$$\mathrm{hol}_\xi : H_1(L; \mathbb{Z}) \to U^*_{\mathbb{K}_R}.$$

If, in the definition (2.3) of the disk potential W_L, we replace $x^{\partial u}$ with $\mathrm{hol}_\xi(\partial u)$, then we get an element of $\mathbb{K}_R$ that we denote by $W_L(\xi)$.

When defining the monotone category $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^n; \mathbb{K}_R)$, one can only take morphisms between objects (L_1, ξ_1) and (L_2, ξ_2) if $W_{L_1}(\xi_1) = W_{L_2}(\xi_2)$, see [27]. This does not impose an additional constraint in our case, due to the following result. It can be interpreted as saying that the monotone Fukaya category of T^*S^n is unobstructed.

Lemma 3.6. *Let $L \subset T^*S^n$ be a compact monotone Lagrangian with a unitary local system ξ on a trivial line bundle. Write $\mathcal{L} = (L, \xi)$. If $HF^*(\mathcal{L}, \mathcal{L}; \mathbb{K}_R) \neq 0$, then $W_L(\xi) = 0$.*

Proof. This follows from [29, equation (9.1)] (see [29, Section 3.17] for a discussion of Lagrangians with local systems). According to that equation, if $HF^*(\mathcal{L}, \mathcal{L}; \mathbb{K}_R) \neq 0$ then $m_0(\mathcal{L}) = W_L(\xi)$ can be computed by applying the open-closed map to the first Chern class of the total space of T^*S^n. The lemma follows from the vanishing of this class. $\quad\square$

Remark 3.7. Let L^n be a monotone Lagrangian torus with disk potential W_L. The critical points of W_L in $(U^*_{\mathbb{K}_R})^n$ correspond to the rank 1 unitary local systems ξ on the trivial $\mathbb{K}_R$-line bundle over L for which $HF^*(\mathcal{L}, \mathcal{L}; \mathbb{K}_R) \neq 0$, where $\mathcal{L} = (L, \xi)$, see [37, Proposition 4.2]. Recall that $HF^*(\mathcal{L}, \mathcal{L}; \mathbb{K}_R) \neq 0$ is equivalent to the non-triviality of $\mathcal{L}$ in $\mathcal{F}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^n; \mathbb{K}_R)$. By Lemma 2.9, the Lagrangian tori $L_C \subset T^*S^2$ have disk potential $W_1 = x_1(1 + x_2)^2$. The critical locus of this potential is given by the condition $x_2 = -1$. Recall also that Lemma 2.13 says that the disk potential of a Lagrangian torus $T_C \subset T^*S^3$ is $W_2 = x_1(1 + x_2)(1 + x_3)$, whose critical locus is given by $x_2 = x_3 = -1$. Observe that, for both tori $L_C \subset T^*S^2$ and $T_C \subset T^*S^3$, the disk potentials vanish on their critical points, which is compatible with Lemma 3.6.

3.5. Yoneda functors

In this section we will be working over the field $\mathbb{K}_{\mathbb{C}} = \mathbb{K}$, since we will use some formality results from Section 5. Let $\mathcal{A}_{\mathbb{K}} := CW^*(F, F; \mathbb{K})$ be the A_∞-algebra of a cotangent fiber in T^*S^n, with $n \geq 2$, and let $\mathrm{mod}^{A_\infty}(\mathcal{A}_{\mathbb{K}})$ be the differential $\mathbb{Z}/2\mathbb{Z}$-graded category of right A_∞-modules over $\mathcal{A}_{\mathbb{K}}$. Given two objects $\mathcal{M}$ and $\mathcal{M}'$ in $\mathrm{mod}^{A_\infty}(\mathcal{A}_{\mathbb{K}})$, the morphism space $\mathrm{hom}_{\mathrm{mod}^{A_\infty}(\mathcal{A}_{\mathbb{K}})}(\mathcal{M}, \mathcal{M}')$ is a chain complex computing $\mathrm{Ext}^*_{\mathcal{A}_{\mathbb{K}}}(\mathcal{M}, \mathcal{M}')$, see [32, Remark 2.15].

There is a Yoneda functor

$$\mathcal{Y} : \mathcal{W}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K}) \to \mathrm{mod}^{A_\infty}(\mathcal{A}_{\mathbb{K}})$$

$$\mathcal{L} \mapsto CW^*(F, \mathcal{L})$$

which restricts to a functor

$$\mathcal{Y}_c : \mathcal{F}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K}) \to \mathrm{mod}_{pr}^{A_\infty}(\mathcal{A}_{\mathbb{K}}),$$

where $\mathrm{mod}_{pr}^{A_\infty}(\mathcal{A}_{\mathbb{K}}) \subset \mathrm{mod}^{A_\infty}(\mathcal{A}_{\mathbb{K}})$ is the subcategory of *proper modules* $\mathcal{M}$, such that $H^*(\mathcal{M})$ is finite dimensional over $\mathbb{K}$ (the subscript in $\mathcal{Y}_c$ stands for 'compact').

Now, let $A_{\mathbb{K}} := H^*(\mathcal{A}_{\mathbb{K}})$ be the cohomology algebra of $\mathcal{A}_{\mathbb{K}}$. Let $\mathrm{mod}(A_{\mathbb{K}})$ be the $\mathbb{Z}/2\mathbb{Z}$-graded category of right $A_{\mathbb{K}}$-modules, such that morphism spaces are $\mathrm{Ext}_{A_{\mathbb{K}}}^*$ groups (respecting the $\mathbb{Z}/2\mathbb{Z}$-gradings). There is a functor

$$H : \mathrm{mod}^{A_\infty}(\mathcal{A}_{\mathbb{K}}) \to \mathrm{mod}(A_{\mathbb{K}})$$

$$\mathcal{M} \mapsto H^*(\mathcal{M})$$

which restricts to

$$H_c : \mathrm{mod}_{pr}^{A_\infty}(\mathcal{A}_{\mathbb{K}}) \to \mathrm{mod}_{pr}(A_{\mathbb{K}})$$

where $\mathrm{mod}_{pr}(A_{\mathbb{K}}) \subset \mathrm{mod}(A_{\mathbb{K}})$ is the subcategory of finite dimensional $\mathbb{Z}/2\mathbb{Z}$-graded modules over $A_{\mathbb{K}}$.

Proposition 4.1 below implies that the functor $\mathcal{Y}$ (hence also $\mathcal{Y}_c$) is cohomologically full and faithful. According to Corollary 5.5 below, H (hence also H_c) is a quasi-equivalence of categories. We conclude the following.

Proposition 3.8. *The composition*

$$Y := H \circ \mathcal{Y} : \mathcal{W}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K}) \to \mathrm{mod}(A_{\mathbb{K}})$$

$$\mathcal{L} \mapsto HW^*(F, \mathcal{L})$$

and its restriction

$$Y_c := H_c \circ \mathcal{Y}_c : \mathcal{F}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K}) \to \mathrm{mod}_{pr}(A_{\mathbb{K}})$$

are cohomologically full and faithful embeddings. □

4. Floer cohomology computations

4.1. The Lagrangians F and N

Recall from Section 2 that the Lagrangian lifts $F_\eta \subset T^*S^n$ and $N_\eta \subset T^*S^3$ are Hamiltonian-isotopic to, respectively, a cotangent fiber (which we denote by F) and the conormal Lagrangian of an unknot in S^3 (which we denote by N).

Proposition 4.1. *The cotangent fiber F generates $\mathcal{W}^{\mathbb{Z}}(T^*S^n; \mathbb{Z})$ and $\mathcal{W}^{\mathbb{Z}}(T^*S^n; \mathbb{K}_R)$, and it split-generates $\mathcal{W}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K}_R)$. When $n = 3$, the Lagrangian N split-generates $\mathcal{W}^{\mathbb{Z}}(T^*S^3; \mathbb{Z})$, $\mathcal{W}^{\mathbb{Z}}(T^*S^3; \mathbb{K}_R)$ and $\mathcal{W}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^3; \mathbb{K}_R)$.*

Proof. The fact that a cotangent fiber generates $\mathcal{W}^{\mathbb{Z}}(T^*S^n; \mathbb{Z})$ is proven in [2], and the result follows for $\mathcal{W}^{\mathbb{Z}}(T^*S^n; \mathbb{K}_R)$. Let us recall the argument: it is first shown that a cotangent fiber split-generates, and this is then extended to a proof of generation. The fact that a cotangent fiber split-generates follows from combining the split-generation criterion of [1] with results about cotangent bundles and loop spaces in [3]. The split-generation criterion of [1] is extended to the monotone wrapped Fukaya category in [29, Theorem 1.1], and can again be combined with results in [3] to conclude that a cotangent fiber split-generates $\mathcal{W}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K}_R)$.

The previous paragraph and Lemma 4.6 below imply the result for N. $\quad\square$

Remark 4.2. It would be interesting to determine if a cotangent fiber generates $\mathcal{W}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K}_R)$, but this is not necessary to prove the results in this article about compact monotone Lagrangians.

Recall that under our assumption that $n \geq 2$ we have $HW^*(F, F; \mathbb{Z}) \cong \mathbb{Z}[u]$, where $\deg(u) = 1 - n$, as follows from [3]. Denote this ring by $A_{\mathbb{Z}}$. Also denote by F_0 a cotangent fiber corresponding to a lift of a path η_0 through the critical value 1 of π_n, and by F' one that is a lift of a path η' through -1, see Fig. 1. Since F_0 and F' are Hamiltonian-isotopic, we have

$$HW^*(F_0, F'; \mathbb{Z}) \cong A_{\mathbb{Z}}. \tag{4.1}$$

On the other hand,

$$HW^*(F_0, F'; \mathbb{Z}) \cong \lim_{i \to \infty} HF^*(F_i, F'; \mathbb{Z}),$$

where the F_i are lifts of the paths η_i illustrated in Fig. 1. In our Morse–Bott model, the cochain complex for $CF^*(F_i, F'; \mathbb{Z})$ is described, as a graded abelian group, as

$$\bigoplus_{k=0}^{i} H^{*+(n-1)(1+2k)}(S^{n-1}; \mathbb{Z}).$$

Lemma 4.3. *The chain level continuation maps*

$$CF^*(F_i, F'; \mathbb{Z}) \to CF^*(F_{i+1}, F'; \mathbb{Z})$$

are inclusions, and the differentials vanish on these chain complexes.

Proof. There is a compactly supported isotopy of $\mathbb{C} \setminus \{\pm 1\}$ taking the path η_{i+1} to a path $\tilde{\eta}_{i+1}$ such that:

- $\tilde{\eta}_{i+1}$ intersects η' transversely at precisely $i + 1$ points (just like η_{i+1}) and
- $\tilde{\eta}_{i+1}$ coincides with η_i in a connected portion of the path starting at $1 \in \mathbb{C}$ and containing all the intersections of η_i with η'.

This isotopy of the base can be lifted to a compactly-supported Hamiltonian isotopy of T^*S^n, taking F_{i+1} to a Lagrangian lift $\tilde{F}_{i+1}$ of the path $\tilde{\eta}_{i+1}$. The continuation map

$$CF^*(F_i, F'; \mathbb{Z}) \to CF^*(\tilde{F}_{i+1}, F'; \mathbb{Z})$$

is clearly an inclusion, and so must be the map in the statement.

As for the vanishing of the differentials, notice that for degree reasons this is only a non-trivial statement when $n = 2$. The fact that

$$A_{\mathbb{Z}} \cong \lim_{i \to \infty} HF^*(F_i, F'; \mathbb{Z})$$

and that the continuation maps are inclusions implies that there can be no non-trivial differentials in the chain complexes $CF^*(F_i, F'; \mathbb{Z})$. This is because the direct limit would not have the correct rank in the degrees related by a non-trivial contribution to the differential. $\square$

The previous result implies the following.

Lemma 4.4. *Up to a factor ± 1, the unit e, resp. the generator u, in $A_{\mathbb{Z}}$ is represented by the minimum, resp. maximum, of the auxiliary Morse function on $F_0 \cap F' \cong S^{n-1}$, thought of as a class in $HF^0(F_0, F'; \mathbb{Z})$, resp. $HF^{1-n}(F_0, F'; \mathbb{Z})$.*

We now consider the Lagrangian N.

Remark 4.5. In the following, we use the cohomological degree shift notation, where $[k]$ corresponds to a shift by $-k$.

Proposition 4.6. *The Lagrangian $N \in T^*S^3$ is quasi-isomorphic to $F \oplus F[1]$ in $\mathcal{W}^{\mathbb{Z}}(T^*S^3; \mathbb{Z})$. In particular, $HW^*(N, N; \mathbb{Z})$ is isomorphic to the graded matrix algebra*

$$B_{\mathbb{Z}} := \begin{pmatrix} A_{\mathbb{Z}} & A_{\mathbb{Z}}[1] \\ A_{\mathbb{Z}}[-1] & A_{\mathbb{Z}} \end{pmatrix}.$$

Hence, $HW^*(N, N; \mathbb{K}_R)$ is isomorphic to $B_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{K}_R$ for any commutative ring R.

Proof. Recall the construction, in [3], of a cohomologically fully faithful A_∞-functor

$$\mathcal{F} : \mathcal{W}^{\mathbb{Z}}(T^*Q; \mathbb{Z}) \to \mathrm{Tw}(\mathcal{P}(Q)),$$

where the target is a category of twisted complexes on a Pontryagin category $\mathcal{P}(Q)$ of a closed spin manifold Q. Objects in $\mathcal{P}(Q)$ are points in Q, with $\hom_{\mathcal{P}(Q)}(q_1, q_2) = C_{-*}(\Omega_{q_1,q_2}(Q); \mathbb{Z})$ (cubical chains) and composition defined via concatenation of paths. Here, $\Omega_{q_1,q_2}(Q)$ is the space of Moore paths in Q that start at q_1 and end at q_1. Write also Ω_q for $\Omega_{q,q}$. Given an object in $\mathcal{W}^{\mathbb{Z}}(T^*Q; \mathbb{Z})$, which is a $\mathbb{Z}$-graded exact spin Lagrangian L in T^*Q, we can assume (up to a Hamiltonian isotopy) that L intersects the zero-section transversely at the points $q_1, \ldots, q_m$. The image of L under $\mathcal{F}$ is a twisted complex supported on a direct sum of grading shifts of the q_i. The differential in the twisted complex is constructed from moduli spaces of Floer strips between Q and L.

Let us use this functor in our setting. The Lagrangian N intersects S^3 cleanly along a copy of S^1. One can deform N by a Hamiltonian isotopy so that it intersects S^3 transversely at exactly two points q_1 and q_2, with consecutive indices. Hence, $\mathcal{F}(N)$ is a twisted complex supported on the sum of shifts of q_1 and of q_2. The differential on this twisted complex is given by a cycle in $C_0(\Omega_{q_1,q_2}(S^3); \mathbb{Z})$. Homologous cycles yield quasi-isomorphic twisted complexes, so the differential on $\mathcal{F}(N)$ is determined by an element $x \in H_0(\Omega_{q_1,q_2}(S^3); \mathbb{Z}) \cong \mathbb{Z}$.

Given $q \in S^3$ and identifying $H_{-*}(\Omega_q(S^3); \mathbb{Z})$ with $HW^*(F_q, F_q; \mathbb{Z})$, we can say that N is quasi-isomorphic to $\mathrm{Cone}(F_q \xrightarrow{x} F_q)$ in a category of twisted complexes over $\mathcal{W}^{\mathbb{Z}}(T^*Q; \mathbb{Z})$, where x is now thought of as a class in $HW^0(F_q, F_q; \mathbb{Z}) \cong \mathbb{Z}$. In particular, up to a degree shift,

$$HF^*(N, S^3; \mathbb{Z}) \cong H^* \left(\mathrm{Cone}(HF^*(F_q, S^3; \mathbb{Z}) \xrightarrow{x} HF^*(F_q, S^3; \mathbb{Z})) \right) \cong$$

$$\cong H^* \left(\mathrm{Cone}(\mathbb{Z} \xrightarrow{x} \mathbb{Z}) \right) \cong \begin{cases} \mathbb{Z}[1] \oplus \mathbb{Z} & \text{if } x = 0 \\ (\mathbb{Z}/x\mathbb{Z}) & \text{otherwise} \end{cases}.$$

On the other hand, one can adapt [28, Proposition 3.4.6] to Floer cohomology with $\mathbb{Z}$-coefficients (instead of $\mathbb{Z}/2\mathbb{Z}$), and conclude that

$$HF^*(N, S^3; \mathbb{Z}) \cong H^*(S^1; \mathbb{Z}),$$

up to a degree shift. Therefore, we conclude that $x = 0$, the differential in the twisted complex $\mathcal{F}(N)$ is trivial, and N is quasi-isomorphic to $F \oplus F[1]$, as wanted. $\square$

Remark 4.7. Strictly speaking, the argument above only implies that $N = F \oplus F[1]$ up to a global degree shift. However, this will be enough for our purposes, since the main application of the previous proposition will be in Lemma 4.17, which is about the $\mathbb{Z}/2\mathbb{Z}$-graded monotone Fukaya category.

The ring $B_{\mathbb{Z}}$ of endomorphisms of $F \oplus F[1]$ can be represented pictorially as follows:

$$\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix} \circlearrowleft F \qquad F[1] \circlearrowright \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$$

Define

$$e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \, e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \, e_{21} := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \, e_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Note that

$$|e_1| = 0 = |e_2|, \, |e_{21}| = 1, \, |e_{12}| = -1.$$

As a graded free abelian group, $B_{\mathbb{Z}}$ has generators in low degrees given by

degree	1	0	-1	-2	-3
generator	e_{21}	e_1, e_2	e_{12}, ue_{21}	ue_1, ue_2	$ue_{12}, u^2 e_{21}$

In a manner similar to what we did above for F, let us give a more explicit description of the Morse–Bott wrapped Floer cohomology of N. Denote by N' the lift of a path η' through -1 and by the N_i lift of a path η_i through 1 that winds i times around the critical values of the Morse–Bott Lefschetz fibration, see Figs. 1 and 3.

By Proposition 4.6, we know that,

$$B_{\mathbb{Z}} \cong HW^*(N_0, N'; \mathbb{Z}) \cong \lim_{i \to \infty} HF^*(N_i, N'; \mathbb{Z}).$$

The Morse–Bott Floer cochain complex for $CF^*(N_i, N'; \mathbb{Z})$ with $i \geq 0$ is given, as a graded abelian group, by

$$\bigoplus_{k=0}^{i} H^{*+2k+1}(T^2; \mathbb{Z}).$$

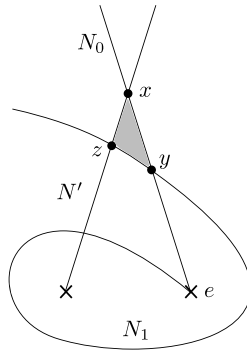Similarly to what is stated in Lemma 4.3 for F, the continuation maps

Fig. 3. N_0 and N_1.

$$CF^*(N_i, N'; \mathbb{Z}) \to CF^*(N_{i+1}, N'; \mathbb{Z})$$

are inclusions and the differentials vanish on these chain complexes.

We also have

$$B_{\mathbb{Z}} \cong HW^*(N_0, N_0; \mathbb{Z}) \cong \lim_{i \to \infty} HF^*(N_i, N_0; \mathbb{Z}).$$

For $i > 0$, the Morse–Bott Floer cochain complex for $CF^*(N_i, N_0; \mathbb{Z})$ is

$$H^*(S^1) \oplus \bigoplus_{k=1}^{i} H^{*+2k}(T^2; \mathbb{Z}) \tag{4.2}$$

and we have again that the continuation maps are inclusions and the differentials vanish.

As we saw after Proposition 4.6, the free abelian group $HW^*(N, N; \mathbb{Z})$ has two generators in degree -2, denoted by ue_1 and ue_2.

Remark 4.8. At several points in this paper, including the proof of the next result, we will explicitly compute certain products μ^2 via counts of holomorphic curves. Since we will always be in a position where we can compute the product on cohomology, and since the relevant holomorphic curves will always project to triangles in $\mathbb{C}$ over which the Lefschetz fibrations of interest are trivial, it will suffice to make all the calculations using a product complex structure, for which the relevant holomorphic curves are regular.

Lemma 4.9.

(1) The fundamental class of S^1 in (4.2), with $i = 1$, represents the class $\pm e = \pm(e_1 + e_2) \in HW^0(N_1, N_0, \mathbb{Z})$, where e is the unit.
(2) The fundamental class of T^2 in (4.2), with $i = 1$, represents the class $\pm ue_1 \pm ue_2 \in HW^{-2}(N_1, N_0; \mathbb{Z})$.

Proof. The statement in (1) follows from the fact that the canonical map $H^*(S^1) \to HW^*(N_0, N_0; \mathbb{Z})$ is a ring map, so it preserves units.

For (2), it is convenient to also consider the Lagrangian N'. The product μ^2 gives a map

$$HF^0(N_0, N'; \mathbb{Z}) \otimes HF^{-2}(N_1, N_0, \mathbb{Z}) \to HF^{-2}(N_1, N'; \mathbb{Z}).$$

Fig. 3 will be useful to understand the map

$$HF^0(N_0, N'; \mathbb{Z}) \to HF^{-2}(N_1, N'; \mathbb{Z}) \tag{4.3}$$

given by right multiplication with the fundamental class of T^2 in $HF^{-2}(N_1, N_0, \mathbb{Z})$, which lies over the intersection point y in Fig. 3. Note that $HF^0(N_0, N'; \mathbb{Z}) \cong HW^0(N_0, N'; \mathbb{Z}) \cong \mathbb{Z}^2$ is generated by classes that lie over the point x in the Figure, and that $HF^{-2}(N_1, N'; \mathbb{Z}) \cong HW^{-2}(N_1, N'; \mathbb{Z}) \cong \mathbb{Z}^2$ is generated by classes that lie over the point z. The product can now be computed by lifting the shaded triangle. Since the fibration is trivial over this triangle, there is a T^2-family of such lifts. Inserting the fundamental class of T^2 over y does not impose any constraint on this family of disks, which implies that the map (4.3) is an isomorphism. Since we are working over $\mathbb{Z}$, this means that the fundamental class of T^2 represents $\pm u e_1 \pm u e_2$, as wanted. $\qquad\square$

Remark 4.10. We are not specifying if the two signs in part (2) above are the same, since that is not necessary where this result is applied later in the paper. Nevertheless, a surgery argument related to Lemma 4.13 below should imply that the signs are equal, hence $\pm u e_1 \pm u e_2 = \pm u e$.

4.2. Computations in T^*S^n

We begin by assuming that n **is odd**. The wrapped Fukaya category $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K}_R)$ contains objects of the form $(S^n, \alpha[pt])$, were $\alpha \in \mathbb{K}_{R,0}$ and $[pt] \in H^n(S^n; \mathbb{K}_R)$ is the class of a point. We want to understand how a cotangent fiber F acts on such an object.

Let F_i and F' be as in the previous section. Given $a \in HF^*(F_i, F'; \mathbb{K}_R)$ and $X \in \mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K}_R)$, define a map

$$\psi_a^X : HF^*(F', X; \mathbb{K}_R) \to HF^{*+\deg(a)}(F_i, X; \mathbb{K}_R)$$
$$x \mapsto \mu^2(x, a)$$

Lemma 4.11.

(1) There is an isomorphism

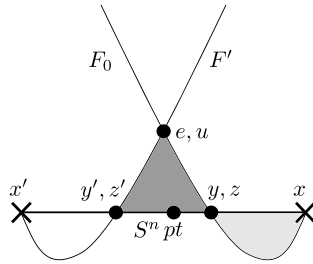$$HF^*(F, (S^n, \alpha[pt]); \mathbb{K}_R) \cong \mathbb{K}_R.$$

Fig. 4. The chain complexes $CF^*(F_0, (S^n, \alpha[pt]))$ and $CF^*(F', (S^n, \alpha[pt]))$.

(2) *Using the identification in Lemma 4.4 of $e \in HF^0(F_0, F'; \mathbb{Z})$ with the class of a point in S^{n-1}, and of $u \in HF^{1-n}(F_0, F'; \mathbb{Z})$ with the fundamental class of S^{n-1}, we have*

$$\psi_u^{(S^n, \alpha[pt])} = \pm \alpha \, \psi_e^{(S^n, \alpha[pt])}.$$

Proof. As we saw, in the Lefschetz fibration description $\pi_n : X_n \to \mathbb{C}$ of T^*S^n the zero section S^n is the Lagrangian lift of the interval $[-1, 1] \subset \mathbb{C}$.

For part (1), we can replace a cotangent fiber F with its Hamiltonian-isotopic Lagrangians F_0 and F', as in the previous section. Recall that these are lifts of paths out of the critical values that intersect the interval $[-1, 1] \subset \mathbb{C}$ transversely and only at one of the endpoints of the interval. Then, $CF^*(F_0, (S^n, \alpha[pt]); \mathbb{K}_R)$ has a single generator in degree 0, and the result follows. The same is true replacing F_0 with F'.

Let us give an alternative argument, with an eye towards part (2). This time, let F_0 and F' be lifts of paths that intersect the interior of $[-1, 1]$, as in Fig. 4. We start with F_0. The chain complex $CF^*(F_0, (S^n, \alpha[pt]); \mathbb{K}_R)$ now has generators x, y, z in degrees $-n$, $1 - n$ and 0, respectively (y is the maximum and z the minimum of an auxiliary Morse function on the component of $S^n \cap F_0$ that is diffeomorphic to S^{n-1}), see Fig. 4. The fact that ∂x is of the form $\pm T^A y$, where A is the σ-area of the lightly shaded bigon (recall the definition of σ in Section 2), follows from the fact that the algebraic count of lifts of the shaded strip is ± 1. That can be seen using the invariance under compactly supported Hamiltonian isotopies in $\mathbb{C}^n$ of $HF^*(\mathbb{R}^n, i\mathbb{R}^n)$, which is of rank 1. It follows that the cohomology is of rank 1, generated by z. There is a similar argument for F' instead of F_0, with z' now being the maximum of an auxiliary Morse function on S^{n-1}.

To prove (2), we use again the representation of F_0 and F' in Fig. 4. The dark triangle in Fig. 4 does not contain critical values of π_n, so the restriction of the Lefschetz fibration to that triangle is trivial. The triangle hence lifts to an S^{n-1}-family of holomorphic triangles with the appropriate Lagrangian boundary conditions. This family can be made rigid by using $e \in CF^*(F_0, F'; \mathbb{K}_R)$ (represented by a minimum) as an input in

$$\psi_e^{(S^n, \alpha[pt])}(z') = \mu^2(z', e) = \pm T^B z,$$

where B is the σ-area of the dark triangle. Similarly, the family of lifted triangles can be rigidified by using the bounding cochain $\alpha[pt]$ as an input in

$$\psi_u^{(S^n, \alpha[pt])}(z') = \mu^3(\alpha[pt], z', u) = \pm T^B \alpha z.$$

The result now follows. $\square$

Consider now the case of **even** n. Recall that we equip S^n with the trivial rank 2 vector bundle of mixed degree $E = \mathbb{K}_R \oplus \mathbb{K}_R[1]$, and with bounding cochains of the form $b_{\alpha,\beta} = \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix}_{[pt]}$, such that $\alpha, \beta \in \mathbb{K}_{R,0}$ and $[pt] \in H^n(S^n; \mathbb{Z})$ is represented by the minimum of a perfect Morse function on S^n. Let F_0, F' be as before.

Lemma 4.12.

(1) There is an isomorphism

$$HF^*(F, (S^n, b_{\alpha,\beta}); \mathbb{K}_R) \cong \mathbb{K}_R \oplus \mathbb{K}_R[1].$$

(2) Using the identification in Lemma 4.4 of $e \in HF^0(F_0, F'; \mathbb{Z})$ with the class of a point in S^{n-1}, and of $u \in HF^{1-n}(F_0, F'; \mathbb{Z})$ with the fundamental class of S^{n-1}, we have

$$\psi_{ue}^{(S^n, b_{\alpha,\beta})} = \pm \begin{pmatrix} 0 & \beta \\ \alpha & 0 \end{pmatrix} \psi_e^{(S^n, b_{\alpha,\beta})}.$$

Proof. The proof of (1) is similar to Lemma 4.11. One can again replace F with either F_0 or F' as in Fig. 4. We obtain a $\mathbb{K}_R$-basis v_0, v_1 for $HF^*(F_0, (S^n, b_{\alpha,\beta}); \mathbb{K}_R)$, where v_0, v_1 is the standard basis for the fiber of $E = \mathbb{K}_R \oplus \mathbb{K}_R[1]$ at z (the fiber minimum) indicated in Fig. 4. Similarly, we denote by v_0', v_1' the analogous basis for $HF^*(F', (S^n, b_{\alpha,\beta}); \mathbb{K}_R)$, with z replaced by z' (the fiber maximum) in Fig. 4.

The result in (2) follows again from the study of lifts of the dark triangle in Fig. 4. Once more, the lifts of the triangle can be rigidified either by taking e as an input in μ^2 or by inputing the bounding cochain $b_{\alpha,\beta}$ in μ^3. Taking bases v_i and v_i' above, we get

$$\psi_e^{(S^n, b_{\alpha,\beta})}(v_i') = \mu^2(v_i', e) = \pm T^B v_i,$$

for $i = 0, 1$, where B is the σ-area of the dark triangle. We also get

$$\psi_u^{(S^n, b_{\alpha,\beta})}(v_0') = \mu^3(b_\alpha, v_0', u) = \pm T^B \alpha v_1$$

and

$$\psi_u^{(S^n, b_{\alpha,\beta})}(v_1') = \mu^3(b_{\alpha,\beta}, v_1', u) = \pm T^B \beta v_0,$$
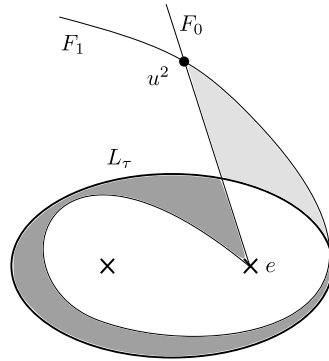
as wanted. $\square$

Fig. 5. L_τ as a cone on morphisms between F_0 and F_1.

We now consider the Lagrangians L_τ, which are diffeomorphic to $S^1 \times S^{n-1}$. Let $U \in U^*_{\mathbb{K}_R}$ be an invertible unitary element in the Novikov field and take $\alpha := T^{-2(n-1)\tau}U^{-1} \in \mathbb{K}_R \setminus \mathbb{K}_{R,0}$. If $n > 2$, write L_α for the Lagrangian L_τ equipped with the unitary local system ξ in the trivial rank 1 $\mathbb{K}_R$-bundle over L_τ, such that U is the holonomy of ξ along a loop that projects in degree 1 to the curve C_τ (recall that we think of C_τ as the curve C in Fig. 1). If $n = 2$, recall that we picked a basis h_1, h_2 for $H_1(L_\tau; \mathbb{Z})$ in Lemma 2.9, to write the disk potential of L_τ. The curve h_1 projects in degree 1 to C_τ and h_2 is a fiber of $\pi|_{L_\tau}$. In Remark 3.7, we observed that the Floer cohomology of (L_τ, ξ) is non-trivial precisely when ξ is a local system with holonomy -1 around h_2. Write L_α for (L_τ, ξ), such that the holonomy of ξ is U around h_1 and -1 around h_2.

If the σ-areas of the two shaded regions in Fig. 5 are the same, then the figure suggests that L_τ should be equivalent to surgery on morphisms supported on the two connected components of the intersection $F_1 \cap F_0 = \{*\} \cup S^{n-1}$. Recall that surgery on an intersection point of two Lagrangians corresponds in the Fukaya category to taking the cone on the morphism given by the intersection point, see Chapter 10 of [15]. This motivates the following result.

Lemma 4.13. *For the appropriate choice of spin structure, L_α is isomorphic in $\mathcal{W}^{\mathbb{Z}/2\mathbb{Z}}_{\mathrm{mon}}(T^*S^n; \mathbb{K}_R)$ to $\mathrm{Cone}(u^2 - \alpha e)$, where $u^2 - \alpha e$ is thought of as a morphism in $HW^{\mathrm{even}}(F, F)$. In particular,*

$$HF^*(F, L_\alpha; \mathbb{K}_R) \cong H^*(S^{n-1}; \mathbb{K}_R),$$

as $\mathbb{Z}/2\mathbb{Z}$-graded free $\mathbb{K}_R$-modules.

Proof. Given a monic polynomial $p(u) = u^d + a_{d-1}u^{d-1} + \ldots + a_0$ in $\mathbb{K}_R[u] \cong HW^*(F, F; \mathbb{K}_R)$, the object $\mathrm{Cone}(p(u))$ is such that $HF^*(F, \mathrm{Cone}(p(u)); \mathbb{K}_R)$ is a free $\mathbb{K}_R$-module of rank d. The right action of u on $HF^*(F, \mathrm{Cone}(p(u)); \mathbb{K}_R)$ is by the transpose of the companion matrix to $p(u)$:
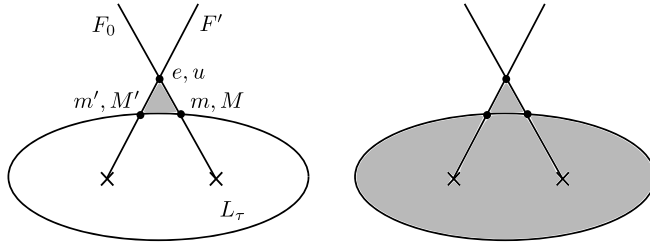
Fig. 6. The action of u on L_α.

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{d-1} \end{pmatrix}.$$

Hence, we want to show that $HF^*(F, L_\alpha; \mathbb{K}_R)$ is a free $\mathbb{K}_R$-module of rank 2, where u acts on the right as

$$\begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix} \tag{4.4}$$

Let us represent the Hamiltonian isotopy class of F by F_0 and by F', as before. In Fig. 6, we see that $F_0 \cap L_\tau$ is diffeomorphic to S^{n-1}. Choosing an auxiliary perfect Morse function on this sphere, we get a chain model for $CF^*(F_0, L_\alpha; \mathbb{K}_R)$ whose generators are the minimum m and the maximum M. We can similarly get generators m', M' for $CF^*(F', L_\alpha; \mathbb{K}_R)$. We will first work with coefficients in $\mathbb{K}_\mathbb{Z}$, and then argue that the case of $\mathbb{K}_\mathbb{C}$-coefficients follows. In particular, we will begin by assuming that $U \in U^*_{\mathbb{K}_\mathbb{Z}}$ and then consider the more general case of $U \in U^*_{\mathbb{K}_\mathbb{C}} = U^*_\mathbb{K}$.

Recall Lemma 4.4. The element $e \in CF^0(F_0, F'; \mathbb{K}_\mathbb{Z})$ (the minimum in its S^{n-1} fiber) acts on $CF^*(F', L_\alpha, \mathbb{K}_\mathbb{Z})$ by

$$\psi_e^{L_\alpha}(M') = \mu^2(M', e) = \pm T^A m,$$

by taking lifts of the shaded triangle on the left in Fig. 6. We denote by A the σ-area of this triangle. Note that the Lefschetz fibration is trivial over the triangle, so it has an S^{n-1}-family of holomorphic lifts. The remaining contributions to right multiplication by e must come from lifts of the shaded triangle on the right in Fig. 6. Since e represents a cohomological unit in $HW^*(F, F; \mathbb{K}_\mathbb{Z})$, it acts by an isomorphism over $\mathbb{K}_\mathbb{Z}$ and we conclude that the lifts of that triangle contribute to

$$\psi_e^{L_\alpha}(m') = \mu^2(m', e) = \pm T^B U M,$$

where B is the σ-area of that right triangle in the plane. Note that these lifted triangles pick up holonomy U. The same holomorphic triangles determine the action of e over

$\mathbb{K} = \mathbb{K}_{\mathbb{C}}$. We conclude that $\psi_e^{L_\alpha}$ is given by the same formulas over $\mathbb{K}$ as over $\mathbb{K}_{\mathbb{Z}}$, even when we take $U \in U_{\mathbb{K}}^*$.

The element $u \in CF^{1-n}(F_0, F'; \mathbb{K}_R)$ is represented by the maximum in the same S^{n-1} fiber as e, and acts on $CF^*(F', L_\alpha; \mathbb{K}_R)$ by

$$\psi_u^{L_\alpha}(M') = \mu^2(M', u) = \pm T^A M,$$

and

$$\psi_u^{L_\alpha}(m') = \mu^2(m', u) = \pm T^A m.$$

In both cases, this corresponds to lifting the triangle on the left in Fig. 6.

Observe that $B = A + 2(n-1)\tau$, so we can write

$$\psi_u^{L_\alpha} = \begin{pmatrix} 0 & 1 \\ \pm\alpha & 0 \end{pmatrix} \psi_e^{L_\alpha}. \tag{4.5}$$

To get a positive sign in α as in (4.4), we note that by changing the spin structure on L_τ we can change the sign of $\psi_e^{L_\alpha}(m')$ (which comes from lifting the shaded triangle on the right in Fig. 6). This has the desired has the effect of replacing α by $-\alpha$ in the matrix in (4.5).

We still need to show that $HF^*(F, L_\alpha; \mathbb{K}_R) \neq 0$. We prove that $HF^*(F', L_\alpha; \mathbb{K}_R) \neq 0$. If n is odd, then this is obvious, since the indices of the generators m', M' have the same parity, so the differential is zero. Observe that the case $n = 2$ is addressed in Remark 3.7. For general even n, we write

$$\mu^1(m) = \kappa_1 M, \qquad \mu^1(M) = \kappa_2 m, \qquad \mu^1(m') = \kappa_1' M', \qquad \mu^1(M') = \kappa_2' m',$$

for some $\kappa_1, \kappa_2, \kappa_1', \kappa_2' \in \mathbb{K}$. The Leibniz rule (and the fact that $\mu^1(u) = 0$) yields

$$\mu^1(\mu^2(m', u)) = \mu^2(\mu^1(m'), u) = \kappa_1' \mu^2(M', u) = \pm\kappa_1' T^A M$$
$$= \pm\mu^1(T^A m) = \pm T^A \kappa_1 M \implies \kappa_1 = \pm\kappa_1'$$

and

$$\mu^1(\mu^2(m', e)) = \mu^2(\mu^1(m'), e) = \kappa_1' \mu^2(M', e) = \pm\kappa_1' T^A m$$
$$= \pm\mu^1(T^B U M) = \pm T^B U \kappa_2 m \implies T^{B-A} U \kappa_2 = \pm\kappa_1' = \pm\kappa_1.$$

Therefore,

$$\mu^1 \circ \mu^1(M) = \kappa_2 \mu^1(m) = \pm T^{B-A} U \kappa_2^2 M.$$

But $\mu^1 \circ \mu^1 = 0$, because F and L_α both have vanishing disk potential. Since $T^{B-A}U \neq 0$, we conclude that $\kappa_2 = \kappa_1 = \kappa_1' = 0$. A similar argument shows that $\kappa_2' = 0$, and implies that $HF^*(F', L_\alpha; \mathbb{K}_R) \neq 0$, as wanted. $\square$

As we saw, Lemma 4.13 can be rephrased as saying that $HF^*(F, L_\alpha; \mathbb{K}_R)$ is isomorphic to $\mathbb{K}_R^2$, if n is odd, and to $\mathbb{K}_R \oplus \mathbb{K}_R[1]$, if n is even, and that the action of u is represented by the matrix (4.4). To relate this with the generation results for modules that will be discussed below, it is convenient to restrict our attention to $\mathbb{K} = \mathbb{K}_{\mathbb{C}}$, which is an algebraically closed field. Since the eigenvalues of the matrix (4.4) are $\pm\sqrt{\alpha}$ (the two square roots of α in $\mathbb{K}$), we conclude the following.

Corollary 4.14. *If n is odd, then $HF^*(F', L_\alpha; \mathbb{K})$ and $HF^*(F_0, L_\alpha; \mathbb{K})$ have bases in which*
$$\psi_{ue}^{L_\alpha} = \begin{pmatrix} \sqrt{\alpha} & 0 \\ 0 & -\sqrt{\alpha} \end{pmatrix} \psi_e^{L_\alpha}.$$

We are now ready to prove the following result, up to Corollary 6.5 below.

Theorem 4.15. *The category $\mathrm{mod}_{pr}(A_{\mathbb{K}})$ is split-generated by the collection of right $A_{\mathbb{K}}$-modules*

- $\{HF^*(F, (S^n, \alpha[pt]); \mathbb{K})\}_{0 \leq \mathrm{val}(\alpha) \leq \infty} \cup \{HF^*(F, L_\alpha; \mathbb{K})\}_{\mathrm{val}(\alpha)<0}$, *if n is odd;*
- $\{HF^*(F, S^n; \mathbb{K})\} \cup \{HF^*(F, (S^n, b_{\alpha,1}); \mathbb{K})\}_{0 \leq \mathrm{val}(\alpha)<\infty} \cup \{HF^*(F, L_\alpha; \mathbb{K})\}_{\mathrm{val}(\alpha)<0}$, *if n is even.*

Proof. In the n odd case, if $\mathrm{val}(\alpha) \geq 0$, then Lemma 4.11 implies that
$$HF^*(F_0, (S^n, \alpha[pt]); \mathbb{K}) \cong S_{\pm\alpha}$$

as right $A_{\mathbb{K}}$-modules, where S_α is the 1-dimensional (over $\mathbb{K}$) right $A_{\mathbb{K}}$-module on which $u \in A_{\mathbb{K}}$ acts as multiplication by α (as in Lemma 6.4 below).

If $\mathrm{val}(\alpha) < 0$, Lemma 4.13 and Corollary 4.14 imply that
$$HF^*(F, L_\alpha; \mathbb{K}) \cong S_{\sqrt{\alpha}} \oplus S_{-\sqrt{\alpha}}$$

as right $A_{\mathbb{K}}$-modules.

Corollary 6.5 below now implies the result when n is odd. The case of n even is analogous, where this time we apply Lemma 4.12 instead of Lemma 4.11 and Corollary 6.8 instead of Corollary 6.5. $\square$

The following is a version of Theorem 1.4 from the Introduction.

Corollary 4.16. *The category $\mathcal{F}_{mon}^{\mathbb{Z}/2\mathbb{Z}}(T^*S^n; \mathbb{K})$ is split-generated by the collection of objects*
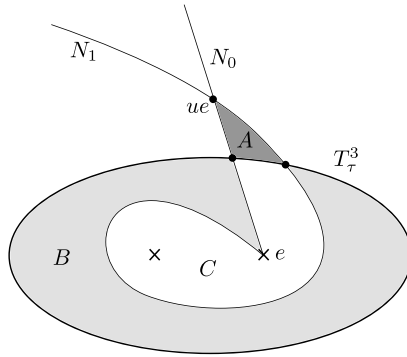
Fig. 7. The action of u on T_α^3.

- $\{(S^n, \alpha[pt])\}_{0 \le \mathrm{val}(\alpha) \le \infty} \cup \{L_\alpha\}_{\mathrm{val}(\alpha) < 0}$, *when n is odd;*
- $\{S^n\} \cup \{(S^n, b_{\alpha,1})\}_{0 \le \mathrm{val}(\alpha) < \infty} \cup \{L_\alpha\}_{\mathrm{val}(\alpha) < 0}$, *when n is even.*

Proof. This follows from Theorem 4.15 and Proposition 3.8. □

4.3. Computations in $T^* S^3$

We now want to study how $u \in HW^*(F, F; \mathbb{K}_R) \cong \mathbb{K}_R[u] = A_{\mathbb{K}_R}$ acts on the tori T_τ^3. Recall from Lemma 2.13 that the disk potential of T_τ^3 can be computed in a basis h_1, h_2, h_3 of $H_1(T_\tau^3; \mathbb{Z})$, where h_1 is a loop projecting bijectively to the curve $C_\tau \subset \mathbb{C} \setminus \{\pm 1\}$ (that T_τ^3 covers), while h_2 and h_3 are vanishing circles that project to points under the fibration. As observed in Remark 3.7, the critical points of the disk potential that belong to $(U_{\mathbb{K}_R}^*)^3$ correspond to unitary local systems on T_τ^3, whose holonomy around h_1 is arbitrary, and whose holonomy around each of h_2 and h_3 is -1.

Given $U \in U_{\mathbb{K}_R}^*$, let $\alpha := T^{-2\tau} U^{-1} \in \mathbb{K}_R \setminus \mathbb{K}_{R,0}$ and denote by T_α the Lagrangian T_τ^3 equipped with a unitary local system ξ in the trivial rank 1 $\mathbb{K}_R$-bundle, whose holonomy around h_1 is U, and whose holonomies around h_2 and h_3 are -1.

Given $a \in HF^*(N_1, N_0; \mathbb{K}_R)$ and $X \in \mathcal{W}_{\mathrm{mon}}^{\mathbb{Z}/2\mathbb{Z}}(T^* S^3; \mathbb{K}_R)$, define a map

$$\phi_a^X \colon HF^*(N_0, L; \mathbb{K}_R) \to HF^*(N_1, L; \mathbb{K}_R)$$
$$x \mapsto \mu^2(x, a)$$

Lemma 4.17. *For the appropriate choice of spin structure on T_α,*

(1) there is an isomorphism

$$HF^*(N, T_\alpha; \mathbb{K}) \cong H^*(T^2; \mathbb{K}),$$

possibly with a degree shift;

(2) *using the identification in Lemma 4.9 of $\pm e \in HF^0(N_1, N_0; \mathbb{Z})$ with the fundamental class of S^1, and of $\pm ue_1 \pm ue_2 \in HF^{-2}(N_1, N_0; \mathbb{Z})$ with the fundamental class of T^2, we can pick bases so that*

$$\phi^{T_\alpha}_{\pm ue_1 \pm ue_2} = \begin{pmatrix} \pm\alpha & & & \\ 0 & \pm\alpha & & \\ 0 & 0 & \pm\alpha & \\ * & 0 & 0 & \pm\alpha \end{pmatrix} \phi^{T_\alpha}_e$$

where the 4×4 matrix above is lower triangular.

Proof. We begin with (1). By Remark 3.7, T_α corresponds to a critical point of the disk potential of the torus T^3_τ. Hence, $HF^*(T_\alpha, T_\alpha; \mathbb{K}) \cong H^*(T^3; \mathbb{K})$ has rank 8. Observe that for the Lagrangian lifts N_i of the paths η_i in Fig. 1, each of the graded $\mathbb{K}$-vector spaces $CF^*(N_i, T_\alpha; \mathbb{K})$ is isomorphic to $H^*(T^2; \mathbb{K})$. This has rank 4, so the rank of $HF^*(N_i, T_\alpha; \mathbb{K})$ can only be 0, 2 or 4.

On the other hand, by Proposition 4.6,

$$HF^*(N_i, T_\alpha, \mathbb{K}) \cong HF^*(F, T_\alpha; \mathbb{K}) \oplus HF^*(F, T_\alpha; \mathbb{K})[1].$$

Since F split-generates the monotone wrapped Fukaya category and T_α is a non-trivial object, we conclude that the rank of $HF^*(F, T_\alpha; \mathbb{K})$ is 1 or 2. Denote by M this $\mathbb{K}[u]$-module. The fully faithfulness of the Yoneda embedding implies that

$$HF^*(T_\alpha, T_\alpha; \mathbb{K}) \cong \mathrm{Ext}^*_{\mathbb{K}[u]}(M, M)$$

and the rank of the right side cannot be 8 if M has rank 1, since the space of endomorphisms of a skyscraper sheaf in the derived category of a smooth curve has rank 2 (see Lemma 6.3 below). We conclude that M has rank 2.[2] Hence, $HF^*(N_i, T_\alpha; \mathbb{K})$ has rank 4 and the differential on $CF^*(N_i, T_\alpha; \mathbb{K})$ vanishes. This implies the statement in (1).

To prove (2), we use Fig. 7. First, we need to determine the image of $\pm e$ and $\pm ue_1 \pm ue_2$ under the functor $\mathcal{G}_1 : \mathcal{W}^\mathbb{Z}(T^*S^3; \mathbb{Z}) \to \mathcal{W}^\mathbb{Z}(T^*S^3; \mathbb{K}_R)$ defined in Section 3.4. By Lemma 4.9, if we pick a perfect Morse function on $N_0 \cap N_1 \cong S^1 \cup T^2$, we know that $\pm e$ is given by the maximum (which we denote by x) on the component S^1, and $\pm ue_1 \pm ue_2$ is given by the maximum (which we denote by y) on the component T^2. Pick primitives f_i for the restriction of the Liouville form λ on T^*S^3 to the N_i, $i \in \{0, 1\}$. Assume that f_0 and f_1 both vanish at x. Then, $\mathcal{G}_1(\pm e) = T^{f_1(x) - f_0(x)}x = x \in CF^0(N_1, N_0; \mathbb{K}_R)$. Similarly, $\mathcal{G}_1(\pm ue_1 \pm ue_2) = T^{f_1(y) - f_0(y)}y \in CF^{-2}(N_1, N_0; \mathbb{K}_R)$. Under our assumptions,

---

[2] The referee suggested the following alternative argument for calculating that the rank of $M = HF^*(F, T_\alpha; \mathbb{K})$ is 2, avoiding the computation of Ext. If one thinks of the Lagrangian T^3_τ as coming from surgery on two cleanly intersecting Lagrangian spheres, as explained in Remark 1.6, then it is clear that T^3_τ intersects a cotangent fiber $T^*_p S^3$ transversely in two points, for all p contained in a non-empty open subset of S^3. Since F is Hamiltonian-isotopic to a cotangent fiber, this implies that the rank of M is 0 or 2. But since T_α has non-trivial Floer cohomology, the rank of M cannot vanish.

$f_1(y) - f_0(y)$ is the negative of the symplectic area of a strip between N_1 and N_0, obtained from lifting the union of the darkly shaded and the white triangles in Fig. 7. We denote this area by $A + C$ in the figure. We can conclude that $\pm u e_1 \pm u e_2$ is represented by $T^{-A-C} y$ in $CF^{-2}(N_1, N_0; \mathbb{K}_R)$.

The intersections $N_i \cap T_\tau^3$ are 2-tori, and we can pick perfect Morse functions on these 2-tori to specify bases for $HF^*(N_i, T_\alpha; \mathbb{K})$. Denote the basis elements $p_0^i, p_1^i, p_2^i, p_3^i$, with Morse indices 0, 1, 1 and 2, respectively. Each of the shaded triangles in the figure lifts to a T^2-family of holomorphic triangles with suitable boundary conditions. Using the fundamental classes of S^1 and T^2 as inputs in μ^2 does not constrain the T^2-families of holomorphic triangles. Therefore, for a suitable labeling of the generators, the contribution of p_i^1 to $\phi_{\pm u e_1 \pm u e_2}^{T_\alpha}(p_i^0)$ is

$$\pm T^{-A-C} T^A = \pm \alpha T^B U, \tag{4.6}$$

since the sum of the σ-areas of the lightly shaded and white triangles in the figure is $B + C = 2\tau$. Similarly, the contribution of p_i^1 to $\phi_e^{T_\alpha}(p_i^0)$ is $\pm T^B U$. This justifies the diagonal terms in the 4×4 matrix in (2). The possibly non-zero off-diagonal term in that matrix is given by the lifts of a large triangle in the base, given by the union of the white triangle and the two shaded triangles. Such lifts come in 4-dimensional families, which is why they contribute to an off-diagonal corner in the matrix. $\square$

Remark 4.18. The fibration is trivial over the darkly shaded triangle in Fig. 7, which is why it has a T^2-family of lifts. The fibration is not trivial over the lightly shaded triangle, because one of its vertices is a critical value. Nevertheless, this triangle still has a T^2-family of lifts. Note that one could also modify N_0 slightly, in a manner similar to what is done in Fig. 4 for the proof of Lemma 4.11, so that the analogue of the lightly shaded triangle now includes no critical points, even at the corners.

We can now prove the following analogue of Theorem 4.15.

Theorem 4.19. *The collection of $A_\mathbb{K}$-modules*

$$\{HF^*(F, (S^3, \alpha[pt]); \mathbb{K})\}_{0 \leq \mathrm{val}(\alpha) \leq \infty} \cup \{HF^*(F, T_\alpha; \mathbb{K})\}_{\mathrm{val}(\alpha) < 0}$$

split-generates the category $\mathrm{mod}_{pr}(A_\mathbb{K})$.

Proof. We just have to show that we can replace the objects supported on the collection of Lagrangians $\{(S^1 \times S^2)_\tau\}_{\tau > 0}$ by the objects supported on the collection $\{T_\tau^3\}_{\tau > 0}$.

By Proposition 4.6, $N \cong F \oplus F[1]$. In the proof of Lemma 4.17, it is shown that the right $A_\mathbb{K}$-module $HF^*(F, T_\alpha; \mathbb{K})$ is of rank 2. It follows from (2) in Lemma 4.17 that this module must be isomorphic (up to degree shifts) to $S_\alpha \oplus S_\alpha$, $S_{-\alpha} \oplus S_{-\alpha}$, $S_\alpha \oplus S_{-\alpha}$ or $M_{\pm\alpha}^2$ (in the notation of Section 6.1 below). Lemma 6.3 below implies that only the first two options (possibly with degree shifts is each of the summands) are compatible with

the fact that $HF^*(T_\alpha, T_\alpha; \mathbb{K}) \cong \mathrm{Ext}^*_{\mathbb{K}[u]}(M, M)$ has rank 8. The result now follows from Corollary 6.5. $\square$

Corollary 4.20. *The category $\mathcal{F}^{\mathbb{Z}/2\mathbb{Z}}_{mon}(T^*S^3; \mathbb{K})$ is split-generated by the collection of objects $\{(S^3, \alpha[pt])\}_{0 \leq \mathrm{val}(\alpha) \leq \infty} \cup \{T_\alpha\}_{\mathrm{val}(\alpha) < 0}$.*

Proof. This follows from Theorem 4.19 and Proposition 3.8. $\square$

Now that we understand the F-modules associated to the Lagrangians $(S^1 \times S^2)_\tau$ and T^3_τ in T^*S^3, we can also prove Theorem 1.3.

Proof of Theorem 1.3. We wish to show that, if we fix $\tau, \tau' > 0$, then $\tau = \tau'$ iff $(S^1 \times S^2)_\tau$ and $T^3_{\tau'}$ can be equipped with local systems such that their Floer cohomology is nontrivial. Let $U \in U^*_{\mathbb{K}}$ and write $\alpha = T^{-4\tau}U^{-1}$. Recall that the minimal Maslov number of $(S^1 \times S^2)_\tau$ is 4, and that $(S^1 \times S^2)_\alpha$ denotes $(S^1 \times S^2)_\tau$ equipped with a rank 1 unitary local system with holonomy specified by U. The proof of Theorem 4.15 implies that $HF^*(F, (S^1 \times S^2)_\alpha; \mathbb{K}) \cong S_{\sqrt{\alpha}} \oplus S_{-\sqrt{\alpha}}$, where $\sqrt{\alpha} = T^{-2\tau}(\sqrt{U})^{-1}$ for some square root $\sqrt{U} \in \mathbb{K}$ of U. Write also $\alpha' = T^{-4\tau'}U^{-1}$ and $\sqrt{\alpha'} = T^{-2\tau'}(\sqrt{U})^{-1}$. The minimal Maslov number of $T^3_{\tau'}$ is 2, and $T_{\sqrt{\alpha'}}$ denotes $T^3_{\tau'}$ with a rank 1 unitary local system of holonomy specified by $\sqrt{U}$. The proof of Theorem 4.19 implies that $HF^*(F, T_{\sqrt{\alpha'}}; \mathbb{K})$ is isomorphic either to $S_{\sqrt{\alpha'}} \oplus S_{\sqrt{\alpha'}}$ or to $S_{-\sqrt{\alpha'}} \oplus S_{-\sqrt{\alpha'}}$ (possibly with degree shifts in the summands).

The result now follows from Proposition 3.8 and Lemma 6.3 below. $\square$

5. Intrinsic formality of algebras and modules

Recall that $A_{\mathbb{K}} = HW^*(F, F; \mathbb{K})$ is isomorphic to the polynomial algebra $\mathbb{K}[u]$, where $\deg(u) = 1 - n$ and $n \geq 2$. From this point on, we will always work over $\mathbb{K}$, and write A instead of $A_{\mathbb{K}}$ to make the notation lighter. We want to show that A is *intrinsically formal*, and will later prove an analogous result for certain types of A-modules. Intrinsic formality of A means that if $\mathcal{B}$ is any A_∞-algebra such that the algebra $H^*(\mathcal{B})$ is isomorphic to A, then $\mathcal{B}$ is quasi-isomorphic to A as A_∞-algebras.

Denoting by $|A|$ the algebra A where we *forget the grading*, we can define the Hochschild cohomology $HH^r(|A|, |A|)$ as the homology of $CC^r(|A|, |A|) := \mathrm{Hom}_{\mathbb{K}}(|A|^{\otimes r}, |A|)$, for $r \geq 0$, with respect to the Hochschild differential, see for instance [39].

To keep track of the grading on A, one can define

$$CC^r(A, A[s]) := \mathrm{Hom}^s_{\mathbb{K}}(A^{\otimes r}, A),$$

which consists of graded homomorphisms that increase the degree by $s \in \mathbb{Z}$ (we continue to use the cohomological convention under which $A[s]$ is obtained by *subtracting s from all degrees in A*), see [35, Section 4b]. The Hochschild differential preserves s, so the

$CC^r(A, A[s])$ are subcomplexes of $CC^r(|A|, |A|)$. For each s we have a direct sum of chain complexes

$$CC^*(|A|, |A|) = CC^*(A, A[s]) \oplus Q^{*,s}$$

where $Q^{r,s} \subset CC^*(|A|, |A|)$ consists of those homomorphisms that have no term of degree s. One can identify $Q^{r,s}$ with the quotient $CC^r(|A|, |A|)/CC^r(A, A[s])$. We can conclude that there are inclusions on cohomology

$$HH^r(A, A[s]) \subset HH^r(|A|, |A|).$$

Remark 5.1. Since A is supported in infinitely many degrees, none of the inclusions

$$\bigoplus_{s \in \mathbb{Z}} CC^r(A, A[s]) \subset CC^r(|A|, |A|) \subset \prod_{s \in \mathbb{Z}} CC^r(A, A[s])$$

is the identity.

By [39, Corollary 9.1.5], $HH^*(|A|, |A|) \cong \mathrm{Ext}^*_{|A|^e}(|A|, |A|)$, where $|A|^e = |A| \otimes_{\mathbb{K}} |A|^{\mathrm{op}}$ (this is isomorphic to $|A| \otimes_{\mathbb{K}} |A|$, since $|A|$ is commutative). Note that $\mathrm{Ext}_{|A|^e}(|A|, |A|)$ can be computed using any projective resolution of $|A|$ as an $|A|^e$-module. We use the *Koszul resolution*

$$0 \to |A|^e \xrightarrow{f} |A|^e \xrightarrow{g} |A| \to 0, \tag{5.1}$$

where $f(a(u) \otimes b(u)) = a(u)u \otimes b(u) - a(u) \otimes ub(u)$ and $g(p(u), q(u)) = p(u)q(u)$.

The existence of this 2-step resolution implies that $HH^r(|A|, |A|) = 0$ if $r \geq 2$. Since $HH^r(A, A[s]) \subset HH^r(|A|, |A|)$, this implies that $HH^r(A, A[s]) = 0$ for all s and for all $r \geq 2$.

It is known that A is intrinsically formal if $HH^r(A, A[2 - r]) = 0$ for all $r \geq 3$, see [21, Corollary 4], [34, Section 3] and [35, Theorem 4.7]. We can thus conclude that the $\mathbb{Z}$-graded algebra A is intrinsically formal. The vanishing for $r = 2$ means that it is also not possible to deform the product structure on A.

The previous argument can be adapted to show that, if we collapse the $\mathbb{Z}$-grading of A to a $\mathbb{Z}/2\mathbb{Z}$-grading, A is still intrinsically formal. More specifically, let $CC^r(A, A[\text{even}]) \subset CC^r(|A|, |A|)$ be the subcomplex of homomorphisms with no odd degree components, and let $CC^r(A, A[\text{odd}]) \subset CC^r(|A|, |A|)$ be the subcomplex of homomorphisms with no even degree components. We get a decomposition

$$HH^r(|A|, |A|) \cong HH^r(A, A[\text{even}]) \oplus HH^r(A, A[\text{odd}]).$$

In the $\mathbb{Z}/2\mathbb{Z}$-graded case, intrinsic formality of A follows from the simultaneous vanishing

$$\begin{cases} HH^r(A, A[\text{even}]) = 0, \text{ for all } r \geq 3 \text{ even} \\ HH^r(A, A[\text{odd}]) = 0, \text{ for all } r \geq 3 \text{ odd} \end{cases}$$

which is again a consequence of the fact that $HH^r(|A|, |A|) = 0$ if $r \notin \{0, 1\}$.

We can conclude the following.

Proposition 5.2. *$A = \mathbb{K}[u]$ is intrinsically formal as a $\mathbb{Z}$-graded algebra and as a $\mathbb{Z}/2\mathbb{Z}$-graded algebra.*

We now discuss right modules over the graded algebra A. In a manner similar to the previous discussion, let $|A|$, $|M|$ and $|N|$ be the result of forgetting the $\mathbb{Z}$-gradings of the algebra A and of right A-modules M and N. Then, $\mathrm{Hom}_\mathbb{K}(|M|, |N|)$ is an A-bimodule and its Hochschild cochain complex is, for $r \geq 0$,

$$CC^r(|A|, \mathrm{Hom}_\mathbb{K}(|M|, |N|)) := \mathrm{Hom}_\mathbb{K}(|A|^{\otimes r}, \mathrm{Hom}_\mathbb{K}(|M|, |N|))$$
$$\cong \mathrm{Hom}_\mathbb{K}(|M| \otimes_\mathbb{K} |A|^{\otimes r}, |N|).$$

Remembering the $\mathbb{Z}$-gradings, we can denote as before the homomorphisms of degree $s \in \mathbb{Z}$ by

$$CC^r(A, \mathrm{Hom}_\mathbb{K}(M, N)[s]) := \mathrm{Hom}^s_\mathbb{K}(A^{\otimes r}, \mathrm{Hom}_\mathbb{K}(M, N)) \cong \mathrm{Hom}^s_\mathbb{K}(M \otimes_\mathbb{K} A^{\otimes r}, N).$$

The Hochschild differential preserves s and we get inclusions on cohomology

$$HH^r(A, \mathrm{Hom}_\mathbb{K}(M, N)[s]) \subset HH^r(|A|, \mathrm{Hom}_\mathbb{K}(|M|, |N|)).$$

Using again [39, Corollary 9.1.5], we get that $HH^*(|A|, \mathrm{Hom}_\mathbb{K}(|M|, |N|)) \cong \mathrm{Ext}^*_{|A|^e}(|A|, \mathrm{Hom}_\mathbb{K}(|M|, |N|))$. The 2-step Koszul resolution (5.1) can be used to show that $HH^r(|A|, \mathrm{Hom}_\mathbb{K}(|M|, |N|)) = 0$ for $r \geq 2$ and for every M, N. Consequently, we get $HH^r(A, \mathrm{Hom}_\mathbb{K}(M, N)[s]) = 0$ for $r \geq 2$ and for all $s \in \mathbb{Z}$.

Remark 5.3. It is worth pointing out that $HH^*(|A|, \mathrm{Hom}_\mathbb{K}(|M|, |N|))$ is also isomorphic to $\mathrm{Ext}^*_{|A|/\mathbb{K}}(|M|, |N|)$, the *relative* Ext, see [39, Lemma 9.1.9].

Say that a $\mathbb{Z}$-graded right A-module M is *intrinsically formal* if, for every $\mathbb{Z}$-graded right A_∞-module $\mathcal{M}$ over A such that the A-module $H^*\mathcal{M}$ is isomorphic to M, we have that $\mathcal{M}$ is quasi-isomorphic to M (as A_∞-modules over A). In an analogous manner to the Hochschild cohomology criterion for intrinsic formality of graded algebras discussed earlier, it can be shown that if

$$HH^r(A, \mathrm{Hom}_\mathbb{K}(M, M)[1 - r]) = 0$$

for all $r \geq 2$, then M is intrinsically formal, see [23, Theorem 3.2]. What we saw above implies that every $\mathbb{Z}$-graded right A-module is intrinsically formal.

The same argument could again be adapted to the case of $\mathbb{Z}/2\mathbb{Z}$-graded modules over A (with the grading of A collapsed to $\mathbb{Z}/2\mathbb{Z}$). If M and N are $\mathbb{Z}/2\mathbb{Z}$-graded right A-modules, we can define cohomology groups $HH^r(A, \mathrm{Hom}_{\mathbb{K}}(M, N)[s])$ with $s \in \{0, 1\}$. This time, we have a decomposition

$$HH^r(|A|, \mathrm{Hom}_{\mathbb{K}}(|M|, |N|)) \cong HH^r(A, \mathrm{Hom}_{\mathbb{K}}(M, N)) \oplus HH^r(A, \mathrm{Hom}_{\mathbb{K}}(M, N)[1]).$$

The sufficient condition for intrinsic formality of M is now given by the simultaneous vanishing

$$\begin{cases} HH^r(A, \mathrm{Hom}_{\mathbb{K}}(M, N)[1]) = 0, \text{ for all } r \geq 2 \text{ even} \\ HH^r(A, \mathrm{Hom}_{\mathbb{K}}(M, N)) = 0, \text{ for all } r \geq 2 \text{ odd} \end{cases}$$

and this criterion is again met by the discussion above. We can conclude the following.

Proposition 5.4. *All $\mathbb{Z}$-graded and all $\mathbb{Z}/2\mathbb{Z}$-graded right A-modules are intrinsically formal.*

As in Section 3.5, denote by $\mathrm{mod}(A)$ a category whose objects are right A-modules (we do not mean A_∞-modules). These modules are $\mathbb{Z}$- or $\mathbb{Z}/2\mathbb{Z}$-graded, depending on the context. Given two modules M, N, define their morphism space to be

$$\mathrm{hom}^*_{\mathrm{mod}(A)}(M, N) = \mathrm{Ext}^*_A(M, N)$$

(instead of usual A-module homomorphisms). The following is a consequence of the results of this section. To make the notation more uniform, we denote the A_∞-algebra $\mathcal{A}_{\mathbb{K}} = CW^*(F, F; \mathbb{K})$ by $\mathcal{A}$.

Corollary 5.5. *Passing to cohomology gives a functor*

$$H : \mathrm{mod}^{A_\infty}(\mathcal{A}) \to \mathrm{mod}(A)$$
$$\mathcal{M} \mapsto H^*(\mathcal{M})$$

which is a quasi-equivalence (meaning that it induces an equivalence of categories on cohomology). The category $\mathrm{mod}(A)$ is equivalent to the cohomology category of $\mathrm{mod}^{A_\infty}(\mathcal{A})$.

Proof. The fact that morphisms on the cohomology category of $\mathrm{mod}^{A_\infty}(\mathcal{A})$ are given by Ext groups is explained in [32, Remark 2.15]. There is a composition of quasi-equivalences of dg-categories

$$\mathrm{mod}(A) \to \mathrm{mod}^{A_\infty}(A) \to \mathrm{mod}^{A_\infty}(\mathcal{A}).$$

The fact that $\mathcal{A}$ is formal (by Proposition 5.2) implies that the functor on the right is a quasi-equivalence, see [32, Section 2f]. The functor on the left is given by inclusion (thinking of $\mathrm{mod}(A)$ as a dg-category with trivial differentials), and it is a quasi-equivalence by Proposition 5.4. The functor H in the statement is a quasi-inverse for this composition. $\square$

6. Generation of categories of modules

Definition 6.1. Let $\mathrm{mod}_{pr}(A)$ be the subcategory of $\mathrm{mod}(A)$, whose objets are finite dimensional right A-modules (the subscript stands for *proper*).

The fact that $\mathbb{C}$ is algebraically closed of characteristic zero implies that $\mathbb{K}$ is also algebraically closed, see [16, Appendix A].[3] This will enable us to study the category $\mathrm{mod}_{pr}(A)$ using Jordan normal forms.

Remark 6.2. One should be able to work over the Novikov field $\mathbb{K}_R$ for a general commutative ring R, by allowing as objects monotone Lagrangians equipped with higher rank local systems. In that setting, one would expect to be able to prove analogues of Corollaries 4.16 and 4.20, showing that every compact Lagrangian object is split-generated by objects supported on S^n and $(S^1 \times S^{n-1})_\tau$ (with the latter being replaceable by T_τ^3, if $n = 3$), without appealing to the Jordan normal form.

Recall that $A = \mathbb{K}[u]$, where $\deg(u) = 1 - n$. Since the monotone Fukaya category is $\mathbb{Z}/2\mathbb{Z}$-graded, we will consider two cases, depending on the parity of n.

6.1. When n is odd

Take an object $M \oplus N$ of $\mathrm{mod}_{pr}(A)$, where M is in degree 0 and N is in degree 1. Since $\mathbb{K}$ is algebraically closed, if $M \neq 0$ then it has a splitting

$$M \cong \bigoplus_{i=1}^m M_{\alpha_i}^{k_i}$$

where $\alpha_i \in \mathbb{K}$, $k_i \in \mathbb{Z}_+$ and M_α^k is the vector space $\mathbb{K}^k$ with a right action of u by the $k \times k$ transposed Jordan block

$$(J_\alpha^k)^T = \begin{pmatrix} \alpha & & & & \\ 1 & \alpha & & & \\ & \ddots & \ddots & & \\ & & 1 & \alpha & \\ & & & 1 & \alpha \end{pmatrix} \tag{6.1}$$

---

[3] In characteristic $p > 0$, the polynomial $x^p - x - T^{-1}$ does not have roots in the Novikov field. See [22] for a discussion of the algebraic closure of the power series field in positive characteristic.

If the module N is non-trivial, then it also has a splitting

$$N \cong \bigoplus_{j=1}^{n} M_{\beta_j}^{l_j}[1]$$

for certain $\beta_j \in \mathbb{K}$ and $l_j \in \mathbb{Z}_+$.

Denote the 1-dimensional module M_α^1 by S_α. The following result was used above in study of the Lagrangian tori $T_\tau^3 \subset T^* S^3$. For the calculation with S_α, one can use the resolution

$$\mathbb{K}[u] \overset{u - \alpha}{\to} \mathbb{K}[u] \to \mathbb{K},$$

and a similar argument works for M_α^2.

Lemma 6.3. *Given $\alpha, \alpha' \in \mathbb{K}$, we have*

$$\mathrm{Ext}^*_{A_{\mathbb{K}}}(S_\alpha, S_{\alpha'}) \cong \begin{cases} \mathbb{K} \oplus \mathbb{K}[1] & \text{if } \alpha = \alpha' \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mathrm{Ext}^*_{A_{\mathbb{K}}}(M_\alpha^2, M_{\alpha'}^2) \cong \begin{cases} \mathbb{K}^2 \oplus \mathbb{K}^2[1] & \text{if } \alpha = \alpha' \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 6.4. *For every $k \in \mathbb{Z}_+$, M_α^k is in the triangulated closure of S_α.*

Proof. Observe that there are A-module homomorphisms

$$\varphi_\alpha^k \colon M_\alpha^k \to S_\alpha$$

obtained by projecting onto the last coordinate. We can think of an A-module homomorphism as a homomorphism of A_∞-modules, and take its cone. Recall that $\mathrm{Cone}(\varphi_\alpha^k)$ is the right A_∞-module over A given by the chain complex

$$(M_\alpha^k[1] \oplus S_\alpha, \mu^1 = \varphi_\alpha^k),$$

with $\mu^2 = (\mu^2_{M_\alpha^k[1]}, \mu^2_{S_\alpha})$ and trivial higher A_∞-maps, see [32, Section (3e)]. We have that $H^* \mathrm{Cone}(\varphi_\alpha^k) \cong M_\alpha^{k-1}[1]$ and so $\mathrm{Cone}(\varphi_\alpha^k)$ is quasi-isomorphic to $M_\alpha^{k-1}[1]$.

We can now argue by induction on k to prove the statement in the lemma. In detail, since there is a distinguished triangle

$$M_\alpha^k \to S_\alpha \to M_\alpha^{k-1}[1] \to M_\alpha^k[1],$$

axiom TR2 for triangulated categories (see for instance [39, Definition 10.2.1]) implies that there is also a distinguished triangle

$$S_\alpha \to M_\alpha^{k-1}[1] \to M_\alpha^k[1] \to S_\alpha[1],$$

and by induction on k we get that M_α^k is in the triangulated closure of S_α for all $k \geq 1$. $\quad\square$

We can now conclude the following.

Corollary 6.5. *The category $\mathrm{mod}_{pr}(A)$ is generated by the collection of modules $\{S_\alpha\}_{\alpha \in \mathbb{K}}$.*

6.2. When n is even

This case is more subtle, because now u acts on A-modules as an operator of odd degree. Take again an object $M \oplus N$ in $\mathrm{mod}_{pr}(A)$, where M is a finite dimensional $\mathbb{K}$-vector space in degree 0 and N is finite dimensional in degree 1. We begin by defining some relevant examples. Given $\alpha \in \mathbb{K}$ and $k \geq 1$, let $\tilde{M}_\alpha^k$ be the right A-module consisting of the $\mathbb{K}$-vector space $\mathbb{K}^k \oplus \mathbb{K}^k[1]$, with u acting on the right on row vectors by the block matrix $\begin{pmatrix} 0 & I_k \\ (J_\alpha^k)^T & 0 \end{pmatrix}$, where $(J_\alpha^k)^T$ is as in (6.1) and I_k is the identity. Denote by $\tilde{S}_\alpha$ the module $\tilde{M}_\alpha^1$. This consists of the $\mathbb{K}$-vector space $\mathbb{K} \oplus \mathbb{K}[1]$, with a right u-action by the matrix $\begin{pmatrix} 0 & 1 \\ \alpha & 0 \end{pmatrix}$.

For $\alpha = 0$, we need to consider another type of right A-module, where the dimensions of the even and odd summands are different. Given $k \geq 1$, let $\tilde{N}_0^k$ be the right A-module consisting of the $\mathbb{K}$-vector space $\mathbb{K}^k \oplus \mathbb{K}^{k-1}[1]$, on which u acts by the following $(2k-1) \times (2k-1)$-matrix, where all the empty blocks are understood to be filled with zeros:

$$\begin{pmatrix} & & \\ \hline & I_k & \\ \hline I_k & & \end{pmatrix}.$$

Denote by S_0 the module $\tilde{N}_0^1$. This is the $\mathbb{K}$-vector space $\mathbb{K}$, on which u acts as multiplication by zero. Note that in the previous section (when n was assumed odd), S_0 was also defined as a 1-dimensional module in even degree with trivial u-action.

Let us now go back to the general case of a right A-module $M \oplus N$, where M and N are finite dimensional, M is in degree 0 and N is in degree 1. Since we will be interested in split-generation of $\mathrm{mod}_{pr}(A)$, we can assume that $\dim_{\mathbb{K}} M = \dim_{\mathbb{K}} N$, by taking a direct sum of M or N with a $\mathbb{K}$-vector space with trivial u-action, if necessary. Since u has odd degree, by picking bases for M and N, the u-action is given by a matrix of the form $\begin{pmatrix} 0 & R \\ S & 0 \end{pmatrix}$, where R and S are square matrices. Since u^2 has even degree, it

acts by endomorphisms of both M and N. As we saw in the case of n odd, we can pick bases for M and N so that the right action of u^2 is represented by the transpose of a Jordan matrix. This means that we can assume that RS and SR consist of finitely many transposed Jordan blocks along the diagonal.

Lemma 6.6. *A suitable choice of bases for M and N induces and isomorphism between $M \oplus N$ and a direct sum of modules of the form $\tilde{M}_\alpha^k$, $\tilde{M}_0^k[1]$, $\tilde{N}_0^k$ or $\tilde{N}_0^k[1]$.*

Proof. If $\alpha \in \mathbb{K}$ is a non-zero eigenvalue of the action of u^2 on M, and if $v_1, \ldots, v_k$ is a Jordan basis for a Jordan block associated to this u^2-action, then $\mathrm{Span}(v_1, \ldots, v_k, v_1 \cdot u, \ldots, v_k \cdot u)$ is a u-invariant subspace of $M \oplus N$, and it is isomorphic to $\tilde{M}_\alpha^k$ as a right A-module. The Jordan blocks associated to $\alpha = 0$ are not invertible matrices, which is why we need to also allow summands of the form $\tilde{M}_0^k[1]$, $\tilde{N}_0^k$ or $\tilde{N}_0^k[1]$. $\square$

Lemma 6.7. *For every $k \in \mathbb{Z}_+$ and every $\alpha \in \mathbb{K}$, $\tilde{M}_\alpha^k$ is in the triangulated closure of $\tilde{S}_\alpha$. Also, $\tilde{M}_0^k$ and $\tilde{N}_0^k$ are in the triangulated closure of S_0.*

Proof. The fact that the $\tilde{M}_\alpha^k$ can be generated by $\tilde{S}_\alpha$ follows inductively from the short exact sequences

$$0 \to \tilde{M}_\alpha^{k-1} \to \tilde{M}_\alpha^k \to \tilde{S}_\alpha \to 0,$$

in a manner analogous to the proof of Lemma 6.4. The short exact sequences

$$0 \to \tilde{N}_0^{k-1} \to \tilde{N}_0^k \to \tilde{S}_0 \to 0$$

can be used to show that the $\tilde{N}_0^k$ can be generated by S_0 and $\tilde{S}_0$. Finally, the short exact sequence

$$0 \to S_0 \to \tilde{S}_0 \to S_0[1] \to 0$$

implies that $\tilde{S}_0$ is in the triangulated closure of S_0. $\square$

We can now conclude the following.

Corollary 6.8. *The category $\mathrm{mod}_{pr}(A)$ is split-generated by the collection of modules $\{S_0\} \cup \{\tilde{S}_\alpha\}_{\alpha \in \mathbb{K} \backslash 0}$.*

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